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Abstract

Based on previous homogenization results for imperfect transmission problems in two-component domains with periodic microstructure, we derive quantitative estimates for the difference between the microscopic and macroscopic solution. This difference is of order ε^ρ, where $\varepsilon > 0$ describes the periodicity of the microstructure and $\rho \in (0, \frac{1}{2}]$ depends on the transmission condition at the interface between the two components. The corrector estimates are proved without assuming additional regularity for the local correctors using the periodic unfolding method.

1 Introduction

This paper considers a class of linear elliptic equations with an imperfect transmission condition modeling, for instance, heat conduction, diffusion, or stationary current flow in a composite material. The macroscopic domain Ω consists of a connected component Ω_1^ε and a second component Ω_2^ε, which is the collection of periodically distributed inclusions or pores. The characteristic length scale of the microstructure, given by the distance between two such pores, is of order ε. On the interface Γ^ε between the two components, the flux is proportional to the jump of the solution across this interface, which models e.g. a contact resistance. The corresponding proportionality factor is of order ε^γ with $\gamma \in \mathbb{R}$. More precisely, we consider for $\varepsilon > 0$ the problem

$$\begin{aligned}
-\operatorname{div}(A^\varepsilon \nabla u_1^\varepsilon) &= f && \text{in } \Omega_1^\varepsilon, \\
-\operatorname{div}(A^\varepsilon \nabla u_2^\varepsilon) &= f && \text{in } \Omega_2^\varepsilon, \\
A^\varepsilon \nabla u_1^\varepsilon \cdot n_1^\varepsilon &= -A^\varepsilon \nabla u_2^\varepsilon \cdot n_2^\varepsilon && \text{on } \Gamma^\varepsilon, \\
-A^\varepsilon \nabla u_1^\varepsilon \cdot n_1^\varepsilon &= \varepsilon^\gamma h^\varepsilon \left(u_1^\varepsilon - u_2^\varepsilon\right) && \text{on } \Gamma^\varepsilon, \\
u_1^\varepsilon &= 0 && \text{on } \partial\Omega,
\end{aligned} \tag{1.1}$$

where n_i^ε is the unit outer normal to Ω_i^ε for $i = 1, 2$. The matrix $A^\varepsilon(x) \in \mathbb{R}^{d\times d}_{\text{sym}}$ and the coefficient $h^\varepsilon(x)$ are ε-periodic and uniformly bounded. Moreover, we suppose that A^ε is uniformly elliptic, h^ε is strictly positive, and the source term f is square-integrable.

For ε tending to zero, the homogenization limit of problem (1.1) has already been well studied in the literature. Based on Tartar's method for oscillating test functions, the homogenization limit was derived for all $\gamma \leq 1$ in [Mon03, DoM04]; see references therein for earlier works. This problem was also treated for two connected components [CaP97], poly crystals [Hum00], for stochastic microstructures [Hei11], and evolution problems [DFM07, Jos09]. Recently, the homogenization limit and strong two-scale convergence of the gradients were proved in [DLNT11] for $\gamma \leq 1$ via the method of periodic unfolding. However, up to now all publications contain qualitative results, whereas, this paper provides quantitative corrector estimates.

In the limit $\varepsilon \to 0$, we obtain one homogenized elliptic equation posed in the whole macroscopic domain

$$\begin{aligned} -\operatorname{div}(A_\gamma^0 \nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

The constant matrix $A_\gamma^0 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ depends on γ, whereby we distinguish the following three cases: (i) for $-1 < \gamma \leq 1$, (ii) for $\gamma = -1$, and (iii) for $\gamma < -1$. The case $\gamma > 1$ is not treated here, since it allows for unbounded solutions as it is shown in [Hum00].

For $\gamma < -1$, the jump $u_1^\varepsilon - u_2^\varepsilon$ across the interface Γ^ε is negligibly small such that A_γ^0 is given via the standard unit cell problem on the whole reference cell, see (6.2) for more details. In other words, the model in (1.1) behaves for $\varepsilon \ll 1$ like a classical Poisson equation with ε-periodic coefficients in a one-component domain.

For $\gamma = -1$, the matrix A_{-1}^0 is obtained by solving a unit cell problem in two sub-domains of the reference cell separated by an interface, see (5.2). In this case, the unit cell problem reflects the structure of (1.1) on the level of the reference cell and the effective matrix takes the imperfect transmission condition into account. Indeed, this is the only case, where the effective matrix depends on the values of the boundary term h^ε.

For $-1 < \gamma \leq 1$, we obtain the same effective matrix A_γ^0 as in [CiP79], cf. (4.2), wherein the homogenization of the Poisson equation is considered in the perforated domain Ω_1^ε with no-flux boundary conditions at the holes. In their situation, the function f is multiplied by the ratio of the volume of the occupied domain Ω_1^ε divided by the total volume of Ω. However, this ratio does not appear in (1.2) which shows that the exchange between the two components is sufficient in order to take into account also the source term in Ω_2^ε.

The main result of this paper are the quantitative error estimates between the microscopic solution $(u_1^\varepsilon, u_2^\varepsilon)$, the macroscopic limit u, and their corresponding correctors for the three different cases: (i) in the Theorems 4.1 and 4.3, (ii) in Theorem 5.1, and (iii) in Theorem 6.3. In the special case $\gamma = 1$, the jump across the interface Γ^ε is of order $O(1)$ and it depends on f, h^ε, and the volume fraction of the component Ω_2^ε. To prove this result in Theorem 4.3, we require additional H^2-regularity for the source term f and that h^ε is constant. In the case $\gamma < -1$, Lipschitz continuity of the matrix A^ε is assumed for technical reasons.

For all $\gamma \leq 1$, the L^2-corrector estimates are of order $\varepsilon^{\rho(\gamma)}$ with $0 < \rho(\gamma) \leq 1/2$, and in the special cases $\gamma \in \{-1, 0, 1\}$ or $\gamma \leq -2$ we recover the maximal convergence rate $\rho = 1/2$. In order to prove these quantitative estimates, we need that the limit u is of higher H^2-regularity, whereas u_1^ε and u_2^ε as well as the local correctors are in general only H^1-regular. The derivation of the corrector estimates relies on the two-scale formulation of the limit (1.2) as in [DLNT11], the periodic unfolding method as in [CDG08, CD*12], and unfolding based error estimates as in [Gri04, Gri05, Rei16]. The key step of the proofs is to construct the correct approximating sequences via defining suitable recovery operators. Those operators recover the oscillations of the gradients in the components Ω_1^ε and Ω_2^ε from the macroscopic limit and the local correctors. Especially for the case (i), the new operator $\mathcal{H}_2^\varepsilon$ is introduced in (3.6) in order to capture the "flatness" of the gradient ∇u_2^ε within the inclusions.

The text is structured as follows. In Section 2, we present the model as well as all necessary assumptions and notations. In Section 3, the periodic unfolding method is introduced and we define the unfolding and folding (averaging) operator and, in particular, the recovery operators in Subsection 3.1. In the case (i), the corrector estimates are given in Section 4. Therein, we distinguish the cases $-1 < \gamma < 1$ (Theorem 4.1) and $\gamma = 1$ (Theorem 4.3) in the Subsections 4.1 and 4.2, respectively, although they share the same effective matrix. The remaining corrector estimates for the cases (ii) and (iii) are given in the Sections 5, and 6, respectively. We conclude the presentation with a brief discussion in Section 7 and a possible application to supercapacitors in Subsection 7.1.

2 The imperfect transmission problem posed in a two-component domain

Throughout the text we postulate the following assumptions on the domain and the periodic microstructure as shown in Figure 1.

(D1) The macroscopic domain Ω is a d-dimensional polytope with $d \geq 2$, i.e. it is

$$\Omega = \prod_{i=1}^{d} [a_i, b_i) \quad \text{with} \quad a_i < b_i \quad \text{and} \quad a_i, b_i \in \mathbb{Z}.$$

(D2) The *reference cell* $Y = [0,1)^d$ is the disjoint union of the subsets Y_1 and $\overline{Y_2}$, where $Y_2 \subset Y$ is open, connected, and satisfies $\operatorname{dist}(Y_2, \partial Y) > 0$. The inner boundary $\Gamma = \partial Y_2$ is Lipschitz continuous and it holds $Y_1 = Y \backslash \overline{Y_2}$.

(D3) Let $K_\varepsilon = \{\lambda \in \mathbb{Z}^d \,|\, \varepsilon\lambda \in \Omega\}$ denote the set of nodal points inside Ω. The two disjoint components Ω_1^ε and Ω_2^ε, and their common boundary Γ^ε are given via

$$\Omega_1^\varepsilon = \bigcup_{\lambda \in K_\varepsilon} \varepsilon(\lambda + Y_1), \quad \Omega_2^\varepsilon = \bigcup_{\lambda \in K_\varepsilon} \varepsilon(\lambda + Y_2), \quad \text{and} \quad \Gamma^\varepsilon = \partial\Omega_2^\varepsilon.$$

(D4) The microscopic period $\varepsilon > 0$ is given via $\varepsilon = 1/n$ with $n \in \mathbb{N}$ such that Ω is the exact union of translated cells $\varepsilon(\lambda + Y)$ with $\lambda \in K_\varepsilon$ for all ε.

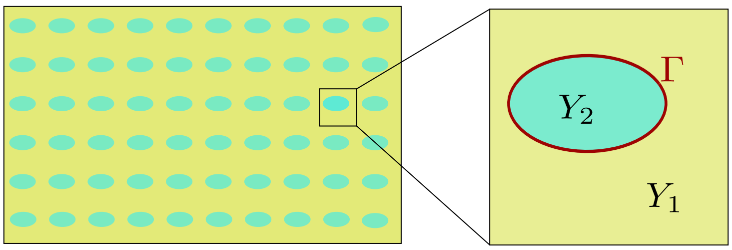

Figure 1: The two-component domain Ω (left) and the reference cell Y (right).

By construction, the set Ω_1^ε is *connected*, bounded, and has a Lipschitz boundary, whereas the set Ω_2^ε consists of isolated inclusions. The former implies the existence of extension operators mapping from $\mathrm{H}^1_\mathrm{D}(\Omega_1^\varepsilon)$ to $\mathrm{H}^1_\mathrm{D}(\Omega)$, where the subscript D indicates homogeneous Dirichlet boundary conditions. Indeed, $\partial\Omega_1^\varepsilon$ is the disjoint union of Γ^ε and $\partial\Omega$, and it is

$$\mathrm{H}^1_\mathrm{D}(\Omega_1^\varepsilon) = \left\{u \in \mathrm{H}^1(\Omega_1^\varepsilon) \,|\, u = 0 \text{ on } \partial\Omega\right\}.$$

The assumption that Y_2 does not touch the boundary of the reference cell Y is essential for the construction of the recovery operator $\mathcal{H}_2^\varepsilon$ in (3.6), see also Remark 3.1. This assumption is also contained in [DLNT11].

Remark 2.1. *The Assumption (D4) significantly simplifies the presentation of the corrector estimates, however, the results remain valid for arbitrary domains Ω with smooth boundary and $\varepsilon \in \mathbb{R}$. In such a case, one considers bigger domains $\widetilde{\Omega}_i^\varepsilon$, $i = 1, 2$, satisfying again (D2)–(D4) and uses Lemma A.6 to control the error at the boundary (cf. [Gri04, Gri05, Rei16]). In any case, we have to avoid cells $\varepsilon(\lambda + Y)$ intersecting the boundary $\partial\Omega$ such that Ω_1^ε is always a Lipschitz domain (cf. also [DLNT11, Fig. 2]).*

In order to obtain unique and bounded solutions for the microscopic respective homogenized problem, we require the following assumptions for the given data. The dot "$\cdot$" always denotes the scalar product in $\mathbb{R}^d$.

(A1) The matrix $A \in \mathrm{L}^\infty(Y; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is Y-periodic, symmetric, and uniformly elliptic, i.e.

$$\exists\, \alpha > 0: \quad A(y)\xi\cdot\xi \geq \alpha|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, \text{ and a.a. } y \in Y.$$

(A2) The boundary term h is a Y-periodic function in $\mathrm{L}^\infty(\Gamma)$ and satisfies

$$\exists\, h_0 > 0: \quad h\,(y) \geq h_0 \quad \text{for a.a. } y \in \Gamma.$$

(A3) It is $f \in \mathrm{L}^2(\Omega)$ and the coefficients of the microscopic problem are given via

$$A^\varepsilon(x) = A\left(\tfrac{x}{\varepsilon}\right) \quad \text{and} \quad h^\varepsilon(x) = h\left(\tfrac{x}{\varepsilon}\right).$$

Under the above assumptions, the Lax–Milgram theorem yields the existence of a unique solution $(u_1^\varepsilon, u_2^\varepsilon)$ for the weak formulation of the microscopic problem (1.1) (cf. [Mon03, Sec. 1]), i.e. find $(u_1^\varepsilon, u_2^\varepsilon) \in \mathrm{H}^1_{\mathrm{D}}(\Omega_1^\varepsilon) \times \mathrm{H}^1(\Omega_2^\varepsilon)$ such that

$$\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon\cdot\nabla\varphi_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon\cdot\nabla\varphi_2 \,\mathrm{d}x + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon\,(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1 - \varphi_2)\,\mathrm{d}\sigma_x = \int_{\Omega_1^\varepsilon} f\varphi_1\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f\varphi_2\,\mathrm{d}x \qquad (2.1)$$

for all admissible test functions $(\varphi_1, \varphi_2) \in \mathrm{H}^1_{\mathrm{D}}(\Omega_1^\varepsilon) \times \mathrm{H}^1(\Omega_2^\varepsilon)$. Moreover, this solution satisfies the following *a priori* bounds.

Proposition 2.2 ([Mon03, Prop. 3.1])**.** *Any solution of the microscopic problem* (1.1) *is bounded for all $\gamma \leq 1$ via*

$$\|u_1^\varepsilon\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{\gamma}{2}}\|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \leq C,$$

where the constant $C > 0$ is independent of ε.

The solution $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$ of the homogenized problem (1.2) is unique and bounded, too. Moreover, u is of higher regularity, since the macroscopic domain Ω is a bounded convex polytope. Indeed, it holds according to [Gri85, Thm. 3.2.1.3]

$$\|u\|_{\mathrm{H}^2(\Omega)} \leq C\|f\|_{\mathrm{L}^2(\Omega)},$$

where $C > 0$ only depends on the effective matrix $A^0_\gamma \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and the domain Ω. The precise definition of A^0_γ depends on the three different regimes (i)–(iii) for γ and it is given in the corresponding section.

3 Periodic unfolding

Following [CDZ06, DLNT11, CD*12], we define the *periodic unfolding operators* $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$, which map one-scale functions on the oscillating domains Ω_1^ε and Ω_2^ε to two-scale functions on the fixed domains $\Omega\times Y_1$ and $\Omega\times Y_2$, respectively. Therefore, let $x=[x]+\{x\}$ denote the standard two-scale decomposition of every $x\in\mathbb{R}^d$ into its integer part $[x]\in\mathbb{Z}^d$ and the remainder $\{x\}:=x-[x]\in Y$. For any Lebesgue measurable function u on Ω_1^ε the periodic unfolding operator $\mathcal{T}_1^\varepsilon$ is given via

$$(\mathcal{T}_1^\varepsilon\, u)(x,y):=u\left(\varepsilon\left[\tfrac{x}{\varepsilon}\right]+\varepsilon y\right)\quad \text{for a.a. } (x,y)\in\Omega\times Y_1$$

and it satisfies the integration formula

$$\int_{\Omega_1^\varepsilon} u\,\mathrm{d}x=\int_{\Omega\times Y_1}\mathcal{T}_1^\varepsilon\, u\,\mathrm{d}x\,\mathrm{d}y \quad \text{for } u\in \mathrm{L}^1(\Omega_1^\varepsilon). \tag{3.1}$$

Within the inclusions, we define for any Lebesgue measurable function u on Ω_2^ε the second periodic unfolding operator $\mathcal{T}_2^\varepsilon$ by (cf. [DLNT11, Def. 2.8])

$$(\mathcal{T}_2^\varepsilon\, u)(x,y):=u\left(\varepsilon\left[\tfrac{x}{\varepsilon}\right]+\varepsilon y\right)\quad \text{for a.a. } (x,y)\in\Omega\times Y_2.$$

In particular, both periodic unfolding operators are well-defined for functions u on Ω via the relation $\mathcal{T}_i^\varepsilon u=\mathcal{T}_i^\varepsilon(\chi_{\Omega_i^\varepsilon}u)$, for $i=1,2$, with $\chi_{\Omega_i^\varepsilon}$ denoting the characteristic function of the set Ω_i^ε. Moreover, the restriction of $\mathcal{T}_i^\varepsilon$ to Lebesgue measurable functions u on Γ^ε is also well-defined and it is $\mathcal{T}_1^\varepsilon\, u=\mathcal{T}_2^\varepsilon\, u$ almost everywhere in $\Omega\times\Gamma$. We recall that for Sobolev functions $u_i\in\mathrm{W}^{1,p}(\Omega_i^\varepsilon)$ the unfolding $\mathcal{T}_i^\varepsilon u_i$ belongs to the space $\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}(Y_i))$, for all $1\le p\le\infty$ and $i=1,2$. Then, if the traces of u_1 and u_2 coincide in $\mathrm{L}^p(\Gamma^\varepsilon)$, so do the traces of $\mathcal{T}_1^\varepsilon\, u_1$ and $\mathcal{T}_2^\varepsilon\, u_2$ in $\mathrm{L}^p(\Omega\times\Gamma)$. There holds the following integration formula for boundary unfolding (cf. [CDZ06, Prop. 5.2])

$$\varepsilon\int_{\Gamma^\varepsilon}u\,\mathrm{d}\sigma_x=\int_{\Omega\times\Gamma}\mathcal{T}_1^\varepsilon u\,\mathrm{d}x\,\mathrm{d}\sigma_y=\int_{\Omega\times\Gamma}\mathcal{T}_2^\varepsilon u\,\mathrm{d}x\,\mathrm{d}\sigma_y \quad \text{for } u\in\mathrm{L}^1(\Gamma^\varepsilon). \tag{3.2}$$

We complete this collection by introducing the *folding operator* (also called averaging operator) $\mathcal{F}_i^\varepsilon$: $\mathrm{L}^p(\Omega\times Y_i)\to\mathrm{L}^p(\Omega_i^\varepsilon)$ [1] for $i=1,2$ and $1\le p<\infty$ via

$$(\mathcal{F}_i^\varepsilon U)(x):=\fint_{\varepsilon\left(\left[\frac{x}{\varepsilon}\right]+Y\right)}U\left(z,\{\tfrac{x}{\varepsilon}\}\right)\,\mathrm{d}z\quad \text{for a.a. } x\in\Omega_i^\varepsilon,$$

where $\fint_{\mathcal{O}}=|\mathcal{O}|^{-1}\int_{\mathcal{O}}$ denotes the usual average over the domain $\mathcal{O}\subset\mathbb{R}^d$. Here, and in the following, $|\mathcal{O}|$ denotes the d-dimensional Lebesgue measure of domains respective the $d{-}1$-dimensional Lebesgue measure of hypersurfaces.

According to [DLNT11, CD*12], the folding operator $\mathcal{F}_1^\varepsilon$ is the adjoint of $\mathcal{T}_1^\varepsilon$, i.e.

$$\int_{\Omega_1^\varepsilon}(\mathcal{F}_1^\varepsilon\, U)v\,\mathrm{d}x=\int_{\Omega\times Y_1}U(\mathcal{T}_1^\varepsilon\, v)\,\mathrm{d}x\,\mathrm{d}y\quad \text{for } U\in\mathrm{L}^2(\Omega\times Y_1),\ v\in\mathrm{L}^2(\Omega_1^\varepsilon).$$

In the same manner, $\mathcal{F}_2^\varepsilon$ is the adjoint of $\mathcal{T}_2^\varepsilon$. Finally, we note that the periodic unfolding respective averaging operator, $\mathcal{T}_\varepsilon$ and $\mathcal{F}_\varepsilon$, as introduced in [CDG08] are given via

$$\mathcal{T}_\varepsilon\, u=\begin{cases}\mathcal{T}_1^\varepsilon\, u & \text{in } \Omega\times Y_1\\ \mathcal{T}_2^\varepsilon\, u & \text{in } \Omega\times Y_2\end{cases}\quad \text{and}\quad \mathcal{F}_\varepsilon\, U=\begin{cases}\mathcal{F}_1^\varepsilon(U|_{\Omega\times Y_1}) & \text{in } \Omega_1^\varepsilon\\ \mathcal{F}_2^\varepsilon(U|_{\Omega\times Y_2}) & \text{in } \Omega_2^\varepsilon\end{cases}.$$

[1] Note that $\mathrm{L}^p(\Omega\times Y)$ and $\mathrm{L}^p(\Omega;\mathrm{L}^p(Y))$ can be identified for $1\le p<\infty$, whereas this fails for $p=\infty$.

3.1 Construction of recovery operators

In this section we introduce two operators, $\mathcal{G}_\varepsilon$ and $\mathcal{H}_2^\varepsilon$, which will help us to construct suitable *recovery* respective *approximating sequences* for the derivation of the corrector estimates. To do so, we define the *scale-splitting operator* $\mathcal{Q}_\varepsilon : \mathrm{H}^1(\Omega) \to \mathrm{W}^{1,\infty}(\mathbb{R}^d)$ following [CDG08, Def. 4.1]. Let $\widetilde{u} \in \mathrm{H}^1(\mathbb{R}^d)$ denote the extension of $u \in \mathrm{H}^1(\Omega)$ according to [Neč67, Thm. 3.9]. For $x \in \varepsilon([x/\varepsilon]+Y)$ and every $\kappa = (\kappa_1, \ldots, \kappa_d) \in \{0,1\}^d$, we set

$$\bar{x}_l^{(\kappa_l)} := \begin{cases} \frac{x_l - \varepsilon[x/\varepsilon]_l}{\varepsilon} & \text{if } \kappa_l = 1 \\ 1 - \frac{x_l - \varepsilon[x/\varepsilon]_l}{\varepsilon} & \text{if } \kappa_l = 0 \end{cases}$$

and

$$(\mathcal{Q}_\varepsilon\, u)(x) := \sum_{\kappa \in \{0,1\}^d} (\mathcal{F}_\varepsilon\, \widetilde{u}) \left(\varepsilon[\tfrac{x}{\varepsilon}] + \varepsilon\kappa\right) \cdot \bar{x}_1^{(\kappa_1)} \cdots \bar{x}_d^{(\kappa_d)}.$$

The function $\mathcal{Q}_\varepsilon\, u$ interpolates the values of $\mathcal{F}_\varepsilon\, u$ at the nodes $\varepsilon[x/\varepsilon]$ via $\mathcal{Q}_1$-Lagrange elements as customary in the finite elements methods. Since $\mathcal{Q}_\varepsilon\, u$ is (weakly) differentiable, in contrast to $\mathcal{F}_\varepsilon\, u$, we can use the scale-splitting operator to construct oscillating one-scale functions that recover global corrector-type functions $\widehat{u}(x,y) = \sum_{i=1}^d \frac{\partial u}{\partial x_i}(x) \cdot \chi_i(y)$. Let $\mathrm{H}^1_{\mathrm{per}}(Y)$ respective $\mathrm{H}^1_{\mathrm{per}}(Y_1)$ denote the space of Y-periodic Sobolev functions, i.e.

$$\mathrm{H}^1_{\mathrm{per}}(Y_1) = \left\{ \varphi \in \mathrm{H}^1(Y_1) \,|\, \varphi \text{ is } Y\text{-periodic} \right\}.$$

For any Y-periodic function, we may identify $x \mapsto \varphi(x)$ with $x \mapsto \varphi(\{x\})$ for all $x \in \mathbb{R}^d$.

For $u \in \mathrm{H}^2(\Omega)$ and $\chi \in \mathrm{H}^1_{\mathrm{per}}(Y)^d := \mathrm{H}^1_{\mathrm{per}}(Y;\mathbb{R}^d)$, the approximating sequence $(\mathcal{G}_\varepsilon\, \widehat{u})_\varepsilon \subset \mathrm{H}^1_{\mathrm{D}}(\Omega)$ is given via (cf. [Gri04])

$$\left(\mathcal{G}_\varepsilon\, \widehat{u}\right)(x) := \varrho_\varepsilon(x) \sum_{i=1}^d \mathcal{Q}_\varepsilon \left(\frac{\partial u}{\partial x_i}\right)(x) \cdot \chi_i\left(\tfrac{x}{\varepsilon}\right). \tag{3.3}$$

The cut-off function $\varrho_\varepsilon \in \mathrm{C}_{\mathrm{c}}^\infty(\Omega;[0,1])$ satisfies $\varrho_\varepsilon(x) \equiv 1$ for all $x \in \Omega$ with $\operatorname{dist}(x,\partial\Omega) > \varepsilon$ and $|\nabla \varrho_\varepsilon| \leq \frac{c_0}{\varepsilon}$; and it guarantees the Dirichlet boundary condition on $\partial\Omega$. We may also call $\mathcal{G}_\varepsilon : \left(\mathrm{H}^2(\Omega), \mathrm{H}^1_{\mathrm{per}}(Y)^d\right) \to \mathrm{H}^1_{\mathrm{D}}(\Omega)$ *recovery* respective *gradient folding operator*, since it holds $\mathcal{T}_\varepsilon(\mathcal{G}_\varepsilon\, \widehat{u}) \to \widehat{u}$ and $\mathcal{T}_\varepsilon[\varepsilon\nabla(\mathcal{G}_\varepsilon\, \widehat{u})] \to \nabla_y \widehat{u}$ in $\mathrm{L}^2(\Omega \times Y)$. The uniform boundedness of the scale-splitting operator $\|\, \mathcal{Q}_\varepsilon\, u\|_{\mathrm{H}^1(\Omega)} \leq C\|u\|_{\mathrm{H}^1(\Omega)}$ according to [CDG08, Prop. 4.5] implies the following bound

$$\|\, \mathcal{G}_\varepsilon\, \widehat{u}\|_{\mathrm{L}^2(\Omega)} + \varepsilon \|\nabla(\mathcal{G}_\varepsilon\, \widehat{u})\|_{\mathrm{L}^2(\Omega)} \leq C\|u\|_{\mathrm{H}^2(\Omega)} \|\chi\|_{\mathrm{H}^1(Y_1)^d}.$$

On the perforated domain Ω_1^ε, we adjust the construction of the approximating sequence as follows: for $\chi^1 \in \mathrm{H}^1_{\mathrm{per}}(Y_1)^d$, the sequence $(\mathcal{G}_1^\varepsilon\, \widehat{u})_\varepsilon \subset \mathrm{H}^1_{\mathrm{D}}(\Omega_1^\varepsilon)$ is given via

$$\left(\mathcal{G}_1^\varepsilon\, \widehat{u}\right)(x) := \varrho_\varepsilon(x) \sum_{i=1}^d \mathcal{Q}_\varepsilon \left(\frac{\partial u}{\partial x_i}\right)\Bigg|_{\Omega_1^\varepsilon}(x) \cdot \chi_i^1\left(\tfrac{x}{\varepsilon}\right). \tag{3.4}$$

In the same manner, we define for $\chi^2 \in \mathrm{H}^1(Y_2)^d$ the sequence $(\mathcal{G}_2^\varepsilon\, \widehat{u})_\varepsilon \subset \mathrm{H}^1(\Omega_2^\varepsilon)$ via

$$\left(\mathcal{G}_2^\varepsilon\, \widehat{u}\right)(x) := \sum_{i=1}^d \mathcal{Q}_\varepsilon \left(\frac{\partial u}{\partial x_i}\right)\Bigg|_{\Omega_2^\varepsilon}(x) \cdot \chi_i^2\left(\left\{\tfrac{x}{\varepsilon}\right\}\right). \tag{3.5}$$

Notice that ϱ_ε is skipped in (3.5), since the inclusions Ω_2^ε is not equipped with Dirichlet boundary conditions. Here, we also use $\{x/\varepsilon\}$, since χ_i^2 is in general not Y-periodic.

We introduce the second recovery operator $\mathcal{H}_2^\varepsilon$ in the inclusions as follows: define

$$\mathcal{H} : \mathrm{H}^2(\Omega) \to \mathrm{H}^1(\Omega; \mathrm{C}^\infty(Y_2)); \qquad (\mathcal{H}u)(x, y) := \nabla u(x)\cdot y$$

and observe that the gradient $\nabla_y(\mathcal{H}u) = \nabla u$ is constant with respect to all $y \in Y_2$. Recall that for every $x \in \Omega_2^\varepsilon$, the two-scale decomposition gives $x = \varepsilon([x/\varepsilon] + \{x/\varepsilon\})$ with $[x/\varepsilon] \in \mathbb{Z}^d$ and $\{x/\varepsilon\} \in Y_2$. With this, the recovery operator $\mathcal{H}_2^\varepsilon$, given via

$$\mathcal{H}_2^\varepsilon : \mathrm{H}^2(\Omega) \to \mathrm{H}^1(\Omega_2^\varepsilon); \qquad (\mathcal{H}_2^\varepsilon\, u)\,(x) := \nabla u(x)\cdot\left\{\tfrac{x}{\varepsilon}\right\}, \tag{3.6}$$

is well-defined and it holds $\|\,\mathcal{H}_2^\varepsilon\, u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon\|\nabla(\mathcal{H}_2^\varepsilon\, u)\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \le 2\|\nabla u\|_{\mathrm{H}^1(\Omega)}$. Moreover, we recover the convergences $\mathcal{T}_2^\varepsilon(\mathcal{H}_2^\varepsilon\, u) \to \mathcal{H}u$ and $\mathcal{T}_2^\varepsilon(\varepsilon\nabla\,\mathcal{H}_2^\varepsilon\, u) \to \nabla u$ in $\mathrm{L}^2(\Omega \times Y_2)$.

Remark 3.1. *We point out that $\mathcal{H}u$ is in general not Y-periodic. However, since it holds $\mathrm{dist}(Y_2, \partial Y) > 0$, we can periodically extend $\mathcal{H}u$ to $\widetilde{\mathcal{H}}u \in \mathrm{H}^1(\Omega; \mathrm{C}^\infty_{\mathrm{per}}(\overline{Y}))$ and, by translation, also to the whole space $\mathrm{C}^\infty(\mathbb{R}^d)$. With this, we may also construct $(\widetilde{\mathcal{H}}_2^\varepsilon u)_\varepsilon \subset \mathrm{H}^1(\Omega)$ on the whole domain Ω. Notice that this construction fails in the case $\partial Y_2 \cap \partial Y \neq \emptyset$.*

Otherwise, if Ω_1^ε and Ω_2^ε are connected for $d \ge 3$, there also exists a suitable extension operator $\mathcal{E}_2^\varepsilon : \mathrm{H}^1_{\mathrm{D}}(\Omega_2^\varepsilon) \to \mathrm{H}^1_{\mathrm{D}}(\Omega)$ and we may treat u_2^ε in a similar manner as u_1^ε.

4 Corrector estimates for $-1 < \gamma \le 1$

We begin with recalling the two-scale convergence of the solutions $(u_1^\varepsilon, u_2^\varepsilon)_\varepsilon$ of the microscopic problem (1.1) as it is shown in [DLNT11, Sec. 3.2 & 4.3]. There exist limit functions $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$ and $\widehat{u}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y_1))$ with $\int_{Y_1} \widehat{u}_1\,\mathrm{d}y = 0$ such that

$$\begin{aligned}&\mathcal{T}_1^\varepsilon\, u_1^\varepsilon \to u \text{ strongly in } \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_1)), \quad \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \to \nabla u + \nabla_y \widehat{u}_1 \text{ strongly in } \mathrm{L}^2(\Omega \times Y_1),\\ &\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \to 0 \text{ strongly in } \mathrm{L}^2(\Omega \times Y_2).\end{aligned}$$

Moreover, we distinguish the following two cases

$$\begin{aligned}\text{if } -1 < \gamma < 1 : &\quad \mathcal{T}_2^\varepsilon\, u_2^\varepsilon \rightharpoonup u \text{ weakly in } \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_2)),\\ \text{if } \gamma = 1 : &\quad \mathcal{T}_2^\varepsilon\, u_2^\varepsilon \rightharpoonup u + \theta f \text{ weakly in } \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_2)),\end{aligned}$$

where $\theta := |Y_2|(\int_\Gamma h\,\mathrm{d}\sigma_y)^{-1}$. The quantity θf characterizes the jump $u_1^\varepsilon - u_2^\varepsilon$ across the interface Γ^ε. In the limit $\varepsilon \to 0$, the pair $(u, \widehat{u}_1)$ solves the weak two-scale formulation

$$\int_{\Omega\times Y_1} A(y)[\nabla u + \nabla_y \widehat{u}_1]\cdot[\nabla\varphi + \nabla_y \Phi]\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f\varphi\,\mathrm{d}x \tag{4.1}$$

for all $\varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$ and $\Phi \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y_1))$. The macroscopic function u is in particular the solution of the homogenized equation (1.2), wherein the effective matrix $A_\gamma^0 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is constant and it is given for all $-1 < \gamma \le 1$ via the formula

$$A_\gamma^0 e_i := \int_{Y_1} A(y)(e_i + \nabla_y \chi_i^1)\,\mathrm{d}y.$$

Here, $\{e_1, \ldots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$ and $\chi_i^1 \in \mathrm{H}^1_{\mathrm{per}}(Y_1)$ are the local correctors. The latter are the solutions of the cell problem for $i = 1, \ldots, d$

$$\begin{aligned} -\operatorname{div}_y \left(A[e_i + \nabla_y \chi_i^1]\right) &= 0 && \text{in } Y_1, \\ A[e_i + \nabla_y \chi_i^1]\cdot n_1 &= 0 && \text{on } \Gamma, \\ \chi_i^1 \text{ is } Y\text{-perioidc}, \quad \textstyle\int_{Y_1} \chi_i^1 \,\mathrm{d}y &= 0, \end{aligned} \tag{4.2}$$

where n_1 denotes the unit outer normal to Y_1. We point out that the effective matrix and the local correctors only depend on the values of $A(y)$ restricted to the subset Y_1. In other words, the values of $A(y)|_{Y_2}$ and $h(y)$ do not enter the limit problem, as if the second component Ω_2^ε contained only "empty space" in the first place. The corresponding global corrector $\widehat{u}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y_1))$ is given via the formula

$$\widehat{u}_1(x, y) := \sum_{i=1}^d \frac{\partial u}{\partial x_i}(x)\cdot \chi_i^1(y).$$

Note that the higher regularity of the limit solution $u \in \mathrm{H}^2(\Omega)$ implies also the higher x-regularity of the global corrector $\widehat{u}_1 \in \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y_1))$.

4.1 The case $-1 < \gamma < 1$

Theorem 4.1. *Let the assumptions (D1)–(D4) on the microstructure and (A1)–(A3) on the data hold true. Then, the solutions* $(u_1^\varepsilon, u_2^\varepsilon)$ *and* u *of the microscopic problem* (1.1) *and the homogenized equation* (1.2)*, respectively, satisfy for* $-1 < \gamma < 1$

$$\|u_1^\varepsilon - u - \varepsilon\, \mathcal{G}_1^\varepsilon\, \widehat{u}_1\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{\gamma}{2}} \|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \le \varepsilon^{\frac{1-|\gamma|}{2}} C, \tag{4.3}$$

where C *is a positive constant independent of* ε*.*

The derivation of the estimates follows the principle idea of the unfolding based estimates in [Gri04, Gri05]. In particular the control of the periodicity defect of $\mathcal{T}_1^\varepsilon\, \varphi \in \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_1))$, which is in general not Y-periodic for arbitrary functions $\varphi \in \mathrm{H}^1(\Omega_1^\varepsilon)$, is proved in these two articles.

Proof. By assumption it holds $|Y| = 1$.

Step 1: Periodicity defect. In the weak formulation (4.1), we choose the two-scale test function Φ^ε according to Theorem A.3 such that it holds

$$\|\, \mathcal{T}_1^\varepsilon(\nabla\varphi) - [\nabla\varphi + \nabla_y \Phi^\varepsilon]\|_{\mathrm{L}^2(Y_1; \mathrm{H}^1(\Omega)^*)} \le (\varepsilon + \varepsilon^{\frac{1}{2}}) C \|\varphi\|_{\mathrm{H}^1(\Omega)},$$

where $C > 0$ only depends on Ω and Y_1. Exploiting this estimate with the higher x-regularity of $A[\nabla u + \nabla_y \widehat{u}_1] \in \mathrm{H}^1(\Omega; \mathrm{L}^2(Y_1))$ [2] as well as the duality of periodic unfolding operator $\mathcal{T}_1^\varepsilon$ and folding operator $\mathcal{F}_1^\varepsilon$ yields with $\mathcal{F}_1^\varepsilon\, A = A^\varepsilon$

$$\left| \int_{\Omega_1^\varepsilon} A^\varepsilon\, \mathcal{F}_1^\varepsilon[\nabla u + \nabla_y \widehat{u}_1]\cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega f\varphi \,\mathrm{d}x \right| \le \varepsilon^{\frac{1}{2}} C \|\varphi\|_{\mathrm{H}^1(\Omega)}.$$

[2] Notice that the spaces $\mathrm{H}^1(\Omega; \mathrm{L}^2(Y_1))$ and $\mathrm{L}^2(Y_1; \mathrm{H}^1(\Omega))$ can be identified.

Using $u \in \mathrm{H}^2(\Omega)$, the definition of $\mathcal{G}_1^\varepsilon$ in (3.4), the boundedness of the linear operator $\mathcal{Q}_\varepsilon$ from $\mathrm{H}^1(\Omega)$ into itself, the assumptions $\varrho_\varepsilon(x) \in [0,1]$ and $|\nabla \varrho_\varepsilon| \le \frac{c_0}{\varepsilon}$, as well as the Lemmas A.1, A.5, and A.6 give

$$\begin{aligned}
&\| \mathcal{F}_1^\varepsilon[\nabla u + \nabla_y \widehat{u}_1] - [\nabla u_1 + \varepsilon \nabla(\mathcal{G}_1^\varepsilon \widehat{u}_1)]\|_{\mathrm{L}^2(\Omega_1^\varepsilon)} \\
&\le \| \mathcal{F}_1^\varepsilon(\nabla u) - \nabla u\|_{\mathrm{L}^2(\Omega_1^\varepsilon)} + c_0 \left\| \sum_{i=1}^d \mathcal{Q}_\varepsilon(\tfrac{\partial u}{\partial x_i}) \cdot \chi_i^1(\tfrac{\cdot}{\varepsilon}) \right\|_{\mathrm{L}^2(\mathcal{N}_\varepsilon(\partial\Omega))} \\
&\quad + \left\| \varepsilon \varrho_\varepsilon \sum_{i=1}^d \nabla \left[\mathcal{Q}_\varepsilon(\tfrac{\partial u}{\partial x_i}) \right] \cdot \chi_i^1(\tfrac{\cdot}{\varepsilon}) \right\|_{\mathrm{L}^2(\Omega_1^\varepsilon)} + \left\| \mathcal{F}_1^\varepsilon(\nabla_y \widehat{u}_1) - \varrho_\varepsilon \sum_{i=1}^d \mathcal{Q}_\varepsilon(\tfrac{\partial u}{\partial x_i}) \cdot \nabla_y \chi_i^1(\tfrac{\cdot}{\varepsilon}) \right\|_{\mathrm{L}^2(\Omega_1^\varepsilon)} \\
&\le \varepsilon^{\frac12} C \|u\|_{\mathrm{H}^2(\Omega)},
\end{aligned}$$

where $\mathcal{N}_\varepsilon(\partial\Omega) = \{x \in \Omega_1^\varepsilon \,|\, \mathrm{dist}(x, \partial\Omega) \le \varepsilon\}$ denotes the ε-neighborhood of the boundary $\partial\Omega$. We finish Step 1 with

$$\left| \int_{\Omega_1^\varepsilon} A^\varepsilon [\nabla u + \varepsilon \nabla(\mathcal{G}_1^\varepsilon \widehat{u}_1)] \cdot \nabla \varphi \, \mathrm{d}x - \int_\Omega f \varphi \, \mathrm{d}x \right| \le \varepsilon^{\frac12} C \|\varphi\|_{\mathrm{H}^1(\Omega)}. \tag{4.4}$$

Step 2: Admissible test functions. We test the weak formulation of the microscopic problem (2.1) with

$$\varphi_1^\varepsilon := u_1^\varepsilon - u - \varepsilon \mathcal{G}_1^\varepsilon \widehat{u}_1 \in \mathrm{H}_\mathrm{D}^1(\Omega_1^\varepsilon) \quad \text{and} \quad \varphi_2^\varepsilon := u_2^\varepsilon - u + \varepsilon \mathcal{H}_2^\varepsilon u \in \mathrm{H}^1(\Omega_2^\varepsilon) \tag{4.5}$$

and arrive at

$$\begin{aligned}
\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \cdot \nabla \varphi_1^\varepsilon \, \mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \cdot \nabla \varphi_2^\varepsilon \, \mathrm{d}x + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon \, (u_1^\varepsilon - u_2^\varepsilon)(\varphi_1^\varepsilon - \varphi_2^\varepsilon) \, \mathrm{d}\sigma_x& \\
- \int_{\Omega_1^\varepsilon} f \varphi_1^\varepsilon \, \mathrm{d}x - \int_{\Omega_2^\varepsilon} f \varphi_2^\varepsilon \, \mathrm{d}x = 0.&
\end{aligned} \tag{4.6}$$

According to Theorem A.7, the extension $\varphi := \mathcal{E}_1^\varepsilon \varphi_1^\varepsilon$ satisfying $\|\varphi_1\|_{\mathrm{H}^1(\Omega)} \le C \|\varphi_1^\varepsilon\|_{\mathrm{H}^1(\Omega_1^\varepsilon)}$ is an admissible test function for the limit problem in (4.4). Subtracting (4.6) from the left-hand side in estimate (4.4) and recalling $\varphi|_{\Omega_1^\varepsilon} = \varphi_1^\varepsilon$ gives

$$\begin{aligned}
&\left| \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla \varphi_1^\varepsilon \cdot \nabla \varphi_1^\varepsilon \, \mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \cdot \nabla \varphi_2^\varepsilon \, \mathrm{d}x + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon \, (u_1^\varepsilon - u_2^\varepsilon)(\varphi_1^\varepsilon - \varphi_2^\varepsilon) \, \mathrm{d}\sigma_x \right| \\
&\le \left| \int_{\Omega_2^\varepsilon} f \, (\varphi - \varphi_2^\varepsilon) \, \mathrm{d}x \right| + \varepsilon^{\frac12} C \|\varphi_1^\varepsilon\|_{\mathrm{H}^1(\Omega_1^\varepsilon)}.
\end{aligned} \tag{4.7}$$

With Young's inequality and $\mu_1 > 0$ to be specified later, it holds

$$\varepsilon^{\frac12} \|\varphi_1^\varepsilon\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} \le \mu_1 \|\varphi_1^\varepsilon\|_{\mathrm{H}^1(\Omega_1^\varepsilon)}^2 + \varepsilon C(\mu_1). \tag{4.8}$$

Step 3: Approximation errors. Inserting $\varphi_1^\varepsilon - \varphi_2^\varepsilon = u_1^\varepsilon - u_2^\varepsilon - \varepsilon(\mathcal{G}_1^\varepsilon \widehat{u}_1 + \mathcal{H}_2^\varepsilon u)$ into (4.7), we estimate the boundary term of lower order with $\mu_2 > 0$ and (A.2) via

$$\begin{aligned}
&\left| \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon \, (u_1^\varepsilon - u_2^\varepsilon) \varepsilon (\mathcal{G}_1^\varepsilon \widehat{u}_1 + \mathcal{H}_2^\varepsilon u) \, \mathrm{d}\sigma_x \right| \\
&\le \mu_2 \varepsilon^\gamma \|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)}^2 + \varepsilon^{2+\gamma} C(\mu_2) \| \mathcal{G}_1^\varepsilon \widehat{u}_1 + \mathcal{H}_2^\varepsilon u\|_{\mathrm{L}^2(\Gamma^\varepsilon)}^2 \\
&\le \mu_2 \varepsilon^\gamma \|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)}^2 + \varepsilon^{1+\gamma} C(\mu_2) \|u\|_{\mathrm{H}^2(\Omega)}^2.
\end{aligned} \tag{4.9}$$

The source term on the right-hand side of (4.7) is estimated with (A.1) and $\mu_3 > 0$

$$\begin{aligned}&\left|\int_{\Omega_2^\varepsilon} f\,[\mathcal{E}_1^\varepsilon\, u_1^\varepsilon - u_2^\varepsilon - \varepsilon(\mathcal{E}_1^\varepsilon(\mathcal{G}_1^\varepsilon\,\widehat{u}_1) + \mathcal{H}_2^\varepsilon\, u)]\,\mathrm{d}x\right| \\ &\leq C\left(\|\,\mathcal{E}_1^\varepsilon\, u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon\|u\|_{\mathrm{H}^1(\Omega)}\right)\|f\|_{\mathrm{L}^2(\Omega)} \\ &\leq C\left(\varepsilon^{\frac{1}{2}}\|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} + \varepsilon\left\{\|\nabla(\mathcal{E}_1^\varepsilon\, u_1^\varepsilon)\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \|\nabla u_2^\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \|u\|_{\mathrm{H}^1(\Omega)}\right\}\right)\|f\|_{\mathrm{L}^2(\Omega)} \\ &\leq \mu_3\varepsilon^\gamma\|u_1^\varepsilon - u_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)} + \varepsilon^{1-\gamma}C(\mu_3) + \varepsilon C. \end{aligned} \tag{4.10}$$

Recalling that $\nabla\varphi_2^\varepsilon = \nabla u_2^\varepsilon - \nabla(u + \varepsilon\,\mathcal{H}_2^\varepsilon\, u)$, we control the ∇u_2^ε-term in (4.7) via

$$\left|\int_{\Omega_2^\varepsilon} A^\varepsilon\nabla u_2^\varepsilon\cdot[\nabla u - \varepsilon\nabla\,\mathcal{H}_2^\varepsilon\, u]\,\mathrm{d}x\right| \leq \mu_4\|\nabla u_2^\varepsilon\|^2_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon C(\mu_4)\|u\|_{\mathrm{H}^2(\Omega)}. \tag{4.11}$$

Here, we used that $\varepsilon\nabla(\mathcal{H}_2^\varepsilon\, u)(x) = \varepsilon\nabla^2 u(x)\{\frac{x}{\varepsilon}\} + \nabla u(x)$ for all $x \in \Omega_2^\varepsilon$ (with ∇^2 denoting the Hessian) according to (3.6) and, hence, it is

$$\|\nabla u - \varepsilon\nabla\,\mathcal{H}_2^\varepsilon\, u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \leq \varepsilon C\|u\|_{\mathrm{H}^2(\Omega)}.$$

Combining the error estimates in (4.9)–(4.11) with (4.7)–(4.8) gives

$$\begin{aligned}&\left|\int_{\Omega_1^\varepsilon} A^\varepsilon\nabla\varphi_1^\varepsilon\cdot\nabla\varphi_1^\varepsilon\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon\nabla u_2^\varepsilon\cdot\nabla u_2^\varepsilon\,\mathrm{d}x + \varepsilon^\gamma\int_{\Gamma^\varepsilon} h^\varepsilon\,(u_1^\varepsilon - u_2^\varepsilon)(u_1^\varepsilon - u_2^\varepsilon)\,\mathrm{d}\sigma_x\right| \\ &\leq \mu_1\|\varphi_1^\varepsilon\|^2_{\mathrm{H}^1(\Omega_1^\varepsilon)} + (\mu_2+\mu_3)\varepsilon^\gamma\|u_1^\varepsilon - u_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)} + \mu_4\|\nabla u_2^\varepsilon\|^2_{\mathrm{L}^2(\Omega_2^\varepsilon)} \\ &\quad + (\varepsilon + \varepsilon^{1+\gamma} + \varepsilon^{1-\gamma})C. \end{aligned} \tag{4.12}$$

Exploiting that A^ε is uniformly elliptic and $h^\varepsilon \geq h_0 > 0$, choosing $\mu_1 = \mu_4 = \alpha/2$ and $\mu_2 = \mu_3 = h_0/4$, as well as applying Poincaré–Friedrich's inequality to $\varphi_1^\varepsilon \in \mathrm{H}^1_{\mathrm{D}}(\Omega_1^\varepsilon)$ yields

$$\frac{\alpha}{2C_{\mathrm{PF}}}\|\varphi_1^\varepsilon\|^2_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \frac{\alpha}{2}\|\nabla u_2^\varepsilon\|^2_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon^\gamma\frac{h_0}{2}\|u_1^\varepsilon - u_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)} \leq \varepsilon^{1-|\gamma|}C.$$

The desired estimate (4.3) follows by taking the square root. □

Remark 4.2. *To see that the* $\mathrm{H}^1(\Omega_1^\varepsilon)$*-estimate in* (4.3) *is analogous to the* $\mathrm{H}^1(\Omega)$*-estimate in [Gri04, Prop. 4.3], we can control the term* $\|\varepsilon(1-\varrho_\varepsilon^{-1})\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1\|_{\mathrm{H}^1(\Omega_1^\varepsilon)}$ *by* $\sqrt{\varepsilon}C\|u\|_{\mathrm{H}^2(\Omega)}$ *as in Step 1 and obtain*

$$\left\|u_1^\varepsilon - u - \varepsilon\sum_{i=1}^d \mathcal{Q}_\varepsilon\left(\tfrac{\partial u}{\partial x_i}\right)\cdot\chi_i^1\left(\tfrac{\cdot}{\varepsilon}\right)\right\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon^\gamma\|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \leq \varepsilon^{\frac{1-|\gamma|}{2}}C.$$

4.2 The case $\gamma = 1$

In order to characterize the jump $u_1^\varepsilon - u_2^\varepsilon$ across the interface Γ^ε, we impose two additional assumptions on the given data, i.e.

$$f \in \mathrm{H}^2(\Omega) \qquad \text{and} \qquad h\,(y) \equiv h_0 \quad \text{for all } y \in \Gamma. \tag{4.13}$$

With this, we simply have $h^\varepsilon\,(x) \equiv h_0$ for all $x \in \Gamma^\varepsilon$ as well as $\theta = |Y_2|(h_0|\Gamma|)^{-1}$. The extra regularity for the source term f is needed to apply the recovery operator $\mathcal{H}_2^\varepsilon$ and concerning h's regularity we refer to Remark 4.4.

Theorem 4.3. *Let the assumptions of Theorem 4.1 as well as in* (4.13) *hold true. Then, there exists a positive constant C independent of ε such that it holds*

$$\|u_1^\varepsilon - u - \varepsilon\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{1}{2}}\|u_1^\varepsilon - u_2^\varepsilon + \theta f\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \leq \varepsilon^{\frac{1}{2}} C. \tag{4.14}$$

Proof. Step 1 of the proof is exactly as in the case $-1 < \gamma < 1$ and in what follows we only outline the modifications in Step 2 and 3.

Step 2: Admissible test functions. For the weak formulation of the microscopic problem, we choose the test functions φ_1^ε as in (4.5) and

$$\varphi_2^\varepsilon := u_2^\varepsilon - u_1 + \varepsilon\,\mathcal{H}_2^\varepsilon\,u - \theta(f - \varepsilon\,\mathcal{H}_2^\varepsilon\,f) \in \mathrm{H}^1(\Omega_2^\varepsilon).$$

Thus, we arrive at

$$\begin{aligned}\Bigg|\int_{\Omega_1^\varepsilon} A^\varepsilon\nabla\varphi_1^\varepsilon\cdot\nabla\varphi_1^\varepsilon\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon\nabla u_2^\varepsilon\cdot\nabla\varphi_2^\varepsilon\,\mathrm{d}x \\ +\varepsilon\int_{\Gamma^\varepsilon} h_0(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1^\varepsilon - \varphi_2^\varepsilon)\,\mathrm{d}\sigma_x + \int_{\Omega_2^\varepsilon} f\,(\mathcal{E}_1^\varepsilon\,\varphi_1^\varepsilon - \varphi_2^\varepsilon)\,\mathrm{d}x\Bigg| \leq \varepsilon^{\frac{1}{2}} C\|\varphi_1^\varepsilon\|_{\mathrm{H}^1(\Omega_1^\varepsilon)}.\end{aligned} \tag{4.15}$$

Step 3: Approximation errors. Inserting $\varphi_1^\varepsilon - \varphi_2^\varepsilon = u_1^\varepsilon - u_2^\varepsilon - \varepsilon(\mathcal{G}_1^\varepsilon\,\widehat{u}_1 + \mathcal{H}_2^\varepsilon\,u) + \theta(f - \varepsilon\,\mathcal{H}_2^\varepsilon\,f)$ into (4.15), we obtain for the boundary term

$$\begin{aligned}\varepsilon\int_{\Gamma^\varepsilon} h^\varepsilon\,(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1^\varepsilon - \varphi_2^\varepsilon)\,\mathrm{d}\sigma_x \geq \varepsilon h_0\|u_1^\varepsilon - u_2^\varepsilon + \theta f\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)} \\ - \varepsilon\int_{\Gamma^\varepsilon} h_0\theta f\,(u_1^\varepsilon - u_2^\varepsilon + \theta f)\,\mathrm{d}\sigma_x \\ - \varepsilon^2\int_{\Gamma^\varepsilon} h_0(u_1^\varepsilon - u_2^\varepsilon)[\mathcal{G}_1^\varepsilon\,\widehat{u}_1 + \mathcal{H}_2^\varepsilon\,u - \theta\,\mathcal{H}_2^\varepsilon\,f]\,\mathrm{d}\sigma_x.\end{aligned}$$

The absolute value of the third term (on the right-hand side) above is bounded by εC as in (4.9). For the source term in (4.15), we obtain

$$\begin{aligned}&\int_{\Omega_2^\varepsilon} f\,(\mathcal{E}_1^\varepsilon\,\varphi_1^\varepsilon - \varphi_2^\varepsilon)\,\mathrm{d}x \\ &= \int_{\Omega_2^\varepsilon} f\,(\mathcal{E}_1^\varepsilon\,u_1^\varepsilon - u_2^\varepsilon + \theta f)\,\mathrm{d}x + \varepsilon\int_{\Omega_2^\varepsilon} f\,[\mathcal{E}_1^\varepsilon(\mathcal{G}_1^\varepsilon\,\widehat{u}_1) - \mathcal{H}_2^\varepsilon\,u - \theta\,\mathcal{H}_2^\varepsilon\,f]\,\mathrm{d}x\end{aligned}$$

and again the absolute value of the second integral is bounded by εC. It remains to control the following difference using the integration formula (3.2)

$$\begin{aligned}&\int_{\Omega_2^\varepsilon} f\,(\mathcal{E}_1^\varepsilon\,u_1^\varepsilon - u_2^\varepsilon + \theta f)\,\mathrm{d}x - \varepsilon\int_{\Gamma^\varepsilon} h_0\theta f\,(u_1^\varepsilon - u_2^\varepsilon + \theta f)\,\mathrm{d}\sigma_x \\ &= \frac{1}{|\Gamma|}\int_{\Omega\times\Gamma} \chi_{\Omega_2^\varepsilon} f\,(\mathcal{E}_1^\varepsilon\,u_1^\varepsilon - u_2^\varepsilon + \theta f)\,\mathrm{d}x\,\mathrm{d}\sigma_y - \int_{\Omega\times\Gamma} h_0\,\mathcal{T}_2^\varepsilon(\theta f)\,\mathcal{T}_2^\varepsilon\left(\mathcal{E}_1^\varepsilon\,u_1^\varepsilon - u_2^\varepsilon - \theta f\right)\,\mathrm{d}x\,\mathrm{d}\sigma_y,\end{aligned}$$

where $\chi_{\Omega_2^\varepsilon}$ denotes the indicator function of the set Ω_2^ε. Recall that the traces of u_1^ε and $\mathcal{E}_1^\varepsilon\,u_1^\varepsilon$ coincide on Γ^ε and, hence, it holds $\mathcal{T}_1^\varepsilon\,u_1^\varepsilon = \mathcal{T}_1^\varepsilon(\mathcal{E}_1^\varepsilon\,u_1^\varepsilon)$ almost everywhere in $\Omega\times\Gamma$. After suitably rearranging the integrands, we get

$$
\begin{aligned}
&\Bigg| \frac{1}{|\Gamma|} \int_{\Omega\times\Gamma} \chi_{\Omega_2^\varepsilon} f \left\{ \mathcal{E}_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon + \theta f - \mathcal{T}_2^\varepsilon(\mathcal{E}_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon + \theta f) \right\} \mathrm{d}x \, \mathrm{d}\sigma_y \\
&\quad + \int_{\Omega\times\Gamma} \left\{ \frac{1}{|\Gamma|} \chi_{\Omega_2^\varepsilon} (f - \mathcal{T}_2^\varepsilon f) + \left(\frac{1}{|\Gamma|} \chi_{\Omega_2^\varepsilon} - h_0\theta \right) \mathcal{T}_2^\varepsilon f \right\} \mathcal{T}_2^\varepsilon(\mathcal{E}_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon + \theta f) \, \mathrm{d}x \, \mathrm{d}\sigma_y \Bigg| \\
&\le C \left(\| \mathcal{E}_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon + \theta f - \mathcal{T}_2^\varepsilon(\mathcal{E}_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon + \theta f) \|_{\mathrm{L}^2(\Omega_2^\varepsilon\times\Gamma)} + \| f - \mathcal{T}_2^\varepsilon f \|_{\mathrm{L}^2(\Omega_2^\varepsilon\times\Gamma)} \right) \\
&\quad + \left| \int_{\Omega\times\Gamma} \left\{ \left(\frac{1}{|\Gamma|} \chi_{\Omega_2^\varepsilon} - h_0\theta \right) \mathcal{T}_2^\varepsilon f \right\} \mathcal{T}_2^\varepsilon(\mathcal{E}_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon + \theta f) \, \mathrm{d}x \, \mathrm{d}\sigma_y \right| \qquad (4.16) \\
&\le \varepsilon C.
\end{aligned}
$$

Here, Lemma A.2 yields the $\mathrm{L}^2(\Omega_2^\varepsilon \times \Gamma)$-estimate for f and $\mathcal{E}_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon + \theta f$ belonging to the space $\mathrm{H}^1(\Omega_2^\varepsilon)$. Moreover, the integral term in (4.16) vanishes as follows: the function $g(x,y) := (\mathcal{T}_2^\varepsilon f)\, \mathcal{T}_2^\varepsilon(\mathcal{E}_1^\varepsilon u_1^\varepsilon - u_2^\varepsilon + \theta f)$ is constant with respect to x in each microscopic cell $\varepsilon(\lambda + Y)$ and it holds

$$
\frac{1}{|\Gamma|} \int_{\varepsilon(\lambda+Y)\times\Gamma} \chi_{\Omega_2^\varepsilon} g \, \mathrm{d}x \, \mathrm{d}\sigma_y = \varepsilon^d \frac{|Y_2|}{|\Gamma|} \int_\Gamma g|_{\varepsilon(\lambda+Y_2)} \, \mathrm{d}\sigma_y \qquad (4.17)
$$

as well as with $\theta = |Y_2|(h_0|\Gamma|)^{-1}$

$$
\int_{\varepsilon(\lambda+Y)\times\Gamma} h_0\theta g \, \mathrm{d}x \, \mathrm{d}\sigma_y = \varepsilon^d \frac{|Y_2|}{|\Gamma|} \int_\Gamma g|_{\varepsilon(\lambda+Y_2)} \, \mathrm{d}\sigma_y. \qquad (4.18)
$$

Since the difference (4.17)$-$(4.18) vanishes on each subset $\varepsilon(\lambda + Y) \times \Gamma \subset \Omega \times \Gamma$ and Ω is the exact union of translated cells, the whole integral vanishes in (4.16). Treating the gradient terms in (4.15) as in (4.8) and (4.11), we overall arrive at

$$
\frac{\alpha}{2C_{\mathrm{PF}}} \|\nabla\varphi_1^\varepsilon\|^2_{\mathrm{L}^2(\Omega_1^\varepsilon)} + \frac{\alpha}{2} \|\nabla u_2^\varepsilon\|^2_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon h_0 \|u_1^\varepsilon - u_2^\varepsilon - \theta f\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)} \le \varepsilon C,
$$

which gives estimate (4.14). ☐

Remark 4.4. *The extra assumption on the boundary function h^ε stems from the fact that the following equality only holds true for constant functions h*

$$
\frac{\int_\Gamma hg \, \mathrm{d}\sigma}{\int_\Gamma h \, \mathrm{d}\sigma} = \frac{1}{|\Gamma|} \int_\Gamma g \, \mathrm{d}\sigma \quad \textit{for all } g \in \mathrm{L}^2(\Gamma).
$$

This identity is needed for the equality of (4.17) *and* (4.18)*. So far, the only generalization for functions h^ε are small perturbations of order ε, i.e. $h^\varepsilon(x) = h_0 + \varepsilon h_1(x)$.*

5 Corrector estimates for $\gamma = -1$

This case is in some sense more special than (i) and (iii), since the limit problem depends indeed on all values of $A(y)$ in the whole reference cell Y *and* the boundary term $h(y)$. So, we recover on the level of the reference cell again an imperfect transmission problem, see (5.2). According to [DLNT11, Sec. 3.4 & 4.2], there exist *three* limit functions $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$ as well as $\widehat{u}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y_1))$ and $\widehat{u}_2 \in \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_2))$ with $\int_{Y_1} \widehat{u}_1 \,\mathrm{d}y = 0$ such that

$$\begin{aligned} \mathcal{T}_1^\varepsilon u_1^\varepsilon &\to u \text{ strongly in } \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_1)), \quad \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \to \nabla u + \nabla_y \widehat{u}_1 \text{ strongly in } \mathrm{L}^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon u_2^\varepsilon &\rightharpoonup u \text{ weakly in } \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_2)), \quad \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \to \nabla u + \nabla_y \widehat{u}_2 \text{ strongly in } \mathrm{L}^2(\Omega \times Y_2). \end{aligned}$$

Moreover, the triple $(u, \widehat{u}_1, \widehat{u}_2)$ solves the weak two-scale formulation

$$\begin{aligned} &\int_{\Omega\times Y_1} A(y)[\nabla u + \nabla_y \widehat{u}_1]\cdot[\nabla\varphi + \nabla_y \Phi_1] \,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{\Omega\times Y_2} A(y)[\nabla u + \nabla_y \widehat{u}_2]\cdot[\nabla\varphi + \nabla_y \Phi_2] \,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{\Omega\times\Gamma} h\,(y)(\widehat{u}_1 - \widehat{u}_2)(\Phi_1 - \Phi_2)\,\mathrm{d}x\,\mathrm{d}\sigma_y = \int_\Omega f\varphi\,\mathrm{d}x \end{aligned} \tag{5.1}$$

for $\varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$, $\Phi_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y_1))$, and $\Phi_2 \in \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_2))$. The macroscopic function u solves indeed the homogenized equation (1.2) and the effective matrix A^0_{-1} is given via

$$A^0_{-1} = A^1 + A^2 \quad \text{with} \quad A^k e_i := \int_{Y_k} A(y)[e_i + \nabla_y \chi_i^k]\,\mathrm{d}y \quad \text{for } k = 1, 2.$$

Here, $\chi_i^1 \in \mathrm{H}^1_{\mathrm{per}}(Y_1)$ and $\chi_i^2 \in \mathrm{H}^1(Y_2)$ solve the following cell problem for $i = 1, \dots, d$

$$\begin{aligned} -\operatorname{div}_y\,(A[e_i + \nabla_y\chi_i^1]) &= 0 && \text{in } Y_1, \\ -\operatorname{div}_y\,(A[e_i + \nabla_y\chi_i^2]) &= 0 && \text{in } Y_2, \\ A\nabla_y\chi_i^1\cdot n_1 &= -A\nabla_y\chi_i^2\cdot n_2 && \text{on } \Gamma, \\ -A[e_i + \nabla_y\chi_i^1]\cdot n_1 &= h\,(\chi_i^1 - \chi_i^2) && \text{on } \Gamma, \\ \chi_i^1 \text{ is } Y\text{-perioidc}, \quad &\textstyle\int_{Y_1} \chi_i^1\,\mathrm{d}y = 0, \end{aligned} \tag{5.2}$$

where, n_1 and n_2 denote the unit outer normal to Y_1 and Y_2, respectively. Thanks to the higher regularity of the limit u, the corresponding global correctors $\widehat{u}_1$ and $\widehat{u}_2$ belong to the spaces $\mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y_1))$ and $\mathrm{H}^1(\Omega; \mathrm{H}^1(Y_2))$, respectively. They are given via

$$\widehat{u}_k(x, y) := \sum_{i=1}^d \frac{\partial u}{\partial x_i}(x)\cdot\chi_i^k(y) \quad \text{for } k = 1, 2.$$

Theorem 5.1. *Let the assumptions of Theorem 4.1 hold true. Then, there exists a positive constant C independent of ε such that it holds*

$$\begin{aligned} \|u_1^\varepsilon - u - \varepsilon\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon - \nabla u - \varepsilon\nabla(\mathcal{G}_2^\varepsilon\,\widehat{u}_2)\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \\ + \varepsilon^{-\frac{1}{2}}\|(u_1^\varepsilon - u_2^\varepsilon) - \varepsilon(\mathcal{G}_1^\varepsilon\,\widehat{u}_1 - \mathcal{G}_2^\varepsilon\,\widehat{u}_2)\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \le \varepsilon^{\frac{1}{2}} C. \end{aligned} \tag{5.3}$$

Proof. *Step 1: Periodicity defect.* In the weak formulation (5.1), we want to choose the two-scale test functions

$$\Phi_1 = \mathcal{T}_1^\varepsilon(\varepsilon^{-1}\varphi) - \nabla\varphi\cdot y \quad \text{and} \quad \Phi_2 = \mathcal{T}_2^\varepsilon(\varepsilon^{-1}\varphi_2^\varepsilon) - \nabla\varphi\cdot y$$

with arbitrary $\varphi_2^\varepsilon \in \mathrm{H}^1(\Omega_2^\varepsilon)$, however, Φ_1 does not respect the Y-periodicity in general. Compensating the periodicity defect with $\Psi^\varepsilon \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y_1))$ according to Theorem A.3 and Remark A.4(b) as well as using the duality of $\mathcal{T}_1^\varepsilon$ and $\mathcal{F}_1^\varepsilon$ respective $\mathcal{T}_2^\varepsilon$ and $\mathcal{F}_2^\varepsilon$ gives

$$\bigg|\int_{\Omega_1^\varepsilon} A^\varepsilon\,\mathcal{F}_1^\varepsilon[\nabla u+\nabla_y\widehat{u}_1]\cdot\nabla\varphi\,\mathrm{d}x+\int_{\Omega_2^\varepsilon} A^\varepsilon\,\mathcal{F}_2^\varepsilon[\nabla u+\nabla_y\widehat{u}_2]\cdot\nabla\varphi_2^\varepsilon\,\mathrm{d}x$$
$$+\int_{\Omega\times\Gamma} h\,(y)(\widehat{u}_1-\widehat{u}_2)[\mathcal{T}_1^\varepsilon(\varepsilon^{-1}\varphi)-\mathcal{T}_2^\varepsilon(\varepsilon^{-1}\varphi_2^\varepsilon)]\,\mathrm{d}x\,\mathrm{d}\sigma_y-\int_\Omega f\varphi\,\mathrm{d}x\bigg|\leq\varepsilon^{\frac12}C\|\varphi\|_{\mathrm{H}^1(\Omega)}. \quad (5.4)$$

For the boundary term, we also used the continuous embedding of $\mathrm{H}^1(Y_1)$ into $\mathrm{L}^2(\Gamma)$ such that it holds for $U=h\,(\widehat{u}_1-\widehat{u}_2)\in\mathrm{H}^1(\Omega;\mathrm{L}^2(\Gamma))$ and $\Phi=\Psi^\varepsilon-\mathcal{T}_1^\varepsilon(\varepsilon^{-1}\varphi)+\nabla\varphi\cdot y\in\mathrm{L}^2(\Omega;\mathrm{H}^1(Y_1))$, with Ψ^ε as in Remark A.4(b), [3]

$$\bigg|\int_{\Omega\times\Gamma}U\Phi\,\mathrm{d}x\,\mathrm{d}\sigma_y\bigg|\leq C_{\mathrm{emb}}\|U\|_{\mathrm{L}^2(\Gamma;\mathrm{H}^1(\Omega))}\|\Phi\|_{\mathrm{H}^1(Y_1;\mathrm{H}^1(\Omega)^*)}\leq\varepsilon^{\frac12}C\|\varphi\|_{\mathrm{H}^1(\Omega)}.$$

Next, we want to replace $\widehat{u}_1-\widehat{u}_2$ with $\mathcal{T}_1^\varepsilon\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1-\mathcal{T}_2^\varepsilon\,\mathcal{G}_2^\varepsilon\,\widehat{u}_2$ (recall (3.5) for $\mathcal{G}_2^\varepsilon$) in the boundary integral in (5.4) via the Lemmas A.1 and A.5. Together with the integration formula (3.1) as well as $\nabla_y(\mathcal{T}_1^\varepsilon\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1)=\mathcal{T}_1^\varepsilon[\varepsilon\nabla(\mathcal{G}_1^\varepsilon\,\widehat{u}_1)]$, we get

$$\begin{aligned}&\|\,\mathcal{T}_1^\varepsilon\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1-\widehat{u}_1\|_{\mathrm{L}^2(\Omega\times\Gamma)}\\&\leq C_{\mathrm{emb}}\big(\|\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1-\mathcal{F}_1^\varepsilon\,\widehat{u}_1\|_{\mathrm{L}^2(\Omega_2^\varepsilon)}+\|\,\mathcal{T}_1^\varepsilon\,\mathcal{F}_1^\varepsilon\,\widehat{u}_1-\widehat{u}_1\|_{\mathrm{L}^2(\Omega\times Y_1)}\\&\quad+\|\varepsilon\nabla(\mathcal{G}_1^\varepsilon\,\widehat{u}_1)-\mathcal{F}_1^\varepsilon(\nabla_y\widehat{u}_1)\|_{\mathrm{L}^2(\Omega_2^\varepsilon)}+\|\,\mathcal{T}_1^\varepsilon\,\mathcal{F}_1^\varepsilon(\nabla_y\widehat{u}_1)-\nabla_y\widehat{u}_1\|_{\mathrm{L}^2(\Omega\times Y_1)}\big)\\&\leq\varepsilon C\|u\|_{\mathrm{H}^2(\Omega)}.\end{aligned}$$

The same estimate holds for $\mathcal{T}_2^\varepsilon\,\mathcal{G}_2^\varepsilon\,\widehat{u}_2-\widehat{u}_2$. Applying the integration formula (3.2) and treating the gradient terms as in the case $-1<\gamma<1$ gives

$$\begin{aligned}&\bigg|\int_{\Omega_1^\varepsilon}A^\varepsilon[\nabla u+\varepsilon\nabla\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1]\cdot\nabla\varphi\,\mathrm{d}x+\int_{\Omega_2^\varepsilon}A^\varepsilon[\nabla u+\varepsilon\nabla\,\mathcal{G}_2^\varepsilon\,\widehat{u}_2]\cdot\nabla\varphi_2^\varepsilon\,\mathrm{d}x\\&\quad+\varepsilon^{-1}\int_{\Gamma^\varepsilon}h^\varepsilon\varepsilon(\mathcal{G}_1^\varepsilon\,\widehat{u}_1-\mathcal{G}_2^\varepsilon\,\widehat{u}_2)(\varphi-\varphi_2^\varepsilon)\,\mathrm{d}\sigma_x-\int_\Omega f\varphi\,\mathrm{d}x\bigg|\\&\leq\mu_1\|\varphi\|^2_{\mathrm{H}^1(\Omega)}+\mu_2\varepsilon^{-1}\|\varphi-\varphi_2^\varepsilon|^2_{\mathrm{L}^2(\Gamma^\varepsilon)}+\varepsilon C(\mu_1,\mu_2).\end{aligned}\quad (5.5)$$

In particular, the integration formula (3.2) implies $\|\,\mathcal{T}_1^\varepsilon(\varepsilon^{-1}\varphi)-\mathcal{T}_2^\varepsilon(\varepsilon^{-1}\varphi_2^\varepsilon)\|^2_{\mathrm{L}^2(\Omega\times\Gamma)}\leq\varepsilon^{-1}\|\varphi-\varphi_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)}$, cf. also [DLNT11, Eq. (2.8)].

Step 2: Admissible test functions. We test the weak formulation (2.1) with

$$\varphi_1^\varepsilon:=u_1^\varepsilon-u-\varepsilon\,\mathcal{G}_1^\varepsilon\,\widehat{u}_1\in\mathrm{H}^1_{\mathrm{D}}(\Omega_1^\varepsilon)\quad\text{and}\quad\varphi_2^\varepsilon:=u_2^\varepsilon-u-\varepsilon\,\mathcal{G}_2^\varepsilon\,\widehat{u}_2\in\mathrm{H}^1(\Omega_2^\varepsilon)$$

and choose $\varphi=\mathcal{E}_1^\varepsilon\,\varphi_1^\varepsilon$ in (5.5) such that the difference between microscopic and reformulated macroscopic weak formulations reads

$$\bigg|\int_{\Omega_1^\varepsilon}A^\varepsilon\nabla\varphi_1^\varepsilon\cdot\nabla\varphi_1^\varepsilon\,\mathrm{d}x+\int_{\Omega_2^\varepsilon}A^\varepsilon\nabla\varphi_2^\varepsilon\cdot\nabla\varphi_2^\varepsilon\,\mathrm{d}x+\,\varepsilon^{-1}\int_{\Gamma^\varepsilon}h^\varepsilon|\varphi_1^\varepsilon-\varphi_2^\varepsilon|^2\,\mathrm{d}\sigma_x\bigg|$$
$$\leq\bigg|\int_{\Omega_2^\varepsilon}f\,(\mathcal{E}_1^\varepsilon\,\varphi_1^\varepsilon-\varphi_2^\varepsilon)\,\mathrm{d}x\bigg|+\mu_1\|\varphi\|^2_{\mathrm{H}^1(\Omega)}+\mu_2\varepsilon^{-1}\|\varphi_1^\varepsilon-\varphi_2^\varepsilon|^2_{\mathrm{L}^2(\Gamma^\varepsilon)}+\varepsilon C(\mu_1,\mu_2).$$

Estimating the Ω_2^ε-integral on the right-hand side as in (4.10), exploiting the uniform ellipticity of A^ε and $h^\varepsilon\geq h_0$, as well as choosing μ_1 and μ_2 suitably gives the desired estimate (5.3). □

[3]Here, we also used the continuous embedding of $\mathrm{L}^2(\Omega)$ into $\mathrm{H}^1(\Omega)^*$ and, hence, $\mathrm{H}^1(Y_1;\mathrm{L}^2(\Omega))\subset\mathrm{H}^1(Y_1;\mathrm{H}^1(\Omega)^*))\subset\mathrm{L}^2(\Gamma;\mathrm{H}^1(\Omega)^*)$.

6 Corrector estimates for $\gamma < -1$

In this regime, we recover in the limit $\varepsilon \to 0$ the standard unit cell problem. Indeed, there exist according to [DLNT11, Sec. 3.3 & 4.1] two limit functions $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$ and $\widehat{u} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$ with $\int_Y \widehat{u}\,\mathrm{d}y = 0$ such that the microscopic solutions $(u_1^\varepsilon, u_2^\varepsilon)_\varepsilon$ satisfy

$$\begin{aligned} \mathcal{T}_1^\varepsilon u_1^\varepsilon \to u \text{ strongly in } \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_1)), \quad \mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \to \nabla u + \nabla_y \widehat{u} \text{ strongly in } \mathrm{L}^2(\Omega \times Y_1), \\ \mathcal{T}_2^\varepsilon u_2^\varepsilon \rightharpoonup u \text{ weakly in } \mathrm{L}^2(\Omega; \mathrm{H}^1(Y_2)), \quad \mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \to \nabla u + \nabla_y \widehat{u} \text{ strongly in } \mathrm{L}^2(\Omega \times Y_2). \end{aligned}$$

In particular, the pair $(u, \widehat{u})$ is the unique weak solution of the two-scale limit problem

$$\int_{\Omega\times Y} A(y)[\nabla u + \nabla_y \widehat{u}]\cdot[\nabla\varphi + \nabla_y \Phi]\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega\times Y} f\varphi\,\mathrm{d}x \tag{6.1}$$

for all $\varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$ and $\Phi \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$. Moreover, u solves the macroscopic equation (1.2), and the effective matrix A^0_γ as well as the global corrector $\widehat{u}$ are given via

$$A^0_\gamma e_i := \int_Y A(y)[e_i + \nabla_y \chi_i]\,\mathrm{d}y \quad \text{and} \quad \widehat{u}(x,y) := \sum_{i=1}^d \frac{\partial u}{\partial x_i}(x)\cdot\chi_i(y),$$

where the local correctors $\chi_i \in \mathrm{H}^1_{\mathrm{per}}(Y)$ solve the standard cell problem for $i = 1, \dots, d$

$$\begin{aligned} -\operatorname{div}_y\left(A[e_i + \nabla_y \chi_i]\right) = 0 \quad \text{in } Y, \\ \chi_i \text{ is } Y\text{-perioidc}, \quad \textstyle\int_Y \chi_i\,\mathrm{d}y = 0. \end{aligned} \tag{6.2}$$

We aim to derive the corrector estimates in the case $\gamma < -1$ in two steps. First, we introduce the standard homogenization problem in the whole domain without any interfaces, so to speak the *perfect* transmission problem: find $w^\varepsilon \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$ such that

$$\int_\Omega A^\varepsilon \nabla w^\varepsilon \cdot \nabla\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x \tag{6.3}$$

for all admissible test functions $\varphi \in \mathrm{H}^1_{\mathrm{D}}(\Omega)$. This classical problem is well-studied in the literature, and there exist error estimates for the difference of the microscopic solution w^ε and the macroscopic solution u of (1.2).

Proposition 6.1 ([Gri04, Prop. 4.3])**.** *Let the assumptions of Theorem 4.1 hold true. Then, there exists a positive constant C independent of ε such that it holds*

$$\|w^\varepsilon - u - \varepsilon\,\mathcal{G}_\varepsilon\,\widehat{u}\|_{\mathrm{H}^1(\Omega)} \le \varepsilon^{\frac{1}{2}} C.$$

In the second step, we control the difference of the solution w^ε of the standard homogenization problem and the solution $(u_1^\varepsilon, u_2^\varepsilon)$ of the imperfect transmission problem. In order to prove such error estimates, we require the extra regularity $A^\varepsilon \in \mathrm{W}^{1,\infty}(\Omega)$ such that the $\mathrm{H}^1(\Omega_2^\varepsilon)$-norm of $A^\varepsilon \nabla w^\varepsilon$ can be controlled.

Theorem 6.2. *Let the assumptions of Theorem 4.1 as well as $A \in \mathrm{W}^{1,\infty}(Y)$ hold true. Then, there exists a positive constant C independent of ε such that it holds*

$$\|w^\varepsilon - u_1^\varepsilon\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \|\nabla w^\varepsilon - \nabla u_2^\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{\gamma}{2}}\|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \le \varepsilon^{\frac{-1-\gamma}{2}} C.$$

Proof. In the weak formulations (6.3) and (2.1), we choose the admissible test functions $\varphi = \mathcal{E}_1^\varepsilon u_1^\varepsilon - w^\varepsilon$ as well as $\varphi_1 = u_1^\varepsilon - w^\varepsilon$ and $\varphi_2 = u_2^\varepsilon - w^\varepsilon$, respectively. Taking the difference of both formulations gives

$$\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla\varphi_1 \cdot \nabla\varphi_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2^\varepsilon \cdot \nabla\varphi_2 - A^\varepsilon \nabla w^\varepsilon \cdot \nabla\varphi \,\mathrm{d}x \\ + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon |u_1^\varepsilon - u_2^\varepsilon|^2 \,\mathrm{d}\sigma_x = \int_{\Omega_2^\varepsilon} f\,(u_2^\varepsilon - \mathcal{E}_1^\varepsilon u_1^\varepsilon)\,\mathrm{d}x.$$

Adding $\pm A^\varepsilon \nabla w^\varepsilon \cdot \nabla u_2^\varepsilon$ under the Ω_2^ε-integral and using partial integration yields

$$\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla\varphi_1 \cdot \nabla\varphi_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla\varphi_2 \cdot \nabla\varphi_2 \,\mathrm{d}x - \int_{\Omega_2^\varepsilon} \mathrm{div}(A^\varepsilon \nabla w^\varepsilon)(u_2^\varepsilon - \mathcal{E}_1^\varepsilon u_1^\varepsilon)\,\mathrm{d}x \\ + \int_{\Gamma^\varepsilon} A^\varepsilon \nabla w^\varepsilon \cdot n_2^\varepsilon\,(u_2^\varepsilon - u_1^\varepsilon)\,\mathrm{d}\sigma_x + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon |u_1^\varepsilon - u_2^\varepsilon|^2 \,\mathrm{d}\sigma_x = \int_{\Omega_2^\varepsilon} f\,(u_2^\varepsilon - \mathcal{E}_1^\varepsilon u_1^\varepsilon)\,\mathrm{d}x.$$

While noting that $-\mathrm{div}(A^\varepsilon \nabla w^\varepsilon) = f$ in $\Omega_2^\varepsilon \subset \Omega$, the two Ω_2^ε-integrals containing the difference $u_2^\varepsilon - \mathcal{E}_1^\varepsilon u_1^\varepsilon$ cancel each other. It remains to control the additional boundary term. Applying Hölder's and Young's inequality with $\mu > 0$ gives

$$\left| \int_{\Gamma^\varepsilon} A^\varepsilon \nabla w^\varepsilon \cdot n_2^\varepsilon (u_2^\varepsilon - u_1^\varepsilon)\,\mathrm{d}\sigma_x \right| \leq \|A^\varepsilon \nabla w^\varepsilon \cdot n_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \\ \leq \varepsilon^{-\gamma} C(\mu) \|A^\varepsilon \nabla w^\varepsilon \cdot n_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)} + \varepsilon^\gamma \mu \|u_1^\varepsilon - u_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)}. \tag{6.4}$$

With estimate (A.2), we arrive at

$$\|A^\varepsilon \nabla w^\varepsilon \cdot n_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)} \leq C \left(\varepsilon^{-1} \|A^\varepsilon \nabla w^\varepsilon\|^2_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon \|\nabla[A^\varepsilon \nabla w^\varepsilon]\|^2_{\mathrm{L}^2(\Omega_2^\varepsilon)} \right).$$

The additional regularity $A^\varepsilon \in \mathrm{W}^{1,\infty}(\Omega)$ implies the higher regularity of the solution $w^\varepsilon \in \mathrm{H}^2(\Omega)$. Revisiting the proofs of the Theorems 3.1.3.3 and 3.2.1.3 in [Gri85] yields the existence of a constant $C > 0$ only depending on the properties of the domain Ω and the ellipticity constant α such that

$$\|w^\varepsilon\|_{\mathrm{H}^2(\Omega)} \leq C\,(1 + M)\,\|f\|_{\mathrm{L}^2(\Omega)} \quad \text{with} \quad M = \|A^\varepsilon\|_{\mathrm{L}^\infty(\Omega)} \|A^\varepsilon\|_{\mathrm{W}^{1,\infty}(\Omega)}.$$

Using $\partial_{x_i} A^\varepsilon(x) = \varepsilon^{-1} \partial_{y_i} A(x/\varepsilon)$, for $i = 1, \ldots, d$, gives $\|\nabla[A^\varepsilon \nabla w^\varepsilon]\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \leq \varepsilon^{-1} C$, which in turn yields $\|A^\varepsilon \nabla w^\varepsilon \cdot n_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)} \leq \varepsilon^{-1} C$. Inserting the latter into (6.4), yields overall

$$\left| \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla\varphi_1 \cdot \nabla\varphi_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla\varphi_2 \cdot \nabla\varphi_2 \,\mathrm{d}x + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon |u_1^\varepsilon - u_2^\varepsilon|^2 \,\mathrm{d}\sigma_x \right| \\ \leq \varepsilon^{-1-\gamma} C(\mu) + \varepsilon^\gamma \mu \|u_1^\varepsilon - u_2^\varepsilon\|^2_{\mathrm{L}^2(\Gamma^\varepsilon)}.$$

Finally, choosing $\mu = h_0/2$ as well as exploiting the uniform ellipticity of A^ε and $h^\varepsilon \geq h_0$ gives the desired error estimate. □

Combining the results of Proposition 6.1 and Theorem 6.2 gives immediately the main result of this Section.

Theorem 6.3. *Let the assumptions of Theorem 6.2 hold true. Then, there exists a positive constant C independent of ε such that it holds for $\gamma < -1$*

$$\|u_1^\varepsilon - u - \varepsilon\,\mathcal{G}_\varepsilon\,\widehat{u}\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon - \nabla u - \varepsilon \nabla(\mathcal{G}_\varepsilon\,\widehat{u})\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{\gamma}{2}} \|u_1^\varepsilon - u_2^\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \\ \leq C \left(\varepsilon^{\frac{|\gamma|-1}{2}} + \varepsilon^{\frac{1}{2}} \right).$$

Remark 6.4. *The Lipschitz continuity of A^ε is indeed a quite restrictive assumption and we expect that it can be generalized in some sense.*

(a) Indeed, the difference $u_1^\varepsilon - u_2^\varepsilon$ belongs to the better space $\mathrm{H}^{1/2}(\Gamma^\varepsilon)$ and one could study the dual paring

$$ {}_{\mathrm{H}^{-1/2}(\Gamma^\varepsilon)}\langle A^\varepsilon \nabla w^\varepsilon \cdot n_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon \rangle_{\mathrm{H}^{1/2}(\Gamma^\varepsilon)} $$

instead of the $\mathrm{L}^2(\Gamma^\varepsilon)$-scalar product in (6.4). Unfortunately, the $\mathrm{H}^{1/2}(\Gamma^\varepsilon)$-norm of $u_1^\varepsilon - u_2^\varepsilon$ is only of order $O(1)$, which can been seen from $\|u\|_{\mathrm{H}^{1/2}(\Gamma^\varepsilon)} \leq C\left(\|u\|_{\mathrm{L}^2(\Gamma^\varepsilon)} + \|\nabla u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)}\right)$. The same problem also occurs when comparing the two solutions $(u_1^\varepsilon, u_2^\varepsilon)$ and u directly.

(b) There arises the question whether one can construct for any sequence $(\xi_\varepsilon)_\varepsilon$, which is uniformly bounded in $\mathrm{H}^{1/2}(\Gamma^\varepsilon)$ and satisfies $\|\xi_\varepsilon\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \lesssim \varepsilon^{-\gamma/2}$, a sequence of extensions $(\widetilde{\xi_\varepsilon})_\varepsilon \subset \mathrm{H}^1(\Omega_2^\varepsilon)$ with $\|\nabla \widetilde{\xi_\varepsilon}\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \lesssim \varepsilon^\rho$ and $\rho > 0$. If this were possible, one could choose more clever test functions in the proof of Theorem 6.2 and would obtain the estimate $\|u^\varepsilon - w^\varepsilon\|_{\mathrm{H}^1(\Omega_1^\varepsilon)} + \|\nabla u_2^\varepsilon - \nabla w^\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \lesssim \varepsilon^\rho$ (only assuming A^ε bounded).

7 Discussion

The present corrector estimates do not require any additional regularity of the microscopic solution or the local correctors. However, we need that the limit satisfies $u \in \mathrm{H}^2(\Omega)$, which is immediate for convex Lipschitz domains or domains whose boundary is of class $\mathcal{C}^2$. In the case $\gamma = 1$, the more restrictive assumptions $f \in \mathrm{H}^2(\Omega)$ and h is constant are necessary in order to characterize the jump across the interface. It remains open whether these assumptions can be relaxed. Anyways, the source term has to be more regular than in all the other cases, since we need $f \in \mathrm{L}^2(\Gamma^\varepsilon)$ for all $\varepsilon > 0$ in estimate (4.14). In the third case $\gamma < -1$, we had to impose the Lipschitz continuity of A^ε to control the fluxes across the interface. It is to expect that this assumption can be relaxed to discontinuous A^ε, however, the proof remains open.

We point out that our corrector estimates recover the convergence rate $\sqrt{\varepsilon}$, in the special cases $\gamma \in \{-1, 0, 1\}$ and $\gamma \leq -2$. This rate seems to be optimal for corrector estimates up to the boundary of the macroscopic domain as it was also obtained in [Gri04, Rei16] for elliptic equations with periodically oscillating coefficients and without interfaces.

In order to treat double porosity models, which include degenerating terms such as $\mathrm{div}(\varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon)$ as in [DoŢ13, Ain15], we can introduce another gradient folding operator. For $U \in \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$, the one-scale function $\widehat{\mathcal{G}}_\varepsilon U := \widehat{u}_\varepsilon$ is given via the solution $\widehat{u}_\varepsilon \in \mathrm{H}^1(\Omega)$ of the elliptic problem (cf. [Han11, MRT14])

$$ \int_\Omega (\widehat{u}_\varepsilon - \mathcal{F}_\varepsilon U)\varphi + (\varepsilon \nabla \widehat{u}_\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U)) \cdot \varepsilon \nabla \varphi \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathrm{H}^1(\Omega). $$

In [Rei15, Rei16] the folding mismatch between the averaging operator $\mathcal{F}_\varepsilon$ and the gradient folding operator $\widehat{\mathcal{G}}_\varepsilon$ is quantified. We believe that the imperfect transmission problem can also be considered with non-homogeneous Dirichlet boundary conditions, as it is done in the paper [Gri13] on error estimates with boundary data in $\mathrm{H}^{1/2}(\partial\Omega)$. It is an open problem whether similar corrector estimates can also be proved for nonlinear transmission conditions as in [DoL15, Le 15]. However, we expect that the present results carry over to systems of coupled semilinear parabolic equations with linear transmission conditions. Previously, unfolding-based estimates for reaction-diffusion systems were proved in [FMP12, Rei15].

7.1 Application to supercapacitors

A prospective application of models containing imperfect transmission conditions is the supercapacitor, which is a small electrochemical device to store energy. The high capacity of the device is obtained by maximizing the surface area to volume ratio, which is achieved by taking a porous electrode, see Figure 2.

Let us study the stationary current flow in such a device. First of all we note that discontinuities of the potential are in principle nonphysical. However, when considering a double-layer[4] $\Sigma_\delta^\varepsilon$, where the thickness of the layer δ is much smaller than the pore size ε, we can reduce the double-layer model to an interface model: In [DGM15], the asymptotic limit $\delta \to 0$ was studied for one single electrode and one obtains that the normal of the electric displacement across the interface Γ^ε is equal to the surface charge density in one single layer[5], i.e. $D \cdot n = Q^{\mathrm{SL}}$. Hereby, $D = \epsilon_{\mathrm{r}}\epsilon_0 E$ depends on the relative permittivity ϵ_{r}, the vacuum permittivity ϵ_0, and the electric field E, where $E = -\nabla\varphi$ is given via the electrostatic potential φ. Using a linearization argument, we obtain that Q^{SL} is proportional to the difference of the electric potential, i.e. $Q^{\mathrm{SL}} = C(\varphi_1 - \varphi_2)$, where the proportionality factor C denotes the capacity per surface element[6]. Thus, we obtain the interface condition $-\epsilon_{\mathrm{r}}\epsilon_0\nabla\varphi_1^\varepsilon \cdot n_1^\varepsilon = C(\varepsilon)(\varphi_1^\varepsilon - \varphi_2^\varepsilon)$ on Γ^ε. Realizing that the total capacity of each electrode $C(\varepsilon)A \sim A/d$ is proportional to the ratio of its surface area A divided by the distance between two electrodes $d \sim \varepsilon$ yields $C(\varepsilon) \sim \varepsilon^{-1}$ as in the case (ii). For Carbon-based electrodes, characteristic pore sizes are $2-50$ nanometers, see e.g. [SiG08], and the macroscopic length scale of the device is about several hundreds micrometers. Hence, the parameter ε is of order $10^{-5} - 10^{-4}$.

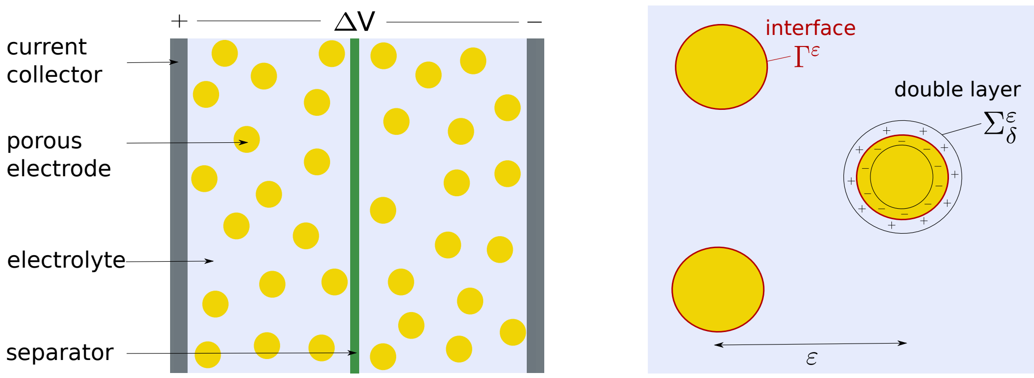

Figure 2: The main components of a supercapacitor (left) and the double layer (right).

A Auxiliary estimates

Lemma A.1. *For all $u \in \mathrm{H}^1(\Omega)$ it holds*

$$\| \mathcal{F}_1^\varepsilon u - u\|_{\mathrm{L}^2(\Omega_1^\varepsilon)} \leq \varepsilon C \|\nabla u\|_{\mathrm{L}^2(\Omega)},$$

where the constant $C > 0$ only depends in the domain Y.

[4]The double-layer $\Sigma_\delta^\varepsilon \subset \mathbb{R}^3$ is given via the δ-neighborhood of the interface Γ^ε as sketched in Figure 2.

[5]The surface charge density $Q^{\mathrm{SL}} = \lim_{\delta\to 0}\int_0^\delta q_\delta \,\mathrm{d}x$ is the line integral of the charge density q_δ.

[6]The relation between current flux and potential difference is in general not linear, since the capacity depends nonlinearly on the electric potential, see e.g. [LGD16].

Proof. For all one-scale functions $u \in \mathrm{L}^2(\Omega)$, the cell-average $\mathcal{F}_1^\varepsilon\, u$ belongs indeed to the space $\mathrm{L}^2(\Omega)$. Recalling that Ω is the union of translated cells $\varepsilon(\lambda+Y)$ with $\lambda \in K_\varepsilon$, we can apply Poincaré–Wirtinger's inequality to each cell $\varepsilon(\lambda+Y)$ and obtain $\| \mathcal{F}_1^\varepsilon\, u - u\|_{\mathrm{L}^2(\Omega)} \leq \varepsilon C \|\nabla u\|_{\mathrm{L}^2(\Omega)}$ as in [Gri04, Sect. 3]. □

Lemma A.2. *For all $u \in \mathrm{H}^1(\Omega_2^\varepsilon)$ it holds*

$$\| \mathcal{T}_2^\varepsilon\, u - u\|_{\mathrm{L}^2(\Omega_2^\varepsilon \times \Gamma)} \leq \varepsilon C \|\nabla u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)},$$

where $C > 0$ only depends on the domains Y_2 and Γ.

Proof. (The generalization of this proof to nonconvex inclusions Y_2 is owed to the anonymous referee.) For $u \in \mathrm{H}^1(\Omega_2^\varepsilon)$, we define the piecewise constant function

$$\bar{u}_\varepsilon(x) := \fint_{\varepsilon([\frac{x}{\varepsilon}]+Y_2)} u(z)\,\mathrm{d}z \quad \text{for a.a. } x \in \Omega_2^\varepsilon.$$

Due to the Poincaré–Wirtinger inequality, it holds

$$\|u - \bar{u}_\varepsilon\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \leq \varepsilon C \|\nabla u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)}.$$

Recalling (3.1) as well as noting $\mathcal{T}_2^\varepsilon\, \bar{u}_\varepsilon = \bar{u}_\varepsilon$ and $\nabla_y\, (\mathcal{T}_2^\varepsilon\, u) = \mathcal{T}_2^\varepsilon(\varepsilon \nabla u)$ implies

$$\| \mathcal{T}_2^\varepsilon\, u - \bar{u}_\varepsilon\|_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y_2))} \leq \varepsilon C \|\nabla u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)}.$$

Exploiting the continuous embedding of $\mathrm{L}^2(\Omega;\mathrm{H}^1(Y_2))$ into $\mathrm{L}^2(\Omega\times\Gamma)$, gives

$$\| \mathcal{T}_2^\varepsilon\, u - u\|_{\mathrm{L}^2(\Omega_2^\varepsilon\times\Gamma)} \leq \| \mathcal{T}_2^\varepsilon\, u - \bar{u}_\varepsilon\|_{\mathrm{L}^2(\Omega\times\Gamma)} + \|\bar{u}_\varepsilon - u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \leq \varepsilon C \|\nabla u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)}$$

and the constant $C > 0$ only depends on the properties of the domains Y_2 and Γ. □

Theorem A.3 (Periodicity defect)**.** *For every $\varphi \in \mathrm{H}^1(\Omega)$, there exists a Y-periodic function $\Phi^\varepsilon \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{per}}(Y_1))$ such that*

$$\|\Phi^\varepsilon\|_{\mathrm{H}^1(Y_1;\mathrm{L}^2(\Omega))} \leq C\|\varphi\|_{\mathrm{H}^1(\Omega)} \text{ and } \| \mathcal{T}_1^\varepsilon(\nabla\varphi) - (\nabla\varphi + \nabla_y\Phi^\varepsilon)\|_{\mathrm{L}^2(Y_1;\mathrm{H}^1(\Omega)^*)} \leq \varepsilon^{\frac{1}{2}} C \|\varphi\|_{\mathrm{H}^1(\Omega)},$$

where the constant $C > 0$ only depends in the domains Ω and Y_1.

Proof. For $\varphi \in \mathrm{H}^1(\Omega)$ the desired estimates hold with $\widehat{\Phi}^\varepsilon \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{per}}(Y))$ according to [Gri05, Thm. 2.3]. Choosing $\Phi^\varepsilon = \widehat{\Phi}^\varepsilon|_{\Omega\times Y_1}$ yields the assertion. □

Remark A.4. *(a) Note that the two-scale function Φ^ε in Theorem A.3 is only unique modulo the addition of a one-scale function $\xi^\varepsilon \in \mathrm{L}^2(\Omega)$. We can choose for instance $\xi^\varepsilon = \int_{Y_1} \Phi^\varepsilon\,\mathrm{d}y$ such that $\Phi^\varepsilon - \xi^\varepsilon$ has vanishing Y_1-mean value.*

(b) Moreover, introducing $\Phi_1 = \mathcal{T}_1^\varepsilon(\varepsilon^{-1}\varphi) - \nabla\varphi\cdot y$, $\xi_1 = \int_{Y_1}\Phi_1\,\mathrm{d}y$, and $\Psi^\varepsilon = \Phi^\varepsilon + \xi_1 - \xi^\varepsilon$ with ξ^ε as in (a), we obtain by Poincaré–Wirtinger's inequality

$$\|\Phi_1 - \Psi^\varepsilon\|_{\mathrm{H}^1(Y;\mathrm{H}^1(\Omega)^*)} \leq C_{\mathrm{PW}} \| \mathcal{T}_1^\varepsilon(\nabla\varphi) - (\nabla\varphi + \nabla_y\Psi^\varepsilon)\|_{\mathrm{L}^2(Y_1;\mathrm{H}^1(\Omega)^*)}.$$

Lemma A.5 (Folding mismatch)**.** *For $u \in \mathrm{H}^1(\Omega)$, $\chi \in \mathrm{L}^2(Y_k)$, and $k = 1, 2$ it holds*

$$\|(\mathcal{F}_k^\varepsilon u - \mathcal{Q}_\varepsilon\, u)\chi(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2(\Omega_k^\varepsilon)} \leq \varepsilon C \|u\|_{\mathrm{H}^1(\Omega)} \|\chi\|_{\mathrm{L}^2(Y_k)},$$

where $C > 0$ only depends on the domains Ω, Y, and Y_k.

Proof. Let $\widetilde{\chi} \in \mathrm{L}^2(Y)$ denote the extension of $\chi \in \mathrm{L}^2(Y_k)$ with zero. Then we have

$$\|(\mathcal{F}_k^\varepsilon u - \mathcal{Q}_\varepsilon u)\chi(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2(\Omega_k^\varepsilon)} = \|(\mathcal{F}_\varepsilon u - \mathcal{Q}_\varepsilon u)\widetilde{\chi}(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2(\Omega)} \leq \varepsilon C \|u\|_{\mathrm{H}^1(\Omega)} \|\widetilde{\chi}\|_{\mathrm{L}^2(Y)}$$

according to [Rei15, Lem. 2.3.9], which is based on [Gri04, Prop. 3.2]. □

Lemma A.6 ([Gri05, Eq. (2.4)] or [Rei15, Lem. 2.3.3]). *Let Ω denote an open, bounded domain with Lipschitz boundary. Moreover, let $\mathcal{N}_\varepsilon(\partial\Omega) \subset \Omega_1^\varepsilon$ denote the ε-neighborhood of the boundary $\mathcal{N}_\varepsilon(\partial\Omega) = \{x \in \Omega_1^\varepsilon \,|\, \mathrm{dist}(x, \partial\Omega) \leq \varepsilon\}$. Then, we have for all $u \in \mathrm{H}^1(\Omega_1^\varepsilon)$*

$$\|u\|_{\mathrm{L}^2(\mathcal{N}_\varepsilon(\partial\Omega))} \leq \varepsilon^{\frac{1}{2}} C \left(\|u\|_{\mathrm{L}^2(\Omega_1^\varepsilon)} + \varepsilon^{\frac{1}{2}} \|\nabla u\|_{\mathrm{L}^2(\Omega_1^\varepsilon)} \right),$$

where $C > 0$ only depends on the properties of the domain Ω.

Theorem A.7 ([CiP79]). *There exists a family of linear operators $\mathcal{E}_1^\varepsilon : \mathrm{H}^1_{\mathrm{D}}(\Omega_1^\varepsilon) \to \mathrm{H}^1_{\mathrm{D}}(\Omega)$ such that for every $u \in \mathrm{H}^1_{\mathrm{D}}(\Omega_1^\varepsilon)$ it holds*

$$(\mathcal{E}_1^\varepsilon u)|_{\Omega_1^\varepsilon} = u \quad \text{and} \quad \|\mathcal{E}_1^\varepsilon u\|_{\mathrm{H}^1(\Omega)} \leq C \|u\|_{\mathrm{H}^1(\Omega_1^\varepsilon)},$$

where $C > 0$ only depends on the domains Ω, Y, and Γ.

Lemma A.8 ([Mon03, Lem. 2.7 & Prop. 2.9]). *For $u \in \mathrm{H}^1(\Omega_2^\varepsilon)$ it holds*

$$\|u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \leq C \left(\varepsilon^{\frac{1}{2}} \|u\|_{\mathrm{L}^2(\Gamma^\varepsilon)} + \varepsilon \|\nabla u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \right), \tag{A.1}$$

$$\|u\|_{\mathrm{L}^2(\Gamma^\varepsilon)} \leq C \left(\varepsilon^{-\frac{1}{2}} \|u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{1}{2}} \|\nabla u\|_{\mathrm{L}^2(\Omega_2^\varepsilon)} \right), \tag{A.2}$$

where $C > 0$ only depends on the domains Y_2 and Γ.

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