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Another phase transition in the Axelrod model

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Abstract

Axelrod's model of cultural dissemination, despite its apparent simplicity, demonstrates complex behavior that has been of much interest in statistical physics. Despite the many variations and extensions of the model that have been investigated, a systematic investigation of the effects of changing the size of the neighborhood on the lattice in which interactions can occur has not been made. Here we investigate the effect of varying the radius R of the von Neumann neighborhood in which agents can interact. We show, in addition to the well-known phase transition at the critical value of q, the number of traits, another phase transition at a critical value of R, and draw a $q - R$ phase diagram for the Axelrod model on a square lattice. In addition, we present a mean-field approximation of the model in which behavior on an infinite lattice can be analyzed.

I. INTRODUCTION

The Axelrod model of cultural dissemination [1] is an apparently simple model of cultural diffusion, in which "culture" is modeled as a discrete vector (of length F), a multivariate property possessed by an agent at each of the N sites on a fully occupied finite square lattice. Agents interact with their lattice neighbors, and the dynamics of the model are based on the two principles of homophily and social influence. The former means that agents prefer to interact with similar others, while the latter means that agents, when they interact, become more similar. Despite this apparent simplicity, in fact the model displays a rich dynamic behavior, and does not inevitably converge to a state in which all agents have the same culture. Rather it will converge either to a monocultural state, or a multicultural state, depending on the model parameters. The Axelrod model has come to be of great interest in statistical physics, with a number of variations and analyses conducted. A review from a statistical physics perspective can be found in Castellano *et al.* [2], and more recent reviews from different perspectives in Kashima *et al.* [3], Sîrbu *et al.* [4].

One of the best-known features of the Axelrod model is the nonequilibrium phase transition between the monocultural (ordered) and multicultural (disordered) states, controlled by the value of q, the number of traits (possible values of each vector element) [5, 6]. A number of variations and extensions of the model have been proposed, including an external field (modeling a "mass media" effect), noise, and interaction via complex networks rather than a lattice.

External influence on culture vectors, in the form of a "generalized other" was first introduced by Shibanai *et al.* [7]. Further work on external influence on culture vectors, or mass media effect, considers an external field which acts to cause features to become more similar to the external culture vector with a certain probability [6, 8–14], or variations such as nonuniform or local fields [15, 16] or fields with adaptive features [17]. Counterintuitively, these mass media effects were found to actually increase cultural diversity rather than result in further homogenization, an effect explained by local homogenizing interactions causing the absorbing state to be less fragmented than when interacting with the external field only, the latter case actually resulting in more, rather than less, diversity [18].

The effect of noise, or "cultural drift", foreshadowed by Axelrod [1, p. 221], in the form of random perturbations of cultural features, has been examined [6, 19–24]. A sufficiently small level of noise actually promotes monoculture, while too high a level of noise prevents stable cultural regions from forming (an "anomic" state, as described by Centola *et al.* [22]). In fact, there is another phase transition induced by the noise rate [19]. Another form of noise, in the form of random error in determining cultural similarity between agents, has also been investigated [23, 24]. Noise is also incorporated in various other extensions of the Axelrod model [10, 11, 23, 25–27].

Rather than interacting with the neighbors on a lattice, neighborhoods defined by complex networks have also been investigated, including both static [25, 28–31] and coevolving networks [22, 32–34]. The use of complex networks rather than a lattice results in the phase transition controlled by the value of q still existing, albeit possibly with a different critical value. The effect of network topology on the phase transition driven by noise has also been investigated [35].

Another extension of the Axelrod model is the incorporation of multilateral influence, that is, interaction between more than two agents [13, 24]. Multilateral influence allows diversity to be sustained in the presence of noise, when with dyadic influence it would collapse to monoculture or anomie [24] — that is, it removes the phase transition controlled by the noise rate described by Klemm *et al.* [19].

Although most investigations of the Axelrod model and its extensions have been purely through computational experiments, a number of papers have used either mean-field analysis, or proved rigorous results mathematically. The original description of the phase transition controlled by q used mean-field analysis [5], as have some other papers [36–38]. A rigorous mathematical analysis is much more challenging, and has so far mostly been restricted to the one-dimensional case [39–42], with the exception of Li [43], who proves results for the usual two-dimensional model. The critical behavior of the order parameter has also been investigated quantitatively for the case of $F = 2$ on the square lattice and small-world networks [31]. Computational experiments have also been used to investigate the relationship between the lattice area and the number of cultures [44] and thermodynamic quantities such as temperature, energy, and entropy [45]. For the one-dimensional case, Gandica *et al.* [6] propose a thermodynamic version of the Axelrod model and demonstrate its equivalence to a coupled Potts model, as well as analyzing its behavior with respect to noise and an external field. An Axelrod-like model with $F = 2$ on a two-dimensional lattice is analyzed in the asymptotic case of $N \to \infty$ by Genzor *et al.* [46].

Other extensions and variations of the Axelrod model include bounded confidence and metric features [23], agent migration [34, 47–49], extended conservativeness (a preference for the last source of cultural information) [50], surface tension [51], cultural repulsion [52], the presence of some agents with constant culture vectors [53, 54], having one or more features constant on some [55] or all [49] agents, using empirical [56, 57] or simulated [57, 58] rather than uniform random initial culture vectors, comparing mass media model predictions to empirical data on a mass media campaign [11], coupling two Axelrod models through global fields [59, 60], combining the Axelrod model with a spatial public goods game [26], modeling diffusion of innovations by adding a new trait on a feature [61], and even using it as a heuristic for an optimization problem [62].

In addition to the earliest phase diagrams showing just q and the order parameter [5] or the noise rate r and the order parameter [19, 24], the following phase diagrams, derived from either simulation experiments, or mean-field analysis (or both), have been drawn for the Axelrod model and various extensions (notation may be changed from the original papers for consistency): $q - B$ where B is external field strength [8, 9, 11, 15, 30]; $r - \nu$ and $B - \nu$ where r is noise rate, and ν is a parameter controlling the network clustering structure [10]; $q - o$ where o is the degree of overlap between the layers of a multilayer network [25]; $\theta - q$ where θ is the "bounded confidence" threshold (minimum cultural similarity required for interaction) [23]; $F - q$ for the one-dimensional case [36]; $\kappa - q$ where κ is the fraction of "persistent agents" or "opinion leaders" (those with a constant culture vector) [53, 54].

Klemm *et al.* [28] show a $p - q$ phase diagram where p is the rewiring probability on small-world network, and also plot the relationship between the order parameter (largest region size) and $k_{\max}/N$ where $k_{\max}$ is maximum node degree in a structured scale-free network. In the small-world network, the phase transition still exists and is shifted by the degree of disorder of the network. In random scale-free networks, the transition disappears in the thermodynamic limit, but in structured scale-free networks the phase transition still exists. Klemm *et al.* [63] examine the nature of the phase transition in the one- and two-dimensional cases, while Hawick [64] investigates in addition three- and four-dimensional systems as well as triangular and hexagonal lattices.

Despite these extensive investigations into various aspects of the Axelrod model and its variants, there has been a surprising lack of systematic investigation of the effect of increasing the neighborhood size, or "range of interaction" [1] on a simple Axelrod model with dyadic interaction on a square lattice. This is despite Axelrod himself discussing the issue briefly [1, p. 213] and conducting experiments with neighborhoods of size 8 and 12, finding that these result in fewer stable regions than the original von Neumann neighborhood (size 4). Flache and Macy [24], in their model with multilateral influence, use a larger von Neumann neighborhood size, justifying it as empirically more plausible and a more conservative test of the preservation of cultural diversity [24, pp. 974-975]. Their extended model makes use of the larger neighborhood as its multilateral social influence uses more than two agents in an interaction, however all their experiments, including those reproducing the dyadic (interpersonal) influence model with noise of Klemm *et al.* [19], fix the radius at $R = 6$, a precedent followed in a subsequent paper [27], while another model using a larger neighborhood for multilateral interactions fixes the radius at $R = 2$ [26].

Vázquez and Redner [37] investigate, for the special case $F = 2$, the Axelrod model on a regular random graph using a mean-field analysis, giving an analytic explanation for the non-monotonic time dependence of the number of active links. Increasing the coordination number may be considered to be similar to increasing the neighborhood size on a lattice with fixed coordination number — in both cases all agents have the same number of "neighbors" (aside from edge effects in the case of finite lattices), which increases monotonically with the coordination number or von Neumann radius respectively. Vázquez and Redner [37] find that larger coordination numbers give better agreement between their master equation and Axelrod model simulations, but do not describe a phase transition controlled by the coordination number.

Here we investigate the effect of varying the radius of the von Neumann neighborhood in which agents can interact, and find another phase transition in the Axelrod model at a critical value of the radius R, as well as the well-known phase transition at a critical value of q [5], and draw a $q - R$ phase diagram for the Axelrod model on a square lattice.

II. MODEL

Each of the N agents on the fully occupied $L \times L$ lattice ($N = L^2$) has an F-dimensional culture vector ($F \geq 2$) $\sigma_i = (\sigma_{i,1}, \ldots, \sigma_{i,F})$ for all $1 \leq i \leq N$. Each entry of the cultural vector represents a feature and takes a single value from 1 to q, so, more precisely, $\sigma_{i,f} \in \{1, \ldots, q\}$ for all $1 \leq i \leq L^2$ and $1 \leq f \leq F$. Each of the F elements is referred to as a "feature", and q is known as the number of "traits". The cultural similarity of two agents is the number of features they have in common. If element f of the culture vector belonging to agent i is $\sigma_{i,f}$, then the cultural similarity $0 \leq c(i,j) \leq 1$ of two agents i and j is a normalized Hamming similarity

$$c(i,j) = \frac{1}{F} \sum_{k=1}^{F} \delta_{\sigma_{i,k},\sigma_{j,k}} \tag{1}$$

where $\delta_{x,y}$ is the Kronecker delta function.

An agent can interact with its neighbors, traditionally (as was originally used by Axelrod [1], for example), defined as the von Neumann neighborhood, that is, the four (north, south, east, west) surrounding cells on the lattice, so the number of potentially interacting agents is the lattice coordination number $g = 5$. Here we extend this to larger von Neumann neighborhoods by increasing the radius R, that is, extending the neighborhood to all cells within a given Manhattan distance, as was done by Flache and Macy [24]. This is illustrated in Figure 1. Hence the number of potentially interacting agents (the focal agent and all its neighbors) in the von Neumann neighborhood with radius R is now $g(R) = 2R(R + 1) + 1$ [65, 66] at most (we do not use periodic boundary conditions).

Initially, the agents are assigned uniform random culture vectors. The dynamics of the model are as follows. A focal agent a is chosen at random, and another agent b from the radius R von Neumann neighborhood is also chosen at random. With probability proportional

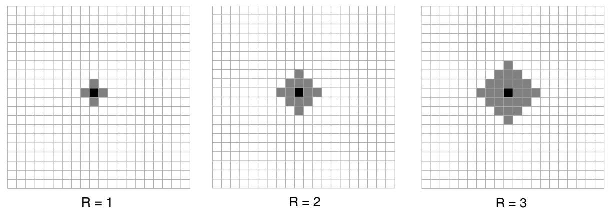

FIG. 1. Von Neumann neighborhoods of radius $R = 1$, $R = 2$, and $R = 3$. The focal agent is shown in black and the von Neumann neighborhood for that agent in gray.

to their cultural similarity (the number of features on which they have identical traits), the two agents a and b interact. This interaction results in a randomly chosen feature on a whose value is different from that on b being changed to b's value. This process is repeated until an absorbing, or frozen, state is reached. In this state, no more change is possible, because all agents' neighbors have either identical or completely distinct (no features in common, so no interaction can occur) culture vectors.

In the absorbing state, the agents form cultural regions, or clusters. Within the cluster, all agents have identical culture vectors. Then the average size of the largest cluster, $\langle S_{\max} \rangle / L^2$ is used as the order parameter [2, 5, 28], separating the ordered and disordered phases. In a monocultural (ordered) state, $\langle S_{\max} \rangle / L^2 \approx 1$, a single cultural region covers almost the entire lattice; in a multicultural (disordered) state, multiple cultural regions exist. Other order parameters that have been used include the number of cultural domains [1, 24], mean density of cultural domains [67], entropy [68], overlap between neighboring sites [63], and activity (number of changes) per agent [69].

Source code for the model (implemented in C++ and Python with MPI [70]) is available from `https://sites.google.com/site/alexdstivala/home/axelrod_qrphase/`.

III. RESULTS

Figure 2 shows the order parameter (largest region size) plotted against q for $F = 5$, on three different lattice sizes. It is apparent that, as the size of the von Neumann neighborhood is increased, the critical value of q also increases. That is, by allowing a larger range of interactions, a larger scope of cultural possibilities is required in order for a multicultural absorbing state to exist. Increasing the lattice size has a similar effect, although, as we shall show in Section IV, there is still a finite critical value of q in the limit of an infinite lattice.

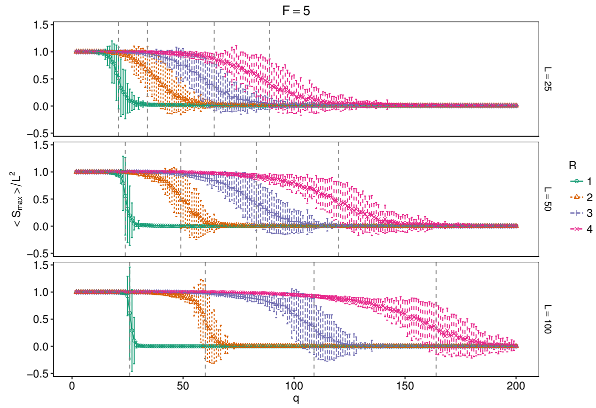

FIG. 2. The order parameter $\langle S_{\max}\rangle / L^2$ (largest region size) plotted against the number of traits q for the Axelrod model for $F = 5$, four different values of the von Neumann radius R, and three lattice sizes. Each data point is the average over 50 independent runs and error bars show the 95% confidence interval. Vertical dashed lines show the critical value of q, where the variance of the order parameter is largest.

Figure 3 shows the order parameter (largest region size) plotted against the von Neumann radius for $F = 5$, various values of q, and three different lattice sizes. In each case (apart from the smallest value of q, in which a monocultural state always prevails), there is a phase transition visible between a multicultural state (for R less than a critical value) and a monocultural state. Note that when R is sufficiently large relative to the lattice size L, every agent has every other agent in its von Neumann neighborhood, and hence the situation is equivalent to a complete graph or a well-mixed population (or "soup" [71, p. 132]). In this situation, it has long been known that heterogeneity cannot be sustained [1, 71]. Fig. 3 shows that there appears to be a phase transition controlled by R, between the multicultural phase and the monocultural phase. As the size of the neighborhood increases, so does the probability of an agent finding another agent with at least one feature in common with which to interact, and hence local convergence can happen in larger neighborhoods, resulting in larger cultural regions. However this does not result, at the absorbing state (for a fixed value of q), in a gradual increase in maximum cultural region size from a completely fragmented state to a monocultural state. Rather, global polarization (a multicultural absorbing state) still occurs for sufficiently small R, but at the critical value of the radius R_c there is a phase transition so that for neighborhoods defined by $R > R_c$ a monocultural state prevails.

This phase transition is further apparent in Figure 4, which shows histograms of the

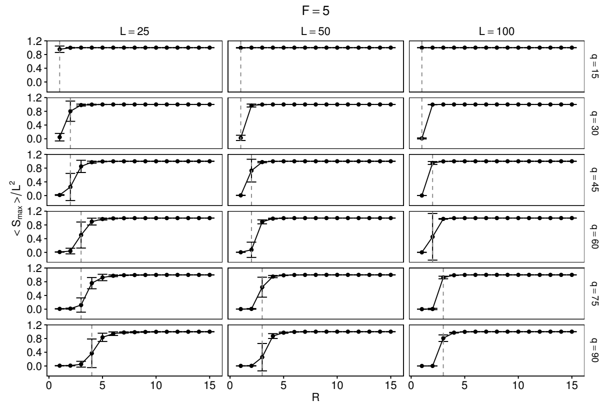

FIG. 3. The order parameter $\langle S_{\max} \rangle / L^2$ (largest region size) plotted against the von Neumann radius R for the Axelrod model for $F = 5$, some different values of q, and three lattice sizes. Each data point is the average over 50 independent runs and error bars show the 95% confidence interval. Vertical dashed lines show the critical value of R, where the variance of the order parameter is largest.

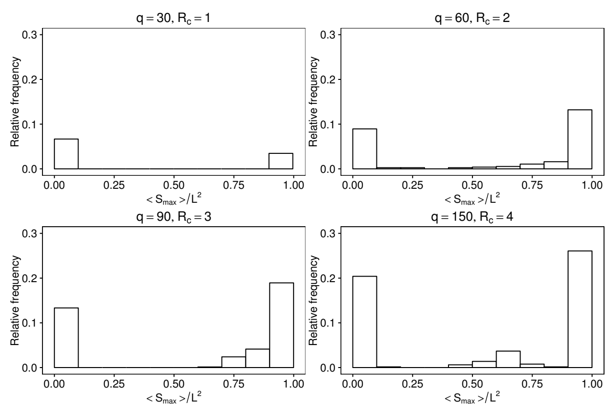

FIG. 4. Distribution of the order parameter at the critical radius for some different values of q, with $F = 5$, $L = 100$. Each distribution is from 50 independent runs.

distribution of the order parameter (largest region size) at the critical radius for some different values of q. That is, for each value of q, the radius R_c at which the variance of the order parameter is greatest. This shows the bistability of the order parameter at the critical radius, where the two extreme values are equally probable [28].

Figure 5 colors points on the $q - R$ plane according to the value of the order parameter, resulting in $q - R$ phase diagram. A multicultural state only results for sufficiently large values of q and small values of R. Figure 6 shows the phase transition more clearly, with

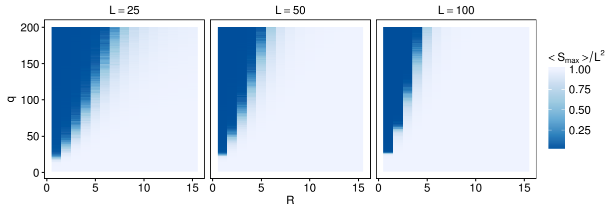

FIG. 5. $q - R$ phase diagram showing the order parameter $\langle S_{\max} \rangle / L^2$ for the Axelrod model for $F = 5$ and three lattice sizes ($L = 25$, $L = 50$, and $L = 100$). Each data point is colored according to the size of the largest region $\langle S_{\max} \rangle / L^2$ averaged over 50 independent runs.

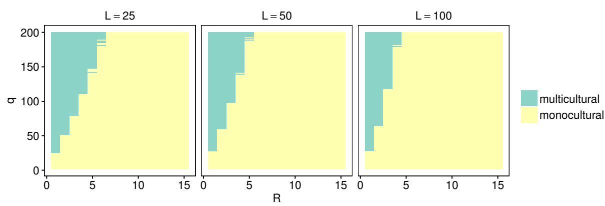

FIG. 6. $q - R$ phase diagram for the Axelrod model for $F = 5$ and three lattice sizes ($L = 25$, $L = 50$, and $L = 100$). As in Klemm *et al.* [28], the arbitrary, but small, value of 0.1 is used as the value of the order parameter to plot the critical value of q separating the monocultural and multicultural region for each value of the von Neumann neighborhood radius R.

the multicultural states in the upper left of the plane and the monocultural states in the bottom right.

IV. MEAN-FIELD ANALYSIS

We detail the mean-field analysis carried out by Castellano *et al.* [5] who gave a differential equation. In the mean-field setting, we focus on the bonds between sites (or agents) located on an infinite lattice, so we can assume that each site and its von Neumann neighborhood consists of exactly $g(R)$ sites. The infinite lattice setting naturally implies that we do not consider edge effects.

For a single, randomly chosen bond between two sites, we let $P_m(t)$ be the probability that the bond is of type m at time t, so both sites of the bond share m common features, while $F - m$ features are different. If the randomly chosen bond is connected to sites i and j, then

$$m = \#\{\sigma_{i,f} = \sigma_{j,f} : f \in \{1, \ldots, q\}\}.$$

At time $t = 0$, we denote by ρ_0 the probability of a single feature of any two sites being common, so $\rho_0 = \mathbb{P}(\sigma_{i,f} = \sigma_{j,f})$. If the features are distributed uniformly from 1 to q, then $\rho_0 = 1/q$. It is sometimes assumed that the features have a Poisson distribution [5, 36, 44, 67, 68] with mean q, so then application of the Skellam distribution gives $\rho_0 = e^{-2q} I_0(2q)$, where I_0 is a modified Bessel function of the first kind. For the single bond, the number of common features is a binomial random variable, so

$$P_m(0) = \binom{F}{m} \rho_0^m (1 - \rho_0)^{F-m}.$$

Castellano *et al.* [5] derived a master equation, also known as a forward equation, given by

$$\frac{dP_m(t)}{dt} = \sum_{k=1}^{F-1} \frac{k}{F} P_k(t) \left[\delta_{m,k+1} - \delta_{m,k} + (g-1) \sum_{n=0}^{F} \left(P_n(t) W_{n,m}^{(k)}(t) - P_m(t) W_{m,n}^{(k)}(t) \right) \right], \quad (2)$$

where $W_{n,m}^{(k)}(t)$ is the probability that an n-type bond becomes an m-type bond due to the updating of a k-type neighbor bond [72]. This equation is only defined for $1 \leq m \leq F$, but naturally the probabilities sum to one, giving

$$P_0(t) = 1 - \sum_{m=1}^{F} P_m(t).$$

For $1 \leq m \leq F$, we show [72] that the master equation or, rather, the set of nonlinear differential equations (2) can be re-written as

$$\begin{aligned} \frac{dP_m(t)}{dt} = & \left[\frac{m-1}{F} P_{m-1}(t) - \frac{m}{F} P_m(t) \right] \\ & + (g-1) \Big[P_{m-1}(t) W_{m-1,m}^{(k)}(t) \\ & - P_m(t) W_{m,m-1}^{(k)}(t) \\ & + P_{m+1}(t) W_{m+1,m}^{(k)}(t) \\ & - P_m(t) W_{m,m+1}^{(k)}(t) \Big] \sum_{k=1}^{F-1} \frac{k}{F} P_k(t), \end{aligned} \quad (3)$$

and zeroth differential equation is

$$\frac{dP_0(t)}{dt} = - \sum_{m=1}^{F} \frac{dP_m(t)}{dt}. \quad (4)$$

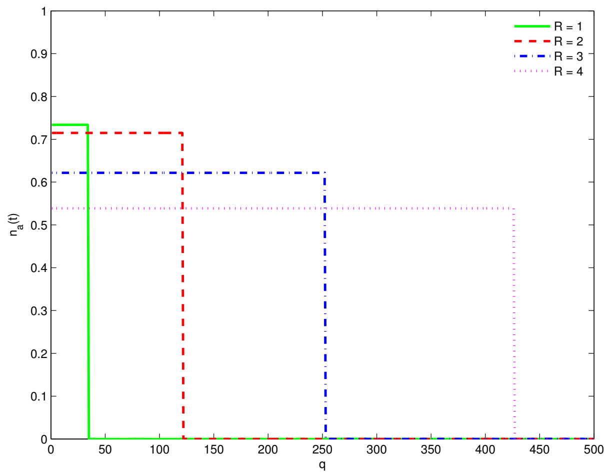

FIG. 7. Phase diagram within the mean-field approximation for some different values of the von Neumann radius R. The value of $n_a(t)$ (shown at $t = 10^3$) is obtained by numerical integration of (3) and (4).

As in Castellano *et al.* [5], we can investigate model dynamics within the mean-field treatment by studying the density $n_a = \sum_{i=1}^{F-1} P_i$ of active bonds, that is a bond across which at least one feature is different and one the same. Hence in an absorbing (frozen) state, $n_a = 0$. In the mean-field analysis, since an infinite lattice is assumed, $n_a = 0$ only when a multicultural absorbing state is reached; as noted by Castellano *et al.* [5], the coarsening process by which a monocultural state is formed lasts indefinitely on an infinite lattice.

Figure 7 plots the number of active bonds against the value of q for some different values of R within the mean-field approximation. It can be seen that the behavior is qualitatively the same as that shown in Fig. 2 for the simulations on finite lattices: the critical value of q is higher for larger neighborhood sizes. On finite lattices, larger lattice sizes also increase the critical value of q for a given neighborhood size, however on an infinite lattice, there is still a finite critical value of q for a given neighborhood size. This suggests that, if the lattice size in the simulation could be increased further (a very computationally demanding process), eventually the critical values would approach those obtained in the mean-field approximation.

V. CONCLUSION

The original Axelrod model had agents only interact with their immediate neighbors on a lattice, modeling the assumption of that geographic proximity largely determines the possibility of interaction. Subsequent work has extended this to neighbors on complex networks, or allowed agent migration, or assumed a well-mixed population (infinite-ranged social interactions) on the assumption that online interactions are making this assumption more realistic [56].

Despite these, and other, increasingly sophisticated modifications of the Axelrod model, however, an examination of the consequences of simply extending the lattice (von Neumann) neighborhood had not been carried out. We have done so, and shown another phase transition in the model, controlled by the von Neumann radius R, as well as the well-known phase transition at the critical value of q, and drawn a $q - R$ phase diagram. We have also used a mean-field analysis to analyze the behavior on an infinite lattice.

These results show that, as well as the value of q, the "scope of cultural possibilities" [1], having a critical value above which a multicultural state prevails, there is also a critical value of the radius of interaction, above which a monocultural state prevails. This simply says that, rather unsurprisingly, a world in which people can only interact with their immediate neighbors is (for a fixed value of q), more likely to remain multicultural than one in which people can interact with those further away. Given this inevitability of a monocultural state for large enough "neighborhoods", it might be more useful to consider alternative measurements of cultural diversity, such as the "long term cultural diversity" measured using the curve plotting the number of final cultural domains against the initial number of connected cultural components, as the bounded confidence threshold is varied, as described by Valori *et al.* [56] (where a well-mixed population was assumed, and hence a monocultural state results for when the bounded confidence threshold is zero). An obvious extension of this work is to examine the behavior of the Axelrod model on complex networks where the neighborhood is extended to all agents within paths of length R on the network.

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[72] See Supplemental Material at [*URL will be inserted by publisher*] for expressions for the $W_{n,m}^{(k)}(t)$ probabilities and the derivation of the differential equations.

Supplemental Material

Another phase transition in the Axelrod model

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1 Introduction

In this Supplemental Material, we show the derivation of the master equation described by Castellano et al. (2000), given by

$$\frac{dP_m(t)}{dt} = \sum_{k=1}^{F-1} \frac{k}{F} P_k(t) \left[\delta_{m,k+1} - \delta_{m,k} + (g-1) \sum_{n=0}^{F} \left(P_n(t) W_{n,m}^{(k)}(t) - P_m(t) W_{m,n}^{(k)}(t) \right) \right], \quad (1)$$

and its simplification

$$\begin{aligned} \frac{dP_m(t)}{dt} = &\left[\frac{m-1}{F} P_{m-1}(t) - \frac{m}{F} P_m(t) \right] \\ &+ (g-1) \Big[P_{m-1}(t) W_{m-1,m}^{(k)}(t) - P_m(t) W_{m,m-1}^{(k)}(t) \\ &+ P_{m+1}(t) W_{m+1,m}^{(k)}(t) - P_m(t) W_{m,m+1}^{(k)}(t) \Big] \sum_{k=1}^{F-1} \frac{k}{F} P_k(t), \end{aligned} \quad (2)$$

where the zeroth differential equation is

$$\frac{dP_0(t)}{dt} = -\sum_{m=1}^{F} \frac{dP_m(t)}{dt}. \tag{3}$$

2 Derivation of differential equations

If $k = F$ or $k = 0$, then by the dynamics of the Axelrod model no change can occur, hence why the first summation in the master equation excludes these two scenarios. To further understand the set of non-linear differential equations, we first consider just the left hand-side, which we see simplifies to

$$\sum_{k=1}^{F-1} \frac{k}{F} P_k(t) \left[\delta_{m,k+1} - \delta_{m,k}\right] = \frac{m-1}{F} P_{m-1}(t) - \frac{m}{F} P_m(t).$$

The term $P_k(t)$ is the probability of selecting a bond of type k, while the probability of selecting one of these k features is k/F, so the probability of two events both happening is the product $kP_k(t)/F$. In light of this, if $k = m-1$, then a new of bond of type m is created, due to a new common feature across the bond arising. Conversely, if $k = m$, then a bond of type m is removed, due to a new common feature across the bond arising.

We now see in the master equation a balance between creation and removal of bonds. But when such a transition occurs, either creation or removal, then it is possible to create or destroy common features across the bonds for all the other sites, by our assumptions, in the von Neumann neighborhood of the transitioned (or culturally influenced) site in the randomly chosen bond. This can occur to $g(R) - 1$ different sites, since the lattice is infinite, and we have again a balance between creation and removal of common features across bonds. For each bond of the $g(R) - 1$ sites, change can occur regardless of how many common features they share, which is reflected by the second summation in the master equation.

For each transition that occurs for the initially chosen bond, a bond (connected to the other neighborhood sites) with n common features may be influenced by the transition of original bond with k common features so the newly influenced bond now has m common features, where we recall that the probability of such a transition is denoted by $W^{(k)}_{n,m}(t)$. But under such a transition, the number of common features can only increase by one, decrease by one, or remain the same, so we see that the right-hand side of the master equation becomes

$$(g-1)\sum_{n=1}^{F}\left(P_n(t)W^{(k)}_{n,m} - P_m(t)W^{(k)}_{m,n}(t)\right) = (g-1)\Big(P_{m-1}(t)W^{(k)}_{m-1,m}(t) - P_m(t)W^{(k)}_{m,m-1}(t)$$
$$+ P_{m+1}(t)W^{(k)}_{m+1,m}(t) - P_m(t)W^{(k)}_{m,m+1}(t)\Big),$$

where the $W^{(k)}_{m,m}(t)$ terms canceled each other out, while all the other values of $W^{(k)}_{n,m}(t)$ equal zero; also see Van Kampen (1992, Chapter VI). This results in

$$\begin{aligned}\frac{dP_m(t)}{dt} &= \left[\frac{m-1}{F}P_{m-1}(t) - \frac{m}{F}P_m(t)\right] \\ &+ (g-1)\sum_{k=1}^{F-1}\frac{k}{F}P_k(t)\Big[P_{m-1}(t)W^{(k)}_{m-1,m}(t) - P_m(t)W^{(k)}_{m,m-1}(t) \\ &+ P_{m+1}(t)W^{(k)}_{m+1,m}(t) - P_m(t)W^{(k)}_{m,m+1}(t)\Big]. \qquad (4)\end{aligned}$$

We now need to derive the transition probabilities $W^{(k)}_{n,m}(t)$, which we will see are independent of k in the mean-field analysis.

3 Derivation of $W^{(k)}_{n,m}(t)$

We consider three sites, which we simply refer to as the first, second and third sites. The first and second sites are connected by the first bond with k common features, while the second and third sites are connected by the second bond with n common features. We will consider the different types of bonds in terms of their common features. By the dynamics of the model, we know features will not change if the number of common features on the first bond is $k = 0$ or $k = F$, so these two cases are excluded. We assume the first bond has k common features, where $0 < k < F$, while the second bond has n common features. For concreteness, we can assume without loss of generality that the culture vector of the first site is a zero vector $(0, 0, \dots, 0)$ with length F, while the second site's culture vector is $(0, \dots, \sigma_{k+1}, \dots, \sigma_F)$, where $\sigma_i \neq 0$ for $i > k$, which ensures that the first and the second sites have only k common features. Similarly, the third site has another cultural vector $(\sigma'_1, \sigma'_2, \dots, \sigma'_F)$, which we can describe with an index set $I_n \subset \{1, \dots, F\}$, so there are n elements such that $\sigma'_i = \sigma_i$ for $i \in I_n$. In other words, there are n entries of the third culture vector that coincide with the entries of the second culture vector, resulting in n common features across the second bond.

We now assume another common feature is created across the first bond, so we set $\sigma_{k+1} = 0$, which may create or remove a common feature across the second bond or have no effect at all. If $n = F$, then $\sigma'_i = 0$ for $i = 1, \ldots, k$ and $\sigma'_i = \sigma_i$ for $i > k$, so a common trait across the second bond, corresponding to σ_{k+1} and σ'_{k+1}, must be removed, implying

$$W^{(k)}_{F,F-1}(t) = 1.$$

For $n = 0$, two possibilities exist: a new common feature is created if $\sigma'_{k+1} = 0$, which we will assume occurs with probability $\rho(t) = \mathbb{P}(\sigma'_{k+1} = 0)$, or no new feature is created, which occurs with probability $1 - \rho(t)$.

If $0 < n < F$, the new feature change will remove a common feature across the second bond if its corresponds to one of the n common features across the third bond. The probability of this event can be reasoned by first noting that there are in total $\binom{n}{F}$ different ways to have the n common features on the second bond. But only one of those common features will be the $(k+1)$ th one, meaning the other $n-1$ common features can be arranged in $\binom{n-1}{F-1}$ different ways. The ratio of these two numbers and the previous transition probabilities give the general expression

$$W^{(k)}_{n,n-1}(t) = \frac{\binom{F-1}{n-1}}{\binom{F}{n}} = \frac{n}{F}, \quad 0 \leq n \leq F.$$

Given that the above event does not happen, the number of common features across the second bond remains or increases by one, so then the probability of n becoming $n+1$ is the probability of $\sigma'_{k+1} = 0$, which we assume is also given by $\rho(t) = \mathbb{P}(\sigma'_{k+1} = 0)$, since $\sigma_{k+1} = 0$, and $\sigma'_{k+1} \neq 0$, then n remains n, which leads to the remaining transition probabilities

$$W^{(k)}_{n,n}(t) = \left[1 - W^{(k)}_{n,n-1}(t)\right][1 - \rho(t)], \quad W^{(k)}_{n,n+1} = \left[1 - W^{(k)}_{n,n-1}(t)\right]\rho(t), \quad 0 \leq n \leq F.$$

Choice of $\rho(t)$ For $\rho(t)$, the original choice was

$$\rho(t) = \frac{1}{F}\sum_{k=1}^{F} kP_k(t),$$

which is the average fraction of common features across a randomly chosen bond. However, ρ may be set to a constant value, giving qualitatively the same results (Castellano et al., 2000). We note that the choice of ρ means that there is no dependence on k in the expressions for the transition probabilities, which simplifies the summation in the master equation.

3.1 Small g analysis

When the coordination number $g = 1$, the nonlinear component of the differential equations (2) disappears giving a tractable coupled set of linear differential equations

$$\frac{dP_m(t)}{dt} = \frac{m-1}{F}P_{m-1}(t) - \frac{m}{F}P_m(t), \quad 1 \leq m \leq F, \tag{5}$$

which can be solved using standard techniques such as Laplace transforms. For $F \geq 3$, the first three solutions are

$$P_1(t) = P_1(0)e^{-t/F} \tag{6}$$

$$P_2(t) = P_1(0)[e^{-t/F} - e^{-2t/F}] + P_2(0)e^{-2t/F} \tag{7}$$

$$P_3(t) = P_1(0)[e^{-t/F} + 2e^{-2t/F} - e^{-3t/F}] + 2P_2(0)[e^{-2t/F} - e^{-3t/F}] + P_3(0)e^{-3t/F}. \tag{8}$$

Now there are naturally no changes induced from other feature updates, and then, for all $m \geq 1$ the probability P_m will rapidly converge to zero, or in other words, P_0 will rapidly converge to one. To ensure non-zero convergence of P_m, the nonlinear term

$$(g-1)\sum_{k=1}^{F-1}\frac{k}{F}P_k(t)\left[P_{m-1}(t)W^{(k)}_{m-1,m}(t) - P_m(t)W^{(k)}_{m,m-1}(t) \right. \\ \left. + P_{m+1}(t)W^{(k)}_{m+1,m}(t) - P_m(t)W^{(k)}_{m,m+1}(t)\right], \tag{9}$$

must be sufficiently large and positive, which can be achieved by suitably varying the parameters g, F, or q, the last model parameter appearing only in the initial values $P_m(0)$ of the differential equations.

4 Numerical solution

The numerical solution of the master equation was carried out using MATLAB®. The MATLAB® code is available from `https://sites.google.com/site/alexdstivala/home/axelrod_qrphase/`.

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