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Single logarithmic conditional stability in determining unknown boundaries

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Abstract

We prove a conditional stability estimate of log-type for determining unknown boundaries from a single Cauchy data taken on an accessible subboundary. Our approach relies on new interior and boundary estimates derived from the Carleman estimate for elliptic equations. A local stability result for target identification of an acoustic sound-soft scatterer from a single far-field pattern is also obtained.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the elliptic differential operator

$$(Au)(x) := -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u, \quad x \in \Omega, \tag{1}$$

where $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, $1 \le i, j \le n$, and there exists a positive constant σ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \sigma \sum_{i=1}^{n} \xi_i^2, \quad \xi_1, ..., \xi_n \in \mathbf{R},\ x \in \Omega. \tag{2}$$

We assume that

$$0 \le c \quad \text{in} \quad \Omega. \tag{3}$$

Let $D \subset \Omega$ be a simply connected star-shaped subdomain such that $\overline{D} \subset \Omega$. Throughout the paper, we define the complement of D in Ω as $D^c := \Omega \setminus \overline{D}$ and suppose that ∂D and $\partial\Omega$ are both of C^4-class. Let $u = u(D)$ be a solution to the Dirichlet boundary value problem

$$Au = 0 \quad \text{in } D^c, \quad u|_{\partial D} = 0.$$

For simplicity we write $\partial_A u = \sum_{i,j=1}^{n} a_{ij}(\partial_j u)\nu_i$, which will be referred to as the Neumann data of u at ∂D where $\nu = (\nu_1, ..., \nu_n)$ denotes the unit outward normal vector at $\partial\Omega$. One aim of this paper concerns a stability estimate of the following inverse problem with a single Cauchy data:

Inverse Problem 1 (IP1): Determine the shape ∂D from knowledge of the Cauchy data $(u, \partial_A u)|_\Gamma$ where $\Gamma \subset \partial\Omega$ is an arbitrarily chosen subboundary.

The above inverse problem arises from, for example, the detection of the inaccessible interior corroded boundary ∂D by the measurement data taken on an accessible outer subboundary Γ. There have been many works on this inverse boundary problem. As for works in view of numerics related to non-destructive testing technique, we refer, for example, to [3,4,22,24,25].

The purpose of this paper is to propose a novel approach based only on Carleman estimates for proving a conditional stability estimate of logarithmic type. Let $D_1, D_2 \subset \Omega$ be two simply connected domains such that $\overline{D_j} \subset \Omega$ and ∂D_j is of C^4-class. Let $u_j = u(D_j)$ satisfy

$$
\begin{aligned}
&Au_j = 0 && \text{in } D_j^c, \\
&u_j = 0 && \text{on } \partial D_j, \\
&u_j = g_j, \quad \partial_A u_j = h_j && \text{on } \Gamma
\end{aligned}
\tag{4}
$$

for $j = 1, 2$, where $g_j \in H^3(\Gamma)$ and $h_j \in H^2(\Gamma)$. Since $c \geq 0$ (see (3)), it is well-known that the above boundary value problem admits a unique solution $u_j \in H^4(D_j^c)$.

For (IP1), we suppose the following conditions hold:

Condition A: There exist $M > 0$ and $\delta > 0$ such that

$$
\begin{aligned}
&|\partial D_j| \leq M, \quad \|u_j\|_{C^1(\overline{D_j^c})} + \|u_j\|_{H^4(D_j^c)} \leq M, \quad j = 1, 2 \\
&\text{dist}\,(\partial D_j, \partial\Omega) \geq \delta > 0
\end{aligned}
\tag{5}
$$

where $|\partial D|$ means the Lebesgue measure of the boundary ∂D.

Condition B:

$$
\inf_{x\in\Gamma} |g_j(x)| > C_0 > 0, \quad j = 1, 2. \tag{6}
$$

We are ready to state the first result of this paper.

Theorem 1.1. *Under the conditions (A) and (B) there exists a constant $\theta \in (0, 1)$ such that*

$$
d(\partial D_1, \partial D_2) \leq C \left(\frac{1}{\log \frac{1}{\|u_1 - u_2\|_{H^3(\Gamma)} + \|\partial_A(u_1 - u_2)\|_{H^2(\Gamma)}}} \right)^{\theta}
$$

provided $\|u_1 - u_2\|_{H^3(\Gamma)} + \|\partial_A(u_1 - u_2)\|_{H^2(\Gamma)}$ is sufficiently small. Here d$(\partial D_1, \partial D_2)$ is the Hausdorff distance.

Remark 1.1. *Our argument may be applied to vectorial elliptic equations such as Lamé system and the Navier-Stokes equations. In an analogous manner we can also establish the single logarithmic conditional stability.*

If the condition (3) is not fulfilled, additional assumptions on the geometry of D are needed in order to get the same stability estimate. In the special case of $a_{ij}(x) \equiv \delta_{ij}$, $b_i = 0$ and $c(x) = -k^2$ for some $k > 0$, the equation $-Au = 0$ reduces to the Helmholtz equation $(\Delta + k^2)u = 0$ which models the time-harmonic acoustic wave propagation in an isotropic homogeneous medium. Hence, our inverse problem (IP1) in this case is closely related to the shape identification problem arising from inverse obstacle scattering with a single incoming wave. Below we present a local stability result for target identification of a sound-soft obstacle from a single far-field pattern with a priori assumptions on the underlying scatterer.

Let $D_1, D_2 \in \mathbb{R}^n$ be two distinct sound-soft obstacles embedded in an isotropic homogeneous medium. Assume an incoming wave of the form $u^{in}(x) = \exp(ik\alpha \cdot x)$, $\alpha \in \mathbb{S}^{n-1}$, is incident onto D_j, where $k > 0$ is the wavenumber. Denote by $u_j = u_j(D_j)$ the total field corresponding to D_j. Then the scattered field $u_j^{sc} := u_j - u^{in}$ satisfies the boundary value problem

$$(\Delta + k^2)u_j^{sc} = 0 \quad \text{in} \quad \mathbb{R}^n \backslash \overline{D}_j, \quad u_j^{sc} = -u^{in} \quad \text{on} \quad \partial D_j, \tag{7}$$

and the Sommerfeld radiation condition

$$\lim_{|x| \to \infty} |x|^{\frac{n-1}{2}} \left\{ \frac{\partial u_j^{sc}}{\partial |x|} - iku_j^{sc} \right\} = 0, \quad j = 1, 2. \tag{8}$$

In particular, the Sommerfeld radiation condition (8) leads to the asymptotic expansion

$$u^{sc}(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}}\, u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^{n/2}}\right), \quad |x| \to +\infty, \tag{9}$$

uniformly in all directions $\hat{x} := x/|x| \in \mathbb{S}^{n-1}$. The function $u^\infty(\hat{x})$ is an analytic function defined on $\mathbb{S}^{n-1}$ and is referred to as the *far-field pattern* or the *scattering amplitude*. The vector $\hat{x} \in \mathbb{S}^{n-1}$ is called the observation direction of the far field. The classical inverse obstacle scattering problem can be stated as

Inverse Problem 2 (IP2): Determine the boundary ∂D from a single far-field pattern $u^\infty(\hat{x})$ for all $\hat{x} \in \mathbb{S}^{n-1}$ with fixed $k > 0$.

It remains a long-standing open problem whether a single Cauchy data (or equivalently, a single far-field pattern) can uniquely determine the boundary of a general sound-soft scatterer; see e.g., Colton and Kress [11, Chapter 5.1]. Local uniqueness results were obtained in [12] and [28] under the *smallness* and *closeness* assumptions. Correspondingly, local stability estimates of logarithmic type were verified in [21] and [27] under these*a priori* assumptions. Note that the arguments of [27] are closest to those of [1] using three spheres inequalities, and that in [27] a sharper upper bound of the *closeness* of two sound-soft obstacles were derived from the Faber-Krahn inequality. As a by-product of the proof of Theorem 1.1, we present a novel approach to the stable determination of the boundary of a soft obstacle from a single far-field pattern.

Let $B_R(z) = \{x \in \mathbb{R}^n : |x - z| = R\}$ and $B_R = B_R(O)$. Clearly, B_1 is the unit ball in $\mathbb{R}^n$. Denote by Vol(D) the volume of D in $\mathbb{R}^n$. We assume one of the following a priori conditions holds:

Condition C:

$$D_j \subset B_R \quad \text{with} \quad kR < \eta_n, \quad n = 1, 2, \tag{10}$$

where η_n denotes the first root of the spherical Bessel function ($n = 3$) or Bessel function ($n = 2$) of the first order.

Condition D: There exist two bounded connected domains $D^\pm \subset \mathbb{R}^n$ such that

$$D^- \subset D_j \subset D^+, \quad \text{Vol}\,(D^+ \backslash D^-) \leq (\frac{\eta_n}{k})^n\, \text{Vol}(B_1), \tag{11}$$

where η_n is defined as the same as in Condition C.

The stability of the inverse problem (IP2) is stated as follows.

Theorem 1.2. *Suppose that D_j ($j = 1, 2$) are sound-soft obstacles with C^4-smooth boundaries which satisfy either the* smallness *Condition C or the* closeness *type Condition D. Then the Hausdorff distance of ∂D_1 and ∂D_2 can be estimated by*

$$\boldsymbol{d}(\partial D_1, \partial D_2) \leq C \left(\frac{1}{\log \frac{1}{||u_1^\infty - u_2^\infty||_{L^2(\mathbb{S}^{n-1})}}} \right)^\theta,$$

provided $||u_1^\infty - u_2^\infty||_{L^2(\mathbb{S}^{n-1})} < 1/e$. Here, the constants $\theta \in (0, 1)$ and $C > 0$ depend on the wavenumber k, the regions $D^\pm$ under Condition D or the radius R under Condition C.

Remark 1.2. *The upper bounds in (10) and (11) are derived from the Faber-Krahn inequality which provides a lower bound for the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of the Laplace equation over a bounded domain $\Omega \subset \mathbb{R}^n$, i.e.,*

$$\lambda_1(\Omega) \geq \lambda_1(B_1) \left(\frac{Vol(B_1)}{Vol(\Omega)} \right)^{2/n}. \tag{12}$$

The inequality (12) has been used in ([14]) to improve the local uniqueness results of inverse obstacle scattering in [12] and [28].

Our arguments rely essentially on new elliptic interior and boundary estimates (see Lemmas 2.1 and 2.3 in Section 2) in combination of quantitative unique continuation (see Lemma 3.1 in Section 3), all of which are verified using Carleman estimates for elliptic equations (see Lemma 2.2). For completeness, we will provide in the appendix a proof of the elliptic Carleman estimate based on the integration by parts only. The proofs of Theorems 1.1 and 1.2 will be carried out in Section 4.

2 Interior and boundary estimates

2.1 Interior stability estimate and elliptic Carleman estimate

We introduce the notation $\Lambda(y, \lambda, \nu)$ before stating our interior estimate. An essential ingredient in our analysis is the solution estimate in a level set $\Lambda(y, \lambda, \nu) + \delta\nu$ defined below. Given $y = (y_1, \cdots, y_n) \in \Omega$, $\lambda > 0$ and a unit vector $\nu \in \mathbb{S}^{n-1}$, we denote by $\Lambda(y, \lambda, \nu)$ a paraboloidal domain with the vertex located at y and the axis parallel to ν which is congruent to $y_n < -\lambda \sum_{j=1}^{n-1} y_j^2$. For $\delta > 0$, set

$$\Lambda(y, \lambda, \nu) + \delta\nu := \{x : x - \delta\nu \in \Lambda(y, \lambda, \nu)\} = \bigcup_{x \in \Lambda(y,\lambda,\nu)} \{x + \delta\nu\}, \tag{13}$$

that is, the translation of $\Lambda(y, \lambda, \nu)$ along the direction ν. Note that there are exactly two paraboloidal domains $\Lambda(y, \lambda, \nu)$ uniquely determined by y, λ and μ. In this paper, $\Lambda(y, \lambda, \nu)$ is always chosen such that $\Lambda(y, \lambda, \nu) + \delta\nu \subset \Lambda(y, \lambda, \nu)$ for any $\delta > 0$. Since $\Lambda(y, \lambda, \nu) \cap \Omega$ may have several disconnected components if Ω is not convex, we make the convention that the paraboloidal domain $\Lambda(y, \lambda, \nu)$ always means the connected component of $\Lambda(y, \lambda, \nu) \cap \Omega$ whose boundary contains y. Analogously, the notation $\Lambda(y, \lambda, \nu) \cap \partial\Omega$ always means the intersection of the boundary of this connected domain with $\partial\Omega$. This convention also applies to the paraboloidal domain $\Lambda(y, \lambda, \nu) + \delta\nu$ for $\delta > 0$.

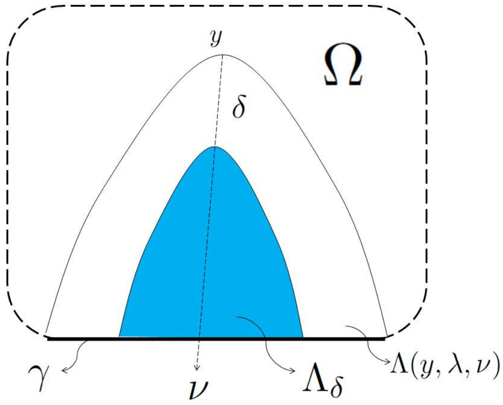

Figure 1: Configurations of Λ, $\Lambda_\delta := (\Lambda(y,\lambda,\nu)+\delta\nu)\cap\Omega$ with $y\in\Omega$ and $\gamma := \partial\Omega\cap\Lambda(y,\lambda,\nu)$.

Lemma 2.1. (interior estimate) *Let $\Omega\subset\mathbb{R}^n$ be a bounded connected domain with the boundary $\partial\Omega$ of C^2-class. Let $y\in\Omega$, $\gamma=\partial\Omega\cap\Lambda(y,\lambda,\nu)$ and $\ell=\min\{t : y+t\nu\in\partial\Omega, t>0\}$. For $0<\delta<\ell$, set $\Lambda_\delta := (\Lambda(y,\lambda,\nu)+\delta\nu)\cap\Omega$ (see Figure 1). Suppose that $u\in H^2(\Omega)$ is a solution to the elliptic equation (1). Then there exist constants $C>0$ and $\kappa\in(0,1)$, which depend on $l,\delta,\lambda,a_{ij},b_i$ and c, such that*

$$
\begin{aligned}
& ||u||_{H^1(\Lambda_\delta)} \\
\leq\ & C\left(||u||_{H^1(\gamma)}+||\partial_\nu u||_{L^2(\gamma)}\right)+C\left(||u||_{H^1(\gamma)}+||\partial_\nu u||_{L^2(\gamma)}\right)^\kappa\ ||u||_{H^1(\Omega)}^{1-\kappa}.
\end{aligned}
$$

Here C and κ do not depend on γ.

Lemma 2.1 yields a stability estimate for u provided that $||u||_{H^1(\Omega)}$ is bounded which is called a conditional stability estimate. Further, it implies that a solution to the elliptic equation (1) with vanishing Cauchy data on an arbitrary non-empty open sub-boundary of $\partial\Omega$ must vanish identically. Lemma 2.1 was proved in [16] by applying the following elliptic Carleman estimate.

Lemma 2.2. *(Carleman estimate) Let $\Omega\subset\mathbb{R}^n$ be a bounded connected domain with the boundary $\partial\Omega$ of C^2-class, and let $D\subset\Omega$ be a domain such that $\overline{D}\subset\Omega$ and ∂D is of C^2-class. Suppose that $d\in C^2(\overline{\Omega})$ satisfy $|\nabla d|\neq 0$ on $\overline{\Omega}$ and set*

$$\varphi(x) := e^{\lambda d(x)}, \qquad x\in\Omega,$$

with a positive parameter $\lambda>0$. There exists positive constants $\lambda_0, s_0(\lambda)$ and $C(s_0,\lambda)$ such that

$$
\begin{aligned}
& \int_D \{s\lambda^2\varphi|\nabla u|^2+s^3\lambda^4\varphi^3u^2\}e^{2s\varphi}dx \\
\leq\ & C\int_D |Au|^2e^{2s\varphi}dx + Ce^{C(\lambda)s}\int_{\partial D}(|\nabla u|^2+|u|^2)\,ds
\end{aligned}
$$

for all $s>s_0$, $\lambda\geq\lambda_0$ and for all $u\in H^2(D)$. Here the constants s_0, C are dependent on λ, but independent of s and the geometry of D, and are bounded provided that $\max_{1\leq i,j\leq n}\|a_{ij}\|_{C^1(\overline{\Omega})}$, $\max_{1\leq i\leq n}\|b_i\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, $\|d\|_{C^2(\overline{\Omega})}$ are bounded.

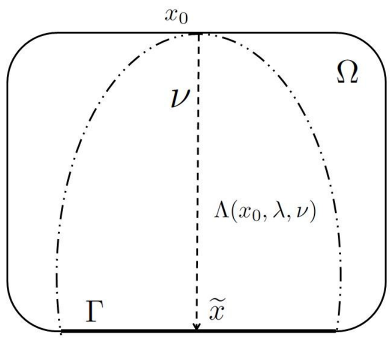

Figure 2: Configurations of $\Lambda(x_0, \lambda, \nu)$ with $x_0 \in \partial\Omega$ and $\Gamma := \partial\Omega \cap \Lambda(x_0, \lambda, \nu)$.

In particular, fixing $\lambda > 0$ sufficiently large, we can rewrite the above estimate as

$$\begin{aligned} &\int_D \{s|\nabla u|^2 + s^3 u^2\} e^{2s\varphi} dx \\ \leq\ & C \int_D |Au|^2 e^{2s\varphi} dx + Ce^{Cs} \int_{\partial D} (|\nabla u|^2 + |u|^2)\, ds \end{aligned} \tag{14}$$

for all $s > s_0$ and all $u \in H^2(D)$.

For clarity we shall present the proof of Lemma 2.2 in the Appendix. We emphasize that the proofs of our interior estimate (see Lemma 2.1) and the estimate at a boundary point (see Lemma 2.3 below) both rely heavily on the Carleman estimate (14).

2.2 Stability at a boundary point

For a boundary point $x_0 \in \partial\Omega$, let $\nu = \nu(x_0)$ be the unit normal vector pointing into the interior of Ω. Given $\lambda > 0$, we denote by $\Lambda(x_0, \lambda, \nu)$ the paraboloidal domain with the vertex located at x_0 and the axis parallel to ν which is congruent to $x_n < -\lambda \sum_{i=1}^{n-1} x_i^2$. Further, one can observe that $\partial\Omega$ intersect with $\Lambda(x_0, \lambda, \nu)$ tangentially at x_0. Moreover, we assume that the surface $\Gamma := \{\Lambda(x_0, \lambda, \nu) \cap \partial\Omega\} \setminus \{x_0\}$ is a non-empty connected relatively open subset of $\partial\Omega$ and there exists $\widetilde{x} \in \Gamma$ such that $\overline{x_0\widetilde{x}}$ is parallel to ν. We set $\ell = |\overline{x_0\widetilde{x}}|$. Assume that $\partial\Omega$ is of C^4-class and $u \in H^4(\Omega)$ is a solution to (1). Next we discuss a conditional stability estimate of u at the boundary point x_0.

Lemma 2.3. *(i) There exist constants $C_2 > 0$ and $\kappa_1 \in (0, 1)$, which depend on ℓ, λ, $\max_{1\leq i,j\leq n} \|a_{ij}\|_{C^1(\overline{\Omega})}$, $\max_{1\leq i\leq n} \|b_i\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, such that*

$$|u(x_0)| \ \leq\ C_2 \max\{1, \|u\|_{H^3(\Omega)}\} \left\{ \left(\frac{1}{|\ln 1/\varrho|} \right)^{\frac{1}{2}} + \varrho^{\kappa_1} \right\}. \tag{15}$$

Here C_2 and κ_1 are independent of choice of x_0, and can be chosen uniformly in $\ell \in [\ell_0, \ell_1]$, where $\ell_0, \ell_1 > 0$ are arbitrarily fixed such that $\ell_0 < \ell_1$.

(ii) If $\varrho \le 1/e$ where $e \approx 2.718281828$, then the estimate in the first assertion can be rewritten as

$$|u(x_0)| \le C_2 \max\{1, \|u\|_{H^3(\Omega)}\} \left(\frac{1}{\ln 1/\varrho}\right)^{\min\{\frac{1}{2},\kappa_1\}}.$$

Proof. (i) By the Sobolev embedding we have $|u(x_0)| \le C_2 \,||u||_{H^3(\Omega)}$, whence the first assertion follows if $\varrho \ge 1$. Hence, it remains to prove the lemma under the assumption that $\varrho \le 1$.

Without loss of generality, after translation and rotation we can define the paraboloidal domain $\Lambda(x_0, \lambda, \nu)$ as

$$\Lambda(x_0, \lambda, \nu) = \{(x', x_n) : x_n < -\lambda \sum_{i=1}^{n-1} x_i^2 + \ell\}, \quad \lambda, \ell > 0$$

with $\nu = (0, \cdots, 0, -1)$, $x_0 = (0, \cdots, 0, \ell)$. Further, we may assume that the point $\tilde{x} \in \Gamma$ coincides with the origin O. Set

$$d(x) = -x_n - \lambda \sum_{i=1}^{n-1} x_i^2 + \ell, \quad D_t := \{x \in \Lambda(x_0, \lambda, \nu) \cap \Omega : d(x) > t\} \quad \text{for } 0 \le t < l/2.$$

We note that $D_{t_2} \subset D_{t_1}$ if $t_1 < t_2$ and $D_t = (\Lambda(x_0, \lambda, \nu) + t\nu) \cap \Omega$. In particular, $D_0 = \Lambda(x_0, \lambda, \nu) \cap \Omega$. We can always choose a cut-off function $\chi_t \in C^\infty(\mathbb{R}^n)$ such that $0 \le \chi_t \le 1$ and

$$\chi_t(x) = \begin{cases} 1, & x \in D_t, \\ 0, & x \in D_0 \setminus D_{t/2}, \end{cases} \qquad \|\chi_t\|_{C^2(\mathbb{R}^n)} \le C_3/t^2, \quad 0 \le t < l/2. \tag{16}$$

In fact, we may choose $\widetilde{\chi} \in C^\infty(\mathbb{R}^n)$ such that $0 \le \widetilde{\chi} \le 1$ and

$$\widetilde{\chi}(\eta) = \begin{cases} 1, & \eta \ge 1, \\ 0, & \eta \le 0. \end{cases}$$

Then the function $\chi_t(x) = \widetilde{\chi}\left(\frac{2d(x)-t}{t}\right)$ satisfies (16). Set $v := \chi_t u$. Using the fact that $D_{2t} \subset D_0$ and applying the Carleman estimate to v in D_0, we obtain

$$\begin{aligned}
&\int_{D_{2t}} (s|\nabla v|^2 + s^3 v^2) e^{2s\varphi} dx \\
\le\ & \int_{D_0} (s|\nabla v|^2 + s^3 v^2) e^{2s\varphi} dx \\
\le\ & C \int_{D_0} \left| \sum_{i,j=1}^n a_{ij}((\partial_i \chi_t)\partial_j u + (\partial_j \chi_t)\partial_i u + (\partial_i \partial_j \chi_t) u) + \sum_{i=1}^n b_i (\partial_i \chi_t) u \right|^2 e^{2s\varphi} dx \\
& + C e^{Cs} \int_\Gamma (|\nabla v|^2 + v^2) ds \\
\le\ & C \int_{D_{t/2} \setminus \overline{D_t}} \left| \sum_{i,j=1}^n a_{ij}((\partial_i \chi_t)\partial_j u + (\partial_j \chi_t)\partial_i u + (\partial_i \partial_j \chi_t) u) + \sum_{i=1}^n b_i (\partial_i \chi_t) u \right|^2 e^{2s\varphi} dx \\
& + C e^{Cs} \int_\Gamma (|\nabla u|^2 + u^2) ds
\end{aligned}$$

where $\varphi = \exp(\lambda d(x))$, $\lambda > 0$ is sufficiently large and $s > s_0$ for some $s_0 > 0$. Since $\varphi(x) \geq \exp(2\lambda t)$ in D_{2t} and $\varphi(x) \leq \exp(\lambda t)$ in $D_{t/2} \backslash D_t$, it can be derived from the previous relation that

$$\|u\|^2_{H^1(D_{2t})} \leq \frac{C_4}{t^4} e^{-2sr(t)} \|u\|^2_{H^1(\Omega)} + C_5 e^{C_0 s} (\|u\|^2_{H^1(\Gamma)} + \|\partial_A u\|^2_{L^2(\Gamma)}) \tag{17}$$

for all $s \geq s_0$, with $r(t) := e^{2\lambda t} - e^{\lambda t}$. Analogously, applying the Carleman estimate to $v_i = \chi_t \partial_i u$ and $v_{ij} = \chi_t \partial_i \partial_j u$, $1 \leq i, j \leq n$ we can obtain

$$\|\nabla u\|^2_{H^1(D_{2t})} + \|\nabla^2 u\|^2_{H^1(D_{2t})} \leq \frac{C_4}{t^4} e^{-2sr(t)} M^2 + C_5 e^{C_0 s} \varrho^2, \quad s \geq s_0, \tag{18}$$

where $\|u\|_{H^3(\Omega)} \leq M$. Combining (17) and (18) gives

$$\|u\|^2_{H^3(D_{2t})} \leq \frac{C_4}{t^4} e^{-2sr(t)} M^2 + C_5 e^{C_0 s} \varrho^2, \quad s \geq s_0, \tag{19}$$

We fix an arbitrary $t_0 \in [0, l/2)$ and $t_0 \leq 1$. By the Sobolev embedding theorem, there exists a constant $C_6 = C_6(t) > 0$ such that

$$\|u\|_{C^1(\overline{D_{2t}})} \leq C_6(t) \, \|u\|_{H^3(D_{2t})}, \quad 0 \leq t \leq t_0.$$

Recalling that D_t is defined by a translation of D_0 and that $D_{2t_0} \neq \emptyset$, $D_{2t_0} \subset D_{2t} \subset D_0$. Choose a constant $C_7 > 0$ uniformly in all $t \in [0, t_0]$ such that

$$\|u\|_{C^1(\overline{D_{2t}})} \leq C_7 \, \|u\|_{H^3(D_{2t})}, \quad \text{for all} \quad 0 \leq t \leq t_0.$$

Hence, it follows from (19) that

$$\|u\|_{C^1(\overline{D_{2t}})} \leq \frac{C_8}{t^2} e^{-sr(t)} M + C_8 e^{C_0 s} \varrho \tag{20}$$

for all $s \geq 0$ and all $t \in [0, t_0]$. We find a value s minimizing the right-hand side of (20) that is, we choose $s \geq s_0$ such that

$$e^{-sr(t)} M = e^{C_0 s} \, \varrho.$$

Consequently we have

$$\|u\|_{C^1(\overline{D_{2t}})} \leq \frac{C_9}{t^2} M^{\frac{C_0}{C_0 + r(t)}} \varrho^{\frac{r(t)}{C_0 + r(t)}} \leq \frac{C_9}{t^2} M_1 \varrho^{\frac{r(t)}{C_0 + r(t)}} \tag{21}$$

for all $0 \leq t \leq t_0$, where we set $M_1 := \max\{M, 1\}$.

For simplicity we write $\partial_n = \partial / \partial_{x_n}$. Since $(0, ..., 0, \ell - 2t) \in \overline{D_{2t}}$, we observe from (2) that

$$|\partial_n u(0, ..., 0, \ell - 2t)| \leq \frac{C_9}{t^2} M_1 \, \varrho^{\frac{r(t)}{C_0 + r(t)}}, \quad 0 \leq t \leq t_0. \tag{22}$$

Using the inequalities

$$e^{2\lambda t} - 2e^{\lambda t} + 1 \geq 0, \quad e^{\lambda t} - \lambda t - 1 \geq 0 \quad \text{for all} \quad t > 0,$$

it is easy to check that

$$\frac{r(t)}{C_0+r(t)} \geq \frac{e^{\lambda t}-1}{C_0+e^{\lambda \ell_0}-e^{\frac{\lambda \ell_0}{2}}} \geq \frac{\lambda}{C_0+e^{\lambda \ell_0}-e^{\frac{\lambda \ell_0}{2}}} t \equiv C_{10} t \tag{23}$$

for some $C_{10}>0$. Since $\varrho \leq 1$, we have by (22) and (23) that

$$|\partial_n u(0,...,0,\ell-2t)| \leq \frac{C_9}{t^2} M_1 \varrho^{C_{10} t}, \quad 0 \leq t \leq t_0.$$

Hence

$$\begin{aligned}
& |\partial_n u(0,...,0,\ell-2t)| \\
= \;& |\partial_n u(0,...,0,\ell-2t)|^{3/4} \, |\partial_n u(0,...,0,\ell-2t)|^{1/4} \\
\leq \;& \|u\|_{C^1(\overline{\Omega})}^{3/4} \left(C_9 \, t^{-2} \, M_1 \varrho^{C_{10} t}\right)^{1/4} \\
\leq \;& M^{3/4} M_1 1/4 \, C_9^{1/4} t^{-1/2} \varrho^{C_{10} t/4} \\
\leq \;& C_{11} M_1 \, t^{-1/2} \, \varrho^{C_{12} t},
\end{aligned}$$

where in the last equality we have used again the Sobolev embedding $\|u\|_{C^1(\overline{\Omega})} \leq C \, \|u\|_{H^3(\Omega)}$. Therefore, by (21) we obtain

$$\begin{aligned}
& |u(x_0)| = |u(0,...,0,\ell)| = \left| u(0,...,0,\ell-2t_0) + \int_{t_0}^{0} \frac{\partial}{\partial t}(u(0,...,0,\ell-2t))dt \right| \\
\leq \;& \|u\|_{C(\overline{D_{2t_0}})} + \int_0^{t_0} 2C_{11} M_1 \, t^{-1/2} \, \varrho^{C_{12} t} dt \\
\leq \;& C_{14} \|u\|_{H^2(D_{2t_0})} + \int_0^{t_0} C_{14} M_1 \, t^{-1/2} \, \exp\left(-\left(C_{12} \log \frac{1}{\varrho}\right) t\right) dt \\
\leq \;& \frac{C_9}{t_0^2} M_1 \varrho^{\frac{r(t_0)}{C_0+r(t_0)}} + C_{14} M_1 \int_0^{\infty} t^{-1/2} \, \exp\left(-\left(C_{12} \log \frac{1}{\varrho}\right) t\right) dt \\
= \;& C_{15} M_1 \varrho^{\kappa_0} + C_{14} M_1 \frac{\Gamma\left(\frac{1}{2}\right)}{\left(C_{12} \log \frac{1}{\varrho}\right)^{\frac{1}{2}}}
\end{aligned}$$

from which the stability estimate (15) follows.

(ii) The second assertion follows straightforwardly from the first assertion in combination with the inequality

$$\varrho \leq \frac{1/e}{\log \frac{1}{\varrho}} < \frac{1}{\log \frac{1}{\varrho}} \quad \text{for all} \quad 0 \leq \varrho \leq \frac{1}{e}.$$

□

3 Quantitative unique continuation

The aim of this section is to verify the quantitative unique continuation for solutions of the elliptic equation $Au = 0$ (see (1)). Set $m = \left[\frac{n}{2}\right] + 2$, where the notation $[a]$ denotes the largest natural number not exceeding $a > 0$.

Lemma 3.1. *(Quantitative unique continuation) Let $Au = 0$ in Ω and $||u||_{H^m(\Omega)} \leq M$, where $M > 0$ is an a priori bound. We assume there exists $z \in \Omega$ such that $|u(z)| > C_0$. Suppose further that*

$$|u(x)| < \delta \quad \textit{for all} \quad x \in B_r(y) \subset \Omega, \tag{24}$$

for some $y \in \Omega$ and $\delta, r > 0$, then an upper bound of the radius r can be estimated by

$$r \leq C/C_0^\kappa\, \delta^\theta,$$

where κ, θ and C are positive constants depending only on the space dimension, the region Ω and the distance between z and $\partial\Omega$.

The unique continuation follows directly from Lemma 3.1.

Corollary 3.1. *Let $Au = 0$ in Ω and $u \equiv 0$ in $B_r(y) \subset \Omega$ for some $r > 0$, $y \in \Omega$. Then $u \equiv 0$.*

Proof. Assume on the contrary that $|u(z)| > C_0 > 0$ for some $z \in \Omega$. Since $u \equiv 0$ in $B_r(y)$, we have $|u(x)| < \delta$ for any $\delta > 0$ and for all $x \in B_r(y)$. Applying Lemma 3.1 we see $r \leq C/C_0^\kappa\, \delta^\theta$ for all $\delta > 0$. Now, letting $\delta \to 0$ yields the relation $r = 0$, which contradicts the fact that $r > 0$. Hence $u \equiv 0$ in Ω. □

Below we carry out the proof of Lemma 3.1, relying on the use of the interior estimate in Lemma 2.1.

Proof of Lemma 3.1. For notational convenience, we write $x' = (x_2, \cdots, x_n)$ so that $x = (x_1, x'), z = (z_1, z') \in \mathbb{R}^n$. Without loss of generality we suppose that y coincides with the origin O, $|z'| = 0$ and $0 < r < 1$. Using the interior estimate (see [13]), it follows from (24) that

$$||\nabla u||_{L^\infty(B_{r/2})} \leq C_1/r||u||_{L^\infty(B_r)} \leq C_1\delta/r, \tag{25}$$

where the constant $C_1 > 0$ is independent of r. Hence,

$$||u||_{W^{1,\infty}(B_{r/2})} \leq C_1\delta(1 + 1/r). \tag{26}$$

We may always choose a paraboloidal domain $\Lambda(y, \lambda, \nu)$ with $y \in \Omega, \nu = (-1, 0, \cdots, 0)$ such that $B_{rr_0}(z) \subset \{\Lambda(y, \lambda, \nu) + \delta_0\nu\} \cap \Omega$ for some $r_0, \delta_0 > 0$. Note that the point y and the parameters λ, r_0 and δ involved are dependent only on the geometry of Ω and the distance between z and $\partial\Omega$. By Lemma 2.1,

$$||u||_{H^1(B_{rr_0}(z))} \leq ||u||_{H^1(\Omega_\delta)} \leq C_2\left(||u||_{H^1(\gamma)} + ||\partial_\nu u||_{L^2(\gamma)}\right)^\kappa \tag{27}$$

for some $\kappa \in (0, 1]$ and $C_2 > 0$ independent of $\gamma = \{\Lambda(y, \lambda, \nu) + \delta_0\nu\} \cap \partial\Omega$. Further, without loss of generality we may suppose that $\gamma \subset \{(0, x') : |x'| < r/2\}$. Otherwise, this can be achieved by constructing a family of paraboloidal domains. Combining the estimates in (26) and (27), we obtain

$$||u||_{H^1(B_{rr_0}(z))} \leq C_2(C_1\delta(1 + 1/r)\, r^{(n-1)/2})^\kappa \leq C_3\, \delta^\kappa\, (1 + r^{n-3})^{\kappa/2}, \tag{28}$$

where $C_3 > 0$ does not depend on δ. Moreover, recalling the inequality $(r^{n-3})^{\frac{\kappa}{2}} \leq Cr^{-\kappa}$ for all $r \in (0, 1]$, it holds that

$$\|u\|_{H^1(B_{rr_0}(z))} \leq C_4\delta^\kappa\, r^{-\kappa}, \quad C_4 > 0.$$

Now, applying Lemma 3.2 below we obtain for $m = [\frac{n}{2}] + 1$ and $\theta = 1/m \in (0,1)$ that

$$\begin{aligned} \|u\|_{L^\infty(B_{rr_0}(z))} &\leq C\,(rr_0)^{-m-n/2}||u||^{\theta}_{H^1(B_{rr_0}(z))} \\ &\leq C\,r_0^{-m-\frac{n}{2}} r^{-m-\frac{n}{2}} r^{-\kappa\theta}\delta^{\kappa\theta} \\ &= C\,r^{-\mu_1}\delta^{\mu_2} \end{aligned}$$

where $\mu_1 = m + \frac{n}{2} + \kappa\theta > 0$ and $\mu_2 = \kappa\theta \in (0,1)$. Since $|u(z)| > C_0 > 0$, we have

$$C_0 \leq \|u_1\|_{L^\infty(B_{rr_0}(z))} < Cr^{-\mu_1}\delta^{\mu_2},$$

leading to the relation

$$r^{\mu_1} \leq C\,C_0^{-1}\delta^{\mu_2}.$$

Finally, an upper bound of r can be estimated by

$$r \leq C/C_0^{1/\mu_1}\delta^{\mu_2/\mu_1}.$$

The proof of the lemma is complete. □

In proving the quantitative unique continuation we have used the following result.

Lemma 3.2. *Let $B_r = B_r(O) \subset \mathbb{R}^n$ for some $r \in (0,1)$. Suppose that*

$$||u||_{H^{m+1}(B_r)} \leq M, \quad m := \big[n/2\big] + 1.$$

Then there exists a constant $C = C(M,n) > 0$ such that

$$||u||_{L^\infty(B_r)} \leq C\ r^{-m-n/2}\,||u||^{1/m}_{H^1(B_r)}. \tag{29}$$

Proof. By the change of variables $y = x/r$ and $\widetilde{u}(y) := u(x)$, we have

$$\int_{B_r}\sum_{|\alpha|\leq m}|\partial^\alpha_x u|^2 dx = \int_{B_1}\sum_{|\alpha|\leq m}|\partial^\alpha_y \widetilde{u}|^2 r^{n-2\alpha}dy.$$

Hence there exist $C_0, C_1 > 0$ independent of $r \in (0,1)$ such that

$$C_0 r^{\frac{n}{2}}\|\widetilde{u}\|_{H^m(B_1)} \leq \|u\|_{H^m(B_r)} \leq C_1\,r^{\frac{n}{2}-m}\|\widetilde{u}\|_{H^m(B_1)}. \tag{30}$$

Let $m = \left[\frac{n}{2}\right] + 1$ and $m' = \left[\frac{n}{2}\right] + 2$. In B_1 we have an interpolation inequality:

$$\|\widetilde{u}\|_{H^m(B_1)} \leq C\|\widetilde{u}\|^{\frac{m'-m}{m'-1}}_{H^1(B_1)}\|\widetilde{u}\|^{\frac{m-1}{m'-1}}_{H^{m'}(B_1)} = C\|\widetilde{u}\|^{1/m}_{H^1(B_1)}\|\widetilde{u}\|^{1-1/m}_{H^{m'}(B_1)}.$$

Making use of (30) we get

$$\|u\|_{H^m(B_r)} \leq C_2\,r^{-m}\|u\|^{1/m}_{H^1(B_r)}\|u\|^{1-1/m}_{H^{m'}(B_r)}. \tag{31}$$

Moreover, applying the Sobolev embedding theorem yields

$$\|\widetilde{u}\|_{L^\infty(B_1)} \leq C_3\|\widetilde{u}\|_{H^m(B_1)}.$$

Together with the first inequality in (30), this implies that

$$\|u\|_{L^\infty(B_r)} \leq C_3\,C_0^{-1}r^{-\frac{n}{2}}\|u\|_{H^m(B_r)}.$$

We use (31) to estimate the right hand side of the previous inequality to obtain

$$\|u\|_{L^\infty(B_r)} \leq C_4\,r^{-m-\frac{n}{2}}\|u\|^{1/m}_{H^1(B_r)}\|u\|^{1-1/m}_{H^{m'}(B_r)} \leq C_5\,r^{-m-\frac{n}{2}}\|u\|^{1/m}_{H^1(B_r)},$$

which proves (29). □

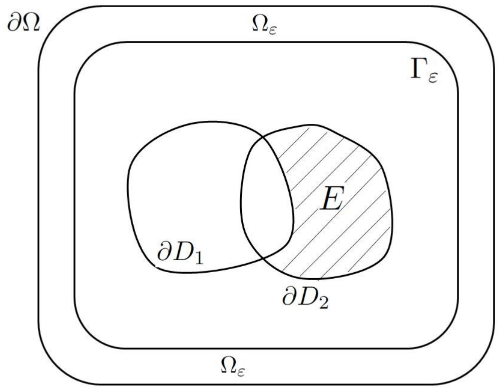

Figure 3: Illustration of two sub-boundaries ∂D_1, ∂D_2 and the domain $E := D_1^c \backslash \overline{D}_2^c$.

4 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Set

$$u = u_1 - u_2 \qquad \text{in } D_1^c \cap D_2^c$$

and

$$\varrho := \|u_1 - u_2\|_{H^3(\Gamma)} + \|\partial_A (u_1 - u_2)\|_{H^2(\Gamma)}.$$

Let $\Omega_\varepsilon = \{x : \text{ dist }(x, \partial\Omega) < \varepsilon\}$ with sufficiently small $\varepsilon > 0$; see Figure 3. Since the parameter $\lambda > 0$ of the parabolic domain $\Lambda(y, \lambda, \nu)$ in Lemma 2.1 can be chosen arbitrarily large, we can always construct a family of paraboloidal domains to prove that

$$\|u\|_{H^1(\Omega_\varepsilon)} \le C_1 \varrho^{\kappa_1},$$

where the constant $\kappa_1 \in (0, 1]$ depend on $\partial\Omega$ and ε only and the constant C_1 relies on the upper bound M involved in Condition A. We set $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \partial\Omega$. By the interpolation inequality and Condition A, we find

$$\|u\|_{H^{7/2}(\Omega_\varepsilon)} \le C \|u\|^{\frac{1}{6}}_{H^1(\Omega_\varepsilon)} \|u\|^{\frac{5}{6}}_{H^4(\Omega_\varepsilon)} \le C_2\, \varrho^{\kappa_2}.$$

Applying the trace theorem gives

$$\|u\|_{H^3(\Gamma_\varepsilon)} + \|\partial_A u\|_{H^2(\Gamma_\varepsilon)} \le C_3\, \varrho^{\kappa_2},$$

where $C_3 > 0$ depends on $\partial\Omega$, ε and M. Let E be any connected component of $D_1^c \backslash \overline{D_2^c}$; see the shadow area in Figure 3. Since ∂D_2 is star-shaped, the boundary $\partial E \cap \partial D_2$ can be connected to Γ in $\Omega \setminus (\overline{D_1 \cup D_2})$. We apply Lemma 2.3 (ii) to the region $\Omega \setminus (\overline{D_1 \cup D_2})$ to obtain an estimate of u on $\partial E \cap \partial D_2$:

$$\|u\|_{L^\infty(\partial E \cap \partial D_2)} \le C_4 \left(\frac{1}{\log \frac{1}{\|u\|_{H^3(\Gamma_\varepsilon)} + \|\partial_A u\|_{H^2(\Gamma_\varepsilon)}}} \right)^{\kappa_3} \le C_4 \left(\frac{1}{\log 1/\varrho} \right)^{\kappa_3},$$

for some $\kappa_3 \in (0, 1/2]$, where $\varrho > 0$ is supposed to be sufficiently small. Since $u_2 = 0$ on ∂D_2, we have

$$\|u_1\|_{L^\infty(\partial E \cap \partial D_2)} \le C_4 \left(\frac{1}{\log\, 1/\varrho} \right)^{\kappa 3}.$$

Using the fact that $u_1 = 0$ on ∂D_1, the previous inequality can be written as

$$\|u_1\|_{L^\infty(\partial E)} \leq C_4 \left(\frac{1}{\log\, 1/\varrho}\right)^{\kappa_3}. \tag{32}$$

We set $B_r(z) = \{x \in \mathbb{R}^n;\ |x - z| < r\}$. Let

$$r_0 = \sup\{r :\ \overline{B_r(z)} \subset E \text{ with some } z \in E\}.$$

That is, r_0 is the radius of the inscribed ball in E. Suppose that $B_{r_0}(z_0) \subset E$ for some $z_0 \in E$. The maximum principle in E yields

$$\|u_1\|_{L^\infty(B_{r_0}(z_0))} \leq \|u_1\|_{L^\infty(E)} \leq C_4 \left(\frac{1}{\log\, 1/\varrho}\right)^{\kappa_3} := \delta_0. \tag{33}$$

On the other hand, it is seen form Condition B that there exist $C_0 > 0$ and $z \in \Gamma$ such that $|u_1(z)| \geq C_0$. Now applying the quantitative unique continuation, we see that

$$r_0 \leq C\, \delta_0^\kappa \leq C \left(\frac{1}{\log\, 1/\varrho}\right)^{\theta} \tag{34}$$

for some $\kappa, \theta \in (0, 1)$. Note that the constant C depends on the a priori bounds involved in Conditions A and B, the region Ω and the upper bounds of the coefficients in equation (1). Since the estimate (34) applies to the radius of the inscribed ball in any connected component of $D_1^c \backslash \overline{D_2^c}$ and $D_1^c \backslash \overline{D_2^c}$, we finish the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Let $D \subset \mathbb{R}^n$ be the unbounded connected component of $(\mathbb{R}^n \backslash \overline{D}_1) \cap (\mathbb{R}^n \backslash \overline{D}_2)$. Analogously to the proof of Theorem 1.1, we set

$$u := u_1 - u_2 \quad \text{ in } \quad D, \quad \varrho := \|u_1^\infty - u_2^\infty\|_{L^2(\mathbb{S}^{n-1})}.$$

We first estimate the near field data in D from the far field pattern. By [21], there exist a radius $R_1 > R$ and a constant $C > 0$ such that

$$||u||_{L^2(B_{R_1+1} \backslash B_{R_1})} \leq C\, \varrho^{\alpha(\varrho)},$$

where the function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\alpha(\varrho) := (1 + \log(-\log \varrho + e))^{-1}.$$

Setting $\Omega := B_{R_1+1/2}$ and $\Gamma = \partial\Omega = \{|x| = R_1 + 1/2\}$, it follows from the elliptic interior estimate that

$$||u||_{H^3(\Gamma)} + ||\partial_\nu u||_{H^2(\Gamma)} \leq C\, \varrho^{\alpha(\varrho)}.$$

Now, we may restrict our discussions to the bounded domain Ω, following the lines in the proof of Theorem 1.1. For this purpose it is necessary to check the Conditions A and B for the inverse problem (IP1). By well-posedness of the forward scattering and the smoothness assumption of ∂D_j, there exist $M, \delta > 0$ such that the relations in (5) hold. On the other hand, Since $|u^{in}(x)| = 1$ in $\mathbb{R}^n$, the boundary $\Gamma = \{|x| = R_1\}$ of $\partial\Omega$ can be chosen depending on the a priori data only such that (see e.g., [26, Corollary 3.3])

$$|u_j(x)| > 1/2, \quad \text{for all} \quad x \in \Gamma, \quad j = 1, 2,$$

which implies the Condition B in (6). Arguing as in the proof of Theorem 1.1, we get (cf. (32))

$$\|u_1\|_{L^\infty(\partial E)} \leq C\,|\alpha(\varrho)\log\,\varrho|^{-\theta} := \delta_0, \quad \theta \in (0,1), \tag{35}$$

where the region $E \subset \Omega$ is defined the same as in the proof of Theorem 1.1. Under the Condition C or D, k^2 is not a Dirichlet eigenvalue of $-\Delta$ in E. Hence the estimate (33) still holds with δ_0 given in (35). Consequently,

$$d(\partial D_1, \partial D_2) \leq C\,|\alpha(\varrho)\log\,\varrho|^{-\theta}, \tag{36}$$

for some $\theta \in (0,1)$. If $\varrho \leq \varrho_0 < 1/e$ for some $\varrho_0 > 0$, we may rewrite (36) as

$$d(\partial D_1, \partial D_2) \leq C\,\left(\frac{1}{\log 1/\varrho}\right)^{\theta'},$$

for some $0 < \theta' < \theta$. □

5 Appendix: Proof of Carleman estimate

Here we give a direct derivation of the Carleman estimate for the elliptic operator A, i.e., Lemma 2.2. There is an approach based on the general theory (e.g., [15, 19, 20]), but we present a direct proof which is based on integration by parts. One can refer to [17, 18] for similar direct derivation of a parabolic Carleman estimate and [5] for a hyperbolic Carleman estimate.

Thanks to the large parameter s, it is sufficient to prove the Carleman estimate in the case of $b_i = c = 0$, $1 \leq i \leq n$, i.e., to verify Lemma 2.2 for the principal part of the elliptic operator A, given by

$$(A_0u)(x) \equiv -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u = f, \quad x \in \Omega.$$

In fact, regarding the lower-order part $\sum_{i=1}^{n} b_i\partial_i u + cu$ as the right-hand side, we can absorb the weighted L^2-norms of the lower-order part into the left-hand side by applying the Carleman estimate for A_0 and taking the parameter $s > 0$ sufficiently large.

Let $D \subset \Omega$ and $\varphi(x) = e^{\lambda d(x)}$ be given as in Lemma 2.2. For notational simplicity we set

$$\sigma(x) = \sum_{i,j=1}^{n} a_{ij}(x)(\partial_i d)(x)(\partial_j d)(x), \quad x \in \overline{D}.$$

Define

$$w(x) := e^{s\varphi(x)}u(x)$$

and

$$Pw(x,t) := e^{s\varphi}A_0(e^{-s\varphi}w) = e^{s\varphi}A_0 u = e^{s\varphi}\,f.$$

Below we give some technical remarks on the proof of the Carleman estimate. The derivation argument consists of three steps:

Step 1: Decomposition of the differential operator P into the sum of P_1 and P_2, where P_1 is composed of the second-order and zeroth-order terms in x, whereas P_2 is composed of first-order terms in x. Here the terms in Pw are classified by the highest order of s, λ and φ.

Step 2: Estimation of $\int_D 2(P_1 w)(P_2 w)dx$ from below.

Step 3: Derivation of another estimate for

$$\int_D Pw \times \text{[the term } u \text{ with second highest order of } s, \lambda, \varphi \text{ among } Pw\text{]}.$$

Moreover the estimate in the second step produces the estimate of u with desirable order of s, λ, φ but not the term of ∇u. This is caused by the different orders of the derivatives of terms under consideration. Therefore another estimate in the third step is necessary. Such kind of double estimates is also used in proving the observability inequality of the time-dependent wave equation by the multiplier method. As for the multiplier method, the two estimates are obtained from (see e.g., Komornik [23, Pages 36-39]):

$$\int_0^T \int_\Omega (\partial_t^2 v - \Delta v)(h(x) \cdot \nabla v) dxdt$$

and

$$\int_0^T \int_\Omega (\partial_t^2 v - \Delta v) v dxdt$$

respectively, with a suitable vector-valued function $h(x)$, and then the estimates are summed up to obtain an L^2-estimate of v. The second estimate for the wave equation via the multiplier method is similar to the third step in our case.

Proof of Lemma 2.2.

Step 1. Let $\nu = \nu(x)$ be the outward unit normal vector to ∂D. Simple calculations show that

$$\begin{aligned} Pw &= -\sum_{i,j=1}^n a_{ij}\,\partial_i\partial_j w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}\,\partial_i d\,\partial_j w \\ &\quad - s^2\lambda^2\varphi^2\sigma w + s\lambda^2\varphi\sigma w + s\lambda\varphi w \sum_{i,j=1}^n a_{ij}\partial_i\partial_j d \end{aligned}$$

in D. Note that in the previous identity we have specified all the dependency of coefficients on s, λ and φ. The last two terms in Pw can be rewritten as $A_1\, w$, where $A_1 = A_1(x; s, \lambda, \varphi, \sigma)$ is defined as

$$\begin{aligned} A_1(x; s, \lambda, \varphi, \sigma) &:= s\lambda^2\varphi\sigma + s\lambda\varphi \sum_{i,j=1}^n a_{ij}\partial_i\partial_j d =: s\lambda^2\varphi\, a_1(x; s, \lambda), \\ a_1(x; s, \lambda) &:= \sigma + (1/\lambda) \sum_{i,j=1}^n a_{ij}\partial_i\partial_j d. \end{aligned}$$

Hence,

$$Pw = -\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j w + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(x)(\partial_i d)\partial_j w - s^2\lambda^2\varphi^2\sigma w + A_1 w.$$

We note that a_1 depends on s and λ, and

$$|a_1(x; s, \lambda)| \le C \quad \text{for } x \in \overline{D} \text{ and all sufficiently large } \lambda > 0 \text{ and } s > 0.$$

Here and henceforth by C, C_1, etc., we denote generic constants which are dependent on λ, but independent of s and the geometry of D, and are bounded provided that $\max_{1\le i,j\le n} \|a_{ij}\|_{C^1(\overline{\Omega})}$, $\max_{1\le i\le n} \|b_i\|_{L^\infty(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, $\|d\|_{C^2(\overline{\Omega})}$ are bounded.

Taking into account the orders of (s, λ, φ), we split P into the sum of P_1 and P_2, where P_1 is composed of second-order and zeroth-order terms in x, whereas P_2 is composed of first-order terms in x. That is,

$$\begin{aligned} P_1 w &:= -\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j w - s^2\lambda^2\varphi^2 w\sigma(x) + A_1 w, \\ P_2 w &:= 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(x)(\partial_i d)\partial_j w. \end{aligned}$$

By $\|fe^{s\varphi}\|^2_{L^2(D)} = \|P_1 w + P_2 w\|^2_{L^2(D)}$, we have

$$2\int_D (P_1 w)(P_2 w)dx \le \int_D f^2 e^{2s\varphi} dx. \tag{37}$$

Step 2: We need to derive a lower bound of the left hand side of (37). Clearly, we have

$$\int_D (P_1 w)(P_2 w)dx = \sum_{k=1}^3 J_k,$$

where

$$\begin{aligned} J_1 &:= -\sum_{i,j=1}^n \int_D a_{ij}(\partial_i\partial_j w) 2s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell}(\partial_k d)(\partial_\ell w)dx, \\ J_2 &:= -\int_D 2s^3\lambda^3\varphi^3\sigma w \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)dx, \\ J_3 &:= \int_D (A_1 w) 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)dx. \end{aligned} \tag{38}$$

Now, applying integration by parts, $a_{ij} = a_{ji}$ and $u \in H^2(D)$ and assuming that $\lambda > 1$ and $s > 1$ are sufficiently large, we reduce all the derivatives of w to $w, \partial_i w$. We continue the estimation of J_k,

$k = 1, 2, 3$ as follows. First,

$$\begin{aligned} J_1 &= -\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\int_D 2s\lambda\varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_\ell w)(\partial_i\partial_j w)dx \\ &= 2s\lambda\int_D\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\lambda(\partial_i d)\varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_\ell w)(\partial_j w)dx \\ &\quad +2s\lambda\int_D\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\varphi\partial_i(a_{ij}a_{k\ell}\partial_k d)(\partial_\ell w)(\partial_i w)dx \\ &\quad +2s\lambda\int_D\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_i\partial_\ell w)(\partial_j w)dx \\ &:= J_1^{(1)} + J_1^{(2)} + J_1^{(3)}. \end{aligned}$$

The first and third terms in J_1 can be estimated by

$$J_1^{(1)} = 2s\lambda^2\int_D\varphi\left|\sum_{i,j=1}^{n}a_{ij}(\partial_i d)(\partial_j w)\right|^2 dx \geq 0,$$

and

$$\begin{aligned} J_1^{(3)} &= \int_D 2s\lambda\sum_{k,\ell=1}^{n}\left(\sum_{i>j}\varphi a_{ij}a_{k\ell}(\partial_k d)\{(\partial_i\partial_\ell w)(\partial_j w) + (\partial_j\partial_\ell w)(\partial_i w)\}dx\right. \\ &\qquad\left. +\sum_{k,\ell=1}^{n}\sum_{i=1}^{n}\varphi a_{ii}a_{k\ell}(\partial_k d)(\partial_i\partial_\ell w)(\partial_i w)\right)dx \\ &= s\lambda\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\int_D\varphi a_{ij}a_{k\ell}(\partial_k d)\partial_\ell((\partial_i w)(\partial_j w))dx \\ &= s\lambda\int_{\partial D}\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\varphi a_{ij}a_{k\ell}(\partial_k d)(\partial_i w)(\partial_j w)\nu_\ell ds \\ &\quad -s\lambda^2\int_D\varphi\sigma\sum_{i,j=1}^{n}a_{ij}(\partial_i w)(\partial_j w)dx \\ &\quad -s\lambda\int_D\varphi\sum_{i,j=1}^{n}\sum_{k,\ell=1}^{n}\partial_\ell(a_{ij}a_{k\ell}\partial_k d)(\partial_i w)(\partial_j w)dx. \end{aligned}$$

Hence, we can estimate J_1 from below by

$$
\begin{aligned}
J_1 \geq & -\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dx \\
& -C\int_D s\lambda\varphi|\nabla w|^2 dx + 2s\lambda^2 \int_D \varphi \left|\sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)\right|^2 dx \\
& +s\lambda \int_{\partial D} \sum_{i,j=1}^n \sum_{k,\ell=1}^n \varphi a_{ij} a_{k\ell}(\partial_k d)(\partial_i w)(\partial_j w)\nu_\ell dS \\
\geq & -\int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dx - C\int_D s\lambda\varphi|\nabla w|^2 dx \\
& -Cs\lambda \int_{\partial D} \varphi|\nabla w|^2 ds. \qquad (39)
\end{aligned}
$$

On the other hand, the other two terms J_2 and J_3 in the integral $\int_D 2(P_1 w)(P_2 w)dx$ can be estimated by

$$
\begin{aligned}
J_2 = & -\int_D 2s^3\lambda^3\varphi^3\sigma w \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)dx \\
= & -\int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^n \sigma a_{ij}(\partial_i d)\partial_j(w^2)dx \\
= & \int_D s^3\lambda^3 \sum_{i,j=1}^n 3\varphi^2\{\lambda(\partial_j d)\varphi\}\sigma a_{ij}(\partial_i d)w^2 dx \\
& +\int_D s^3\lambda^3\varphi^3 \sum_{i,j=1}^n \partial_j(\sigma a_{ij}\partial_i d)w^2 dx - \int_{\partial D} \sum_{i,j=1}^n s^3\lambda^3\varphi^3\sigma a_{ij}(\partial_i d)w^2\nu_j dS \\
\geq & \int_D 3s^3\lambda^4\varphi^3\sigma^2 w^2 dx - C\int_D s^3\lambda^3\varphi^3 w^2 dx - C\int_{\partial D} s^3\lambda^3\varphi^3 w^2 ds \qquad (40)
\end{aligned}
$$

and

$$
\begin{aligned}
|J_3| = & \left|\int_D s\lambda^2\varphi a_1 \times 2s\lambda\varphi w \sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)dx\right| \\
= & \left|\int_D 2a_1 s^2\lambda^3\varphi^2 \sum_{i,j=1}^n a_{ij}(\partial_i d)w(\partial_j w)dx\right| \\
= & \left|\int_D a_1 s^2\lambda^3\varphi^2 \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j(w^2)dx\right| \\
= & \left|-\int_D \sum_{i,j=1}^n \partial_j(a_1 s^2\lambda^3\varphi^2 a_{ij}(\partial_i d))w^2 dx + \int_{\partial D} \sum_{i,j=1}^n a_1 s^2\lambda^3\varphi^2 a_{ij}(\partial_i d)w^2\nu_j dS\right| \\
\leq & C\int_D s^2\lambda^4\varphi^2 w^2 dx + C\int_{\partial D} s^2\lambda^3\varphi^2 w^2 ds. \qquad (41)
\end{aligned}
$$

Hence, combining (38)-(41) we obtain

$$\begin{aligned}\int_D (P_1 w)(P_2 w)dx \;\; &\geq \;\; 3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dx - \int_D s\lambda^2\varphi\sigma \sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dx \\ &\quad -C\int_D s\lambda\varphi|\nabla w|^2 dx - C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2 dx \\ &\quad -C\int_{\partial D} s\lambda\varphi|\nabla w|^2 dS - C\int_{\partial D}(s^3\lambda^3\varphi^3 + s^2\lambda^3\varphi^2)w^2 ds.\end{aligned}$$

Rearranging the terms in the previous inequality yields

$$\begin{aligned} & 3\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dx - \int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dx \\ \leq \;\; & \frac{1}{2}\int_D f^2 e^{2s\varphi}dx + C\int_D s\lambda\varphi|\nabla w|^2 dx \\ & +C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2 dx \\ & +C\int_{\partial D}(s\lambda\varphi|\nabla w|^2 + (s^3\lambda^3\varphi^3 + s^2\lambda^3\varphi^2)w^2)ds. \end{aligned} \tag{42}$$

Step 3. The first and the second terms on the left-hand side of (42) have different signs, so we need another estimate. In this step will obtain another estimation of

$$\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dx$$

by means of

$$\int_D (P_1 w + P_2 w)\times(s\lambda^2\varphi\sigma w)dx.$$

Here the factor $s\lambda^2\varphi\sigma w$ is necessary for obtaining the term of $|\nabla w|^2$ with the desirable (s,λ,φ)-factor $s\lambda^2\varphi$. That is, multiplying $s\lambda^2\varphi\sigma w$ to both sides of the equation

$$2s\lambda\varphi\sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w) - \sum_{i,j=1}^n a_{ij}\partial_i\partial_j w - s^2\lambda^2\varphi^2\sigma w + A_1 w = fe^{s\varphi},$$

we obtain

$$\int_D fe^{s\varphi}s\lambda^2\varphi\sigma w dx = \sum_{k=1}^4 I_k, \tag{43}$$

where

$$\begin{aligned} & I_1 := \int_D 2s\lambda\varphi\sum_{i,j=1}^n a_{ij}(\partial_i d)(\partial_j w)s\lambda^2\varphi\sigma w dx,\; I_2 := -\int_D\left(\sum_{i,j=1}^n a_{ij}\partial_i\partial_j w\right)s\lambda^2\varphi\sigma w dx, \\ & I_3 := -\int_D s^3\lambda^4\varphi^3\sigma^2 w^2 dx, \qquad I_4 := \int_D (A_1 w)(s\lambda^2\varphi\sigma w)dx. \end{aligned}$$

Now, using integration by parts and the relation $\partial_i\varphi = \lambda(\partial_i d)\varphi$, we estimate the terms I_j ($j = 1, 2, 3, 4$) as follows.

$$
\begin{aligned}
|I_1| &= \left| \int_D s^2\lambda^3\varphi^2\sigma \sum_{i,j=1}^n a_{ij}(\partial_i d)\partial_j(w^2)dx \right| \\
&= \Bigg| -\int_D \sum_{i,j=1}^n s^2\lambda^3\{2\lambda(\partial_j d)\varphi^2\}\sigma a_{ij}(\partial_i d)w^2dx \\
&\quad -\sum_{i,j=1}^n s^2\lambda^3\varphi^2\partial_j(\sigma a_{ij}(\partial_i d))w^2dx + \int_{\partial D}\sum_{i,j=1}^n s^2\lambda^3\varphi^2\sigma a_{ij}(\partial_i d)w^2\nu_j dS \Bigg| \\
&\leq C\int_D s^2\lambda^4\varphi^2w^2dx + C\int_{\partial D} s^2\lambda^3\varphi^2w^2ds; \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= -\int_D s\lambda^2\sum_{i,j=1}^n \varphi\sigma a_{ij}w(\partial_i\partial_j w)dx \\
&= \int_D s\lambda^2\sum_{i,j=1}^n \varphi\sigma a_{ij}(\partial_i w)(\partial_j w)dx + \int_D s\lambda^2\sum_{i,j=1}^n \partial_i(\varphi\sigma a_{ij})w(\partial_j w)dx \\
&\quad -\int_{\partial D} s\lambda^2\sum_{i,j=1}^n \varphi\sigma a_{ij}w(\partial_j w)\nu_i ds \\
&\geq \int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dx - C\int_D s\lambda^3\varphi|\nabla w||w|dx \\
&\quad -C\int_{\partial D} s\lambda^2\varphi|w||\nabla w|ds; \qquad (45)
\end{aligned}
$$

$$
I_3 = -\int_D s^3\lambda^4\varphi^3\sigma^2w^2dx; \qquad (46)
$$

$$
|I_4| \leq C\left|\int_D s\lambda^2\varphi \times s\lambda^2\varphi\sigma w^2dx\right| \leq C\int_D s^2\lambda^4\varphi^2w^2dx. \qquad (47)
$$

Hence, by (43)-(47) we obtain

$$
\begin{aligned}
&\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dx - \int_D s^3\lambda^4\varphi^3\sigma^2w^2dx \\
\leq\ & C\int_D |fe^{s\varphi}s\lambda^2\varphi\sigma w|dx + C\int_D s^2\lambda^4\varphi^2w^2dx + C\int_D s\lambda^3\varphi|\nabla w||w|dx \\
&+C\int_{\partial D}(s^2\lambda^3\varphi^2w^2 + s\lambda^2\varphi|w||\nabla w|)ds. \qquad (48)
\end{aligned}
$$

Since

$$
s\lambda^3\varphi|\nabla w||w| = (s\lambda^2\varphi|w|)(\lambda|\nabla w|) \leq 1/2s^2\lambda^4\varphi^2w^2 + 1/2\lambda^2|\nabla w|^2,
$$

we have

$$\int_D s\lambda^3\varphi|\nabla w||w|dx \le 1/2\int_D (s^2\lambda^4\varphi^2w^2 + \lambda^2|\nabla w|^2)dx. \tag{49}$$

Furthermore, using the inequalities

$$\begin{aligned}
s\lambda^2\varphi|w||\nabla w| &= (s^{1/2}\lambda^{1/2}\varphi^{1/2}|\nabla w|)(s^{1/2}\lambda^{\frac{3}{2}}\varphi^{1/2}w) \\
&\le 1/2s\lambda\varphi|\nabla w|^2 + 1/2s\lambda^3\varphi w^2, \\
|fe^{s\varphi}s\lambda^2\varphi\sigma w| &\le 1/2f^2e^{2s\varphi} + 1/2s^2\lambda^4\varphi^2\sigma^2w^2 \\
&\le 1/2f^2e^{2s\varphi} + Cs^2\lambda^4\varphi^2w^2,
\end{aligned}$$

it follows from (48) and (49) that

$$\begin{aligned}
&\int_D s\lambda^2\varphi\sigma\sum_{i,j=1}^n a_{ij}(\partial_i w)(\partial_j w)dx - \int_D s^3\lambda^4\varphi^3\sigma^2w^2dx \\
\le\ & C\int_D f^2e^{2s\varphi}dx + C\int_D s^2\lambda^4\varphi^2w^2dx + C\int_D \lambda^2|\nabla w|^2dx \\
&+C\int_{\partial D}(s\lambda\varphi|\nabla w|^2 + (s\lambda^3\varphi + s^2\lambda^3\varphi^2)w^2)ds.
\end{aligned} \tag{50}$$

End of the proof. Multiplying (50) by two, adding the resulting expression to (42), and making use of (2) and the relation $\sigma_0 \equiv \inf_{(x,t)\in Q}\sigma(x,t) > 0$, we obtain

$$\begin{aligned}
&\int_D s^3\lambda^4\varphi^3\sigma_0^2w^2dx + \int_D s\lambda^2\varphi|\nabla w|^2dx \\
\le\ & C\int_D f^2e^{2s\varphi}dx + C\int_D (s\lambda\varphi + \lambda^2)|\nabla w|^2dx \\
&+C\int_D (s^3\lambda^3\varphi^3 + s^2\lambda^4\varphi^2)w^2dx \\
&+C\int_{\partial D}(s\lambda\varphi|\nabla w|^2 + (s^3\lambda^3\varphi^3 + s^2\lambda^3\varphi^2 + s\lambda^3\varphi)w^2)ds.
\end{aligned} \tag{51}$$

Therefore, taking $\lambda > 0$ and $s > 0$ sufficiently large, we can absorb the second and the third terms on the right-hand side of (51) into the left-hand side. Consequently, it follows that

$$\begin{aligned}
&\int_D s^3\lambda^4\varphi^3w^2dx + \int_D s\lambda^2\varphi|\nabla w|^2dx \\
\le\ & C\int_D f^2e^{2s\varphi}dx + C\int_{\partial D}(s\lambda\varphi|\nabla w|^2 + s^3\lambda^3\varphi^3w^2)ds.
\end{aligned}$$

Noting $w = ue^{s\varphi}$, we have

$$\begin{aligned}
&\int_D (s\lambda^2\varphi|\nabla u|^2 + s^3\lambda^4\varphi^3u^2)e^{2s\varphi}dx \\
\le\ & C\int_D f^2e^{2s\varphi}dxd + Ce^{C(\lambda)s}\int_{\partial D}(|\nabla u|^2 + u^2)ds,
\end{aligned}$$

which finishes the proof of the Carleman estimate. □

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