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Improvement of flatness for nonlocal phase transitions

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ABSTRACT. We prove an improvement of flatness result for nonlocal phase transitions. For a class of nonlocal equations that includes $(-\Delta)^{s/2}u = u - u^3$, with $s \in (0,1)$, we obtain a result in the same spirit of a celebrated theorem of Savin [30] for the equation $-\Delta u = u - u^3$. As a consequence, we deduce that entire solutions to $(-\Delta)^{s/2}u = u - u^3$ with asymptotically flat level sets are 1D when $s \in (0,1)$.

The results presented are completely new even for the case of the fractional Laplacian, but the robustness of the proofs allows us to treat also more general, possibly anisotropic, integro-differential operators.

Analogous and complementary improvement of flatness results for $(-\Delta)^{s/2}u = u - u^3$ with $s \in [1,2)$ have been (only very recently) obtained by Savin in [32]. The proofs in [32] follow a more robust version of the proof in [30], which has been introduced (also very recently and by Savin) in [31].

We remark that the proofs in [31, 32] are valid only for the "mildly nonlocal case" in which $s \in [1,2)$, while we focus here on the "genuinely nonlocal case" in which $s \in (0,1)$, which presents fundamental differences with respect to the previous ones.

CONTENTS

1. INTRODUCTION

In this paper we establish an improvement of flatness result for a generalized nonlocal Allen-Cahn equation

$$Lu = f(u) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where L is an elliptic scaling invariant operator of order $s \in (0,1)$, of the form

$$Lu(x) := \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) \frac{\mu\big(y/|y|\big)}{|y|^{n+s}}\, dy. \tag{1.2}$$

The typical example of nonlinearity that we take into account is when f is (minus) the derivative of a double-well potential W. Also, throughout the paper, we assume that the measure μ in (1.2), which is often called in jargon the "spectral measure", satisfies

$$\mu(z) = \mu(-z) \quad \text{and} \quad 0 < \lambda \leqslant \mu(z) \leqslant \Lambda < +\infty \quad \text{for all } z \in S^{n-1}. \tag{H0}$$

Equations as in (1.1) naturally arise in several contexts, such as phase transitions, atom dislocations in crystals, mathematical biology, etc. (see e.g. Section 2 in [21], the Appendix in [18], the Introduction in [11], and also [6] and the references therein for a series of motivations under different perspectives).

Motivated by a celebrated conjecture by Ennio De Giorgi in [19], a natural problem in phase transitions is whether or not all the solutions (under appropriate energetic or geometric assumptions) are 1D profiles, i.e. increasing functions of only one Euclidean variable.

The goal this paper is to address this symmetry problem for the nonlocal phase transition equation $(-\Delta)^{s/2}u = u - u^3$, and, even more generally, for equations as in (1.1), in the *genuinely nonlocal regime* in which $s \in (0,1)$.

We use the wording *genuinely nonlocal regime* for the following reason. Although for all $s \in (0,2)$ the diffusion operators $(-\Delta)^{s/2}$ are nonlocal, when studying asymptotic properties (at large scales) of the nonlocal phase transition equations one finds very different behaviors in the two regimes $s \in [1,2)$ and $s \in (0,1)$.

Namely, when $s \in [1,2)$, the interfaces between the two phases exhibit a local behavior at a large scale (in spite of the nonlocal character of the problem) and the phase separation is related to the minimization of the classical perimeter functional (similarly as in the case of the Laplacian $s = 2$ as given by the classical $\Gamma-$convergence result of Modica and Mortola [28]). Conversely, in the genuinely nonlocal regime $s \in (0,1)$, the interface maintains a nonlocal character at any scale and the phase separation is described by a nonlocal perimeter functional introduced by Caffarelli, Roquejoffre and Savin in [12]. A rigorous statement of the previous heuristic description was proven by Savin and the third author in [33] through $\Gamma-$convergence results.

In this paper we prove that asymptotically flat phase transitions are 1D, i.e. depend only on one Euclidean variable. As a consequence we obtain several new classification results for entire minimizers of nonlocal phase transitions along the lines of the conjecture of De Giorgi. These main results will be collected in Theorems 1.2, 1.3, 1.4, 1.5 and 1.6 and discussed in details in the forthcoming Subsection 1.4.

As a matter of fact, the cornerstone for these theorems is an "improvement of flatness" result, which is contained in Theorem 1.1 and which will be presented in Subsection 1.3.

Next we introduce the mathematical framework in which we work, by listing the precise assumptions on the operator L and on the nonlinearity f which are involved in the main equation (1.1). This is done in Subsections 1.1 and 1.2. Let us however tell in advance that $L = (-\Delta)^{s/2}$ and $f(u) = u - u^3$ trivially satisfy the assumptions given in Subsections 1.1 and 1.2 and thus the reader who is interested only in this model equation can skip Subsections 1.1 and 1.2 and go straight to Subsections 1.3 and 1.4 to read the main results.

1.1. **Further hypotheses on L.** Let us introduce the following notation, that we will use throughout the paper, for the fractional Laplacian in dimension 1 (without normalization constants). Given a bounded $\psi \in C^2(\mathbb{R})$, we define

$$\mathcal{L}\psi(z) := \int_{-\infty}^{\infty} \frac{\psi(z) - \psi(z+\zeta)}{|\zeta|^{1+s}}\, d\zeta, \quad z \in \mathbb{R}. \tag{1.3}$$

For ψ as above, $\omega \in S^{n-1}$ and $h > 0$, we define, for any $x \in \mathbb{R}^n$,

$$\bar{\psi}_{\omega,h}(x) := \psi\left(\omega \cdot \frac{x}{h}\right). \tag{1.4}$$

Then, for each operator L of the form (1.2), let $h_L : S^{n-1} \to (0, \infty)$ be defined as follows. We set $h_L(\omega) := h$, where $h > 0$ satisfies

$$L\bar{\psi}_{\omega,h}(x) = \mathcal{L}\psi\left(\omega \cdot \frac{x}{h}\right) \text{ for all } \psi \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{1.5}$$

Using the function h_L, we define the closed convex set

$$\mathcal{C} = \mathcal{C}_L := \bigcap_{\omega \in S^{n-1}} \big\{x \in \mathbb{R}^n \,:\, x \cdot \omega \leqslant h_L(\omega)\big\}. \tag{1.6}$$

We notice also that, since L is even, both h_L and $\mathcal{C}_L$ are even, i.e. symmetric with respect to the origin. In addition, we remark that, when $L = (-\Delta)^{s/2}$, $\mathcal{C}_L$ is a ball (centered at 0).

Our assumption on L is that

$$\partial\,\mathcal{C}_L \text{ is } C^{1,1} \text{ and strictly convex.} \tag{1.7}$$

More quantitatively, we assume that there exist $\rho' > \rho > 0$ such that

$$\text{the curvatures of } \partial\,\mathcal{C}_L \text{ are bounded above by } \frac{1}{\rho} \text{ and below by } \frac{1}{\rho'}. \tag{H1}$$

This assumption is equivalent to saying that for all $x \in \partial\,\mathcal{C}_L$ there exist $\rho' > \rho > 0$ and $y,\, z \in \mathbb{R}^n$ such that $B_\rho(y) \subset \mathcal{C}_L \subset B_{\rho'}(z)$, and $x \in \partial B_\rho(y) \cap \partial B_{\rho'}(z)$.

We remark that the definition of h_L in (1.5) is well posed, and indeed an explicit expression of $h_L(\omega)$ is obtained through the formula

$$h_L(\omega) = \left(\frac{1}{2}\int_{S^{n-1}} |\omega \cdot \theta|^s \, \mu(\theta)\, d\theta\right)^{1/s}. \tag{1.8}$$

To prove (1.8), we proceed as follows

$$\begin{aligned}
L\bar{\psi}_{\omega,h}(x) &= \int_{\mathbb{R}^n} \left(\psi\left(\omega \cdot \frac{x}{h}\right) - \psi\left(\omega \cdot \frac{x+y}{h}\right)\right) \frac{\mu\big(y/|y|\big)}{|y|^{n+s}}\, dy \\
&= \int_0^{+\infty} d\varrho \int_{S^{n-1}} d\theta\, \left(\psi\left(\omega \cdot \frac{x}{h}\right) - \psi\left(\omega \cdot \frac{x}{h} + \omega \cdot \frac{\varrho\theta}{h}\right)\right) \frac{\mu(\theta)}{\varrho^{1+s}} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\varrho \int_{S^{n-1}} d\theta\, \left(\psi\left(\omega \cdot \frac{x}{h}\right) - \psi\left(\omega \cdot \frac{x}{h} + \omega \cdot \frac{\varrho\theta}{h}\right)\right) \frac{\mu(\theta)}{|\varrho|^{1+s}} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\zeta \int_{S^{n-1}} d\theta\, \left(\psi\left(\omega \cdot \frac{x}{h}\right) - \psi\left(\omega \cdot \frac{x}{h} + \zeta\right)\right) \frac{|\omega \cdot \theta|^s\, \mu(\theta)}{h^s\, |\zeta|^{1+s}},
\end{aligned}$$

where we used the change of variables $\zeta = \frac{\varrho\omega\cdot\theta}{h}$. Hence, if $h = h_L(\omega)$ is given by (1.8),

$$L\bar{\psi}_{\omega,h}(x) = \int_{-\infty}^{+\infty} \left(\psi\left(\omega \cdot \frac{x}{h}\right) - \psi\left(\omega \cdot \frac{x}{h} + \zeta\right)\right) \frac{d\zeta}{|\zeta|^{1+s}} = \mathcal{L}\psi\left(\omega \cdot \frac{x}{h}\right),$$

that is (1.5).

A special case of (1.8) occurs when the spectral measure is induced by a convex set, namely when

$$\frac{\mu\big(y/|y|\big)}{|y|^{n+s}} = \frac{1}{\|y\|_K^{n+s}}$$

for some convex set K, where $\|\cdot\|_K$ is the norm with unit ball K, that is, for any $p \in \mathbb{R}^n$,

$$\|p\|_K := \sup\{t > 0 \text{ s.t. } p/t \notin K\}. \tag{1.9}$$

Then, in this case, an integration in polar coordinates yields

$$h_L(\omega) = \left(\frac{1}{2}\int_{S^{n-1}} d\theta \frac{|\omega\cdot\theta|^s}{\|\theta\|_K^{n+s}}\right)^{1/s} = \left(\frac{n+s}{2}\int_{S^{n-1}} d\theta \int_0^{1/\|\theta\|_K} d\varrho\, |\omega\cdot\theta|^s\, \varrho^{n+s-1}\right)^{1/s}$$
$$= \left(\frac{n+s}{2}\int_K |\omega\cdot x|^s dx\right)^{1/s}.$$

As pointed out to us by M. Ludwig, to whom we are indebted for this comment and the interesting references provided, the convex body associated to this support function is the so called "L_p-intersection body" of K. These convex bodies are well studied in convex geometry, in relation to the important Busemann-Petty problem, see [3] and references therein for more information on this subject.

As shown in [3], for any given convex set K (bounded and with nonempty interior) which is symmetric with respect to the origin, the function h_L is strictly convex in all the nonradial directions. Also, from (1.8) it follows that h_L is $C^{1,1}$ in $\mathbb{R}^n \setminus \{0\}$ when μ is $C^{1,1}$. Actually $\mu \in C^{2-s+\varepsilon}$ suffices since the "kernel" $|\omega\cdot\theta|^s$ is C^s and this yields a regularity improvement.

When K is any $C^{1,1}$ convex set, the previous observations imply that the set

$$\mathcal{C}_L^* := \{h_L = 1\}$$

is a $C^{1,1}$, even with respect to 0, strictly convex set. Noting that $\mathcal{C}_L$ and $\mathcal{C}_L^*$ are one the polar body of the other, one can show that $\mathcal{C}_L$ is also a $C^{1,1}$, even, strictly convex set. Indeed, since $\mathcal{C}_L^*$ is a $C^{1,1}$, even, strictly convex set, any point of its boundary can be touched by two even ellipsoids, one contained in, and the other one containing, $\mathcal{C}_L^*$. Considering the polar transformations of these ellipsoids we show the same property for $\mathcal{C}_L$.

1.2. **Hypotheses on f.** Our assumptions on f, precisely stated below, are satisfied when $f = -W'$, with W being a C^2 double-well potential with wells (i.e. minima) at ± 1 and satisfying that $W'' > 0$ near ± 1.

More precisely, and somehow more generally, we assume that f belongs to $C^1\big([-1,1]\big)$ and satisfies, for some $\kappa > 0$ and $c_\kappa > 0$,

$$f(-1) = f(1) = 0 \quad \text{and} \quad f'(t) < -c_\kappa \quad \text{for } t \in [-1, -1+\kappa] \cup [1-\kappa, 1]. \tag{H2}$$

Moreover, we assume that

$$\text{there exists } \phi_0 \text{ satisfying} \quad \begin{cases} \mathcal{L}\phi_0 = f(\phi_0) & \text{in } \mathbb{R}, \\ \phi_0' > 0 & \text{in } \mathbb{R}, \\ \phi_0(0) = 0, \\ \displaystyle\lim_{x\to\pm\infty} \phi_0 = \pm 1. \end{cases} \tag{H3}$$

We recall that $\mathcal{L}$ denotes the one-dimensional fractional Laplacian in (1.3).

When $f = -W'$ and W is a C^2 double-well potential with wells at ± 1, the existence of the previous one-dimensional heteroclinic solution is proven in [29, 9] (see also [17] for the case of general kernels). Thus, condition (H3) is satisfied in this case. Also, condition (H2) is satisfied when $W'' > 0$ near ± 1, and this model case, which has concrete realizations in phase transition models, is for us the main motivating example.

In order to precisely identify the quantities on which the constants in the estimates of the paper depend, let us define

$$l_\kappa := \inf\big\{\, l > 0 \;:\; \phi_0\big([-l, l]\big) \supset [-1+\kappa, 1-\kappa]\big\}. \tag{1.10}$$

More informally, l_κ is (half of) the length of the symmetric interval where the transition of ϕ_0 essentially occurs.

1.3. **The improvement of flatness result.** In the framework that we have just introduced, we are now in the position of stating our main result as follows.

Throughout the paper, we take $s \in (0,1)$ and we call a constant *universal* if it depends only on n, s, λ, Λ, ρ, ρ', κ, c_κ and l_κ, see Subsections 1.1 and 1.2. In particular, universal constants depend only on n, L, and f.

In the statement of the next theorem, for fixed $\alpha_0 > 0$, given $a \in (0,1)$ we define

$$j_a := \left\lfloor \frac{\log a}{\log(2^{-\alpha_0})} \right\rfloor . \tag{1.11}$$

Note that j_a is a nonnegative integer and that $2^{\alpha_0 j_a}$ is comparable to $1/a$.

Theorem 1.1. *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*. Then there exist universal constants $\alpha_0 \in (0, s/2)$, $p_0 \in (2, \infty)$ and $a_0 \in (0, 1/4)$ such that the following statement holds.*

Let $a \in (0, a_0]$ and $\varepsilon \in (0, a^{p_0}]$. Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of $Lu = \varepsilon^{-s} f(u)$ in $B_{2^{j_a}}$ such that $0 \in \{-1+\kappa \leqslant u \leqslant 1-\kappa\}$ and

$$\{\omega_j \cdot x \leqslant -a2^{j(1+\alpha_0)}\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \{\omega_j \cdot x \leqslant a2^{j(1+\alpha_0)}\} \quad \text{in } B_{2^j},$$

for $0 \leqslant j \leqslant j_a$, where $\omega_j \in S^{n-1}$.

Then,

$$\left\{\omega \cdot x \leqslant -\frac{a}{2^{1+\alpha_0}}\right\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \left\{\omega \cdot x \leqslant \frac{a}{2^{1+\alpha_0}}\right\} \quad \text{in } B_{1/2},$$

for some $\omega \in S^{n-1}$.

We observe that Theorem 1.1 is related to the improvement of flatness in [30], with some important differences: besides the nonlinear dependence of ε from a, which is different from the classical case —on which we will comment later on —we remark that our Theorem 1.1 is valid for solutions, and not only for minimizers of the energy functional. This is due to the fact that our methods bypass the use of density estimates.

In order to explain more intuitively the content of Theorem 1.1, let us introduce some (informal) terminology. We call *transition level sets* (of u) all the level sets $\{u = \theta\}$ for $\theta \in (-1+\kappa, 1+\kappa)$. We say that the transition level sets are flat at a scale R if they are trapped, after some rotation, in a cylinder $B'_R \times (-aR, aR)$. We call *flatness* the adimensional quantity a.

With this terminology, Theorem 1.1 says that if the transition level sets are flat enough at a very large scale, then its flatness improves geometrically at the half scale. This result is suited for iteration. However, as we will see in more detail in Section 7, the geometric improvement of the flatness cannot be done up to scale 1 but only up to some (still huge) mesoscale. This is an important difference with respect to the result in [30] and we will comment more on it later on.

Another way to look at the result in Theorem 1.1 is as an approximate $C^{1,\alpha}$ regularity result for level sets. Namely, if the transition level sets of the solution of $Lu = \varepsilon^{-s} f(u)$ in B_1 are trapped between two parallel planes close enough to the origin, and ε is small enough, then the transition occurs essentially on a $C^{1,\alpha}$ graph in $B_{1/2}$ up to errors that decay algebraically (in ε) as $\varepsilon \downarrow 0$. The limit case as $\varepsilon \downarrow 0$ of this result plays a crucial role in the regularity theory of nonlocal minimal surfaces; see Theorem 6.8 in [12].

1.4. 1**D symmetry of asymptotically flat entire solutions and consequences.** An important consequence of Theorem 1.1 is a general rigidity result. It states that entire solutions of nonlocal Allen-Cahn equations satisfying an “appropriately weak” asymptotic flatness condition for its transition level sets possess one-dimensional symmetry, i.e. their level sets are hyperplanes.

Combining Theorem 1.1 with asymptotic results such that the Γ-convergence results of Savin and the third author in [33] and the classification of minimizing cones for the fractional perimeter [34, 16, 26], we obtain new 1D symmetry results for entire solutions of $(-\Delta)^{s/2}u = f(u)$ in dimension $n \leqslant 8$. In this section we list these new results.

As explained at the beginning of the introduction, these symmetry properties for solutions of equations which model phase transitions are a classical topic of research that goes back to a famous conjecture posed by De Giorgi in [19]. The problem has been widely studied in the local setting (see e.g. [2, 1, 27, 4, 30, 38, 20, 24, 25]) and some results have been obtained also in the nonlocal case (see [10, 37, 7, 8, 36]). See also the very recent results in [31, 32], in which the techniques invented in [30] have been suitably extended and modified in order to deal with the equation $(-\Delta)^{s/2}u = u - u^3$ with $s \in (1, 2)$ (notice that the range of s dealt with in [31, 32] is complementary with the range treated in this paper, which takes into account the genuinely nonlocal case).

To state our results, it is convenient to give the following definition: we say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is 1D if it depends only on one Euclidean variable, up to a rotation, namely if there exist $\bar{u} : \mathbb{R} \to \mathbb{R}$ and $\bar{\omega} \in S^{n-1}$ such that $u(x) = \bar{u}(\bar{\omega} \cdot x)$ for any $x \in \mathbb{R}^n$.

The following general theorem will lead to new rigidity results in different concrete situations: for minimizers or monotone solutions of fractional Allen Cahn equations, for stable solutions, etc. We write it in this general form so that it can be neatly applied to all these situations.

Theorem 1.2 (One-dimensional symmetry for asymptotically flat solutions)**.** *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*.*

Let u be a solution of $Lu = f(u)$ in $\mathbb{R}^n$.

Assume that there exist $R_0 \geqslant 1$ and $a : (R_0, +\infty) \to (0, 1]$ such that $a(R) \downarrow 0$ as $R \uparrow +\infty$ and such that, for all $R > R_0$, we have

$$\{\omega \cdot x \leqslant -a(R)R\} \subset \{u \leqslant -1 + \kappa\} \subset \{u \leqslant 1 - \kappa\} \subset \{\omega \cdot x \leqslant a(R)R\} \quad \text{in } B_R, \tag{1.12}$$

for some $\omega \in S^{n-1}$, which may depend on R.

Then, u is 1D.

Let us point out that Theorem 1.2 will be a consequence of Theorem 1.1, but it does not immediately follows from it. Its proof is nontrivial because in Theorem 1.1 we (need to) assume that $\varepsilon \leqslant a^{p_0}$, instead of $\varepsilon \leqslant ca$ as in [30]. As a consequence, applying iteratively Theorem 1.1 to a flat enough interface in a large ball B_R, we improve geometrically the flatness, but only up to a mesoscale B_r, where $r = R^{1-\delta}$. If instead we had a condition like $\varepsilon \leqslant ca$, we could improve the flatness right away up to scale 1 and the Theorem 1.2 would be an immediate consequence of Theorem 1.1.

For this reason, the proof of Theorem 1.2 requires a suitable multiscale iteration of Theorem 1.1, combined with the use of the sliding method of Berestycki, Caffarelli and Nirenberg [4, 5], appropriately modified to treat the nonlocal case (see e.g. [23]). See Subsection 1.5 for further details on the proofs.

Now we consider the concrete case of minimizing solutions of the nonlocal Allen-Cahn equation $(-\Delta)^{s/2}u = u - u^3$, with $s \in (0, 1)$. We remark that the problem is variational, with associate energy functional given by

$$\mathcal{E}(u, \Omega) := \mathcal{E}^{\rm Dir}(u, \Omega) + \int_\Omega (1 - u^2(x))^2 \, dx,$$

where, for some appropriate constant $C_{n,s} > 0$,

$$\mathcal{E}^{\rm Dir}(u, \Omega) := C_{n,s} \iint_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+s}} \, dx \, dy. \tag{1.13}$$

We say that a solution u of $(-\Delta)^{s/2}u = u - u^3$ is a *minimizer* of $\mathcal{E}$ in $\mathbb{R}^n$ if

$$\mathcal{E}(u, B) \leqslant \mathcal{E}(u + \varphi, B),$$

for any ball $B \subset \mathbb{R}^n$ and any $\varphi \in C_0^\infty(B)$ (notice that, for simplicity, we are dropping the normalization constant in the fractional Laplace framework).

In this setting, we have:

Theorem 1.3 (One-dimensional symmetry in the plane)**.** *Let u be a minimizer of $\mathcal{E}$ in $\mathbb{R}^2$.*

Then, u is 1D.

Theorem 1.3 has been also proved, by different methods, in [9, 37]. On the other hand, the following results are, as far as we know, completely new, since they deal with higher-dimensional spaces (indeed, the only symmetry results known for the fractional Allen-Cahn equation are the ones in [7, 8], which hold in dimension $n = 3$ with $s \in [1, 2)$, while we will consider now the case $n \geqslant 3$ and $s \in (0, 1)$, under different assumptions).

Theorem 1.4 (One-dimensional symmetry for monotone solutions in $\mathbb{R}^3$)**.** *Let $n \leqslant 3$ and u be a solution of $(-\Delta)^{s/2}u = u - u^3$ in $\mathbb{R}^n$.*

Suppose that

$$\frac{\partial u}{\partial x_n}(x) > 0 \quad \textit{for any } x \in \mathbb{R}^n$$

and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

Then, u is 1D.

The next two results deal with the case in which the fractional parameter s is sufficiently close to 1 (that is, roughly speaking, when the nonlocal diffusive operator is sufficiently close to $\sqrt{-\Delta}$). In this case, it is known that the minimizers of the corresponding geometric problem of fractional perimeters are close to the classical minimal surfaces (see [16]). This fact provides an additional rigidity of the interfaces that we can exploit in order to obtain symmetry results.

Theorem 1.5 (One-dimensional symmetry when s is close to 1)**.** *Let $n \leqslant 7$. Then, there exists $\eta_n \in (0, 1)$ such that for any $s \in [1 - \eta_n,\, 1)$ the following statement holds true.*

Let u be a minimizer of $\mathcal{E}$ in $\mathbb{R}^n$. Then, u is 1D.

Theorem 1.6 (One-dimensional symmetry for monotone solutions in $\mathbb{R}^8$ when s is close to 1)**.** *Let $n \leqslant 8$. Then, there exists $\eta_n \in (0, 1)$ such that for any $s \in [1 - \eta_n,\, 1)$ the following statement holds true.*

Let u be a solution of $(-\Delta)^{s/2}u = u - u^3$ in $\mathbb{R}^n$.

Suppose that

$$\frac{\partial u}{\partial x_n}(x) > 0 \quad \textit{for any } x \in \mathbb{R}^n \tag{1.14}$$

and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{1.15}$$

Then, u is 1D.

1.5. **Overview of the proofs and organization of the paper.** At a very high level, our proof of Theorem 1.1 follows the classical "improvement of flatness strategy" that goes back to De Giorgi (see e.g. the retrospective in [14]) and that was pioneered in [30] for the case of level sets of classical phase transitions and suitably modified in [12] in the context of nonlocal minimal surfaces. Our argument uses all the ingredients of the previous literature, but needs to go beyond them. The delicate proof of our version of the improvement of flatness result relies on compactness and extends from Section 2 to Section 6. Let us give next the "big picture" of it.

Very roughly, we take a sequence u_a of solutions of $(-\Delta)^{s/2}u_a = \varepsilon^{-s}f(u_a)$ such that the transition level sets of u_a are trapped in (a sequence of) very flat cylinders of flatness a. We assume that $\varepsilon < a^{p_0}$ for p_0 very large and we show that $u_a \approx \pm 1$ outside of, essentially, a $n-1$ dimensional surface (very flat but possibly very irregular). We then consider "vertical rescalings", that is, we consider the change of variables

$$(x', x_n) \mapsto \left(x', \frac{x_n}{a}\right)$$

of these "transition" surfaces (clearly here we assume that they are flat in the direction x_n).

A main step in the proof consists in proving that these vertical rescalings are compact and converge to a graph. To achieve this compactness we need a "Hölder type" estimate, or improvement of oscillation, for vertical rescalings. The proof of this improvement of oscillation estimate requires to build fine barriers for the semilinear equation. We thus obtain that the vertical rescalings converge to a Hölder continuous graph $g : \mathbb{R}^{n-1} \to \mathbb{R}$. Moreover, we prove that g is a viscosity solution of the linear translation invariant elliptic equation $(-\Delta)^{\frac{1+s}{2}} g = 0$ in $\mathbb{R}^{n-1}$. Finally we deduce the improvement of flatness of the transition level sets from the $C^{1,\alpha}$ interior regularity of the equation $(-\Delta)^{\frac{1+s}{2}} g = 0$.

The rest of the paper, namely Sections 7 and 8, is devoted to the proof of Theorem 1.2 and its consequences. As explained before, Theorem 1.2 follows from Theorem 1.1 but not in a straightforward way. Let us summarize next the main steps of its proof.

We use two different iterations of Theorem 1.1. The first iteration, that we informally call "preservation of flatness", is given in Corollary 7.1. The second iteration, really a geometric "improvement of flatness" is given in Corollary 7.2. Corollary 7.2 is stronger in the sense that the flatness is improved geometrically in a sequence of dyadic balls, but only up to a large mesoscale. In Corollary 7.1 the flatness does not improve but is just preserved across scales but, as a counterpart, it gives information up to scale 1.

To prove Theorem 1.2 we need to combine Corollary 7.1 with a multi-scale application of Corollary 7.2. Doing so, we prove that the transition level sets are trapped, in all of $\mathbb{R}^n$, between a Lipschitz graph and a finite vertical translation of it. Then, we need to use the sliding method (in its full strength) to conclude that the level sets of the solution are indeed flat.

It is worth to emphasize again that, differently from the classical case of [30], and from the mildly nonlocal case of [32], in our genuinely nonlocal setting the improvement of flatness result is not strong enough to imply the 1D symmetry of entire solutions right away. Only with the suitably tailored argument outlined above we can conclude the proof of the symmetry result.

The paper is organized as follows.

In Section 2 we introduce a method to build fine barriers for the semilinear equation $Lu = \varepsilon^{-s}f(u)$. The main idea is to model the solutions using the (anisotropic) distance function given by the norm $\|\cdot\|_{\mathcal{C}}$ to a very flat convex graph.

In Section 3 we give some simple (though very useful) auxiliary results on the decay of the solutions. We prove that a solution u decays to $+1$ (resp. to -1) like $(d/\varepsilon)^{-\gamma_0}$, where d is the distance to $\{u \leqslant 1-\kappa\}$ (resp. to $\{u \geqslant -1+\kappa\}$).

In Section 4 we state and prove the improvement of oscillation result for vertical rescaling, thus obtaining the compactness of the vertical rescalings.

In Section 5 we prove that the limit of the vertical rescalings satisfies a linear translations invariant nonlocal elliptic equation.

In Section 6 we complete the proof of Theorem 1.1.

In Section 7 we prove Corollaries 7.1 and 7.2 and give the proof of the Theorem 1.2, that is based on a suitable application of the sliding method.

In Section 8, we then give the proofs of the symmetry results in Theorems 1.3, 1.4, 1.5 and 1.6.

Notation. For the convenience of the reader we gather here the notation that we will follow throughout all the paper. The following list of notations is just for quick reference and all the notations are introduced (again) within the text at their first appearance.

- L, f are the nonlocal elliptic operator and the nonlinearity, respectively, see (1.1).
- n, s, λ, Λ are, respectively, the dimension, the fractional parameter (or the order of the operator) and the ellipticity constants of L, see (1.2) and (H0).
- $\mathcal{L}$ denotes the one-dimensional fractional Laplacian as in (1.3).
- $\mathcal{C} = \mathcal{C}_L$ is the convex body with support function h_L, and $\rho' > \rho > 0$ are the two constants in its curvature bounds, see Subsection 1.2 and in particular (H1).
- κ, c_κ and l_κ are the constants in the quantitative assumptions of f, see Subsection 1.3.
- We will call a constant *universal* if it depends only on n, s, λ, Λ, ρ, ρ', κ, c_κ and l_κ. In particular, universal constants depend only on n, L, and f.
- We write
$$X \subset Y \quad \text{in } B \qquad \text{if} \qquad X \cap B \subset Y \cap B.$$
- We denote by $\| \cdot \|_{\mathcal{C}}$ the norm with unit ball $\mathcal{C}$. We also denote by $\mathcal{C}_r(y)$ the ball of radius r and center y with respect to this norm, namely
$$\mathcal{C}_r(y) := y + r\mathcal{C}.$$
Notice that when L is the fractional Laplacian $\mathcal{C}_r(y)$ is simply $B_r(y)$.
- Points in $\mathbb{R}^{n-1}$ will be denoted by x' and $x = (x', x_n)$ denotes a point in $\mathbb{R}^n$ with n-th coordinate x_n. From now on, we also denote by B'_r the $(n-1)$-dimensional ball of radius $r > 0$.
- ξ denotes the function $\xi : \mathbb{R}^{n-1} \to \mathbb{R}$ which is defined by
$$\xi(x') = \xi(|x'|) := \big(1 + |x'|^2\big)^{\frac{1+\alpha}{2}} - 1. \tag{1.16}$$
- Given $b > 0$, we denote by d_b the signed distance function to the set $\{x_n \geqslant b\,\xi(x')\}$ with respect to the norm $\| \cdot \|_{\mathcal{C}}$, that is,
$$d_b(x) = \begin{cases} +\inf\big\{\|z-x\|_{\mathcal{C}} \ : \ z_n = b\,\xi(z')\big\}, & \text{for } x_n \geqslant b\,\xi(x'), \\ -\inf\big\{\|z-x\|_{\mathcal{C}} \ : \ z_n = b\,\xi(z')\big\}, & \text{for } x_n \leqslant b\,\xi(x'). \end{cases}$$
- Given $\phi : \mathbb{R} \to (-1,1)$, for any $x \in \mathbb{R}$, we set
$$\phi^b(x) := \phi\big(d_b(x)\big).$$
Notice that $\phi^b : \mathbb{R} \to (-1,1)$, and it may be seen as a "rearrangement" of the layer solution ϕ with respect to the signed distance function.

In addition to the previous notations we use also the following very standard ones.

- Given $r \in \mathbb{R}$, we denote by $r_+ := \max\{r, 0\}$ and $r_- := \max\{-r, 0\}$.

- Given a measurable function $f : X_1 \times \cdots \times X_m \to \mathbb{R}$, we use the repeated integral notation
$$\int_{X_1} dx_1 \ldots \int_{X_m} dx_m\, f(x_1, \ldots, x_m) := \int_{X_1} \left[\ldots \int_{X_m} f(x_1, \ldots, x_m)\, dx_m \ldots \right] dx_1.$$

2. Approximate solutions via deformation of level sets

In this section we construct approximate solutions in B_1 by deforming (slightly curving) the flat level sets of a one-dimensional solution.

2.1. A layer cake formula. We start with a simple layer cake representation for the integro-differential operators. We use the notation

$$\chi_{[a_1,a_2]}(\theta) := \begin{cases} 1 & \text{if } a_1 \leqslant a_2 \text{ and } \theta \in [a_1, a_2], \\ 0 & \begin{array}{c}\text{if either } a_1 > a_2, \\ \text{or } a_1 \leqslant a_2 \text{ and } \theta \notin [a_1, a_2].\end{array} \end{cases} \tag{2.1}$$

Using this, we have the following simple layer cake type representation for nonlocal operators:

Lemma 2.1. *It holds that*

$$Lv(x) = \int_{\mathbb{R}^n} dy \int_{\mathbb{R}} d\theta \Big(\chi_{[v(x+y),v(x)]}(\theta) - \chi_{[v(x),v(x+y)]}(\theta) \Big) \frac{\mu(y/|y|)}{|y|^{n+s}}. \tag{2.2}$$

Furthermore, if $x \in \mathbb{R}^n$ is such that $v(x) = w(x)$, then

$$Lv(x) - Lw(x) = \int_{\mathbb{R}^n} dy \int_{\mathbb{R}} d\theta \Big(\chi_{[v(x+y),w(x+y)]}(\theta) - \chi_{[w(x+y),v(x+y)]}(\theta) \Big) \frac{\mu(y/|y|)}{|y|^{n+s}}. \tag{2.3}$$

Proof. By (2.1),

$$(a_1 - a_2)_- = (a_2 - a_1)_+ = \int_{\mathbb{R}} \chi_{[a_1,a_2]}(\theta)\, d\theta$$

and therefore

$$\begin{aligned} v(x) - v(x+y) &= \big(v(x) - v(x+y)\big)_+ - \big(v(x) - v(x+y)\big)_- \\ &= \int_{\mathbb{R}} \chi_{[v(x+y),v(x)]}(\theta)\, d\theta - \int_{\mathbb{R}} \chi_{[v(x),v(x+y)]}(\theta)\, d\theta. \end{aligned}$$

So, we integrate and we find (2.2).

Similarly, we write

$$w(x+y) - v(x+y) = \int_{\mathbb{R}} \chi_{[v(x+y),w(x+y)]}(\theta)\, d\theta - \int_{\mathbb{R}} \chi_{[w(x+y),v(x+y)]}(\theta)\, d\theta,$$

which gives (2.3) after integration. □

2.2. Touching the level sets of the distance function by concentric spheres. This section discuss some geometric features related to the signed anisotropic distance function to a convex set. To this aim, we recall some basic properties of the support function h_L defined in (1.8). First of all, for any $x, y \in \mathbb{R}^n$, the following inequality of Cauchy-Schwarz type holds true

$$x \cdot y \leqslant h_L(y)\, \|x\|_{\mathcal{C}}. \tag{2.4}$$

See e.g. Lemma 2.1 in [22] for an elementary proof.

As a counterpart of (2.4), we have that equality holds when one of the two vectors is normal to the sphere to which the other vector belongs. More precisely, we have that if $z_0 \in \mathbb{R}^n$, $R > 0$, $z \in \partial\mathcal{C}_R(z_0)$ and $\omega_0 \in S^{n-1}$ is the inner normal of $\partial\mathcal{C}_R(z_0)$ at the point z, then

$$\omega_0 \cdot (z_0 - z) = R\, h_L(\omega_0), \tag{2.5}$$

see for example Lemma 2.3 in [22].

Moreover, it is useful to recall that h_L is the "support function" of the convex body $\mathcal{C}$, namely for any $\omega \in S^{n-1}$ we have that

$$h_L(\omega) = \sup_{x\in\mathcal{C}} x \cdot \omega, \tag{2.6}$$

see for instance Lemma 2.2 in [22].

We recall also here that both h_L and $\mathcal{C}$ are even.

Given a nonempty, closed and convex set $K \subset \mathbb{R}^n$, we define the anisotropic signed distance function from K as

$$d_K(x) := \inf\big\{\ell(x) \,:\, \ell(x) = \omega\cdot x + c, \quad h_L(\omega) = 1, \quad c\in\mathbb{R} \quad \text{and} \quad \ell \geqslant 0 \ \text{ in all of } K\big\}. \tag{2.7}$$

Notice that d_K is a concave function, since it is the infimum of affine functions. Moreover, as shown for instance in Proposition 2.7 of [22], it holds that

$$d_K(x) = \begin{cases} +\inf\big\{\|z-x\|_{\mathcal{C}} \,:\, z\in\partial K\big\} & \text{for } x\in K, \\ -\inf\big\{\|z-x\|_{\mathcal{C}} \,:\, z\in\partial K\big\} & \text{for } x\in\mathbb{R}^n\setminus K. \end{cases} \tag{2.8}$$

We have that d_K is a Lipschitz function, with Lipschitz constant 1 with respect to the anisotropic norm, namely, for any $p, q \in \mathbb{R}^n$,

$$|d_K(p) - d_K(q)| \leqslant \|p-q\|_{\mathcal{C}}, \tag{2.9}$$

see e.g. Lemma 2.4 in [22].

With this setting, we can now prove that the level sets of d_K are touched by appropriate concentric anisotropic spheres:

Lemma 2.2. *Let $z_0 \in K = \{d_K > 0\}$. Assume that $\mathcal{C}_R(z_0) \subset \{d_K > 0\}$ touches $\partial K = \{d_K = 0\}$ at some point $\bar z \in \{d_K = 0\}$.*

Then, for any $t \in (-\infty, R)$,

$$\begin{aligned} &\textit{the set } \mathcal{C}_{R-t}(z_0) \textit{ is contained in } \{d_K > t\} \\ &\textit{and touches } \{d_K = t\} \textit{ at the point} \\ &z := z_0 + \frac{R-t}{R}(z - z_0) \in \big(\partial\mathcal{C}_{R-t}(z_0)\big) \cap \{d_K = t\}. \end{aligned} \tag{2.10}$$

Furthermore, if we denote by $\omega_0 \in S^{n-1}$ the inner normal of $\partial\mathcal{C}_R(z_0)$ at the point $\bar z$, it holds that

$$\begin{aligned} &R - \|x - z_0\|_{\mathcal{C}} \leqslant d_K(x) \leqslant \frac{\omega_0}{h_L(\omega_0)}\cdot(x - \bar z) \ \textit{for any } x\in\mathbb{R}^n, \\ &\textit{and equalities hold when } \ x = z_0 + \frac{R-t}{R}(z - z_0), \ \textit{for some } t\in(-\infty, R). \end{aligned} \tag{2.11}$$

In particular,

$$d_K\left(z_0 + \frac{R-t}{R}(\bar z - z_0)\right) = t. \tag{2.12}$$

In addition, if $\tau \in (-\infty, R)$ *and* $z_\tau := z_0 + \frac{R-\tau}{R}(z - z_0)$*, then*

$$\begin{aligned} &\mathcal{C}_{|t-\tau|}(z_\tau) \textit{ is tangent from the outside to the set} \\ &\left\{ x \in \mathbb{R}^n \textit{ s.t. } d_K(x) \leqslant t \leqslant \frac{\omega_0}{h_L(\omega_0)} \cdot (x - z) \right\} \\ &\textit{at the point } z. \end{aligned} \tag{2.13}$$

Proof. The geometric setting of Lemma 2.2 is depicted in Figure 1.

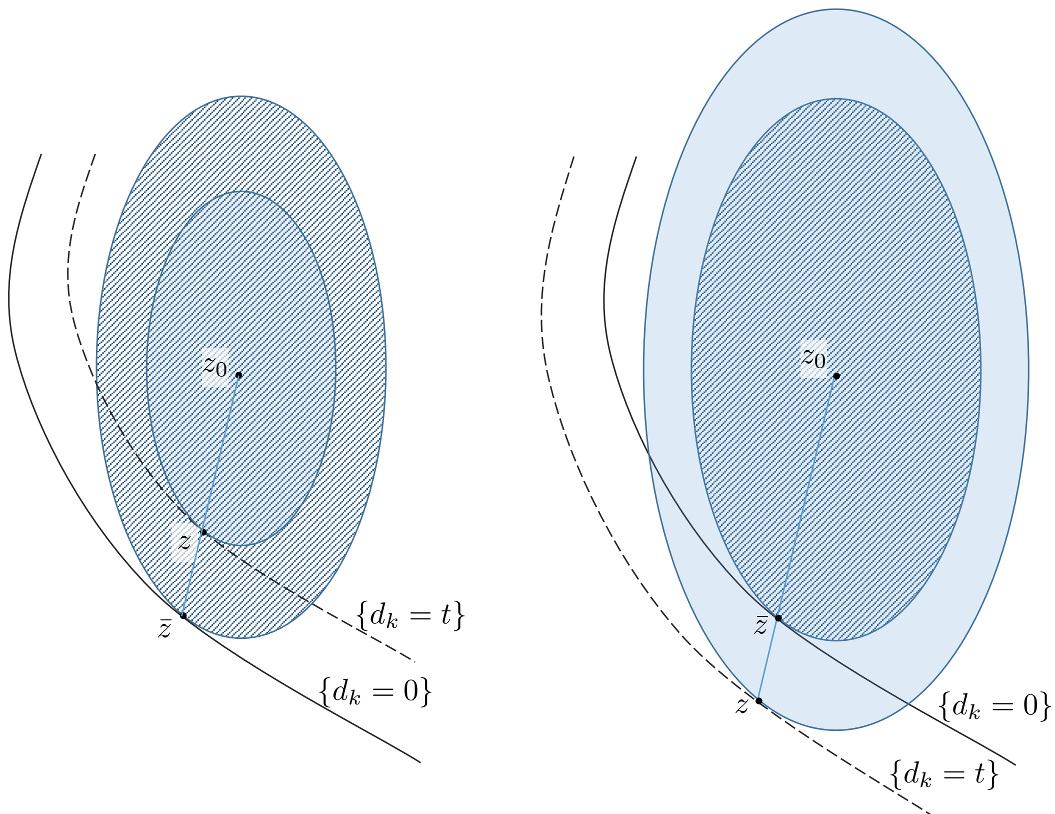

FIGURE 1. *The geometry of Lemma 2.2 when* $t \in (0, R)$ *and when* $t \in (-\infty, 0)$.

The proof goes like this. For every $t \in (-\infty, R)$, we have that

$$\|z - z_0\|_{\mathcal{C}} = \frac{R-t}{R}\|\bar{z} - z_0\|_{\mathcal{C}} = R - t, \tag{2.14}$$

and therefore

$$z \in \partial\mathcal{C}_{R-t}(z_0). \tag{2.15}$$

In addition, we point out that, for every $t \in (-\infty, R)$,

$$\mathcal{C}_{R-t}(z_0) \subset \{d_K \geqslant t\}. \tag{2.16}$$

To check this, we distinguish two cases: either $t \geqslant 0$ (i.e. $t \in [0, R)$) or $t < 0$. If $t \geqslant 0$, we argue as follows. Let $p \in \mathcal{C}_{R-t}(z_0)$. Then, for any q with $\|q\|_{\mathcal{C}} \leqslant t$ we have that $p + q \in \mathcal{C}_R(z_0) \subset \{d_K \geqslant 0\}$.

Consequently, in light of (2.8), for any affine function $\ell(x) = \omega \cdot x + c$, with $h_L(\omega) = 1$, $c \in \mathbb{R}$, and such that $\ell \geqslant 0$ in $\{d_K > 0\}$, it holds that

$$\ell(p + q) \geqslant 0. \tag{2.17}$$

Therefore, we slide the halfspace with inner normal $\frac{\omega}{|\omega|}$ till it touches $\partial\mathcal{C}$ and we take this touching point q. Namely, we have $q \in \partial\mathcal{C}_t$, with $\frac{\omega}{|\omega|}$ as inner normal of $\partial\mathcal{C}_t$ at q. Hence, by (2.5),

$$-\frac{\omega}{|\omega|}\cdot q = t\,h_L\left(\frac{\omega}{|\omega|}\right) = \frac{t\,h_L(\omega)}{|\omega|} = \frac{t}{|\omega|}.$$

This and (2.17) give that

$$0 \leqslant \ell(p+q) = \omega\cdot p + c + \omega\cdot q = \omega\cdot p + c - t.$$

This shows that $\ell(p) \geqslant t$ and so, in view of (2.22), that $d_K(p) \geqslant t$, that establishes (2.16) in this case.

So, we now check (2.16) in the case in which $t<0$. For this, let $p \in \mathcal{C}_{R-t}(z_0)$. If $p \in \mathcal{C}_R(z_0)$, then $d_K(p) \geqslant 0 \geqslant t$, and we are done, so we can suppose that $p \in \mathcal{C}_{R-t}(z_0)\setminus\mathcal{C}_R(z_0)$, hence

$$\|p-z_0\|_{\mathcal{C}} \in \big[R,\,R-t\big].$$

We take

$$q := z_0 + \frac{R\,(p-z_0)}{\|p-z_0\|_{\mathcal{C}}}.$$

Notice that $\|q-z_0\|_{\mathcal{C}} = R$, hence $q \in \mathcal{C}_R(z_0) \subset \{d_K \geqslant 0\}$. This and (2.9) imply that

$$-d_K(p) \leqslant d_K(q) - d_K(p) \leqslant \|q-p\|_{\mathcal{C}} = \big|R - \|p-z_0\|_{\mathcal{C}}\big| = \|p-z_0\|_{\mathcal{C}} - R \leqslant (R-t) - R,$$

that gives $d_K(p) \geqslant t$, as desired. This completes the proof of (2.16).

Now we check that

$$d_K(z) = t. \tag{2.18}$$

To this aim, we observe that

$$z \in \mathcal{C}_{R-t}(z_0) \subset \{d_K \geqslant t\},$$

thanks to (2.15) and (2.16). Consequently, to establish (2.18), we only need to prove that

$$d_K(z) \leqslant t. \tag{2.19}$$

To this goal, if $t \geqslant 0$ we use (2.9) and we see that

$$d_K(z) = d_K(z) - d_K(\bar z) \leqslant \|z-\bar z\|_{\mathcal{C}} = \frac{t}{R}\,\|\bar z - z_0\|_{\mathcal{C}} \leqslant t,$$

which is (2.18) in this case.

If instead $t<0$, we denote by $\omega_0 \in S^{n-1}$ the inner normal of $\partial\mathcal{C}_R(z_0)$ at the point z, and we exploit (2.5) (recall also (2.38)) to see that

$$\begin{aligned} d_K(z) &\leqslant \frac{\omega_0}{h_L(\omega_0)}\cdot(z-\bar z) = \frac{\omega_0}{h_L(\omega_0)}\cdot\left(z_0 - \bar z + \frac{R-t}{R}(\bar z - z_0)\right) \\ &= \frac{t}{R}\frac{\omega_0}{h_L(\omega_0)}\cdot(z_0-\bar z) = t. \end{aligned} \tag{2.20}$$

This finishes the proof of (2.18).

Then, (2.10) follows from (2.15), (2.16) and (2.18). In turn, (2.10) also implies (2.12).

We also observe that, from the previous considerations, (2.11) follows in a straightforward way using (2.8).

Now we prove (2.13). First of all, we notice that $\|z_\tau - \bar z\|_{\mathcal{C}} = |t-\tau|$, due to (2.14), so $\bar z$ lies on $\partial\mathcal{C}_{|t-\tau|}(z_\tau)$. Thus, to prove the result in (2.13), we need to show that

$$\left\{x \in \mathbb{R}^n \text{ s.t. } \|x - z_\tau\|_{\mathcal{C}} < |t-\tau| \text{ and } d_K(x) \leqslant t \leqslant \frac{\omega_0}{h_L(\omega_0)}\cdot(x-z)\right\} = \varnothing. \tag{2.21}$$

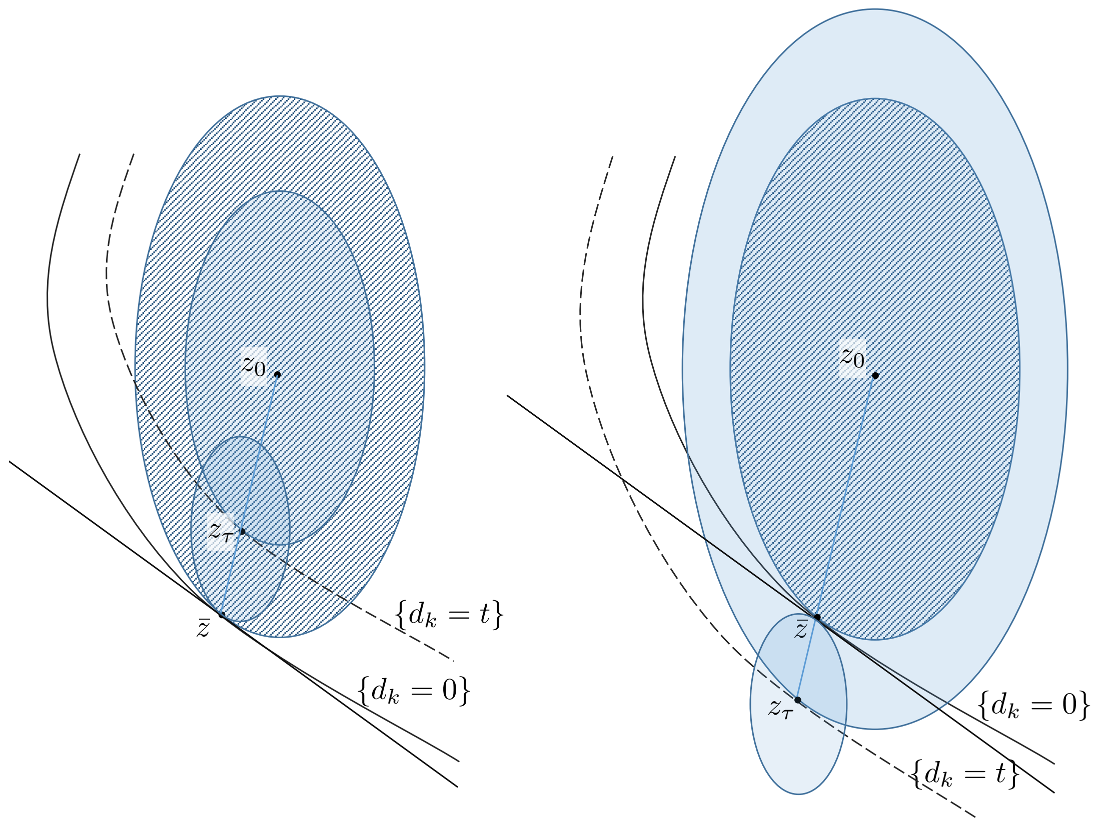

FIGURE 2. *Proof of* (2.21) *when* $\tau \geqslant t$ *and when* $\tau < t$.

For this, we refer to Figure 2, we argue by contradiction and we suppose that there exists x in the set on the left hand side of (2.21). Then, we distinguish two cases, either $\tau \geqslant t$ or $\tau < t$. If $\tau \geqslant t$, we use (2.12) to see that $d_K(z_\tau) = \tau$ and so, exploiting (2.9),

$$0 \leqslant t - d_K(x) = t - \tau + d_K(z_\tau) - d_K(x) \leqslant -|t-\tau| + \|x - z_\tau\|_{\mathcal{C}} < 0,$$

which is a contradiction. If instead $\tau < t$, using (2.4) and (2.5) we find that

$$\begin{aligned} t &\leqslant \frac{\omega_0}{h_L(\omega_0)} \cdot (x - \bar{z}) = \frac{\omega_0}{h_L(\omega_0)} \cdot (z_\tau - \bar{z}) + \frac{\omega_0}{h_L(\omega_0)} \cdot (x - z_\tau) \\ &\leqslant \tau \frac{\omega_0}{h_L(\omega_0)} \cdot \frac{z_0 - \bar{z}}{R} + \|x - z_\tau\|_{\mathcal{C}} = \tau + \|x - z_\tau\|_{\mathcal{C}} < \tau + |t - \tau| = t, \end{aligned}$$

which is a contradiction. This proves (2.21), which in turn gives (2.13). □

2.3. **Distance function from a convex graph.** Here, we look at the special case of the distance function from a sufficiently flat graph with an appropriate growth. For this, let $\alpha \in (0, s)$ be a fixed constant. Let us introduce the function $\xi : \mathbb{R}^{n-1} \to \mathbb{R}$ defined by

$$\xi(x') = \left(1 + |x'|^2\right)^{\frac{1+\alpha}{2}} - 1.$$

Note that $\xi(0) = 0$ and that ξ is convex with

$$D^2 \xi = \operatorname{diag}\left(\frac{1 + \alpha r^2}{1 + r^2}, 1, 1, \ldots, 1\right)(1+\alpha)(1+r^2)^{\frac{\alpha-1}{2}} \geqslant 0$$

in a coordinate system with the first axis pointing in the radial direction.

Given some orthonormal coordinates $x = (x', x_n)$ in $\mathbb{R}^n$ and $b > 0$, let us define

$$\Gamma_b := \left\{ x_n \geqslant b\,\xi(x') \right\}.$$

From the convex set Γ_b we define the following anisotropic signed distance function

$$d_b(x) := \inf \left\{ \ell(x) \;:\; \ell(x) = \omega \cdot x + c, \quad h_L(\omega) = 1, \quad c \in \mathbb{R}, \text{ and } \quad \ell \geqslant 0 \text{ in all of } \Gamma_b \right\}. \tag{2.22}$$

By comparing with (2.7), we have that d_b coincides with d_K with the particular choice $K := \Gamma_b$. Hence, in view of (2.8), it holds that

$$d_b(x) = \begin{cases} +\inf \left\{ \|x - z\|_{\mathcal{C}} \;:\; z \in \partial\Gamma_b \right\} & \text{for } x \in \Gamma_b, \\ -\inf \left\{ \|x - z\|_{\mathcal{C}} \;:\; z \in \partial\Gamma_b \right\} & \text{for } x \in \mathbb{R}^n \setminus \Gamma_b, \end{cases} \tag{2.23}$$

where $\| \cdot \|_{\mathcal{C}}$ denotes the norm with unit ball $\mathcal{C}$; for this, we use the notations in (1.6) and (1.9) and we recall that, throughout the paper, $\mathcal{C} = \mathcal{C}_L$ is the convex body associated to L and, for any $r > 0$ and any $y \in \mathbb{R}^n$, we set

$$\mathcal{C}_r(y) := y + r\mathcal{C}. \tag{2.24}$$

The following result states that under the hypothesis (H1), and for b small enough, all the level sets of d_b passing close enough to the origin are $C^{1,1}$ graphs, with their second derivatives bounded by Cb near the origin and with growth at infinity controlled by $Cb|x|^{1+\alpha}$.

Lemma 2.3. *There exist $b_0 > 0$ and $C_0 > 0$, depending only on α, ρ and ρ', such that for any $b \in (0, b_0)$ and any $t \in \mathbb{R}$, with $\{d_b = t\} \cap \mathcal{C}_{4/\rho} \neq \varnothing$, we have that*

$$\{d_b = t\} \;=\; \{y_n = G(y')\}$$

where $G : \mathbb{R}^{n-1} \to \mathbb{R}$ is a suitable convex function satisfying

$$\left| D^2 G \right| \leqslant C_0 b \quad \text{in } B'_{4\rho'/\rho} \tag{2.25}$$

and

$$\left| G(y') - G(0) \right| \leqslant C_0 b (1 + |y'|)^{1+\alpha} \quad \text{for all } y' \in \mathbb{R}^{n-1}. \tag{2.26}$$

To prove Lemma 2.3 we need the following simple preliminary result:

Lemma 2.4. *We have the following inequalities between the anisotropic and the Euclidean norm*

$$\frac{1}{\rho'} | \cdot | \leqslant \| \cdot \|_{\mathcal{C}} \leqslant \frac{1}{\rho} | \cdot |. \tag{2.27}$$

Proof. By (H1), we have

$$B_\rho \subset \mathcal{C} \subset B_{\rho'}.$$

Therefore, recalling (1.9),

$$\|x\|_{\mathcal{C}} = \sup\{t > 0 \text{ s.t. } x/t \notin \mathcal{C}\} \leqslant \sup\{t > 0 \text{ s.t. } x/t \notin B_\rho\} = \frac{1}{\rho}|x|,$$

which proves the second inequality in (2.27). The second inequality is proven likewise. □

Proof of Lemma 2.3. We have

$$\left| D^2 \xi(x') \right| \leqslant C \left(1 + |x'|^2\right)^{\frac{\alpha-1}{2}}, \tag{2.28}$$

for some $C > 0$ depending only on α.

Using that $0 \in \partial\Gamma_b = \{d_b = 0\}$ and that, by assumption, there exists $p \in \mathcal{C}_{4/\rho}$ such that $p \in \{d_b = t\}$, we have that

$$|t| \leqslant \|p - 0\|_{\mathcal{C}} \leqslant \frac{4}{\rho}. \tag{2.29}$$

Choose $y \in \{d_b = t\}$. Recalling Lemma 2.4, let $\bar{y}$ be a point on $\partial\Gamma_b$ for which

$$\frac{1}{\rho'}|\bar{y} - y| \leqslant \|\bar{y} - y\|_{\mathcal{C}} = |d_b(y)| = |t|.$$

By (2.28) there exists a ball of radius $R \geqslant c/b$ contained in Γ_b and touching $\partial\Gamma_b$ at the point y, where $c > 0$ depends only on α. Since $\mathcal{C}_r \subset B_{r\rho'}$ there exists z_0 in Γ_b such that

$$\mathcal{C}_{R/\rho'}(y_0) \subset \Gamma_b \quad \text{and touches } \partial\Gamma_b \text{ at } \bar{y}. \tag{2.30}$$

Then, by Lemma 2.2 we have that

$$\mathcal{C}_{R/\rho'-t}(y_0) \subset \{d_b > t\} \quad \text{and touches } \{d_b = t\} \text{ at } y. \tag{2.31}$$

Since $\mathcal{C}$ is assumed to be $C^{1,1}$, this shows that the boundary of the convex set $\{d_b > t\}$ is $C^{1,1}$.

Let us prove that, indeed, the boundary of $\{d_b > t\}$ is a graph and control the gradient and the second derivatives of this graph. We assume that b_0 is small enough so that

$$R/\rho' - t \geqslant \frac{c}{b\rho'} - \frac{4}{\rho} \geqslant \frac{c}{b}$$

where c denotes a positive universal constant (that may change each time).

Now, denoting $y = (y', y_n)$ and $\bar{y} = (\bar{y}', \bar{y}_n)$, we have

$$|\bar{y}'| \leqslant |y'| + |y - \bar{y}| \leqslant |y'| + \rho'|t| \leqslant |y'| + \frac{4}{\rho} \leqslant |y'| + C.$$

The tangent plane to $\mathcal{C}_{R/\rho'-t}(y_0)$ at $\bar{y}$ is parallel to the tangent plane to $\mathcal{C}_{R/\rho'}(y_0)$ at y and, by (2.30), this slope is given by

$$b(1+\alpha)|\bar{y}'|(1+|\bar{y}'|^2)^{\frac{\alpha-1}{2}} \leqslant 2(1+|\bar{y}'|^2)^{\frac{\alpha}{2}} \leqslant C_0(1+|y'|^2)^{\frac{\alpha}{2}},$$

where C_0 is a universal constant and where we have used that

$$(b\xi(r))' = b(1+\alpha)r(1+r^2)^{\frac{\alpha-1}{2}}.$$

Since the point y can be chosen arbitrarily on the surface $\{d_b = t\}$, this proves that this surface is an entire graph. Namely, that

$$\{d_b = t\} \; = \; \{y_n = G(y')\} \qquad \text{where } |DG(y')| \leqslant C_0(1+|\bar{y}|^2)^{\frac{\alpha}{2}}.$$

Finally, the estimate for the second derivative in (2.25) follows from (2.31) recalling that $R \geqslant cb$. On the other hand, (2.26) follows from the fact that

$$\big|G(y') - G(0)\big| \leqslant \sup_{|z'|\leqslant|y'|} |DG(z')||y'| \leqslant C_0|y'|b(1+|y'|^2)^{\frac{\alpha}{2}} \leqslant C_0 b(1+|y'|)^{1+\alpha} \quad \text{for all } y' \in \mathbb{R}^{n-1}.$$

This completes the proof of Lemma 2.3. $\square$

2.4. **Modeling solutions with the distance function.** We now construct useful barriers by using the level sets of the distance function as a profile and controlling the error produced in the equation by such procedure. For this, we let $\phi : \mathbb{R} \to (-1, 1)$ be a C^2 and increasing function with

$$\lim_{z\to\pm\infty} \phi(z) = \pm 1.$$

Note that any such ϕ solves an equation of the type

$$A\phi = f_\phi(\phi) \quad \text{in } \mathbb{R},$$

where $f_\phi : (-1,1) \to \mathbb{R}$ is defined by

$$f_\phi := (A\phi) \circ \phi^{-1}. \tag{2.32}$$

Now we define a suitable rearrangement procedure that produces a function $\phi_b : \mathbb{R}^n \to (-1,1)$ from any given ϕ as above and modeled along the level sets of the distance function d_b, as introduced in (2.22). Namely, we set

$$\phi^b(x) := \phi\big(d_b(x)\big). \tag{2.33}$$

Then, we have that ϕ^b is "almost" a solution of the equation with nonlinearity f_ϕ, as given by the following result:

Lemma 2.5. *Let L satisfy* (H1). *Then, there exist positive quantities b_0 and C_0 depending only on n, s, λ, Λ, ρ and ρ' (and thus independent of ϕ), such that the following holds.*

Assume that

$$[-1+\delta, 1-\delta] \subset \phi\left(\left[-\frac{1}{\rho'}, \frac{1}{\rho'}\right]\right). \tag{2.34}$$

Then, for all $\omega \in S^{n-1}$ and $b \in (0, b_0)$ we have

$$0 \leqslant L\phi^b - f_\phi\big(\phi^b\big) \leqslant C_0(b+\delta) \quad \text{in } B_1. \tag{2.35}$$

Proof. Let us fix $z \in B_1$. Let $\theta_0 = \phi^b(z)$ be the level of ϕ^b at z. By (2.33), we know that $d_b(z) = \phi^{-1}(\phi^b(z)) = \phi^{-1}(\theta_0) =: t_0$.

We also recall that h_L was introduced in (1.5) (or, equivalently, in (1.8)) and we let ω be the unit vector normal to $\{d_b = t_0\}$ at z and pointing towards $\{d_b > t_0\}$. Then, we define

$$\tilde{d}(x) := \frac{\omega}{h_L(\omega)} \cdot (x-z) + t_0. \tag{2.36}$$

We also set $\tilde{\phi} := \phi \circ \tilde{d}$. Using the notation in (1.4), we have that

$$\tilde{\phi}(x) = \phi\left(\frac{\omega}{h_L(\omega)} \cdot (x-z) + t_0\right) = \phi\left(\frac{\omega}{h_L(\omega)} \cdot \big(x - z + \omega\, h_L(\omega)\, t_0\big)\right) = \bar{\phi}_{\omega,h}(x - z + \omega\, h\, t_0),$$

with $h := h_L(\omega)$.

Consequently, by (1.5) and (2.32), for any $x \in \mathbb{R}^n$,

$$\begin{aligned} L\tilde{\phi}(x) &= L\bar{\phi}_{\omega,h}(x - z - \omega\, h\, t_0) = A\phi\left(\frac{\omega}{h} \cdot (x - z + \omega\, h\, t_0)\right) \\ &= f_\phi\left(\phi\left(\frac{\omega}{h} \cdot (x - z + \omega\, h\, t_0)\right)\right) = f_\phi\left(\phi\left(\frac{\omega}{h} \cdot (x-z) + t_0\right)\right) = f_\phi(\tilde{\phi}(x)). \end{aligned} \tag{2.37}$$

Now, by (2.11) in Lemma 2.2 we have

$$d_b \leqslant \tilde{d} \quad \text{in } \mathbb{R}^n, \tag{2.38}$$

see also Lemma 2.6 in [22] for the elementary proof of this and related facts. Moreover,

$$d_b = \tilde{d} \quad \text{along the ray} \quad \mathcal{R} := \{z_0 + t'(z - z_0),\ t' \geqslant 0\}. \tag{2.39}$$

From the observations in (2.38) and (2.39) it follows that

$$\{d_b = t\} \quad \text{is tangent to} \quad \{\tilde{d} = t\} \quad \text{at some point on } \mathcal{R}. \tag{2.40}$$

Notice that, by construction,

$$\phi^b(z) = \phi(t_0) = \tilde{\phi}(z) \tag{2.41}$$

and, by (2.38) and the monotonicity of ϕ, it holds that $\phi^b \leqslant \tilde{\phi}$. Accordingly, $L\phi^b(z) - L\tilde{\phi}(z) \geqslant 0$. Thus, we apply the layer cake formula in (2.3) of Lemma 2.1 and use that the image of ϕ is contained in $[-1,1]$ to conclude that

$$\begin{aligned} 0 \leqslant L\phi^b(z) - L\tilde{\phi}(z) &= \int_{\mathbb{R}^n} dy \int_{\mathbb{R}} d\theta \chi_{[\phi^b(z+y),\, \tilde{\phi}(z+y)]}(\theta)\, \frac{\mu(y/|y|)}{|y|^{n+s}} \\ &= \int_{-1}^{1} d\theta \int_{\mathbb{R}^n} dy\, \frac{\mu(y/|y|)}{|y|^{n+s}}\, \chi_{S_\theta}(z+y) = \int_{-1}^{1} d\theta\, I_z(\theta) \end{aligned} \tag{2.42}$$

where

$$S_\theta := \left\{x \in \mathbb{R}^n \ : \ \phi^b(x) \leqslant \theta \leqslant \tilde{\phi}(x)\right\} = \left\{x \in \mathbb{R}^n \ : \ d_b(x) \leqslant \phi^{-1}(\theta) \leqslant \tilde{d}(x)\right\}$$

and

$$I_z(\theta) := \int_{\mathbb{R}^n} \frac{\mu(y/|y|)}{|y|^{n+s}} \, \chi_{S_\theta}(z+y)\, dy.$$

Now we recall (2.37) and (2.41) to see that $L\tilde{\phi}(z) = f_\phi(\tilde{\phi}(z)) = f_\phi(\phi^b(z))$ and so we can rewrite (2.42) as

$$0 \leqslant L\phi^b(z) - f_\phi(\phi^b(z)) = \int_{-1}^{1} d\theta \, I_z(\theta). \tag{2.43}$$

Now, given $\theta \in (-1,1)$, let us define

$$t_\theta := \phi^{-1}(\theta).$$

In the next steps of the proof we will establish different estimates for $I_z(\theta)$ by distinguishing the two cases $\{d_b = t_\theta\} \cap \mathcal{C}_{3/\rho}(z) = \varnothing$ and $\{d_b = t_\theta\} \cap \mathcal{C}_{3/\rho}(z) \neq \varnothing$.

Case 1. Let $\{d_b = t_\theta\} \cap \mathcal{C}_{3/\rho}(z) = \varnothing$. We take $b \in (0, b_0)$ with b_0 small enough, depending only on ρ and ρ', and we claim that we have that

$$S_\theta \cap B_2 = \varnothing. \tag{2.44}$$

Indeed, by (2.13), $\{d_b = t_\theta\} \cap \mathcal{C}_{3/\rho}(z) = \varnothing$ implies that $S_\theta \cap \mathcal{C}_{3/\rho}(z) = \varnothing$. Hence, recalling that $z \in B_1$, we have that $B_2 \subset B_3(z) \subset \mathcal{C}_{3/\rho}(z)$ and hence (2.44) follows.

Thus, since $z \in B_1$, using (2.44) we conclude that

$$I_z(\theta) = \int_{\mathbb{R}^n \setminus B_1} \frac{\mu(y/|y|)}{|y|^{n+s}} \, \chi_{S_\theta}(z+y)\, dy \leqslant \int_{\mathbb{R}^n \setminus B_1} \frac{\Lambda}{|y|^{n+s}} \, dy \leqslant C, \tag{2.45}$$

for some $C > 0$.

Now we claim that in this case we have

$$\theta \in [-1, \, -1+\delta) \cup (1-\delta, \, 1]. \tag{2.46}$$

Indeed, if not, by (2.34),

$$\theta \in [-1+\delta, 1-\delta] \subset \phi\left(\left[-\frac{1}{\rho'}, \frac{1}{\rho'}\right]\right)$$

and so

$$t_\theta = \phi^{-1}(\theta) \in \left[-\frac{1}{\rho'}, \frac{1}{\rho'}\right].$$

Then, using that $0 \in \{d_b = 0\}$ we find that

$$\inf\left\{\frac{1}{\rho'}|y-0| \ : \ y \in \{d_b = t_\theta\}\right\} \leqslant \inf\left\{\|y-0\|_{\mathcal{C}} \ : \ y \in \{d_b = t_\theta\}\right\} = |t_\theta| \leqslant \frac{1}{\rho'}$$

and thus $\{d_b = t_\theta\}$ intersects B_2, which is a contradiction. This proves (2.46).

Case 2. Now we deal with the case $\{d_b = t_\theta\} \cap \mathcal{C}_{3/\rho}(z) \neq \varnothing$ and $b \in (0, b_0)$, with b_0 small enough. Note that we have $\{d_b = t_\theta\} \cap \mathcal{C}_{4/\rho} \varnothing$ since $z \in B_1 \subset \mathcal{C}_{1/\rho}$.

In this case, we recall (2.39) and (2.40) and we take $\bar{z} = (\bar{z}', \bar{z}_n)$ to be the triple intersection point described there, that is

$$\bar{z} \in \{d_b = t_\theta\} \cap \{\tilde{d} = t_\theta\} \cap \mathcal{R}. \tag{2.47}$$

With this notation, we can write the set S_θ as a suitable portion of space trapped between a linear function and a convex one with small detachment one from the other. For this, we exploit Lemma 2.3 to see that

$$\{d_b = t_\theta\} \ = \ \{y_n = G(y')\} \tag{2.48}$$

with G convex and satisfying

$$|DG(y')| \leqslant C_0 b \quad \text{in } |y'| < \frac{4\rho'}{\rho} \quad \text{and} \quad \big|G(y') - G(0)\big| \leqslant C_0 b(1 + |y'|)^{1+\alpha} \quad \text{for all } y'. \tag{2.49}$$

Therefore, the condition $d_b(x) \leqslant t_\theta$ is equivalent to the fact that the point x lies below the graph of G, namely that $x_n \leqslant G(x')$. Similarly, from (2.47), we have that ω is normal to both $\{\tilde{d} = t_\theta\}$ and $\{d_b = t_\theta\}$ at $\bar{z}$ and so, by (2.48), the condition that $t_\theta \leqslant \tilde{d}(x)$ is equivalent to

$$x_n \geqslant G(\bar{z}') + \nabla G(\bar{z}') \cdot (x' - \bar{z}').$$

In consequence of these observations, we have that

$$S_\theta \;=\; \big\{G(\bar{z}') + \nabla G(\bar{z}') \cdot (x' - \bar{z}') \leqslant x_n \leqslant G(x')\big\}. \tag{2.50}$$

Next we observe that, as a consequence of (2.13), for $r = \|z - \bar{z}\|_{\mathcal{C}}$, we have

$$\mathcal{C}_r(z) \subset \mathbb{R}^n \setminus S_\theta. \tag{2.51}$$

Therefore, for all y in S_θ, recalling Lemma 2.4,

$$|y - \bar{z}| \leqslant \rho' \|y - \bar{z}\|_{\mathcal{C}} \leqslant C\big(\|y - z\|_{\mathcal{C}} + r\big) \leqslant C|y - z|.$$

Accordingly, if $z + y \in S_\theta$, then $|z + y - \bar{z}| \leqslant C|y|$. As a consequence of this and (2.50), we have that, for any fixed $y' \in \mathbb{R}^{n-1}$,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\chi_{S_\theta}(z+y)}{|y|^{n+s}}\, dy_n &\leqslant C \int_{\mathbb{R}} \frac{\chi_{S_\theta}(z+y)}{|z+y-\bar{z}|^{n+s}}\, dy_n \leqslant C \int_{\mathbb{R}} \frac{\chi_{S_\theta}(z+y)}{|z'+y'-\bar{z}'|^{n+s}}\, dy_n \\
&= C \int_{\{G(\bar{z}')+\nabla G(\bar{z}')\cdot(z'+y'-\bar{z}') \leqslant z_n + y_n \leqslant G(z'+y')\}} \frac{dy_n}{|z'+y'-\bar{z}'|^{n+s}} \\
&= C\, \frac{G(z'+y') - G(\bar{z}') - \nabla G(\bar{z}') \cdot (z'+y'-\bar{z}')}{|z'+y'-\bar{z}'|^{n+s}}.
\end{aligned}$$

Hence, if we integrate in $y' \in \mathbb{R}^{n-1}$ and use the change of variable $Y' := z' + y' - \bar{z}'$, up to renaming $C > 0$ we have that

$$\begin{aligned}
I_z(\theta) &\leqslant C \int_{\mathbb{R}^n} \frac{\chi_{S_\theta}(z+y)}{|y|^{n+s}}\, dy \leqslant C \int_{\mathbb{R}^{n-1}} \frac{G(z'+y') - G(\bar{z}') - \nabla G(\bar{z}') \cdot (z'+y'-\bar{z}')}{|z'+y'-\bar{z}'|^{n+s}}\, dy' \\
&= C \int_{\mathbb{R}^{n-1}} \frac{G(Y'+\bar{z}') - G(\bar{z}') - \nabla G(\bar{z}') \cdot Y'}{|Y'|^{n+s}}\, dY' \leqslant Cb,
\end{aligned} \tag{2.52}$$

where (2.49) has been used in the last estimate —note that $\bar{z} \in \mathcal{C}_{3/\rho}(z)$ and thus

$$|\bar{z}'| \leqslant |\bar{z}| \leqslant \rho'\big(\|z - \bar{z}\|_{\mathcal{C}} + \|z\|_{\mathcal{C}}\big) \leqslant \rho'\big(3/\rho + 1/\rho\big) \leqslant 4\rho'/\rho.$$

Final estimate. We recall that, from (2.43),

$$0 \leqslant L\phi^b(z) - f_\phi(\phi^b(z)) = \int_{-1}^{1} d\theta\, I_z(\theta) = \int_{\mathcal{A}} d\theta\, I_z(\theta) + \int_{\mathcal{B}} d\theta\, I_z(\theta),$$

where $\mathcal{A}$ is the set of levels θ as in *Case 1* and $\mathcal{B}$ is the set of levels θ as in *Case 2*. Then, on the one hand, (2.46) implies that $|\mathcal{A}| \leqslant 2\delta$, and, for each $\theta \in \mathcal{A}$, we have that $I_z(\theta) \leqslant C$. On the other hand, (2.52) yields that, for each $\theta \in \mathcal{B}$, we have that $I_z(\theta) \leqslant Cb$. Therefore,

$$0 \leqslant L\phi^b(z) - f_\phi(\phi^b(z)) = \int_{\mathcal{A}} d\theta\, I_z(\theta) + \int_{\mathcal{B}} d\theta\, I_z(\theta) \leqslant C\delta + Cb,$$

which proves (2.35), as desired. $\square$

3. Decay estimates for solutions

The goal of this section is to provide suitable decay estimates for our solutions. For this, we start with a preliminary result:

Lemma 3.1. *Let w be such that $Lw \leqslant -kw$ in B_R, where $R \in [2,\infty)$ and $k \in [1,\infty)$. Suppose that $0 \leqslant w \leqslant 2$ in all of $\mathbb{R}^n$, then*

$$0 \leqslant w \leqslant \frac{C}{(k^{1/s}R)^{\gamma_0}} \quad \text{in } B_1,$$

where C, $\gamma_0 > 0$ depend only on n, s, and on the ellipticity constants.

Proof. The idea of the proof is to use a barrier argument at the different scales. For the reader's convenience, we split the proof into three steps.

Step 1. We prove the following statement. *Assume that $L\bar{w} \leqslant -\bar{w}$ in B_1 and*

$$0 \leqslant \bar{w} \leqslant 2^{\gamma_0 j} \quad \text{in } B_{2^j} \tag{3.1}$$

for all $j \geqslant 0$. Then, (3.1) *holds also for $j = -1$.*

For this, we take $\eta \in C_0^\infty(B_{3/4})$ radially nonincreasing, with $\eta = 1$ in $B_{1/2}$. Let also $\gamma_0 \in (0,1)$, to be taken appropriately small, and set $h_0 := 1 - 2^{-\gamma_0} > 0$. We define the function

$$\phi := (1 - h_0\eta)\chi_{B_1} + \sum_{j=1}^{\infty} 2^{\gamma_0 j}\chi_{B_{2^j}\setminus B_{2^{j-1}}}.$$

We observe that $\phi = 1 - h_0\eta$ in B_1 and $\phi = 2^{\gamma_0 j}$ in $B_{2^j} \setminus B_{2^{j-1}}$ for any $j \geqslant 1$. As a consequence, for any $x \in B_{3/4}$,

$$\begin{aligned}
-L\phi(x) \;&=\; \int_{B_1} \frac{(1-h_0\eta)(z) - (1-h_0\eta)(x)}{|z-x|^{n+s}}\,\mu\left(\frac{z-x}{|z-x|}\right)dz \\
&\qquad + \sum_{j=1}^{+\infty}\int_{B_{2^j}\setminus B_{2^{j-1}}} \frac{2^{\gamma_0 j} - (1-h_0\eta)(x)}{|z-x|^{n+s}}\,\mu\left(\frac{z-x}{|z-x|}\right)dz \\
&\leqslant\; h_0\left[\left|\int_{B_1}\frac{\eta(x)-\eta(z)}{|z-x|^{n+s}}\,\mu\left(\frac{z-x}{|z-x|}\right)dz\right| + \sum_{j=1}^{+\infty}\left|\int_{B_{2^j}\setminus B_{2^{j-1}}}\frac{2^{\gamma_0 j} - 1 - h_0}{|z-x|^{n+s}}\,\mu\left(\frac{z-x}{|z-x|}\right)dz\right|\right] \\
&\leqslant\; Ch_0 + C\sum_{1\leqslant j\leqslant \gamma_0^{-1/3}}(2^{\gamma_0 j} - 1) + C\sum_{j\geqslant \gamma_0^{-1/3}}\frac{2^{\gamma_0 j}}{2^{j(1+s)}} \\
&\leqslant\; Ch_0 + C\frac{2^{\gamma_0^{2/3}} - 1}{\gamma_0^{1/3}} + \frac{C}{2^{\frac{1+s}{2\gamma_0^{1/3}}}},
\end{aligned}$$

with $C > 0$ possibly varying from line to line. In particular, when γ_0 (and so h_0) is small, we have that $-L\phi \leqslant 1/2 \leqslant \phi$ in $B_{3/4}$.

Since also $\phi \geqslant \bar{w}$ outside $B_{3/4}$, using the maximum principle we have that $\bar{w} \leqslant \phi$ in $B_{3/4}$. Consequently, $\bar{w} \leqslant 1 - h_0 = 2^{-\gamma_0}$ in $B_{1/2}$. This completes the proof of the statement in *Step 1*.

Step 2. Now we prove the following statement. *Let $\tilde{w}$ be such that $L\tilde{w} \leqslant -\tilde{w}$ in $B_{\tilde{R}}$, where $\tilde{R} \geqslant 1$. Suppose that $0 \leqslant \tilde{w} \leqslant 2$ in all of $\mathbb{R}^n$, then, for any $\tilde{\rho} \in \left[\frac{1}{2},\, \tilde{R}\right)$, we have*

$$0 \leqslant \tilde{w} \leqslant C \left(\frac{\tilde{\rho}}{\tilde{R}}\right)^{\gamma_0} \quad \textit{in } B_{\tilde{\rho}},$$

for some C, $\gamma_0 > 0$.

The proof of this claim is an iteration of *Step 1*. Namely, we take $N \in \mathbb{N}$ such that $2^N \leqslant \tilde{R} < 2^{N+1}$. For any $i \in \mathbb{N}$, $i \in [1,\, N+1]$, we set

$$\bar{w}_i(x) = 2^{(i-1)\gamma_0 - 1}\, \tilde{w}(2^{N-i+1}x). \tag{3.2}$$

Notice that, by construction,

$$L\bar{w}_i \leqslant -2^{(N-i+1)s}\, \bar{w}_i \leqslant -\bar{w}_i \text{ in } B_{2^{i-1}} \supset B_1 \tag{3.3}$$

and, if $i \in \mathbb{N}$, $i \in [1,\, N]$,

$$\bar{w}_{i+1}(x) = 2^{\gamma_0}\, \bar{w}_i(x/2). \tag{3.4}$$

We claim that

$$\text{for any } 0 \leqslant j \leqslant i-1, \text{ we have that } \bar{w}_i \leqslant 2^{(j-1)\gamma_0} \text{ in } B_{2^{j-1}}. \tag{3.5}$$

The proof of (3.5) is by induction. First, we observe that, for any $j \geqslant 0$, in B_{2^j} we have that

$$\bar{w}_1 \leqslant 2^{-1} \sup_{\mathbb{R}^n} \tilde{w} \leqslant 1 \leqslant 2^j.$$

From this and (3.3), we can use *Step 1* with $\bar{w} := \bar{w}_1$ and find that $\bar{w}_1 \leqslant 2^{-\gamma_0}$ in $B_{1/2}$. This is (3.5) when $i = 1$.

Now, we suppose that (3.5) holds true for the index $i \in [1,\, N]$, and we prove it for the index $i+1$. To this aim, we claim that, for any $j \geqslant 0$,

$$\bar{w}_{i+1} \leqslant 2^{\gamma_0 j} \text{ in } B_{2^j}. \tag{3.6}$$

To check this, we distinguish two cases. If $j \geqslant i$, then we recall (3.2) and we see that

$$\sup_{B_{2^j}} \bar{w}_{i+1} \leqslant 2^{i\gamma_0 - 1} \sup_{\mathbb{R}^n} \tilde{w} \leqslant 2^{i\gamma_0} \leqslant 2^{j\gamma_0},$$

as desired. If instead $j \leqslant i-1$, then we exploit (3.5) with index i together with (3.4) and we obtain

$$\sup_{B_{2^j}} \bar{w}_{i+1} = 2^{\gamma_0} \sup_{B_{2^{j-1}}} \bar{w}_i \leqslant 2^{\gamma_0} \cdot 2^{(j-1)\gamma_0} = 2^{j\gamma_0}.$$

This proves (3.6).

So, by (3.3) and (3.6), we can use *Step 1* with $\bar{w} := \bar{w}_{i+1}$ and conclude that $\bar{w}_{i+1} \leqslant 2^{-\gamma_0}$ in $B_{1/2}$. This inequality and (3.6) imply that

$$\text{for any } 0 \leqslant j \leqslant i, \text{ we have that } \bar{w}_{i+1} \leqslant 2^{(j-1)\gamma_0} \text{ in } B_{2^{j-1}},$$

that is (3.5) for the index $i+1$, as desired. This completes the inductive proof of (3.5).

Hence, using the notation $m := i - j$, we deduce from (3.5) that

$$\sup_{B_{2^{N-m}}} \tilde{w} \leqslant 2^{1-m\gamma_0}, \tag{3.7}$$

for any $m \in \mathbb{Z}$ with $m \leqslant N+1$.

Now we take $M \in \mathbb{Z}$ such that $2^{-M-1} \leqslant 2^{-N}\tilde{\rho} < 2^{-M}$. Notice that

$$\frac{1}{2} \leqslant \tilde{\rho} \leqslant 2^{N-M},$$

hence $M \leqslant N+1$. Then, we can apply (3.7) with $m := M$ and we obtain that

$$\sup_{B_{\tilde{\rho}}} \tilde{w} \leqslant \sup_{B_{2^{N-M}}} \tilde{w} = \leqslant 2^{1-M\gamma_0} = \frac{2^{1+2\gamma_0} \cdot 2^{(N-M-1)\gamma_0}}{2^{(N+1)\gamma_0}} \leqslant \frac{2^{1+2\gamma_0} \cdot \tilde{\rho}^{\gamma_0}}{\tilde{R}^{\gamma_0}}.$$

This establishes the claim in *Step 2*.

Step 3. Now we complete the proof of Lemma 3.1 scaling the statement proven in *Step 2*. To this aim, we take w as in the statement of Lemma 3.1 and $p \in B_1$. We define $\tilde{R} := (R-1)k^{1/s}$ and

$$\tilde{w}(x) := w\left(p + \frac{x}{k^{1/s}}\right).$$

Notice that $\tilde{R} \geqslant k^{1/s} \geqslant 1$. Furthermore, for any $x \in B_{\tilde{R}}$ we have that

$$\left|p + \frac{x}{k^{1/s}}\right| \leqslant |p| + \frac{|x|}{k^{1/s}} \leqslant 1 + \frac{\tilde{R}}{k^{1/s}} = R,$$

and therefore, for any $x \in B_{\tilde{R}}$,

$$L\tilde{w}(x) = \frac{1}{k} Lw\left(p + \frac{x}{k^{1/s}}\right) \leqslant -w\left(p + \frac{x}{k^{1/s}}\right) = -\tilde{w}(x).$$

So, we can use *Step 2* with $\tilde{\rho} := 1/2$ and obtain that

$$w(p) = \tilde{w}(0) \leqslant \sup_{B_{1/2}} \tilde{w} \leqslant \frac{C}{(2\tilde{R})^{\gamma_0}} = \frac{C}{(2(R-1)k^{1/s})^{\gamma_0}} \leqslant \frac{C}{(Rk^{1/s})^{\gamma_0}},$$

which is the desired result. $\square$

As a consequence of the previous preliminary result, we have:

Lemma 3.2. *Let $R \geqslant 2$ and $\varepsilon \in (0,1]$. Let $u : \mathbb{R}^n \to [-1,1]$ be a solution of $Lu = \varepsilon^{-s} f(u)$ in $\mathbb{R}^n$. Then, if ε is sufficiently small,*

$$u(x) \geqslant 1 - C\left(\frac{\varepsilon}{R}\right)^{\gamma_0} \quad \textit{whenever} \quad B_R(x) \subset \{u \geqslant 1-\kappa\}$$

and

$$u(x) \leqslant -1 + C\left(\frac{\varepsilon}{R}\right)^{\gamma_0} \quad \textit{whenever} \quad B_R(x) \subset \{u \leqslant -1+\kappa\},$$

for some C, $\gamma_0 > 0$.

In particular, for $n=1$, the profile ϕ_0 satisfies

$$\big|\phi_0 - (-1)\big| \leqslant C_f |x|^{-\gamma_0} \textit{ in } (-\infty,-1] \qquad \textit{and} \qquad \big|\phi_0 - 1\big| \leqslant C_f |x|^{-\gamma_0} \textit{ in } [1,+\infty). \tag{3.8}$$

Proof. Using assumption (H2) we have

$$-f(u) = f(1) - f(u) \leqslant -c_\kappa(1-u) \quad \text{for } u \geqslant 1-\kappa$$

and therefore

$$L(1-u) = -Lu = -\varepsilon^{-s} f(u) \leqslant -\varepsilon^{-s} c_\kappa (1-u) \quad \text{in } \{u \geqslant 1-\kappa\}.$$

Thus, from Lemma 3.1 with $w := 1-u$ and $k := \varepsilon^{-s} c_\kappa$ we obtain the desired decay estimates. $\square$

4. Improvement of oscillation for level sets of solutions

The goal of this section is to establish the following improvement of oscillation result for level sets, which is one of the cornerstones of this paper. This result is crucial since it gives compactness of sequences of vertical rescaling of the level sets.

For fixed $\alpha \in (0, s)$, $m_0 \in \mathbb{N}$ and $a > 0$, let us introduce

$$k_a := \left\lfloor \frac{\log a}{\log(2^{-\alpha})} \right\rfloor - m_0, \quad \text{which belongs to } \mathbb{N} \text{ for } a \text{ small.} \tag{4.1}$$

Notice that $k_a \uparrow +\infty$ as $a \downarrow 0$, and

$$\frac{1}{2} 2^{-\alpha m_0} 2^{-\alpha k_a} \leqslant a \leqslant 2^{-\alpha m_0} 2^{-\alpha k_a}. \tag{4.2}$$

Theorem 4.1. *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*. Then, given $\alpha \in (0, s)$ there exist $p_0 \in (2, \infty)$, $a_0 \in (0, 1/4)$, and $\eta_0 \in (0, 1)$, depending only on α, m_0, and on the universal constants, such that the following statement holds.*

Let $a \in (0, a_0)$ and $\varepsilon \in (0, a^{p_0})$. Let $u : \mathbb{R}^n \to (-1, 1)$ be a solution of $Lu = \varepsilon^{-s} f(u)$ in $B'_{2^{k_a}} \times (-2^{k_a}, 2^{k_a})$ such that

$$\{x_n \leqslant -a 2^{j(1+\alpha)}\} \subset \{u \leqslant -1 + \kappa\} \subset \{u \leqslant 1 - \kappa\} \subset \{x_n \leqslant a 2^{j(1+\alpha)}\} \quad \text{in } B'_{2^j} \times (-2^{k_a}, 2^{k_a}),$$

for $j = \{0, 1, 2, \ldots, k_a\}$.

Then, either

$$\{x_n \leqslant -a(1 - \eta_0)\} \subset \{u \leqslant -1 + \kappa\} \quad \text{in } B'_{1/2} \times (-2^{k_a}, 2^{k_a})$$

or

$$\{u \leqslant 1 - \kappa\} \subset \{x_n \leqslant a(1 - \eta_0)\} \quad \text{in } B'_{1/2} \times (-2^{k_a}, 2^{k_a}).$$

We will deduce Theorem 4.1 from the following result:

Proposition 4.2. *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*. Then, given $\alpha \in (0, s)$ there exist $p_0 \in (2, \infty)$, $a_0 \in (0, 1/4)$, and $\eta_0 \in (0, 1)$, depending only on α, m_0, and on the universal constants, such that the following statement holds.*

Let $a \in (0, a_0)$ and $\varepsilon \in (0, a^{p_0})$. Let $u : \mathbb{R}^n \to (-1, 1)$ be a solution of $Lu = \varepsilon^{-s} f(u)$ in $B'_{2^{k_a}} \times (-2^{k_a}, 2^{k_a})$ such that

$$\{u \leqslant 1 - \kappa\} \subset \left\{x_n \leqslant a 2^{j(1+\alpha)}\right\} \qquad \text{in } B'_{2^j} \times (-2^{k_a}, 2^{k_a}) \tag{4.3}$$

for $j = \{0, 1, 2, \ldots, k_a\}$, and

$$\int_{B_2} u \, dx \geqslant 0. \tag{4.4}$$

Then, we have that

$$\{u \leqslant 1 - \kappa\} \subset \{x_n \leqslant a(1 - \eta_0)\} \quad \text{in } B'_{1/2} \times (-2^{k_a}, 2^{k_a}). \tag{4.5}$$

For its use in the proof of Proposition 4.2, we recall the following maximum principle:

Lemma 4.3. *There exists $\theta > 0$, depending only on n, s, λ and Λ, such that the following statement holds true.*

Let $w \in C^2(B_4)$ satisfy

$$\begin{cases} Lw \geqslant -\theta \quad \text{in } B_4 \cap \{w \leqslant 0\}, \\[2ex] \displaystyle\int_{\mathbb{R}^n} w_-(y)(1+|y|)^{-n-s}\,dy \leqslant \theta, \\[2ex] \displaystyle\int_{B_4} w_+(y)\,dy \geqslant 1. \end{cases}$$

Then $w > 0$ in B_2.

Proof. See Lemma 6.2 in [13]. ☐

In order to prove Proposition 4.2 (and so Theorem 4.1), we also need the following observation:

Lemma 4.4. *Let $\phi := \phi_0(\,\cdot\,/\varepsilon)$ and $\phi^b := \phi \circ d_b$, where d_b is defined in* (2.22) *(see also* (2.23)*).*

Then,

$$\left|L\phi^b - \varepsilon^{-s} f(\phi^b)\right| \leqslant C(b + \varepsilon^{\gamma_0}) \quad \text{in } B_4,$$

where $C > 0$ is a universal constant and $\gamma_0 > 0$ is the constant given by Lemma 3.2.

Proof. By (3.8), we have that (2.34) is satisfied with $\delta := C\varepsilon^{\gamma_0}$. Hence, using Lemma 2.5 (scaled to B_4 and with $f_\phi := \varepsilon^{-s} f$), we obtain that $\left|L\phi^b - \varepsilon^{-s} f(\phi^b)\right| \leqslant C(b+\delta)$. The desired result now plainly follows. ☐

With this, we are in the position of proving Proposition 4.2.

Proof of Proposition 4.2. In all the proof we denote

$$C_r := B'_r \times (-2^{k_a}, 2^{k_a}).$$

Fix $z' \in B'_{1/2}$ and let

$$\bar{u}(x) := u(x' - z', x_n). \tag{4.6}$$

By assumptions, we have

$$\{\bar{u} \leqslant 1 - \kappa\} \subset \left\{x_n \leqslant a + \frac{1}{2}\, b\, \xi(x')\right\} \quad \text{in } C_{2^{k_a}} \tag{4.7}$$

for

$$b := Ca, \tag{4.8}$$

where $C > 0$ depends only on α and ξ was defined in (1.16).

Throughout the proof, we use the notations

$$\phi(t) := \phi_0\left(\frac{t}{\varepsilon}\right) \quad \text{and} \quad \phi^b(x) := \phi \circ d_b(x). \tag{4.9}$$

The idea of the proof is to consider the infimum h_* among all the $h \geqslant 0$ such that

$$\min_{x \in \overline{B_1}} \left(\bar{u}(x) - \phi^b(x - he_n)\right) \geqslant 0. \tag{4.10}$$

We will indeed observe that such h_* is well defined. Then, we will show that

$$h_* < a(1-\eta) \tag{4.11}$$

for a suitable and universal $\eta \in (0,1)$. The proof of (4.11) will be done by contradiction (namely, we will show that the inequality $h_* \geqslant a - \eta a$ leads to a contradiction). Then, from the inequality in (4.11), the claim in Proposition 4.2 will follow in a straightforward way.

Step 1. Let us show first that *if* $h \geqslant a+3$ *then* (4.10) *holds true*.

First, we claim that

$$\phi^b(x - he_n) \leqslant -1 + \frac{C\varepsilon^{\gamma_0}}{\big(x_n - h - b\xi(x')\big)_-^{\gamma_0}} \quad \text{for all } x \in C_{2^{k_a - 1}} \tag{4.12}$$

and

$$\bar{u}(x) \geqslant 1 - \frac{C\varepsilon^{\gamma_0}}{\big(x_n - a - \frac{1}{2}\, b\,\xi(x')\big)_+^{\gamma_0}} \quad \text{for all } x \in C_{2^{k_a - 1}}. \tag{4.13}$$

To prove (4.12) and (4.13), it is important to observe that, by (4.2),

$$|\nabla(b\xi)(z')| \leqslant Ca(1+|z'|^2)^{\frac{\alpha-1}{2}}|z'| \leqslant Ca2^{k_a} \leqslant C2^{-m_0} \quad \text{for all } z' \in B'_{2^{k_a}}. \tag{4.14}$$

Now, to show (4.12), we use the decay properties of ϕ_0 in Lemma 3.2, which imply that, for all $h \geqslant 0$,

$$\phi^b(x - he_n) = \phi_0\left(\frac{d_b(x-he_n)}{\varepsilon}\right) \leqslant -1 + \frac{C\varepsilon^{\gamma_0}}{\big(d_b(x-he_n)\big)_-^{\gamma_0}}. \tag{4.15}$$

Also, as a consequence of (4.14), we see that, for all $y \in B'_{2^{k_a}} \times \mathbb{R}$,

$$\big(d_b(y)\big)_- \geqslant c\left(y_n - b\xi(y')\right)_-, \tag{4.16}$$

for some $c>0$ depending only on ρ and ρ'. (for more details see Lemma 3.2 in [22]).

Now, making use of (4.15) and (4.16) (with $x \in B_{2^{k_a}}$ and $y := x - he_n$), we deduce (4.12).

Let us now prove (4.13). To do it, given $x \in C_{2^{k_a-1}}$, define $R = R(x)$ to be the the largest radius for which

$$B_R(x) \subset C_{2^{k_a}} \cap \left\{ y_n > a + \frac{1}{2}\, b\,\xi(y') \right\}.$$

By (4.7), we know that $u(y) \geqslant 1-\kappa$ for any $y \in B_{2^{k_a}}$ with $y_n > a + \frac{1}{2}\, b\,\xi(y')$ and by assumption u solves $Lu = \varepsilon^{-s}u$ in $C_{2^{k_a}}$. Hence, using Lemma 3.2 we obtain

$$u(x) \geqslant 1 - \frac{C\varepsilon^{\gamma_0}}{R^{\gamma_0}}. \tag{4.17}$$

Now we observe that, by (4.14), for any $x \in C_{2^{k_a}/2}$ with $x_n > a + \frac{1}{2}\, b\,\xi(x')$ we have

$$R(x) \geqslant c\left(x_n - a - \frac{1}{2}\, b\,\xi(x')\right)_+,$$

as long as $c>0$ is sufficiently small. Hence, (4.13) follows.

Now we remark that

$$\big(x_n - a - \tfrac{1}{2}\, b\xi(x')\big) - \big(x_n - h - b\,\xi(x')\big) = h - a + \frac{b}{2}\,\xi(x'). \tag{4.18}$$

Hence, since we are now assuming that $h - a \geqslant 3 > 2$, we deduce from (4.18) that

$$\big(x_n - a - \tfrac{1}{2}\, b\xi(x')\big) - \big(x_n - h - b\,\xi(x')\big) \geqslant 1 + \frac{b}{2}\,\xi(x') \geqslant 1.$$

Consequently,

$$\text{either} \quad \big(x_n - a - \frac{1}{2}\, b\, \xi(x')\big) \geqslant 1 \tag{4.19}$$

$$\text{or} \quad \big(x_n - h - b\xi(x')\big) \leqslant -1. \tag{4.20}$$

Now we claim that

$$\bar{u}(x) - \phi^b(x - he_n) \geqslant -C\varepsilon^{\gamma_0} \quad \text{for any } x \in C_{2^{k_a-1}}. \tag{4.21}$$

For this, we distinguish two cases, according to (4.19) and (4.20). If (4.19) is satisfied, then we exploit (4.13) and the fact that $\phi^b \leqslant 1$ to find that

$$\bar{u}(x) - \phi^b(x - he_n) \geqslant \bar{u}(x) - 1 \geqslant -\frac{C\varepsilon^{\gamma_0}}{\big(x_n - a - \frac{1}{2}\, b\, \xi(x')\big)_+^{\gamma_0}} \geqslant -C\varepsilon^{\gamma_0},$$

up to renaming $C > 0$, which gives (4.21) in this case.

If instead the inequality in (4.20) holds true, we use (4.12) and the fact that $\bar{u} \geqslant -1$ to see that

$$\bar{u}(x) - \phi^b(x - he_n) \geqslant \bar{u}(x) + 1 - \frac{C\varepsilon^{\gamma_0}}{\big(x_n - h - b\xi(x')\big)_-^{\gamma_0}} \geqslant -\frac{C\varepsilon^{\gamma_0}}{\big(x_n - h - b\xi(x')\big)_-^{\gamma_0}} \geqslant -C\varepsilon^{\gamma_0},$$

up to renaming constants, and this completes the proof of (4.21).

Furthermore, since ξ is a nonnegative function with $\xi(0) = 0$, the affine function $\ell(x) := x_n/\tilde{c}$, with $\tilde{c} = h_L(e_n) > 0$, is admissible in (2.22). As a consequence, we obtain that $d_b(x) \leqslant x_n/\tilde{c}$. Accordingly, from the monotonicity of ϕ, we have that

$$\phi_b(x) = \phi(d_b(x)) \leqslant \phi(x_n/\tilde{c}) \text{ for all } x \in \mathbb{R}^n. \tag{4.22}$$

Now, since in this case $h \geqslant a + 3 \geqslant 3$, we observe that, for any $x \in B_2$,

$$\frac{x_n - h}{\varepsilon} \leqslant \frac{2-3}{\varepsilon} = -\frac{1}{\varepsilon}$$

and so, if ε is large enough,

$$\sup_{B_2} \phi_0\left(\frac{x_n - h}{\tilde{c}\varepsilon}\right) \leqslant -\frac{1}{2}.$$

Therefore, recalling the assumption (4.4) and (4.22),

$$\begin{aligned}\int_{B_2} \bar{u}(x) - \phi^b(x - he_n)\, dx &\geqslant \int_{B_2} \bar{u}(x) - \phi(\tilde{c}(x_n - h))\, dx \\ &= \int_{B_2} \bar{u}(x) - \phi_0\left(\tilde{c}\,\frac{x_n - h}{\tilde{c}\varepsilon}\right)\, dx \geqslant 0 - \int_{B_2} \phi_0\left(\frac{x_n - h}{\tilde{c}\varepsilon}\right)\, dx \geqslant c,\end{aligned} \tag{4.23}$$

where $c > 0$ is a universal constant.

We consider now the function $w(x) := \bar{u}(x) - \phi^b(x - he_n)$. Let us show that

$$Lw \geqslant -C(b + \varepsilon^{\gamma_0}) \quad \text{in } \{w \leqslant 0\} \cap B_4. \tag{4.24}$$

Indeed, let

$$\Omega := \{w \leqslant 0\} \cap \big(\{u \geqslant 1 - \kappa\} \cup \{\phi^b(\cdot - he_n) \leqslant -1 + \kappa\}\big).$$

To start with, we will show that

$$\big(\{w \leqslant 0\} \cap B_4\big) \setminus \Omega = \varnothing. \tag{4.25}$$

Indeed, suppose, by contradiction, that there exists a point $y \in \big(\{w \leqslant 0\} \cap B_4\big) \setminus \Omega$. Then,

$$\bar{u}(y) < 1 - \kappa \quad \text{and} \quad \phi^b(y - he_n) > -1 + \kappa. \tag{4.26}$$

Thus, by (4.7), we see that

$$0 \geqslant y_n - a - \frac{1}{2}b\xi(y') = y_n - h + h - a - \frac{1}{2}b\xi(y') \geqslant y_n - h + 3 - \frac{1}{2}b\xi(y').$$

Therefore

$$y_n - h - b\xi(y') = y_n - h + 3 - \frac{1}{2}b\xi(y') - 3 - \frac{1}{2}b\xi(y') \leqslant 0 - 3 - \frac{1}{2}b\xi(y') < 0.$$

Hence, we can use (4.12), which gives that

$$\phi^b(y_n - he_n) \leqslant -1 + C\varepsilon^{-\gamma_0},$$

up to renaming $C > 0$. Thus, for ε small, we deduce that $\phi^b(y - he_n) \leqslant -1 + \kappa$, which gives that the second inequality in (4.26) cannot occur. This contradiction establishes (4.25).

Hence, in view of (4.25), to complete the proof of (4.24), we only need to show that (4.24) holds true in $\Omega \cap B_4$. To this aim, we take $y \in \Omega \cap B_4$. Then, $w(y) \leqslant 0$ and so $\bar{u}(y) \leqslant \phi^b(y - he_n)$. Therefore, using Lemma 4.4,

$$\begin{aligned} Lw(y) = L\bar{u}(y) - L\phi^b(y - he_n) &\geqslant \varepsilon^{-s} f\big(\bar{u}(y)\big) - \varepsilon^{-s} f\big(\phi^b(y - he_n)\big) - C\,(b + \varepsilon^{\gamma_0}) \\ &\geqslant \varepsilon^{-s} f'(\xi)\, w(y) - C\,(b + \varepsilon^{\gamma_0}), \end{aligned} \tag{4.27}$$

where $C > 0$ and $\xi = \xi(y)$ belongs to the real interval $\big[\bar{u}(y),\, \phi^b(y \cdot - he_n)\big]$.

We also recall that by (H2) we have that $f' \leqslant 0$ in $[-1, -1 + \kappa] \cup [1 - \kappa, 1]$. Moreover, by the definition of Ω, we have that either $1 - \kappa \leqslant \bar{u}(y) < \phi^b(y - he_n) \leqslant 1$ or $-1 \leqslant \bar{u}(y) < \phi^b(y - he_n) \leqslant -1 + \kappa$. In any case, we have that $f'(\xi) \leqslant 0$ and so (4.24) follows from (4.27).

Now, putting together (4.24), (4.21) and (4.23), we have proven that w satisfies

$$\begin{cases} Lw \geqslant -C(b + \varepsilon^{\gamma_0}) \quad \text{in } B_4 \cap \{w < 0\}, \\ w \geqslant -C\varepsilon^{\gamma_0} \quad \text{in } C_{2^{k_a - 1}}, \\ w \geqslant -2 \quad \text{in } \mathbb{R}^n \setminus C_{2^{k_a - 1}}, \\ \displaystyle\int_{B_2} w(y)\, dy \geqslant c. \end{cases}$$

Note that

$$\int_{\mathbb{R}^n} \frac{w^-(y)}{(1 + |y|)^{n+s}}\, dy \leqslant C\varepsilon^{\gamma_0} + \int_{|y| \geqslant 2^{k_a - 1}} \frac{2dy}{|y|^{n+s}} \leqslant C\varepsilon^{\gamma_0} + C2^{-sk_a}.$$

Then, choosing a_0 small enough (that corresponds to k_a large in view of (4.1)), we fall under the assumptions of Lemma 4.3, which yields that $w > 0$ in B_2. This plainly implies the desired statement for *Step 1*.

Step 2. Let

$$h_* := \inf\ \big\{\ h \geqslant 0\ :\ \text{(4.10) holds}\ \big\}.$$

Notice that the infimum is taken over a nonempty set, thanks to *Step 1*, and indeed $h_* \leqslant a + 3 < +\infty$. We next show that

$$h_* < a - \eta a \quad \text{as long as } \eta > 0 \text{ is sufficiently small.} \tag{4.28}$$

The proof of (4.28) will be by contradiction, namely we will show that the two conditions $h_* \geqslant a - \eta a$ and η small enough lead to a contradiction (for an appropriately small a_0).

To this aim, we define

$$\phi_*(x) := \phi^b(x - h_* e_n).$$

We observe that, by the definition of h_*, we have that $u - \phi_* \geqslant 0$ in B_1.

Under this assumption, we will prove that

$$\bar{u} - \phi_* > 0 \quad \text{in } B_2, \tag{4.29}$$

which contradicts the definitions of h_* and ϕ_*.

Indeed, using the contradictory assumption that $h_* \geqslant a - \eta a$, we have

$$\big(x_n - a - \tfrac{1}{2}\, b\,\xi(x')\big) - \big(x_n - h_* - b\,\xi(x')\big) = h_* - a + \frac{b}{2}\,\xi(x') \geqslant \frac{b}{2}\,\xi(x') - \eta a.$$

Then, if η is small enough we have, for all $x \in C_{2^{k_a-1}} \setminus B_1$,

$$\big(x_n - a - \tfrac{1}{2}\, b\,\xi(x')\big) - \big(x_n - h_* - b\,\xi(x')\big) \geqslant \frac{b}{2}\,\xi(1/2) - \eta a \geqslant \frac{b}{8}$$

where we have used that $b = Ca$ (recall (4.8)).

Therefore, for all $x \in C_{2^{k_a-1}} \setminus B_1$,

$$\text{either} \quad \big(x_n - a - \tfrac{1}{2}\, b\, g(x')\big) \geqslant \frac{b}{16} \quad \text{or} \quad \big(x_n - b\, g(x')\big) \leqslant -\frac{b}{16}.$$

Thus, similarly as in *Step 1*, using either (4.12) and the fact that $\bar u \geqslant -1$, or (4.13) and $\phi_* \leqslant 1$, we obtain that

$$\bar u - \phi_* \geqslant -C(\varepsilon/b)^{\gamma_0} \quad \text{in } C_{2^{k_a-1}},$$

for some $C > 0$.

Next, similarly as in *Step 1*, the function $w := u - \phi_*$ satisfies

$$\begin{cases} Lw \geqslant -C(b + \varepsilon^{\gamma_0}) \quad \text{in } B_4 \cap \{w \leqslant 0\}, \\ w \geqslant -C(\varepsilon/b)^{\gamma_0} \quad \text{in } B_{2^{k_a-1}} \setminus B_4, \\ w \geqslant -2 \quad \text{in } \mathbb{R}^n \setminus B_{2^{k_a-1}}, \\ \displaystyle\int_{B_2} w(y)\, dy \geqslant c\, h_* \geqslant ca, \end{cases} \tag{4.30}$$

up to renaming $c > 0$.

Notice now that, recalling (4.8),

$$\begin{aligned} \frac{1}{a}\int_{\mathbb{R}^n} \frac{w^-(y)}{(1+|y|)^{n+s}}\, dy &\leqslant \frac{C}{a}\left(\frac{\varepsilon}{b}\right)^{\gamma_0} + \int_{|y|\geqslant 2^{k_a-1}} \frac{2dy}{|y|^{n+s}} \leqslant \frac{C}{a}\left(\frac{\varepsilon}{a}\right)^{\gamma_0} + \frac{C}{a} 2^{-sk_a} \\ &\leqslant \frac{C}{a}\left(a^{p_0-1}\right)^{\gamma_0} + C2^{-(s-\alpha)k_a}\,\frac{2^{-\alpha k_a}}{a} \\ &\leqslant C\big(a^{(p_0-1)\gamma_0-1} + C_{m_0} 2^{-(s-\alpha)k_a}\big) \to 0 \qquad \text{as } a \downarrow 0, \end{aligned}$$

where $C_{m_0} > 0$ depends on m_0. Similarly,

$$\frac{C}{a}\left(b + \varepsilon^{\gamma_0} + \left(\frac{\varepsilon}{b}\right)^{\gamma_0}\right) \leqslant Ca^{(p_0-1)\gamma_0-1} \to 0 \qquad \text{as } a \downarrow 0.$$

Then, choosing a_0 small enough, we can apply Lemma 4.3 to show that $w > 0$ in B_2, thus proving (4.29),

Now, by the definition of h_*, we know that there exists a point $x_* \in \overline{B_1}$ such that $w(x_*) = \bar u(x_*) - \phi_*(x_*) = 0$. This is in contradiction with (4.29). Therefore, we have proved (4.28) and completed the proof of *Step 2*.

Step 3. We now complete the proof of Proposition 4.2. For this, we recall the definition of $\bar u$ in (4.6) and we prove that

$$\{\bar u \leqslant 1 - \kappa\} \subset \left\{x_n \leqslant a\,\left(1 - \frac{\eta}{2}\right)\right\} \quad \text{on } \{0\} \times (-1, 1). \tag{4.31}$$

Indeed, by *Step 2*, we know that

$$\bar u(x) - \phi^b(x - a(1-\eta)e_n) \geqslant 0.$$

Moreover (see e.g. Lemma 3.1 in [22]), we have that, on $\{x'=0\}\times(-1,1)$,

$$d_b(x-a(1-\eta)e_n)\geqslant \frac{x_n-a(1-\eta)}{\tilde c}$$

for some $\tilde c>0$, and so

$$\phi^b(x-a(1-\eta)e_n)\geqslant \phi_0\left(\frac{x_n-a(1-\eta)}{\tilde c\varepsilon}\right)$$

on $\{x'=0\}\times(-1,1)$. Therefore, we have that

$$\begin{aligned}\{x_n\in(-1,1)\,:\,\bar u(0,x_n)\leqslant 1-\kappa\}&\subset\left\{x_n\in(-1,1)\,:\,\phi_0\left(\frac{x_n-a(1-\eta)}{\tilde c\varepsilon}\right)\leqslant 1-\kappa\right\}\\ &\subset\left\{\frac{x_n-a(1-\eta)}{\tilde c\varepsilon}\in(-\infty,l_\kappa)\right\}\subset\left\{\frac{x_n}{a}<\frac{\tilde c\varepsilon l_k}{a}+(1-\eta)\right\}\subset\left\{\frac{x_n}{a}<1-\frac{\eta}{2}\right\},\end{aligned}$$

where l_κ has been introduced in (1.10), and the last inclusion holds since ε/a is as small as desired. This estimate establishes (4.31), as desired.

Now, from (4.6) and (4.31), we obtain that

$$\{x_n\in(-1,1)\,:\,u(x',x_n)\leqslant 1-\kappa\}\subset\left\{x_n\leqslant a\left(1-\frac{\eta}{2}\right)\right\},\tag{4.32}$$

where $x'\in B'_{1/2}$ is arbitrary.

Now, to complete the proof of Proposition 4.2, let $x=(x',x_n)\in\{u\leqslant 1-\kappa\}$, with $|x'|<1/2$ and $|x_n|<2^{k_a}$. Then, using (4.3) with $j=0$, we obtain that

$$x_n\leqslant a<1.\tag{4.33}$$

Now, if $x_n\leqslant 0$, then (4.5) is obviously true, so we may assume that $x_n>0$. Thanks to this and (4.33), we are in position of using (4.32), which in turn implies (4.5), as desired. ☐

With this, we are now in the position of completing the proof of Theorem 4.1.

Proof of Theorem 4.1. If (4.4) holds true, the claim follows from Proposition 4.2. If instead the opposite inequality in (4.4) holds, we look at $\tilde u:=-u$, which satisfies

$$L\tilde u=-\varepsilon^{-s}f(-\tilde u)=:\varepsilon^{-s}\tilde f(\tilde u).$$

Since $\tilde f$ satisfies the same structural conditions as f in (H2) and (H3), and now $\tilde u$ satisfies (4.4), we can apply Proposition 4.2 to $\tilde u$ and obtain the desired result. ☐

Rescaling and iterating Theorem 4.1 we obtain the following result:

Corollary 4.5. *There exist constants $a_0>0$, $p_0>2$, $\sigma>0$ and $C>0$, depending only on α, m_0, and on universal constants, with σ satisfying $\alpha(1+\sigma)<s$, such the the following statement holds.*

Let $a\in(0,a_0)$ and $\varepsilon\in(0,a^{p_0})$. Let k_a be given by (4.1). *Assume that $u_a:\mathbb{R}^n\to(-1,1)$ is a solution of $Lu=\varepsilon^{-s}f(u)$ in $B'_{2^{k_a}}\times(-2^{k_a},2^{k_a})$ such that*

$$\{x_n\leqslant -a2^{j(1+\alpha)}\}\subset\{u_a\leqslant -1+\kappa\}\subset\{u_a\leqslant 1-\kappa\}\subset\{x_n\leqslant a2^{j(1+\alpha)}\}\quad\text{in } B'_{2^j}\times(-2^{k_a},2^{k_a})\tag{4.34}$$

for $0\leqslant j\leqslant k_a$.

Then, there exist two functions $g_a=g_a(x')$ and $g^a=g^a(x')$ belonging to $C^\sigma(B'_{2^{k_a-1}})$ and satisfying $g_a\leqslant g^a$ such that, for all $R\in[1,2^{k_a-1}]$, we have

$$\|g_a\|_{L^\infty(B_R)}+R^\sigma[g_a]_{C^\sigma(B_R)}\leqslant CR^{1+\alpha(1+\sigma)},\quad \|g^a\|_{L^\infty(B_R)}+R^\sigma[g^a]_{C^\sigma(B_R)}\leqslant CR^{1+\alpha(1+\sigma)},\tag{4.35}$$

$$\|g_a - g^a\|_{L^\infty(B_R)} \leqslant C R^{1+\alpha(1+\sigma)} a^{1+\sigma}, \tag{4.36}$$

and

$$\{x_n \leqslant a g_a(x')\} \subset \{u_a \leqslant -1+\kappa\} \subset \{u_a \leqslant 1-\kappa\} \subset \{x_n \leqslant a g^a(x')\} \quad \text{in } B'_{2^{k_a-1}} \times (-2^{k_a}, 2^{k_a}).$$

In particular, the two functions g_a and g^a converge locally uniformly as $a \to 0$ to some Hölder continuous function g satisfying the growth control $g(x') \leqslant C(1+|x'|)^{1+\alpha(1+\sigma)}$.

Proof. The proof of this result follows from iterating and rescaling the Harnack inequality of Theorem 4.1; see [30, 12] for similar arguments.

Step 1. We first prove the following claim which states that the transition region is trapped near the origin between two Hölder functions that are separated by a very small distance near the origin.

Throughout the proof we denote by $C_r := B'_r \times (-2^{k_a}, 2^{k_a})$.

Claim. *For some $(0, z_n) \in \{-1+\kappa \leqslant u_a \leqslant 1-\kappa\}$ we have*

$$\big\{x_n \leqslant z_n - aC(|x'|^\sigma + r)\big\} \subset \{u_a \leqslant -1+\kappa\} \subset \{u_a \leqslant 1-\kappa\} \subset \big\{x_n \leqslant z_n + aC(|x'|^\sigma + r)\big\} \quad \text{in } C_1, \tag{4.37}$$

for

$$r := 8\,(a_0)^{-\frac{1}{1-\sigma}}\, a^{\frac{1}{1-\sigma}-1},$$

where $a_0 > 0$ is the small constant in Theorem 4.1 and where $C > 0$ and $\sigma \in (0,1)$ depend only on α, m_0, and on universal constants.

Let us prove that for every integer $l \geqslant 0$, satisfying

$$a 2^{(1-\sigma)l} < a_0, \tag{4.38}$$

we have that

$$\{x_n \leqslant c_l - a2^{-\sigma l}\} \subset \{u_a \leqslant -1+\kappa\} \subset \{u_a \leqslant 1-\kappa\} \subset \{x_n \leqslant c_l + a2^{-\sigma l}\} \quad \text{in } C_{2^{-l}} \tag{4.39}$$

where $c_l \in \mathbb{R}$ satisfy

$$c_l - a2^{-\sigma l} \leqslant c_{l+1} - a2^{-\sigma(l+1)} \leqslant c_{l+1} + a2^{-\sigma(l+1)} \leqslant c_l + a2^{-\sigma l}. \tag{4.40}$$

The proof is by induction over the integer l. Indeed, it follows from (4.34) that (4.39) holds true for $l = 0$, with

$$c_0 = 0 \tag{4.41}$$

Assume now that (4.39) holds true for $0 \leqslant l \leqslant l_0$, and let us prove that (4.39) is also satisfied for $l = l_0 + 1$. For this, let

$$U(x) := u_a\big(2^{-l_0}x', 2^{-l_0}x_n + c_{l_0}\big).$$

We have

$$LU = \left(\frac{\varepsilon}{2^{-l_0}}\right)^{-s} f(U) \quad \text{in } C_1. \tag{4.42}$$

To abbreviate the notation we define

$$\mathcal{A} := \{U \leqslant -1+\kappa\} \quad \text{and} \quad \mathcal{B} := \{U \leqslant 1-\kappa\}.$$

We claim that

$$\{x_n \leqslant -a2^{(1-\sigma)l_0}2^{j(1+\alpha)}\} \subset \mathcal{A} \subset \mathcal{B} \subset \{x_n \leqslant a2^{(1-\sigma)l_0}2^{j(1+\alpha)}\} \quad \text{in } C_{2^j}, \tag{4.43}$$

for $j = 0, \ldots, k_a$. As a matter of fact, to prove (4.43), we first show that it holds for $j = 0$, then for $j = 1, \ldots, l_0$ and then we complete the argument by showing that (4.43) holds also for $j = l_0+1, \ldots, k_a$.

To this aim, we observe that, since (4.39) holds for $0 \leqslant l \leqslant l_0$, we have

$$\{x_n \leqslant 2^{l_0}(c_l - c_{l_0}) - a2^{l_0-\sigma l}\} \subset \mathcal{A} \subset \mathcal{B} \subset \{x_n \leqslant 2^{l_0}(c_l - c_{l_0}) + a2^{l_0-\sigma l}\} \quad \text{in } C_{2^{l_0-l}}, \tag{4.44}$$

for any $0 \leqslant l \leqslant l_0$. This, when $l = l_0$, gives (4.43) for $j = 0$.

Hence, we focus now on the proof of (4.43) when $j = 1, \dots, l_0$. For this, we can suppose that

$$l_0 \geqslant 1, \tag{4.45}$$

otherwise this case is void, and we will use (4.44) with $l = 0, \dots, l_0 - 1$. We remark that the inequalities in (4.40) imply that, for any $0 \leqslant l \leqslant l_0 - 1$,

$$c_l - a2^{-\sigma l} \leqslant c_{l_0} - a2^{-\sigma l_0} \leqslant c_{l_0} + a2^{-\sigma l_0} \leqslant c_l + a2^{-\sigma l}.$$

Therefore

$$\begin{aligned} & c_l \leqslant c_{l_0} - a2^{-\sigma l_0} + a2^{-\sigma l} \leqslant c_{l_0} + a2^{-\sigma l} \\ \text{and} \qquad & c_{l_0} \leqslant c_l + a2^{-\sigma l} - a2^{-\sigma l_0} \leqslant c_l + a2^{-\sigma l}. \end{aligned}$$

Accordingly, we have that, for $0 \leqslant l < l_0 - 1$,

$$|c_l - c_{l_0}| \leqslant a2^{-\sigma l}$$

and so

$$2^{l_0}|c_l - c_{l_0}| + a2^{l_0 - \sigma l} \leqslant 2a2^{l_0 - \sigma l} = a2^{(1-\sigma)l_0}2^{\sigma(l_0 - l)+1}.$$

From this and (4.44), using the notation $j := l_0 - l$, we see that, for any $j = 1, \dots, l_0$,

$$\{x_n \leqslant -a2^{(1-\sigma)l_0}2^{\sigma j+1}\} \subset \mathcal{A} \subset \mathcal{B} \subset \{x_n \leqslant a2^{(1-\sigma)l_0}2^{\sigma j+1}\} \quad \text{in } C_{2^j}, \tag{4.46}$$

We also observe that, for any $j = 1, \dots, l_0$, taking $\sigma \leqslant \alpha$, we have that

$$(\sigma j + 1) - (1+\alpha)j \leqslant (\alpha j + 1) - (1+\alpha)j = 1 - j \leqslant 0$$

and thus

$$2^{\sigma j+1} \leqslant 2^{(1+\alpha)j}.$$

So, we insert this into (4.46) and we complete the proof of (4.43) for $j = 1, \dots l_0$.

To complete the proof of (4.43), we have now to take into account the case $j = l_0 + 1, \dots, k_a$. For this, we recall assumption (4.34) (used here with the index i) and we obtain that

$$\{x_n \leqslant -2^{l_0}c_{l_0} - a2^{l_0 + i(1+\alpha)}\} \subset \mathcal{A} \subset \mathcal{B} \subset \{x_n \leqslant -2^{l_0}c_{l_0} + a2^{l_0 + i(1+\alpha)}\} \quad \text{in } C_{2^{l_0+i}} \tag{4.47}$$

for $i = 0, \dots, k_a$ (in our setting, we will then take $j = l_0 + i$, with $i = 1, \dots, k_a - l_0$). Now, we point out that

$$\begin{aligned} & c_{l_0} + a2^{-\sigma l_0} \leqslant c_0 + a2^{-\sigma \cdot 0} = a \\ \text{and} \qquad & -a = c_0 - a2^{-\sigma \cdot 0} \leqslant c_{l_0} - a2^{-\sigma l_0}, \end{aligned}$$

thanks to (4.40) and (4.41). Consequently, we have that $|c_{l_0}| \leqslant a$ and so

$$2^{l_0}|c_{l_0}| + a2^{l_0 + i(1+\alpha)} \leqslant a2^{l_0}(1 + 2^{i(1+\alpha)}) \leqslant a2^{l_0 + 1 + i(1+\alpha)}. \tag{4.48}$$

We also observe that, taking $\sigma \leqslant \alpha$ and using (4.45),

$$\begin{aligned} l_0 + 1 + i(1+\alpha) &= 1 + (\sigma - 1 - \alpha)l_0 + (1-\sigma)l_0 + (i + l_0)(1+\alpha) \\ &\leqslant 1 - l_0 + (1-\sigma)l_0 + (i+l_0)(1+\alpha) \leqslant (1-\sigma)l_0 + (i+l_0)(1+\alpha). \end{aligned}$$

This and (4.48) give that

$$2^{l_0}|c_{l_0}| + a2^{l_0 + i(1+\alpha)} \leqslant a2^{(1-\sigma)l_0 + (i+l_0)(1+\alpha)}.$$

Plugging this into (4.47) with $i = j - l_0$, we obtain (4.43) for $j = l_0 + 1, \dots, k_a$.

These considerations complete the proof of (4.43). Next, in view of (4.43), we may apply Theorem 4.1 with u replaced by U, with a replaced by

$$\bar{a} := a2^{(1-\sigma)l_0},$$

and with ε replaced by

$$\bar{\varepsilon} := \frac{\varepsilon}{2^{-l_0}}.$$

Note that, since we assume that $\varepsilon < a^{p_0}$, the condition $\bar{\varepsilon} < \bar{a}^{p_0}$ holds whenever

$$\frac{a^{p_0}}{2^{-l_0}} < \big(a2^{(1-\sigma)l_0}\big)^{p_0}.$$

This is equivalent to

$$1 < 2^{((1-\sigma)p_0-1)l_0}$$

which is always satisfied when $p_0 > 2$ and σ is taken small.

We recall however that, in order to apply Theorem 4.1, we must have that $\bar{a}$ is less than the small universal constant a_0. This is the reason why we need condition (4.38) to continue the iteration.

Thanks to these observations and (4.43), we can thus apply Theorem 4.1. In this way, we have proved that (4.39) holds whenever (4.38) holds, which immediately implies the statement of the claim.

Step 2. To complete the proof of Corollary 4.5, let us fix a nonnegative integer $l \leqslant k_a - 1$ and $z' \in B'_{2^l}$. Here, we define

$$U(x) := u\big(z' + 2^l x', 2^l x_n\big).$$

Then, rescaling (4.34) we find

$$\{x_n \leqslant -2^{-l}a2^{(l+i+1)(1+\alpha)}\} \subset \{U \leqslant -1+\kappa\} \subset \{U \leqslant 1-\kappa\} \subset \{x_n \leqslant 2^{-l}2^{(l+i+1)(1+\alpha)}\} \tag{4.49}$$

in $B'_{2^i} \times (-2^{k_a-l}, 2^{k_a-l})$, for $0 \leqslant i \leqslant k_a - l - 1$.

Let us denote

$$\bar{a} := 2^{-l}2^{(l+1)(1+\alpha)}a = 2^{(l+1)\alpha+1}a.$$

Observe that, recalling the definition of k_a in (4.1), we have

$$k_{\bar{a}} < k_a - l - 1.$$

Thus, (4.49) implies that

$$\{x_n \leqslant -\bar{a}2^{i(1+\alpha)}\} \subset \{U \leqslant -1+\kappa\} \subset \{U \leqslant 1-\kappa\} \subset \{x_n \leqslant \bar{a}2^{i(1+\alpha)}\} \tag{4.50}$$

in $B'_{2^i} \times (-2^{k_a-l}, 2^{k_a-l})$. We note also that U solves $LU = \bar{\varepsilon}^{-s} f(U)$ for

$$\bar{\varepsilon} := 2^l\varepsilon < 2^l a^{p_0} \leqslant \frac{2^l}{(2^{\alpha l})^{p_0}}\bar{a}^{p_0}$$

and hence the inequality $\bar{\varepsilon} < \bar{a}^{p_0}$ is satisfied provided that we choose p_0 large enough.

Thus, the claim in *Step 1* yields that, for a suitable $\bar{z}_n \in \mathbb{R}$,

$$\big\{\bar{x}_n \leqslant \bar{z}_n - \bar{a}C(|\bar{x}'|^\sigma + \bar{r})\big\} \subset \{U \leqslant -1+\kappa\} \subset \{U \leqslant 1-\kappa\} \subset \big\{\bar{x}_n \leqslant z_n + \bar{a}C(|\bar{x}'|^\sigma + \bar{r})\big\} \tag{4.51}$$

in $B'_1 \times (-2^{k_a-l}, 2^{k_a-l})$, for $\bar{r} = C(\bar{a})^{\frac{1}{1-\sigma}-1}$.

After rescaling, and setting $x = 2^l\bar{x}$, $z_n = 2^l\bar{z}_n$ and $r := 2^l\bar{r}$, we obtain

$$\begin{aligned}\left\{\frac{x_n}{2^l} \leqslant \frac{z_n}{2^l} - C2^{l\alpha}a\left(\frac{|x'|^\sigma}{2^{l\sigma}} + \frac{r}{2^l}\right)\right\} &\subset \{U \leqslant -1+\kappa\}\\ &\subset \{U \leqslant 1-\kappa\} \subset \left\{\frac{x_n}{2^l} \leqslant \frac{z_n}{2^l} + C2^{l\alpha}a\left(\frac{|x'|^\sigma}{2^{l\sigma}} + \frac{r}{2^l}\right)\right\}\end{aligned}$$

in $B'_{2^l}(z') \times (-2^{k_a}, 2^{k_a})$, for

$$r = 2^l\bar{r} = C2^l\bar{a}^{\left(\frac{1}{1-\sigma}-1\right)} = C(2^l)^{1+\alpha\left(\frac{1}{1-\sigma}-1\right)}a^{\left(\frac{1}{1-\sigma}-1\right)} \leqslant C(2^l)^{1+\alpha(1+\sigma)}a^\sigma. \tag{4.52}$$

Now, given $z' \in B'_{2^{k_a-1}}$, let us denote $R_{z'} := 2^l$, where $l := \min\{l' \,:\, 2^{l'} \geqslant |z'|\}$. In view of (4.52), we define also

$$r_{z'} := C\,(R_{z'})^{1+\alpha(1+\sigma)}\,a^\sigma$$

and the function $\Psi_{z'} : \mathbb{R}^{n-1} \to [0, +\infty]$, given by

$$\Psi_{z'}(x') := \begin{cases} CR_{z'}^{1+\alpha}\left(R_{z'}^{\alpha\sigma}\frac{|x'-z'|^\sigma}{R_{z'}^\sigma} + \frac{r_{z'}}{R_{z'}}\right) & \text{for } |x'| \leqslant R_{z'}, \\ +\infty & \text{for } |x'| > R_{z'}. \end{cases}$$

Hence, from (4.51), we have that

$$\{x_n \leqslant z_n - a\Psi_{z'}(x')\} \subset \{U \leqslant -1+\kappa\} \subset \{U \leqslant 1-\kappa\} \subset \left\{\frac{x_n}{2^l} \leqslant z_n + -a\Psi z'(x')\right)$$

in $B'_{2^{k_a-1}} \times (-2^{k_a}, 2^{k_a})$.

Furthermore, we notice that

$$\Psi_{z'}(z') \leqslant CR_{z'}^\alpha R_{z'}^{1+\alpha(1+\sigma)} a^\sigma$$

and

$$\|\Psi_{z'}\|_{L^\infty}(B_{R_{z'}}(z')) + R_{z'}^\sigma[\Psi_{z'}]_{C^\sigma}(B_{R_{z'}}(z')) = CR_{z'}^{1+\alpha(1+\sigma)}.$$

We then define

$$g^a(x') := \min_{z' \in \overline{B'_{2^{k_a-1}}}} \big(z_n(z') + \Psi_{z'}(x')\big) \quad \text{and} \quad g_a(x') := \max_{z' \in \overline{B'_{2^{k_a-1}}}} \big(z_n(z') - \Psi_{z'}(x')\big).$$

It is now straightforward to verify that these two functions satisfy the requirements in the statement of Corollary 4.5, as desired. □

We state a further consequence of Corollary 4.5 and Lemma 3.2 for its use in the next section.

Corollary 4.6. *With the same assumptions as in Corollary 4.5, the following statement holds true. Given $\theta \in (-1,1)$, we have that*

$$\{x_n \leqslant ag(x') - Ca^{1+\sigma}(1+|x|)^{1+\alpha(1+\sigma)} - C(1+|x|)^{\alpha\sigma}d\} \subset \{u_a \leqslant \theta\}$$

and

$$\{u_a \leqslant \theta\} \subset \{x_n \leqslant ag(x') + Ca^{1+\sigma}(1+|x|)^{1+\alpha(1+\sigma)} + C(1+|x|)^{\alpha\sigma}d\}$$

in $C_{2^{k_a-1}}$, for all $d > 0$ satisfying

$$\left(\frac{\varepsilon}{d}\right)^{\gamma_0} \leqslant 1 - |\theta|.$$

Proof. This is a direct consequence of Corollary 4.5 and the decay estimates of Lemma 3.2. □

5. Viscosity equation for the limit of vertical rescalings

In this section we will prove that the limiting graph g given by Corollary 4.5 satisfies the equation

$$\bar{L}g = 0 \quad \text{in } \mathbb{R}^{n-1} \tag{5.1}$$

where

$$\bar{L}h(x') := \int_{\mathbb{R}^{n-1}} \big(h(x') + \nabla h(x')\cdot(y'-x') - h(y')\big)\,\mathcal{K}(x'-y',0)\,dy', \qquad x' \in \mathbb{R}^{n-1}, \tag{5.2}$$

and

$$\mathcal{K}(y) := \frac{\mu\,(y/|y|)}{|y|^{n+s}}.$$

We introduce $\mathcal{K}$ both to simplify the notation and because the results of this part are also valid for more general kernels. The definition of $\bar{L}h(x')$ is valid for functions h which are C^2 in a neighborhood of x' and satisfying
$$\int_{\mathbb{R}^{n-1}} |h(x')|\,(1+|x'|)^{-n-s}\,dx' < +\infty.$$
We also point out that (5.1) is a linear and translation invariant equation.

The strategy that we have in mind is the following: once we have proved that g is an entire solution of (5.1), satisfying the growth control $g(x') \leqslant C(1+|x'|)^{1+\alpha(1+\sigma)}$ (as given by Corollary 4.5), we will deduce that g is affine. This will be an immediate consequence of the interior regularity estimates for the equation (5.1).

This set of ideas is indeed the content of the following result:

Proposition 5.1. *The limit function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ given by Corollary 4.5 satisfies* (5.1) *in the viscosity sense. As a consequence, g is affine.*

In all this section we assume that u_a is a solution of $Lu_a = \varepsilon^{-s} f(u)$ in $B_{2^{k_a}}$, where $\varepsilon \in (0, a^{p_0})$ with p_0 large enough. We denote by g the limiting graph as $a \to 0$ of the vertical rescalings of the level set, see Corollary 4.5. We recall that this graph satisfies the growth control
$$\big|g(x')\big| \leqslant C(1+|x'|)^{1+\alpha(1+\sigma)}. \tag{5.3}$$
Moreover, as a consequence of Corollary 4.6 we may assume that, for any given $\theta \in (-1,1)$,
$$\{x_n \leqslant ag(x') - C(a^{1+\sigma}+d)(1+|x|)^{1+\alpha(1+\sigma)}\} \subset \{u_a \leqslant \theta\} \tag{5.4}$$
and
$$\{u_a \leqslant \theta\} \subset \{x_n \leqslant ag(x') + C(a^{1+\sigma}+d)(1+|x|)^{1+\alpha(1+\sigma)}\} \tag{5.5}$$
for all $d > 0$ satisfying
$$\left(\frac{\varepsilon}{d}\right)^{\gamma_0} \leqslant 1 - |\theta|. \tag{5.6}$$
In all the section and in the rest of the paper we will fix constant $\alpha, \sigma > 0$ satisfying
$$\alpha(1+\sigma) < s \qquad \text{and} \qquad \alpha < \sigma$$
For concreteness we may take, here and in the rest of the paper,
$$\alpha = \frac{s}{4} \qquad \text{and} \qquad and \sigma = 1.$$

To prove that g is a viscosity solution of (5.1), we will argue by contradiction. Indeed, we will assume that g is touched by above by a convex paraboloid at x_0 and that the operator computed at a test function h that is built (from g) by replacing g with the paraboloid in a tiny neighborhood of x_0 gives the wrong sign. Using this contradictory assumption, we will be able to build a supersolution of $Lu = \varepsilon^{-s} f(u)$ touching u_a from above at some interior point near x_0. This will give the desired contradiction.

In all the section, we assume that Q is a fixed convex quadratic polynomial and, up to a rigid motion, we can take the touching point x_0 to be the origin. We also let d_a be the anisotropic signed distance function to $\{x_n \geqslant aQ(x')\}$, i.e. we use the setting in (2.7), with $K := K_a := \{x_n \geqslant aQ(x')\}$. More explicitly
$$d_a(x) \;:=\; \inf\big\{\ell(x) \;:\; \ell \text{ affine},\, h_L(\nabla\ell) = 1, \text{ and } \ell \geqslant 0 \text{ in } K_a\big\}. \tag{5.7}$$
Then, we will consider the following functions:
$$\tilde{u}_a(x) := \phi_0\left(\frac{d_a(x)}{\varepsilon}\right)\chi_{\mathcal{Q}_\delta} + u_a(x', ax_n)\chi_{\mathbb{R}^n\setminus\mathcal{Q}_\delta} \tag{5.8}$$
and
$$v_a(x) := \phi_0\left(\frac{d_a(x)}{\varepsilon}\right)\chi_{\mathcal{Q}_\delta} + \operatorname{sign}(x_n - ag(x'))\chi_{\mathbb{R}^n\setminus\mathcal{Q}_\delta}, \tag{5.9}$$

where $\delta > 0$,

$$\mathcal{Q}_\delta := B'_\delta \times (-\delta, \delta), \tag{5.10}$$

and ϕ_0 is the 1D profile in (H3). In a sense, u_a and v_a have "very flat level sets" and we will compute the action of the operator L on such functions.

By explicit computations and error estimates, we will prove that not only $L\tilde{u}_a - \varepsilon^{-s} f(\tilde{u}_a) \to 0$ and $Lv_a - \varepsilon^{-s} f(v_a) \to 0$ as $a \to 0$ in a neighborhood of 0, but we also provide the behavior of the next order in an expansion in the variable a. Namely, for a small enough, we will show that

$$\frac{1}{a}\big(L\tilde{u}_a - \varepsilon^{-s} f(\tilde{u}_a)\big) \approx \frac{1}{a}\big(Lv_a - \varepsilon^{-s} f(v_a)\big) \approx \bar{-}Lh(0)$$

in neighborhood of 0 in $\mathbb{R}^n$ (we recall that h is the test function built from the touching paraboloid before (5.7)).

To prove this, we will use our previous idea of "subtracting the tangent 1D profile"

$$\tilde{\phi}(x) = \phi_0(\tilde{d}/\varepsilon), \tag{5.11}$$

where $\tilde{d}$ will be the signed anisotropic distance function to some appropriate tangent plane to the zero level set of u_a.

More precisely, in order to compute $Lv_a - \varepsilon^{-s} f(v_a)$ at a point $z \in B_{\delta/4}$, we introduce the "tangent profile" at z defined as (5.11) with

$$\tilde{d}(x) := \frac{\omega}{h_L(\omega)} \cdot (x - z) + t_0, \quad \text{where } t_0 = d_a(z) \tag{5.12}$$

and $\omega \in S^{n-1}$ is the unit normal vector to $\{d_a = t_0\}$ pointing towards $\{d_a > t_0\}$.

Using the layer cake decomposition in Lemma 2.1, we will compute the difference $Lv_a - \varepsilon^{-s} f(v_a)$ as the integral

$$\begin{aligned} Lv_a(z) - \varepsilon^{-s} f(v_a(z)) &= Lv_a(z) - L\tilde{\phi}(z) \\ &= \int_{-1}^{1} d\theta \int_{\mathbb{R}^n} \big(\chi_{S_\theta}(y) - \chi_{T_\theta}(y)\big)\mathcal{K}(z - y)\, dy \end{aligned} \tag{5.13}$$

where

$$S_\theta := \big\{v_a \leqslant \theta \leqslant \tilde{\phi}\big\} \quad \text{and} \quad T_\theta := \big\{\tilde{\phi} \leqslant \theta \leqslant v_a\big\}. \tag{5.14}$$

However, in this section we will obtain more information by introducing the vertical rescaling (or change of variables)

$$(y', y_n) = (\bar{y}', a\bar{y}_n)$$

which allows us to compute

$$\frac{1}{a}\big(Lv_a(z) - f(v_a(z))\big) = \int_{-1}^{1} d\theta \int_{\mathbb{R}^n} \big(\chi_{\bar{S}_\theta}(\bar{y}) - \chi_{\bar{T}_\theta}(\bar{y})\big)\mathcal{K}(z' - \bar{y}', z_n - a\bar{y}_n)\, d\bar{y}$$

where

$$\bar{S}_\theta := \big\{(\bar{x}', \bar{x}_n) \ : \ (\bar{x}', a\bar{x}_n) \in S_\theta\big\} \quad \text{and} \quad \bar{T}_\theta := \big\{(\bar{x}', \bar{x}_n) \ : \ (\bar{x}', a\bar{x}_n) \in T_\theta\big\}. \tag{5.15}$$

We will see that for all the level sets outside a set of "small" measure $2a^2$, namely for

$$\theta \in \left(-1 + a^2, 1 - a^2\right),$$

we have

$$\begin{aligned} \bar{S}_\theta \ &= \ \big\{\, \bar{y} = (\bar{y}', \bar{y}_n) \ : \ \ h_\theta(\bar{y}') \leqslant \bar{y}_n \leqslant h_\theta(\bar{z}') + \nabla h_\theta(\bar{z}') \cdot (\bar{y}' - \bar{z}') \,\big\} \\ \text{and} \quad \bar{T}_\theta \ &= \ \big\{\, \bar{y} = (\bar{y}', \bar{y}_n) \ : \ \ h_\theta(\bar{z}') + \nabla h_\theta(\bar{z}') \cdot (\bar{y}' - \bar{z}') \leqslant \bar{y}_n \leqslant h_\theta(\bar{y}') \,\big\}, \end{aligned} \tag{5.16}$$

where, given $\beta \in (0, 1)$, we have, for some $\eta > 0$,

$$\|h_\theta - h\|_{C^{1,\beta}(B'_\delta)} \leqslant Ca^\eta \quad \text{and} \quad h_\theta = h \quad \text{in } \mathbb{R}^n \setminus B'_\delta. \tag{5.17}$$

This will imply that when $|z'|$, $|z_n|$ and a are all converging to 0, we have

$$
\begin{aligned}
&\frac{1}{a}\big(L\tilde{u}_a(z) - \varepsilon^{-s} f(\tilde{u}_a(z))\big)\\
\overset{1}{\approx}\; &\frac{1}{a}\big(Lv_a(z) - \varepsilon^{-s} f(v_a(z))\big)\\
\overset{2}{=}\; &\int_{-1}^{1} d\theta \int_{\mathbb{R}^n} \big(\chi_{\bar{S}_\theta}(\bar{y}) - \chi_{\bar{T}_\theta}(\bar{y})\big)\mathcal{K}(z'-\bar{y}', z_n - a\bar{y}_n)\, d\bar{y}\\
\overset{3}{\approx}\; &\int_{-1+a^2}^{1-a^2} d\theta \int_{\mathbb{R}^n} \big(\chi_{\bar{S}_\theta}(\bar{y}) - \chi_{\bar{T}_\theta}(\bar{y})\big)\mathcal{K}(z'-\bar{y}', z_n - a\bar{y}_n)\, d\bar{y}\\
\overset{4}{\approx}\; &\int_{-1+a^2}^{1-a^2} d\theta \int_{\mathbb{R}^n} \big(\chi_{\bar{S}_\theta}(\bar{y}) - \chi_{\bar{T}_\theta}(\bar{y})\big)\mathcal{K}(z'-\bar{y}', 0)\, d\bar{y}\\
\overset{5}{=}\; &\int_{-1+a^2}^{1-a^2} -\bar{L}h_\theta(z')\, d\theta\\
\overset{6}{\approx}\; &-\bar{L}h(0).
\end{aligned}
\tag{5.18}
$$

In the next six lemmas, corresponding to the numbers appearing in (5.18), we prove the claimed equalities and we control the errors in the previous chain of approximations.

Lemma 5.2 (Approximation 1)**.** *We have*

$$
\lim_{a\to 0}\sup_{z\in B_{\delta/4}} \left|\frac{1}{a}\big(L\tilde{u}_a(z) - f(\tilde{u}_a(z))\big) - \frac{1}{a}\big(Lv_a(z) - f(v_a(z))\big)\right| \;=\; 0.
$$

Proof. We observe that $\tilde{u}_a = v_a$ in $\mathcal{Q}_\delta$. Then, using the layer cake formula in (2.3) of Lemma 2.1,

$$
|L\tilde{u}_a(z) - Lv_a(z)| \leqslant Ca^2 + \int_{-1+a^2}^{1-a^2} d\theta \int_{\mathbb{R}^n\setminus\mathcal{Q}_\delta} \chi_{\{\tilde{u}_a\leqslant\theta\leqslant v_a\}\cup\{v_a\leqslant\theta\leqslant\tilde{u}_a\}}(y)\; |y-z|^{-n-s}\, dy. \tag{5.19}
$$

We also remark that, by the definition of v_a, we have that, for all $\theta\in(-1,1)$,

$$
\{v_a \geqslant \theta\} = \{x_n \geqslant ag(x')\} \quad \text{in } \mathbb{R}^n\setminus\mathcal{Q}_\delta. \tag{5.20}
$$

Hence, if $\theta\in(-1+a^2, 1-a^2)$, we use (5.4), (5.5) and (5.6) and we find that

$$
\begin{aligned}
&\{\tilde{u}_a \leqslant \theta \leqslant v_a\}\cup\{v_a\leqslant\theta\leqslant\tilde{u}_a\}\\
&\qquad \subset \big\{ag(x') - C(a^{1+\sigma}+d)(1+|x|)^{1+\alpha(1+\sigma)} \leqslant x_n \leqslant ag(x') + C(a^{1+\sigma}+d)(1+|x|)^{1+\alpha(1+\sigma)}\big\}
\end{aligned}
\tag{5.21}
$$

in $B_{2^{k_a-1}}\setminus\mathcal{Q}_\delta$, whenever

$$
(\varepsilon/d)^{\gamma_0} \leqslant a^2. \tag{5.22}
$$

For p_0 chosen large enough (recall that we assume $\varepsilon < a^{p_0}$), we may take

$$
d := a^{1+\sigma} \tag{5.23}
$$

and satisfy (5.22). Hence, with the setting in (5.23), we get from (5.21) that

$$
\begin{aligned}
&\{\tilde{u}_a \leqslant \theta \leqslant v_a\}\cup\{v_a\leqslant\theta\leqslant\tilde{u}_a\}\\
&\qquad \subset \big\{|x_n - ag(x')| \leqslant Ca^{1+\sigma}(1+|x|)^{1+\alpha(1+\sigma)}\big\} \quad \text{in } B_{2^{k_a-1}}.
\end{aligned}
$$

It then follows that, for all $\theta \in (-1+a^2, 1-a^2)$,

$$\begin{aligned} &\int_{\mathbb{R}^n\setminus\mathcal{Q}_\delta} \chi_{\{\tilde u_a\leqslant\theta\leqslant v_a\}\cup\{v_a\leqslant\theta\leqslant\tilde u_a\}}(y)\;|y-z|^{-n-s}\,dy \\ &\leqslant \int_{\mathbb{R}^n\setminus B_{2^{k_a-1}}} |y-z|^{-n-s}\,dy + C_\delta \int_1^{2^{k_a-1}} a^{1+\sigma}\,\frac{r^{1+\alpha(1+\sigma)+n-2}}{r^{n+s}}dr \\ &\leqslant C_\delta(a^{s/\alpha}+a^{1+\sigma}), \end{aligned} \tag{5.24}$$

where we have used that σ is chosen small so that $\alpha(1+\sigma)<s$ (recall the setting of Corollary 4.5). The desired result then follows immediately from (5.19) and (5.24). ☐

Lemma 5.3 (Equality 2)**.** *Let $z\in B_{\delta/4}$. Then*

$$\frac{1}{a}\big(Lv_a(z)-f(v_a(z))\big)=\int_{-1}^{1}d\theta\int_{\mathbb{R}^n}\big(\chi_{\bar S_\theta}(\bar y)-\chi_{\bar T_\theta}(\bar y)\big)\mathcal{K}(z'-\bar y', z_n-a\bar y_n)\,d\bar y$$

where $\bar S_\theta$ and $\bar T_\theta$ are defined in (5.15).

Proof. From the layer cake formula in (2.3) of Lemma 2.1 and the idea of "subtracting the tangent 1D profile" at z (exactly as in the proof of Lemma 2.5) we obtain that (5.13) and (5.14) hold, where $\tilde\phi$ is defined by (5.11) and (5.12). Then, the result simply follows by performing the change of variables $(y',y_n)=(\bar y', a\bar y_n)$. ☐

Lemma 5.4 (Approximation 3)**.** *Let $z\in B_{\delta/4}$. If a is small enough, then for all $\theta\in(-1,1)$ with $|\theta|\geqslant 1-a^2$ we have*

$$\left|\int_{\mathbb{R}^n}\big(\chi_{\bar S_\theta}(\bar y)-\chi_{\bar T_\theta}(\bar y)\big)\mathcal{K}(z'-\bar y', z_n-a\bar y_n)\,d\bar y\right|\leqslant\frac{C}{a},$$

for some $C>0$.

Proof. To prove this result, it is convenient to look at the statement with the integrals written with respect to the original variables $(y',y_n)=(\bar y', a\bar y_n)$. In this setting, we have to show that

$$I_1:=\left|\int_{\mathbb{R}^n}\big(\chi_{S_\theta}(y)-\chi_{T_\theta}(y)\big)\mathcal{K}(z-y)\,dy\right|\leqslant C. \tag{5.25}$$

To prove this, we actually do not need the condition $|\theta|\geqslant 1-a^2$, although the result will be used only for these values of θ.

Note that in $\mathcal{Q}_\delta$ we have that $v_a=\phi_0(d/\varepsilon)$ and $\tilde\phi=\phi_0(\tilde d/\varepsilon)$. Recalling the definition of T_θ in (5.14) and the facts that, by construction, the level sets of d are convex, and the level sets of $\tilde d$ are tangent hyperplanes to the level sets of d, we obtain that

$$T_\theta\cap\mathcal{Q}_\delta=\varnothing \tag{5.26}$$

for all θ.

Now, to prove (5.25), we distinguish the two cases $S_\theta\cap\mathcal{Q}_{\delta/2}=\varnothing$ and $S_\theta\cap\mathcal{Q}_{\delta/2}\neq\varnothing$.

In the first case in which

$$S_\theta\cap\mathcal{Q}_{\delta/2}=\varnothing, \tag{5.27}$$

we claim that

$$|z-y|\geqslant\frac{\delta}{4}\text{ for all }y\in S_\theta\cup T_\theta. \tag{5.28}$$

To check this, let $y\in S_\theta\cup T_\theta$. Then, by (5.26) and (5.27), we have that $y\notin\mathcal{Q}_{\delta/2}$. This, together with the fact that $z\in\mathcal{Q}_{\delta/4}$, proves (5.28).

Therefore, in light of (5.28), we have that

$$I_1 \leqslant C_\delta \int_{\mathbb{R}^n} \frac{dy}{(\delta+|y|)^{n+s}} \leqslant C.$$

This proves (5.25) in this case.

In the second case in which

$$S_\theta \cap \mathcal{Q}_{\delta/2} \neq \varnothing,$$

we use the fact that $\{v_a = \theta\} \cap \mathcal{Q}_\delta$ is the level set of the anisotropic distance function to the parabola $x_n = Q_a(x') := aQ(x')$. Hence, exactly as in Lemma 2.3, we have that $\{v_a = \theta\} \cap \mathcal{Q}_\delta$ is a convex $C^{1,1}$ graph with $C^{1,1}$ norm bounded by Ca (and thus by C). Therefore, recalling also (5.26),

$$\left|\int_{B_{\delta/4}(z)} \big(\chi_{S_\theta}(y) - \chi_{T_\theta}(y)\big)\mathcal{K}(z-y)\,dy\right| \leqslant \int_{B_{\delta/4}(z)\cap S_\theta} \mathcal{K}(z-y)\,dy \leqslant C.$$

Consequently, we conclude that

$$I_1 \leqslant C + \int_{\mathbb{R}^n\setminus B_{\delta/4}(z)} \frac{dy}{|z-y|^{n+s}} \leqslant C,$$

up to renaming $C>0$, and so (5.25) follows also in this second case, as desired. $\square$

Lemma 5.5 (Approximation 4)**.** *For all $\theta \in (-1,1)$ with $|\theta| \leqslant 1-a^2$ we have*

$$\left|\int_{\mathbb{R}^n} \big(\chi_{\bar S_\theta}(\bar y) - \chi_{\bar T_\theta}(\bar y)\big)\mathcal{K}(z'-\bar y', z_n - a\bar y_n)\,d\bar y - \int_{\mathbb{R}^n} \big(\chi_{\bar S_\theta}(\bar y) - \chi_{\bar T_\theta}(\bar y)\big)\mathcal{K}(z'-\bar y', 0)\,d\bar y\right| \to 0$$

as $(|a|+|z_n|) \to 0$ whenever $|z'| \leqslant \delta/4$.

To prove Lemma 5.5, we need the following pivotal result:

Lemma 5.6. *For all $\theta \in (-1+a^2, 1-a^2)$ there exists a function $h_\theta : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$h_\theta = h = g \text{ outside } B'_\delta, \tag{5.29}$$

$h_\theta \in C^{1,1}(B'_\delta)$ and (5.16) holds true. Namely,

$$\begin{aligned} \bar S_\theta &= \big\{\, \bar y = (\bar y', \bar y_n) \ :\ h_\theta(\bar y') \leqslant \bar y_n \leqslant h_\theta(\bar z') + \nabla h_\theta(\bar z')\cdot(\bar y' - \bar z') \,\big\} \\ \text{and}\quad \bar T_\theta &= \big\{\, \bar y = (\bar y', \bar y_n) \ :\ h_\theta(\bar z') + \nabla h_\theta(\bar z')\cdot(\bar y' - \bar z') \leqslant \bar y_n \leqslant h_\theta(\bar y') \,\big\}. \end{aligned} \tag{5.30}$$

Moreover,

$$\|h_\theta - h\|_{L^\infty(B'_\delta)} \leqslant Ca \quad \text{and} \quad \|h_\theta - h\|_{C^{1,1}(B'_\delta)} \leqslant C \tag{5.31}$$

for some $C>0$. In particular, (5.17) holds true for $\eta = \frac{1-\beta}{2}$.

Proof. If θ is as in the statement of Lemma 5.6, we take $t_\theta := \varepsilon\phi_0^{-1}(\theta)$. Then, using (3.8), we have that

$$a^2 \leqslant 1-|\theta| = 1 - \left|\phi_0\left(\frac{t_\theta}{\varepsilon}\right)\right| \leqslant \frac{C}{1+\left(\frac{|t_\theta|}{\varepsilon}\right)^{\gamma_0}}.$$

Hence (assuming $\varepsilon < a^{p_0}$ and p_0 conveniently large), we find that

$$|t_\theta| \leqslant \frac{C\varepsilon}{a^{2/\gamma_0}} \leqslant a^2. \tag{5.32}$$

Then, by the definition of v_a, we have

$$\{v_a = \theta\} = \{d_a = t_\theta\} \quad \text{in } \mathcal{Q}_\delta.$$

Now, since $\{d_a = 0\} = \{x_n = aQ(x')\}$, by exactly the same argument of Lemma 2.3, we obtain that

$$\{d_a = t_\theta\} = \{x_n = G_\theta(x')\}$$

for some G_θ satisfying

$$|D^2 G_\theta| \leqslant Ca \quad \text{in } B_1'.$$

Notice also that, by (5.32), the graph of G_θ in B_δ' lies in a Ca^2-neighborhood of the graph of aQ (that is ah, recall the construction of the touching test function before (5.7)).

We now recall that the tangent profile at z, that we denoted by $\tilde\phi$, is built in such a way that

$$\{\tilde\phi = \theta\} = \{\tilde d = t_\theta\}$$

is the tangent plane to $\{x_n = ag(x')\}$ at the point $z = (z', z_n)$.

These observations and (5.20) imply that

$$\begin{aligned} S_\theta \;&=\; \big\{\, y = (y', y_n) \;:\; \tilde h_\theta(y') \leqslant y_n \leqslant \tilde h_\theta(\bar z') + \nabla \tilde h_\theta(z') \cdot (y' - z') \,\big\} \\ \text{and} \quad T_\theta \;&=\; \big\{\, y = (y', y_n) \;:\; \tilde h_\theta(z') + \nabla \tilde h_\theta(z') \cdot (y' - z') \leqslant y_n \leqslant \tilde h_\theta(y') \,\big\}, \end{aligned}$$

for a suitable function $\tilde h_\theta$, with

$$\sup_{y' \in B_\delta'} |D^2 \tilde h_\theta(y')| \leqslant Ca \tag{5.33}$$

and $\tilde h_\theta = ag$ outside B_δ'. In addition,

$$\text{the graph of } \tilde h_\theta \text{ in } B_\delta' \text{ lies in a } Ca^2\text{-neighborhood of the graph of } ah. \tag{5.34}$$

Now, the desired result in (5.30) follows from the change of variables $(y', y_n) = (\bar y', a\bar y_n)$, by taking

$$h_\theta := \tilde h_\theta / a.$$

To check (5.31), we observe that the estimate in $C^{1,1}(B_\delta')$ follows from the bound in (5.33) and the fact that h is a given paraboloid in B_δ'. Also, the uniform bound in (5.31) is a consequence of (5.34).

These observations establish (5.31). We also remark that (5.17) follows from (5.31) by interpolation. ☐

Proof of Lemma 5.5. We claim that the map

$$\mathbb{R}^n \ni \bar y = (\bar y', \bar y_n) \mapsto \mathcal{J}(\bar y) := \frac{\chi_{\bar S_\theta}(\bar y) + \chi_{\bar T_\theta}(\bar y)}{|z' - \bar y'|^{n+s}} \;\text{ belongs to }\; L^1(\mathbb{R}^n). \tag{5.35}$$

For this, we use Lemma 5.6 to see that

$$\begin{aligned} &\int_{B_{\delta/4}'(z) \times (-\infty,\infty)} \mathcal{J}(\bar y)\, d\bar y \\ &\qquad \leqslant C \int_{\mathbb{R}} d\bar y_n \int_{S^{n-2}} d\omega \int_0^\delta dr\; \frac{r^{n-2}\big(\chi_{\bar S_\theta}(z' + r\omega, \bar y_n) + \chi_{\bar T_\theta}(z' + r\omega, \bar y_n)\big)}{r^{n+s}} \\ &\qquad \leqslant C \int_0^\delta \frac{r^{n-2}\, r^2}{r^{n+s}}\, dr \leqslant C\delta^{1-s} \leqslant C, \end{aligned} \tag{5.36}$$

up to renaming $C > 0$. On the other hand, recalling (5.3) and (5.29), we deduce from (5.30) that $\bar{S}_\theta$ and $\bar{T}_\theta$ are controlled at infinity by a function with growth $C|\bar{y}'|^{1+\alpha}$. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \left(B'_{\delta/4}(z)\times(-\infty,\infty)\right)} \mathcal{J}(\bar{y})\,d\bar{y} \\
&\leqslant C \int_{\mathbb{R}} d\bar{y}_n \int_{S^{n-2}} d\omega \int_{\delta/4}^{+\infty} dr\, \frac{r^{n-2}\left(\chi_{\bar{S}_\theta}(z'+r\omega,\bar{y}_n)+\chi_{\bar{T}_\theta}(z'+r\omega,\bar{y}_n)\right)}{r^{n+s}} \\
&\leqslant C \int_{\delta/4}^{+\infty} \frac{r^{n-2}\,r^{1+\alpha}}{r^{n+s}}\,dr \leqslant C\delta^{\alpha-s} \leqslant C.
\end{aligned}$$

This and (5.36) imply (5.35), as desired.

Then, using (5.35) and the fact that $K(z'-\bar{y}', z_n - a\bar{y}_n) \to K(z'-\bar{y}',0)$ almost everywhere in $\mathbb{R}^n$ as $(|a|+|z_n|) \to 0$, we see that the result in Lemma 5.5 follows by the dominated convergence theorem. ☐

Lemma 5.7 (Equality 5)**.** *For all $\theta \in (-1+a^2, 1-a^2)$ we have*

$$\int_{\mathbb{R}^n} \left(\chi_{\bar{S}_\theta}(\bar{y}) - \chi_{\bar{T}_\theta}(\bar{y})\right)\mathcal{K}(z'-\bar{y}',0)\,d\bar{y} = -\bar{L}h_\theta(z')$$

where $h_\theta \in C^{1,1}(B'_\delta)$ is given in Lemma 5.6.

Proof. From (5.30), we see that

$$\int_{\mathbb{R}^n} \left(\chi_{\bar{S}_\theta}(\bar{y}) - \chi_{\bar{T}_\theta}(\bar{y})\right)\mathcal{K}(z'-\bar{y}',0)\,d\bar{y} = \int_{\mathbb{R}^{n-1}} \left(h_\theta(\bar{y}') - \nabla h_\theta(z')(\bar{y}'-z') - h_\theta(z')\right)\mathcal{K}(z'-\bar{y}',0)\,d\bar{y}'.$$

This and (5.2) give the desired result. ☐

Lemma 5.8 (Approximation 6)**.** *For all $\theta \in (-1+a^2, 1-a^2)$ we have*

$$\left|\bar{L}h_\theta(z') - \bar{L}h(0)\right| \to 0$$

as $(|a|+|z'|) \to 0$.

Proof. It is standard using that (5.17) holds, as given by Lemma 5.6. ☐

Let us give now an elementary result that will be useful in the proof of Proposition 5.1.

Lemma 5.9. *Given $r > 0$, there exists $\delta > 0$, depending only on n, s, ellipticity constants and r, such that the following holds.*

Assume that $Lw \geqslant a > 0$ in $B_r \cap \{w \leqslant 0\}$ and $w \geqslant -\delta a$ in all of $\mathbb{R}^n$.

Then, $w > 0$ in $\overline{B_{r/2}}$.

Proof. The proof is standard, we give the details for the convenience of the reader. We consider the function $\tilde{w} := w + \delta a(1-\eta(x/r))$, where $\eta \in C^2_0(B_1)$ is a smooth radial cutoff with $\eta = 1$ in $B_{1/2}$. If, by contradiction, $w \leqslant 0$ at some point in $B_{r/2}$, then $\tilde{w}$ attains an absolute minimum at some point x_0 in B_r. Thus,

$$0 \geqslant L\tilde{w}(x_0) \geqslant Lw - C\delta a r^{-s} \geqslant a - C\delta a r^{-s} \geqslant a/2 > 0,$$

which gives a contradiction if δ is taken small enough. ☐

With this preliminary work, we can finally complete the proof of Proposition 5.1, by arguing as follows.

Proof of Proposition 5.1. Up to a translation, we can test the definition of viscosity solution for a smooth function touching g by above at the point $x_0 = 0$ (the argument to take care of the touching by below is similar).

Let $U' \subset \mathbb{R}^{n-1}$ be a neighborhood of the origin and $\psi \in C^2(U')$. Assume that ψ touches by above g in U' at 0. Assume by contradiction that $\tilde{\psi} := \psi\chi_{U'} + g\chi_{\mathbb{R}^n\setminus U'}$ satisfies $\bar{L}\tilde{\psi}(0) > 0$.

Then (see, for instance, Section 3 in [15]), we know that there exist $\delta > 0$ small and two concave polynomials, denoted by Q and $\tilde{Q}$, satisfying

$$Q(0) = \tilde{Q}(0) = g(0) \quad \text{and} \quad Q > \tilde{Q} \geqslant g \quad \text{in } B'_\delta \setminus \{0\} \tag{5.37}$$

and such that, if we define $Q^t := Q + t$ and $h := Q^t\chi_{B'_\delta} + g\chi_{\mathbb{R}^n\setminus B'_\delta}$, it holds that

$$\bar{L}h(0) > 0,$$

for all $t \in (-\delta^3, \delta^3)$.

Let us now consider the function $\tilde{u}_{a,t}$ defined as in (5.8), with d_a replaced by the distance from aQ^t, namely,

$$\tilde{u}_{a,t}(x) := \phi_0\left(\frac{d_a(x)}{\varepsilon}\right)\chi_{\mathcal{Q}_\delta} + u_a(x)\chi_{\mathbb{R}^n\setminus\mathcal{Q}_\delta} \tag{5.38}$$

where now d_a is the anisotropic signed distance function to $\{x_n \geqslant aQ^t(x')\}$ and $\mathcal{Q}_\delta$ was defined in (5.10).

By (5.18) (which has been proved in Lemmas 5.2, 5.3, 5.4, 5.5, 5.7 and 5.8), we obtain that

$$L\tilde{u}_{a,t} - \varepsilon^{-s}f(\tilde{u}_{a,t}) \leqslant -ca \quad \text{in } B_r, \tag{5.39}$$

for some $r > 0$ and $c > 0$, whenever a is small enough and $t \in [-\delta^3, \delta^3]$. By possibly reducing $r > 0$, we will suppose that

$$r \in (0, \delta). \tag{5.40}$$

We note that, in this setting, r and c depend on $\bar{L}h(0)$.

Next we show that, for $t = \delta^3$ and a small enough, we have

$$u_a - \tilde{u}_{a,t} > 0 \quad \text{in } \overline{B_{r/2}}. \tag{5.41}$$

To prove this, we recall that, by Corollary 4.6 (used here with $d := a^2$), we have

$$\{x_n \leqslant ag(x') - Ca^{1+\sigma}\} \subset \{u_a \leqslant \theta\} \subset \{x_n \leqslant ag(x') + Ca^{1+\sigma}\} \tag{5.42}$$

in $B'_1 \times (-1, 1)$, provided that $(\varepsilon/a^2)^{\gamma_0} \leqslant 1 - |\theta|$. On the other hand, by definition $\tilde{u}_{a,t} = \phi_0(d_a/\varepsilon)$ in $\mathcal{Q}_\delta$. Therefore,

$$\{x_n \leqslant aQ^t(x') - Ca^2\} \subset \{\tilde{u}_{a,t} \leqslant \theta\} \subset \{x_n \leqslant aQ^t(x') + Ca^2\} \tag{5.43}$$

in $\mathcal{Q}_\delta$, also provided that $(\varepsilon/a^2)^{\gamma_0} \leqslant 1 - |\theta|$ (with γ_0 given by (3.8)).

We remark that, roughly speaking, (5.42) says that the "transition level sets" of u_a lie essentially on the surface $\{x_n = ag(x')\}$, while (5.43) says that the "transition level sets" of $\tilde{u}_{a,t}$ lie essentially on the surface $\{x_n = aQ^t(x')\}$, up to small errors of size $a^{1+\sigma}$.

Then, since $Q \geqslant g$ in B'_δ by (5.37), for $t = \delta^3$ (or any other fixed positive number), if we assume that $\varepsilon \leqslant a^{p_0}$ with p_0 large enough, we can use (5.42) with $\theta := 1 - a^2$ and (5.43) with $\theta := -1 + a^2$, take a small enough and conclude that

$$\{u_a \leqslant 1 - a^2\} \subset \{\tilde{u}_{a,t} \leqslant -1 + a^2\} \quad \text{in } \mathcal{Q}_\delta. \tag{5.44}$$

In particular, by (5.40), we obtain that

$$\{u_a \leqslant 1 - \kappa\} \subset \{\tilde{u}_{a,t} \leqslant -1 + \kappa\} \quad \text{in } B_r. \tag{5.45}$$

Now we observe that

$$u_a - \tilde{u}_{a,t} > -a^2 \quad \text{in all of } \mathbb{R}^n. \tag{5.46}$$

Indeed, if $x \in \mathcal{Q}_\delta$, we distinguish two cases: either $u_a(x) > 1 - a^2$ or $u_a(x) \leqslant 1 - a^2$. In the first case, we have that

$$u_a(x) - \tilde{u}_{a,t}(x) > (1 - a^2) - 1 = -a^2.$$

In the second case, we can use (5.44) and obtain that $\tilde{u}_{a,t}(x) \leqslant -1 + a^2$ and, consequently

$$u_a(x) - \tilde{u}_{a,t}(x) > -1 - (-1 + a^2) = -a^2.$$

These observations prove (5.46) when $x \in \mathcal{Q}_\delta$. If instead $x \in \mathbb{R}^n \setminus \mathcal{Q}_\delta$, we recall (5.38) and we have that $\tilde{u}_{a,t}(x) = u_a(x)$, and this implies (5.46) also in this case.

Now, we observe that

$$f(u_a) \geqslant f(\tilde{u}_{a,t}) \text{ in } B_r \cap \{u_a - \tilde{u}_{a,t} \leqslant 0\}. \tag{5.47}$$

To check this we take $x \in B_r \cap \{u_a - \tilde{u}_{a,t} \leqslant 0\}$ and we distinguish two cases, either $u_a(x) \leqslant 1 - \kappa$ or $u_a(x) > 1 - \kappa$. In the first case, we exploit (5.45) and we obtain that $\tilde{u}_{a,t}(x) \leqslant -1 + \kappa$ and thus

$$u_a(x) \leqslant \tilde{u}_{a,t}(x) \leqslant -1 + \kappa.$$

This and the monotonicity of f in (H2) imply (5.47) in this case.

If instead $u_a(x) > 1 - \kappa$, we have

$$1 - \kappa < u_a(x) \leqslant \tilde{u}_{a,t}(x),$$

and once again the monotonicity of f in (H2) implies (5.47), as desired.

Now, from (5.39) and (5.47) it follows that

$$L(u_a - \tilde{u}_{a,t}) \geqslant \varepsilon^{-s}\big(f(u_a) - f(\tilde{u}_{a,t})\big) + ca \geqslant ca \quad \text{in } B_r \cap \{u_a - \tilde{u}_{a,t} \leqslant 0\}.$$

Then, Lemma 5.9 applied to $w := u_a - \tilde{u}_{a,t}$ gives that (5.41) holds for $t = \delta^3$.

Also, using (5.42) with $\theta := 0$, we have that

$$(0, \dots, 0,\, ag(0) - Ca^{1+\sigma}) \in \{u_a \leqslant 0\} \quad \text{and} \quad (0, \dots, 0,\, ag(0) + Ca^{1+\sigma}) \in \{u_a \geqslant 0\}.$$

Therefore there exists $\tau \in [g(0) - Ca^\sigma,\, g(0) + Ca^\sigma]$ such that the point $p_a = (p'_a, p_{a,n}) := (0, \dots, 0, a\tau)$ satisfies

$$u_a(p_a) = 0. \tag{5.48}$$

We claim that, for every fixed $t < 0$, taking a small enough (possibly in dependence of t), we have

$$u_a - \tilde{u}_{a,t} \leqslant 0 \quad \text{at the point } p_a. \tag{5.49}$$

To this end, we recall (5.37) and we observe that

$$\begin{aligned} p_{a,n} - aQ^t(p'_a) - Ca^2 &= a\tau - aQ^t(0) - Ca^2 \geqslant a\big(g(0) - Ca^\sigma\big) - aQ(0) - at - Ca^2 \\ &= -Ca^{1+\sigma} - at - Ca^2 > 0, \end{aligned}$$

since $t < 0$, as long as a is small enough (possibly depending on t). From this and (5.43) (applied here with $\theta := 0$), we conclude that

$$p_a \in \{x_n > aQ^t(x') - Ca^2\} \subset \{\tilde{u}_{a,t} \geqslant 0\}.$$

This and (5.48) give that

$$u_a(p_a) - \tilde{u}_{a,t}(p_a) \leqslant u_a(p_a) = 0,$$

which proves (5.49).

Now we let $t_* = t_*(a)$ be the infimum of the $t \in \mathbb{R}$ such that (5.41) holds. Notice that, by (5.41) and (5.49), we know that

$$\liminf_{a \to 0} t_*(a) = 0. \tag{5.50}$$

Next, by (5.37) we have

$$Q - g \geqslant c_0 > 0 \quad \text{for any } x' \text{ outside } B'_{r/8}, \tag{5.51}$$

where c_0 depends only on Q and $\tilde{Q}$.

Also, in view of (5.50), if a is small enough, we may assume that $t_* > -c_0/2$. Thus, by (5.51), we have that

$$Q^{t_*} - g = Q + t_* - g \geqslant c_0/2 > 0 \quad \text{for any } x' \text{ outside } B'_{r/8}.$$

Hence, using again (5.42) and (5.43), we obtain that

$$\{u_a \leqslant 1-\kappa\} \subset \{\tilde{u}_{a,t} \leqslant -1+\kappa\} \quad \text{in } \mathcal{Q}_\delta \setminus B_{r/2}.$$

Hence, as before, using that $\tilde{u}_{a,t} = u_a$ outside of $\mathcal{Q}_\delta$, we conclude that

$$u_a - \tilde{u}_{a,t_*} > -a^2 \quad \text{in } \mathbb{R}^n \setminus B_{r/2}.$$

Using again (5.39) and assumption (H2), it follows that, for a small enough,

$$L(u_a - \tilde{u}_{a,t_*}) \geqslant \varepsilon^{-s}\big(f(u_a) - f(\tilde{u}_{a,t_*})\big) + ca \geqslant ca \quad \text{in } (B_r \setminus B_{r/2}) \cap \{u_a - \tilde{u}_{a,t_*} \leqslant 0\}. \tag{5.52}$$

On the other hand, by the definition of t_*, we have that $u_a - \tilde{u}_{a,t_*} \geqslant 0$ in $B_{r/2}$ and hence formula (5.52) holds true by replacing $(B_r \setminus B_{r/2})$ with B_r (since the contribution in $B_{r/2}$ is void).

Then, Lemma 5.9, applied to $w := u_a - \tilde{u}_{a,t_*}$, yields that $u_a - \tilde{u}_{a,t_*} > 0$ in $\overline{B_{r/2}}$, which is a contradiction with the definition of t_*. □

6. Completion of the proof of Theorem 1.1

Using the techniques developed till now, we are in the position to prove Theorem 1.1.

We need an auxiliary result, a geometric observation. It says that if in a sequence of dyadic balls a set is trapped in a sequence of slabs with possibly varying orientations, then it is also trapped in a sequence of parallel slabs.

Lemma 6.1. *Let $\alpha \in (0,1)$. Assume that, for some $a \in (0,1)$ and $X \subset \mathbb{R}^n$, we have*

$$\big\{x \cdot \omega_j \leqslant -a\, 2^{j(1+\alpha)}\big\} \subset X \subset \big\{x \cdot \omega_j \leqslant a\, 2^{j(1+\alpha)}\big\} \quad \text{in } B_{2^j} \tag{6.1}$$

for all

$$j = \left\{0, 1, 2, \dots, j_a := \left\lfloor \frac{\log a}{\log(2^{-\alpha})} \right\rfloor \right\}$$

where $\omega_j \in S^{n-1}$.

Then, for some $m_0 \in \mathbb{N}$, with $m_0 \leqslant j_a$, and $C > 0$, depending only on α, we have[1] that

$$\big\{x \cdot \omega_0 \leqslant -C\theta\, a\, 2^{j(1+\alpha)}\big\} \subset X \subset \big\{x \cdot \omega_0 \leqslant C\theta\, a\, 2^{j(1+\alpha)}\big\} \quad \text{in } B'_{\theta 2^j} \times (-2^{k_a}, 2^{k_a}) \tag{6.2}$$

for every $j \in \mathbb{N}$, with $0 \leqslant j \leqslant j_a - m_0$.

Proof. We have, for all $j \in \{0, 1, \dots, j_a\}$,

$$\big\{x \cdot \omega_{j+1} \leqslant -a 2^{(j+1)(1+\alpha)}\big\} \subset X \subset \big\{x \cdot \omega_j \leqslant a 2^{(j+1)(1+\alpha)}\big\} \quad \text{in } B_{2^j}.$$

Thus, rescaling by a factor 2^{-j}, we obtain that

$$\big\{x \cdot \omega_{j+1} \leqslant -a 2^{j\alpha+1+\alpha}\big\} \subset \big\{x \cdot \omega_j \leqslant a 2^{j\alpha}\big\} \quad \text{in } B_1. \tag{6.3}$$

Also, for all $j \in \{0, 1, \dots, j_a - 1\}$, we have that

$$a 2^{(j+1)\alpha} \leqslant a 2^{j_a \alpha} \leqslant 1. \tag{6.4}$$

Hence,

$$\delta_j := a 2^{-j\alpha} \leqslant 2^{-j-1-\alpha} < 1. \tag{6.5}$$

[1] We stress that ω_0 in (6.2) is simply ω_j with $j := 0$.

Notice that, with this notation, (6.3) implies that

$$\big\{x\cdot\omega_{j+1}\leqslant -4\delta_j\big\}\subset\big\{x\cdot\omega_j\leqslant\delta_j\big\}\quad\text{in } B_1. \tag{6.6}$$

Observe now that

$$|\omega_{j+1}-\omega_j|\leqslant 32\delta_j. \tag{6.7}$$

Now, from (6.7), summing a geometric series, we deduce that

$$|\omega_j-\omega_0|\leqslant\sum_{i=0}^{j-1}|\omega_{i+1}-\omega_i|\leqslant C\sum_{i=0}^{j-1}\delta_i=Ca\sum_{i=0}^{j-1}2^{i\alpha}=\frac{Ca\,2^{j\alpha}}{2^\alpha-1}\leqslant Ca\,2^{j\alpha},$$

up to renaming $C>0$.

From this, and up to renaming C once again, we obtain that

$$\begin{aligned}&\big\{x\cdot\omega_0\leqslant -Ca\,2^{j(1+\alpha)}\big\}\subset\big\{x\cdot\omega_j\leqslant -a\,2^{j(1+\alpha)}\big\}\\ \text{and}\qquad &\big\{x\cdot\omega_j\leqslant a\,2^{j(1+\alpha)}\big\}\subset\big\{x\cdot\omega_0\leqslant Ca\,2^{j(1+\alpha)}\big\}\quad\text{in } B_{2^j},\end{aligned}$$

which implies the desired result (if m_0 is sufficiently large). □

Now we are in the position of completing the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us denote $u=u_a$ to emphasize the dependence of the statement on a. By Lemma 6.1 we have that, in a suitable coordinate system such that the axis x_n is parallel to ω_0,

$$\{x_n\leqslant -a2^{j(1+\alpha_0)}\}\subset\{u_a\leqslant -1+\kappa\}\subset\{u_a\leqslant 1-\kappa\}\subset\{x_n\leqslant a2^{j(1+\alpha_0)}\}\quad\text{in } B'_{2^j}\times(-2^{k_a},2^{k_a})$$

for $0\leqslant j\leqslant k_a$, where $k_a=j_a-m_0$ and where $m_0=m_0(\alpha_0)$ is the constant of Lemma 6.1.

Then, by Corollaries 4.5 and 4.6, combined with Proposition 5.1, we find that

$$\{x_n\leqslant ag(x')-Ca^{1+\sigma}\}\subset\{u_a\leqslant -1+\kappa\}\subset\{u_a\leqslant 1-\kappa\}\subset\{x_n\leqslant ag(x')+Ca^{1+\sigma}\}$$

in $B'_1\times(-2^{k_a},2^{k_a})$, where g is affine. The assumption $0\in\{-1+\kappa\leqslant u_a\leqslant 1-\kappa\}$ guarantees that

$$g(0)=0.$$

Then, if a is small enough, this implies that

$$\left\{\omega\cdot x\leqslant -\frac{a}{2^{1+\alpha_0}}\right\}\subset\{u_a\leqslant -1+\kappa\}\subset\{u_a\leqslant 1-\kappa\}\subset\left\{\omega\cdot x\leqslant\frac{a}{2^{1+\alpha_0}}\right\}\quad\text{in } B_{1/2},$$

for some $\omega\in S^{n-1}$, and thus Theorem 1.1 follows. □

7. Proof of Theorem 1.2

Now we give the proof of Theorem 1.2, by applying a suitable iteration of Theorem 1.1 at any scale and the sliding method. For this, we point out two useful rescaled iterations of Theorem 1.1. The first, in Corollary 7.1, is a "preservation of flatness" iteration up to scale 1, while the second, in Corollary 7.2, is a "improvement of flatness" iteration up to a mesoscale.

We first give the

Corollary 7.1 ("preservation of flatness"). *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*. Then there exist universal constants $\alpha_0\in(0,s/2)$, $p_0\in(2,\infty)$ and $a_0\in(0,1/4)$ such that the following statement holds.*

Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of $Lu = f(u)$ in $\mathbb{R}^n$, such that $0 \in \{-1+\kappa \leqslant u \leqslant 1-\kappa\}$. Let $k \geqslant j \in \mathbb{N}$ and suppose that

$$j \geqslant \frac{p_0 \, |\log a_0|}{\log 2}. \tag{7.1}$$

Assume that

$$\{\omega_i \cdot x \leqslant -a_0 2^i\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \{\omega_i \cdot x \leqslant a_0 2^i\} \quad \textit{in } B_{2^i}, \tag{7.2}$$

for every $i \geqslant k$, where $\omega_i \in S^{n-1}$.

Then, for every $i \in \mathbb{N}$, with $j \leqslant i \leqslant k$, it holds that

$$\{\omega_i \cdot x \leqslant -a_0 2^i\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \{\omega_i \cdot x \leqslant a_0 2^i\} \quad \textit{in } B_{2^i}, \tag{7.3}$$

for some $\omega_i \in S^{n-1}$.

Proof. We prove (7.3) for all indices i of the form $i = k - \ell$, with $\ell \in \{0, \dots, k-j\}$. The argument is by induction over ℓ. Indeed, when $\ell = 0$, then (7.3) is a consequence of (7.2). Hence, recursively, we assume that the interface of u in $B_{2^{k-q}}$ is contained in a slab of size $a_0 2^{k-q}$, with $q \in \{0, \dots, \ell-1\}$, and we prove that the same holds for $q = \ell$. To this aim, we set $\tilde{u}(x) := u(2^{k-\ell+1}x)$ and $\varepsilon := \frac{1}{2^{k-\ell+1}}$. Notice that $L\tilde{u} = \varepsilon^{-s} f(\tilde{u})$ and

$$\frac{\varepsilon}{a_0^{p_0}} = \frac{1}{a_0^{p_0}\, 2^{k-\ell+1}} \leqslant \frac{1}{a_0^{p_0}\, 2^{j+1}} \leqslant 1, \tag{7.4}$$

thanks to (7.1). In addition, we claim that

$$\text{for any } i \in \mathbb{N}, \text{ the interface of } \tilde{u} \text{ in } B_{2^i} \text{ is trapped in a slab of size } a_0\, 2^{i(1+\alpha_0)}. \tag{7.5}$$

For this, we distinguish the cases $i \geqslant \ell$ and $i \in \{0, \dots, \ell-1\}$. First, suppose that $i \geqslant \ell$. Then, if x lies in the interface of $\tilde{u}$ in B_{2^i}, then $y := 2^{k-\ell+1}x$ lies in the interface of u in $B_{2^{k-\ell+1+i}}$. Accordingly, by (7.2), we know that y is trapped in a slab of size $a_0\, 2^{k-\ell+1+i}$. As a consequence, x is trapped in a slab of size $a_0\, 2^i \leqslant a_0\, 2^{i(1+\alpha_0)}$.

This is (7.5) in this case, so we can now focus on the case in which $i \in \{0, \dots, \ell-1\}$. For this, we take x in the interface of $\tilde{u}$ in B_{2^i}, and we observe that $y := 2^{k-\ell+1}x$ lies in the interface of u in $B_{2^{k-\ell+1+i}} = B_{2^{k-(\ell-1-i)}}$. Then, from the inductive assumption, we know that y is trapped in a slab of size $a_0\, 2^{k-(\ell-1-i)} = a_0\, 2^{k-\ell+1+i}$. Scaling back, it follows that x is trapped in a slab of size $a_0\, 2^i$, which implies (7.5) also in this case.

So, in light of (7.4) and (7.5), we can apply Theorem 1.1 to $\tilde{u}$ and find that the interface of $\tilde{u}$ in $B_{1/2}$ is trapped in a slab of size $\frac{a_0}{2^{1+\alpha_0}}$.

That is, scaling back, the interface of u in $B_{2^{k-\ell}}$ is trapped in a slab of size $\frac{a_0\, 2^{k-\ell+1}}{2^{1+\alpha_0}} \leqslant a_0 2^{k-\ell}$, which gives the desired step of the induction. □

We next give the

Corollary 7.2 ("improvement of flatness"). *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3). *Then there exist universal constants $\alpha_0 \in (0, s/2)$, $p_0 \in (2, \infty)$ and $a_0 \in (0, 1/4)$ such that the following statement holds.*

Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of $Lu = f(u)$ in $\mathbb{R}^n$, such that $0 \in \{-1+\kappa \leqslant u \leqslant 1-\kappa\}$. Let $k, l \in \mathbb{N}$ be such that

$$l \leqslant \frac{k}{\alpha_0 p_0 + 1} + 1 + \frac{p_0 \log a_0}{(\alpha_0 p_0 + 1) \log 2}. \tag{7.6}$$

Assume that

$$\{\omega_j \cdot x \leqslant -a_0 2^j\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \{\omega_j \cdot x \leqslant a_0 2^j\} \quad \textit{in } B_{2^j}, \tag{7.7}$$

for every $j \geqslant k$, where $\omega_j \in S^{n-1}$.

Then, for every $i \in \{0,\dots,l\}$, it holds that

$$\left\{\omega_i \cdot x \leqslant -\frac{a_0\,2^{k-i}}{2^{\alpha_0\,i}}\right\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \left\{\omega_i \cdot x \leqslant \frac{a_0\,2^{k-i}}{2^{\alpha_0 i}}\right\} \quad \textit{in } B_{2^{k-i}}, \tag{7.8}$$

for some $\omega_i \in S^{n-1}$.

Proof. The proof is by induction over i. When $i=0$, we have that (7.8) follows from (7.7) with $j=k$.

Now, we assume that (7.8) holds true for all $i \in \{0,\dots,i_0-1\}$, with $1 \leqslant i_0 \leqslant l$, and we prove it for i_0. To this aim, we set

$$\tilde{u}(x) := u(2^{k-i_0+1}x), \qquad \tilde{\varepsilon} := \frac{1}{2^{k-i_0+1}}, \qquad \tilde{a} := \frac{a_0}{2^{\alpha_0(i_0-1)}}.$$

Our goal is to use Theorem 1.1 in this setting (namely, the triple (u,ε,a) in the statement of Theorem 1.1 becomes here $(\tilde{u},\tilde{\varepsilon},\tilde{a})$). For this, we need to check that $(\tilde{u},\tilde{\varepsilon},\tilde{a})$ satisfy the assumptions of Theorem 1.1. First of all, we notice that $\tilde{a} \leqslant a_0$ and

$$\frac{\tilde{\varepsilon}}{\tilde{a}^{p_0}} = \frac{2^{\alpha_0 p_0(i_0-1)}}{a_0^{p_0}\,2^{k-i_0+1}} = \frac{2^{(\alpha_0 p_0+1)i_0}}{a_0^{p_0}\,2^{\alpha_0 p_0+k+1}} \leqslant \frac{2^{(\alpha_0 p_0+1)l}}{a_0^{p_0}\,2^{\alpha_0 p_0+k+1}} \leqslant 1, \tag{7.9}$$

thanks to (7.6).

Now we claim that, for any $j \geqslant 0$,

$$\text{the interface of } \tilde{u} \text{ in } B_{2^j} \text{ is trapped in a slab of width } \tilde{a}2^{j(1+\alpha_0)}. \tag{7.10}$$

For this, we distinguish two cases, either $j \geqslant i_0$ or $j \in \{0,\dots,i_0-1\}$. In the first case, we take $x \in B_{2^j}$ belonging to the interface of $\tilde{u}$, and we observe that $y := 2^{k-i_0+1}x \in B_{2^{j+k-i_0+1}}$ belongs to the interface of u: then, we can use (7.7) and find that y is trapped in a slab of size

$$a_0 2^{j+k-i_0+1} = \tilde{a}2^{\alpha_0(i_0-1)+j+k-i_0+1}.$$

Scaling back, this says that x is trapped in a slab of size

$$\tilde{a}2^{\alpha_0(i_0-1)+j} \leqslant \tilde{a}2^{\alpha_0(j-1)+j} \leqslant \tilde{a}2^{j(1+\alpha_0)}.$$

This proves (7.10) in this case, and now we focus on the case in which $j \in \{0,\dots,i_0-1\}$. For this, let us take $x \in B_{2^j}$ in the interface of $\tilde{u}$. Then, we have that $y := 2^{k-i_0+1}x \in B_{2^{j+k-i_0+1}} = B_{2^{k-(i_0-j-1)}}$ belongs to the interface of u and hence, in view of the inductive assumption, is trapped in a slab of width

$$\frac{a_0\,2^{k-(i_0-j-1)}}{2^{\alpha_0(i_0-j-1)}} = \tilde{a}2^{\alpha_0 j+k-i_0+1+j}.$$

Thus, scaling back, we find that x is trapped in a slab of width $\tilde{a}2^{\alpha_0 j+j}$, which establishes (7.10).

In light of (7.9) and (7.10), we can apply Theorem 1.1 (with (u,ε,a) replaced here by $(\tilde{u},\tilde{\varepsilon},\tilde{a})$): in this way, we conclude that the interface of $\tilde{u}$ in $B_{1/2}$ is trapped in a slab of width $\frac{\tilde{a}}{2^{1+\alpha_0}}$. That is, scaling back, the interface of u in $B_{2^{k-i_0}}$ is trapped in a slab of width

$$\frac{\tilde{a}\,2^{k-i_0+1}}{2^{1+\alpha_0}} = \frac{a_0}{2^{\alpha_0(i_0-1)}} \cdot \frac{2^{k-i_0+1}}{2^{1+\alpha_0}} = a_0\,2^{k-i_0-\alpha_0 i_0},$$

which is (7.8) for i_0. This completes the inductive step. □

For the proof of Theorem 1.2, it is also useful to have the following maximum principle:

Lemma 7.3. *Assume that w is continuous and bounded from below, and satisfies, in the viscosity sense, $Lw \geqslant -cw$ in $\{w<0\}$, for some $c>0$. Then $w \geqslant 0$ in $\mathbb{R}^n$.*

Proof. Assume, by contradiction, that $\{w<0\}\neq\varnothing$. Then, up to a translation, we may assume that $w(0)<0$. Let also $C_o\geqslant 0$ be such that $w\geqslant -C_o$ in $\mathbb{R}^n$. Fix $\eta\in C^\infty(\mathbb{R}^n,\,[0,1])$ with $\eta=0$ in $B_{1/2}$ and $\eta=1$ in $\mathbb{R}^n\setminus B_1$. For any $\delta>0$, we define

$$w_\delta(x):=w(x)+C_o\eta(\delta x).$$

Notice that

$$\inf_{\mathbb{R}^n} w_\delta\leqslant w(0)+C_o\eta(0)=w(0)<0. \tag{7.11}$$

Moreover, if $x\in\mathbb{R}^n\setminus B_1$, then

$$w_\delta(x)=w(x)+C_o\geqslant 0.$$

This and (7.11) imply that

$$\inf_{\mathbb{R}^n} w_\delta=\min_{\overline{B_1}} w_\delta=w_\delta(x_\delta),$$

for a suitable $x_\delta\in\overline{B_1}$.

We remark that $w_\delta(x_\delta)\leqslant w_\delta(0)=w(0)<0$, and so $w(x_\delta)=w_\delta(x_\delta)-C_o\eta(\delta x_\delta)<0$. Hence

$$0\geqslant Lw_\delta(x_\delta)=Lw(x_\delta)+C_o\,L\big(\eta(\delta x_\delta)\big)\geqslant -cw(x_\delta)-C\delta^s,$$

for some $C>0$. Consequently,

$$\inf_{\mathbb{R}^n} w_\delta=w(x_\delta)+C_o\eta(\delta x_\delta)\geqslant -\frac{C\delta^s}{c}+C_o\eta(\delta x_\delta).$$

That is, for any $x\in\mathbb{R}^n$,

$$w(x)+C_o\eta(\delta x)\geqslant -\frac{C\delta^s}{c}+C_o\eta(\delta x_\delta).$$

Taking limit in δ, we thus conclude that, for any $x\in\mathbb{R}^n$,

$$w(x)=w(x)+C_o\eta(0)\geqslant 0,$$

against our initial assumption. □

With this, we can now complete the proof of Theorem 1.2, with the following argument:

Proof of Theorem 1.2. Step 1. We prove that in an appropriate orthonormal coordinate system we have

$$\{x_n\leqslant z_n-C2^{j(1-\delta)}\}\subset\{u\leqslant -1+\kappa\}\subset\{u\leqslant 1-\kappa\}\subset\{x_n\leqslant z_n+C2^{j(1-\delta)}\}\quad\text{in } B_{2^j}(z) \tag{7.12}$$

for all $z\in\{-1+\kappa\leqslant u\leqslant 1-\kappa\}$ and $j\in\mathbb{N}$, for a suitable $\delta\in(0,1)$.

Let $a_0>0$ be the constant in Theorem 1.1. First we claim that there exists $k_0\geqslant 1$ universal such that, for any $z\in\{-1+\kappa\leqslant u\leqslant 1-\kappa\}$ and $k\geqslant k_0$, we have

$$\big\{\omega\cdot(x-z)\leqslant -a_0\,2^k\big\}\subset\{u\leqslant -1+\kappa\}\subset\{u\leqslant 1-\kappa\}\subset\big\{\omega\cdot(x-z)\leqslant a_0\,2^k\big\}\quad\text{in } B_{2^k}(z), \tag{7.13}$$

where $\omega\in S^{n-1}$ may depend on z and k.

To prove (7.13), we use (1.12), to see that, if k is sufficiently large (depending on a_0),

$$\big\{\omega\cdot x\leqslant -a_0 2^{k-1}\big\}\subset\{u\leqslant -1+\kappa\}\subset\{u\leqslant 1-\kappa\}\subset\big\{\omega\cdot x\leqslant a_0 2^{k-1}\big\}\quad\text{in } B_{2^{k+1}}, \tag{7.14}$$

for some $\omega\in S^{n-1}$ possibly depending on k. Then, if k is also large enough (depending on z) in such a way that $|z|\leqslant k$, we can suppose that $B_{2^k}(z)\subset B_{2^{k+1}}$ and

$$a_0 2^{k-1}+|z|\leqslant a_0 2^{k-1}+k\leqslant a_0 2^k.$$

These observations and (7.14) give that, if k is sufficiently large, possibly depending on a_0 and z, then

$$\big\{\omega\cdot(x-z)\leqslant -a_0 2^k\big\}\subset\{u\leqslant -1+\kappa\}\subset\{u\leqslant 1-\kappa\}\subset\big\{\omega\cdot(x-z)\leqslant a_0 2^k\big\}\quad\text{in } B_{2^k}(z).$$

Hence, in light of Corollary 7.1 (centered here at the point z), we can conclude that (7.13) holds true (we stress indeed that condition (7.1) gives a universal lower threshold for the validity of (7.13)).

Our goal is now to use (7.13) to prove (7.12). For this, we need to pick up the exponent δ in (7.12) which will imply the "stabilization" of the direction ω from one scale to another. To this aim, fixed j large enough, we take

$$k := \left\lfloor \frac{\alpha_0 p_0 + 1}{\alpha_0 p_0} j + \frac{\log a_0}{\alpha_0 \log 2} \right\rfloor$$

and $l := k - j$. We observe that

$$l \geqslant \frac{\alpha_0 p_0 + 1}{\alpha_0 p_0} j + \frac{\log a_0}{\alpha_0 \log 2} - 1 - j = \frac{j}{\alpha_0 p_0} + \frac{\log a_0}{\alpha_0 \log 2} - 1. \tag{7.15}$$

In this setting, we have that

$$l - \frac{k}{\alpha_0 p_0 + 1} = \frac{\alpha_0 p_0 k}{\alpha_0 p_0 + 1} - j \leqslant \frac{\alpha_0 p_0}{\alpha_0 p_0 + 1} \left(\frac{\alpha_0 p_0 + 1}{\alpha_0 p_0} j + \frac{\log a_0}{\alpha_0 \log 2} \right) - j = \frac{p_0 \log a_0}{(\alpha_0 p_0 + 1) \log 2}.$$

This says that (7.6) is satisfied. Also, condition (7.7) (here, centered at the point z) follows from (7.13). Consequently, in view of (7.8) (centered here at the point z), we conclude that the interface of u in $B_{2^j} = B_{2^{k-m}}$ is trapped in a slab of size

$$\frac{a_0\, 2^{k-l}}{2^{\alpha_0 m}} = \frac{a_0\, 2^j}{2^{\alpha_0 l}} \leqslant \frac{a_1\, 2^j}{2^{\frac{j}{p_0}}} = a_1\, 2^{j(1-\delta)},$$

for some $a_1 > 0$, where $\delta := \frac{1}{p_0}$, and (7.15) has been exploited.

In formulas, this says that

$$\begin{aligned} &\{\omega_{z,j} \cdot (x - z) \leqslant -a_1 2^{j(1-\delta)}\} \subset \{u \leqslant -1 + \kappa\} \\ &\quad \subset \{u \leqslant 1 - \kappa\} \subset \{\omega_{z,j} \cdot (x - z) \leqslant a_1 2^{j(1-\delta)}\} \quad \text{in } B_{2^j}(z), \end{aligned} \tag{7.16}$$

for any $j \geqslant j_0$ large enough, for suitable $\omega_{z,j} \in S^{n-1}$.

Next we improve (7.16) by finding a direction which is independent of j and z. For this, we start to get rid of the dependence of j: namely, we use (7.16) in two consecutive dyadic scales (say, j and $j+1$) and we obtain, similarly as in the proof of Lemma 6.1, that

$$|\omega_{z,j+1} - \omega_{z,j}| \leqslant C 2^{-j\delta}.$$

This implies that

$$\lim_{j \to +\infty} \omega_{z,j} = \omega_{z,\infty}, \tag{7.17}$$

for each fixed z.

We will make this statement more precise, by showing that the limit is independent of z, namely we claim that

$$\lim_{j \to +\infty} \omega_{z,j} = \omega_\infty, \tag{7.18}$$

for some $\omega_\infty \in S^{n-1}$. For this, we observe that, for any z, $\bar{z} \in \{-1 + \kappa \leqslant u \leqslant 1 - \kappa\}$,

$$\{\omega_{z,j} \cdot (x - z) \leqslant -a_1 2^{j(1-\delta)}\} \subset \{u \leqslant -1 + \kappa\} \subset \{u \leqslant 1 - \kappa\} \subset \{\omega_{\bar{z},j} \cdot (x - \bar{z}) \leqslant a_1 2^{j(1-\delta)}\}$$

in $B_{2^j}(z) \cap B_{2^j}(\bar{z})$, thanks to (7.16). This implies that

$$|\omega_{z,j} - \omega_{\bar{z},j}| \to 0 \quad \text{as } j \to \infty.$$

From this and (7.17), we deduce (7.18), as desired.

Let us choose now an orthonormal coordinate system in which $\omega_\infty = (0, 0, \dots, 0, 1)$. Then, (7.16) and (7.18) imply that (7.12) holds true for all $j \geqslant j_0$ universal. Also, for $j < j_0$, (7.12) holds true simply by choosing C large enough, hence we have proved the desired claim in (7.12) for all $j \in \mathbb{N}$.

In addition, for our purposes, it is interesting to observe that, as as consequence of (7.12), we have

$$\{x_n \leqslant G(x') - C\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \{x_n \leqslant G(x') + C\} \tag{7.19}$$

in all of $\mathbb{R}^n$, for some $G \in \mathrm{Lip}(\mathbb{R}^{n-1})$ with Lipschitz seminorm universally bounded and such that

$$|G(x') - G(y')| \leqslant \bar{C}\,(|x'-y'|^{1-\delta} + 1), \tag{7.20}$$

for a suitable $\bar{C} > 0$.

Step 2. We now use (7.19) and a sliding method (which is somehow related to the one in [23]) to conclude that u has 1D symmetry. Indeed, given $(e'_o, 0) \in S^{n-1} \cap \{x_n = 0\}$ and $\varepsilon > 0$ we consider

$$u^t(x) := u(x - et)$$

where

$$e = (e', e_n) := \frac{(e'_o, \varepsilon)}{\sqrt{1+\varepsilon^2}}. \tag{7.21}$$

Our goal is to prove that

$$u^t \leqslant u \quad \text{in all of } \mathbb{R}^n \quad \text{and for all } t > 0. \tag{7.22}$$

From the fact that e'_o and ε are arbitrary it will follow immediately that $u = u(x_n)$ is a $1D$ function.

To prove (7.22), we first observe that, if we take t large enough (depending on ε), we have that

$$\{u \leqslant 1-\kappa\} \subset \{u^t \leqslant -1+\kappa\}. \tag{7.23}$$

To check this, let $x \in \{u \leqslant 1-\kappa\}$. Then, by (7.19), we know that $x_n \leqslant G(x') + C$. Hence, in view of (7.20), we have that

$$\begin{aligned}
(x-et)_n - G((x-et)') + C &= x_n - \frac{\varepsilon t}{\sqrt{1+\varepsilon^2}} - G\left(x' - \frac{e'_o\, t}{\sqrt{1+\varepsilon^2}}\right) + C \\
&\leqslant G(x') - \frac{\varepsilon t}{\sqrt{1+\varepsilon^2}} - G\left(x' - \frac{e'_o\, t}{\sqrt{1+\varepsilon^2}}\right) + 2C \\
&\leqslant \bar{C}\left[\left(\frac{t}{\sqrt{1+\varepsilon^2}}\right)^{1-\delta} + 1\right] - \frac{\varepsilon t}{\sqrt{1+\varepsilon^2}} + 2C \leqslant 0,
\end{aligned}$$

as long as t is large enough (possibly in dependence of ε). Hence, by (7.19),

$$u^t(x) = u(x-et) \leqslant -1+\kappa,$$

that proves (7.23).

Now we define $I_- := (-1, -1+\kappa]$ and $I_+ := [1-\kappa, 1)$ and we observe that, for large t,

$$\text{if } x \in \mathbb{R}^n, \text{ and } u^t(x) \geqslant u(x), \text{ then either } u^t(x), u(x) \in I_- \text{ or } u^t(x), u(x) \in I_+. \tag{7.24}$$

To prove it, let x be such that

$$u^t(x) \geqslant u(x). \tag{7.25}$$

We distinguish two cases,

$$\text{either } u(x) \in I_+, \tag{7.26}$$

$$\text{or } u(x) \in (-1,1) \setminus I_+. \tag{7.27}$$

If (7.26) holds, then (7.25) gives that $u^t(x) \in I_+$, and we are done. If instead (7.27) holds, then (7.23) gives that $u^t(x) \in I_-$. This and (7.25) imply that $u(x) \in I_-$, and this concludes the proof of (7.24).

Now we claim that

$$u^t \leqslant u \quad \text{for all } t \text{ large enough (possibly in dependence of } \varepsilon). \tag{7.28}$$

To prove this, let $w := u - u^t$. We claim that

$$Lw \geqslant -c_\kappa w \quad \text{in} \quad \{w \leqslant 0\}. \tag{7.29}$$

Indeed, from (7.24) and the monotonicity of f in $I_- \cup I_+$ given in (H2), we have that, if $x \in \{w \leqslant 0\} = \{u^t \geqslant u\}$,

$$-Lw(x) = Lu^t(x) - Lu(x) = f(u^t(x)) - f(u(x)) = \int_{u(x)}^{u^t(x)} f'(\tau)\,d\tau \leqslant -c_\kappa\,(u^t(x) - u(x)) = c_\kappa w(x),$$

thus establishing (7.29).

Then, from (7.29) and Lemma 7.3, we deduce that $w \geqslant 0$. This concludes the proof of (7.28).

Now, to complete the proof of (7.22), we perform a sliding method to check that $u^t \leqslant u$ also when t decreases, up to $t = 0$. To this aim, we first check the touching points inside the tubular neighborhood described by the function G in (7.19). Namely, we let G and C be as in (7.19), we let $t_0 > 0$ be a fixed, suitably large, t for which (7.24) holds true, and we define

$$C' := C + t_0\,\|\nabla G\|_{L^\infty(\mathbb{R}^{n-1})}. \tag{7.30}$$

Let also

$$\mathcal{G} := \{x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } |x_n - G(x')| \leqslant C'\}. \tag{7.31}$$

and the set $\mathcal{G}$ is somehow the cornerstone of the sliding strategy that we follow here, since

$$\text{if } t > 0 \text{ and } u^t \leqslant u \text{ in } \mathcal{G}, \text{ then } u^t \leqslant u \text{ in the whole of } \mathbb{R}^n. \tag{7.32}$$

Notice that, from the discussion before (7.30), we already know that $u^t \leqslant u$ in the whole of $\mathbb{R}^n$ for $t \geqslant t_0$, so, to establish (7.32), we can focus on the case $t \in [0, t_0)$. To this objective, we claim that (7.24) holds true also in this setting (we stress that the original statement in (7.24) was proved only for large t). To prove it, let x be such that

$$u^t(x) > u(x). \tag{7.33}$$

We distinguish two cases, namely

$$\text{either } u(x) \in I_+, \tag{7.34}$$

$$\text{or } u(x) \in (-1, 1) \setminus I_+. \tag{7.35}$$

If (7.34) is satisfied, then (7.33) implies that $u^t(x)$ also lies in I_+, which gives (7.24). So, we can focus on the case in which (7.35) holds true. Then, from the assumption in (7.32), we know that $u^t \leqslant u$ in $\mathcal{G}$. This and (7.33) imply that x lies outside $\mathcal{G}$. This and (7.35) give that x lies below $\mathcal{G}$, that is, recalling (7.31),

$$x_n \leqslant G(x') - C'.$$

Hence, in light of (7.30),

$$\begin{aligned}(x - et)_n - G((x - et)') &\leqslant x_n - G(x') + t\,\|\nabla G\|_{L^\infty(\mathbb{R}^{n-1})}\\ &\leqslant x_n - G(x') + t_0\,\|\nabla G\|_{L^\infty(\mathbb{R}^{n-1})} = x_n - G(x') + C' - C \leqslant -C.\end{aligned}$$

This and (7.19) imply that $x - te \in \{u \leqslant -1 + \kappa\}$. That is $u^t(x) \in I_-$. This proves that (7.24) holds true also in this setting. From this and the assumption in (7.32), it follows that $u^t \leqslant u$, by arguing exactly as in the proof of (7.28). This completes the proof of (7.32).

Now, in view of (7.32), to complete the proof of (7.22), it is enough to show that

$$\text{for any } t > 0, \text{ it holds that } u^t \leqslant u \text{ in } \mathcal{G}. \tag{7.36}$$

To this aim, we let

$$\bar{t} := \inf\{t \geqslant 0 \text{ s.t. } u^t \leqslant u \text{ in } \mathcal{G}\}.$$

Notice that $\bar t \leqslant t_0$, thanks to the discussion before (7.30). We claim that, in fact,

$$\bar t = 0. \tag{7.37}$$

To this aim, we assume, by contradiction, that $\bar t > 0$. Then, we have that $u^{\bar t} \leqslant u$ in $\mathcal{G}$, and there exists a sequence of points

$$x_j \in \mathcal{G} \tag{7.38}$$

such that $u(x_j) - u^{\bar t}(x_j) \leqslant 1/j$. So, we set $u_j(x) := u(x + x_j)$, $u^{\bar t}_j(x) := u^{\bar t}(x + x_j)$ and $w_j(x) := u^{\bar t}_j(x) - u_j(x)$, and we see that $w_j(0) \geqslant -1/j$, $w_j(x) \leqslant 0$ for any $x \in \mathbb{R}^n$ with $x + x_j \in \mathcal{G}$, and

$$Lw_j(x) = f(u^{\bar t}_j(x)) - f(u_j(x)) \qquad \text{in } \mathbb{R}^n.$$

That is, from the Theorem of Ascoli, passing to the limit as $j \to +\infty$, we find that there exist $\bar u$, $\bar u^{\bar t}$ and $\bar w$ (which are the locally uniform limits of u_j, $u^{\bar t}_j$ and w_j, respectively) and $\bar{\mathcal{G}}$ (which is a tubular neighborhood obtained as the limit of $\mathcal{G} - x_j$) such that $\bar w(0) = 0$ and

$$\bar u(x - \bar t e) - \bar u(x) = \bar u^{\bar t}(x) - \bar u(x) = \bar w(x) \leqslant 0$$

for any $x \in \bar{\mathcal{G}}$. Consequently, we infer that

$$\bar w(x) \leqslant 0 \text{ for any } x \in \mathbb{R}^n, \tag{7.39}$$

thanks to (7.32) (applied here to $\bar u$, which solves the equation $L\bar u = f(\bar u)$).

Notice that

$$L\bar w = f(\bar u^{\bar t}) - f(\bar u) \qquad \text{in } \mathbb{R}^n$$

and so

$$L\bar w(0) = f(\bar u^{\bar t}(0)) - f(\bar u(0)) = 0.$$

This and (7.39) imply that $\bar w$ vanishes identically in $\mathbb{R}^n$. As a consequence, for any $x \in \mathbb{R}^n$,

$$\bar u(x) = \bar u^{\bar t}(x) = \lim_{j\to+\infty} u^{\bar t}(x + x_j) = \lim_{j\to+\infty} u(x + x_j - e\bar t) = \lim_{j\to+\infty} u_j(x - e\bar t) = \bar u(x - e\bar t), \tag{7.40}$$

which means that $\bar u$ is periodic (of period $\bar t$ in direction e). Also, from (7.19) and (7.38), moving in the vertical direction, we know that there exists $\tilde x_j$ that is at distance at most $2C'$ from x_j and such that $u(\tilde x_j) = 0$. So we write $\tilde x_j = x_j + \hat x_j$, with $|\hat x_j| \leqslant 2C'$, and we find, up to a subsequence, that $\hat x_j$ converges to some $\hat x$ and

$$0 = \lim_{j\to+\infty} u(\tilde x_j) = \lim_{j\to+\infty} u(x_j + \hat x_j) = \lim_{j\to+\infty} u_j(\hat x_j) = \bar u(\hat x). \tag{7.41}$$

We also claim that

$$\{\bar u = 0\} \subset \{x_n \geqslant -C_o\,(|x'|^{1-\delta} + 1)\}, \tag{7.42}$$

for some $C_o > 0$, where $\delta \in (0,1)$ is as in (7.20). To check this, we use the notation $x_j = (x'_j, x_{j,n}) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we set $G_j(x') := G(x' + x'_j) - x_{j,n}$ and we see that if $p \in \{\bar u = 0\}$, then, for j large enough, we have that $p \in \{|u_j| < 1 - \kappa\}$, that is $p + x_j \in \{|u| < 1 - \kappa\} \subset \{x_n \geqslant G(x') - C\}$, thanks to (7.19). This gives that $p_n + x_{j,n} \geqslant G(p' + x'_j) - C$. Since $x_j \in \mathcal{G}$, we have that $x_{j,n} - G(x'_j) \leqslant C'$. Hence, recalling (7.20), we find that

$$p_n \geqslant G(p' + x'_j) - x_{j,n} - C \geqslant G(p' + x'_j) - G(x'_j) - C - C' \geqslant -\bar C(|p'|^{1-\delta} + 1) - C - C'.$$

This completes the proof of (7.42).

Now, from (7.40) and (7.41), we know that $\hat{x}-\ell e\bar{t} \in \{\bar{u}=0\}$ for any $\ell \in \mathbb{N}$. This and (7.42) imply that $\hat{x}-\ell e\bar{t} \in \{x_n \geqslant -C_o\left(|x'|^{1-\delta}+1\right)\}$, for any $\ell \in \mathbb{N}$. That is, recalling (7.21),

$$\begin{aligned} 0 &\leqslant \lim_{\ell\to+\infty}(\hat{x}-\ell e\bar{t})_n + C_o\left(|(\hat{x}-\ell e\bar{t})'|^{1-\delta}+1\right) \\ &= \lim_{\ell\to+\infty}\hat{x}_n - \frac{\ell\varepsilon\bar{t}}{\sqrt{1+\varepsilon^2}} + C_o\left(\left|\hat{x}'-\frac{\ell e_o'\,\bar{t}}{\sqrt{1+\varepsilon^2}}\right|^{1-\delta}+1\right) \\ &= -\infty. \end{aligned}$$

This is a contradiction and so (7.37) is proved. Notice that (7.37) implies (7.36), which in turn implies (7.22), thanks to (7.32)

Finally, from (7.22) we obtain that $D_e u \geqslant 0$ in all of $\mathbb{R}^n$ for all e of the form (7.21) where $\varepsilon > 0$ is arbitrary.

Accordingly, we have that $D_{(e_o',0)}u \geqslant 0$ for any $e_o' \in S^{n-1}\cap\{x_n=0\}$. Hence, exchanging e_o' with $-e_o'$, we obtain that $D_{(e_o',0)}u$ vanishes identically. It thus follows that $u(x)=u(x_n)$, that is u has 1D symmetry. ☐

8. Proof of Theorems 1.3, 1.4, 1.5 and 1.6

As a first step towards the proof of Theorems 1.3, 1.4, 1.5 and 1.6, we recall that the limit interface of the minimizers is a nonlocal minimal surface.

More precisely, we say that $E \subset \mathbb{R}^n$ is s-minimal in $\mathbb{R}^n$ if its characteristic function is a minimizer for the functional in (1.13), that is if $\mathcal{E}^{\mathrm{Dir}}(\chi_E,B) < +\infty$ and

$$\mathcal{E}^{\mathrm{Dir}}(\chi_E,B) \leqslant \mathcal{E}^{\mathrm{Dir}}(\chi_F,B),$$

for any ball $B\subset\mathbb{R}^n$ and any $F\subset\mathbb{R}^n$ such that $F\setminus B = E\setminus B$.

These nonlocal minimal surfaces have been introduced in [12] and widely studied in the recent literature. In this setting, we have (see Corollary 1.7 in [35]):

Lemma 8.1. *Let u be a minimal solution of $(-\Delta)^{s/2}u = u-u^3$ in $\mathbb{R}^n$. For any $\varepsilon>0$, let $u_\varepsilon(x) := u(x/\varepsilon)$. Then there exists $E\subset\mathbb{R}^n$ which is s-minimal in $\mathbb{R}^n$ and, up to a subsequence, $u_\varepsilon \to \chi_E - \chi_{\mathbb{R}^n\setminus E}$ a.e. in $\mathbb{R}^n$. Also, $\{|u_\varepsilon| \leqslant 1-\kappa\}$ converges locally uniformly to ∂E.*

By a standard sliding method (see e.g. Lemma 9.1 in [38]), one also sees that monotone solutions are minimal:

Lemma 8.2. *Let u be a solution of $(-\Delta)^{s/2}u = u-u^3$ in $\mathbb{R}^n$. Suppose that*

$$\frac{\partial u}{\partial x_n}(x) > 0 \quad \textit{for any } x\in\mathbb{R}^n$$

and

$$\lim_{x_n\to\pm\infty} u(x',x_n) = \pm 1.$$

Then, u is a minimal solution.

The result in Lemma 8.1 can be better specified for monotone solutions, by obtaining that the limit interface is a graph. The precise statement that we need is the following:

Lemma 8.3. *Let u be a minimal solution of $(-\Delta)^{s/2}u = u-u^3$ in $\mathbb{R}^n$.*

Suppose that

$$\frac{\partial u}{\partial x_n}(x) > 0 \quad \textit{for any } x\in\mathbb{R}^n$$

and

$$\lim_{x_n\to\pm\infty} u(x',x_n) = \pm 1.$$

For any $\varepsilon > 0$, let $u_\varepsilon(x) := u(x/\varepsilon)$. Then there exist $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ and $E \subset \mathbb{R}^n$ which is s-minimal in $\mathbb{R}^n$, such that $E = \{x_n > \gamma(x'),\ x' \in \mathbb{R}^{n-1}\}$, and, up to a subsequence, $u_\varepsilon \to \chi_E - \chi_{\mathbb{R}^n\setminus E}$ a.e. in $\mathbb{R}^n$.

Proof. In view of Lemma 8.1, we only have to check that ∂E is the graph of some function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$. As a matter of fact, one has that $E = \{x_n > \gamma(x'),\ x' \in \mathbb{R}^{n-1}\}$ up to sets of measure zero, for a suitable $\gamma : \mathbb{R}^{n-1} \to [-\infty, +\infty]$ (see e.g. the argument from (9.3) on in [38]). That is, ∂E is a nonlocal minimal surface, which is a graph, with possibly vertical portions.

By sliding E in the vertical direction and using the comparison principle, one sees that either $\gamma(\mathbb{R}^{n-1}) \in \{\pm\infty\}$ or else ∂E is a graph.

In the first case E is a minimizing cone in dimension $n-1$ and hence for $n-1 \leqslant 7$ it is a half space. In the second case, since E is a cone it is automatically a Lipschitz graph and we conclude anyway that E is flat using the Bernstein type result in [26]. ☐

With these preliminary results, we can now complete the proofs of Theorems 1.3, 1.4, 1.5 and 1.6.

Proof of Theorems 1.3 and 1.5. From Lemma 8.1, we know that the level sets of u_ε approach locally uniformly ∂E, and E is s-minimal in $\mathbb{R}^n$. Then we use either [34] (in case we are in $\mathbb{R}^2$ and we want to prove Theorem 1.3) or [16] (in case we are in $\mathbb{R}^n$ with $n \leqslant 7$, s is close to 1 and we want to prove Theorem 1.5) and we see that ∂E is a hyperplane.

Hence, we are in the setting of Theorem 1.2, which implies that u is 1D. ☐

Proof of Theorems 1.4 and 1.6. By Lemma 8.2 we know that u is a minimal solution and by Lemma 8.3 we conclude that the level set of u_ε approach an s-minimal set E which is a complete graph (as a matter of fact, in view of Lemma 8.1, we also know that these level sets approach ∂E locally uniformly).

Then, when $n = 3$ and we want to prove Theorem 1.4, we make use of Corollary 1.3 in [26]; similarly, when $n \leqslant 8$, s is close to 1 and we want to prove Theorem 1.6, we make use of Theorem 1.2 in [26] combined with [16]. In any case, we conclude that E is a halfspace. This enables us to exploit Theorem 1.2, which implies that u is 1D. ☐

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