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A nonlocal concave-convex problem with nonlocal mixed boundary data

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ABSTRACT. The aim of this paper is to study the following problem

$$P_\lambda \equiv \left\{ \begin{array}{rcll} (-\Delta)^s u &=& \lambda u^q + u^p & \text{in } \Omega, \\ u &>& 0 & \text{in } \Omega, \\ \mathcal{B}_s u &=& 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{array} \right.$$

with $0 < q < 1 < p$, $N > 2s$, $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain,

$$(-\Delta)^s u(x) = a_{N,s}\ P.V. \int\limits_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

$a_{N,s}$ is a normalizing constant, and $\mathcal{B}_s u = u\chi_{\Sigma_1} + \mathcal{N}_s u \chi_{\Sigma_2}$. Here, Σ_1 and Σ_2 are open sets in $\mathbb{R}^N \backslash \Omega$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\overline{\Sigma}_1 \cup \overline{\Sigma}_2 = \mathbb{R}^N \backslash \Omega$.

In this setting, $\mathcal{N}_s u$ can be seen as a Neumann condition of nonlocal type that is compatible with the probabilistic interpretation of the fractional Laplacian, as introduced in [15], and $\mathcal{B}_s u$ is a mixed Dirichlet-Neumann exterior datum. The main purpose of this work is to prove existence, nonexistence and multiplicity of positive energy solutions to problem (P_λ) for suitable range of λ and p and to understand the interaction between the concave-convex nonlinearity and the Dirichlet-Neumann data.

CONTENTS

1. INTRODUCTION

In [15], the authors introduced a new nonlocal Neumann condition, which is compatible with the probabilistic interpretation of the nonlocal setting related to some Lévy process in $\mathbb{R}^N$. Motivated by this, we aim in this work to study a semilinear nonlocal elliptic problem with mixed Dirichlet-Neumann data. More precisely, we study existence and multiplicity of positive solutions to the following problem:

$$P_\lambda \equiv \left\{ \begin{array}{rcll} (-\Delta)^s u &=& \lambda u^q + u^p & \text{in } \Omega, \\ u &>& 0 & \text{in } \Omega, \\ \mathcal{B}_s u &=& 0 & \text{in } \mathbb{R}^N \backslash \Omega\,, \end{array} \right.$$

with $0 < q < 1 < p$, $N > 2s$, $\lambda > 0$.

In our setting, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $(-\Delta)^s$ is the fractional Laplacian operator, defined as

$$(-\Delta)^s u(x) = a_{N,s}\ P.V. \int\limits_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy.$$

See e.g. [18], [19], [13] and the references therein for more information about this operator. In this framework $a_{N,s} > 0$ is a suitable normalization constant and the exterior condition

$$\mathcal{B}_s u = u\chi_{\Sigma_1} + \mathcal{N}_s u \chi_{\Sigma_2} \tag{1.1}$$

can be seen as a nonlocal version of the classical Dirichlet-Neumann mixed boundary condition. As a matter of fact, here $\mathcal{N}_s$ is the non-local normal derivative introduced in [15], given by

$$\mathcal{N}_s u(x) = a_{N,s}\int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy, \quad x\in\mathbb{R}^N\backslash\overline{\Omega}. \tag{1.2}$$

Also, Σ_1 and Σ_2 are open sets in $\mathbb{R}^N\backslash\Omega$ such that $\Sigma_1\cap\Sigma_2=\emptyset$ and $\overline{\Sigma}_1\cup\overline{\Sigma}_2=\mathbb{R}^N\backslash\Omega$. As customary, in (1.1) we denoted by χ_A the characteristic function of a set A.

Using an integration by parts formula stated in [15], one sees that problem (P_λ) can be set in a variational setting, since the requested solutions can be seen as critical points of the functional

$$J_\lambda(u)=\frac{1}{2}\|u\|^2-\frac{\lambda}{q+1}\|u_+\|_{q+1}^{q+1}-\frac{1}{p+1}\|u_+\|_{p+1}^{p+1}, \tag{1.3}$$

where $\|v\|_r^r=\displaystyle\int_\Omega |v|^r\,dx$ and $u_+=\max(u,0)$.

Such problem, in the local case of the classical Laplacian, was extensively studied in the literature, especially after the seminal work of Ambrosetti, Brezis and Cerami [3]. Similar problems with a Dirichlet-Neumann datum were studied, for the subcritical case, in [12] and, in the critical case, in [17].

In the nonlocal framework, ($s<1$), with Dirichlet data, the problem was dealt with in [8] for the subcritical case and in [7] for the critical case. See also [23], [24] and [14].

Also, in [8] the authors uses an extension method, which allows them to reduce the problem to a local one, see [10]. We stress that, in our case, because of the nonlocal Neumann part, we cannot use such extension and then we deal with the problem in an appropriate purely nonlocal, and somehow more general, framework. Moreover, to obtain our multiplicity result, we have to use an additional argument which was classically developed by Alama in [1].

Our main results are the following:

Theorem 1. *Let $0<s<1$, $0<q<1<p$. Then there exist $\Lambda>0$, such that:*

1 *For all $\lambda\in(0,\Lambda)$, problem (P_λ) has a minimal solution u_λ such that $J_\lambda(u_\lambda)<0$. Moreover, these solutions are ordered, namely: if $\lambda_1<\lambda_2$ then $u_{\lambda_1}<u_{\lambda_2}$.*
2 *If $\lambda>\Lambda$, problem (P_λ) has no positive weak solutions.*
3 *If $\lambda=\Lambda$, problem (P_λ) has at least one positive solution.*

Theorem 2. *For all $0<s<1$, $0<q<1<p<\frac{N+2s}{N-2s}$, $\lambda\in(0,\Lambda)$, problem (P_λ) has a second solution $v_\lambda>u_\lambda$.*

The paper is organized as follows: In Section 2, we introduce the functional setting to deal with problem (P_λ), as well as the notion of solution we will work with and some auxiliary results. Section 3 is devoted to prove existence of minimal and extremal solutions. Finally in Section 4 we prove the existence of a second solution using Alama's argument.

2. Preliminaries and functional setting.

We introduce in this section a natural functional framework for our problems and we give some related properties and some useful embedding results needed when we deal with problem (P_λ). According to the definition of the fractional Laplacian, see [13], [23], and the integration by parts formula, see [15] , it is natural to introduce the following spaces. We denote by $H^s(\mathbb{R}^N)$ the classical Sobolev spaces,

$$H^s(\mathbb{R}^N)=\left\{u\in L^2(\mathbb{R}^N): \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}}\in L^2(\mathbb{R}^N\times\mathbb{R}^N)\right\}, \tag{2.1}$$

endowed with the norm

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \|u\|^2_{L^2(\mathbb{R}^N)} + \int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy. \tag{2.2}$$

Definition 3. *Let Ω be a bounded domain of $\mathbb{R}^N$. For $0<s<1$, we note*

$$\mathbb{H}^s(\Omega,\Sigma_1) = \left\{u \in H^s(\mathbb{R}^N) :\ u = 0 \text{ in } \Sigma_1\right\}.$$

Endowed with the norm,

$$\|u\|^2 = a_{N,s}\int\int_{\mathcal{D}_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,$$

where $\mathcal{D}_\Omega = (\mathbb{R}^N\times\mathbb{R}^N)\setminus(\Omega^c\times\Omega^c)$.

Notice that $\|\,.\,\|$ is an equivalent norm to the one induced by $H^s(\mathbb{R}^N)$. The following result justifies our choices of $\|\,.\,\|$.

Proposition 4. *Let $s\in(0,1)$, for all $u,\ v\in\mathbb{H}^s(\Omega,\Sigma_1)$ we have,*

$$\int_\Omega v(-\Delta)^s u\,dx = \frac{a_{N,s}}{2}\int\int_{\mathcal{D}_\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_{\Sigma_2} v\mathcal{N}_s u\,dx.$$

The proof of this result is a direct application of the integration by parts formula, see Lemma 3.3 in [15].

The space $(\mathbb{H}^s(\Omega,\Sigma_1),\langle\,,\,\rangle)$ has good analytic properties. In particular:

Proposition 5. $(\mathbb{H}^s(\Omega,\Sigma_1),\langle\,,\,\rangle)$ *is a Hilbert space, with scalar product*

$$\langle u,v\rangle = a_{N,s}\int\int_{\mathcal{D}_\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy.$$

In the rest of the paper, for the simplicity of typing, we shall denote the functional space introduced in definition (3) by $\mathbb{H}^s$ and we shall normalize

(2.3) the constant $a_{N,s}$ to be equal to 2.

Now we give a Sobolev-type result for function in $\mathbb{H}^s$. To this end, we recall the classical Sobolev inequality,

Proposition 6. *Let $s\in(0,1)$ and $N>2s$. There exist a constant $S=S(N,s)$ such that, for any function $u\in H^s(\mathbb{R}^N)$, we have*

$$S\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leqslant \int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy$$

where $2^*_s = \frac{2N}{N-2s}$.

See e.g. [19], [13] and the references therein for the proof of Proposition 6.

Corollary 7. *Let $s\in(0,1)$ and $N>2s$. There exists a constant $C=C(N,s,\Omega,\Sigma_2)$ such that, for any function $u\in\mathbb{H}^s$,*

$$\|u\|^2_{L^r(\Omega)} \leqslant C\|u\|^2,$$

for all $1<r\leqslant 2^*_s$.

Now we consider the standard truncation functions given by

$$T_k(u) = \max\big\{ -k,\ \min\{k,u\}\big\}$$

and $G_k(u) = u - T_k(u)$. In this setting, the following are some useful properties of $\mathbb{H}^s$-functions which are needed to get some regularity results for some elliptic problems in $\mathbb{H}^s$ (see also Theorem 13 below).

Proposition 8. *Let u be a function in $\mathbb{H}^s$, then*

1 *if $\Phi \in Lip(\mathbb{R})$ is such that $\Phi(0) = 0$, then $\Phi(u) \in \mathbb{H}^s$. In particular for any $k > 0$, $T_k(u),\ G_k(u) \in \mathbb{H}^s$.*
2 *For any $k \geqslant 0$*

$$\|G_k(u)\|^2 \leqslant \int_\Omega G_k(u)(-\Delta)^s u\,dx + \int_{\Sigma_2} G_k(u)\mathcal{N}_s u\,dx$$

3 *For any $k \geqslant 0$*

$$\|T_k(u)\|^2 \leqslant \int_\Omega T_k(u)(-\Delta)^s u\,dx + \int_{\Sigma_2} T_k(u)\mathcal{N}_s u\,dx$$

Proof. The claim in (1) follows from the setting of the norm given in Definition 3. As for (2) and (3), we claim that, for any $a, b \geqslant 0$ and any $x \in \mathbb{R}^N$,

$$a\left(G_k(u)(-\Delta)^s T_k(u)\right)(x) + b\left(G_k(u)\mathcal{N}_s T_k(u)\right)(x) \geqslant 0. \tag{2.4}$$

To check this, we can take $x \in \{G_k(u) \neq 0\}$, otherwise (2.4) is obvious. Then, if $x \in \{G_k(u) > 0\}$ we have that $T_k(u)(x) = k$, which is the maximum value that $T_k(u)$ attains, and therefore $(-\Delta)^s T_k(u)(x) \geqslant 0$ and $\mathcal{N}_s T_k(u)(x) \geqslant 0$. Conversely, if $x \in \{G_k(u) < 0\}$ we have that $T_k(u)(x) = -k$, which is the minimum value that $T_k(u)$ attains, and therefore $(-\Delta)^s T_k(u)(x) \leqslant 0$ and $\mathcal{N}_s T_k(u)(x) \leqslant 0$. By combining these observations, we obtain (2.4). From (2.4) and Proposition 4 it follows that

$$\begin{aligned}&\int_\Omega T_k(u)(-\Delta)^s G_k(u)\,dx + \int_{\Sigma_2} T_k(u)\mathcal{N}_s G_k(u)\,dx\\ &\qquad = \int_\Omega G_k(u)(-\Delta)^s T_k(u)\,dx + \int_{\Sigma_2} G_k(u)\mathcal{N}_s T_k(u)\,dx \geqslant 0.\end{aligned} \tag{2.5}$$

Also, using (2.3) and Propositions 4 and 5, we see that

$$\begin{aligned}\|G_k(u)\|^2 &= \iint_{\mathcal{D}_\Omega} \frac{(G_k(u)(x) - G_k(u)(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ &= \int_\Omega G_k(u)(-\Delta)^s G_k(u)\,dx + \int_{\Sigma_2} G_k(u)\mathcal{N}_s G_k(u)\,dx\\ &= \int_\Omega G_k(u)(-\Delta)^s\big(u - T_k(u)\big)\,dx + \int_{\Sigma_2} G_k(u)\mathcal{N}_s\big(u - T_k(u)\big)\,dx\end{aligned} \tag{2.6}$$

and, similarly,

$$\|T_k(u)\|^2 = \int_\Omega T_k(u)(-\Delta)^s\big(u - G_k(u)\big)\,dx + \int_{\Sigma_2} T_k(u)\mathcal{N}_s\big(u - G_k(u)\,dx. \tag{2.7}$$

Then, the claim in (2) follows from (2.6) and (2.5), while the claim in (3) follows from (2.7) and (2.5). □

Let us now consider the following problem,

$$\left\{\begin{array}{rcll}(-\Delta)^s u &=& f & \text{in } \Omega,\\ \mathcal{B}_s u &=& 0 & \text{in } \mathbb{R}^N\backslash\Omega\,,\end{array}\right. \tag{2.8}$$

where Ω is a bounded regular domain of $\mathbb{R}^N$, $N > 2s$, $\mathbb{H}^{-s}$ is the dual space of $\mathbb{H}^s$ and $f \in \mathbb{H}^{-s}$.

Definition 9. *We say that $u \in \mathbb{H}^s$ is an energy solution to (2.8) if*

$$\int\int_{\mathcal{D}_\Omega} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+2s}}\,dx\,dy = (f,\varphi) \quad \forall \varphi \in \mathbb{H}^s \tag{2.9}$$

where $(\ ,\ )$ represent the duality between $\mathbb{H}^s$ and $\mathbb{H}^{-s}$.

Notice that the existence and uniqueness of energy solutions to problem (2.8) follow from the Lax-Milgram Theorem. Furthermore if $f \geqslant 0$ then $u \geqslant 0$. Indeed for $u \in \mathbb{H}^s$, thanks to Lemma 8, we know that $u_- = \min(u,0) \in \mathbb{H}^s$. Taking u_- as a test function in (2.9) it follows that $u_- = 0$.

A supersolution (respectively, subsolution) is a function that verifies (2.9) with equality replaced by "$\geqslant$"(respectively, "$\leqslant$") for every non-negative test function in $\mathbb{H}^s$. Using a standard iterative argument we can easily prove the following result.

Lemma 10. *Assume that problem (2.8) has a sub solution $\underline{w}$ and a super solution $\overline{w}$, verifying $\underline{w} \leqslant \overline{w}$ then there exist a solution w satisfying $\underline{w} \leqslant w \leqslant \overline{w}$.*

Here we prove some regularity results when f satisfies some minimal integrability condition. To prove the boundedness of the solution we follows the idea of Stampacchia for second order elliptic equations with bounded coefficients. The interior Hölder regularity is a consequence of continuities properties, see [15], and the regularities results in [25].

Lemma 11. *Let u be a solution to problem (2.8). If $f \in L^q(\Omega)$, $q > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$.*

Proof. We follow here a related argument presented in [19]. See also [25] and [14] for related results. Let $k > 0$ and take $\varphi = G_k(u)$ as a test function in (2.9). Hence, thanks to Proposition 8, we get

$$\|G_k(u)\|^2 \leqslant \int_{A_k} G_k(u) f\,dx + \int_{\Sigma_2} G_k(u)\mathcal{N}_s u\,dx$$

where $A_k = \{x \in \Omega : u > k\}$. Applying Corollary 7 in the left hand side and Hölder inequality in the right hand side,

$$S^2\|G_k(u)\|^2_{L^{2^*_s}(\Omega)} \leqslant \|G_k(u)\|^2 \leqslant \|f\|_{L^m(\Omega)}\|G_k(u)\|_{L^{2^*_s}(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}$$

we have that

$$S^2\|G_k(u)\|^2_{L^{2^*_s}(\Omega)} \leqslant \|f\|_{L^m(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}$$

thus,

$$S^2(h-k)|A_h|^{\frac{1}{2^*_s}} \leqslant \|f\|_{L^m(\Omega)}|A_k|^{1-\frac{1}{2^*_s}-\frac{1}{m}}$$

and then,

$$|A_h| \leqslant S^{2^*_s-2}\frac{\|f\|^{2^*_s}_{L^m(\Omega)}|A_k|^{2^*_s(1-\frac{1}{2^*_s}-\frac{1}{m})}}{(h-k)^{2^*_s}}$$

Since $m > \frac{N}{2s}$ we have that

$$2^*_s\left(1-\frac{1}{2^*_s}-\frac{1}{m}\right) > 1$$

Hence we apply Lemma 14 in [19] with $\psi(\sigma) = |A_\sigma|$ and the result follows. $\square$

Corollary 12. *Let u be an energy solution of (2.8) and suppose that $f \in L^\infty(\Omega)$. Then $u \in C^\gamma(\overline{\Omega})$, for some $\gamma \in (0,1)$.*

Proof. We claim that u is bounded in $\mathbb{R}^N$. Then one could apply interior regularity results for the solutions to $(-\Delta)^s u = 0 \in \Omega$ and $u = g$ in Ω^c. See e.g. [25] and [21].

To check the claim, recalling Lemma 11, we have to consider only the case $x \in \overline{\Sigma}_2$. Then, by (1.2)

$$u(x) = c(N,s)^{-1} \int_\Omega \frac{u(y)}{|x-y|^{N+2s}}\,dy \text{ , where } c(N,s) = \int_\Omega \frac{1}{|x-y|^{N+2s}} \tag{2.10}$$

Hence,

$$|u(x)| \leqslant \|u\|_{L^\infty(\Omega)} \text{ for all } x \in \overline{\Sigma}_2. \tag{2.11}$$

Also, if Σ_2 is unbounded, using Proposition 3.13 in [15], we have

$$\lim_{x\to\infty,\, x\in\overline{\Sigma}_2} u(x) = \frac{1}{|\Omega|}\int_\Omega u(y)\,dy \tag{2.12}$$

Then the claim follows from Lemma 11, (2.11) and (2.12).

□

As a variation of Lemma 11, we point out that if $f = f(x,u)$ and f has the following growth

$$|f(x,s)| \leqslant c(1+|s|^p) \text{ where } p \leqslant \frac{N+2s}{N-2s} \tag{2.13}$$

then, using a Moser iterative scheme, we can prove that:

Theorem 13. *If u is an energy solution to problem (2.8) with f as in (2.13) then $u \in L^\infty(\Omega)$.*

The following is a strong maximum principle for semi-linear equations, it will be used to separate minimal solution of problem (P_λ) for different values of the parameter λ, see [20] .

Proposition 14. *Let $N \geqslant 1$, $0 < s < 1$ and let $f_1, f_2 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be two continuous functions. Let Ω be a domain in $\mathbb{R}^N$ and $v, w \in L^\infty(\mathbb{R}^N) \cap C^{2s+\gamma}$, for some $\gamma > 0$, be such that*

$$\begin{cases} (-\Delta)^s v \geqslant f_1(x,v), & \text{in } \Omega, \\ (-\Delta)^s w \leqslant f_2(x,w), & \text{in } \Omega, \\ v \geqslant w & \text{in } \mathbb{R}^N. \end{cases}$$

Suppose furthermore that

$$f_2(x,w(x)) \leqslant f_1(x,w(x)) \text{ for any } x \in \Omega. \tag{2.14}$$

If there exists a point $x_0 \in \Omega$ at which $v(x_0) = w(x_0)$, then $v = w$ in the whole Ω.

Proof. Let $\phi = v - w$ and set

$$Z_\phi = \{x \in \Omega : \phi(x) = 0\}$$

By assumption $x_0 \in Z_\phi$. Moreover, thanks to the continuity of ϕ, we know that Z_ϕ is closed. We claim now that Z_ϕ is also open. Indeed, let $\bar{x} \in Z_\phi$. Clearly $\phi \geqslant 0$ in $\mathbb{R}^N$, $\phi(\bar{x}) = 0$ and

$$(-\Delta)^s\phi(\bar{x}) \geqslant f_1(\bar{x}, v(\bar{x})) - f_2(\bar{x}, w(\bar{x})) = f_1(\bar{x}, w(\bar{x})) - f_2(\bar{x}, w(\bar{x})) \geqslant 0,$$

in view of (2.14). Accordingly,

$$\begin{aligned} 0 \;\leqslant\; (-\Delta)^s\phi(\bar{x}) \;&=\; \frac{1}{2}\int_{\mathbb{R}^N} \frac{2\phi(\bar{x}) - \phi(\bar{x}+z) - \phi(\bar{x}-z)}{|z|^{N+2s}}\,dz \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \frac{-\phi(\bar{x}+z) - \phi(\bar{x}-z)}{|z|^{N+2s}}\,dz \leqslant 0. \end{aligned}$$

Hence ϕ vanishes identically in $B_\varepsilon(\bar{x})$ and then, for ε small, $B_\varepsilon(\bar{x}) \subseteq Z_\phi$. That is, we have proved that Z_ϕ is open, and so, by the connectedness of Ω, we get that $Z_\phi = \Omega$. □

Now we establish two important results for our purposes. The first result is a Picone-type inequality and the second is a Brezis-Kamin comparison principle for concave nonlinearities.

Theorem 15. *Consider $u,\ v \in \mathbb{H}^s$, suppose that $(-\Delta)^s u \geqslant 0$ is a bounded Radon measure in Ω, $u \geqslant 0$ and not identically zero, then,*

$$\int_{\Sigma_2} \frac{|v|^2}{u} \mathcal{N}_s u \, dx + \int_\Omega \frac{|v|^2}{u} (-\Delta)^s u \, dx \leqslant \int\int_{\mathcal{D}_\Omega} \frac{\big(v(x) - v(y)\big)^2}{|x-y|^{N+2s}} \, dx \, dy$$

The proof of this result is based on a punctual inequality and follows in the same way as in [19]. As a consequence, we have the next comparison principle that extends to the fractional framework the classical one obtained by Brezis and Kamin, see [9].

Lemma 16. *Let $f(x,\sigma)$ be a Carathéodory function such that $\frac{f(x,\sigma)}{\sigma}$ is deceasing in σ, uniformly with respect to $x \in \Omega$. Suppose that $u, v \in \mathbb{H}^s$, with $0 < s < 1$, are such that*

$$\begin{cases} (-\Delta)^s u \geqslant f(x,u), & u > 0 \quad \text{in } \Omega, \\ (-\Delta)^s v \leqslant f(x,v), & v > 0 \quad \text{in } \Omega. \end{cases}$$

Then $u \geqslant v$ in Ω.

The proof of this result is a slight modification of the proof of Theorem 20 in [19]. Finally, we will use the following compactness lemma to get strong convergence in the space $\mathbb{H}^s$.

Lemma 17. *Let $\{v_n\}_n$ be a sequence of non-negative functions such that $\{v_n\}_n$ is bounded in $\mathbb{H}^s$, $v_n \rightharpoonup v$ in $\mathbb{H}^s$ and $v_n \leqslant v$.*
Assume that $(-\Delta)^s v_n \geqslant 0$ then, $v_n \to v$ strongly in $\mathbb{H}^s$.

Proof. Since $v_n \leqslant v$, then using the fact that $(-\Delta)^s v_n \geqslant 0$ it follows that

$$\int_\Omega (-\Delta)^s v_n (v - v_n) \, dx \quad \geqslant \quad 0.$$

Hence

$$\int_\Omega (-\Delta)^s v_n v \, dx \quad \geqslant \quad \int_\Omega (-\Delta)^s v_n v_n \, dx.$$

From this and Young's inequality, we obtain that

$$\int\int_{\mathcal{D}_\Omega} \frac{\big(v_n(x) - v_n(y)\big)^2}{|x-y|^{N+2s}} \, dx \, dy \quad \leqslant \quad \int\int_{\mathcal{D}_\Omega} \frac{\big(v(x) - v(y)\big)^2}{|x-y|^{N+2s}} \, dx \, dy$$

Thus

$$\limsup_{n\to\infty} \|v_n\| \quad \leqslant \quad \|v\|.$$

Since $v_n \rightharpoonup v$ in $\mathbb{H}^s$ then, by the last inequality,

$$\limsup_{n\to\infty} \|v_n\| \leqslant \|v\| \leqslant \liminf_{n\to\infty} \|v_n\|.$$

As a consequence,

$$\lim_{n\to\infty} \|v_n\| = \|v\|$$

and so

$$\limsup_{n\to\infty} \|v_n - v\|^2 = \limsup_{n\to\infty} \|v_n\|^2 + \|v\|^2 - 2\langle v_n, v\rangle$$
$$2\|v\|^2 - 2\limsup_{n\to\infty} \langle v_n, v\rangle = 2\limsup_{n\to\infty} \langle v - v_n, v\rangle \leqslant 2\|v\| \limsup_{n\to\infty} \|v_n - v\|,$$

which gives that $v_n \to v$ strongly in $\mathbb{H}^s$. □

3. Proof of Theorem 1

In this section we prove Theorem 1. We observe that problem (P_λ) has a variational structure, indeed it is the Euler-Lagrange equation of the energy functional in (1.3). We note that J_λ is well defined and differentiable on $\mathbb{H}^s$ and for any $\varphi \in \mathbb{H}^s$,

$$(J_\lambda^{'}(u),\, \varphi) = \langle u,\, \varphi\rangle - \lambda \int_\Omega |u|^q \varphi\, dx - \int_\Omega |u|^p \varphi\, dx$$

Thus critical points of the functional J_λ are solutions to (P_λ).

We split the proof of Theorem 1 into several auxiliary lemmas.

Lemma 18. *Let Λ be defined by*

$$\Lambda = \sup\{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a solution }\}$$

Then $0 < \Lambda < \infty$.

Proof. Let λ be such that problem (P_λ) has a solution $\bar{u}_\lambda$. We consider the following problem

$$\left\{\begin{array}{rcll} (-\Delta)^s z &=& z^q & \text{in } \Omega, \\ z &>& 0 & \text{in } \Omega, \\ \mathcal{B}_s z &=& 0 & \text{in } \mathbb{R}^N \backslash \Omega\,. \end{array}\right. \tag{3.1}$$

To find a solution of problem (3.1), we consider the following minimization problem

$$\min\left\{\frac{1}{2}\|w\|^2 - \frac{\lambda}{q+1}\int_\Omega |w|^{q+1},\ w \in \mathbb{H}^s\right\},$$

and we denote by z the associated minimizer. By Lemma 16, we have that $z \geqslant 0$, and, by Proposition 14, it follows that $z > 0$ and it is unique. In particular, z is the desired solution of problem (3.1). Also, using Theorem 13 we have $z \in L^\infty(\Omega)$. Now if $\bar{z} = cz$ then $\bar{z}$ is a solution to

$$\left\{\begin{array}{rcll} (-\Delta)^s \bar{z} &=& \lambda \bar{z}^q & \text{in } \Omega, \\ \mathcal{B}_s \bar{z} &=& 0 & \text{in } \mathbb{R}^N \backslash \Omega\,, \end{array}\right. \tag{3.2}$$

with $\lambda = c^{1-q}$. By Lemma 16 $\bar{z} \leqslant \bar{u}_\lambda$. Let $\phi \in \mathbb{H}^s$, then using Picone's inequality we get

$$\begin{array}{rcl} \displaystyle\int\int_{\mathcal{D}_\Omega} \frac{\big(\phi(x)-\phi(y)\big)^2}{|x-y|^{N+2s}}\, dx\, dy &\geqslant& \displaystyle\int_\Omega \frac{\phi^2}{\bar{u}_\lambda}(-\Delta)^s \bar{u}_\lambda\, dx \\ &\geqslant& \displaystyle\int_\Omega \phi^2(\lambda \bar{u}_\lambda^{q-1} + \bar{u}_\lambda^{p-1})\, dx \\ &\geqslant& \displaystyle\int_\Omega \phi^2(\lambda \bar{z}_\lambda^{q-1} + \bar{z}_\lambda^{p-1})\, dx \\ &\geqslant& \displaystyle\int_\Omega \bar{z}^{p-1}\phi^2\, dx \\ &\geqslant& \displaystyle\lambda^{\frac{p-1}{1-q}} \int_\Omega z^{p-1}\phi^2\, dx \end{array}$$

Hence

$$\lambda^{\frac{p-1}{1-q}} \leqslant \inf_{\phi \in \mathbb{H}^s} \frac{\displaystyle\int\int_{\mathcal{D}_\Omega} \frac{\big(\phi(x)-\phi(y)\big)^2}{|x-y|^{N+2s}}\, dx\, dy}{\displaystyle\int_\Omega z^{p-1}\phi^2\, dx} = \Lambda^* \tag{3.3}$$

consequently $\Lambda \leqslant (\Lambda^*)^{\frac{1-q}{p-1}} < \infty$. □

We notice that if λ is small, using the fact that

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx \\ &\geqslant \frac{1}{2}\|u\|^2 - \lambda C_1\|u\|^{\frac{q+1}{2}} - C_2\|u\|^{\frac{p+1}{2}} \end{aligned}$$

for some positive constants C_1 and C_2 we get the existence of two solutions. The first solution is obtained by minimization and the second by the mountain pass theorem. This method is based on the geometry of the function $h(t) = \frac{1}{2}t^2 - \lambda C_1 t^{\frac{q+1}{2}} - C_2 t^{\frac{p+1}{2}}$, see [4] and [2]. We observe now that

(3.4) $$S = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a solution}\} \text{ is an interval.}$$

To prove this, we argue as follows:

- First, we show that if $\mu_1,\ \mu_2 \in S$ are such that $\mu_1 < \mu_2$ then for all $\mu \in (\mu_1, \mu_2)$ we have that the solution of (P_{μ_1}), that we denote by v_{μ_1}, and the solution of (P_{μ_2}), that we denote by v_{μ_2}, are respectively sub and super-solution to (P_μ);
- Then, by Lemma 16, we obtain that $v_{\mu_1} \leqslant v_{\mu_2}$;
- Finally, by Lemma 10, we get the existence of a solution v_μ to problem (P_μ) for $\mu \in (\mu_1, \mu_2)$, and then $\mu \in S$, which establishes (3.4).

Now we discuss the energy properties of the positive solutions.

Lemma 19. *If problem (P_λ) has a positive solution for $0 < \lambda < \Lambda$, then it has a minimal solution u_λ such that $J_\lambda(u_\lambda) < 0$. Moreover the family u_λ of minimal solutions is increasing with respect to λ.*

Proof. Suppose that (P_λ) has a solution v_λ for a given λ. Then there exists a sequence v_n such that $v_0 = \bar{z}$,

$$\left\{ \begin{array}{rcll} (-\Delta)^s v_n &=& \lambda v_{n-1}^q + v_{n-1}^p & \text{in } \Omega, \\ v_n &\geqslant& 0 & \text{in } \Omega, \\ \mathcal{B}_s v_n &=& 0 & \text{in } \mathbb{R}^N \backslash \Omega\,, \end{array} \right.$$

where $\bar{z}$ is as in the proof of Lemma 18. By Lemma 16, we have that $\bar{z} \leqslant \ldots \leqslant v_{n-1} \leqslant v_n \leqslant v_\lambda$ and then, by Proposition 14, it follows that $\bar{z} < v_n < v_\lambda$.

So, using v_n as a test function, we get $\|v_n\| \leqslant \|v_\lambda\|$. Hence there exists $u_\lambda \in \mathbb{H}^s$ such that $v_n \rightharpoonup u_\lambda$. Accordingly, since $(-\Delta)^s v_n \geqslant 0$, using Lemma 17, we conclude that $v_n \to u_\lambda$ strongly in $\mathbb{H}^s$ and $u_\lambda \leqslant v_\lambda$. This shows that u_λ is a minimal solution.

Then, by Lemma 16 and Proposition 14, we obtain the monotonicity of the family $\{u_\lambda,\ \ \lambda \in (0, \Lambda)\}$. Henceforth, given $\lambda \in (0, \Lambda)$, we use the notation u_λ for the minimal solution. Let us define $a(x) = \lambda q u_\lambda^{q-1} + p u_\lambda^{p-1}$ and let μ_1 be the first eigenvalue of the following the problem:

(3.5) $$\left\{ \begin{array}{rcll} (-\Delta)^s \phi - a(x)\phi &=& \mu_1 \phi & \text{in } \Omega, \\ \phi &>& 0 & \text{in } \Omega, \\ \mathcal{B}_s \phi &=& 0 & \text{in } \mathbb{R}^N \backslash \Omega\,. \end{array} \right.$$

We claim that

(3.6) $$\mu_1 \geqslant 0.$$

Indeed, let $\bar{z}$ be as in the proof of Lemma 18, then $\bar{z}$ is subsolution to (P_λ), the claim in (3.6) follows using the same argument as in the proof of Lemma 3.5 in [3].

Now we notice that (3.6) is equivalent to

(3.7) $$\|\phi\|^2 \geqslant \int_\Omega a(x)\phi^2 dx \quad \forall \phi \in \mathbb{H}^s.$$

Also, since u_λ is a solution to (P_λ), testing the equation against u_λ itself we find that

$$\|u_\lambda\|^2 = \lambda\|u_\lambda\|_{q+1}^{q+1} + \|u_\lambda\|_{p+1}^{p+1}.$$

Thus, by (3.7), taking $\phi = u_\lambda$ we get

$$\|u_\lambda\|^2 - \lambda q\|u_\lambda\|_{q+1}^{q+1} - p\|u_\lambda\|_{p+1}^{p+1} \geqslant 0.$$

By inserting these relations into (1.3), we obtain that $J_\lambda(u_\lambda) < 0$, as desired. □

We remark that Lemma 19 gives point (1) in Theorem 1, and point (2) is a direct consequence of Lemma 18. Thus, to complete the proof of Theorem 1, we can now focus on the proof of point (3). To this end, we have the following result:

Lemma 20. *Problem (P_λ) has at least one solution if $\lambda = \Lambda$.*

Proof. Let $\{\lambda_n\}$ be a sequence such that $\lambda_n \nearrow \Lambda$. We denote by v_n the minimal solution to problem (P_{λ_n}). Since $J_{\lambda_n}(v_n) < 0$, we have

$$\begin{aligned} 0 &> J_\lambda(v_n) - \frac{1}{p+1}J'_\lambda(v_n) \\ &\geqslant (\frac{1}{2} - \frac{1}{p+1})\|v_n\|^2 + \lambda(\frac{1}{p+1} - \frac{1}{q+1})\|v_n\|_{q+1}^{q+1} \\ &\geqslant (\frac{1}{2} - \frac{1}{p+1})\|v_n\|^2 - \lambda(\frac{1}{q+1} - \frac{1}{p+1})\|v_n\|^{q+1}. \end{aligned}$$

Then, it follows that $\{v_n\}$ is bounded in $\mathbb{H}^s$. Accordingly, we have that $v_n \rightharpoonup v^*$ in $\mathbb{H}^s$, for some $v^* \in \mathbb{H}^s$. From this and the fact that $(-\Delta)^s v_n \geqslant 0$, recalling Lemma 17 we conclude that $v_n \to v^*$ strongly in $\mathbb{H}^s$. As a consequence, v^* is a solution of (P_λ) for $\lambda = \Lambda$. □

Remark 21. If $p \leqslant 2_s^* - 1$ then using Theorem 13, we can easily see that $v^* \in L^\infty(\Omega)$, that is v^* is a regular extremal solution.

In view of Lemma 20, we obtain point (3) of Theorem 1. The proof of Theorem 1 is thus complete.

4. Proof of Theorem 2

In this section we prove the existence of a second positive solution to (P_λ). As the proof uses a mountain pass-type argument, we need to restrict the range of p, more precisely we ask $p < \frac{N+2s}{N-2s}$. The proof of Theorem 2 goes as follows. As in the local case, we can prove that the problem has a second positive solution for λ small. This follows using the mountain pass theorem. For this purpose it is essential to have a first solution which is a local minimum in $\mathbb{H}^s$. Let

$$f_\lambda(r) = \begin{cases} \lambda r^q + r^p, & \text{if } r \geqslant 0, \\ 0, & \text{if } r < 0 \end{cases}$$

and

$$F_\lambda(u) = \int_0^u f_\lambda(r)\,dr.$$

We define the functional $J_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F_\lambda(u)$. Critical points of J_λ correspond to solutions of (P_λ). Define the set

$$A = \{\lambda > 0 \ : \ J_\lambda \text{ has a local minimum} u_{0,\lambda}\}.$$

It is clear that if $\lambda \in A$ and w_λ is a minimum of J_λ in $\mathbb{H}^s$, then $v = 0$ is a local minimum of the functional

$$\hat{J}_\lambda(v) = \frac{1}{2}||v||^2 - \int_\Omega G_\lambda(v)dx, \tag{4.1}$$

where

$$G_\lambda(v) = \int_0^v g_\lambda(r)\,dr$$

and

$$g_\lambda(r) = \begin{cases} \lambda\left((u_{0,\lambda}(x)+r)^q - u_{0,\lambda}(x)^q\right) + (u_{0,\lambda}(x)+r)^p - u_{0,\lambda}(x)^p, & \text{if } r \geqslant 0, \\ 0, & \text{if } r < 0 \end{cases}.$$

We can see that $\hat{J}_\lambda$ possesses the mountain pass geometry. Thus, let $v_0 \in \mathbb{H}^s$ be such that $\hat{J}_\lambda(v_0) < 0$ and define

$$\Gamma = \{\gamma : [0,1] \to \mathbb{H}^s\ \gamma(0) = 0,\, \gamma(1) = v_0\} \text{ and } c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi_\lambda\left(\gamma(t)\right).$$

We have that $c \geqslant 0$ and since $p < 2_s^* - 1$, then $\hat{J}_\lambda$ satisfies the Palais-Smale condition. If $c > 0$, then using the Ambrosetti-Rabinowitz theorem we reach a non trivial critical point. If $c = 0$, then we use the Ghoussoub-Preiss Theorem, see [16]. As a consequence if we start with a local minimum of the functional $\hat{J}_\lambda$, then we obtain a second critical point of $\hat{J}_\lambda$, and hence a second solution to (P_λ).

Next, to show that problem (P_λ) has a second solution for all $\lambda \in (0,\Lambda)$, we follow some arguments similar to those developed by Alama in [1] taking into consideration the nonlocal nature of the operator.

We prove first, using a variational formulation of the Perron's method, that the functional has a constrained minimum and then that this minimum is a local minimum in the whole $\mathbb{H}^s$. To this end, we use a truncation technique and some energy estimates.

Fix $\lambda_0 \in (0,\Lambda)$ and let $\lambda_0 < \bar{\lambda} < \Lambda$. Define $u_0,\ \bar{u}$ to be the minimal solutions to problem (P_λ) with $\lambda = \lambda_0$ and $\lambda = \bar{\lambda}$ respectively. By definition we obtain that $u_0 < \bar{u}$. Let us define

$$M = \{u \in \mathbb{H}^s(\Omega) : 0 \leqslant u \leqslant \bar{u}\}.$$

It is clear that $u_0 \in M$ and that M is a convex closed subset of $\mathbb{H}^s$. Since J_{λ_0} is bounded from below in M and lower semi-continuous, then we get the existence of $\vartheta \in M$ such that

$$J_{\lambda_0}(\vartheta) = \inf_{u\in M} J_{\lambda_0}(u).$$

Let v be the unique solution to

$$\begin{cases} (-\Delta)^s u &= \lambda_0 u^q & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ \mathcal{B}_s u &= 0 & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases}$$

We have that $J_{\lambda_0}(v) < 0$, and then $\vartheta \neq 0$. As in Theorem 2.4 in [27], page 17, we conclude that ϑ is a solution to problem (P_λ).

If $\vartheta \neq u_{\lambda_0}$, then the proof of Theorem 2 is complete. Accordingly, we can assume that $\vartheta = u_0$. We show that

$$\vartheta \text{ is a local minimum of } J_{\lambda_0}. \tag{4.2}$$

For this, we argue by contradiction, and we assume that ϑ is not a local minimum of J_{λ_0}. Then there exists a sequence $\{v_n\} \subset \mathbb{H}^s$ such that $\|v_n - \vartheta\|_{\mathbb{H}^s} \to 0$ and

$$J_{\lambda_0}(v_n) < J_{\lambda_0}(\vartheta). \tag{4.3}$$

We define $w_n = (v_n - \bar{u})_+$ and $u_n = \max\{0, \min\{v_n, \bar{u}\}\}$. It is clear that $u_n \in M$ and

$$u_n(x) = \begin{cases} 0 & \text{if } v_n(x) \leqslant 0, \\ v_n(x) & \text{if } 0 \leqslant v_n(x) \leqslant \bar{u}(x), \\ \bar{u}(x) & \text{if } \bar{u}(x) \leqslant u_n(x). \end{cases}$$

Thus $u_n = v_n^- + w_n$. Let $T_n = \{x \in \Omega :\ u_n(x) = v_n(x)\}$ and $S_n = \text{supp}\ \ w_n \cap \Omega$. Notice that supp $\ v_n^+ \cap \Omega = T_n \cup S_n$. We claim that

$$|S_n| \to 0 \text{ as } n \to \infty. \tag{4.4}$$

To this end, let $\varepsilon > 0$,

$$\begin{aligned} & E_n = \{x \in \Omega :\ v_n(x) \geqslant \bar{u}(x) > \vartheta(x) + \delta\} \\ \text{and} \qquad & F_n = \{x \in \Omega :\ v_n(x) \geqslant \bar{u}(x) \text{ and } \bar{u}(x) \leqslant \vartheta(x) + \delta\}, \end{aligned}$$

where δ is to be suitably chosen. Since

$$\begin{aligned} 0 &= |\{x \in \Omega :\ \bar{u}(x) < \vartheta(x)\}| = \left| \bigcap_{j=1}^{\infty} \left\{ x \in \Omega :\ \bar{u}(x) \leqslant \vartheta(x) + \frac{1}{j} \right\} \right| \\ &= \lim_{j\to\infty} \left| \left\{ x \in \Omega :\ \bar{u}(x) \leqslant \vartheta(x) + \frac{1}{j} \right\} \right|, \end{aligned}$$

then we get the existence of a suitable $\delta_0 = \frac{1}{j_0}$ such that if $\delta < \delta_0$, then

$$|\{x \in \Omega :\ \bar{u}(x) \leqslant \vartheta(x) + \delta\}| \leqslant \frac{\varepsilon}{2}.$$

Thus $|F_n| \leqslant \frac{\varepsilon}{2}$. Since $||u_n - v_0||_{L^2(\Omega)} \to 0$ as $n \to \infty$, we get that for $\eta = \frac{\delta^2 \varepsilon}{2}$, if $n \geqslant n_0$, we have that

$$\frac{\delta^2 \varepsilon}{2} \geqslant \int_\Omega |v_n - \vartheta|^2 dx \geqslant \int_{E_n} |v_n - \vartheta|^2 dx \geqslant \delta^2 |E_n|.$$

Hence $|E_n| \leqslant \frac{\varepsilon}{2}$. Since $S_n \subset F_n \cup E_n$, we conclude that $|S_n| \leqslant \varepsilon$ for $n \leqslant n_0$ and then the claim in (4.4) follows.

Now we define

$$H(u) = \frac{\lambda_0}{q+1} u_+^{q+1} + \frac{u_+^{p+1}}{p+1}.$$

Using the fact that

$$\|v_n\|^2 \geqslant \|v_n^+\|^2 + \|v_n^-\|^2,$$

we obtain that

$$\begin{aligned} J_{\lambda_0}(v_n) &= \frac{1}{2}\|v_n\|^2 - \int\limits_\Omega H(v_n) dx \\ &\geqslant \frac{1}{2}\|v_n^+\|^2 - \int\limits_\Omega H(v_n) dx + \frac{1}{2}\|v_n^-\|^2 \\ &= \frac{1}{2}\|v_n^+\|^2 - \int_{T_n} H(u_n) dx - \int_{S_n} H(v_n) dx + \frac{1}{2}\|v_n^-\|^2 \\ &= \frac{1}{2}\|v_n^+\|^2 - \int_{T_n} H(u_n) dx - \int_{S_n} H(w_n + \bar{u}) dx + \frac{1}{2}\|v_n^-\|^2 \\ &= J_{\lambda_0}(u_n) + \frac{1}{2}\Big(\|v_n^+\|^2 - \|u_n\|^2\Big) + \frac{1}{2}\|v_n^-\|^2 - \int_{S_n} \Big(H(w_n + \bar{u}) - H(\bar{u})\Big) dx, \end{aligned}$$

where we have used the fact that

$$\int\limits_\Omega H(u_n) dx = \int_{T_n} H(u_n) dx + \int_{S_n} H(\bar{u}) dx.$$

Also, since $v_n^+ = u_n + w_n$, then

$$\frac{1}{2}\Big(\|v_n^+\|^2 - \|u_n\|^2\Big) = \frac{1}{2}\|w_n\|^2 + \langle u_n, w_n \rangle.$$

Using that

$$\{w_n \neq 0\} = \{u_n = \bar{u}\},$$

we see that

$$\langle u_n, w_n\rangle \geqslant \int_\Omega (-\Delta)^s \bar{u} w_n \geqslant \lambda \int_{S_n} \bar{u}^q w_n + \int_{S_n} \bar{u}^p w_n.$$

Therefore, recalling that $\bar{u}$ is a supersolution to problem (P_λ) for $\lambda = \lambda_0$, we conclude that

$$\begin{aligned} J_{\lambda_0}(v_n) &\geqslant J_{\lambda_0}(\vartheta) + \frac{1}{2}\|w_n\|^2_{\mathbb{H}^s} + \frac{1}{2}\|(v_n)_-\|^2_{\mathbb{H}^s} \\ &- \int_{S_n} \Big\{H(w_n+\bar{u}) - H(\bar{u}) - \lambda_0 \bar{u}^q w_n - \bar{u}^p w_n\Big\}dx. \end{aligned}$$

Taking into account that

$$0 \leqslant \frac{1}{q+1}(w_n+\bar{u})^{q+1} - \frac{1}{q+1}\bar{u}^{q+1} - \bar{u}^q w_n \leqslant \frac{q}{2}\frac{w_n^2}{\bar{u}^{1-q}},$$

and using the Picone inequality in Theorem 15, we find that

$$\bar{\lambda}\int_\Omega \frac{w_n^2}{\bar{u}^{1-q}}dx \leqslant \int_\Omega \frac{w_n^2}{\bar{u}}(-\Delta)^s\bar{u} \leqslant \|w_n\|^2_{\mathbb{H}^s}.$$

Then, we obtain that

$$\lambda_0 \int_\Omega \frac{1}{q+1}(w_n+\bar{u})^{q+1} - \frac{1}{q+1}\bar{u}^{q+1} - \bar{u}^q w_n \leqslant \frac{q}{2}\frac{w_n^2}{\bar{u}^{1-q}} \leqslant \frac{q}{2}\|w_n\|^2_{\mathbb{H}^s}.$$

Moreover, since $2 \leqslant p+1$,

$$0 \leqslant \frac{1}{p+1}(w_n+\bar{u})^{p+1} - \frac{1}{p+1}\bar{u}^{p+1} - \bar{u}^r w_n \leqslant \frac{p}{2}w_n^2(w_n+\bar{u})^{p-1} \leqslant C(\bar{u}^{p-1}w_n^2 + w_n^{p+1}).$$

Hence, using the Sobolev inequality and the fact that $|S_n| \to 0$ as $n \to \infty$, we reach that

$$\int_\Omega \Big\{\frac{1}{p+1}(w_n+\bar{u})^{p+1} - \frac{1}{p+1}\bar{u}^{p+1} - \bar{u}^p w_n\Big\}dx \leqslant o(1)\|w_n\|^2_{\mathbb{H}^s}.$$

Hence

$$\begin{aligned} J_{\lambda_0}(v_n) &\geqslant J_{\lambda_0}(\vartheta) + \frac{1}{2}\|w_n\|^2_{\mathbb{H}^s}(1-q-o(1)) + \frac{1}{2}\|v_n^-\|^2_{\mathbb{H}^s} \\ &\geqslant J_{\lambda_0}(\vartheta) + \frac{1}{2}\|w_n\|^2_{\mathbb{H}^s}(1-q-o(1)) + o(1). \end{aligned}$$

So we get that

$$0 > J_{\lambda_0}(v_n) - J_{\lambda_0}(\vartheta) \geqslant \frac{1}{2}\|w_n\|^2_{\mathbb{H}^s}(1-q-o(1)) + \frac{1}{2}\|v_n^-\|^2_{\mathbb{H}^s}.$$

Since $q<1$, we conclude that $w_n = v_n^- = 0$ for n large, so $v_n \in M$ and then

$$J_{\lambda_0}(v_n) \geqslant J_{\lambda_0}(\vartheta),$$

which is in contradiction with (4.3).

This completes the proof of (4.2). From this, we have that ϑ is a local minimum for J_{λ_0}, and $\hat{J}_{\lambda_0}$ has $u=0$ as a local minimum and then $\hat{J}_{\lambda_0}$ has a nontrivial critical point $\hat{u}$. As a consequence, $u = \vartheta + \hat{u}$ is a solution, different from ϑ, of problem (P_λ). This concludes the proof of Theorem 2.

Remark 22. If we consider the odd symmetric version of problem (P_λ), namely,

(4.5) $$\begin{cases} (-\Delta)^s u &= \lambda |u|^{q-1}u + |u|^{p-1}u \quad \text{in } \Omega, \\ \mathcal{B}_s u &= 0 \qquad\qquad\qquad\qquad \text{in } \mathbb{R}^N \backslash \Omega\,, \end{cases}$$

the associated functional

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\|u\|_{q+1}^{q+1} - \frac{1}{p}\|u\|_{p+1}^{p+1}$$

is even. Then, for $p < \frac{N+2s}{N-2s}$, by using the Lusternik-Schnirelman min-max argument, it is possible to prove that problem (4.5) has infinitely many solutions with negative energy, see [3] and [6], and following closely the arguments in [4], [3] the same holds for solutions with positive energy.

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