Weierstraß-Institut

für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Nonlocal phase transitions: Rigidity results and anisotropic geometry

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submitted: November 14, 2016

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No. 2334
Berlin 2016

2010 *Mathematics Subject Classification.* 35R11, 60G22, 82B26.

Key words and phrases. Nonlocal phase transitions, rigidity results, sliding methods.

This work has been supported by ERC grant 277749 “EPSILON Elliptic PDE’s and Symmetry of Interfaces and Layers for Odd Nonlinearities”.

Edited by
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ABSTRACT. We provide a series of rigidity results for a nonlocal phase transition equation. The prototype equation that we consider is of the form

$$(-\Delta)^{s/2}u = u - u^3,$$

with $s \in (0,1)$. More generally, we can take into account equations like

$$Lu = f(u),$$

where f is a bistable nonlinearity and L is an integro-differential operator, possibly of anisotropic type.

The results that we obtain are an improvement of flatness theorem and a series of theorems concerning the one-dimensional symmetry for monotone and minimal solutions, in the research line dictated by a classical conjecture of E. De Giorgi in [10].

Here, we collect a series of pivotal results, of geometric type, which are exploited in the proofs of the main results in [12].

1. INTRODUCTION AND MAIN RESULTS

In phase coexistence models, a classical question, which was posed in [10], is whether or not “typical solutions” possess one-dimensional symmetry. In the models driven by semilinear partial differential equations, this type of problems has a long history, see e.g. [17, 2, 1, 18, 9] and the references therein. Related problems arise in the theory of quasilinear equations, see e.g. [8, 13, 15], and find applications in dynamical systems, see [16]. We refer to [14] for a review on this topic.

Recently, similar questions have been posed for a phase transition model in which the long-range particle interaction is described by a nonlocal operator of fractional type, see [6, 21, 5, 3, 4]. Similar models describe also the atom dislocation in some crystals, see e.g. Section 2 in [11], and some phenomena in mathematical biology, see e.g. [7]. The goal of this paper is to present a series of rigidity and symmetry results for semilinear problems driven by nonlocal operators. The results are so general that they can be applied also in a non-isotropic medium (but, as far as we know, they are also new in the isotropic case).

More precisely, we consider a nonlocal Allen-Cahn equation of the type

$$Lu = f(u) \quad \text{in } \mathbb{R}^n,$$

where L is an operator of the form

$$Lu(x) := \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big) \frac{\mu\big(y/|y|\big)}{|y|^{n+s}}\, dy, \qquad x \in \mathbb{R}^n,$$

with $s \in (0,1)$. The typical example of operator comprised by our setting is the fractional Laplacian (in this case $L := (-\Delta)^{s/2}$). The basic nonlinearity f that we take into account is when f is “bistable”, i.e. it is minus the derivative of a double-well potential (e.g., $f(u) = u - u^3$). We assume that the measure μ (which is often called in jargon the “spectral measure”) satisfies

$$\mu(z) = \mu(-z) \quad \text{and} \quad \lambda \leqslant \mu(z) \leqslant \Lambda \quad \text{for all } z \in S^{n-1},$$

for some $\Lambda \geqslant \lambda > 0$. Given a bounded $\psi \in C^2(\mathbb{R})$ we define

$$A\psi(z) := \int_{-\infty}^{+\infty} \frac{\psi(z) - \psi(z+\zeta)}{|\zeta|^{1+s}}\, d\zeta, \qquad z \in \mathbb{R}. \tag{1.1}$$

Roughly speaking, the operator A plays a role of the one-dimensional fractional Laplacian. In order to take into account the possible anisotropy of the operator L, we need to scale A appropriately in any fixed direction. To this aim, for ψ as above, $\omega \in S^{n-1}$, and $h > 0$ we define, for $x \in \mathbb{R}^n$,

$$\bar\psi_{\omega,h}(x) := \psi\left(\omega \cdot \frac{x}{h}\right).$$

We set $h_L(\omega) := h$ where $h > 0$ satisfies

$$L\bar{\psi}_{\omega,h}(x) = A\psi\left(\omega \cdot \frac{x}{h}\right) \text{ for all } \psi \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

We also define

$$\mathscr{C} = \mathscr{C}_L := \bigcap_{\omega \in S^{n-1}} \left\{x \in \mathbb{R}^n \,:\, x \cdot \omega \leqslant h_L(\omega)\right\} \tag{1.2}$$

and assume that

$$\partial\mathscr{C}_L \text{ is } C^{1,1} \text{ and strictly convex.}$$

More quantitatively, we assume that there exist $\rho' > \rho > 0$ such that

$$\text{the curvatures of } \partial\mathscr{C}_L \text{ are bounded above by } \frac{1}{\rho} \text{ and below by } \frac{1}{\rho'}. \tag{H1}$$

Concerning the nonlinearity f, we assume that $f \in C^1\big([-1,1]\big)$ and, for some $\kappa > 0$ and $c_\kappa > 0$,

$$f(-1) = f(1) = 0 \quad \text{and} \quad f'(t) < -c_\kappa \quad \text{for } t \in [-1,-1+\kappa] \cup [1-\kappa,1]. \tag{H2}$$

Moreover, recalling the setting in (1.1), we assume that

$$\text{there exists } \phi_0 \text{ satisfying } \begin{cases} A\phi_0 = f(\phi_0) & \text{in } \mathbb{R}, \\ \phi_0' > 0 & \text{in } \mathbb{R}, \\ \phi_0(0) = 0, \\ \lim\limits_{x \to \pm\infty} \phi_0 = \pm 1. \end{cases} \tag{H3}$$

The main result obtained in [12] is the following improvement of flatness:

Theorem 1.1. *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*. Then there exist universal constants $\alpha_0 \in (0, s/2)$, $p_0 \in (2,\infty)$ and $a_0 \in (0,1/4)$ such that the following statement holds.*

Let $a \in (0,a_0)$ and $\varepsilon \in (0, a^{p_0})$. Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of

$$Lu = \varepsilon^{-s} f(u) \quad \text{in} \quad B_{j_a},$$

with

$$j_a := \left\lfloor \frac{\log a}{\log(2^{-\alpha_0})} \right\rfloor.$$

Assume that $0 \in \{-1+\kappa \leqslant u \leqslant 1-\kappa\}$ and that

$$\{\omega_j \cdot x \leqslant -a2^{j(1+\alpha_0)}\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \{\omega_j \cdot x \leqslant a2^{j(1+\alpha_0)}\} \quad \text{in } B_{2^j},$$

for any $j = \{0,1,2,\dots,j_a\}$ and for some $\omega_j \in S^{n-1}$.

Then,

$$\left\{\omega \cdot x \leqslant -\frac{a}{2^{1+\alpha_0}}\right\} \subset \{u \leqslant -1+\kappa\} \subset \{u \leqslant 1-\kappa\} \subset \left\{\omega \cdot x \leqslant \frac{a}{2^{1+\alpha_0}}\right\} \quad \text{in } B_{1/2},$$

for some $\omega \in S^{n-1}$.

Theorem 1.1 says that if the level sets of the solution are C^{1,α_0}-flat from infinity up to B_1, then they are also C^{1,α_0}-flat up to $B_{1/2}$, and so one can dilate the picture once again and repeat the argument at any small scale towards the origin (as a matter of fact, suitable scaled iterations of Theorem 1.1 are given in Corollaries 7.1 and 7.2 of [12]). An important consequence of Theorem 1.1 is related to the one-dimensional symmetry properties of the solutions. For this, we say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is 1D if there exist $\bar{u} : \mathbb{R} \to \mathbb{R}$ and $\bar{\omega} \in S^{n-1}$ such that $u(x) = \bar{u}(\bar{\omega} \cdot x)$ for any $x \in \mathbb{R}^n$.

Then, we have the following consequences of Theorem 1.1:

Theorem 1.2 (One-dimensional symmetry for asymptotically flat solutions)**.** *Assume that L satisfies* (H1) *and that f satisfies* (H2) *and* (H3)*.*

Let u be a solution of $Lu = f(u)$ in $\mathbb{R}^n$.

Assume that there exists $a : (1, \infty) \to (0, 1]$ such that $a(R) \searrow 0$ as $R \nearrow +\infty$ and such that, for all $R > 0$, we have that

$$\{\omega \cdot x \leqslant -a(R)R\} \subset \{u \leqslant -1 + \kappa\} \subset \{u \leqslant 1 - \kappa\} \subset \{\omega \cdot x \leqslant a(R)R\} \quad \textit{in } B_R,$$

for some $\omega \in S^{n-1}$, which may depend on R. Then, u is 1D.

We stress that all these results, as far as we know, are new even for the equation $(-\Delta)^{s/2}u = u - u^3$, with $s \in (0, 1)$, which is a particular case of our setting.

As a matter of fact, we can consider the concrete case of minimal solutions of the nonlocal Allen-Cahn equation $(-\Delta)^{s/2}u = u - u^3$, with $s \in (0, 1)$. We remark that the energy functional related to such equation is

$$\mathscr{E}(u, \Omega) := \mathscr{E}^{\rm Dir}(u, \Omega) + \int_\Omega (1 - u^2(x))^2 \, dx,$$

where

$$\mathscr{E}^{\rm Dir}(u, \Omega) := C_{n,s} \iint_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+s}} \, dx \, dy, \tag{1.3}$$

for a suitable normalization constant $C_{n,s} > 0$. In this setting, we say that a solution u of $(-\Delta)^{s/2}u = u - u^3$ is a *minimizer* of $\mathscr{E}$ in $\mathbb{R}^n$ if

$$\mathscr{E}(u, B) \leqslant \mathscr{E}(u + \varphi, B),$$

for any ball $B \subset \mathbb{R}^n$ and any $\varphi \in C_0^\infty(B)$. In this setting, the following results hold true:

Theorem 1.3 (One-dimensional symmetry in the plane)**.** *Let u be a minimizer of $\mathscr{E}$ in $\mathbb{R}^2$. Then, u is 1D.*

Theorem 1.4 (One-dimensional symmetry for monotone solutions in $\mathbb{R}^3$)**.** *Let $n \leqslant 3$ and u be a solution of $(-\Delta)^{s/2}u = u - u^3$ in $\mathbb{R}^n$.*

Suppose that

$$\frac{\partial u}{\partial x_n}(x) > 0 \quad \textit{for any } x \in \mathbb{R}^n \qquad \textit{and} \qquad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

Then, u is 1D.

Theorem 1.5 (One-dimensional symmetry when s is close to 1)**.** *Let $n \leqslant 7$. Then, there exists $\eta_n \in (0, 1)$ such that for any $s \in [1 - \eta_n, \, 1)$ the following statement holds true.*

Let u be minimizer of $\mathscr{E}$ in $\mathbb{R}^n$. Then, u is 1D.

Theorem 1.6 (One-dimensional symmetry for monotone solutions in $\mathbb{R}^8$ when s is close to 1)**.** *Let $n \leqslant 8$. Then, there exists $\eta_n \in (0, 1)$ such that for any $s \in [1 - \eta_n, \, 1)$ the following statement holds true.*

Let u be a solution of $(-\Delta)^{s/2}u = u - u^3$ in $\mathbb{R}^n$.

Suppose that

$$\frac{\partial u}{\partial x_n}(x) > 0 \quad \textit{for any } x \in \mathbb{R}^n \qquad \textit{and} \qquad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

Then, u is 1D.

The full proofs of the results mentioned above are given in [12]. Here, we present the details of a series of pivotal results of geometric type that will be exploited in [12].

Related results on symmetry problems with possible applications to nonlocal phase transitions have been recently announced in [19] and obtained in [20] (we stress that the range of fractional parameter dealt with in [19, 20] is complementary to the one of this paper and [12]).

2. Some useful facts on the distance function

We collect here some ancillary results of elementary nature from the theory of convex sets and anisotropic distance functions. For the convenience of the reader, we give full details of the results we need, that are stated in a convenient form for their use in the forthcoming paper [12].

For this, we consider a continuous and degree 1 positively homogeneous function $h:\mathbb{R}^n\to[0,+\infty)$ and the convex set in (1.2). We assume that the boundary of $\mathscr{C}$ is of class C^1. The set $\mathscr{C}$ in our setting plays the role of an anisotropic ball, and so, for any $r>0$, we set

$$\mathscr{C}_r(y):=y+r\mathscr{C}. \tag{2.1}$$

This anisotropic ball induces naturally a norm, defined, for any $p\in\mathbb{R}^n$, by the following formula:

$$\|p\|_{\mathscr{C}}:=\frac{1}{\sup\{\tau>0 \text{ s.t. } \tau p\in\mathscr{C}\}}. \tag{2.2}$$

We observe that, in view of (1.2), (2.1), and (2.2), for any $R>0$ and $z_0\in\mathbb{R}^n$,

$$\mathscr{C}_R(z_0)=\big\{x\in\mathbb{R}^n \text{ s.t. } \|x-z_0\|_{\mathscr{C}}\leqslant R\big\}. \tag{2.3}$$

Also, we have the following elementary inequality of Cauchy-Schwarz type:

Lemma 2.1. *For any $x,y\in\mathbb{R}^n$, we have that $x\cdot y\leqslant h(y)\,\|x\|_{\mathscr{C}}$.*

Proof. If either $x=0$ or $y=0$ we are done, so we can suppose that $y\neq0$ and $x\neq0$. We set $\omega:=\frac{y}{|y|}\in S^{n-1}$ and $\eta:=\frac{x}{\|x\|_{\mathscr{C}}}$. Then $\eta\in\mathscr{C}$, thus by (1.2)

$$\frac{x}{\|x\|_{\mathscr{C}}}\cdot y=\eta\cdot\omega\,|y|\leqslant h(\omega)\,|y|=h\left(\frac{y}{|y|}\right)\,|y|=h(y),$$

which gives the desired result. ☐

Now we show that, in the terminology of convex geometry, the function h is the "support function" of the convex body $\mathscr{C}$.

Lemma 2.2. *For any $\omega\in S^{n-1}$,*

$$h(\omega)=\sup_{x\in\mathscr{C}}x\cdot\omega. \tag{2.4}$$

Proof. From (1.2), we have that, for any $x\in\mathscr{C}$ and any $\omega\in S^{n-1}$, $x\cdot\omega\leqslant h(\omega)$, and so

$$\sup_{x\in\mathscr{C}}x\cdot\omega\leqslant h(\omega), \tag{2.5}$$

for any $\omega\in S^{n-1}$. In particular, this implies that $\mathscr{C}$ is bounded in any direction. Therefore, to check the opposite inequality to the one in (2.5), and thus to complete the proof of the desired result, we can fix $\omega\in S^{n-1}$ and slide a hyperplane with normal direction ω till it touches $\mathscr{C}$ at some point $P\in\partial\mathscr{C}$. That is, we have that for any $x\in\mathscr{C}$ it holds that $\omega\cdot(x-P)\leqslant0$ and so

$$\sup_{x\in\mathscr{C}}\omega\cdot x=\omega\cdot P. \tag{2.6}$$

Also, since $P \in \partial\mathscr{C}$, we deduce from (1.2) that there exists $\varpi \in S^{n-1}$ for which

$$\varpi \cdot P = h(\varpi). \tag{2.7}$$

Notice that $\{\varpi \cdot (x - P) = 0\}$ is a supporting hyperplane for $\mathscr{C}$, since, for any $x \in \mathscr{C}$,

$$\varpi \cdot (x - P) = \varpi \cdot x - h(\varpi) \leqslant 0,$$

thanks to (1.2).

Since $\partial\mathscr{C}$ has been assumed to be a C^1 manifold, the two supporting hyperplanes at P, namely $\{\omega \cdot (x - P) = 0\}$ and $\{\varpi \cdot (x - P) = 0\}$, must coincide, and so $\omega = \varpi$.

As a consequence of this, recalling (2.6) and (2.7), we obtain that

$$\sup_{x \in \mathscr{C}} \omega \cdot x = \omega \cdot P = \varpi \cdot P = h(\varpi) = h(\omega),$$

as desired. □

As a counterpart of Lemma 2.1, we also have

Lemma 2.3. *Let $z_0 \in \mathbb{R}^n$, $R > 0$ and $z \in \partial\mathscr{C}_R(z_0)$. Let $\omega_0 \in S^{n-1}$ be the inner normal of $\partial\mathscr{C}_R(z_0)$ at the point z. Then*

$$\omega_0 \cdot (z_0 - z) = R\, h(\omega_0).$$

Proof. Let

$$\zeta := \frac{z - z_0}{R}.$$

By Lemma 2.3, we know that

$$\zeta \in \partial\mathscr{C}. \tag{2.8}$$

Also, since $\mathscr{C}_R(z_0)$ is convex, we know that $\mathscr{C}_R(z_0) \subset \{x \in \mathbb{R}^n \text{ s.t. } \omega_0 \cdot (x - z) \geqslant 0\}$ and so

$$\mathscr{C} \subset \{y \in \mathbb{R}^n \text{ s.t. } \omega_0 \cdot (y - \zeta) \geqslant 0\}.$$

Hence, by (2.4),

$$h(\omega_0) = h(-\omega_0) = \sup_{y \in \mathscr{C}} (-y \cdot \omega_0) \leqslant -\zeta \cdot \omega_0.$$

On the other hand, by Lemma 2.1,

$$-\zeta \cdot \omega_0 \leqslant h(\omega_0)\, \| - \zeta\|_{\mathscr{C}} = h(\omega_0),$$

and so

$$h(\omega_0) = -\zeta \cdot \omega_0 = \frac{(z_0 - z) \cdot \omega_0}{R},$$

as desired. □

Given a nonempty, closed and convex set $K \subset \mathbb{R}^n$, we define the anisotropic signed distance function from K as

$$\begin{aligned} d_K(x) := \inf \big\{ \ell(x) \,:\, \ell(x) = \omega \cdot x + c, \quad h_L(\omega) = 1, \\ c \in \mathbb{R} \quad \text{and} \quad \ell \geqslant 0 \text{ in all of } K \big\}. \end{aligned} \tag{2.9}$$

Notice that d_K is a concave function, since it is the infimum of affine functions. Also, we have that d_K is a Lipschitz function, with Lipschitz constant 1 with respect to the anisotropic norm, as stated in the following result:

Lemma 2.4. *For any $p, q \in \mathbb{R}^n$,*

$$|d_K(p) - d_K(q)| \leqslant \|p - q\|_{\mathscr{C}}.$$

Proof. Up to exchanging p and q, we suppose that $d_K(p) \geqslant d_K(q)$. Fixed $\delta > 0$, we let $\ell_\delta(x) = \omega_\delta \cdot x + c_\delta$ be such that $h(\omega_\delta) = 1$, $c_\delta \in \mathbb{R}$, $\ell_\delta(x) \geqslant 0$ for any $x \in K$, and with $d_K(q) \geqslant \ell_\delta(q) - \delta$. Then, we have that $d_K(p) \leqslant \ell_\delta(p)$ and so, by Lemma 2.1,

$$\begin{aligned}|d_K(p) - d_K(q)| &= d_K(p) - d_K(q) \leqslant \ell_\delta(p) - \ell_\delta(q) + \delta \\ &= \omega_\delta \cdot (p-q) + \delta \leqslant h(\omega_\delta)\,\|p-q\|_{\mathscr{C}} + \delta = \|p-q\|_{\mathscr{C}} + \delta.\end{aligned}$$

Hence, taking δ arbitrarily close to 0 we obtain the desired result. ☐

It is also useful to observe that the minimum in (2.9) is attained, namely:

Lemma 2.5. *For any $p \in \mathbb{R}^n$ there exists an affine function ℓ_p, of the form $\ell_p(x) = \omega_p \cdot x + c_p$, with $h(\omega_p) = 1$, $c_p \in \mathbb{R}$, such that $\ell_p \geqslant 0$ in K and $d_K(p) = \ell(p)$.*

Moreover, if $t_0 \in \mathbb{R}$ and $z_0 \in \{d_K > t_0\}$ are such that $p \in \partial\mathscr{C}_R(z_0) \cap \{d_K = t_0\}$, with $\mathscr{C}_R(z_0) \subset \{d_K \geqslant t_0\}$, and ω_0 is the interior normal of $\mathscr{C}_R(z_0)$ at p, we have that

$$\omega_p = \frac{\omega_0}{h(\omega_0)} \tag{2.10}$$

$$\text{and} \qquad c_p = t_0 - \frac{\omega_0}{h(\omega_0)} \cdot p. \tag{2.11}$$

Proof. The existence of the optimal affine function ℓ_p follows from the direct methods of the calculus of variations, so we focus on the proof of the second claim. We have that, for any $x \in \mathscr{C}_R(z_0)$,

$$\omega_p \cdot p + c_p = d_K(p) = t_0 \leqslant d_K(x) \leqslant \omega_p \cdot x + c_p,$$

that is

$$\min_{x \in \mathscr{C}_R(z_0)} \omega_p \cdot x = \omega_p \cdot p.$$

Hence, by Lagrange multipliers, the gradient of the map $\omega_p \cdot x$ is parallel to (and in the same direction of) ω_0, that is

$$\omega_p = c\omega_0, \tag{2.12}$$

for some $c \geqslant 0$. Hence, since h is homogeneous,

$$1 = h(\omega_p) = c\, h(\omega_0).$$

This gives that $c = \frac{1}{h(\omega_0)}$, which, combined with (2.12), proves (2.10). Then, we write $t_0 = d_K(p) = \omega_p \cdot p + c_p$ and we obtain (2.11). ☐

In case of tangent anisotropic spheres to level sets of the anisotropic distance function, a useful comparison occurs with respect to Euclidean hyperplanes, as stated in the following result:

Lemma 2.6. *Let K be convex, $z_0 \in \{d_K > t_0\}$, $t_0 \in \mathbb{R}$. Suppose that $\mathscr{C}_R(z_0) \subset \{d_K \geqslant t_0\}$ and let $z \in \partial\mathscr{C}_R(z_0) \cap \{d_K = t_0\}$.*

Let ω_0 be the interior normal of $\mathscr{C}_R(z_0)$ at z and $\{d_K \geqslant t_0\} \subset \{x \in \mathbb{R}^n \text{ s.t. } \omega_0 \cdot (x-z) \geqslant 0\}$. Then, for any $x \in \mathbb{R}^n$ it holds that

$$d_K(x) \leqslant \frac{\omega_0}{h(\omega_0)} \cdot (x-z) + t_0.$$

Proof. We let

$$\tilde{d}(x) := \frac{\omega_0}{h(\omega_0)} \cdot (x-z) + t_0. \tag{2.13}$$

We claim that

$$\tilde{d}(x) \geqslant 0 \text{ for any } x \in K. \tag{2.14}$$

For this, we use Lemma 2.5 (with $p = z$, according to which the affine function

$$\ell_z(x) := \omega_z \cdot x + c_z,$$

with $\omega_z = \frac{\omega_0}{h(\omega_0)}$ and $c_p = t_0 - \frac{\omega_0}{h(\omega_0)} \cdot z$ satisfies $\ell_z(z) = d_K(z)$ and $\ell_z(x) \geqslant 0$ for any $x \in K$. In particular, for any $x \in K$, we have that $\tilde{d}(x) = \ell_z(x) \geqslant 0$, which proves (2.14).

In addition, from the homogeneity of h we have that

$$h\left(\frac{\omega_0}{h(\omega_0)}\right) = \frac{h(\omega_0)}{h(\omega_0)} = 1.$$

Using this and (2.14), we obtain the desired result from (2.9). ☐

Now we show that the function d_K, as defined in (2.9), coincides with the signed distance from the boundary of K, namely:

Proposition 2.7. *Let $K \subset \mathbb{R}^n$ be nonempty, closed and convex. Then it holds that*

$$d_K(x) = \begin{cases} +\inf\left\{\|z-x\|_{\mathscr{C}} \ : \ z \in \partial K\right\} & \text{for } x \in K \\ -\inf\left\{\|z-x\|_{\mathscr{C}} \ : \ z \in \partial K\right\} & \text{for } x \in \mathbb{R}^n \setminus K. \end{cases} \tag{2.15}$$

Proof. First, we show that

$$d_K \geqslant 0 \text{ in } K. \tag{2.16}$$

For this, let ℓ be any affine function in (2.9). Since $\ell \geqslant 0$ in K, the claim in (2.16) plainly follows.

Now we show that

$$d_K \leqslant 0 \text{ in } \mathbb{R}^n \setminus K. \tag{2.17}$$

To this aim, let $p \in \mathbb{R}^n \setminus K$. Since K is convex, we can separate it from p, namely there exists an affine function $\ell_o(x) = \omega_o \cdot x + c_o$, for suitable $\omega_o \in \mathbb{R}^n \setminus \{0\}$ and $c_o \in \mathbb{R}$, such that $\ell_o \geqslant 0$ in K and $\ell_o(p) \leqslant 0$. So, we define

$$\omega := \frac{\omega_o}{h(\omega_o)}, \qquad c := \frac{c}{h(\omega_o)} \qquad \text{and} \qquad \ell(x) := \omega \cdot x + c.$$

In this way, we have that $\ell(x) = \frac{\ell_o(x)}{h(\omega_o)} \geqslant 0$ for any $x \in K$, and $\ell(p) \leqslant 0$. In addition, we have that $h(\omega) = 1$ and so ℓ is an admissible affine function in (2.9). This implies that $d_K(p) \leqslant \ell(p) \leqslant 0$, which gives (2.17).

From (2.16), (2.17) and the continuity of d_K (recall Lemma 2.4), it follows that $d_K = 0$ along ∂K. Hence, to complete the proof of (2.15), we can restrict to the case in which $x \notin \partial K$. Hence, it suffices to check that, for any $P \notin \partial K$,

$$|d_K(P)| = \inf\left\{\|z-P\|_{\mathscr{C}} \ : \ z \in \partial K\right\}. \tag{2.18}$$

To check this, we first observe that, from Lemma 2.4, for any $z \in \partial K$,

$$|d_K(P)| = |d_K(P) - d_K(z)| \leqslant \|z-P\|_{\mathscr{C}}$$

and therefore

$$|d_K(P)| \leqslant \inf\left\{\|z-P\|_{\mathscr{C}} \ : \ z \in \partial K\right\}.$$

Thus, to complete the proof of (2.18), we only need to show that

$$|d_K(P)| \geqslant \inf\left\{\|z-P\|_{\mathscr{C}} \ : \ z \in \partial K\right\}. \tag{2.19}$$

For this, we set $R(P) := \inf\left\{\|z-P\|_{\mathscr{C}} \ : \ z \in \partial K\right\} > 0$ and we notice that $\mathscr{C}_{R(P)}(P)$ is contained either in K (if $P \in K$) or in the closure of the complement of K (if $P \in \mathbb{R}^n \setminus K$), and there exists $p \in \partial K$ with $\|p-P\|_{\mathscr{C}} = R(P)$.

So, if $P \in K$, we use Lemma 2.5 to find see that the affine function $\ell_p(x) := \omega_p \cdot x + c_p$, with $\omega_p = \frac{\omega_0}{h(\omega_0)}$ and $c_p = -\frac{\omega_0}{h(\omega_0)} \cdot p$, satisfies $\ell_p \geqslant 0$ in K and $d_K(p) = \ell(p)$. Accordingly, by (2.9) and Lemma 2.3,

$$|d_K(P)| = d_K(P) \geqslant \ell(P) = \frac{\omega_0}{h(\omega_0)} \cdot (P - p) = \|P - p\|_{\mathscr{C}} \geqslant \inf \big\{ \|z - P\|_{\mathscr{C}} \;:\; z \in \partial K \big\}.$$

This proves (2.19) when P lies inside K, so we now deal with the case in which P lies in $\mathbb{R}^n \setminus K$.

For this, we let $p \in K \cap \partial \mathscr{C}_{R(P)}(P)$ and we denote by $\omega_0 \in S^{n-1}$ the inner normal of $\partial \mathscr{C}_{R(P)}(P)$ at p. Then, since K is convex, we have that $\omega_0 \cdot (x - p) \leqslant 0$ for any $x \in K$. Hence, the affine function

$$\ell(x) := \frac{-\omega_0}{h(\omega_0)} \cdot (x - p)$$

satisfies $\ell \geqslant 0$ in K and so it is admissible in (2.9). Consequently, by Lemma 2.3,

$$-|d_K(P)| = d_K(P) \leqslant \ell(P) = -\frac{\omega_0}{h(\omega_0)} \cdot (P - p) = -R(P)$$

and so

$$|d_K(P)| \geqslant R(p) \geqslant \inf \big\{ \|z - P\|_{\mathscr{C}} \;:\; z \in \partial K \big\}.$$

This completes the proof of (2.19), as desired. □

3. The distance function from a graph

For convenience, we give here two results on the Euclidean distance function from a graph (the anisotropic case follows also from this results directly, up to changing constants, thanks to the equivalency of the norms). For this, we denote by d_* the distance function d_K in (2.9) when h is identically 1 (hence $\mathscr{C}$ in (1.2) is the Euclidean unit ball B_1) and K is the portion of space lying above a function $\zeta \in C^1(\mathbb{R}^{n-1})$, that is $K := \{x_n \geqslant \zeta(x')\}$.

Notice that, in this case, the anisotropic norm $\|\cdot\|_{\mathscr{C}}$ in (2.2) is simply the Euclidean norm and, by (2.15), d_* is simply the signed distance function from the graph of ζ.

Then we have the following results:

Lemma 3.1. *Let $b \in \big(0, \frac{1}{2}\big)$. Assume that $\zeta(0) = 0$ and that $|\nabla \zeta(x')| \leqslant b$ for every $x' \in \mathbb{R}^{n-1}$ with $|x'| \leqslant 2$. Then, for any $x \in B_1$ with $x_n \geqslant \zeta(x')$*

$$d_*(x) \geqslant \frac{1}{2} \big(x_n - \zeta(x')\big).$$

Proof. We let $R := d_*(x) \geqslant 0$ and we observe that $B_R(x)$ lies above the graph of ζ and it is tangent to it at some point $z = (z', z_n) \in \partial B_R(x)$ with $z_n = \zeta(z')$. We also denote by ω the interior normal of $B_R(x)$ at z. Then, by construction,

$$\frac{x - z}{R} = \omega = \frac{\big(-\nabla \zeta(z'),\, 1\big)}{\sqrt{1 + |\nabla \zeta(z')|^2}}. \tag{3.1}$$

Also, since the origin belongs to the graph of ζ, we have that $R = d_*(x) \leqslant |x| \leqslant 1$. Therefore $|z| \leqslant |z - x| + |x| = R + |x| \leqslant 2$. Accordingly, we deduce from (3.1) that

$$\frac{|x' - z'|}{R} = \frac{|\nabla \zeta(z')|}{\sqrt{1 + |\nabla \zeta(z')|^2}} \leqslant |\nabla \zeta(z')| \leqslant b.$$

Thus, using again (3.1),

$$1 \geqslant \frac{1}{\sqrt{1+|\nabla\zeta(z')|^2}} = \frac{x_n - z_n}{R} = \frac{x_n - \zeta(x') + \zeta(x') - \zeta(z')}{R}$$
$$\geqslant \frac{x_n - \zeta(x') - b\,|x'-z'|}{R} \geqslant \frac{x_n - \zeta(x') - b^2 R}{R}.$$

Therefore

$$x_n - \zeta(x') \leqslant (1+b^2)R \leqslant 2R = 2d_*(x). \qquad \square$$

Given $r \in \mathbb{R}$, we use the notation $r_- := \max\{-r, 0\}$.

Lemma 3.2. *Let $\alpha \in (0,1)$, $b \in \left(0, \frac{1}{2}\right)$ and $r \geqslant 1$, with $br^\alpha \leqslant \frac{1}{2}$. Assume that $\zeta(0) = 0$ and that $|\nabla\zeta(x')| \leqslant br^\alpha$ for every $x' \in \mathbb{R}^{n-1}$ with $|x'| \leqslant 3r$.*

Suppose also that

$$\zeta(x') \geqslant 0 \text{ for every } x' \in \mathbb{R}^{n-1}. \tag{3.2}$$

Let $x \in \mathbb{R}^n$ with $|x'| \leqslant r$. Then

$$\big(d_*(x)\big)_- \geqslant 4\left(x_n - \zeta(x')\right)_-.$$

Proof. We can suppose that $x_n < \zeta(x')$, otherwise $\big(x_n - \zeta(x')\big)_- = 0$ and the desired claim is obvious.

Then, we take $R := -d_*(x) > 0$ and we consider the ball $B_R(x)$. By construction, $B_R(x)$ lies below the graph of ζ and it is tangent to it at some point $z = (z', z_n) \in \partial B_R(x)$ with $z_n = \zeta(z')$. Notice that, in view of (3.2),

$$z_n \geqslant 0. \tag{3.3}$$

We also denote by ω the interior normal of $B_R(x)$ at z, and so

$$\frac{x-z}{R} = \omega = \frac{\big(\nabla\zeta(z'),\,-1\big)}{\sqrt{1+|\nabla\zeta(z')|^2}}. \tag{3.4}$$

We claim that

$$z_n x_n \leqslant br^{\alpha+1}\, z_n. \tag{3.5}$$

Indeed, if $x_n \leqslant 0$, then (3.5) follows from (3.3). If instead $x_n > 0$, then we know that

$$\xi(x') = \xi(x') - \xi(0) \leqslant br^\alpha\,|x'| \leqslant br^{\alpha+1}$$

and thus $x_n \in (0, \xi(x')) \subseteq (0, br^{\alpha+1})$, which gives (3.5).

From (3.3) and (3.5) we obtain that

$$z_n^2 - 2x_n z_n \geqslant z_n^2 - 2br^{\alpha+1} z_n \geqslant \inf_{t\geqslant 0} t^2 - 2br^{\alpha+1}t = -b^2 r^{2(\alpha+1)}.$$

Consequently, using that the origin lies on the graph of ζ,

$$r^2 + x_n^2 \geqslant |x'|^2 + x_n^2 = |x|^2 \geqslant |d_*(x)|^2 = |x-z|^2$$
$$= |x'-z'|^2 + |x_n - z_n|^2 = |x'-z'|^2 + x_n^2 + z_n^2 - 2x_n z_n$$
$$\geqslant |x'-z'|^2 + x_n^2 - b^2 r^{2(\alpha+1)},$$

and thus $|x'-z'|^2 \leqslant r^2 + b^2 r^{2(\alpha+1)} \leqslant 2r^2$.

Therefore $|z'| \leqslant |x'| + |x'-z'| \leqslant r + \sqrt{2}\,r \leqslant 3r$. Hence, we deduce from (3.4) that

$$\frac{|x'-z'|}{R} = \frac{|\nabla\zeta(z')|}{\sqrt{1+|\nabla\zeta(z')|^2}} \leqslant |\nabla\zeta(z')| \leqslant br^\alpha$$

and

$$\frac{1}{2} \leqslant \frac{1}{\sqrt{1+b^2r^2}} \leqslant \frac{1}{\sqrt{1+|\nabla\zeta(z')|^2}} = \frac{z_n - x_n}{R} = \frac{\zeta(z')-\zeta(x')+\zeta(x')-x_n}{R}$$
$$\leqslant \frac{br^\alpha\,|z'-x'|+\zeta(x')-x_n}{R} \leqslant \frac{b^2r^{2\alpha}\,R+\zeta(x')-x_n}{R}.$$

That is,

$$\frac{\big(d_*(x)\big)_-}{2} \leqslant \frac{R}{2} \leqslant (1-b^2r^{2\alpha})R \leqslant 2(\zeta(x')-x_n) = 2\big(x_n-\zeta(x')\big)_-,$$

which gives the desired result. □

References

[1] G. Alberti, L. Ambrosio, X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. 65(1-3) (2001), 9–33.

[2] H. Berestycki, L. Caffarelli, L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) 25(1-2) (1998), 69–94 (1997).

[3] X. Cabré, E. Cinti, *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, Discrete Contin. Dyn. Syst. 28(3) (2010), 1179–1206.

[4] X. Cabré, E. Cinti, *Sharp energy estimates for nonlinear fractional diffusion equations*, Calc. Var. Partial Differential Equations 49(1-2) (2014), 233–269.

[5] X. Cabré, Y. Sire, *Nonlinear equations for fractional Laplacians II: Existence, uniqueness, and qualitative properties of solutions*, Trans. Amer. Math. Soc. 367(2) (2015), 911–941.

[6] X. Cabré, J. Solá-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. 58(12) (2005), 1678–1732.

[7] L. Caffarelli, S. Dipierro, E. Valdinoci, *A logistic equation with nonlocal interactions*, Kinet. Relat. Models 10 (2017), no. 1, doi:10.3934/krm.2017006.

[8] L. Caffarelli, N. Garofalo, F. Segála, *A gradient bound for entire solutions of quasi-linear equations and its consequences*, Comm. Pure Appl. Math. 47(11) (1994), 1457–1473.

[9] M. del Pino, M. Kowalczyk, J. Wei, *On a conjecture by De Giorgi in dimensions 9 and higher*. Symmetry for elliptic PDEs, 115–137, Contemp. Math., 528, Amer. Math. Soc., Providence, RI, 2010.

[10] E. De Giorgi, *Convergence problems for functionals and operators*. Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978). Pitagora, Bologna, 1979, pp. 131–188.

[11] S. Dipierro, G. Palatucci, E. Valdinoci, *Dislocation dynamics in crystals: a macroscopic theory in a fractional Laplace setting*, Comm. Math. Phys. 333 (2015), no. 2, 1061–1105.

[12] S. Dipierro, J. Serra, E. Valdinoci, *Improvement of flatness for nonlocal phase transitions*, preprint (2016).

[13] A. Farina, B. Sciunzi, E. Valdinoci, *Bernstein and De Giorgi type problems: new results via a geometric approach*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7(4) (2008), 741–791.

[14] A. Farina, E. Valdinoci, *The state of the art for a conjecture of De Giorgi and related problems*. Recent progress on reaction-diffusion systems and viscosity solutions, 74–96, World Sci. Publ., Hackensack, NJ, 2009.

[15] A. Farina, E. Valdinoci, *1D symmetry for solutions of semilinear and quasilinear elliptic equations*, Trans. Amer. Math. Soc. 363(2) (2011), 579–609.

[16] A. Farina, E. Valdinoci, *Some results on minimizers and stable solutions of a variational problem*, Ergodic Theory Dynam. Systems 32(4) (2012), 1302–1312.

[17] N. Ghoussoub, C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. 311(3) (1998), 481–491.

[18] O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. (2) 169(1) (2009), 41–78.

[19] O. Savin, *Some remarks on the classification of global solutions with asymptotically flat level sets*, preprint (2016).

[20] O. Savin, *Rigidity of minimizers in nonlocal phase transitions*, preprint (2016).

[21] Y. Sire, E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. 256(6) (2009), 1842–1864.

[22] E. Valdinoci, B. Sciunzi, V. O. Savin, *Flat level set regularity of p-Laplace phase transitions*, Mem. Amer. Math. Soc. 182(858) (2006), vi+144 pp.