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Forced symmetry breaking perturbations for periodic solutions

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Abstract

Using the formalism defined by R. Lauterbach and M. Roberts [21], we develop a geometric approach for the problem of forced symmetry breaking for periodic orbits in G-equivariant systems of ODE's. We show that this problem can be studied as the perturbation of the identity mapping on the double coset space $L\backslash G/K$ where K is the maximal subgroup of G acting on the periodic orbit and L the symmetry of the perturbation. We exhibit some example where this kind of symmetry breaking allows to show the existence of heteroclinic cycles between periodic solutions.

1 Introduction

The modelisation of physical phenomena leads often to take in account symmetries which are "first order" symmetries of the physical system i.e. symmetries of an *ideal* model bigger than those of the real physical system. However if the real physical system is a small perturbation of its ideal model, we can expect that the study of the ideal model gives a "first oder" description of the real model. For instance, the spherical Bénard problem can be a model for the problem of onset of convection in stars. This "ideal" model as described in [7], assumes that the star is perfectly spherical. Furthermore, it is often supposed to be not rotating ([4, 9]). It is clear that these assumptions are rough descriptions of the reality and a more realistic model should include at least a small rotation of the star and the induced slight flatness of poles. This forces to break the symmetry from O(3) to SO(2). A natural question is then how to relate the various solutions of the ideal G-symmetric model with those of a more realistic model where the symmetry group is a subgroup H of G. Several approaches where used to investigate the persistence of equilibria solutions [15, 5, 16, 6]. More recent works ([21, 19, 20, 14]) consider the possible dynamics induced by forced symmetry breaking on relative equilibria and show in particular the existence of heteroclinic cycles. In this paper, we use the formalism defined in [21] to investigate the possible dynamics resulting from a forced symmetry breaking on G-orbits of periodic orbits.

In order to motivate this work, we recall in section 2 some general facts concerning the perturbation of normally hyperbolic, flow and group invariant submanifolds. We restrict our attention to the simplest flow and G invariant manifolds, namely to group orbits of a time orbit of a point. In section 3 we give generalities about this kind of invariant manifolds. We restrict the study in 3.2 to a G-orbit M of a time periodic orbit and show that M can be provided with a natural fibration by S^1. We also investigate the geometry of L-equivariant flows on M for a subgroup $L \subset G$ and the corresponding orbit space $L\backslash M$. In section 4.5 results similar to results given in [21] are obtained (see propositions 4.22, 4.23 and 4.24). They pave the way to localize heteroclinic cycles between periodic orbits after L-equivariant perturbation of the G-equivariant flow. In the last section, we exhibit an example of symmetry breaking for periodic solution in SO(3)-equivariant problems leading to the possibility of heteroclinic cycles between periodic orbits.

2 Perturbation of G-invariant manifolds

As we emphasized in the introduction, the question of perturbation of G-invariant manifolds had been investigated from a theoretical point of view by R. Lauterbach and M. Roberts ([21]). Their results were applied to orbits of fixed points with SO(3) symmetry ([21, 20]). In this section, we recall without proofs some of their results.

Let X be a smooth, finite dimensional manifold provided with a smooth action of a compact Lie group G

$$\begin{array}{rcl} \mathrm{G} \times \mathrm{X} & \mapsto & \mathrm{X} \\ (g, x) & \to & gx \end{array} \tag{2.1}$$

Let $f : \mathrm{X} \mapsto \mathrm{TX}$ be a G-equivariant vector field on X and $\Phi : \mathrm{X} \times \mathbb{R} \mapsto \mathrm{X}$ the corresponding flow. Let M be a G and Φ invariant submanifold of X. M is supposed to be normally hyperbolic ([17]), i.e. roughly speaking, the dynamics near M is governed by dynamics transversal to M). Note that this condition is not very restrictive. For example, if M is a relative periodic orbit, we have the following result :

Proposition 2.1 *([13] Prop. 8.2) Let $\Phi : \mathrm{X} \times \mathbb{R} \mapsto \mathrm{X}$ be a smooth and G-equivariant flows on X and $\mathrm{M} \subset \mathrm{X}$ be a relative periodic orbit of Φ. There exist arbitrary smooth perturbations Φ' of Φ such that M is a normally hyperbolic relative periodic orbit of Φ'.*

The essential result for this approach of symmetry breaking perturbations states :

Theorem 2.2 *([21] Prop. 1.1) Assume $f : \mathrm{X} \mapsto \mathrm{TX}$ is a G-equivariant C^r-vector field on X, $r \geq 1$. Let $\mathrm{M} \subset \mathrm{X}$ be a compact submanifold which is invariant under the flow Φ corresponding to f and under the action of G. Assume that M is normally hyperbolic. Let $\mathrm{L} \subset \mathrm{G}$ be a subgroup and $g : \mathrm{X} \mapsto \mathrm{TX}$ be a L-equivariant C^r-vector field with $\left\|g - f\right\|_{\mathrm{C}^r} < \epsilon$. Then, if ϵ is sufficiently small, there exists a unique C^r-manifold M_ϵ near M which is invariant under the flow Φ_ϵ corresponding to w. Moreover there exists a C^r diffeomorphism $\mathrm{M} \to \mathrm{M}_\epsilon$ which is L-equivariant.*

This allows the study of the perturbed dynamic by means of L equivariant vectors fields on M instead of L-equivariant on the a priori unknown manifold M_ϵ. Since M is G-invariant it could be easily characterized in some simple cases like in the case of relative equilibria. A natural question is whether it is possible or not to realise a L-equivariant flow on M as the restriction of a L-equivariant perturbation of a G-equivariant flow on X. The answer is given by the following proposition :

Proposition 2.3 *([21] Proposition 1.3) Let Φ be a L-equivariant flow on X, Φ_{M} be its restriction to M. For each compact neighborhood W of M and for all $\epsilon > 0$ there exists a $\delta > 0$ such that for any flow Ψ_{M} on M with*

$$\| \Psi_{\mathrm{M}} - \Phi_{\mathrm{M}} \|_{\mathrm{C}^1} < \delta$$

there exists a L-equivariant flow Ψ on X such that $\Psi|_{\mathrm{M}} = \Psi_{\mathrm{M}}$ and $\| \Psi - \Phi \|_{\mathrm{C}^1(\mathrm{W})} < \epsilon$.

3 Simple G-invariant manifolds

3.1 A general setup

In this section, we give some general results concerning simple flow and G-invariant subspaces. By "simple", we mean G-orbit of a unique integral manifold of a vector field. We show that there are three classes of such subspaces : direct product, relative equilibria and relative periodic orbits. Most of the material of this section can be read "between the lines" in [12] and [13].
In the following, $f : \mathrm{X} \longrightarrow \mathrm{TX}$ is a G-equivariant vector field on X and $\Phi : \mathrm{X} \times \mathbb{R} \rightarrow \mathrm{X}$ is the corresponding flow on X. By equivariance of f, we have

$$g\Phi(x,t) = \Phi(gx,t)\ \forall g \in \mathrm{G},\ \forall t \in \mathbb{R}$$

We denote U_x the integral curve of Φ passing through $x \in \mathrm{X}$ at the time $t = 0$, M_x its group orbit

$$\mathrm{M}_x = \{gy \,/\, y \in U_x, g \in \mathrm{G}\}$$

Finally, we denote by $\mathrm{H}_x \subset \mathrm{G}$ the isotropy subgroup of x relatively to the action (2.1) and $\mathrm{K}^x \subset \mathrm{G}$ the maximal subgroup acting on U_x, i.e.

$$\mathrm{K}^x = \{k \in \mathrm{G}/\ \exists\ t_k \in \mathbb{R}, k\ x = \Phi(x,t_k)\}$$

Proposition 3.4 *If $y \in \mathrm{M}_x$ then there is $g \in \mathrm{G}$ such that*

$$\mathrm{H}_x = g^{-1}\mathrm{H}_y g \quad \text{and} \quad \mathrm{K}^x = g^{-1}\mathrm{K}^y g$$

Proof : By flow invariance of $\mathrm{Fix}(\mathrm{H}_x)$, $\mathrm{H}_{x'} = \mathrm{H}_x$ for any $x' \in U_x$. Let $y = \Phi(g\,x,t)$. Then

$$g'\,y = y \Longleftrightarrow g'\Phi(g\,x_0,t) = \Phi(g\,x_0,t) \Longleftrightarrow g' \in g^{-1}\mathrm{H}_x g$$

and

$$g'y = \Phi(y,t') \Longleftrightarrow \Phi(g'g\,x_0,t) = \Phi(g\,x_0,t+t') \Longleftrightarrow g' \in g^{-1}\mathrm{K}_x g$$

∎

Since f is a smooth vector field, the integral line U_{x_0} for any given $x_0 \in X$ corresponds either a fixed point, a periodic orbit or a non compact trajectory. Respectively to these three cases, M_{x_0} can be provided with a left action of $\mathrm{G} \times \mathrm{P}$ with $\mathrm{P} = \mathbb{I}, \mathrm{S}^1, \mathbb{R}$

$$\begin{array}{ccc} \mathrm{G} \times \mathrm{P} \times \mathrm{M}_{x_0} & \longrightarrow & \mathrm{M}_{x_0} \\ ((g,p),x) & \longrightarrow & (g,p)\cdot x = \phi(g\,x,p) \end{array} \tag{3.2}$$

This action is clearly transitive.

Since K_{x_0} acts on U_{x_0}, there is a group homomorphism

$$\theta : \mathrm{K}_{x_0} \to \mathrm{P} \tag{3.3}$$

such that for any $k \in \mathrm{K}_{x_0}$, $kx_0 = \phi(x_0, \theta(k))$.

Lemma 3.5 *The isotropy subgroup $\Gamma_{x_0} \subset \mathrm{G} \times \mathrm{P}$ of x_0 for the action (3.2) is*

$$\Gamma_{x_0} = \{(k, \theta(k^{-1})), k \in \mathrm{K}_{\mathrm{x}_0}\}$$

Furthermore, if $x \in \mathrm{M}_{x_0}$, then $\Gamma_x \simeq \Gamma_{x_0}$.

Proof : Let $(g,p) \in \Gamma_{x_0}$. Then $(g,p) \cdot x_0 = \Phi(gx_0, \theta(p)) = x_0$. By definition, g is an element of K_{x_0}. We can write

$$(g,p) \cdot x_0 = \Phi(gx_0, p) = \Phi(x_0, \theta(g) + p)$$

and, since θ is a group homomorphism, $t = \theta(g^{-1})$. The group $\mathrm{G} \times \mathrm{P}$ acts transitively on M_{x_0}. Since Γ_x is the isotropy subgroup of x for this action, Γ_x is conjugated to Γ_{x_0}. ▮

Simple G-orbit can be characterised as follows :

Lemma 3.6 *The G-orbit M_{x_0} can be characterized by :*

1) $\mathrm{K}_{x_0}/\mathrm{H}_{x_0} = \mathbb{I}$ *and M_{x_0} is diffeomorphic to the direct product* $\mathrm{G}/\mathrm{H} \times U_{x_0}$

2) $\mathrm{K}_{x_0}/\mathrm{H}_{x_0} \neq \mathbb{I}$. *Then M_{x_0} is a compact manifold and either $\mathrm{M}_{x_0}/\mathrm{G} \simeq \mathbb{I}$ and M_{x_0} corresponds to a relative equilibrium or a $\mathrm{M}_{x_0}/\mathrm{G} \simeq \mathrm{S}^1$ and M_{x_0} is a relative periodic orbit*

Proof : If $\mathrm{K}_{x_0}/\mathrm{H}_{x_0} \simeq \mathbb{I}$, then the isotropy subgroup of x_0 with respect to the action of $\mathrm{G} \times \mathrm{P}$ is $\mathrm{H}_{x_0} \times \mathbb{I}$. Then

$$\mathrm{M}_{x_0} \simeq (\mathrm{G} \times \mathrm{P})/(\mathrm{H}_{x_0} \times \mathbb{I}) \simeq \mathrm{G}/\mathrm{H} \times \mathrm{P} \simeq \mathrm{G}/\mathrm{H} \times U_{x_0}$$

the free action of P on U_{x_0} yielding the last identity.
Let us now assume that $\mathrm{K}_{x_0}/\mathrm{H}_{x_0} \not\simeq \mathbb{I}$ and let $k \in \mathrm{K} - \mathrm{H}$. Then there is an element t_k in $\mathbb{R}$ such that $kx_0 = \Phi(x_0, t_k)$. Clearly, for any element $x \in U_{x_0}$ there is $n \in \mathbb{Z}$ such that $x \in \Phi(x_0, [nt_k, (n+1)t_k])$ and M_{x_0} is the G-orbit of the subset $V = \Phi(x_0, [0, t_k])$ of U_{x_0}. Using theorem 3.12 in [20], $G_{x_0} \cap V$ is discrete or all of V (theorem 3.12 in [20] is stated for V homeomorphic to S^1 but it is true for any compact V). If $\pi : \mathrm{M}_{x_0} \longrightarrow \mathrm{M}_{x_0}/\mathrm{G}$ is the canonical projection, then M_{x_0} is the saturation of V, i.e. $\mathrm{M}_{x_0} = \pi^{-1} \circ \pi(V)$. If $\mathrm{G}x_0 \cap V$ is all of V then $V/\mathrm{G} \simeq \mathbb{I}$ and $\mathrm{M}_{x_0}/\mathrm{G} \simeq \mathbb{I}$. Thus M_{x_0} is a relative equilibrium. If If $\mathrm{G}x_0 \cap V$ is discrete, then $V/\mathrm{G} \simeq \mathrm{S}^1 \simeq \mathrm{M}_{x_0}$ and M_{x_0} corresponds to a relative periodic orbit. ▮

Remark that the case where $\mathrm{M}_{x_0} \simeq \mathrm{G}/\mathrm{H} \times U_{x_0}$ can be useful for the study of symmetry breaking for heteroclinic connections. A classification of the possible dynamics on relative equilibria and relative periodic orbit can be found in papers of M. Field [12, 13].

3.2 The case of G-orbit of periodic orbits

In this section, we investigate more precisely the geometric structure of group orbits of periodic solutions.
Let $f : \mathrm{X} \mapsto \mathrm{TX}$ a G-equivariant vector field on X and $\Phi : \mathrm{X} \times \mathrm{IR} \to \mathrm{X}$ a flow associated to the system

$$\frac{dx}{dt} = f(x) \tag{3.4}$$

We assume the existence of a point $x_o \in \mathrm{X}$ and a strictly positive real number T such that $x_0 = \Phi(x_o, T)$. Clearly

$$U_{x_0} = \{\Phi(x_o, t), t \in \mathrm{IR}\}$$

Up to a scaling of time in (3.4), we assume T=1. We denote $\mathrm{M} = \mathrm{M}_{x_0}$, $\mathrm{H} = \mathrm{H}_{x_0}$ and $\mathrm{K} = \mathrm{K}^{x_0}$. In 3.1, we defined a transitive left $\mathrm{G} \times \mathrm{S}^1$ action on M:

$$\begin{array}{ccc} \mathrm{G} \times \mathrm{S}^1 \times \mathrm{M} & \mapsto & \mathrm{M} \\ (g, \alpha, x) & \to & (g, \alpha) \cdot x = \Phi(gx, \alpha) \end{array} \tag{3.5}$$

Since the kernel of the homomorphism (3.3) $\theta : \mathrm{K} \longrightarrow \mathrm{S}^1$ is equal to H, $\theta(\mathrm{K})$ is isomorphic to a closed subgroup of SO(2), i.e. $\mathrm{K/H} \simeq \mathbb{I}, \mathbb{Z}_m$ for any integer m or $\mathrm{K/H} \simeq \mathrm{SO(2)}$. In the last case, U_{x_0} is included in $\mathrm{G}x_0$ and M_{x_0} corresponds to a relative equilibrium. We restrict our attention to the case $\mathrm{K/H} \not\simeq \mathrm{SO(2)}$. The isotropy subgroup $\Gamma_{x_0} \subset \mathrm{G} \times \mathrm{S}^1$ of x_0 under the action (3.5) is given by

$$\Gamma_{x_0} = \{(k, \theta(k^{-1})), k \in \mathrm{K}\}$$

We can now give a more precise characterization of M :

Proposition 3.7 *The manifold* M *is diffeomorphic to the quotient manifold* $\mathrm{G} \times \mathrm{S}^1/\Gamma_{x_0}$.

Proof : This is a standard result (see [3] proposition 4.2). ∎

This diffeomorphism is given by :

$$\begin{array}{rccl} \mu_{x_0} : & \mathrm{G} \times \mathrm{S}^1/\Gamma_{x_0} & \to & \mathrm{M} \\ & [g, \alpha] & \to & (g, \alpha) \cdot x_0 = \Phi(g\, x_0, \alpha) \end{array} \tag{3.6}$$

where $[g, \alpha]$ is a representative of the coset of (g, α) in the quotient.

We can consider $\mathrm{G} \times \mathrm{S}^1$ as a right K-space with the action defined by

$$(g, \alpha)\, k = (gk, \theta(k^{-1}) \cdot \alpha) \tag{3.7}$$

Then $\mathrm{G} \times \mathrm{S}^1/\Gamma_{x_0}$ is precisely the orbit space of K acting by (3.7) on $\mathrm{G} \times \mathrm{S}^1$ and this is by definition ([10] page 32) the fiber product $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$. The canonical projection $\mathrm{G} \times \mathrm{S}^1 \to \mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ realizes the identification :

$$[gk, \alpha] = [g, \theta(k) \cdot \alpha] \tag{3.8}$$

For a general study of fiber bundles and fiber products, see [11] or [24]. More concise expositions can be found in [2] or [3]. In the following, we note fibre bundles $p : \mathrm{E} \to \mathrm{B}$ with fibre F and structure group G in the form (E,B,p,F,G) or simply (E,B,p,G) for a principal G-bundle.

We recall the following result :

Proposition 3.8 *Let* G *be a Lie group and* K *a closed subgroup. The operation of* K *on* G *by right multiplication defines a differentiable* K*-principal bundle* $(\mathrm{G}, \mathrm{G}/\mathrm{K}, \mathrm{p}, \mathrm{K})$. *In particular,* G/K *is a differentiable manifold and the projection* $p : \mathrm{G} \to \mathrm{G}/\mathrm{K}$ *is a submersion.*

Proof : See [3] theorem 4.3 or [11] (16.14.1.1). ∎

Recall that, by definition, for a family $\{U_j, j \in J\}$ is of open sets covering the base G/K, there exists a bundle atlas $\{\psi_j : U_j \times \mathrm{K} \to p^{-1}(U_j), j \in J\}$. If we view $U_j \times \mathrm{K}$ as a right K-space with K acting by $(u, h) \cdot k = (u, hk)$, then each ψ_j is an equivariant diffeomorphism i.e $\psi_j(u, h)k = \psi(u, hk)$. Two bundle charts ψ_i and ψ_j give rise to a transition function $\beta_{i,j} : (U_i \cap U_j) \to \mathrm{K}$ such that

$$\psi_i(u, k) = \psi_j(u, k\beta_{i,j}(u))$$

We can now characterize more precisely the geometric structure of M.

Theorem 3.9 *If p is the* K*-principal bundle defined in the previous proposition, then the mapping* $p \circ \mu_{x_0}^{-1} : \mathrm{M} \to \mathrm{G}/\mathrm{K}$ *defines a differentiable* 1*-sphere bundle with structure group* K*, i.e. a fibre bundle* $(\mathrm{M}, \mathrm{G}/\mathrm{K}, p \circ \mu_{x_0}^{-1}, \mathrm{S}^1, \mathrm{K})$.

Proof : Since the mapping $\mu_{x_0}^{-1} : \mathrm{M} \to \mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ is a diffeomorphism, we only have to show that there is a mapping π such that $(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1, \pi, \mathrm{S}^1, \mathrm{K})$ is a differentiable fibre bundle. This is a well known result since $\pi : \mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1 \to \mathrm{G}/\mathrm{K}$ is a fiber bundle associated with the K-principal bundle G/K (see [11] 16.14.7). Let us just recall the main points. The mapping $p : \mathrm{G} \to \mathrm{G}/\mathrm{K}$ defines the differentiable K-principal bundle $(\mathrm{G}, \mathrm{G}/\mathrm{K}, p, \mathrm{K})$. Then $\pi : \mathrm{S}^1 \times_{\mathrm{K}} \mathrm{G} \mapsto \mathrm{G}/\mathrm{K}$ given by $\pi([\alpha, g]) = p(g)$ defines the differentiable fiber bundle.

Indeed, π is a submersion. Furthermore, if $\{U_j, j \in J\}$ and $\{\psi_j, j \in J\}$ are defined as before, then a bundle atlas for $\pi : \mathrm{S}^1 \times_K \mathrm{G} \mapsto \mathrm{G/K}$ is given by $\{\tilde{\psi}_j, j \in J\}$ such that

$$\begin{array}{rcccl} \tilde{\psi}_j & : & U_i \times \mathrm{S}^1 & \rightarrow & \pi^{-1}(\mathrm{U}_j) \\ & & (u, \alpha) & \rightarrow & [\psi_j(u, e), \alpha] \end{array}$$

where e is the identity in S^1.
Each $\tilde{\psi}_j$ defines a diffeomorphism. Indeed $\tilde{\psi}_j$ is differentiable and its inverse, given by

$$\begin{array}{rcccl} \tilde{\psi}_j^{-1} & : & \pi^{-1}(U_j) & \rightarrow & U_j \times \mathrm{S}^1 \\ & & [u, \alpha] & \rightarrow & (\pi(u, \alpha), \alpha) \end{array}$$

is also differentiable.
Now if $\tilde{\psi}_i$ and $\tilde{\psi}_j$ are two charts then we have

$$\begin{aligned} \tilde{\psi}_j(u, \alpha) &= [\psi_j(u, e), \alpha] = [\psi_i(u, e\,\beta_{ij}(u)), \alpha] \\ &= [\psi_i(u, e)\,\beta_{ij}(u), \alpha] = [\psi_i(u, e), \theta(\beta_{ij}(u)) \cdot \alpha] \\ &= \tilde{\psi}_i(u, \theta(\beta_{ij}(u)) \cdot \alpha) \end{aligned}$$

and K is the structure group of the fibre bundle. Now $\pi \circ \mu_{x_0}^{-1} : \mathrm{M} \rightarrow \mathrm{G/K}$ defines a bundle projection and the bundle chart over $U_i \subset \mathrm{G/K}$ is given by $\mu_{x_0} \circ \psi_i$. Clearly, the transition functions are the same for this bundle as for the bundle $(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1, \pi, \mathrm{S}^1, \mathrm{K})$.

∎

The fibers in M correspond to periodic orbits. Indeed, if $g\mathrm{K}$ is an element of G/K, then

$$\pi^{-1}(gK) = [g, \mathrm{S}^1]$$

and $\mu_{x_0}([g, \mathrm{S}^1]) = \Phi(g\,x_0, \mathrm{S}^1)$

Corollary 3.10 *The manifold* M *is orientable.*

Proof : Let U_i and U_j two open subsets of G/K provided with the two bundle charts $\tilde{\psi}_i$ and $\tilde{\psi}_j$. Let now u be a point in $U_i \cap U_j$. We have seen that for any element α in S^1 we have

$$\tilde{\psi}_i(u, \alpha) = \tilde{\psi}_j(u, \theta(\beta_{i,j}(u)) \cdot \alpha)$$

with $\beta_{i,j}(u) \in \mathrm{K}$. The point $\tilde{\psi}_i(u, \alpha)$ is expressed in local coordinates in $U_i \cap U_j$ either as (u, α) or as $(u, \theta(\beta_{i,j}(u)) \cdot \alpha)$ and the change of chart is the diffeomorphism

$$\begin{array}{rcccl} \tilde{\psi}_j^{-1} \circ \tilde{\psi}_i & : & U_i \times \mathrm{S}^1 & \rightarrow & U_j \times \mathrm{S}^1 \\ & & (u, \alpha) & \rightarrow & (u, \theta(\beta_{i,j}(u))\alpha) \end{array}$$

Since Im $\theta \subset$SO(2) and SO(2) acts by rotations on S^1, the Jacobian of $\tilde{\psi}_j^{-1} \circ \tilde{\psi}_i$ is equal to 1. As a consequence, M is orientable. ∎

It is well known that groups of the form G$\times S^1$ are relevant for the study of equivariant Hopf bifurcations with symmetry group G. This allows to consider periodic solutions that exhibit spatial or temporal symmetry as well as a mixing of both kinds of symmetries.The group G provides spatial symmetries and S^1 temporal symmetries. The standard reference for this topic is Golubitsky *et al.* [16], chapter XVI.

There exists an alternative approach adopted by M. Field [13] based on the fibration $(G \times_{N(H)} S^1, G/H, p, S^1, N(H)/H)$. Following this approach, M is fibered by group orbits. In our model, it is fibered by periodic orbits. Our idea is to use the sections of the fiber bundle to obtain a "Poincaré section" and a "Poincaré mapping" which allow the study of perturbations of the flow on M.

Before we end this section, we will consider the simple case where K/H= $\mathbb{I}$. We also consider the case K/H= SO(2). Even if this case does not fit with the definition of relative periodic orbit, the results obtained previously remain correct. This example allows us to recover the case of relative equilibria.

- If K/H= $\mathbb{I}$, then M is diffeomorphic to $G \times S^1/(H, \mathbb{I})$. Thus for two elements (g_1, α_1) and (g_2, α_2), the identification (3.8) yields $[g_1, \alpha_1] = [g_2, \alpha_2]$ iff there exists $h \in H$ such that $g_1 = g_2 h$ and $\alpha_1 = \alpha_2$. Thus $M \simeq G/H \times S^1$.

- If K/H=SO(2), then K can be written $K = H \times \widetilde{SO(2)}$ with $\widetilde{SO(2)} \subset G$ isomorphic to SO(2) and each $k \in K$ can be written in a unique way as $k = h\alpha$ with $h \in H$ and $\alpha \in \widetilde{SO(2)}$. Furthermore, the mapping $\tilde{\theta} = \theta|_{\widetilde{SO(2)}}$ is an isomorphism and for any $k \in K$, $\theta(k) = \theta(h\alpha) = \tilde{\theta}(\alpha)$. Now the subgroup Σ_{x_0} has the form $\Sigma_{x_0} = \{(h\alpha, \tilde{\theta}(\alpha^{-1})), h \in H, \alpha \in \widetilde{SO(2)}\}$. Let us consider the following mapping

$$\begin{array}{rccc} q: & G \times S^1 & \longrightarrow & G = p_1(G \times S^1) \\ & (g, \tilde{\theta}(\alpha)) & \longrightarrow & g\alpha^{-1} = p_1(g\alpha^{-1}, 0) \end{array} \tag{3.9}$$

with p_1 the projection on the first component. q is clearly a diffeomorphism. Let $\pi_0 : G \times S^1 \longrightarrow (G \times S^1)/\Gamma_{x_0}$ and $\pi_1 : G \longrightarrow G/H$ be the canonical projections and $\tilde{q}$ the mapping

$$\begin{array}{rccc} \tilde{q}: & (G \times S^1)/\Gamma_{x_0} & \longrightarrow & G/H \\ & [g, \tilde{\theta}(\alpha)] & \longrightarrow & [g\alpha^{-1}]_H \end{array} \tag{3.10}$$

It is easily checked that $\tilde{q}$ is well defined and bijective. Let $f : G/H \longrightarrow \mathbb{R}$ be a smooth mapping (i.e. $f \circ \pi_1$ is smooth). Since $f \circ \pi_1 \circ q = f \circ \tilde{q} \circ \pi_0$ with the left-hand side smooth, $\tilde{q}$ is smooth (see section 4.2). The same result is true for $\tilde{q}^{-1}$ and $\tilde{q}$ is a diffeomorphism.

4 Perturbation of periodic orbits

Let $f : X \mapsto TX$ be the G-equivariant vector field defined in the previous section and L be a closed subgroup of G. If M is a normally hyperbolic, flow invariant and G-invariant submanifold of X, then by the theorem (2.2), we know that for any small enough L-equivariant perturbation f_ϵ of f, there exist a flow and L-invariant manifold M_ϵ closed to M. In [21], R. Lauterbach and M. Roberts have shown that if M is a normally hyperbolic relative equilibrium, the study of possible dynamics on M_ϵ can be advantageously brought back to the study of strata preserving vector fields on the quotient manifold $L\backslash G/H$. In this section, we obtain a similar kind of reduction in the case where M is a normally hyperbolic group orbit of periodic orbits of (3.4)

4.1 Action of subgroups of G on $G \times_K S^1$

We first characterize the action of subgroups of G on $G \times_K S^1$. Remark that although the frame is slightly different, the results obtained are quite similar to those obtained in [21] section 1.2.

Let H, K and M defined as in the previous section. The mapping μ_{x_0} defined in (3.6) is a diffeomorphism from $G \times_K S^1$ to M. We can define a left G-action on $G \times_K S^1$ by

$$g \cdot [g_0, \alpha] \to [gg_0, \alpha] \tag{4.11}$$

and we consider its restriction to a subgroup $L \subset G$.

Proposition 4.11 *The action of* L *on* M_ϵ *and* $G \times_K S^1$ *are equivariantly diffeomorphic.*

Proof : By (2.2), there exists a L-equivariant diffeomorphism μ_ϵ from M to M_ϵ and the action of L on M_ϵ is induced by the action of G on X. For g_0 in G, we have :

$$g_0 \cdot \mu_{x_0}([g, \alpha]) = g \cdot \Phi(gx_0, \alpha) = \Phi(g_0 g x_0, \alpha) = \mu_{x_0}([g_0 g, \alpha])$$

and $\mu_{x_0} \circ g = g \circ \mu_{x_0}$.
Thus μ_{x_0} is G-equivariant and the diffeomorphism $\mu_\epsilon \circ \mu_{x_0} : G \times_K S^1 \to M_\epsilon$ is L-equivariant. ∎

Now, we shall show that the study of the dynamics depends only on conjugacy classes of groups. In the following we will denote $[g, \alpha]$ the elements of $G \times_K S^1$ and $[g, \alpha]_{K'}$ the elements of $G \times_{K'} S^1$

Proposition 4.12 *Let* K *and* $K' = g_1 K g_1^{-1}$, L *and* $L' = g_1 L g_1^{-1}$ *be subgroups of* G, *with* $g_0, g_1 \in G$. *Then* $G \times_K S^1$ *and* $G \times_{K'} S^1$ *are isomorphic as fibre bundles and the mapping*

$$\begin{aligned} \nu \ : \ & G \times_K S^1 \to G \times_{K'} S^1 \\ & [g, \alpha] \to [g_1 \cdot g \cdot g_1^{-1}, \alpha]_{K'} \end{aligned}$$

realizes a diffeomorphism. The actions of L *on* $S^1 \times_K G$ *and of* L′ *on* $S^1 \times_{K'} G$ *are diffeomorphic and there exists a diffeomorphism* $\tau_{g_0} : K \mapsto K'$ *such that :*

$$\nu(k \cdot [\alpha, g]) = \tau_{g_0}(k)\nu([\alpha, g])$$

Proof : First, it is clear that if H is an isotropy subgroup of K then for any $g \in G$, $H' = gHg^{-1}$ is also an isotropy subgroup and $N(H') = gN(H)g^{-1}$. For any $k \in K$, let $k' = g_0\, k\, g_0^{-1} \in K'$. We have

$$\begin{aligned} g_0 k\, x_0 &= \Phi(g_0\, x_0, \theta(k)) \\ k' g_0\, x_0 &= \Phi(g_0\, x_0, \theta(k')) \end{aligned}$$

and $\theta(k) = \theta(k')$.
Let ν be the mapping

$$\begin{array}{rcccc} \nu & : & G \times_K S^1 & \to & G \times_{K'} S^1 \\ & & [g, \alpha] & \to & [g_0\, g\, {g_0}^{-1}, \alpha]_{K'} \end{array}$$

Then ν is a well defined diffeomorphism. Indeed let

$$[g, \alpha] = [g_0 g' g_0^{-1}, \alpha']$$

then

$$\begin{aligned} [g_0 g' g_0^{-1}, \alpha']_{K'} &= [g_0 g k^{-1} g_0^{-1}, \theta(k)\alpha]_{K'} = [g_0 g g_0^{-1} g_0 k^{-1} g_0^{-1}, \theta(k)\alpha]_{K'} \\ &= [g_o g g_o^{-1}, \theta(g_0 k^{-1} g_0^{-1})\theta(k)\alpha]_{K'} = [g_0 g g_0^{-1}, \alpha]_{K'} \end{aligned}$$

where the last equality is due to the fact that $\theta(g_0 k^{-1} g_0^{-1}) = \theta(k^{-1})$.

In the same way, we can show that the mapping

$$\begin{array}{rcccc} \tilde{\nu} & : & G/K & \to & G/K' \\ & & [g]_K & \to & [g_0 g g_0^{-1}]_{K'} \end{array}$$

is a well defined diffeomorphism. Now if $\pi : G \times_K S^1 \to G/K$ and $\pi' : G \times_{K'} S^1 \to G/K'$, then the following diagram commutes

$$\begin{tikzcd} G \times_K S^1 \arrow[r, "\nu"] \arrow[d, "\pi"'] & G \times_{K'} S^1 \arrow[d, "\pi'"] \\ G/K \arrow[r, "\tilde{\nu}"] & G/K' \end{tikzcd}$$

and $(\nu, \tilde{\nu})$ defines a fibre bundle isomorphism.

Let $L \subset G$ act on $G \times_K S^1$ and $L' = g_1 L g_1^{-1}$ act on $G \times_{K'} S^1$.

$$\nu(k[g, \alpha]) = [g_0 k g g_0^{-1}, \alpha]_{K'} = [g_0 k g_0^{-1} g_0 g g_0^{-1}, \alpha]_{K'} = g_0 k g_0^{-1} [g_0 g g_0^{-1}, \alpha]_{K'}$$

If we take $\tau_{g_0}(k) = g_0 k g_0^{-1}$, we obtain

$$\nu(k[\alpha, g]) = \tau_{g_0}\nu([\alpha, g])$$

∎

We will now determine some properties of the action of a subgroup $\mathrm{L} \subset \mathrm{G}$ on the manifold $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ similar to those given in proposition 1.7 and 1.8 in [21]. We recall that if L_1 and L_2 are two subgroups of G, the set $\mathrm{N}(\mathrm{L}_1, \mathrm{L}_2)$ introduced in Ihrig & Golubitsky [18] is defined by

$$\mathrm{N}(\mathrm{L}_1, \mathrm{L}_2) = \{g \in \mathrm{G} | \mathrm{L}_1 \subset g\mathrm{L}_2 g^{-1}\}$$

if $\mathrm{N}(\mathrm{L}_1, \mathrm{L}_2) \neq \emptyset$ then L_1 is said to be *subconjugated* to L_2

Proposition 4.13 *Let* L *act on* $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ *by the restriction of the action (4.11) and let* L' *be a subgroup of* L. *We have :*

(a) $\mathrm{Fix}(\mathrm{L}') \neq \emptyset \Longleftrightarrow \mathrm{L}'$ *subconjugated to* H

(b) $\mathrm{Fix}(\mathrm{L}') = [\mathrm{N}(\mathrm{L}', \mathrm{H}), \mathrm{S}^1]$

The isotropy subgroup $\mathrm{stab}([g, \alpha])$ *of the element* $[g, \alpha]$ *is given by*

$$\mathrm{stab}([g, \alpha]) = \mathrm{L} \cap g\mathrm{H}g^{-1}$$

Proof : Let $l \in \mathrm{L}'$ and $[\alpha, g] \in \mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$. Then $l \cdot [\alpha, g] = [\alpha, g]$ if and only if there exist $k \in \mathrm{K}$ such that

$$\begin{array}{rcl} l \cdot g &=& g \cdot k \\ \alpha &=& \theta(k^{-1}) \cdot \alpha \end{array}$$

The second condition is equivalent to $k \in \mathrm{H}$ since $\mathrm{H} = \ker \theta$. Then the first condition yields $l = g \cdot k \cdot g^{-1}$ and thus $l \in \mathrm{L} \cap g\mathrm{H}g^{-1}$. ∎

Note that this result is formally identical to the proposition 1.7 obtained in [21].
The flow of the perturbed vector field f_ϵ on M_ϵ can be studied on $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ through the diffeormorphism $\mu_{x_0} \circ \mu_\epsilon$. If the fibers of the bundle $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1 \rightarrow \mathrm{G}/\mathrm{K}$ correspond to periodic orbits of f in M, this is no longer the case for f_ϵ on the manifold M_ϵ. However, there is a "remainder" of this structure since all points of a given fiber are in the same fixed point subspace after the perturbation. More precisely, we have the following corollary of the previous theorem :

Corollary 4.14 *For a fixed* $g \in \mathrm{G}$, *all points* $[g, \alpha] \in \mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ *possess the same isotropy subgroup respectively to the action of a subgroup* $\mathrm{L} \subset \mathrm{G}$ *on* M.

Proof : This is immediate since $\mathrm{stab}([g, \alpha])$ doesn't depend on α. ∎

4.2 Orbit spaces and stratum preserving vector fields

Our strategy to study L-equivariant vector fields on the manifold $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ is to define cross-sections to the bundle $\pi : \mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1 \to \mathrm{G}/\mathrm{K}$ and to use them as Poincaré sections for the initial G-equivariant flow to define a Poincaré mapping. However, the study on $\mathrm{G}\times_{\mathrm{K}}\mathrm{S}^1$ leads to consider "L-equivariant" Poincaré mapping and their lift to L-equivariant flows on $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ could be tricky. Instead, we will use an orbit space projection with respect to the action of L to avoid such complications since only strata preserving flows and maps have to be considered. We will show that the orbit space $\mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1)$ inherits a smooth fibration. More precisely, the mapping $p : \mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1) \to \mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ defines a smooth 1-sphere bundle with structure group K.

Let us recall that if M is a smooth left G-manifold, then the orbit space $\mathrm{G}\backslash\mathrm{M}$ is in general not a smooth manifold but rather a semi-algebraic variety (see [2, 1, 23]). Let $p : \mathrm{M} \to \mathrm{G}\backslash\mathrm{M}$ be the canonical projection (the orbit map). If $\mathrm{H} \subset \mathrm{G}$ is an isotropy subgroup for this action, we denote $\mathrm{M}_{(\mathrm{H})}$ the union of all orbits of type H and let $(\mathrm{G}\backslash\mathrm{M})_{(\mathrm{H})} = p(\mathrm{M}_{(\mathrm{H})})$. Then $(\mathrm{G}\backslash\mathrm{M})_{(\mathrm{H})}$ is naturally a smooth manifold and the restriction $\tilde{p} = p|_{((\mathrm{M})_{(\mathrm{H})})}$ defines a smooth principal G bundle $((\mathrm{M})_{(\mathrm{H})}, (\mathrm{G}\backslash\mathrm{M})_{(H)}, \tilde{p}, \mathrm{G})$. $\mathrm{G}\backslash\mathrm{M}$ possesses a natural smooth stratification which corresponds to the stratification by isotropy types.
A smooth structure can be given on $\mathrm{G}\backslash\mathrm{M}$ using the following definitions : a function $f : \mathrm{G}\backslash\mathrm{M} \to \mathbb{R}$ is smooth if its composition with the orbit map is smooth. If M and N are two smooth G-manifolds, a mapping $f : \mathrm{M} \to \mathrm{N}$ is smooth if its composition with any smooth function on $\mathrm{G}\backslash\mathrm{N}$ is smooth.
If we denote by $\mathcal{X}^\infty(\mathrm{M})^{\mathrm{G}}$ the set of G-equivariant vector fields on M and $\mathcal{X}^\infty(\mathrm{G}\backslash\mathrm{M})$ the set of strata preserving vector fields on $\mathrm{G}\backslash\mathrm{M}$, then there exists a well defined and surjective map : $\mathcal{X}^\infty(\mathrm{M})^{\mathrm{G}} \to \mathcal{X}^\infty(\mathrm{G}\backslash\mathrm{M})$ (see [23]). In other words, every smooth strata preserving vector field on $\mathrm{G}\backslash\mathrm{M}$ can be lifted to a G-equivariant vector field on M. An important point is that this projection is continuous. This allows to study perturbations of equivariant vector fields on M as perturbations of stratum preserving vector fields on $\mathrm{G}\backslash\mathrm{M}$.

4.3 Structure of the orbit space

In this section,we investigate the structure of the orbit space $\mathrm{L}\backslash\mathrm{M}$. The main result of this section is theorem (4.17). Before we state it, we need two elementary results.

Proposition 4.15 *Let* X *and* Y *be right and left* G*-spaces respectively. Let* Z *be the fibre product* $\mathrm{Z} = \mathrm{X} \times_{\mathrm{G}} \mathrm{Y}$. *If* X *is reduced to a single point with a trivial action of* G *then* Z *is diffeomorphic to* $\mathrm{G}\backslash\mathrm{Y}$.

Proof : Two points (x_1, y_1) and (x_2, y_2) in $\mathrm{X}\times\mathrm{Y}$ are identified in Z is there is an element $g \in G$ such that

$$x_1 = x_2 g \text{ and } y_1 = g^{-1} y_2$$

If X is reduced to a single point x_0 with the trivial action of G, then clearly (x_0, y_1) and (x_0, y_2) are identified in Z if and only if there are in the same G-orbit. ∎

Proposition 4.16 *([2] Proposition 2.3 p. 74) Let* X *be a right* H*-space ,* Y *be a left* H*- and right* K*-space and* Z *a left* K*-space. Then there is a canonical homeomorphism*

$$(\mathrm{X} \times_{\mathrm{H}} \mathrm{Y}) \times_{\mathrm{K}} \mathrm{Z} \longrightarrow \mathrm{X} \times_{\mathrm{H}} (\mathrm{Y} \times_{\mathrm{K}} \mathrm{Z})$$

given by $[[x, y], z] = [x, [y, z]]$.

We can now characterize the structure of $\mathrm{L}\backslash\mathrm{M}$:

Theorem 4.17 *The mapping* $\pi_0 : \mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1) \to \mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ *defines a* 1*-sphere bundle with structure group* K.

Proof : Let X be a right L space provided with the action

$$l \circ x = x\ l^{-1}$$

This action extends trivially to a left action on X×G defined by

$$l \circ (x, g) = (x\ l^{-1}, lg)$$

The orbit space of X×G under this action is $\mathrm{X}\times_{\mathrm{L}}\mathrm{G}$. Let us consider the space $\mathrm{X}\times\mathrm{G}\times\mathrm{S}^1$ provided with the action of L on X×G and the action of K on $\mathrm{G}\times\mathrm{S}^1$. Then by projecting successively on the orbit spaces, we obtain the two homeomorphic spaces

$$\mathrm{X} \times_{\mathrm{L}} (\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1) \simeq (\mathrm{X} \times_{\mathrm{L}} \mathrm{G}) \times_{\mathrm{K}} \mathrm{S}^1$$

Now reducing X to a single point x_0 with the trivial action of L yields

$$\mathrm{x}_0 \times_{\mathrm{L}} (\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1) \simeq \mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1)$$
$$(\mathrm{x}_0 \times_{\mathrm{L}} \mathrm{G}) \times_{\mathrm{K}} \mathrm{S}^1 \simeq (\mathrm{L}\backslash\mathrm{G}) \times_{\mathrm{K}} \mathrm{S}^1$$

Thus there is an homeomorphism $f : \mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1) \longrightarrow (\mathrm{L}\backslash\mathrm{G}) \times_{\mathrm{K}} \mathrm{S}^1$ which is given by

$$_{L}[g, \alpha] \to [\,_{L}[g], \alpha]$$

and $p : \mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1) \to \mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ possesses a structure of a fibre bundle with fibre S^1 and structure group K induced by that of the fiber bundle $\pi_1 : (\mathrm{L}\backslash\mathrm{G}) \times_{\mathrm{K}} \mathrm{S}^1 \to \mathrm{L}\backslash\mathrm{G}/\mathrm{K}$. ∎

The previous fibre bundle is a priori a continuous one. We will show that it is in fact smooth.

Theorem 4.18 *The* 1*-sphere bundle* $p : \mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1) \to \mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ *is smooth.*

Proof : Let us consider the following diagram

$$\begin{array}{ccc} G \times_K S^1 & \xrightarrow{\pi} & G/K \\ {\scriptstyle\pi_0}\big\downarrow & & \big\downarrow{\scriptstyle\pi_1} \\ L\backslash(G \times_K S^1) & \xrightarrow{p} & L\backslash G/K \end{array} \tag{4.12}$$

where π_0 and π_1 are the canonical orbit maps. This diagram is clearly commutative. Let now $f : L\backslash G/K \to \mathbb{R}$ be a smooth function. This means that $f \circ \pi_1$ is smooth. Since π is a smooth fibre bundle, the composition $f \circ \pi_1 \circ \pi$ is smooth. By commutativity of the diagram, $f \circ \pi_1 \circ \pi = f \circ p \circ \pi_0$ and p is a smooth mapping. ∎

4.4 Projection of the unperturbed vector field on the orbit space

We would like to study L-equivariant vector fields on M closed to the restriction $f_{|M}$ of the G-equivariant vector field (3.4). First, we will show that the projection of $f_{|M}$ on the orbit space $L\backslash(G \times_K S^1)$ defines a vertical vector field, i.e a vector field tangent to fibre in each point. Let us just recall that if X is a smooth G-manifold, then the ring of real valued polynomial invariants $P_G(X)$ is finitely generated ([1, 22, 23]). Its generators $\rho_1, \cdots, \rho_r$ can be chosen homogeneous. The mapping $\rho : X \to \mathbb{R}^r$ induces an homeomorphism of X/G with V=Im ρ which is smooth on each stratum [1]. If $\pi : X \to X/G$ is the canonical projection, then $\rho \circ \pi^{-1}$ realises this homeomorphism. Let $\dot{x} = F(x)$ a vector field on X. Then its projection on V is given by

$$\left\{ \begin{array}{rcl} \dot{\rho}_1 & = & (\mathrm{grad}_x \rho_1 | \, \dot{x}) \\ & \vdots & \\ \dot{\rho}_r & = & (\mathrm{grad}_x \rho_r | \, \dot{x}) \end{array} \right.$$

Let X be the manifold M where is defined the G-equivariant vector field (3.4). Let $\rho_1, \cdots, \rho_r$ be the generators of $P_L(M)$ for the action of L on M, $\pi_2 : M \to L\backslash M$ and $\rho : M \to V$ the canonical projection. We denote h the homeomorphism $h = \rho \circ \pi_2^{-1}$. Since $p : L\backslash M \to L\backslash G/K$ defines a fibre bundle, the mapping $\tilde{p} = p \circ h^{-1} : V \to L\backslash G/K$ induces clearly a natural fibration with fibre S^1 and structure group K on V. Furthermore, $\tilde{p}$ is smooth over each stratum. In particular the image by h of a fibre in $L\backslash M$ is a fiber in V. In the following, we denote $\tilde{f}$ the projection of the restriction $f|_M$ of the unperturbed vector field. This unperturbed vector field $f|_M$ satisfies the following property :

Theorem 4.19 *The projection $\tilde{f}$ of the vector field $f|_M$ on V is a vertical vector field.*

Proof : Two elements $[g, \alpha_1]$ and $[g, \alpha_2]$ in M are projected in $L\backslash$ M on the same element ${}_L[g]_K$ in $L\backslash G/K$. Then a variation of α in an element $[g, \alpha]$ corresponds in $L\backslash M$

to a variation along a fiber. Equivalently, a variation of α in $\rho([g,\alpha])$ corresponds to a variation along a fiber in V. Now the unperturbed vector field on M is vertical. It can formally be written in the form

$$\left\{ \begin{array}{lcl} \dot{g} & = & 0 \\ \dot{\alpha} & = & q(g,\alpha) \end{array} \right. \tag{4.13}$$

Its projection on V satisfies

$$\dot{\rho}_i = (\mathrm{grad}\rho_i, x) = \frac{\partial \rho_i}{\partial g} \cdot \dot{g} + \frac{\partial \rho_i}{\partial \alpha} \cdot \dot{\alpha} = \frac{\partial \rho_i}{\partial \alpha} \cdot \dot{\alpha}$$

Then, it is vertical in V. ∎

The fact that solutions of $f|_{\mathrm{M}}$ are T-periodic is also reflected in the orbit space by the following fact :

Proposition 4.20 *Let n be the order of the cyclic group $\theta(\mathrm{L} \cap \mathrm{K})$. Then every solution of $\tilde{f}$ is T/n periodic.*

Proof : Let k be an generator of $\mathrm{L} \cup \mathrm{K}$ and Φ the flow associated to f. Let x be any point in the periodic orbit P. Since θ is a group homomorphism, we have

$$k^n \cdot \Phi(x,t) = \Phi(x, \theta(k^n).t) = \Phi(x, n \cdot \theta(k) + t) = \Phi(x,t)$$

Clearly, if T is the period of P, then $\theta(k) = \mathrm{T}/n$. Let now $\tilde{\Phi}$ be the flow associated to $\tilde{f}$. Since $\tilde{f}$ lifts back to f under the tangent map ρ_* of ρ, then $\tilde{\Phi} \circ \rho = \rho \circ \Phi$. For any element x in P, $k \cdot x = \Phi(x, \theta(k))$ and $\rho(x) = \rho(\Phi(x, \theta(k))$. Thus $\rho(x) = \tilde{\Phi}(\rho(x), \theta(k))$. If t_1 and t_2 are two elements in $I = [0, \mathrm{T}/n[$ then $\rho(\Phi(x_0,t_1)) \neq \rho(\Phi(x_0,t_2))$ and the vector field on $L \backslash M$ is not trivial. Since all solutions of $f|_{\mathrm{M}}$ are T-periodic, all solutions of $\tilde{f}$ are T/n periodic. ∎

Remark that in the case of $\Theta(K)$ =SO(2), then the orbit space take the form

$$\mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{H}\times\mathrm{SO}(2)} \mathrm{S}^1) \simeq \mathrm{L}\backslash\mathrm{G}/\mathrm{H}$$

and perturbations of the G-equivariant vector field can be studied as perturbations of the initial strata preserving vector fields on L\G/H. We recover here the result of R. Lauterbach and M. Roberts [21]. However, in this case, the orbit space $\mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1)$ is no more a 1-sphere bundle since all periodic orbits in $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ are collapsed to one point.

4.5 Perturbation of periodic orbits

Let us summarize the results of the previous sections. Let P be a T-periodic solution of the system of the G-equivariant system

$$\dot{x} = f(x) \tag{4.14}$$

defined in (3.4) and H be the isotropy subgroup of points in P. Let K be the maximal subgroup of G acting on P. Its action is isomorphic to the action of a cyclic subgroup $\theta(K) \simeq \mathbb{Z}^m$ on S^1. The G-orbit M of P is diffeomorphic to $\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1$ and the mapping $\pi : \mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1 \to \mathrm{G}/\mathrm{K}$ is a smooth 1-sphere bundle. The restriction to M of the vector field corresponding to (4.14) is vertical, without fixed point. Let L be a subgroup of G. The mapping $p : \mathrm{L}\backslash\mathrm{M} \longrightarrow \mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ is a smooth 1-sphere bundle. The projection $\tilde{f}$ of f on $\mathrm{L}\backslash\mathrm{M}$ defines a vertical and strata preserving vector field where all solutions are T/n-periodic with $n = |\theta(\mathrm{L} \cap \mathrm{K})|$. Small L-equivariant perturbations of (4.14) can be studied as small strata preserving perturbations of the vector field $\tilde{f}$.

The fact that the unperturbed vector field is vertical in $\mathrm{L}\backslash\mathrm{M}$ could allow in some cases a further reduction of dimension for the study of its perturbations. The idea is to use differentiable cross-sections of the fiber bundle $p : \mathrm{L}\backslash\mathrm{M} \longrightarrow \mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ over each stratum to define a Poincaré section and a Poincaré mapping. The projection of the Poincaré mapping to the base of the bundle leads to the study of stratum preserving perturbations of the identity mapping on $\mathrm{L}\backslash\mathrm{G}/\mathrm{K}$. Of course, it is probably not possible to obtain a global result concerning mappings $f : \mathrm{L}\backslash\mathrm{G}/\mathrm{K} \mapsto \mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ when the dimension d of $\mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ exceeds 2. Even it the case $d = 2$, it is not clear whether there does or does not exist a cross-section for p and the study should be done case by case. The answer to this question depends on the characteristic class of the bundle and thus strongly of the particular action of G and its subgroups K and L. For example, there is no global cross-section for the oriented 1-sphere bundle $p : \mathrm{S}^3 \to \mathrm{S}^2$ which constitutes the Hopf fibration of S^3. Since $\mathrm{SO}(3)/\mathrm{SO}(2) \simeq \mathrm{S}^2$, this configuration could occur for perturbations with trivial symmetry of periodic solutions with SO(2) symmetry in SO(3)-equivariant problems. On an other hand, the study of flows restricted to 1 and 2 dimensional strata in $\mathrm{L}\backslash\mathrm{M}$, i.e. strata that are preimages by p of 0 or 1 dimensional stratum in $\mathrm{L}\backslash\mathrm{G}/\mathrm{K}$ can be useful. Similarly to [21] and [20], this restriction allows the localisation of interesting dynamics such as heteroclinic cycles. In this case, these cycles will involve periodic orbits. In the remainder of this section, we restrict our attention to these cases.

Let us consider the set $\mathcal{A}_{(\mathrm{L},\mathrm{G}/\mathrm{K})}$ of connected components of the strata for the action of L on G/K. We denote Γ the following set

$$\Gamma = \bigcup A \text{ such that } A \in \mathcal{A} \text{ and } \dim A = 0 \text{ or } \dim A = 1$$

Our goal is to construct a cross-section to the bundle p over Γ, i.e. a map $s : \Gamma \to \mathrm{L}\backslash\mathrm{M}$ such that $s \circ p = \mathbb{I}_\Gamma$. We will see that it is possible to find a differentiable section over Γ even if this is not always true over strata of dimension greater than one.

Proposition 4.21 *There exists a smooth cross-section s to the fiber p over Γ.*

Proof: First, we point out that since p is a smooth bundle, then a continuous section over Γ, if it exists, can be approximated with a smooth section ([24], p. 25). Now, the usual way to prove the existence of continuous cross sections for a fibre bundle $b : \mathrm{X} \to B$ with fibre F is the following ([24],ch. III) : Find a triangulation of B. Let $U = \{U^0, U^1, \cdots\}$ where each U^k is the corresponding k-skeleton, i.e. the union of k-dimensional simplicies. Determine a cross-section $s_0 : U^0 \to \mathrm{X}$. Such a cross-section can obviously always be found. Try to extend s_0 to a continuous cross-section $s_1 : U^1 \to \mathrm{X}$. If it is possible, try to extend s_1 to a continuous cross-section s_2 and so on. It is possible to see that the extension of s_k to s_{k+1} is possible as long the k-th homotopy group of the fibre $\pi_k(F)$ is trivial. If this is not the case, nothing can be concluded. This is the primary obstruction to the prolongation of the cross-section. In our case, the fibre is S^1 and its homotopy groups satisfy ([24]) :

$$\begin{cases} \pi_1(S^1) &= \mathbb{Z} \\ \pi_k(S^1) &= \mathbb{I} \text{ for } k \neq 1 \end{cases} \tag{4.15}$$

So it is only possible to ensure existence of a cross-section over a 1-dimensional skeleton of L\G/K. Now, by a result of C.T. Yang ([25]), for any G-manifold M, the orbit space G\L can be triangulated in such a way that all points in the interior of a simplex belong to the same orbit type. In our case, we can find a triangulation $U = \{U^0, U^1, \cdots\}$ of L\G/K such that any 0-dimensional (resp. 1-dimensional) element of $\mathcal{A}$ corresponds to an element of U^0 (resp. U_1). Applying the previous construction, we can find a cross-section s_1 over U_1. The restriction of s_1 to Γ gives a cross-section $s : \Gamma \to \mathrm{L}\backslash\mathrm{M}$ continuous over each connected component of Γ. ∎

Now, by differentiability of s, its graph g_s is a surface which is transversal to each line of the flow associated to (4.14) and it defines a *global* Poincaré section for the unperturbed flow over Γ. We denote $P^s_\Phi : g_s \longrightarrow g_s$ the corresponding Poincaré mapping for a vertical flow Φ without fixed point on M. Remark that P^s_Φ is a diffeomorphism.

We assume now that the manifold M is normally hyperbolic and let $g : \mathrm{X} \to \mathrm{TX}$ a smooth L-equivariant vector field such that $\| g - f \|_{C^\infty} < \epsilon$, for $\epsilon \in \mathbb{R}_+$ small enough. Using the results of section 2, there exist a unique smooth manifold M_ϵ near M which is invariant under the L-equivariant flow Φ_ϵ associated to g. This flow can be studied as a small L-equivariant perturbation on M of the flow Φ associated to f. The flow Φ_ϵ induced a smooth strata preserving flow on the orbit space L\M which is close to the flow induced by Φ. We have then the following obvious result :

Proposition 4.22 *Let Φ a vertical flow on* M *without fixed points. For each $\epsilon > 0$, there is a $\delta > 0$ such that for each flow Φ_δ on* M *with $\| \Phi - \Phi_\delta \|_{C^1} < \delta$, there exists a strata preserving diffeomorphism $d_\delta : \Gamma \longrightarrow \Gamma$ with $\| \mathbb{I}_\Gamma - d_\delta \|_{C^1} < \epsilon$ which is smoothly conjugated to the Poincaré mapping $P^s_{\Phi_\delta}$.*

Proof : We use the convention $\Phi = \Phi_0$. Let $d_\delta : \Gamma \longrightarrow \Gamma$ be defined by $d_\delta = p \circ P^s_{\Phi_\delta} \circ s$. Clearly, $d_0 = \mathrm{I\!I}_\Gamma$. By smoothness of p, $P^s_{\Phi_\delta}$ and s, d_δ is strata preserving and we can choose δ small enough such that $\| \mathrm{I\!I}_\Gamma - d_\delta \|_{C^1} < \epsilon$. ∎

The converse to this proposition is given by the following result :

Proposition 4.23 *Let Φ be a vertical vector field without fixed points on* M *and d : $\Gamma \longrightarrow \Gamma$ the mapping associated P^s_Φ. For each $\epsilon > 0$, there is a $\delta > 0$ such that for each diffeomorphism $d_\delta : \Gamma \longrightarrow \Gamma$ with $\| d - d_\delta \|_{C^1} < \delta$, there exists a smooth flow Φ_δ on M such that $\| \Phi - \Phi_\delta \|_{C^1} < \epsilon$ and such that $P^s_{\Phi_\delta}$ is smoothly conjugated to d_δ.*

Remark that since Φ_δ is smooth, it is necessarily strata preserving on Γ.

Proof : Since $\| d - d_\delta \|_{C^1} < \delta$, there is a smooth isotopy joining from d to d_δ, i.e. a smooth mapping $\Delta : \Gamma \times [0,1] \longrightarrow \Gamma$ such that $\Delta(x,0) = d(x)$ and $\Delta(x,1) = d_\delta(x)$. Let now $\widetilde{\Delta}$ the following mapping :

$$\widetilde{\Delta} : \begin{cases} g_s \times [0,1] & \longrightarrow \quad g_s \\ \quad (x,t) & \longrightarrow \quad s \circ \Delta(p(x),t) \end{cases} \tag{4.16}$$

and consider the flow Φ_δ defines by $\Phi_\delta(x,t) = \Phi(\widetilde{\Delta}(x,t),t)$. Clearly Φ_δ is smooth and then strata preserving. Furthermore, $\Phi_\delta(x,0) = x$ and $\Phi_\delta(x,1) = \Phi_\delta(s \circ d_\delta(x),1) = s \circ d_\delta(x)$. The mapping d_δ is then associated to the Poincaré mapping $P^s_{\Phi_\delta}$. By smoothness of all maps involved in the construction of Φ_δ, we can choose δ small enough in such a way that the condition $\| \Phi - \Phi_\delta \|_{C^1} < \epsilon$ holds. ∎

The question whether the flow Φ_ϵ given in the previous proposition can or can not be realised as the restriction of a perturbation Ψ : L\M $\longrightarrow$ T(L\M) of Φ is given by the following result which is an extension theorem in the same spirit as the proposition 1.3 in [21] :

Theorem 4.24 *Let Φ a flow on* L\M *and Φ_Γ its restriction to $p^{-1}(\Gamma)$. For each neighborhood $\widetilde{U}$ of $p^{-1}(\Gamma)$ and for all $\epsilon > 0$, there exist a $\delta > 0$ such that for any flow Ψ_δ on $p^{-1}(\Gamma)$ with*

$$\| \Phi_\Gamma - \Psi_\delta \|_{C^1} < \delta$$

there is a flow Ψ on L\M *such that $\Psi|_{p^{-1}(\Gamma)} = \Psi_\delta$ and*

$$\| \Phi - \Psi \|_{C^1} < \epsilon$$

Proof : In section (4.4), we defined a homeomorphism h : L\M $\longrightarrow$V where V is a semi-algebraic variety embedded in $\mathrm{I\!R}^n$ for a sufficiently big n. The mapping $\tilde{p} = p \circ h$ is a smooth fiber bundle. Furthermore, a smooth flow Φ on L\M induces a smooth flow $\widetilde{\Phi}$ on V with $h \circ \Phi = \widetilde{\Phi} \circ h$. For simplicity, we identify V with L\M and $\widetilde{\Phi}$ with Φ.

Since V is compact, there exists a finite set I and a covering of V by open subsets $\{U_\alpha, \alpha \in I\}$. Let $U = \{\overline{U}_\alpha, \alpha \in I]$ and $\{\phi_\alpha : \overline{U}_\alpha \longrightarrow \mathbb{R}, \alpha \in I\}$ be a smooth partition of the identity associated to U. We define now $I' \subset I$, such that $I' = \{\alpha, \exists\, x \in \widetilde{p}^{-1}(\Gamma), x \in U_\alpha\}$, $\widetilde{U} = \{\widetilde{U}_\alpha = U_\alpha \cap p^{-1}(\Gamma), \alpha \in I'\}$ and $P = \{\widetilde{\phi}_\alpha = \phi|_{\widetilde{U}_\alpha}, \alpha \in I'\}$. The set P is a smooth partition of the identity associated to the closed neighborhood $\widetilde{U}$ of $\widetilde{p}^{-1}(\Gamma)$ in V. Let us suppose now, that a flow Φ is defined on V and that a flow Φ_δ is defined on $\widetilde{p}^{-1}(\Gamma)$ with $\| \Phi_\Gamma - \Psi_\delta \|_{C^1} < \delta$. The flow Ψ_δ defines a cross section over $\widetilde{p}^{-1}(\Gamma)$ for the tangent bundle $TV \longrightarrow V$. By the same arguments used in the proof of proposition 4.21, this section can be extended to a smooth cross-section over V and then Ψ_δ can be extended to a smooth vector field $\widetilde{\Psi}$ on V. Let us define the flow Ψ by :

$$\left\{ \begin{array}{lcll} \Psi(x) & = & \Phi(x) & \text{if } x \notin \widetilde{U} \\ \Psi(x) & = & \Phi(x) + \widetilde{\phi}_\alpha(x)(\widetilde{\Psi}(x) - \Phi(x)) & \text{if } x \in \widetilde{U}_\alpha \end{array} \right. \tag{4.17}$$

Clearly, this flow coincides with Ψ_δ on $\widetilde{p}^{-1}(\Gamma)$ and for a given $\epsilon > 0$, we can choose δ small enough such that $\| \Phi - \Psi \|_{C^1} < \epsilon$.

∎

5 Applications

We consider now some examples where symmetry breaking perturbations for periodic orbits can lead to heteroclinic cycles involving periodic orbits. Our purpose here is not to prove the existence of such cycles but just to localize configurations where they could exist. Furthermore, we don't take into account the problem of their structural stability.

Let us consider the L-equivariant perturbations of a H-symmetric periodic orbit for G-equivariant problem where K$\subset$N(H) is the maximal subgroup acting on the periodic orbit. Using the notations of the previous section, we write the set Γ with the form $\Gamma = \Gamma_0 \cup \Gamma_1$ where $\Gamma_i = \{A \in \Gamma, \dim A = i\}$. We consider now the flow Φ on $\mathrm{L}\backslash\mathrm{M} = \mathrm{L}\backslash(\mathrm{G} \times_{\mathrm{K}} \mathrm{S}^1)$ and Φ_δ a flow on L\M such that $\| \Phi - \Phi_\delta \|_{C^1} < \delta$ for a small $\delta > 0$. Let $d_\delta : \Gamma \longrightarrow \Gamma$ the corresponding mapping given in proposition 4.22.

We can realize a heteroclinic cycle in the following way : let $P = \{e_0, \cdots, e_{n-1}\}$ a set of elements of Γ_0 and $C = \{\eta_0, \cdots, \eta_{n-1}\}$ a set of elements in Γ_1 such that $e_i \cup e_{i+1} \mod n \subset \overline{\eta}_i$ in Γ. Assume now that for all $i = 0 \cdots, n-1$, η_i is the unstable manifold of e_i and the unstable manifold of e_{i+1} for the mapping d_δ and the points $\{e_0, \cdots, e_{n-1}\}$ the only fixed point for d_δ on $P \cup C$. Then $P \cup C \subset \Gamma$ will realize a heteroclinic cycle for d_δ and the corresponding flow Φ_δ will show a heteroclinic cycle between periodic solutions. Following Lauterbach *and al.* [20], such a heteroclinic cycle will be called *forced heteroclinic cycle* if $n = 0$.

We will consider problems with G=SO(3). Let us just recall that the subgroups of SO(3) are either *planar* or *exceptional* subgroups. The planar subgroups are conjugated to $\mathbb{Z}_n$, $\mathbb{D}_n$, SO(2) and O(2). The subgroup $\mathbb{Z}_n$ is generated by a rotation R_n of angle $2\pi/n$

around an arbitrary axis α in $\mathbb{R}^3$. $\mathbb{D}_n$ is generated by R_n and a rotation κ of angle π around an axis β perpendicular to α. SO(2) is the group containing all rotations around α. O(2) is obtained by adding of κ to SO(2). The exceptional subgroups are conjugated to the tetrahedral group $\mathbb{T}$, the octahedral group $\mathbb{O}$ and the icosahedral group $\mathbb{I}$ corresponding respectively to the groups of rotational symmetries of the tetrahedron, the octahedron and the icosahedron. For a more detailed account of SO(3) and its subgroups, see [18, 16] or [8].
Remark that formally, the occurrence of a forced heteroclinic cycle for this problem is equivalent to the occurrence of a forced heteroclinic cycle in the problem of L-equivariant perturbations for K-symmetric relative equilibria in G-equivariant problems. Thus, even although both problems are quite different, the geometry of the group action imposes a strong similarity between the two problems. Using this remark and the classification of forced heteroclinic cycles for relative equilibria performed in [20] yields the following result :

Proposition 5.25 *Let $f : \mathrm{X} \longrightarrow \mathrm{TX}$ a* SO(3) *equivariant vector field on* X. *We assume that the system*

$$\dot{z} = f(z) \tag{5.18}$$

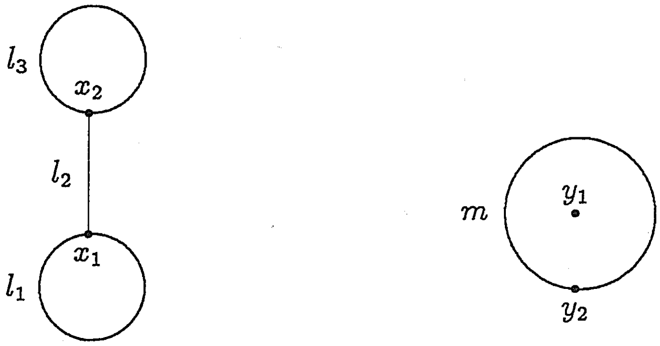
possesses a periodic solution and let M *its* SO(3) *orbit which is supposed to be normally hyperbolic. Let* K *be the maximal subgroup of* G *acting on the periodic orbit. Let $f_\epsilon : \mathrm{X} \longrightarrow \mathrm{TX}$ be a* L *equivariant vector field for any subgroup* L *of* SO(3) *which is close to f in the C^1 topology. We note M_ϵ the flow invariant manifold close to* M. *Then forced heteroclinic cycles can only occur for f_ϵ on M_ϵ if $(\mathrm{L},\mathrm{K}) \in \{(\mathbb{T},\mathbb{T}),(\mathbb{T},\mathbb{O}),(\mathbb{O},\mathbb{T}),(\mathbb{T},\mathrm{O}(2)),(\mathrm{O}(2),\mathbb{T})\}$. The periodic solutions involved in these cycles would be $\mathbb{T}$ symmetric if $(\mathrm{L},\mathrm{K}) = (\mathbb{T},\mathbb{T})$, $\mathbb{T}$ or $\mathbb{D}_2$ symmetric if $(\mathrm{L},\mathrm{K}) \in \{(\mathbb{T},\mathbb{O}),(\mathbb{O},\mathbb{T})\}$ and $\mathbb{D}_2$ symmetric in the remaining cases.*

Proof : See theorem 4.30 in [20]. We just draw in figure 5.1 the sets Γ, the corresponding isotropy types and possible diffeomorphisms realising heteroclinic cycles.

In the left case in figure 5.1, a cycle can be realized if there is a diffeomorphism $d : \Gamma \longrightarrow \Gamma$ close to the identity and without fixed point in l_1 or in l_3. In the second case, a cycle can be realized with a diffeomorphism $d : \Gamma \longrightarrow \Gamma$ close to the identity without fixed points in m.

∎

Let us remark that the heteroclinic cycles given by the previous proposition are not the only possible heteroclinic cycles but only the *forced* ones. More complicated possible cycles can be found. If $\mathrm{K} = \mathrm{O}(2)$ and $\mathrm{L} = \mathbb{D}_2$ a the corresponding set Γ is pictured in figure . An heteroclinic cycle will occur with a diffeomorphism without fixed point in the three strata of type $\mathbb{Z}_2$ and such that each fixed point possesses a stable and an unstable manifold.



(L, K)	x_1	x_2	l_1	l_2	l_3
$(\mathbb{T},\mathbb{T})$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
$(\mathbb{T},\mathbb{O})$	$\mathbb{T}$	$\mathbb{D}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$(\mathbb{O},\mathbb{T})$	$\mathbb{T}$	$\mathbb{D}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

(L, K)	y_1	y_2	m
$(\mathbb{T},\mathrm{O}(2))$	$\mathbb{Z}_3$	$\mathbb{D}_2$	$\mathbb{Z}_2$
$(\mathrm{O}(2),\mathbb{T})$	$\mathbb{Z}_3$	$\mathbb{D}_2$	$\mathbb{Z}_2$

Figure 5.1: *The set* Γ *for* $\mathrm{L},\mathrm{K} \in \{(\mathbb{T},\mathbb{T}),(\mathbb{T},\mathbb{O}),(\mathbb{O},\mathbb{T}),(\mathrm{O}(2),\mathbb{T}),(\mathbb{T},\mathrm{O}(2))\}$

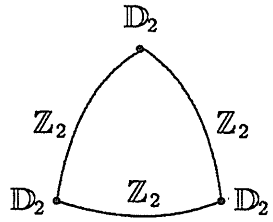


Figure 5.2: The set Γ in $\mathbb{D}_2\backslash\mathrm{SO}(3)/\mathrm{O}(2)$

5.1 A particular case

Let us consider the case G=SO(3), K=O(2) and L $=\mathbb{T}$. We will show that it is possible to find a diffeomorphism close to the identity on L\M=$\mathbb{T}$\SO(3)/O(2) realising an heteroclinic cycle. Let us first remark that the orbit space L\M is contractible. As a consequence, the space M is the direct product $\mathrm{M} \simeq \mathrm{L}\backslash\mathrm{G}/\mathrm{K} \times \mathrm{S}^1$ ([24] prop 11.4). We can then define a global section for the unperturbed flow f on M ([24]). In other word, any mapping closed to the identity on the whole base L\M can be lifted as a smooth vector field on M.

We want to prove that there is a smooth diffeomorphism d_δ on L\M such that the restriction on Γ is close to the identity on L\M and realizes a heteroclinic cycle. We know (section 4.4) that L\M can be embedded in $\mathbb{R}^r$ as a semi-algebraic variety V via an homeomorphism $\tilde{p}^{-1} : \mathrm{V} \longrightarrow \mathrm{L}\backslash\mathrm{M}$. The integer r is the dimension of a Hilbert basis of the ring $P_{\mathrm{L}}(\mathrm{M})$ of invariant polynomials for the action of L on M. Furthermore, the smooth structure on L\M is induced by that of $\mathbb{R}^r$. If we refer to the figure 5.1, let C be the disk bordered by $m \cup y_2$. The points y_1 and y_2 correspond respectively to the isotropy types $\mathbb{Z}_3$ and $\mathbb{D}_2$, m to the type $\mathbb{Z}_2$ and $C-(m \cup y_1 \cup y_2)$ to the trivial isotropy stratum. The variety V is clearly defined once we know it in a neighborhood of the points $\tilde{p}^{-1}(y_1)$ and $\tilde{p}^{-1}(y_2)$. Let x be a point in the L-manifold M and S a differentiable slice for its orbit Lx. Let $\psi_1, \cdots, \psi_k$ a Hilbert basis for the action of the isotropy group L_x on S. Then there exists a real analytic isomorphism of a neighborhood of the origin in the space $\psi(S) = (\psi_1(S), \cdots, \psi_k(S)) \subset \mathbb{R}^k$ with a neighborhood of the image $\pi(x) \subset L\backslash M$ ([1],proposition 1). In our case, since L is discrete, so is the orbit of x and a slice in x is just a small enough open neighborhood of x in M. Now, using the picture 5.3, we see that the slice action of $\mathbb{D}_2$ in y_1 is generated the two elements

$$\rho_1 : z \mapsto \overline{z}, \quad \kappa : z \mapsto -z$$

and the slice action of $\mathbb{Z}_3$ in y_2 is generated by a rotation

$$\rho_2 : z \mapsto e^{2\pi/3}$$

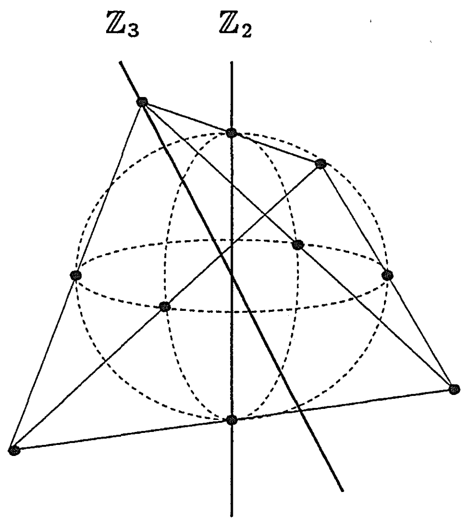


Figure 5.3: Action of $\mathbb{T}$ on SO(3)/O(2)$\simeq \mathbb{RP}^2$ before the identification of opposite points of the sphere

The corresponding rings of invariants are given by

$$P_{\mathbb{Z}_3}(V) =< \theta_1 = z\overline{z},\ \theta_2 = z^3 + \overline{z}^3,\ \theta_2 = z^3 - \overline{z}^3 >$$

and

$$P_{\mathbb{D}_2}(V) =< \theta_1' = z\overline{z},\ \theta_2' = z^2 + \overline{z}^2 >$$

respectively. In a neighborhood of $\tilde{p}(y_1)$, the semi-algebraic variety V is defined by the system

$$\begin{cases} \theta_1 & > & 0 \\ \theta_1^3 & = & \theta_2^2 + \theta_3^2, \end{cases}$$

and it is a cusp.
In a neighborhood of $\tilde{p}(y_2)$, V is defined by the system

$$\left\{ \begin{array}{lcl} \theta'_1 & > & 0 \\ \theta'_2 & > & 0 \end{array} \right.$$

and it is the neighborhood of the origin in the positive quadrant.
Let us recall that we denote by m the $\mathbb{Z}_2$ stratum the in L\M, by y_2 the $\mathbb{D}_2$ stratum and by $\tilde{p}$: L\M $\longrightarrow$ V the embedding of the orbit space in $\mathbb{R}^n$. Then we have the following result :

Proposition 5.26 *For any $\delta > 0$, there exists a diffeomorphism $d : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that the restriction d_δ to V is stratum preserving without fixed point on $\tilde{p}(m)$ and such that*

$$\| \mathbb{I}_V - d_\delta \|_{C^0} < \delta$$

Proof : The mapping δ can easily be found using the time-1 mapping of a smooth flow possessing an homoclinic connection. For example, let us consider the system

$$\dot{z} = f(z) \tag{5.19}$$

which is a extension in $\mathbb{R}^n$ of an oscillator equation with a cubic non-linearity , i.e

$$\left\{ \begin{array}{lcl} \dot{x} & = & y \\ \dot{y} & = & x - x^3 \\ \dot{z}_i & = & -z_i \text{ for } i = 3, \ldots, n \end{array} \right. \tag{5.20}$$

For this system, the (x, y) plane is invariant and the vector field restricted to this plane exhibits two connections $C_+ \subset \mathbb{R}^+ \times \mathbb{R}$ and $C_- \subset \mathbb{R}^- \times \mathbb{R}$ homoclinic to the hyperbolic point 0. Find now in $\mathbb{R}^n$ two tubular neighborhoods U_1 and U_2 of C_+ which are respectively close and open with $U_1 \subset U_2$ and a function $h : U_2 \longrightarrow \mathbb{R}$ satisfying

$$\left\{ \begin{array}{ll} h(x) = 1 & \text{if } x \in U_1 \\ 0 \leq h(x) \leq 1 & \text{if } x \in U_2 - U_1 \\ h(x) = 0 & \text{if } x \notin U_2. \end{array} \right. \tag{5.21}$$

Let $\phi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be the flow corresponding to the vector field $\tilde{f}(x) = h(x) f(x)$ and $\tilde{d} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the time-one diffeomorphism associated to it i.e. $\tilde{d}(x) = \phi(x, 1)$. Clearly C_+ is a homoclinic connection for $\tilde{d}$ and this diffeomorphism is non-zero only in U_1. Let now W_1 be an open tubular neighborhood of $\tilde{p}(y_2 \cup m) \in \mathbb{R}^n$. Since a neighborhood of $\tilde{p}(y_2)$ in V is isomorphic to a neighborhood of the origin of the positive quadrant in $\mathbb{R}^2$, there exists a diffeomorphism $s : U_1 \longrightarrow W_1$ mapping 0 on y_2 and C_+ onto m. Then by choosing correct sizes for U_1 and W_1, we can take $d_\delta = s^{-1} \circ \tilde{d} \circ s$ (remark that s can be chosen such that V is d_δ invariant).

∎

The heteroclinic cycle of the previous example is a structurally stable phenomenon (in the class of $\mathbb{T}$-equivariant perturbations). Indeed, the only fixed point on $y_2 \cap m$ is hyperbolic and then will persist under small enough $\mathbb{T}$-equivariant perturbations.

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