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Global behaviour of a reaction–diffusion system modelling chemotaxis

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Dedicated to Professor Dr. Konrad Gröger
on the occasion of his sixtieth birthday

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Contents

ABSTRACT. Using Lyapunov functionals the global behaviour of the solutions of a reaction–diffusion system modelling chemotaxis is studied for bounded piecewise smooth domains in the plane. Geometric criteria can be given that this dynamical system tends to a (not necessarily trivial) stationary state.

1. Introduction

Chemotaxis is the oriented migration of organisms under the influence of chemical substances (see [Ha], [Mu] for the biological background). Mathematical models of such processes are given by partial differential equations of reaction–diffusion type ([KS]) . As a prototype we take from [JL] the system for $U = U(t,x), V = V(t,x)$

$$\left.\begin{aligned} \frac{\partial U}{\partial t} &= \Delta U - \chi \nabla \cdot (U \nabla V) \\ \frac{\partial V}{\partial t} &= \alpha \Delta V - \beta V + \delta U \end{aligned}\right\} \text{ on } \mathbb{R}_+ \times \Omega, \tag{1.1}$$

where $\mathbb{R}_+$ are the positive reals and Ω is a bounded domain in $\mathbb{R}^N, N \geq 2$, with piecewise smooth boundary $\Gamma = \partial\Omega$. For the system are given initial values and no–flux boundary conditions (homogeneous Neumann conditions)

$$\begin{aligned} U(0,\cdot) &= U_0,\ V(0,\cdot) = V_0 \ \text{ on } \Omega,\ U_0 \geq 0,\ V_0 \geq 0, \\ \nu \cdot \nabla U &= \nu \cdot \nabla V = 0 \ \text{ on } \mathbb{R}_+ \times \Gamma. \end{aligned}$$

Here α, β, δ, χ are positive constants, ν is the outer unit normal on Γ . System (1.1) models the dynamics of a population (concentration U) moving in Ω driven by the gradient of a chemotactic agens (concentration V) produced by the population. There is some similarity between the model (1.1) and the drift–diffusion models of microelectronics ([GG]). But whereas equally charged particles in microelectronics repulse each other, chemotaxis has an attractive effect which leads to agglomeration of particles. This causes mathematical difficulties, and one cannot expect that (1.1) has global solutions for arbitrary parameters α, β, δ, χ and arbitrary initial values.
For a reduced variant of (1.1) where the second equation is the stationary equation

$$\alpha \Delta V - \beta V + \delta U = 0$$

the problem has been considered recently by [DN]. They show local (in time) existence of a solution under homogeneous Dirichlet boundary conditions for the component U for bounded domains in $\mathbb{R}^N$ with smooth boundaries. If a certain (smallness) condition on the initial values U_0 is satisfied, they prove global (in time) existence and decay to the trivial stationary solution $(0,0)$ for $t \to \infty$.
For the same reduced variant of (1.1) on a disk in $\mathbb{R}^2$ it has been proved by [JL] that under homogeneous Neumann boundary conditions there are initial values U_0, V_0 for which U explodes in finite time in the center of the disc. A refined study of the blow–up mechanism has been given by [HV1] for the disk starting from radially symmetric initial values. The same authors also considered the blow–up for the full chemotactic system (1.1) in the case of radial symmetry ([HV2], [HV3]).
This paper is devoted mainly to the study of the global behaviour of solutions to the dynamical system (1.1) in generally nonsmooth domains $\Omega \subset \mathbb{R}^2$. Our interest in this topic originates in numerical experiments ([GJK]) showing the strong influence of the geometry of Ω . As an example we mention that initial values concentrated around the centre of a square may cause blow–up in a corner. Our main point is the observation

that the system (1.1) possesses Lyapunov functionals, i.e., functionals decreasing along solutions as time increases. Using this tool we formulate in the case $N = 2$ a condition involving the data of the system and the geometry of the domain ensuring decay to a homogeneous state (Theorem 4.2) and a weaker condition excluding blow–up in finite time (Theorem 4.3). This condition, which numerical evidence suggests to be sharp, allows non–trivial stationary states. In fact, we prove (Theorem 5.2) that the solutions to (a transformed version (1.4), (1.5) of) (1.1) asymptotically approximate the (generally) non–trivial solutions $(u^\star, v^\star)$ of the problem

$$-\alpha\Delta v^\star + \beta v^\star = \gamma(u^\star - 1) \quad \text{on} \quad \Omega, \quad \frac{\partial v^\star}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \quad u^\star = \frac{|\Omega| e^{v^\star}}{\int_\Omega e^{v^\star} d\Omega}, \tag{1.2}$$

where $\gamma = \chi\delta\overline{U_0}$ with $\overline{U_0}$ the spatial mean of the initial value U_0.
Stationary problems of this type were studied in [Sch] and appear also in other fields (e.g. [Mo1]). As can be seen from (1.2) there is a hidden exponential nonlinearity in the system (1.1). This is the reason why our considerations are restricted essentially to the two–dimensional case $N = 2$ where the Orlicz norm associated to the exponential function is controlled by the Dirichlet integral.
As to the case $N = 3$, it should be possible to prove some of our results for somewhat relaxed chemotaxis models (comp. [Sch]) where ∇V in (1.4) is replaced by $\nabla(\Phi(V))$ with a "sensitivity function" Φ like

$$\log(V + c), \; V/(1 + cV), \quad \text{or} \quad V^2/(1 + cV^2)$$

which leads to $\exp(\Phi(v^\star))$ instead of $\exp(v^\star)$ in (1.2).

It is convenient to transform the system (1.1) in the following manner. Introducing the spatial mean of a function h on $\mathbb{R}_+ \times \Omega$ by

$$\overline{h}(t) = \frac{1}{|\Omega|}\int_\Omega h(t,x)dx, \qquad |\Omega| = meas\,(\Omega),$$

we obtain by integration in (1.1)

$$\overline{U}(t) = \overline{U_0}, \quad \frac{d\overline{V}}{dt} + \beta\overline{V} = \delta\overline{U_0}, \quad \overline{V}(0) = \overline{V_0}.$$

We introduce new unknown functions u, v by

$$u(t,x) = \frac{U(t,x)}{\overline{U_0}}, \quad v(t,x) = \chi(V(t,x) - \overline{V}(t))$$

and a new constant γ by

$$\gamma = \chi\delta\overline{U_0} \tag{1.3}$$

and arrive at the transformed system of (1.1) to be studied in the following:

$$\left.\begin{aligned} u_t &= \Delta u - \nabla\cdot(u\nabla v) \\ v_t &= \alpha\Delta v - \beta v + \gamma(u-1) \end{aligned}\right\} \text{ on } \mathbb{R}_+ \times \Omega, \tag{1.4}$$

completed by the initial and boundary conditions

$$\left.\begin{array}{l} u(0,\cdot)=u_0,\ \ v(0,\cdot)=v_0 \ \text{ on } \Omega, \\ \nu\cdot\nabla u=\nu\cdot\nabla v\ =\ 0 \ \text{ on } \mathbb{R}_+\times\Gamma, \end{array}\right\} \tag{1.5}$$

where

$$u_0=\frac{U_0}{\overline{U_0}},\quad v_0=\chi(V_0-\overline{V_0}) \tag{1.6}$$

with

$$\overline{u_0}=1,\quad \overline{v_0}=0.$$

Obviously, $\overline{u}(t)=1,\ \overline{v}(t)=0$ for $t\geq 0$. The first component u of the solution (u,v), should be positive (being a normed concentration) whereas the second component v as the deviation from a spatial mean may change sign.

2. Preliminaries

We consider the two–dimensional case $N=2$ and assume that $\Omega\subset\mathbb{R}^2$ is a bounded, finitely connected domain with closure $\overline{\Omega}$ and boundary $\Gamma=\partial\Omega$ which we assume as piecewise smooth, more exactly as piecewise C^2 with a finite number of vertices with non–vanishing interior angles. We denote by $C^k=C^k(\overline{\Omega})$ ($k\geq 0$ an integer , $C^0=C$) the usual spaces of continuously differentiable functions. By $L_p=L_p(\Omega)$, $H^k_p=H^k_p(\Omega)$ for $p\geq 1$ we denote the Lebesgue spaces and Sobolev spaces of functions on Ω with the usual norms $\|\cdot\|_p$, $\|\cdot\|_{k,p}$, $\|\cdot\|_2=\|\cdot\|$ and we write $H^1(\Omega)=H^1_2(\Omega)$ ([A],[KJF],[GT]). For the space $L_\infty(\Omega)$ we denote by $L^+_\infty(\Omega)$ the cone of non-negative elements. For a Banach space X we denote its dual by X^* , the dual pairing between $f\in X^*$, $g\in X$ will be denoted by $\langle f,g\rangle$. If X is a Banach space with norm $\|\cdot\|_X$, we denote for $T>0$ by $L_p(0,T;X)$ $(1\leq p\leq\infty)$ the Banach space of all (equivalence classes of) Bochner measurable functions $u:(0,T)\longrightarrow X$ such that $\|u(\cdot)\|_X\in L_p(0,T)$. Correspondingly, if S is an interval of the reals, we denote by $C(S;X)$ the space of continuous functions on S with values in X , especially, $C([0,T];X)$ is a Banach space. Occasionally we (ab)use the notation $L_p(0,T;X^+)$ to denote the set of L_p–Bochner–integrable functions on $(0,T)$ with values in the positive cone X^+ of a Banach space X . We have the continuous and dense imbeddings

$$X=H^1\subset L_2\subset(H^1)^*=X^*$$

and identify $L_2(0,T;L_2)=L_2(Q_T)$, $Q_T=(0,T)\times\Omega$. For functions $u\in L_2(0,T;X)$ with time derivative $u'\in L_2(0,T;X^*)$ (understood in the sense of distributions from $(0,T)$ with values in $(H^1)^*$) we have the imbedding

$$W^1_2(0,T;X)=\{u:u\in L_2(0,T;X),\ u'\in L_2(0,T;X^*)\}\subset C([0,T];L_2)$$

and the rule of partial integration ([LM],[GGZ])

$$\frac{1}{2}\left(\|u(t)\|^2-\|u(s)\|^2\right)=\int_t^s\langle u'(\tau),u(\tau)\rangle\,d\tau\,,\quad t,s\in[0,T].$$

Lemma 2.1. *Let* $h \in H^1$, $\overline{h} = 0$. *Then there is a constant* $\lambda > 0$ *such that*

$$\lambda ||h||^2 \leq ||\nabla h||^2, \tag{2.1}$$

Let $h \in H_1^1$, $\overline{h} = 0$. *Then there is a constant* $\mu > 0$ *such that*

$$\mu ||h||^2 \leq ||h||_{1,1}^2. \tag{2.2}$$

Proof. This is a simple consequence of the Sobolev imbedding theorem. □

Remark 2.1. In (2.2) we take

$$||h||_{1,1}^2 = ||h_x||_1^2 + ||h_y||_1^2.$$

We put $\mathcal{W} = \{h \in H_1^1 : \overline{h} = 0\}$, $\mathcal{H} = \{h \in H^1 : \overline{h} = 0\}$. Obviously $\mathcal{H} \subset \mathcal{W}$ and for the best imbedding constants $\bar{\lambda}, \bar{\mu}$, defined by

$$\bar{\lambda} = \inf_{h \in \mathcal{H}, h \neq 0} \frac{||\nabla h||^2}{||h||^2} \; , \quad \bar{\mu} = \inf_{h \in \mathcal{W}, h \neq 0} \frac{||h||_{1,1}^2}{||h||^2} \; ,$$

holds $\bar{\mu} \leq |\Omega| \bar{\lambda}$ by the Schwarz inequality.

Remark 2.2. In the special case of a rectangular domain $\Omega = (0,a) \times (0,b)$ the following estimate can be proved for $h \in H_1^1$ (see Appendix):

$$||h||^2 \leq 2\{|\Omega|(\overline{h})^2 + A(||h_x||_1^2 + ||h_y||_1^2)\}, \tag{2.3}$$

where

$$A = \frac{1}{4}\left(\frac{b}{a} + \frac{a}{b} + \sqrt{\left(\frac{b}{a} - \frac{a}{b}\right)^2 + 16}\right). \tag{2.4}$$

Lemma 2.2. *Let* $h \in H_1^1$, $\overline{h} = 1$, $h \geq 0$ *a.e.* . *Then*

$$||h - 1||^2 \leq 4k ||\nabla h^{1/2}||^2, \quad k = |\Omega|/\mu. \tag{2.5}$$

Proof. From (2.2) we get

$$\begin{aligned} \mu ||h - \overline{h}||^2 &\leq ||h_x||_1^2 + ||h_y||_1^2 = 4(||h^{1/2}(h^{1/2})_x||_1^2 + ||h^{1/2}(h^{1/2})_y||_1^2) \\ &\leq 4||h||_1 ||\nabla h^{1/2}||^2 \end{aligned}$$

and with $\overline{h} = 1$ the assertion follows. □

We need some elementary facts about Orlicz spaces (see e.g. [A],[KJF]). For the couple of complementary Young functions (or N–functions)

$$\Phi(s) = (s+1)\log(s+1) - s\,, \quad \Psi(s) = \exp(s) - s - 1, \quad s \geq 0,$$

holds Young's inequality

$$st \leq \Phi(s) + \Psi(t), \quad s \geq 0,\ t \geq 0. \tag{2.6}$$

We define the corresponding Orlicz spaces by

$$L_\Phi = \{g \in L_1 : \|g\|_\Phi < \infty\}\,, \quad L_\Psi = \{h \in L_1 : \|g\|_\Psi < \infty\},$$

where

$$\begin{aligned} \|g\|_\Phi &= \sup_h \left\{ \left| \int_\Omega gh\, d\Omega \right| : \int_\Omega \Psi(|h|)\, d\Omega < 1 \right\}, \\ \|h\|_\Psi &= \inf \left\{ c > 0 : \int_\Omega \Psi\left(\frac{|h|}{c}\right) d\Omega < 1 \right\}. \end{aligned}$$

L_Φ and L_Ψ are Banach spaces. Since Φ satisfies the so–called Δ_2–condition, i.e., there exists a positive constant $k > 0$ (in our case $k = 4$) such that for every $s \geq 0$

$$\Phi(2s) \leq k\Phi(s)\,, \tag{2.7}$$

we have the following

Lemma 2.3. *A sequence* $h_n \subset L_\Phi$ *converges to* $h \in L_\Phi$ *if and only if*

$$\lim_{n\to\infty} \int_\Omega \Phi(|h_n - h|)\, d\Omega = 0.$$

Proof. See [A], Section 8.13 , or [KJF], Section 3.10. □

Lemma 2.4. *There exists the imbedding* $H^1 \subset L_\Psi$ *.*

Proof. This is a special case of a more general result of Trudinger [T] on the limiting case of the Sobolev imbedding theorem. See also [Mo2], [A], Section 8.25 , or [KJF], Section 7.2. □

Lemma 2.5. *Suppose that* $\Omega \subset \mathbb{R}^2$ *is a piecewise* C^2 *, bounded, finitely connected domain with finite number of vertices. Let* Θ *be the minimum interior angle at the vertices of* Ω *. Then there exists a constant* c_Ω *such that for all* $h \in H^1$ *with*

$$\int_\Omega |\nabla h|^2\, d\Omega \leq 1, \quad \overline{h} = 0,$$

we have

$$\int_\Omega e^{2\Theta h^2}\, d\Omega \leq c_\Omega. \tag{2.8}$$

Proof. We quote here Proposition 2.3. of [ChY], formulated there for $h \in C^1(\overline{\Omega})$. By the usual density arguments the assertion can be extended to $h \in H^1$. For a similar result in the case of Dirichlet boundary conditions see [Mo1]. □

Corollary 2.6. *For arbitrary* $h \in H^1$ *one has*

$$\int_\Omega e^{|h|}\, d\Omega \le c_\Omega \exp\left[\frac{1}{8\Theta}\int_\Omega |\nabla h|^2\, d\Omega + |\overline{h}|\right]. \tag{2.9}$$

Proof. From

$$|h| \le |h-\overline{h}| + |\overline{h}| \le \frac{2\Theta}{a}|h-\overline{h}|^2 + \frac{a}{8\Theta} + |\overline{h}|, \quad a = \int_\Omega |\nabla h|^2\, d\Omega,$$

the assertion follows by (2.8). Another proof is given in [KW], formula (3.5). □

3. Local existence in time, uniqueness

In this paper we are interested mainly in the global behaviour of solutions to (1.4), (1.5) in nonsmooth domains. To this end we need adequate local existence results. Unfortunately, we could not find in the literature the results on parabolic systems we need. Hence, for the sake of completeness, we sketch in the following a proof of the existence of (weak) solutions.

Definition 3.1. A pair of functions (u,v) with

$$\begin{aligned} &u \in L_\infty(0,T;L_\infty^+) \cap L_2(0,T;H^1), \quad u_t \in L_2(0,T;(H^1)^*),\\ &v \in L_\infty(0,T;L_\infty) \cap C(0,T;H^1), \quad v_t \in L_2(0,T;L_2) \end{aligned}$$

is called a weak solution of (1.4), (1.5) if $\forall h \in L_2(0,T;H^1)$ the following identities hold:

$$\begin{aligned} \int_0^T \langle u_t, h\rangle\, dt + \int_0^T\int_\Omega (\nabla u - u\nabla v)\cdot\nabla h\, d\Omega\, dt &= 0,\\ \int_0^T (v_t,h)\, dt + \int_0^T\int_\Omega \{\alpha\nabla v\cdot\nabla h + (\beta v - \gamma(u-1))\, h\}\, d\Omega\, dt &= 0. \end{aligned}$$

Remark 3.1. This definition is equivalent with

$$\begin{aligned} \langle u_t, g\rangle + \int_\Omega (\nabla u - u\nabla v)\cdot\nabla g\, d\Omega &= 0,\\ (v_t,g) + \int_\Omega \{\alpha\nabla v\cdot\nabla g + (\beta v - \gamma(u-1))\, g\}\, d\Omega &= 0 \end{aligned}$$

for almost all $t \in (0,T)$ and $\forall g \in H^1$.

Theorem 3.2. *For* $u_0 \in L_\infty^+$ *and* $v_0 \in H_p^1$, $p > 2$ *and appropriate* $T > 0$ *there is a unique weak solution of* (1.4), (1.5) *with* $u(0) = u_0$, $v(0) = v_0$. *Moreover, for* $0 \le t < T$ *holds* $t \longrightarrow u(t) \in L_\infty^+$ *and the function* $t \longrightarrow \|\nabla v(t)\|^2$ *is absolutely continuous on* $[0, T]$.

Proof. 1. Existence. Function arguments are sometimes omitted.

(i) We introduce a new unknown function w by

$$w = e^{-v} u$$

and transform (1.4), (1.5) into

$$\left.\begin{array}{l} v_t - \alpha \Delta v + \beta v = \gamma(e^v w - 1) \\ (e^v w)_t - \nabla \cdot (e^v \nabla w) = 0 \end{array}\right\} \text{ on } (0,T) \times \Omega, \tag{3.1}$$

with the initial and boundary conditions

$$v(0) = v_0, \;\; w(0) = w_0 = e^{-v_0} u_0 \;\text{ on } \Omega, \quad \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \;\text{ on } (0,T) \times \Gamma. \tag{3.2}$$

(ii) We want to apply Schauder's fixed point theorem. To this end we define for appropriate $K > 0, T > 0$ the map

$$A_K \in (X \longrightarrow X), \; X = L_2(0,T; L_p^+), \; p > 2, \quad \text{by } f \longrightarrow w := A_K f$$

where w is the solution of

$$\left.\begin{array}{ll} v_t - \alpha \Delta v + \beta v = \gamma(e^{v_K} f - 1), & v(0) = v_0, \; v_K = \operatorname{sgn}(v) \min(|v|, K), \\ (e^{v_K} w)_t - \nabla \cdot (e^{v_K} \nabla w) = 0, & w(0) = e^{-v_0} u_0. \end{array}\right\} \tag{3.3}$$

In the spirit of [LSU], Chap.III, Chap.V there exists for given $f \in X$ a solution v of the first equation with

$$v \in L_2(0,T; H^1), \; v_t \in L_2(0,T; L_2), \quad \text{i.e. } v_t \in L_2(Q_T).$$

and a solution w of the second equation with

$$e^{v_K} w \in L_2(0,T; H^1 \cap L_2^+), \; (e^{v_K} w)_t \in L_2(Q_T).$$

So the map A_K is well defined. Moreover (see e.g. [B]), the function $t \longrightarrow \|\nabla v(t)\|^2$ is absolutely continuous on $[0, T]$.
We have only to add the proof of the non–negativity of w . Testing the second equation of (3.3) with $w^- = \max(-w, 0)$ we find by the rules of calculus in Sobolev spaces

$$\int_\Omega (e^{v_K} w)_t w^- d\Omega + \int_\Omega e^{v_K} \nabla w \cdot \nabla w^- d\Omega = \int_\Omega (e^{v_K} w)_t w^- d\Omega + \int_\Omega e^{v_K} |\nabla w^-|^2 d\Omega = 0.$$

With the identity

$$\frac{d}{dt} \int_\Omega \left(e^{v_K} w^-\right)^2 e^{-v_K} d\Omega = 2 \int_\Omega (e^{v_K} w)_t w^- d\Omega - \int_\Omega \left(e^{v_K} w^-\right)^2 e^{-v_K} (v_K)_t d\Omega$$

we find

$$\frac{d}{dt}\int_\Omega \left(e^{v_K}w^-\right)^2 e^{-v_K}d\Omega + \int_\Omega \left(e^{v_K}w^-\right)^2 e^{-v_K}(v_K)_t d\Omega + 2\int_\Omega e^{v_K}|\nabla w^-|^2 d\Omega = 0.$$

Integration on $[0,t]$, taking into account that $w^-(0)=0$, gives

$$\int_\Omega e^{v_K}\left(w^-\right)^2 d\Omega + 2\int_0^t\int_\Omega e^{v_K}|\nabla w^-|^2 d\Omega ds \le \int_0^t\int_\Omega e^{v_K}\left(w^-\right)^2 |(v_K)_t|\, d\Omega ds. \tag{3.4}$$

By Gagliardo–Nirenberg's inequality (for a proof see e.g. [He])

$$\|w^-\|_4^2 \le c\|w^-\|_{H^1}\|w^-\| \le c(\|\nabla w^-\| + \|w^-\|)\|w^-\|$$

and with $\|(v_K)_t\| \le \|v_t\|$ we estimate on the right hand side of (3.4)

$$\begin{aligned}\int_\Omega e^{v_K}\left(w^-\right)^2 |(v_K)_t|\, d\Omega &\le c\,e^K\|(v_K)_t\|\,\|w^-\|_4^2 \le ce^K\|v_t\|(\|\nabla w^-\| + \|w^-\|)\|w^-\| \\ &\le C(K,\delta)\left(\|v_t\|^2+1\right)\|w^-\|^2 + \delta\|\nabla w^-\|^2\end{aligned}$$

and get

$$\begin{aligned}e^{-K}\left(\int_\Omega \left(w^-(t)\right)^2 d\Omega + 2\int_0^t \|\nabla w^-\|^2 ds\right) &\le C(K,\delta)\int_0^t \left(\|v_t\|^2+1\right)\|w^-\|^2 ds \\ &\quad +\delta\int_0^t \|\nabla w^-\|^2 ds.\end{aligned}$$

With the choice $\delta < 2e^{-K}$ we obtain the Gronwall–type estimate

$$\left\|w^-(t)\right\|^2 \le C(K)\int_0^t \left(\|v_t\|^2+1\right)\|w^-\|^2 ds$$

from which follows $\|w^-(t)\| = 0\ \forall t\in[0,T)$, i.e. $w(t)\in L_2^+$.

(*iii*) Next we show that the map A_K sends the ball $B=\{f : \|f\|_X \le R\}$ into itself for appropriate $R>0$ and sufficiently small $T>0$. We test the first equation of (3.3) with v_t on $Q_t = (0,t)\times\Omega$ and obtain

$$\begin{aligned}\|v_t\|_{Q_t}^2 + \frac{\alpha}{2}\left(\|\nabla v(t)\|^2 - \|\nabla v(0)\|^2\right) &+ \frac{\beta}{2}\left(\|v(t)\|^2 - \|v(0)\|^2\right) \\ &= \gamma\int_0^t\int_\Omega \left(e^{v_K}f-1\right)v_t d\Omega\, ds.\end{aligned}$$

We estimate the right hand side by

$$\left|\gamma\int_0^t\int_\Omega (e^{v_K}f-1)v_t d\Omega\, ds\right| \le \gamma\left(e^K\|f\|_{Q_t} + |\Omega|^{1/2}t^{1/2}\right)\|v_t\|_{Q_t}$$

and get with an appropriate constant $C(K,R)$

$$\|v_t\|_{Q_T}^2 + \sup_{[0,T]}\|\nabla v(t)\|^2 \le C(K,R). \tag{3.5}$$

Testing the second equation of (3.3) with w gives

$$\int_\Omega w(e^{v_K}w)_t d\Omega + \int_\Omega e^{v_K}|\nabla w|^2 d\Omega = 0$$

or, after some calculation,

$$\frac{d}{dt}\left\|e^{v_K/2}w\right\|^2 + 2\int_\Omega e^{v_K}|\nabla w|^2 d\Omega = \int_\Omega e^{v_K}(v_K)_t w^2 d\Omega.$$

By arguments similar to those used in (ii) we obtain

$$\left|\int_\Omega e^{v_K}(v_K)_t w^2 d\Omega\right| \le C(K,\delta)\left(\|v_t\|^2+1\right)\|w\|^2 + \delta\|\nabla w\|^2.$$

We take $\delta = e^{-K}$ and get

$$\frac{d}{dt}\left\|e^{v_K/2}w\right\|^2 + 2\int_\Omega e^{v_K}|\nabla w|^2 d\Omega \le C(K)\left(\|v_t\|^2+1\right)\|w\|^2 + e^{-K}\|\nabla w\|^2,$$

from which we find (by integrating and multiplying with e^K) the Gronwall–type estimate

$$\|w(t)\|^2 + \int_0^t \|\nabla w(s)\|^2 ds \le e^K\left(\|w(0)\|^2 + \int_0^t (\|v_t\|^2+1)\|w\|^2 ds\right)$$

and, with Gronwall's lemma and (3.5)

$$\|w(t)\|^2 + \int_0^t \|\nabla w(s)\|^2 ds \le e^K\|w(0)\|^2 \exp\left(C(K)\int_0^t (\|v_t\|^2+1)\,ds\right) \le C_1(K,R).$$

This implies (by the Gagliardo–Nirenberg and Hölder inequality)

$$\begin{aligned} \|w\|_X^2 &= \int_0^T \|w(s)\|_p^2 ds \le c\int_0^T \|w(s)\|_{H^1}^{\frac{2(p-2)}{p}}\|w(s)\|^{4/p} ds \\ &\le c\left(\int_0^T \|w(s)\|_{H^1}^2 ds\right)^{\frac{p-2}{p}}\left(\int_0^T \|w(s)\|^2 ds\right)^{\frac{2}{p}} \le C_2(K,R)T^{2/p}. \end{aligned}$$

Now, choosing $T>0$ sufficiently small, we can assure that $\|w\|_X \le R$.

(iv) In order to show compactness of the map A_K it will be sufficient to have a bound for $\|w_t\|_Y$, where $Y = L_2(0,T;(H^1)^*)$.
For $h \in L_2(0,T;H_p^1)$ we have with the second equation (3.3)

$$\begin{aligned} (w_t,h) &= (e^{v_K}w_t, e^{-v_K}h) = \left((e^{v_K}w)_t, e^{-v_K}h\right) - ((v_K)_t w, h) \\ &= -\left(e^{v_K}\nabla w, \nabla(e^{-v_K}h)\right) - ((v_K)_t w, h) = -(\nabla w, \nabla h - h\nabla v_K) - ((v_K)_t w, h) \end{aligned}$$

which we estimate as

$$|(w_t,h)| \le \|\nabla w\|(\|\nabla h\| + \|h\|_\infty\|\nabla v\|) + \|v_t\|\,\|w\|\,\|h\|_\infty.$$

By Schwarz's inequality

$$\int_0^T \|\nabla w(s)\| \|\nabla h(s)\| ds \leq \|w\|_{L_2(0,T;H^1)} \|h\|_{L_2(0,T;H^1)}$$

and, similarly,

$$\int_0^T \|\nabla w(s)\| \|h(s)\|_\infty \|\nabla v(s)\| ds \leq \|\nabla v\|_{L_\infty(0,T;L_2)} \|w\|_{L_2(0,T;H^1)} \|h\|_{L_2(0,T;L_\infty)}$$

$$\int_0^T \|v_t(s)\| \|h(s)\|_\infty \|w(s)\| ds \leq \|w\|_{L_\infty(0,T;L_2)} \|v_t\|_{L_2(Q_T)} \|h\|_{L_2(0,T;L_\infty)}$$

we get

$$\begin{aligned}\int_0^T |(w_t, h)| ds \leq \quad & \|w\|_{L_2(0,T;H^1)} \left(\|h\|_{L_2(0,T;H^1)} + \|h\|_{L_2(0,T;L_\infty)} \|\nabla v\|_{L_\infty(0,T;L_2)} \right) \\ & + \|v_t\|_{L_2(Q_T)} \|w\|_{L_\infty(0,T;L_2)} \|h\|_{L_2(0,T;L_\infty)}.\end{aligned}$$

For $p > 2$ and $\Omega \subset \mathbb{R}^2$ we have the imbeddings

$$L_2(0,T;H_p^1) \subset L_2(0,T;H^1); \quad L_2(0,T;H_p^1) \subset L_2(0,T;L_\infty)$$

and get with the estimates proved in *(iii)*

$$\int_0^T |(w_t, h)| ds \leq C \|h\|_{L_2(0,T;H_p^1)} \quad \text{or} \quad \|w_t\|_Y \leq C.$$

(v) For $1 < p < \infty$ we have the imbedding $H_p^1 \subset L_p \subset (H_p^1)^*$ with compact imbedding $H_p^1 \subset L_p$. Then (as a consequence of Theorem 5.1 in [L]) the imbedding of $W = \{w : w \in L_2(0,T;H^1),\ w_t \in L_2(0,T;(H^1)^*)\}$ in $L_2(0,T;L_p)$ is compact. This implies that $A_K(B)$ is a precompact set of $L_2(0,T;L_p)$. By similar arguments as before one can show that $A_K : L_2(0,T;L_p) \longrightarrow L_2(0,T;L_p)$ is continuous. Therefore Schauder's fixed point theorem guarantees the existence of a fixed point $w \in L_2(0,T;L_p)$ such that $A_K w = w$.
To get ride of the cutoff introduced in (3.3) we denote by v the solution of the first equation (3.3) with $f = w$ and by z the solution of

$$z_t - \alpha \Delta z + \beta z = \gamma(e^K w - 1), \ z(0) = v_0.$$

Standard regularity results on linear parabolic equations ([LSU], Chap. III, Theorem 7.1) show that $v, z \in L_\infty(Q_T)$. The difference $g = z - v$ satisfies a.e. on $(0,T)$

$$(g_t, h) + \alpha(\nabla g, \nabla h) + \beta(g, h) = \gamma\left((e^K - e^{v_K})w, h\right) \quad \forall h \in H^1, \ g(0) = 0.$$

Testing with $h = -g^- = \min(0, g)$ and taking into account that $w \geq 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + \alpha\|\nabla h\|^2 + \beta\|h\|^2 \leq 0, \quad h(0) = 0.$$

The usual argumentation yields $g^- = 0$, i.e., $g \geq 0$. Choosing now $K > \|v_0\|_\infty$, we have for sufficiently small $t_0 > 0$

$$\|v(t)\|_\infty \leq \|z(t)\|_\infty \leq K, \quad 0 \leq t \leq t_0.$$

This implies $v_K = v$ on $[0, t_0]$. This means that v, w is a solution of (3.1) for sufficiently small $T = t_0 > 0$.

(vi) From $u \in L_2(0,T;L_2)$ and $v_t \in L_2(0,T;L_2)$ it follows that the function $t \longrightarrow \|\nabla v(t)\|^2$ is absolutely continuous on $[0,T]$. (See [B], Lemme 3.3, p. 73.)

(vii) It remains to show that

$$u = e^v w \in L_\infty(Q_T).$$

By the arguments of [LSU], Chap.III, §7 this holds if

$$|\nabla v| \in L_q(0,T;L_p),\ p > 2,\ q > \frac{2p}{p-2}\,. \tag{3.6}$$

In order to prove (3.6) we test the second equation of (3.1) by w^{p-1} for $p \in [2,\infty)$ and obtain after some calculation

$$\frac{d}{dt}\left\|e^{v/2}w^{p/2}\right\|^2 + \frac{4(p-1)}{p}\int_\Omega e^v\left|\nabla(w^{p/2})\right|^2 d\Omega = -(p-1)\int_\Omega v_t e^v w^p d\Omega.$$

We estimate the right hand side similarly to (iii) using $\|v\|_{L_\infty(Q_T)} \leq K$ and the Gagliardo–Nirenberg inequality $\|f\|_4^2 \leq C\|f\|_{H^1}\|f\|$ as follows

$$\int_\Omega v_t e^v w^p d\Omega \leq \|v_t\|\left(\int_\Omega e^{2v}w^{2p}d\Omega\right)^{1/2} \leq e^K\|v_t\|\left\|w^{p/2}\right\|_4^2 \leq Ce^K\|v_t\|\left\|w^{p/2}\right\|_{H^1}\left\|w^{p/2}\right\|$$

and get

$$\begin{aligned}\frac{d}{dt}\left\|e^{v/2}w^{p/2}\right\|^2 &+ \frac{4(p-1)}{p}\int_\Omega e^v\left|\nabla(w^{p/2})\right|^2 d\Omega \leq C(p,K)\|v_t\|\left\|w^{p/2}\right\|_{H^1}\left\|w^{p/2}\right\| \\ &\leq C(p,K)\|v_t\|\left(\left\|\nabla(w^{p/2})\right\| + \left\|w^{p/2}\right\|\right)\left\|w^{p/2}\right\| \\ &\leq C_1(p)\left(\|v_t\|^2+1\right)\left\|w^{p/2}\right\|^2 + \frac{4(p-1)}{p}e^{-K}\left\|\nabla(w^{p/2})\right\|^2.\end{aligned}$$

This shows that

$$\frac{d}{dt}\left\|e^{v/2}w^{p/2}\right\|^2 \leq C_0\left(\|v_t\|^2+1\right)\left\|w^{p/2}\right\|^2 \leq C_0e^{K/2}\left(\|v_t\|^2+1\right)\left\|e^{v/2}w^{p/2}\right\|^2$$

and by Gronwall's lemma we obtain

$$\sup_{t\in[0,T]}\int_\Omega w^p(t,\cdot)d\Omega \leq C, \quad \text{i.e., } w \in L_\infty(0,T;L_p).$$

Now, by a result of [Gr], for some $p > 2$ the mapping $f \longrightarrow z$ defined by

$$z_t - \alpha\Delta z + \beta z = f, \quad z(0) = v_0$$

is continuous from $L_p(0,T;L_p)$ to $L_p(0,T;H_p^1)$ and hence, by a result of [D] on L_p–regularity of evolution equations, continuous also from $L_q(0,T;L_p)$ to $L_q(0,T;H_p^1)$ for

$q \in [p, \infty)$. Applying this result with $f = \gamma(u-1) = \gamma(e^v w - 1)$ yields $v \in L_q(0,T;H^1_p)$ which implies (3.6) for suitably chosen q .
2. Uniqueness. Let (u_i, v_i), $(i = 1,2)$ be solutions to (1.4), (1.5) with the same initial values $(u_i(0), v_i(0)) = (u_0, v_0)$. From

$$\frac{d}{dt}\left\{u_1(\log u_1 - 1) + u_2(\log u_2 - 1) - (u_1+u_2)\left(\log\left(\frac{u_1+u_2}{2}\right) - 1\right)\right\}$$
$$= u_{1t}\log\frac{2u_1}{u_1+u_2} + u_{2t}\log\frac{2u_2}{u_1+u_2}$$

we obtain with the help of the equations for u_i

$$\int_\Omega \left\{u_1(\log u_1 - 1) + u_2(\log u_2 - 1) - (u_1+u_2)\left[\log\left(\frac{u_1+u_2}{2}\right) - 1\right]\right\}(t)d\Omega$$
$$= \int_0^t\int_\Omega \left[u_{1t}\log\frac{2u_1}{u_1+u_2} + u_{2t}\log\frac{2u_2}{u_1+u_2}\right]d\Omega ds$$
$$= -\int_0^t\int_\Omega \frac{1}{u_1+u_2}\Big[(\nabla u_1 - u_1\nabla v_1)\cdot\left(\frac{u_2}{u_1}\nabla u_1 - \nabla u_2\right) +$$
$$(\nabla u_2 - u_2\nabla v_2)\cdot\left(\frac{u_1}{u_2}\nabla u_2 - \nabla u_1\right)\Big]d\Omega ds$$
$$= -\int_0^t\int_\Omega \frac{u_1u_2}{u_1+u_2}\left[\nabla\left(\log\frac{u_1}{u_2} - (v_1 - v_2)\right)\cdot\nabla\left(\log\frac{u_1}{u_2}\right)\right]d\Omega ds.$$

We estimate the left hand side by Lemma 6.5 (Appendix) and the right hand side by the Schwarz inequality and the arithmetic–geometric mean inequality:

$$\frac{1}{4}\|(\sqrt{u_1} - \sqrt{u_2})(t)\|^2 \le -\frac{1}{2}\int_0^t\int_\Omega \frac{u_1u_2}{u_1+u_2}\left[\left|\nabla\log\left(\frac{u_1}{u_2}\right)\right|^2 - |\nabla(v_1-v_2)|^2\right]d\Omega ds$$
$$\le \frac{1}{8}\|u_1+u_2\|_{L_\infty(Q_t)}\int_0^t \|\nabla(v_1-v_2)\|^2 ds.$$

On the other hand we have (using the absolute continuity of $t \longrightarrow \|\nabla v(t)\|^2$)

$$\int_0^t \|(v_1-v_2)_t\|^2 ds + \frac{\alpha}{2}\|\nabla(v_1-v_2)(t)\|^2 + \frac{\beta}{2}\|(v_1-v_2)(t)\|^2 =$$
$$= \gamma\int_0^t\int_\Omega (u_1-u_2)(v_1-v_2)_t d\Omega ds$$
$$\le \int_0^t\left(\frac{\gamma^2}{4}\|u_1-u_2\|^2 + \|(v_1-v_2)_t\|^2\right)ds$$
$$\le \int_0^t\left(\frac{\gamma^2}{4}\|\sqrt{u_1}+\sqrt{u_2}\|^2_\infty\|\sqrt{u_1}-\sqrt{u_2}\|^2 + \|(v_1-v_2)_t\|^2\right)ds$$
$$\le \frac{\gamma^2}{2}\|u_1+u_2\|_{L_\infty(Q_t)}\int_0^t\|\sqrt{u_1}-\sqrt{u_2}\|^2 ds + \int_0^t\|(v_1-v_2)_t\|^2 ds.$$

Combining this with the estimate above we get

$$\|(\sqrt{u_1}-\sqrt{u_2})(t)\|^2 + \|\nabla(v_1-v_2)(t)\|^2$$
$$\le \max\left(\frac{\gamma^2}{\alpha}, \frac{1}{2}\right) \|u_1+u_2\|_{L_\infty(Q_t)} \int_0^t \left(\|\sqrt{u_1}-\sqrt{u_2}\|^2 + \|\nabla(v_1-v_2)(t)\|^2\right) ds.$$

Uniqueness follows now by Gronwall's lemma. □

4. Estimates of Lyapunov functionals

Let (u, v) be a solution of (1.4) with $u \ge 0$. We introduce the two Lyapunov functionals:

$$G(u,v) = \int_\Omega \left[\frac{\alpha^2}{k}(u(\log u - 1) + 1) + \frac{1}{2}\left(\alpha|\nabla v|^2 + \beta\left(1+\frac{\alpha}{k\gamma}\right)v^2\right)\right] d\Omega, \tag{4.1}$$

$$F(u,v) = \int_\Omega \left[\frac{1}{2\gamma}\left(\alpha|\nabla v|^2 + \beta v^2\right) + u(\log u - 1) + 1 - (u-1)v\right] d\Omega. \tag{4.2}$$

Lemma 4.1. *Let* $\frac{k\gamma}{\alpha} < 1$, $k = \frac{|\Omega|}{\mu}$ *and denote by* λ *the smallest positive eigenvalue of the operator* $h \longrightarrow -\alpha\Delta h + \beta h$ *under homogeneous Neumann conditions. Define*

$$C_0 = \frac{1}{4}\min\left(2, \left(\frac{\alpha}{k\gamma}\right)^2 - 1\right), \quad C_1 = \frac{k\gamma^2}{\alpha^2}C_0, \quad C_2 = \frac{C_0\lambda k\gamma + \alpha\beta}{\alpha + k\gamma}.$$

Then the following estimate holds:

$$\frac{d}{dt}G(u,v) \le -aG(u,v), \quad \textit{where} \quad a = \min(C_1, C_2). \tag{4.3}$$

Proof. Because of $u(\log u - 1) + 1 \ge 0$ for $u \ge 0$ we have $G(u,v) \ge 0$. Differentiation gives

$$\frac{d}{dt}G(u,v) = \frac{\alpha}{2}\frac{d}{dt}\|\nabla v\|^2 + \int_\Omega \left[\frac{\alpha^2}{k}(u_t \log u) + \beta\left(1+\frac{\alpha}{k\gamma}\right) v\, v_t\right] d\Omega.$$

Using the first equation of the system (1.4) (in its weak form) we have

$$\int_\Omega u_t \log u \, d\Omega = -\int_\Omega (\nabla u - (u\nabla v)) \cdot \nabla \log u \, d\Omega.$$

With $\nabla(\sqrt{u}) = \frac{\nabla u}{2\sqrt{u}}$ and $\nabla(\log u) = \frac{\nabla u}{u}$ we get

$$\int_\Omega u_t \log u \, d\Omega = -4\int_\Omega |\nabla(\sqrt{u})|^2 d\Omega + \int_\Omega \nabla u \cdot \nabla v \, d\Omega.$$

Using the second equation of the system (1.4) analogously we have

$$\frac{\alpha}{2}\frac{d}{dt}\|\nabla v\|^2 = -\int_\Omega v_t(v_t + \beta\, v - \gamma(u-1))\, d\Omega,$$

$$\int_\Omega v\, v_t\, d\Omega = \int_\Omega (-\alpha|\nabla v|^2 - \beta\, v^2 + \gamma\, v\,(u-1))\, d\Omega,$$

hence, putting $w = u - 1$,

$$\begin{aligned}\frac{\alpha}{2}\frac{d}{dt}\|\nabla v\|^2 &+ \int_\Omega \beta(1+\frac{\alpha}{k\gamma})v\, v_t\, d\Omega \\ &= \int_\Omega \left[-{v_t}^2 + \gamma\, w\, v_t - \frac{\alpha\beta}{k\gamma}(\alpha|\nabla v|^2 + \beta v^2 - \gamma v w)\right] d\Omega.\end{aligned}$$

Testing appropriately ($h = 1,\ h = u$) we get

$$\int_\Omega (v_t + \beta v - \gamma w)\, d\Omega = 0$$

and

$$\alpha \int_\Omega \nabla u \cdot \nabla v\, d\Omega = -\int_\Omega u(v_t + \beta v - \gamma w)\, d\Omega = \int_\Omega w(\gamma w - v_t - \beta v)\, d\Omega.$$

So we obtain

$$\frac{d}{dt}G(u,v) = \int_\Omega \left[\frac{\alpha}{k}\left(\gamma w^2 - 4\alpha|\nabla(\sqrt{u})|^2\right) + \left(\gamma - \frac{\alpha}{k}\right) v_t\, w - {v_t}^2 - \frac{\alpha\beta}{k\gamma}\left(\alpha|\nabla v|^2 + \beta v^2\right)\right] d\Omega.$$

With the estimate

$$\left|\left(\gamma - \frac{\alpha}{k}\right) v_t\, w\right| \le \frac{1}{2}{v_t}^2 + \frac{1}{2}\left(\gamma - \frac{\alpha}{k}\right)^2 w^2 \quad \text{and (2.5):} \quad \|w\|^2 \le 4k\|\nabla(\sqrt{u})\|^2$$

we find

$$\begin{aligned}\frac{d}{dt}G(u,v) &\le -\int_\Omega \left[\frac{1}{2}\left(\left(\frac{\alpha}{k\gamma}\right)^2 - 1\right)\gamma^2 w^2 + \frac{1}{2}{v_t}^2 + \frac{\alpha\beta}{k\gamma}\left(\alpha|\nabla v|^2 + \beta v^2\right)\right] d\Omega \\ &\le -\int_\Omega \left[\frac{1}{2}\left(\left(\frac{\alpha}{k\gamma}\right)^2 - 1\right)\gamma^2 w^2 + C_0{v_t}^2 + \frac{\alpha\beta}{k\gamma}\left(\alpha|\nabla v|^2 + \beta v^2\right)\right] d\Omega\end{aligned}$$

for $0 < C_0 \le 1/2$. The already used identity $\alpha\|\nabla v\|^2 + \beta\|v\|^2 = (\gamma w - v_t, v)$ yields the estimate

$$\alpha\|\nabla v\|^2 + \beta\|v\|^2 \le \frac{\lambda}{2}\|v\|^2 + \frac{1}{\lambda}\left(\gamma^2\|w\|^2 + \|v_t\|^2\right)$$

for any $\lambda > 0$. We take for λ the smallest positive eigenvalue of the elliptic operator $h \longrightarrow -\alpha\Delta h + \beta h$ under homogeneous Neumann conditions. By the well–known variational characterization of this value we have

$$\lambda\|v\|^2 \le \alpha\|\nabla v\|^2 + \beta\|v\|^2$$

and from the foregoing estimate

$$\alpha\|\nabla v\|^2 + \beta\|v\|^2 \le \frac{2}{\lambda}\left(\gamma^2\|w\|^2 + \|v_t\|^2\right).$$

This gives for the Lyapunov functional

$$\frac{d}{dt}G(u,v) \le \int_\Omega \left\{ \left(C_0 - \frac{1}{2}\left[\left(\frac{\alpha}{k\gamma}\right)^2 - 1\right]\right)\gamma^2 w^2 - \left(\frac{\alpha\beta}{k\gamma} + \frac{C_0\lambda}{2}\right)\left(\alpha|\nabla v|^2 + \beta v^2\right)\right\} d\Omega.$$

With the choice

$$C_0 = \frac{1}{4}\min\left(2, \left(\frac{\alpha}{k\gamma}\right)^2 - 1\right), \quad C_1 = \frac{k\gamma^2 C_0}{\alpha^2}, \quad C_2 = \frac{C_0\lambda k\gamma + 2\alpha\beta}{\alpha + k\gamma}$$

and using the elementary inequality $u(\log u - 1) + 1 \le (u-1)^2 = w^2$ for $u \ge 0$ (see Appendix, Lemma 6.3) we obtain

$$\frac{d}{dt}G(u,v) \le -\int_\Omega \left[C_1\frac{\alpha^2}{k}\left(u(\log u - 1) + 1\right) + \frac{C_2}{2}\left(\alpha|\nabla v|^2 + \beta(1 + \frac{\alpha}{k\gamma})v^2\right)\right] d\Omega.$$

Taking $a = \min(C_1, C_2)$ ends the proof.□

Theorem 4.2. *Under the conditions of Lemma* 4.1 *holds*

$$G(u(t), v(t)) \le e^{-at}G(u_0, v_0). \tag{4.4}$$

Moreover, exponentially

$$u(t) \longrightarrow u^\star = 1 \ \text{in } L_\Phi, \quad v(t) \longrightarrow v^\star = 0 \ \text{in } H^1 \cap L_\Psi \quad \text{as} \quad t \to \infty. \tag{4.5}$$

Proof. The decay result follows directly from (4.3). A simple calculation shows that

$$\int_\Omega \Phi(|u-1|)d\Omega + \|v\|^2_{H^1} \le c\,G(u,v).$$

Indeed,

$$\|v\|^2_{H^1} = \int_\Omega (|\nabla v|^2 + v^2)d\Omega \le \frac{G(u,v)}{\min\left(\frac{\alpha}{2}, \frac{\beta}{2}(1 + \frac{\alpha}{k\gamma})\right)}$$

and, because of $\Phi(|u-1|) \le u(\log u - 1) + 1$ (see Appendix, Lemma 6.4),

$$\int_\Omega \Phi(|u-1|)d\Omega \le \frac{k}{\alpha^2}G(u,v).$$

Hence (4.5) follows from Lemma 4.1 and Lemma 2.4.□

The following result states a sufficient condition for global boundedness of the trajectories $t \to (u(t), v(t))$ defined by the solutions of (1.4), (1.5) and excludes blow–up in appropriate norms.

Theorem 4.3. *Suppose*

$$\frac{\gamma|\Omega|}{4\alpha\Theta} < 1. \tag{4.6}$$

Then

$$\|\nabla v(t)\| + \|u(t)\|_\Phi \leq C < +\infty \quad \textit{for all} \quad t \geq 0.$$

Moreover, for $1 \leq p < \infty$ *there are constants* $c_1 = c_1(p)$, $c_2 = c_2(p)$ *such that*

$$\|u(t)\|_p \leq c_1 \exp(c_2 t) \quad \textit{for all} \quad t \geq 0$$

and a nondecreasing finite function $t \longrightarrow c(t)$ *such that*

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq c(t).$$

Proof. This theorem is is a simple compilation of assertion (4.10) of Lemma 4.4 and Lemma 4.5 – Lemma 4.8 proved below. □

Before proving these lemmas we give some comments on the rôle of the conditions mentioned in Lemma 4.1 and Theorem 4.3.

Remark 4.1. The condition

$$\kappa_1 = \frac{k\gamma}{\alpha} < 1 \tag{4.7}$$

of Lemma 4.1 and the condition

$$\kappa_2 = \frac{\gamma|\Omega|}{4\alpha\Theta} < 1 \tag{4.8}$$

of Theorem 4.3 are two smallness conditions to control the large time behaviour of the dynamical system (1.4), (1.5). Both conditions can be satisfied if the initial value U_0 has a sufficiently small L_1–norm (see (1.3)). For domains we consider one can show that (see Appendix, Lemma 6.2)

$$\kappa_2 \leq \kappa_1,$$

hence the first condition (4.7) is more restrictive.

Remark 4.2. Numerical experiments give some hints that the second condition (4.8) may be sharp, i.e., violating this condition gives blow–up of the solution in finite time. In the experiments we used the following construction. Let be Ω the rhombic domain with an acute opening angle $\Theta \leq \pi/2$ defined by

$$\Omega = \left\{ (x,y) : \frac{|x|}{a} + \frac{|y|}{b} < 1, \ a = \sqrt{\frac{\tan(\Theta/2)}{2}}, \ b = \frac{1}{\sqrt{2\tan(\Theta/2)}} \right\}.$$

We have $|\Omega| = 1$ and with $R^2 = a^2 + b^2$ we find $R \geq 1$ for the side length of the rhombus. With $\alpha = \delta = \chi = 1$ we get from (1.3) $\gamma = \overline{U_0}$, hence $c_2 = \overline{U_0}/(4\Theta)$ for the left hand side of (4.8). We take the tip $(0, b)$ as the origin for polar coordinates

(r,φ) with the ray $\{(x,y): x=0,\ y\leq b\}$ as polar axis $\varphi=0$. For the initial element $u_0=U_0/\overline{U}_0$ of u in (1.4) we choose the radially symmetric function

$$U_0=U_0(r)=\frac{8(1\pm\sigma)}{\sigma}\ \exp\left(-\frac{r^2}{\sigma}\right) \tag{4.9}$$

where $0<\sigma<1$. The corresponding initial v_0 of v is chosen as the solution of

$$\Delta v_0+\gamma(u_0-1)=0 \text{ in } \Omega, \quad \frac{\partial v_0}{\partial n}=0 \text{ on } \partial\Omega .$$

Simple estimates show that

$$\overline{U_0}=\int_\Omega U_0\,d\Omega\leq\int_{-\Theta/2}^{\Theta/2}\int_0^\infty U_0(r)r dr d\varphi=\frac{8(1\pm\sigma)}{\sigma}\Theta\int_0^\infty \exp\left(-\frac{r^2}{\sigma}\right)r dr=4\Theta(1\pm\sigma)$$

and

$$\overline{U_0}\geq\int_{-\Theta/2}^{\Theta/2}\int_0^R U_0(r)r dr d\varphi=4\Theta(1\pm\sigma)\left[1-\exp\left(-\frac{R^2}{\sigma}\right)\right].$$

With the obvious inequalities

$$1-\exp\left(-\frac{R^2}{\sigma}\right)\geq1-\exp\left(-\frac{1}{\sigma}\right)\geq\frac{1}{1+\sigma}$$

we obtain for the positive sign in (4.9) a violation of condition (4.8):

$$\kappa_2=\frac{\overline{U_0}}{4\Theta}>1,$$

whereas for the negative sign we have

$$\kappa_2=\frac{\overline{U_0}}{4\Theta}<1.$$

The numerical experiments show that even for quite small values of σ the switching between the signs in (4.9) leads to a dramatic change in the behaviour of the solutions. For the positive sign we make the observation that

$$F(u(t),v(t))\longrightarrow-\infty \text{ as well as } \int_\Omega u(t)\log u(t)\,d\Omega\longrightarrow\infty \text{ in finite time,}$$

whereas for the negative sign the same quantities remain bounded.
Compared with rigorous results condition (4.8) is sharp at least up to a factor 2. In fact, it has been proved recently ([HV2]) that radially symmetric solutions to (1.4), (1.5) in a disk blow up in finite time if $\kappa_2>2$.

As the following Lemma 4.4 shows, the functional F decreases with increasing time for any positive constants α,β,γ and remains bounded from below for constants satisfying the conditions of Lemma 4.1.

Lemma 4.4. *We have*

$$\frac{d}{dt}F(u,v) \;=\; -D(u,v) \;\leq\; 0, \tag{4.10}$$

where

$$D(u,v) = \frac{\|v_t\|^2}{\gamma} + \int_\Omega u|\nabla(\log u - v)|^2 d\Omega. \tag{4.11}$$

For $\frac{k\gamma}{\alpha} < 1$ *there is a constant* $C > 0$ *such that*

$$F(u(t), v(t)) > -C \quad \textit{for all} \quad t \geq 0.$$

Proof. Differentiation gives

$$\frac{d}{dt}F(u,v) = \frac{\alpha}{2\gamma}\frac{d}{dt}\|\nabla v\|^2 + \int_\Omega \left[\frac{\beta}{\gamma}vv_t - (u-1)\,v_t + (\log u - v)\,u_t\right] d\Omega.$$

Since (see the proof of Lemma 4.1)

$$\frac{\alpha}{2}\frac{d}{dt}\|\nabla v\|^2 = -\int_\Omega v_t\,(v_t + \beta\, v - \gamma(u-1))\, d\Omega,$$

we obtain

$$\frac{d}{dt}F(u,v) = \int_\Omega \left[-\frac{1}{\gamma}{v_t}^2 + (\log u - v)\,u_t\right] d\Omega.$$

Using the first equation of the system (1.4) (in weak form)

$$\int_\Omega (\log u - v)\,u_t d\Omega = -\int_\Omega \nabla(\log u - v)\cdot(\nabla u - u\nabla v) d\Omega = -\int_\Omega u|\nabla(\log u - v)|^2 d\Omega$$

proves the first part of the lemma.

To prove the second part, we use Young's inequality (2.6) and an estimate from the Appendix (Lemma 6.4) to get

$$|u-1|\,|v| \;\leq \Psi(|v|) + \Phi(|u-1|) \leq \Psi(|v|) + u(\log u - 1) + 1.$$

Inserted into (4.2) this gives

$$F(u,v) \geq \int_\Omega \left[\frac{1}{2\gamma}\left(\alpha|\nabla v|^2 + \beta v^2\right) - \Psi(|v|)\right] d\Omega \geq \int_\Omega (-\exp(|v|)) d\Omega.$$

From Corollary 2.6 follows

$$\int_\Omega \exp(|v(t,\cdot)|) d\Omega \leq c_\Omega \exp\left(\frac{1}{8\Theta}\int_\Omega |\nabla v(t,\cdot)|^2 d\Omega\right).$$

By Lemma 4.1 we have

$$G(u(t), v(t)) \leq G(u_0, v_0) = A \text{ for } t \geq 0,$$

hence

$$\int_\Omega |\nabla v(t,\cdot)|^2 d\Omega \leq \frac{2A}{\alpha} \text{ and consequently } \int_\Omega \exp(|v(t,\cdot)|) d\Omega \leq c_\Omega \exp\left(\frac{A}{4\alpha\Theta}\right) = C.$$

It follows the assertion $F(u(t), v(t)) \;\geq\; -C$ for all $t \geq 0$. □

Lemma 4.5. *For* $t \geq 0$ *we have*

$$f(v(t)) \leq F(u(t), v(t)) \,, \tag{4.12}$$

where

$$f(v) = \frac{1}{2\gamma} \int_\Omega (\alpha |\nabla v|^2 + \beta v^2)\, d\Omega - |\Omega| \log \frac{\int_\Omega e^v \, d\Omega}{|\Omega|}. \tag{4.13}$$

Proof. For $w \in H^1(\Omega)$, $v \in H^1(\Omega)$ with $w \geq 0$, $\overline{w} = 1$, $\overline{v} = 0$ we consider the functional

$$g(w,v) = \int_\Omega [\, w\,(\log w - 1) + 1 - (w-1)v\,]\, d\Omega.$$

We take $w^\star = \exp(v - c)$ and with $\overline{w^\star} = 1$ we obtain

$$c = \log \frac{\int_\Omega e^v \, d\Omega}{|\Omega|} \quad \text{and} \quad g(w^\star, v) = -c\,|\Omega|.$$

For this element $w^\star$ we can show the minimum property $g(u,v) \geq g(w^\star, v)$. Indeed, we have

$$\begin{aligned} g(u,v) - g(w^\star, v) &= \int_\Omega [\, u \log u - w^\star \log w^\star - (u - w^\star) v\,]\, d\Omega \\ &= \int_\Omega \left[\int_{w^\star}^{u} \log s \, ds - (u - w^\star)(\log w^\star + \lambda) \right] d\Omega \\ &= \int_\Omega \left(\int_{w^\star}^{u} \log \frac{s}{w^\star} \, ds \right) d\Omega \geq 0 \end{aligned}$$

which proves the Lemma. □

Remark 4.3. Assertion (4.10), (4.11) of Lemma 4.4 and Lemma 4.5 hold independently of the dimension N.

Lemma 4.6. *If* $\dfrac{\gamma|\Omega|}{4\alpha\Theta} < 1$, *then*

$$\|\nabla v(t)\| + \|u(t)\|_\Phi + \int_0^t D(u(s), v(s))\, ds \;\leq\; C \;<\; +\infty \quad \textit{for all} \quad t \geq 0.$$

Proof. From (4.10), (4.12) follows

$$\begin{aligned} f(v(t)) &\leq F(u(t), v(t)) = F(u_0, v_0) + \int_0^t \frac{d}{ds} F(u(s), v(s))\, ds \\ &= F(u_0, v_0) - \int_0^t D(u(s), v(s))\, ds \leq F(u_0, v_0) = C_1. \end{aligned}$$

By estimate (2.9) we have

$$\log\left(\int_\Omega e^{v(t)}\,d\Omega\right) \le \log c_\Omega + \frac{1}{8\Theta}\|\nabla v(t)\|^2$$

and with (4.13)

$$C_1 \ge f(v(t)) \ge \frac{\alpha}{2\gamma}\|\nabla v(t)\|^2 - |\Omega|\{\log c_\Omega + \frac{1}{8\Theta}\|\nabla v(t)\|^2 + \log|\Omega|\}$$

and, collecting constants

$$C_1 \ge f(v(t)) \ge \left(\frac{\alpha}{2\gamma} - \frac{|\Omega|}{8\Theta}\right)\|\nabla v(t)\|^2 - C_2.$$

With (4.6) we find

$$\|\nabla v(t)\|^2 \le C. \tag{4.14}$$

Moreover, from (4.10) – (4.12) we have

$$\int_0^t D(u(s),v(s))\,ds = F(u_0,v_0) - F(u(t),v(t)) \le F(u_0,v_0) - f(v(t)) \le 2C_3.$$

From the definition (4.2) we conclude

$$\begin{aligned}\int_\Omega [u(t)(\log u(t) - 1) + 1]\,d\Omega &\le F(u(t),v(t)) + \int_\Omega (u(t)-1)v(t)\,d\Omega\\ &\le F(u_0,v_0) + \int_\Omega \frac{1}{2}|u(t)-1||2v(t)|\,d\Omega.\end{aligned}$$

Using Lemma 6.4, Young's inequality (2.6), (2.9), and (4.14) we can estimate

$$\begin{aligned}\int_\Omega \Phi(|u(t)-1|)\,d\Omega &\le C_1 + \int_\Omega \left[\Phi(\frac{1}{2}|u(t)-1|) + \Psi(|2v(t)|)\right]d\Omega\\ &\le \frac{1}{2}\int_\Omega \Phi(|u(t)-1|)\,d\Omega + C_4\,, \qquad \text{hence}\\ \int_\Omega \Phi(|u(t)-1|)\,d\Omega &\le 2C_4.\end{aligned}$$

The monotonicity and convexity of Φ and the Δ_2–condition (2.7) give

$$\begin{aligned}\int_\Omega \Phi(|u(t)|)\,d\Omega &\le \int_\Omega \Phi\left(\frac{1}{2}(2|u(t)-1|+2)\right)d\Omega \le \frac{1}{2}\int_\Omega [\Phi(2|u(t)-1|) + \Phi(2)]\,d\Omega\\ &\le \frac{k}{2}\int_\Omega \Phi(|u(t)-1|)\,d\Omega + C_5 \le kC_4 + C_5.\end{aligned}$$

With the estimate (see [KJF], Section 3.6)

$$\|u(t)\|_\Phi \le \int_\Omega \Phi(|u(t)|)\,d\Omega + 1$$

we finish the proof. □

In the following we use the properties of the Lyapunov functional to find growth estimates (with respect to time) of integral norms of the solution (u,v) with $u \ge 0$. We assume $\beta = 0$ – to simplify the calculations.

Lemma 4.7. *Let*

$$\|u(t)\|_\Phi + \int_0^t \|v_t(s)\|^2\, ds \le C_0 \ < +\infty \quad \textit{for all} \quad t \ge 0.$$

Then there is a constant C_1 *such that*

$$\int_0^t \|(u(s)\|^2\, ds \le C_1(1+t) \quad \textit{for all} \quad t \ge 0.$$

Proof. We multiply the equation for u by $\log u$ and integrate over Ω. A calculation as in the proof of Lemma 4.1 and integration by parts gives

$$\int_\Omega u_t \log u \, d\Omega = \frac{d}{dt}\int_\Omega u\,(\log u - 1)\, d\Omega = -\int_\Omega (4\,|\nabla(\sqrt{u})|^2 + \nabla u \cdot \nabla v)\, d\Omega.$$

Using the equation for v we get the identity (written as scalar product)

$$\frac{d}{dt}(u, \log u - 1) + 4\,\|\nabla(\sqrt{u})\|^2 = \frac{1}{\alpha}\,(\gamma\,(u-1) - v_t, u)$$

and, since $u \ge 0$, the estimate

$$\frac{d}{dt}(u, \log u - 1) + 4\,\|\nabla(\sqrt{u})\|^2 \le \frac{1}{\alpha}\,(\gamma\, u - v_t, u) \le C\,(\|u\|^2 + \|v_t\|^2) \tag{4.15}$$

with an appropriate constant C (which, like c, may change during the proof).

Put $w = u+1,\ z_\nu = w\,(\log w)^\nu$. Then

$$\|u\|^2 \le \|w\|^2 = \|w\ \exp{(\log w)}\|_1 = \left\| w \sum_{\nu=0}^{\infty} \frac{(\log w)^\nu}{\nu!} \right\|_1 \le \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\|z_\nu\|_1.$$

For the indices $\nu = 0, 1$ we obviously have (see the proof of Lemma 4.6)

$$\|w\|_1 \le c, \quad \|w \log w\|_1 \le \int_\Omega \Phi(u)\, d\Omega + c \le C \quad \text{and consequently}$$

$$\|u\|^2 \le C + \sum_{\nu=2}^{\infty} \frac{1}{\nu!}\|{z_\nu}^{1/4}\|_4^4\,.$$

To estimate the L_4–norm we use again the Gagliardo–Nirenberg inequality, i.e., for $f \in H^1(\Omega)$ and $4 > q \ge 1$ there is a constant $c = c\,(q, \Omega)$ such that

$$\|f\|_4^4 \le c\,\|f\|_{H^1}^{4-q}\,\|f\|_q^q.$$

So we can continue by estimating

$$\|u\|^2 \le C + \sum_{\nu=2}^{\infty} c(q,\Omega)\frac{1}{\nu!}\|{z_\nu}^{1/4}\|_{H^1}^{4-q}\,\|{z_\nu}^{1/4}\|_q^q. \tag{4.16}$$

We use Poincaré's inequality

$$\|z_\nu{}^{1/4}\|_{H^1} \le C(\|\nabla(z_\nu{}^{1/4})\| + \|z_\nu{}^{1/4}\|_1)$$

and remark that, by definition,

$$z_\nu{}^{1/4} = w\log w \left(\frac{(\log w)^{\nu-4}}{w^3}\right)^{1/4} \quad \text{and} \quad \nabla(z_\nu{}^{1/4}) = \frac{\nabla w}{4}\left(\frac{(\log w)^{\nu-4}}{w^3}\right)^{1/4}(\nu + \log w).$$

For $k > 0,\ \beta > 0$ obviously holds

$$\sup_{w\ge 1} \frac{(\log w)^k}{w^\beta} \le \left(\frac{k}{\beta e}\right)^k \tag{4.17}$$

and so we can estimate

$$\|z_\nu{}^{1/4}\|_1 \le \left[\left(\frac{\nu-4}{3e}\right)^{\nu-4}\right]^{1/4} \|w\log w\|_1 \le C\left(\frac{\nu}{e}\right)^{\nu/4}(\|\Phi(u)\|_1 + c) \le C\left(\frac{\nu}{e}\right)^{\nu/4}.$$

To estimate the gradient part in Poincaré's inequality we remark that

$$|\nabla w| = 2\sqrt{u}\,|\nabla\sqrt{u}\,| \le w^{1/2}\,|\nabla\sqrt{u}\,|$$

and obtain

$$|\nabla(z_\nu{}^{1/4})| \le 2\,|\nabla\sqrt{u}\,|\left(\nu\,\frac{(\log w)^{\nu/4-1}}{w^{1/4}} + \frac{(\log w)^{\nu/4}}{w^{1/4}}\right).$$

Generous use of estimate (4.17) gives

$$|\nabla(z_\nu{}^{1/4})| \le C\left(\frac{\nu}{e}\right)^{\nu/4}|\nabla\sqrt{u}\,|, \quad \text{hence} \quad \|z_\nu{}^{1/4}\|_{H^1} \le C\left(\frac{\nu}{e}\right)^{\nu/4}(\|\nabla\sqrt{u}\,\| + 1).$$

In the same way we get

$$\|z_\nu{}^{1/4}\|_q^q = \|w\log w\,\frac{(\log w)^{\nu q/4-1}}{w^{1-q/4}}\|_1 \ \le\ \sup_{w\ge 1}\frac{(\log w)^{\nu q/4-1}}{w^{1-q/4}}\,\|w\log w\|_1$$

and by (4.17)

$$\|z_\nu{}^{1/4}\|_q^q \le C\left(\frac{\nu}{e}\right)^{\nu q/4-1}\left(\frac{q}{4-q}\right)^{\nu q/4-1}.$$

We use these estimates in (4.16) and find

$$\|u\|^2 \le C + \sum_{\nu=2}^{\infty} c(q,\Omega)\,\frac{1}{\nu!}\,C^{5-q}\left(\frac{\nu}{e}\right)^{\nu-1}\left(\frac{q}{4-q}\right)^{\nu q/4-1}(\|\nabla\sqrt{u}\,\| + 1)^{4-q}$$

and, with the help of Stirling's formula

$$\frac{1}{\nu!}\left(\frac{\nu}{e}\right)^{\nu} \le \frac{1}{\sqrt{2\pi\nu}} \quad \text{or} \quad \frac{1}{\nu!}\left(\frac{\nu}{e}\right)^{\nu-1} \le \frac{e}{\sqrt{2\pi}\nu^{3/2}} \quad \text{for } \nu \ge 1$$

by absorbing the constant factor $e/\sqrt{2\pi}$ into the coefficients $c(q,\Omega)$

$$\|u\|^2 \le C + \sum_{\nu=2}^{\infty} c(q,\Omega)\, C^{5-q} \frac{1}{\sqrt{\nu}} \left(\frac{q}{4-q}\right)^{\nu q/4-1} \frac{(\|\nabla\sqrt{u}\,\|+1)^{4-q}}{\nu}. \tag{4.18}$$

By Young's inequality with $\alpha = 2/(4-q)$, $\beta = 2/(q-2)$ ($\lambda > 0$ will be chosen later) we have

$$XY \le \frac{(\lambda X)^\alpha}{\alpha} + \frac{1}{\beta}\left(\frac{Y}{\lambda}\right)^\beta \quad \text{and with } X = \frac{(\|\nabla\sqrt{u}\,\|+1)^{4-q}}{\nu},\ Y = 1$$

we get

$$\frac{(\|\nabla\sqrt{u}\,\|+1)^{4-q}}{\nu} \le \frac{4-q}{2}\left(\frac{\lambda}{\nu}\right)^{2/(4-q)} (\|\nabla\sqrt{u}\,\|+1)^2 + \left(\frac{q-2}{2}\right)\frac{1}{\lambda^{2/(q-2)}}.$$

We put $q = 2\left(1+\dfrac{1}{\nu}\right)$, then $2 < q \le 3$ for $\nu \ge 2$ and from the foregoing estimate we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{\nu}}\left(\frac{q}{4-q}\right)^{\frac{\nu q}{4}-1} \frac{(\|\nabla\sqrt{u}\|+1)^{4-q}}{\nu} \\
&\qquad \le \frac{1}{\sqrt{\nu}}\left(\frac{\nu+1}{\nu-1}\right)^{\frac{\nu-1}{2}} \left[\left(1-\frac{1}{\nu}\right)\left(\frac{\lambda}{\nu}\right)^{\frac{\nu}{\nu-1}} (\|\nabla\sqrt{u}\|+1)^2 + \frac{1}{\nu\lambda^\nu}\right] \\
&\qquad \le \frac{C}{\sqrt{\nu}}\left[\left(\frac{\lambda}{\nu}\right)^{\frac{\nu}{\nu-1}} (\|\nabla\sqrt{u}\,\|+1)^2 + \frac{1}{\nu\lambda^\nu}\right]
\end{aligned}$$

with an appropriate constant C . We can choose a common bound for $c(q,\Omega)\, C^{5-q}$ and eventually get from (4.18) (taking $\lambda \le 1$ and again modifying the constant C)

$$\|u\|^2 \le C\left\{1 + \sum_{\nu=2}^{\infty}\left(\frac{1}{\sqrt{\nu}}\left(\frac{\lambda}{\nu}\right)^{\frac{\nu}{\nu-1}} \|\nabla\sqrt{u}\,\|^2 + \frac{1}{\nu\lambda^\nu}\right)\right\}.$$

Let $1 \ge \varepsilon > 0$ be given. Split the right hand side according to $\|u\|^2 \le C + T_1 + T_2$ where

$$\begin{aligned}
T_1 &= C\sum_{\nu=2}^{\nu_0-1}\left(\frac{1}{\sqrt{\nu}}\left(\frac{\lambda}{\nu}\right)^{\frac{\nu}{\nu-1}} \|\nabla\sqrt{u}\,\|^2 + \frac{1}{\nu\lambda^\nu}\right), \\
T_2 &= C\sum_{\nu=\nu_0}^{\infty}\left(\frac{1}{\sqrt{\nu}}\left(\frac{\lambda}{\nu}\right)^{\frac{\nu}{\nu-1}} \|\nabla\sqrt{u}\,\|^2 + \frac{1}{\nu\lambda^\nu}\right).
\end{aligned}$$

In the term T_1 take $\lambda = \dfrac{\varepsilon}{(C+1)\nu_0}$; an appropriate choice for ν_0 will be made immediately. We obtain

$$T_1 = C\sum_{\nu=2}^{\nu_0-1}\frac{1}{\sqrt{\nu}}\left\{\left(\frac{\varepsilon}{(C+1)\nu_0\nu}\right)^{\frac{\nu}{\nu-1}} \|\nabla\sqrt{u}\,\|^2 + \frac{1}{\nu}\left(\frac{(C+1)\nu_0}{\varepsilon}\right)^{\nu}\right\}$$

which can be majorized by

$$T_1 \leq \varepsilon \|\nabla\sqrt{u}\|^2 + C(\varepsilon).$$

In the term T_2 take $\lambda = 1$ and estimate

$$T_2 \leq C\|\nabla\sqrt{u}\|^2 \sum_{\nu_0}^{\infty} \frac{1}{\nu^{3/2}} + C\sum_{1}^{\infty} \frac{1}{\nu^{3/2}} \tag{4.19}$$

Now we choose ν_0 so that

$$C\sum_{\nu_0}^{\infty} \frac{1}{\nu^{5/4}} \leq \varepsilon \nu_0^{1/4}. \tag{4.20}$$

This is possible because

$$\sum_{\nu_0}^{\infty} \frac{1}{\nu^{5/4}} \leq \int_{\nu_0-1}^{\infty} \frac{dx}{x^{5/4}} = \frac{4}{(\nu_0-1)^{1/4}} \leq 4\left(\frac{2}{\nu_0}\right)^{1/4},$$

and (4.20) is satisfied for $\nu_0 \geq \dfrac{32C}{\varepsilon^2}$. Using this in (4.19) we find

$$T_2 \;\leq\; C\|\nabla\sqrt{u}\|^2 \sum_{\nu_0}^{\infty} \frac{1}{\nu^{5/4+1/4}} + C \;\leq\; \frac{C\|\nabla\sqrt{u}\|^2}{\nu_0^{1/4}} \sum_{\nu_0}^{\infty} \frac{1}{\nu^{5/4}} + C \;\leq\; \varepsilon\|\nabla\sqrt{u}\|^2 + C$$

and, finally,

$$\|u\|^2 \leq 2\,\varepsilon \|\nabla\sqrt{u}\|^2 + C(\varepsilon). \tag{4.21}$$

We use this estimate with an appropriate $1 \geq \varepsilon > 0$ in (4.15) and by the assumption of the Lemma we get

$$\int_0^t \left\|\nabla\sqrt{u(s)}\right\|^2 ds \leq C(1+t) \text{ and with (4.21) } \int_0^t \|u(s)\|^2 ds \leq C(1+t)$$

which finishes the proof.□

Lemma 4.8. *Suppose*

$$\int_0^t \|\gamma(u(s)-1) - v_t(s)\|^2 ds \leq C(1+t) \quad \textit{for all} \quad t \geq 0.$$

Then there are constants $c_1 = c_1(p)$, $c_2 = c_2(p)$ *and a nondecreasing finite function* $t \longrightarrow c(t)$ *such that*

$$\|u(t)\|_p \leq c_1 \exp(c_2 t) \quad \textit{for} \quad 1 \leq p < \infty, \quad \|u(t)\|_\infty + \|v(t)\|_\infty \leq c(t).$$

Proof. We multiply the equation for u by $p\,u^{p-1}$ and integrate over Ω :

$$p(u_t, u^{p-1}) = -p\,(\nabla u^{p-1}, \nabla u - u\nabla v).$$

Put $w = u^{p/2}$ and remark that

$$p(u_t, u^{p-1}) = \frac{d}{dt}\|w\|^2, \quad p(\nabla u^{p-1}, \nabla u) = \frac{4(p-1)}{p}\|\nabla w\|^2$$

and

$$p(\nabla u^{p-1}, u\nabla v) = p(p-1)(\nabla v, u^{p-1}\nabla u) = (p-1)(\nabla v, \nabla u^p).$$

With the equation for v we get

$$\frac{d}{dt}\|w\|^2 + \frac{4(p-1)}{p}\|\nabla w\|^2 = \frac{(p-1)}{\alpha}(\gamma(u-1) - v_t, u^p).$$

We put $z = \gamma(u-1) - v_t$, $\overline{w} = \frac{1}{|\Omega|}\int_\Omega w d\Omega$ and estimate

$$\begin{aligned}
\frac{d}{dt}\|w\|^2 + \frac{4(p-1)}{p}\|\nabla w\|^2 &= \frac{(p-1)}{\alpha}(z, w^2) = \frac{(p-1)}{\alpha}(z, w^2 - \overline{w}^2) \\
&= \frac{(p-1)}{\alpha}(z, (w-\overline{w})^2 + 2\overline{w}(w-\overline{w})) \\
&\le \frac{(p-1)}{\alpha}\left\{\|z\|\|w-\overline{w}\|_4^2 + 2|\overline{w}|\|z\|\|w-\overline{w}\|\right\}.
\end{aligned}$$

We use again Gagliardo's and Poincaré's inequality

$$\|w-\overline{w}\|_4^2 \le C\|\nabla w\|\|w\|, \quad \|w-\overline{w}\| \le C\|\nabla w\| \text{ and } |\overline{w}| \le C\|w\|$$

and obtain with Young's inequality

$$\begin{aligned}
\frac{d}{dt}\|w\|^2 + \frac{4(p-1)}{p}\|\nabla w\|^2 &\le C\frac{(p-1)}{\alpha}\|\nabla w\|\,\|w\|\,\|z\| \\
&\le \frac{2(p-1)}{p}\|\nabla w\|^2 + C(p,\alpha)\|w\|^2\|z\|^2.
\end{aligned}$$

This leads to the typical constellation of Gronwall's Lemma:

$$\frac{d}{dt}\|w\|^2 \le C\|w\|^2\|z\|^2 \text{ or } \|w(t)\|^2 \le \|w(0)\|^2 + C\int_0^t \|w(s)\|^2\|z(s)\|^2 ds$$

which gives

$$\|w(t)\|^2 \le \|w(0)\|^2 \exp\left(C\int_0^t \|z(s)\|^2 ds\right)$$

and, with the assumption

$$\int_0^t \|z(s)\|^2 ds \le c_0(1+t)$$

the estimate

$$\|u(t)\|_p^p = \|w(t)\|^2 \le \|w(0)\|^2 \exp(c_0 C(1+t)) = c_1 \exp(c_2 t).$$

From this the L_∞–boundedness of v follows from standard results on linear parabolic equations (see [LSU], Chap. III, Theorem 7.1). Moreover, by the same arguments as used in the last part of the proof of the local existence theorem it follows the L_∞–boundedness of u . □

The foregoing estimates give some hints concerning the blow–up behaviour for the case that the Lyapunov functional F has no lower bound.

Proposition 4.9. *If there is a solution* (u, v) *such that*

$$F(u(t_n), v(t_n)) \longrightarrow -\infty \quad as \quad t_n \to t_0, \quad then \quad \|u(t_n)\| \longrightarrow \infty.$$

Proof. Because of $\overline{v(t)} = 0$ we have by Poincaré's inequality $\|v(t)\| \le C\|\nabla v(t)\|$ and consequently

$$\int_\Omega u(t)v(t)d\Omega \ge -\frac{1}{2\gamma}\|\nabla v(t)\|^2 - C(\gamma)\|u(t)\|^2.$$

Using this and

$$\int_\Omega u(t)\log u(t)d\Omega \ge -\frac{|\Omega|}{e}$$

we get in (4.2)

$$F(u(t_n), v(t_n)) \ge -C(\gamma)\|u(t_n)\|^2, \quad \text{i.e.} \quad \|u(t_n)\|^2 \ge -\frac{1}{C(\gamma)}F(u(t_n), v(t_n)) \longrightarrow \infty.$$

So we have blow–up of $\|u(t_n)\|$ as $t_n \to t_0$. □

5. Stationary states

The condition $\kappa_1 < 1$ by Theorem 4.2 ensures the convergence of trajectories $(u(t),\, v(t))$ to the trivial (homogeneous) stationary state $(u^\star = 1,\; v^\star = 0)$ as $t \to \infty$. In this section we show that the condition $\kappa_2 < 1$ implies convergence of (at least) subsequences $(u(t_k),\, v(t_k))$ to in general non–trivial stationary states $(u^\star, v^\star)$ as $t_k \to \infty$. To this we need the following result.

Lemma 5.1. *Suppose*

$$\|v(t)\| \le c \qquad for\ all \quad t \ge 0.$$

Then

$$\|\exp(-v(t))\|_\infty \le C \quad for\ all \quad t \ge 0.$$

Proof. We use the De Giorgi–technique. For $k \geq k_0 \geq 0$ with $k_0 > \|v_0\|_\infty$ we put

$$w_k = (v+k)^- = \max(-(v+k), 0)$$

and denote by Ω_k the set :

$$\Omega_k = \Omega_k(t) = \{x \in \Omega : -(v(t,x)+k) > 0 \ \ a.e.\ \} = \{x \in \Omega : v(t,x) < -k \ \ a.e.\ \}$$

Obviously, $\Omega_l(t) \subset \Omega_k(t)$ for $l > k$ and all $t \geq 0$, hence $|\Omega_l| \leq |\Omega_k| \leq |\Omega|$, i.e. $|\Omega_k|$ is a non–increasing function of k, $\forall t \geq 0$. Testing the equation

$$v_t - \alpha \Delta v + \beta v - \gamma(u-1) = 0$$

with $h = -w_k = \min(v+k, 0)$ we obtain

$$(v_t, h) + \alpha(\nabla v, \nabla h) + \beta(v, h) = \gamma(u-1, h)$$

or, using well–known rules for calculus in Sobolev spaces (see e.g. [KSt],[LSU]),

$$\frac{1}{2}\frac{d}{dt}\|w_k\|^2 + \alpha\|\nabla w_k\|^2 + \beta\int_{\Omega_k} v(v+k)\, d\Omega = \gamma\int_{\Omega_k}(1-u)w_k\, d\Omega.$$

Taking into account that by $u \geq 0,\ w_k \geq 0$ we have

$$\int_{\Omega_k} v(v+k)\, d\Omega \geq 0, \qquad \int_{\Omega_k}(1-u)w_k\, d\Omega \leq \int_{\Omega_k} w_k\, d\Omega,$$

we estimate

$$\frac{1}{2}\frac{d}{dt}\|w_k\|^2 + \alpha\|\nabla w_k\|^2 \leq \gamma\int_{\Omega_k} w_k\, d\Omega. \tag{5.1}$$

By Hölder's inequality

$$\int_{\Omega_k} w_k\, d\Omega \leq \left(\int_{\Omega_k} w_k^4\, d\Omega\right)^{1/4} |\Omega_k|^{3/4} \leq \varepsilon\|w_k\|_4^2 + C(\varepsilon)|\Omega_k|^{3/2}$$

combined with Gagliardo–Nirenberg's inequality

$$\|w_k\|_4^2 \leq C\|w_k\|\,\|w_k\|_{H^1} \leq 2C(\|w_k\|^2 + \|\nabla w_k\|^2)$$

we find (after the usual modification of constants)

$$\frac{1}{2}\frac{d}{dt}\|w_k\|^2 + \alpha\|\nabla w_k\|^2 \leq \varepsilon(\|w_k\|^2 + \|\nabla w_k\|^2) + C(\varepsilon,\gamma)|\Omega_k|^{3/2}.$$

Again by Hölder's and Gagliardo–Nirenberg's inequality we have

$$\|w_k\|^2 \leq \|w_k\|_5^2\,|\Omega_k|^{3/5} \text{ and } \|w_k\|_5^2 \leq C\|w_k\|_{H^1}^{6/5}\|w_k\|^{4/5}, \text{ i.e.}$$
$$\|w_k\|^2 \leq C\|w_k\|_{H^1}^{6/5}\|w_k\|^{4/5}|\Omega_k|^{3/5},$$

and with Young's inequality

$$ab \leq \frac{1}{2}a^p + C(p,q)\,b^q \quad \text{for} \quad p = \frac{5}{3},\ q = \frac{5}{2},\ a = \|w_k\|_{H^1}^{6/5}$$

we obtain

$$\|w_k\|^2 \leq \|\nabla w_k\|^2 + C\|w_k\|^2|\Omega_k|^{3/2}.$$

On the set Ω_k obviously holds $|w_k| \leq |v|$ and consequently $\|w_k\| \leq \|v\| \leq C$ by assumption. So we have

$$\|w_k\|^2 \leq \|\nabla w_k\|^2 + C|\Omega_k|^{3/2} \tag{5.2}$$

We use this estimate in (5.1) and choose $\varepsilon > 0$ appropriately to get

$$\frac{d}{dt}\|w_k\|^2 + \alpha\|\nabla w_k\|^2 \leq C|\Omega_k|^{3/2}$$

or, again with (5.2),

$$\frac{d}{dt}\|w_k\|^2 + \alpha\|w_k\|^2 \leq C|\Omega_k|^{3/2}.$$

From this differential inequality we obtain

$$\|w_k(t)\|^2 \leq \|w_k(0)\|^2 \exp(-\alpha t) + \int_0^t \exp(-\alpha(t-s))|\Omega_k(s)|^{3/2} ds.$$

As assumed at the beginning of the proof we have $\|w_k(0)\| = 0$ for $k \geq k_0$, so it follows that

$$\|w_k(t)\|^2 \leq \frac{1}{\alpha}(1 - e^{-\alpha t}) \sup_{0 \leq t < \infty} |\Omega_k(t)|^{3/2} \leq (1/\alpha) \sup_{0 \leq t < \infty} |\Omega_k(t)|^{3/2}. \tag{5.3}$$

On the other hand, for $l > k \geq k_0$ holds

$$(l-k)^2|\Omega_l(t)| \leq \|w_k(t)\|^2. \tag{5.4}$$

This can be seen as follow: We have $\Omega_l \subset \Omega_k$ and consequently

$$\|w_k(t)\|^2 = \int_{\Omega_k} (v+k)^2 d\Omega = \left(\int_{\Omega_l} + \int_{\Omega_k \setminus \Omega_l}\right)(v+k)^2 d\Omega \geq \int_{\Omega_l} (v+k)^2 d\Omega.$$

Now on $\Omega_l(t) = \{x \in \Omega : v(t,x) < -l \ \ a.e. \}$ we have $v + k < k - l < 0$ from which (5.4) follows. From the estimates (5.3), (5.4) we obtain

$$(l-k)^2 \sup_{0 \leq t < \infty} |\Omega_l(t)| \leq (1/\alpha) \sup_{0 \leq t < \infty} |\Omega_k(t)|^{3/2}$$

or, putting

$$\Phi(k) = \sup_{0 \leq t < \infty} |\Omega_k(t)|^{1/2},$$

the key estimate of the DeGiorgi–technique

$$(l-k)\,\Phi(l) \leq \frac{1}{\sqrt{\alpha}}\,\Phi^{3/2}(k) \quad \text{for } l > k \geq k_0$$

follows. Since the function $k \longrightarrow \Phi(k)$, $k \geq k_0$ is obviously non–increasing, we conclude (see e.g. Lemma B.1. in [KSt]) that there is a sufficiently large k_1 such that $\Phi(k_1) = 0$. This means that for all $t \geq 0$ we have $v(t,\cdot) \geq -k_1$ a.e. in Ω and the Lemma is proved. □

Theorem 5.2. *Let*

$$\frac{\gamma|\Omega|}{4\alpha\Theta} < 1.$$

Then there exist a sequence $t_k \longrightarrow \infty$ *and functions* $u^\star, v^\star$ *such that*

$$u(t_k) \longrightarrow u^\star \ \text{in} \ L_2(\Omega),\ v(t_k) \rightharpoonup v^\star \ \text{in} \ H^1(\Omega),\ \ F(u(t_k), v(t_k)) \longrightarrow F(u^\star, v^\star).$$

Moreover, it holds

$$u^\star = \frac{|\Omega| e^{v^\star}}{\int_\Omega e^{v^\star} d\Omega}$$

and $v^\star$ *is the solution of the boundary value problem*

$$-\alpha\Delta v^\star + \beta v^\star = \gamma(u^\star - 1) \ \ on \ \ \Omega,\ \nu\cdot\nabla v = 0 \ \ on \ \ \partial\Omega. \tag{5.5}$$

Proof. Function arguments are sometimes omitted.
(*i*) We define $w = \exp(-v)\,u$. Note that by Lemma 5.1

$$u|\nabla(\log u - v)|^2 = 4\,e^v|\nabla\sqrt{w}|^2 \geq C|\nabla\sqrt{w}|^2. \tag{5.6}$$

(*ii*) For $\varrho > 0$ we set

$$I_\varrho(t) = t(\|v_t\|^2 + \|\nabla\sqrt{w}\|^2) + \int_\Omega (e^v|\nabla\sqrt{w}|^2 + \varrho u^2) d\Omega.$$

By Lemma 4.6, Lemma 4.7 and (5.6) we have

$$\int_0^t I_\varrho(s)\, ds \leq C_\varrho(1+t).$$

Hence there exists a sequence $t_k \longrightarrow \infty$ such that

$$I_\varrho(t_k) \leq 2\,C_\varrho. \tag{5.7}$$

Indeed, the assumption $I_\varrho(t) > 2\,C_\varrho \quad \forall t \geq t_0$ implies the contradiction

$$C_\varrho(1+t) \geq \int_{t_0}^t I_\varrho(s)\, ds \geq 2\,C_\varrho(t - t_0) \quad \forall t \geq t_0.$$

(*iii*) Now we find

$$\begin{aligned}
\int_\Omega e^v|\nabla\sqrt{w}|^2 d\Omega &= \frac{1}{4}\int_\Omega \left\{4|\nabla\sqrt{u}|^2 - 2\nabla u\cdot\nabla v + u|\nabla v|^2\right\} d\Omega \\
&\geq \|\nabla\sqrt{u}\|^2 - (1/2)\int_\Omega \nabla u\cdot\nabla v d\Omega \\
&= \|\nabla\sqrt{u}\|^2 + (1/2\alpha)\int_\Omega \{v_t + \beta v - \gamma(u-1)\}u\, d\Omega \\
&\geq \|\nabla\sqrt{u}\|^2 - \frac{1}{2\alpha}(\varepsilon\|v_t\|^2 + C(\varepsilon)\|u\|^2) - \frac{\beta}{2\alpha}(\|u\|^2 + \|v\|^2) - \frac{\gamma}{2\alpha}\|u\|^2.
\end{aligned}$$

We choose $\varepsilon = 2\alpha$ and remark that by Lemma 4.6 $\|\nabla v(t)\| \leq C$ and, using $\overline{v}(t) = 0$ and Poincaré's inequality, also $\|v(t)\| \leq C \quad \forall t \geq 0$. We conclude

$$\|\nabla\sqrt{u}\|^2 - \|v_t\|^2 \leq \int_\Omega e^{v}|\nabla\sqrt{w}|^2 d\Omega + C(\alpha,\beta,\gamma)\,\|u\|^2 + C$$

and, taking $\varrho = C(\alpha,\beta,\gamma)$,

$$(t-1)\,\|v_t\|^2 + t\,\|\nabla\sqrt{w}\|^2 + \|\nabla\sqrt{u}\|^2 \leq I_\varrho(t) + C.$$

By (5.7) we have along the sequence $\{t_k\}$

$$(t_k - 1)\,\|v_t(t_k)\|^2 + \; t_k \left\|\nabla\sqrt{w(t_k)}\right\|^2 + \left\|\nabla\sqrt{u(t_k)}\right\|^2 \leq C \quad \forall k$$

and consequently

$$\|v_t(t_k)\| + \left\|\nabla\sqrt{w(t_k)}\right\| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \tag{5.8}$$

and, again with Lemma 4.6

$$\left\|\nabla\sqrt{u(t_k)}\right\| + \|\nabla v(t_k)\| \leq C.$$

(*iv*) By the compactness of the imbedding of $H^1(\Omega)$ into $L_p(\Omega)$ for $1 \leq p < \infty$ we can assume without loss of generality that

$$u(t_k) \longrightarrow u^\star \text{ in } L_p(\Omega), \quad v(t_k) \rightharpoonup v^\star \text{ in } H^1(\Omega) \quad \text{as } k \longrightarrow \infty \tag{5.9}$$

which has the consequence that

$$w(t_k) = \exp(-v(t_k))\,u(t_k) \longrightarrow \exp(-v^\star)\,u^\star = w^\star \;\text{ in }\; L_2(\Omega) \;\text{ as }\; k \longrightarrow \infty.$$

Indeed, we can estimate

$$\|w(t_k) - w^\star\| \leq \|\exp(-v(t_k))\,(u(t_k) - u^\star)\| + \|(\exp(-v(t_k)) - \exp(-v^\star))\,u^\star\|.$$

By Lemma 5.1 and (5.9) the first term on the right hand side goes to zero as $k \longrightarrow \infty$. To show the same for the second term, we use an idea from [KW]. Put $v_k = v(t_k)$ and estimate

$$\|(\exp(-v(t_k)) - \exp(-v^\star))\,u^\star\|^2 = \int_\Omega |e^{-v_k} - e^{-v^\star}|^2\,|u^\star|^2\,d\Omega \leq \|u^\star\|_4^2\,\|e^{-v_k} - e^{-v^\star}\|_4^2.$$

With $|e^t - 1| \leq |t|\,e^{|t|}$ we get from Hölder's inequality

$$\begin{aligned}
\|e^{-v_k} - e^{-v^\star}\|_4^2 &= \int_\Omega |e^{-v_k} - e^{-v^\star}|^4\,d\Omega = \int_\Omega e^{-4v^\star}|e^{-(v_k - v^\star)} - 1|^4\,d\Omega \\
&\leq \int_\Omega e^{-4v^\star}\,e^{4|v_k - v^\star|}|v_k - v^\star|^4\,d\Omega \\
&\leq \left(\int_\Omega e^{-16v^\star}d\Omega\right)^{1/4}\left(\int_\Omega e^{16|v_k - v^\star|}d\Omega\right)^{1/4}\left(\int_\Omega |v_k - v^\star|^8 d\Omega\right)^{1/2}.
\end{aligned}$$

The first two factors on the right hand side are bounded by Corollary 2.6 and the last one tends to zero by (5.9) and the compactness of the imbedding of $H^1(\Omega)$ into $L_8(\Omega)$. Another consequence of (5.8) is

$$\nabla\sqrt{w^\star} = 0, \quad \text{i.e.} \quad w^\star = \exp(-v^\star)\, u^\star = C$$

where the constant follows from $\overline{u^\star} = 1$ and gives

$$u^\star = \frac{|\Omega| e^{v^\star}}{\int_\Omega e^{v^\star}\, d\Omega}. \tag{5.10}$$

The Theorem follows now from (5.8), (5.10) by taking the limit $k \longrightarrow \infty$ in the (weak form of the) v – equation. □

Proposition 5.3. *Consider the rectangle* $\Omega = \{(x,y) : 0 < x < a, 0 < y < b\}$ *with*

$$ab < \frac{2\pi\alpha}{\gamma} \qquad \textit{and} \qquad a^2 > \frac{\pi^2\,\alpha}{2\gamma\,(\log 4 - 1) - \beta} > 0. \tag{5.11}$$

Let be $(u(t), v(t))$ *the solution of* (1.4) *satisfying the initial condition*

$$u(0,x) = u_0(x) = 1 + \cos\frac{\pi x}{a}, \quad v(0,x) = v_0(x) = \cos\frac{\pi x}{a}.$$

Then there exists a sequence $t_k \longrightarrow \infty$ *such that* $(u(t_k), v(t_k))$ *converges to a non–trivial steady state* $(u^\star, v^\star)$ *.*

Proof. With the identity

$$\int_0^\pi (1+\cos x)\log(1+\cos x)\,dx = \pi(1-\log 2)$$

and the conditions (5.11) (the first one comes from $\kappa_2 < 1$ (see (4.8)) for the rectangle $|\Omega| = ab,\ \Theta = \pi/2$) we find

$$F(u_0, v_0) < 0.$$

By Lemma 4.4 and Theorem 5.2 we have

$$F(u^\star, v^\star) \ \le\ F(u(t_k), v(t_k)) \ \le\ F(u_0, v_0) \ <\ 0 \ =\ F(1,0).$$

Consequently, the steady state $(u^\star, v^\star)$ cannot be the trivial state $(1,0)$. □

Remark 5.1. This situation is quite different from the case of homogeneous Dirichlet boundary conditions considered in [DN].

Remark 5.2. The nonlinear operator on the right hand side of (5.5)

$$v \longrightarrow R(v) = \frac{|\Omega| e^v}{\int_\Omega e^v d\Omega} - 1$$

has at $v = 0$ the formal derivative

$$R'(v)\, h = \frac{d}{ds} R\,(v + sh)|_{s=0} = h,$$

so we obtain by formal linearization of (5.5) at $v = 0$ the linear boundary value problem

$$-\alpha\Delta h + \beta h = \gamma\, h \quad \text{on} \quad \Omega, \qquad \frac{\partial h}{\partial \nu} = 0 \quad \text{on} \quad \Gamma.$$

For the rectangle Ω the smallest non–trivial eigenvalue of the operator $h \longrightarrow -\alpha\Delta h + \beta h$ under homogeneous Neumann boundary conditions is

$$\lambda_1 = \left(\frac{\pi}{\max(a,b)}\right)^2 + \beta.$$

From the second condition (5.11) follows

$$\gamma > \frac{\lambda_1}{2\log 4 - 1} > \lambda_1,$$

i.e., we are in a constellation where (5.5) may show bifurcation.

6. Appendix

Here we sketch the proofs of several estimates used above.

Lemma 6.1. *Suppose* $\Omega = (0,a)\times(0,b)$. *Then for* $h \in H_1^1$ *holds*

$$\|h\|^2 \;\leq\; 2\{|\Omega|(\overline{h})^2 + A(\|h_x\|_1^2 + \|h_y\|_1^2)\}, \qquad \textit{where}$$

$$A \;=\; \frac{1}{4}\left(\frac{b}{a} + \frac{a}{b} + \sqrt{\left(\frac{b}{a} - \frac{a}{b}\right)^2 + 16}\right).$$

Proof. Since $C^1(\overline{\Omega})$ is dense in H_1^1 , we assume $h \in C^1(\overline{\Omega})$. There are mean values $0 \leq x_m(y) \leq a$, $0 \leq y_m(x) \leq b$ such that

$$h(x_m(y), y) = \frac{1}{a}\int_0^a h(x,y)\,dx, \quad h(x, y_m(x)) = \frac{1}{b}\int_0^b h(x,y)\,dy.$$

Now we get

$$\begin{aligned} h(x,y) &= \frac{1}{a}\int_0^a h(s,y)\,ds + \int_{x_m(y)}^x h_x(s,y)\,ds \\ &= \frac{1}{b}\int_0^b h(x,s)\,ds + \int_{y_m(x)}^y h_y(x,s)\,ds \end{aligned}$$

and consequently

$\int_\Omega h^2 d\Omega = \int_\Omega h(x,y)h(x,y)\,d\Omega = \frac{1}{ab}\left(\int_\Omega h\,d\Omega\right)^2 +$

$$+\int_\Omega\left[\frac{1}{a}\int_0^a h\,ds\int_{y_m(x)}^y h_y\,ds + \frac{1}{b}\int_0^b h\,ds\int_{x_m(y)}^x h_x\,ds + \int_{x_m(y)}^x h_x\,ds\int_{y_m(x)}^y h_y\,ds\right]d\Omega.$$

We estimate

$\int_\Omega h^2 d\Omega \le |\Omega|\overline{h}^2 +$

$$\begin{aligned} &+ \int_\Omega\left[\frac{1}{a}\int_0^a |h|\,dx\int_0^b |h_y|\,dy + \frac{1}{b}\int_0^b |h|\,dy\int_0^a |h_x|\,dx + \int_0^a |h_x|\,dx\int_0^b |h_y|\,dy\right]d\Omega \\ &= |\Omega|\overline{h}^2 + \int_\Omega |h|\,d\Omega\left[\frac{1}{a}\,||h_y||_1 + \frac{1}{b}\,||h_x||_1\right] + ||h_x||_1||h_y||_1 \\ &\le |\Omega|\overline{h}^2 + \frac{1}{2}\left[||h||^2 + \frac{b}{a}||h_y||_1^2 + \frac{a}{b}||h_x||_1^2 + 4||h_x||_1||h_y||_1\right] \\ &\le |\Omega|\overline{h}^2 + \frac{1}{2}||h||^2 + A\,(||h_x||_1^2 + ||h_y||_1^2), \end{aligned}$$

where A is an appropriate constant, and obtain

$$\frac{1}{2}||h||^2 \le |\Omega|\overline{h}^2 + A\,(||h_x||_1^2 + ||h_y||_1^2)$$

which proves the Lemma. To show that A can be chosen "optimal" as indicated, we put $X = ||h_x||_1$, $Y = ||h_y||_1$, $k = a/b$ and take

$$A = \sup_{X\ge 0, Y\ge 0, X^2+Y^2\ne 0} \frac{kX^2 + \frac{1}{k}Y^2 + 4XY}{X^2+Y^2}.$$

By homogeneity we have with $X = \cos\alpha$, $Y = \sin\alpha$:

$$A = \sup_{X^2+Y^2=1}\left\{kX^2 + \frac{1}{k}Y^2 + 4XY\right\} = \sup_{0\le\alpha\le\pi/2}\left\{k\cos^2\alpha + \frac{1}{k}\sin^2\alpha + 4\cos\alpha\sin\alpha\right\}.$$

A simple exercise in calculus gives the indicated value for A . □

Lemma 6.2. *For the constants κ_1, κ_2 of Remark 4.1 holds*

$$\kappa_2 \le \kappa_1.$$

Proof. For the rectangle and the disk the Lemma can be proved by elementary constructions. For the general case we use ideas from [ChY], [Gi]. We have (see Remark 2.1) with the best imbedding constant $\bar{\mu}$

$$\frac{\kappa_2}{\kappa_1} = \frac{\bar{\mu}}{4\Theta} \leq \frac{||h||_{1,1}^2}{4\Theta||h||^2} \tag{6.1}$$

for $h \in \mathcal{W},\ h \neq 0$. We show that for any $\varepsilon > 0$ we can construct a function h such that

$$\frac{||h||_{1,1}^2}{4\Theta||h||^2} \leq 1 + \varepsilon. \tag{6.2}$$

Let $P \in \partial\Omega$ be a corner point where the interior angle Θ is minimal. For $\delta > 0$ denote

$$B_\delta = \{x \in \Omega : |x - P| \leq \delta\}, \quad A_\delta = B_\delta \cap \Omega, \quad L_\delta = |\partial A_\delta \cap \Omega|.$$

If $\delta > 0$ is sufficiently small, L_δ is the length of the circular arc belonging to Ω centered in P. We define

$$h_\delta = -1 + \frac{|\Omega|}{|A_\delta|}\varphi_{A_\delta}(x)$$

where

$$\varphi_{A_\delta}(x) = \begin{cases} 0 & x \in \Omega \setminus A_\delta \\ 1 & x \in A_\delta \end{cases}$$

is the characteristic function of A_δ. This function has bounded variation (see [Gi], Example 1.4), i.e., $h_\delta \in BV(\Omega)$ and

$$\int_\Omega h_\delta d\Omega = 0, \quad ||h_\delta||^2 = \frac{|\Omega|^2}{|A_\delta|}\left(1 - \frac{|A_\delta|}{|\Omega|}\right), \quad \int_\Omega |Dh_\delta| = \frac{|\Omega| L_\delta}{|A_\delta|}$$

where $\int_\Omega |Df|$ is the total variation of Df for the function f. For sufficiently small $\delta > 0$ we have

$$|A_\delta| = \frac{1}{2}\Theta\delta^2 + o(\delta^2), \quad L_\delta = \Theta\delta + o(\delta).$$

Consequently, for $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\frac{\left(\int_\Omega |Dh_\delta|\right)^2}{||h_\delta||^2} \leq 2\Theta(1 + \frac{\varepsilon}{2}).$$

On the other hand, we can for $h_\delta \in BV(\Omega)$ find a sequence $\{h_{\delta j}\}$ of smooth functions in $C^\infty(\Omega)$ such that (see [Gi], Theorem 1.17)

$$\lim_{j\to\infty} \int_\Omega |h_{\delta j} - h_\delta|\, d\Omega = 0, \quad \lim_{j\to\infty} \int_\Omega |Dh_{\delta j}|\, d\Omega = \lim_{j\to\infty} \int_\Omega |\operatorname{grad} h_{\delta j}|\, d\Omega = \int_\Omega |Dh_\delta|.$$

We can assume $\int_\Omega h_{\delta j} d\Omega = 0$ and choose as the appropriate function h in (6.2) an approximating function $h_{\delta j}$ for which

$$\frac{\left(\int_\Omega |\mathrm{grad}\, h|\, d\Omega\right)^2}{\|h\|^2} \leq 2\Theta(1+\varepsilon).$$

With the estimate $\|h\|_{1,1} \leq \sqrt{2} \int_\Omega |\mathrm{grad}\, h|\, d\Omega$ the Lemma is proved. □

Lemma 6.3. *For* $u \geq 0$ *holds* $u(\log u - 1) + 1 \leq (u-1)^2$.

Proof. Consider the function

$$f(u) = (u-1)^2 - u(\log u - 1) - 1$$

for $u \geq 0$. Obviously,

$$f(1) = 0,\ f(0) = \lim_{u \to 0} f(u) = 0,\ f'(u) = 2(u-1) - \log u.$$

For $u \geq 1$ we have

$$\log u = \int_1^u \frac{dt}{t} \leq \int_1^u dt = u - 1 \leq 2(u-1),$$

hence $f'(u) \geq 0$ for $u \geq 1$, i.e. $f(u) \geq f(1) = 0$.
For $0 < u \leq 1$ put $u = 1/w$ and consider

$$f(1/w) = \frac{1}{w^2}(w \log w - w + 1) \quad \text{on} \quad w \geq 1.$$

Because of

$$w \log w - w + 1 = \int_1^w \log s\, ds \geq 0 \quad \text{for} \quad w > 0$$

the Lemma is proved. □

Lemma 6.4. *The Young function* $\Phi : s \longrightarrow (s+1)\log(s+1) - s$ *is monotone and convex for* $s \geq 0$ *and satisfies*

$$\Phi(|s-1|) \leq s(\log s - 1) + 1\,,\ \Phi(\lambda s) \leq \lambda \Phi(s) \quad \textit{for } s \geq 0, \quad 0 \leq \lambda \leq 1.$$

Proof. Monotonicity and convexity are obvious. Consider the function

$$h(s) = s(\log s - 1) + 1 - \Phi(|s-1|).$$

For $s \geq 1$ we have $h(s) = 0$ and the assertion is trivially correct.
For $0 \leq s < 1$ we have $|s-1| = 1-s$ and

$$h(s) = s\log s + 2 - 2s - (2-s)\log(2-s), \quad h'(s) = \log(s(2-s)),$$

hence $h'(s) < 0$ for $0 < s < 1$ because of $s(2-s) = 1-(1-s)^2 < 1$. Consequently, by the mean value theorem

$$h(s) - h(1) = h'(\eta)(s-1) > 0, \quad 0 < s < \eta < 1.$$

So we get $h(s) > h(1) = 0$ which proves the first estimate.
Consider

$$g(s) = \lambda\Phi(s) - \Phi(\lambda s) = \lambda(s+1)\log(s+1) - (\lambda s+1)\log(\lambda s+1).$$

We have

$$g(0) = 0 \quad \text{and} \quad g'(s) = \lambda\,[\log(s+1) - \log(\lambda s+1)] \geq 0$$

by the monotonicity of *log* which proves the second estimate. □

Lemma 6.5. *The function* $f(x) = x(\log x - 1),\, x \geq 0$ *, satisfies*

$$f(x) - 2f\left(\frac{x+y}{2}\right) + f(y) \;\geq\; \frac{1}{4}(\sqrt{x}-\sqrt{y})^2, \quad x,\, y \geq 0.$$

Proof. We can assume $x \geq y > 0$, put $x/y = t$. We have $t \geq 1$ and, obviously,

$$(\sqrt{t}-1)^2 = \left(\frac{t-1}{\sqrt{t}+1}\right)^2 \;\leq\; \frac{(t-1)^2}{t+1} \;\leq\; \frac{(t-1)^2}{t} = \frac{1}{t}\int_1^t \left(\int_1^t dr\right) ds$$

$$\leq\; 2\int_1^t \left(\int_1^t \frac{dr}{r+s}\right) ds$$

which gives the estimate

$$(\sqrt{t}-1)^2 \leq 4\,\{t\,\log t - (t+1)\log(t+1) + (t+1)\log 2\}$$

equivalent to the assertion.□

Remark 6.1. A more subtle estimation shows that the factor $1/4$ in Lemma 6.5 can be improved to the best possible factor $\log 2$.

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