Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Local approximation of arbitrary functions by solutions of nonlocal equations

Serena Dipierro[1,2], Ovidiu Savin[3], Enrico Valdinoci[1,2,4,5,6]

submitted: September 16, 2016

[1] School of Mathematics and Statistics
University of Melbourne
Richard Berry Building
Parkville VIC 3010
Australia

[2] School of Mathematics and Statistics
University of Western Australia
35 Stirling Highway Crawley
Perth WA 6009
Australia
E-Mail: serydipierro@yahoo.it

[3] Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
USA
E-Mail: savin@math.columbia.edu

[4] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: enrico.valdinoci@wias-berlin.de

[5] Dipartimento di Matematica
Università degli studi di Milano
Via Saldini 50
20133 Milan
Italy
E-Mail: enrico@mat.uniroma3.it

[6] Istituto di Matematica
Applicata e Tecnologie Informatiche
Consiglio Nazionale delle Ricerche
Via Ferrata 1
27100 Pavia
Italy

No. 2303

Berlin 2016

2010 *Mathematics Subject Classification.* 35R11, 60G22, 35A35, 34A08.

Key words and phrases. Density properties, approximation, s-caloric functions.

The first author has been supported by Alexander von Humboldt Foundation. The second author has been supported by NSF grant DMS-1200701. The third author has been supported by ERC grant 277749 "EPSILON Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities" and PRIN grant 201274FYK7 "Aspetti variazionali e perturbativi nei problemi differenziali nonlineari".

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We show that any function can be locally approximated by solutions of prescribed linear equations of nonlocal type. In particular, we show that every function is locally s-caloric, up to a small error. The case of non-elliptic and non-parabolic operators is taken into account as well.

1 Introduction

In this paper, we will show that an arbitrary function can be locally approximated, in the smooth sense, by s-caloric functions, i.e. by solutions of the fractional heat equation in which the diffusion is due to the s-power of the Laplacian, with $s \in (0,1)$.

The precise result obtained is the following:

Theorem 1. *Let $B_1 \subset \mathbb{R}^n$ be the unit ball, $s \in (0,1)$, $k \in \mathbb{N}$ and $f : B_1 \times (-1,1) \to \mathbb{R}$, with $f \in C^k(\overline{B_1} \times [-1,1])$.*

Fix $\varepsilon > 0$. Then there exists $u_\varepsilon = u \in C^\infty(B_1 \times (-1,1)) \cap C(\mathbb{R}^{n+1})$ which is compactly supported in $\mathbb{R}^{n+1}$ and such that the following properties hold true:

$$\partial_t u + (-\Delta)^s u = 0 \text{ in } B_1 \times (-1,1) \tag{1}$$

$$\text{and} \qquad \|u - f\|_{C^k(B_1 \times (-1,1))} \leqslant \varepsilon. \tag{2}$$

We remark that the approximation result in Theorem 1 reflects a purely nonlocal phenomenon, since in the local case the solutions of the classical heat equation are particularly "rigid". For example, solutions of the classical heat equation (i.e. solutions of equation (1) when $s = 1$) satisfy a local Harnack inequality which prevents arbitrary oscillations (in particular, these solutions cannot approximate a given function which does not satisfy these oscillation constraints).

On the contrary, in the nonlocal setting, solutions of linear equations are flexible enough to approximate any given function, and this approximation results hold true in a very general context. As a matter of fact, in our setting, Theorem 1 is just a particular case of a much more general result that we provide in the forthcoming Theorem 2.

To state this general theorem, we introduce now some specific notation. We will often use small fonts to denote "local variables", capital fonts to denote "nonlocal variables", and Greek fonts to denote the set of local and nonlocal variables altogether, namely[1] given $d \in \mathbb{N}$, with $d \geqslant 0$, and $N \in \mathbb{N}$, with $N \geqslant 1$, we consider $x := (x_1, \dots, x_d) \in \mathbb{R}^d$ and $X := (X_1, \dots, X_N) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N}$ and we let $(x, X) \in \mathbb{R}^\nu$, with $\nu := d + n_1 + \dots + n_N$. To avoid confusions, when necessary, the k-dimensional unit ball will be denoted by B_1^k (of course, when no confusion is possible, we will adopt the usual notation B_1).

Given $m = (m_1, \dots, m_d) \in \mathbb{N}^d$ and $(a_1, \dots, a_d) \in \mathbb{R}^d \setminus \{0\}$, we consider the local operator

$$\ell := \sum_{j=1}^{d} a_j \partial_{x_j}^{m_j}.$$

[1] If $d = 0$, simply there are no "local variables" $(x_1, \dots, x_d)$ to take into account and $\nu = n_1 + \dots + n_N$.

Also, given $s = (s_1, \dots, s_N) \in (0,1)^N$ and $A = (A_1, \dots, A_N) \in \mathbb{R}^N \setminus \{0\}$, we consider the nonlocal operator

$$\mathcal{L} := \sum_{j=1}^{N} A_j \, (-\Delta_{X_j})^{s_j},$$

where we denoted by $(-\Delta_{X_j})^{s_j}$ the fractional Laplacian of order $s_j \in (0,1)$ in the set of variables $X_j \in \mathbb{R}^{n_j}$, namely

$$\begin{aligned}&(-\Delta_{X_j})^{s_j} u(x, X_1, \dots, X_j, \dots, X_N)\\ &\quad := C(n_j, s_j) \lim_{\varrho \searrow 0} \int_{Y \in \mathbb{R}^{n_j} \setminus B_\varrho^{n_j}} \frac{u(x, X_1, \dots, X_j, \dots, X_N) - u(x, X_1, \dots, X_j + Y, \dots, X_N)}{|Y|^{n_j + 2s_j}} \, dY,\end{aligned}$$

where we used the normalized constant

$$C(n_j, s_j) := \frac{4^{s_j} \, s_j \, \Gamma\left(\frac{n_j}{2} + s_j\right)}{\pi^{\frac{n_j}{2}} \, \Gamma\left(1 - s_j\right)},$$

being Γ the Euler's Γ-function.

Then, we deal with the superposition[2] of the local and the nonlocal operators, given by

$$\Lambda := \ell + \mathcal{L} \tag{3}$$

and we establish that *all functions are locally Λ-harmonic up to a small error*, i.e. the functions in the kernel of the operator Λ are locally dense in C^k. The precise result goes as follows:

Theorem 2. *Let $k \in \mathbb{N}$ and $f : \mathbb{R}^\nu \to \mathbb{R}$, with $f \in C^k(\overline{B_1^\nu})$. Fix $\varepsilon > 0$. Then there exist $u \in C^\infty(B_1^\nu) \cap C(\mathbb{R}^\nu)$ and $R > 1$ such that the following properties hold true:*

$$\Lambda u = 0 \ \textit{ in } B_1^\nu, \tag{4}$$

$$\|u - f\|_{C^k(B_1^\nu)} \leqslant \varepsilon \tag{5}$$

$$\textit{and} \qquad u = 0 \ \textit{ in } \mathbb{R}^\nu \setminus B_R^\nu. \tag{6}$$

It is interesting to remark that not only Theorem 2 immediately implies Theorem 1 as a particular case, but also that Theorem 2 does not require any ellipticity or parabolicity on the operator, which is perhaps a rather surprising fact. Indeed, we stress that Theorem 2 is valid also for operators with hyperbolic structures, and comprises the cases when

$$\Lambda = \sum_{j=1}^{d} \partial^2_{x_j} + (-\Delta_{X_1})^{s_1}$$

and when

$$\Lambda = (-\Delta_{X_1})^{s_1} - (-\Delta_{X_2})^{s_2}$$

with $s_1,\, s_2 \in (0,1)$. In this sense, the nonlocal features of the fractional Laplacian in some variables dominate the possible elliptic/parabolic/hyperbolic structure of the operator.

The first result in the direction of Theorem 2 has been recently obtained in [4], where Theorem 2 was proved in the special case in which $d = 0$ and $N = 1$ (that is, when there are no "local variables" and only one "nonlocal variable"). Results related to that in [4] have been obtained in [2] for other types of nonlocal operators, such as the ones driven by the Caputo derivative.

We also observe that these "abstract" approximation results have also "concrete" applications, for instance in mathematical biology: for example, they show that biological species with nonlocal strategies can better plan their distribution in order to exhaust a given resource in a strategic region, thus avoiding any unnecessary waste of resource, see e.g. [3, 5].

In this sense, we mention the following application of Theorem 1:

[2] Of course, if $d = 0$, i.e. if there are no "local variables", the operator Λ in (3) coincides with the purely nonlocal operator $\mathcal{L}$.

Theorem 3. *Let $s \in (0,1)$ and $k \in \mathbb{N}$. Fix $\varepsilon > 0$. Let $\sigma \in C^k\big(\overline{B_1} \times [-1,1],\ (0,+\infty)\big)$. Then, there exists $u_\varepsilon \in C^\infty\big(B_1 \times (-1,1),\ (0,+\infty)\big) \cap C(\mathbb{R}^{n+1})$ which is compactly supported and such that*

$$\partial_t u_\varepsilon + (-\Delta)^s u_\varepsilon = (\sigma_\varepsilon - u_\varepsilon) u_\varepsilon \text{ in } B_1 \times (-1,1), \tag{7}$$

$$u_\varepsilon = \sigma_\varepsilon \text{ in } B_1 \times (-1,1) \tag{8}$$

$$\text{and} \qquad \|\sigma - \sigma_\varepsilon\|_{C^k\left(B_1 \times (-1,1)\right)} \leqslant \varepsilon. \tag{9}$$

The biological interpretation of Theorem 3 is that u_ε represents the distribution of a population, which satisfies a logistic equation as in (7). The function σ can be thought as a resource (which in turn produces a birth rate proportional to it). The meaning of Theorem 3 is that, possibly replacing the original resource with a slightly different one (as prescribed quantitatively by (9)), the population can consume all the resource (as given by (8)).

Notice that, in our setting, Theorem 3 is a simple consequence of Theorem 1 (by taking there $f := \sigma$). More general interactions can also be considered, see e.g. Theorem 1.8 in [3].

The rest of the paper is organized as follows. In Section 2 we give a precise boundary behavior of solutions of nonlocal equations (these estimates depend in turn on some technical boundary asymptotics of the Green function of the fractional Laplacian, whose proof is deferred to the end of the paper, in Section 7).

Section 3 contains the main argument towards the proof of Theorem 2, that is that solutions of nonlocal equations can span the largest possible space with their derivatives (we remark that this is a purely nonlocal argument, since, for instance, harmonic functions obviously cannot span strictly positive second derivatives). The argument to prove this fact is based on a “separation of variables” method. Namely, we will look for solutions of nonlocal equations in the form of products of functions depending on “local” and “nonlocal variables”. The nonlocal part of the function is built by the eigenfunctions of the nonlocal operators (whose boundary behavior is somehow singular and can be quantified by the estimates of the previous sections), while the local part of the function is constructed by an ordinary differential equation which is designed to compensate all the coefficients of the operator in the appropriate way.

The proof of Theorem 2 is then discussed step by step, first in Section 4, where f is supposed to be a monomial, then in Section 5, where f is supposed to be a polynomial, and finally completed in the general case in Section 6.

2 Boundary behavior of solutions of fractional Laplace equations

In this section, we detect the exact boundary behavior of solutions of fractional Laplace equations in a ball with Dirichlet data. For estimates in general domains, see e.g. [6] and the references therein. Let us remark that, in our context, we do not only obtain bounds from above and below, but also a precise asymptotics in the limits which approach the boundary.

In order to obtain our bounds, we make use of the fractional Green function, whose setting goes as follows. Given $s \in (0,1)$ and $x, z \in B_1$, we consider the function

$$G(x,z) := |z-x|^{2s-n} \int_0^{r_0(x,z)} \frac{t^{s-1}\,dt}{(t+1)^{\frac{n}{2}}}, \tag{10}$$

with[3]

$$r_0(x,z) := \frac{(1-|x|^2)\,(1-|z|^2)}{|z-x|^2}. \tag{11}$$

[3] Though we will not use this, it is interesting to point out that when $n = 2s$, i.e. when $n = 1$ and $s = \frac{1}{2}$, the function G can be written explicitly, up to constants, as

$$G(x,z) = \log \frac{1 - xz + \sqrt{(1-|x|^2)\,(1-|z|^2)}}{|z-x|}.$$

This follows by computing the integral

$$\int \frac{dt}{\sqrt{t\,(t+1)}} = 2\log(\sqrt{t} + \sqrt{t+1}) + \text{const}.$$

Up to normalization factors, the function G plays the role of a Green function in the fractional setting, as discussed for instance in [1] and in the references therein.

If x lies in an ε-neighborhood of ∂B_1, then G is of order ε^s, as stated precisely in the next result:

Lemma 4. *Let $e \in \partial B_1$, $\varepsilon_o > 0$ and $\omega \in \partial B_1$. Assume that $e + \varepsilon\omega \in B_1$ for all $\varepsilon \in (0, \varepsilon_o]$. Let $f \in C^\alpha(\mathbb{R}^d)$ for some $\alpha \in (0, 1)$, with $f = 0$ outside B_1.*

Then

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-s} \int_{B_1} f(z)\, G(e + \varepsilon\omega, z)\, dz = \int_{B_1} f(z)\, \frac{(-2\, e \cdot \omega)^s\, (1 - |z|^2)^s}{s\, |z - e|^n}\, dz. \tag{12}$$

The rather technical proof of Lemma 4 is postponed to Section 7, for the facility of the reader. Here, we deduce from Lemma 4 the boundary estimates needed to the proof of our main result:

Proposition 5. *Let $e \in \partial B_1$, $\varepsilon_o > 0$ and $\omega \in \partial B_1$. Assume that $e+\varepsilon\omega \in B_1$ for all $\varepsilon \in (0, \varepsilon_o]$. Let $f \in C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$, with $f = 0$ outside B_1.*

Let u be a weak solution of

$$\begin{cases} (-\Delta)^s u = f & \text{in } B_1, \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

Then

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-s}\, u(e + \varepsilon\omega) = \kappa(n, s)\, (-2\, e \cdot \omega)^s \int_{B_1} f(z)\, \frac{(1 - |z|^2)^s}{s\, |z - e|^n}\, dz,$$

where

$$\kappa(n, s) := \frac{\Gamma\left(\frac{n}{2}\right)}{4^s\, \pi^{\frac{n}{2}}\, \Gamma^2(s)},$$

being Γ the Euler's Γ-function.

Proof. We know from Theorems 1 and 2 in [8] that u is actually continuous in $\mathbb{R}^n$ and it is a viscosity solution of the equation. Also, by the fractional Green Representation Theorem (see e.g. Theorem 3.2 in [1] and the references therein), we have that

$$u(e + \varepsilon\omega) = \kappa(n, s) \int_{B_1} f(z)\, G(e + \varepsilon\omega, z)\, dz,$$

with G as in (10). Hence, the desired result follows from (12). □

As a simple consequence, we can characterize the boundary behavior of the first eigenfunction for the fractional Laplacian with Dirichlet data (see e.g. Appendix A in [7] for a discussion on fractional eigenvalues).

Corollary 6. *Let $e \in \partial B_1$. Let $\phi_\star$ be the first eigenfunction for $(-\Delta)^s$, normalized to be positive and such that $\|\phi_\star\|_{L^2(B_1)} = 1$, and let $\lambda_\star > 0$ be the corresponding eigenvalue. Then,*

$$\|\phi_\star\|_{C^s(\mathbb{R}^n)} \leqslant C, \tag{13}$$

for some $C > 0$ depending only on n and s, and

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-s}\, \phi_\star(e + \varepsilon\omega) = \kappa_\star\, \lambda_\star\, (-\, e \cdot \omega)^s_+. \tag{14}$$

where

$$\kappa_\star := 2^s\, \kappa(n, s) \int_{B_1} \phi_\star(z)\, \frac{(1 - |z|^2)^s}{s\, |z - e|^n}\, dz \in (0, +\infty). \tag{15}$$

Proof. The idea is that, since $(-\Delta)^s\phi_\star = \lambda_\star\,\phi_\star$, we can use Proposition 5 and get the desired result. More precisely, we have that $\phi_\star$ is $C^s(B_1)$ (see the proof of Corollary 8 in [8] to obtain the continuity and then Proposition 1.1 in [6] to get the Hölder estimate in (13)).

Notice that, by (13), we have that the quantity $\kappa_\star$ defined in (15) is finite, while the positivity of $\phi_\star$ implies that $\kappa_\star > 0$.

Also, the Hölder estimate in (13) allows to use Proposition 5 with $f := \lambda_\star\,\phi_\star$. Accordingly, for any $\omega \in \partial B_1$ for which there exists $\varepsilon_o > 0$ such that $e + \varepsilon\omega \in B_1$ for all $\varepsilon \in (0, \varepsilon_o]$, we have that

$$\lim_{\varepsilon\searrow 0} \varepsilon^{-s}\,\phi_\star(e+\varepsilon\omega) = \kappa_\star\,\lambda_\star\,(-\,e\cdot\omega)^s. \tag{16}$$

Now, we distinguish two cases: if $e\cdot\omega < 0$, then

$$|e+\varepsilon\omega|^2 = 1 + 2\varepsilon e\cdot\omega + \varepsilon^2 < 1$$

for small ε, and so $e+\varepsilon\omega \in B_1$ for small ε, hence (14) follows from (16).

If instead $e\cdot\omega \geqslant 0$, then $e+\varepsilon\omega \in \mathbb{R}^n \setminus B_1$, thus $\phi_\star(e+\varepsilon\omega) = 0$, which obviously implies (14) in this case. □

For our purposes, it is useful to deduce the following integral estimate from Corollary 6:

Corollary 7. *Let $e \in \partial B_1$. Let $\phi_\star$ be the first eigenfunction for $(-\Delta)^s$, normalized to be positive and such that $\|\phi_\star\|_{L^2(B_1)} = 1$, and let $\lambda_\star > 0$ be the corresponding eigenvalue.*

Let $\kappa_\star$ be as in (15)*. Then*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{|\alpha|-s}\partial^\alpha\phi_\star(e+\varepsilon X) = (-1)^{|\alpha|}\,\kappa_\star\,\lambda_\star\,s\,(s-1)\ldots(s-|\alpha|+1)\,e_1^{\alpha_1}\ldots e_n^{\alpha_n}\,(-e\cdot X)_+^{s-|\alpha|}$$

in the sense of distribution, for any $\alpha \in \mathbb{N}^n$.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^n)$. We write $X = \rho\omega$, with $\rho \geqslant 0$ and $\omega \in S^{n-1}$. Notice that $(\varepsilon\rho)^{-s}\,|\phi_\star(e+\varepsilon\rho\omega)| \leqslant C$, thanks to (13).

So, we use (16) and the Dominated Convergence Theorem to see that

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0} \varepsilon^{|\alpha|-s} \int_{\mathbb{R}^n} \partial^\alpha\phi_\star(e+\varepsilon X)\,\psi(X)\,dX = \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^n} \partial_X^\alpha\big(\varepsilon^{-s}\phi_\star(e+\varepsilon X)\big)\,\psi(X)\,dX \\
&\quad= (-1)^{|\alpha|} \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^n} \varepsilon^{-s}\phi_\star(e+\varepsilon X)\,\partial^\alpha\psi(X)\,dX \\
&\quad= (-1)^{|\alpha|} \lim_{\varepsilon\searrow 0} \int_0^{+\infty} d\rho \int_{S^{n-1}} d\omega\,\rho^{n-1}\rho^s\,(\varepsilon\rho)^{-s}\phi_\star(e+\varepsilon\rho\omega)\,\partial^\alpha\psi(\rho\omega) \\
&\quad= (-1)^{|\alpha|}\,\kappa_\star\,\lambda_\star \int_0^{+\infty} d\rho \int_{S^{n-1}} d\omega\,\rho^{n-1}\rho^s\,(-e\cdot\omega)_+^s\,\partial^\alpha\psi(\rho\omega) \\
&\quad= (-1)^{|\alpha|}\,\kappa_\star\,\lambda_\star \int_0^{+\infty} d\rho \int_{S^{n-1}} d\omega\,\rho^{n-1}(-e\cdot\rho\omega)_+^s\,\partial^\alpha\psi(\rho\omega) \\
&\quad= (-1)^{|\alpha|}\,\kappa_\star\,\lambda_\star \int_{\mathbb{R}^n} (-e\cdot X)_+^s\,\partial^\alpha\psi(X)\,dX \\
&\quad= \kappa_\star\,\lambda_\star \int_{\mathbb{R}^n} \partial_X^\alpha(-e\cdot X)_+^s\,\psi(X)\,dX \\
&\quad= (-1)^{|\alpha|}\,\kappa_\star\,\lambda_\star\,s\,(s-1)\ldots(s-|\alpha|+1)\,e_1^{\alpha_1}\ldots e_n^{\alpha_n} \int_{\mathbb{R}^n} (-e\cdot X)_+^{s-|\alpha|}\,\psi(X)\,dX,
\end{aligned}$$

and this gives the desired result, since ψ is an arbitrary test function. □

3 Spanning the whole of the Euclidean space with Λ-harmonic functions

Here we show that Λ-harmonic functions span the whole of the Euclidean space (this is a purely nonlocal phenomenon, since, for instance, the second derivatives of harmonic functions have to satisfy a linear equation, and therefore are forced to lie in a proper subspace).

To this goal, we consider here multi-indices $i = (i_1, \dots, i_d) \in \mathbb{N}^d$ and $I = (I_1, \dots, I_N) \in \mathbb{N}^{n_1} \times \dots \times \mathbb{N}^{n_N}$. We will use the notation

$$\iota := (i, I) = (i_1, \dots, i_d, I_1, \dots, I_N) \in \mathbb{N}^{\nu}. \tag{17}$$

As usual, we set $|\iota| := i_1 + \dots + i_d + |I_1| + \dots |I_N|$, where $|I_1| := I_{1,1} + \dots + I_{1,n_1}$, and so on. We also write

$$\partial^{\iota} w := \partial^{i_1}_{x_1} \dots \partial^{i_d}_{x_d} \partial^{I_1}_{X_1} \dots \partial^{I_N}_{X_N} w.$$

We consider the span of the derivatives of Λ-harmonic functions, with derivatives up to order K. For this, we denote by $\partial^K w$ the vector field collecting in its entry all the derivatives of the form $\partial^{\iota} w$ with $|\iota| \leqslant K$ (in some prescribed order). Notice that $\partial^K w$ is a vector field on the Euclidean space $\mathbb{R}^{K'}$ for some $K' \in \mathbb{N}$ (of course, K' depends on K).

Then we denote by $\mathcal{H}$ the family of all functions $w \in C(\mathbb{R}^{\nu})$ that are compactly supported in $\mathbb{R}^{\nu}$ and for which there exists a neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^{\nu}$ such that $w \in C^{\infty}(\mathcal{N})$ and $\Lambda w = 0$ in $\mathcal{N}$.

Finally, we define the set

$$\mathcal{V}_K \ := \ \big\{ \partial^K w(0) \text{ for all } w \in \mathcal{H} \big\}. \tag{18}$$

By construction $\mathcal{V}_K \subseteq \mathbb{R}^{K'}$, and we have:

Lemma 8. *It holds that $\mathcal{V}_K = \mathbb{R}^{K'}$.*

Proof. First, we consider the case in which $d \neq 0$ (hence, we are taking into account the case in which the ambient space possesses both "local" and "nonlocal variables"; the case $d = 0$ will be then discussed at the end of the proof).

Since Λ is a linear operator, we have that $\mathcal{V}_K$ is a vector space, hence a linear subspace of $\mathbb{R}^{K'}$. So, we argue by contradiction: if $\mathcal{V}_K$ does not exhaust the whole of $\mathbb{R}^{K'}$, then it must lie in a proper subspace. Accordingly, there exists

$$\vartheta \in \partial B_1^{K'} \tag{19}$$

such that

$$\mathcal{V}_K \subseteq \big\{ \zeta \in \mathbb{R}^{K'} \text{ s.t. } \vartheta \cdot \zeta = 0 \big\}. \tag{20}$$

Now, for any $j \in \{1, \dots, N\}$ we denote by $\tilde{\phi}_{\star,j} \in C(\mathbb{R}^{n_j})$ the first eigenfunction of $(-\Delta)^{s_j}$ in $B_1^{n_j}$ with Dirichlet datum outside $B_1^{n_j}$ (and normalized to have unit norm in $L^2(\mathbb{R}^{n_j})$). The corresponding eigenvalue will be denoted by $\lambda_{\star,j} > 0$.

We also fix a set of free parameters $t_1, \dots, t_d \in \mathbb{R}$. Up to reordering the variables and possibly taking the operators to the other side of the equation, we suppose that $A_N > 0$ and we set $\lambda_j := \lambda_{\star,j}$ for any $j \in \{1, \dots, N-1\}$ and

$$\lambda_N := \frac{1}{A_N} \left(\sum_{j=1}^{d} |a_j| t_j^{m_j} - \sum_{j=1}^{N-1} A_j \lambda_j \right).$$

We also consider the set

$$\mathcal{P} := \left\{ t = (t_1, \dots, t_d) \in \mathbb{R}^d \text{ s.t. } \sum_{j=1}^{d} |a_j| t_j^{m_j} - \sum_{j=1}^{N-1} A_j \lambda_j > 0 \right\}.$$

Notice that $\mathcal{P}$ is open and non-void (since it contains any point t with large coordinates $t_1, \dots, t_d$). We also remark that for any $t \in \mathcal{P}$ we have $\lambda_N > 0$.

Moreover, by construction

$$\sum_{j=1}^{d} |a_j| t_j^{m_j} - \sum_{j=1}^{N} A_j \lambda_j = 0. \tag{21}$$

We also set

$$r_j := \frac{\lambda_{\star,j}^{1/2s_j}}{\lambda_j^{1/2s_j}}$$

and we see that, for any $j \in \{1, \dots, N\}$, the function

$$\phi_j(X_j) := \tilde{\phi}_{\star,j}\left(\frac{X_j}{r_j}\right) = \tilde{\phi}_{\star,j}\left(\frac{\lambda_j^{1/2s_j} X_j}{\lambda_{\star,j}^{1/2s_j}}\right) \tag{22}$$

is an eigenfunction of $(-\Delta)^{s_j}$ in $B_{r_j}^{n_j}$, with Dirichlet datum outside $B_{r_j}^{n_j}$ and eigenfunction equal to λ_j, that is

$$(-\Delta)^{s_j} \phi_j = \lambda_j \phi_j \text{ in } B_{r_j}^{n_j}. \tag{23}$$

Now, we define, for any $j \in \{1, \dots, d\}$,

$$\bar{a}_j := \begin{cases} a_j/|a_j| & \text{if } a_j \neq 0, \\ 1 & \text{if } a_j = 0. \end{cases}$$

We stress that

$$\bar{a}_j \neq 0. \tag{24}$$

Now we consider, for any $j \in \{1, \dots, d\}$, the solution of the Cauchy problem

$$\begin{cases} \partial_{x_j}^{m_j} \bar{v}_j = -\bar{a}_j \bar{v}_j, \\ \partial_{x_j}^{i} \bar{v}_j(0) = 1 \text{ for every } i \in \{0, \dots, m_j - 1\}. \end{cases} \tag{25}$$

Notice that the solution $\bar{v}_j$ is well defined at least in an interval of the form $[-\rho_j, \rho_j]$ for a suitable $\rho_j > 0$, and we define

$$\rho := \min_{j \in \{1, \dots, d\}} \rho_j.$$

We take $\bar{\tau} \in C_0^\infty(B_\rho^d)$, with $\bar{\tau} = 1$ in $B_{\rho/2}^d$, and we set $\tau(x) = \tau(x_1, \dots, x_d) := \bar{\tau}(t_1 x_1, \dots, t_d x_d)$. Moreover, we introduce the function

$$v_j(x_j) := \bar{v}_j(t_j x_j).$$

Notice that

$$a_j \partial_{x_j}^{m_j} v_j = -|a_j| t_j^{m_j} v_j. \tag{26}$$

Now, we take $e_1, \dots, e_N$, with

$$e_j \in \partial B_{r_j}^{n_j} = \partial B^{n_j}_{\lambda_{\star,j}^{1/2s_j}/\lambda_j^{1/2s_j}}. \tag{27}$$

We introduce an additional set of free parameters $Y_1, \dots, Y_N$, with $Y_j \in \mathbb{R}^{n_j}$ and $e_j \cdot Y_j < 0$. We also take $\varepsilon > 0$ (to be taken as small as we wish in the sequel, possibly in dependence of $e_1, \dots, e_N$ and $Y_1, \dots, Y_N$), and we define

$$\begin{aligned} w(x, X) &:= \tau(x)\, v_1(x_1) \dots v_d(x_d)\, \phi_1(X_1 + e_1 + \varepsilon Y_1) \dots \phi_N(X_N + e_N + \varepsilon Y_N) \\ &= \tau(x)\, v(x)\, \phi(X), \\ \text{where} \quad v(x) &:= v_1(x_1) \dots v_d(x_d) \\ \text{and} \quad \phi(X) &:= \phi_1(X_1 + e_1 + \varepsilon Y_1) \dots \phi_N(X_N + e_N + \varepsilon Y_N). \end{aligned}$$

Notice that w is compactly supported in $\mathbb{R}^\nu$. Moreover, in light of (23) and (26), if (x,X) is sufficiently close to the origin, we have that

$$\ell w(x,X) = \sum_{j=1}^{d} a_j \partial_{x_j}^{m_j} \big(\tau(x)\, v(x)\, \phi(X)\big) = -\sum_{j=1}^{d} |a_j| t_j^{m_j}\, \tau(x)\, v(x)\, \phi(X)$$

and $$\mathcal{L}w(x,X) = \sum_{j=1}^{N} A_j (-\Delta)_{X_j}^{s_j} \Big(\tau(x)\, v(x)\, \phi(X)\Big) = \sum_{j=1}^{N} A_j \lambda_j\, \tau(x)\, v(x)\, \phi(X).$$

Hence, by (21),

$$\Lambda w(x,X) = \ell w(x,X) + \mathcal{L}w(x,X) = 0$$

if (x,X) is sufficiently close to the origin. Consequently, $w \in \mathcal{H}$. Thus, in view of (18) and (20), we have that

$$\begin{aligned} 0 = \vartheta \cdot \partial^K w(0) &= \sum_{|\iota| \leqslant K} \vartheta_\iota \partial^\iota w(0) \\ &= \sum_{|i_1|+\dots+|i_d|+|I_1|+\dots+|I_N| \leqslant K} \vartheta_{i_1,\dots,i_d,I_1,\dots,I_N} \partial_{x_1}^{i_1} \dots \partial_{x_d}^{i_d} \partial_{X_1}^{I_1} \dots \partial_{X_N}^{I_N} w(0). \end{aligned} \tag{28}$$

We claim that, for any $i \in \mathbb{N}$,

$$\varpi_{ij} := \partial_{x_j}^{i} \bar{v}_j(0) \neq 0. \tag{29}$$

The proof of this can be done by induction. Indeed, if $i \in \{0,\dots,m_j-1\}$, then (29) is true, thanks to the initial condition in (25). Suppose now that

$$\text{the claim in (29) holds true for all } i \in \{0,\dots,i_o\}, \text{ for some } i_o \geqslant m_j - 1. \tag{30}$$

Then, using the equation in (25) we have that

$$\partial_{x_j}^{i_o+1} \bar{v}_j = \partial_{x_j}^{i_o+1-m_j} \partial_{x_j}^{m_j} \bar{v}_j = -\bar{a}_j \partial_{x_j}^{i_o+1-m_j} \bar{v}_j. \tag{31}$$

By (30), we know that $\partial_{x_j}^{i_o+1-m_j} \bar{v}_j(0) \neq 0$. This, (24) and (31) imply that $\partial_{x_j}^{i_o+1} \bar{v}_j(0) \neq 0$. This proves (29).

Now, from (29) we have that

$$\partial_{x_j}^{i_j} v(0) = t_j^{i_j} \varpi_{i_j\, j} \neq 0.$$

Hence, we write (28) as

$$\begin{aligned} 0 &= \sum_{|i_1|+\dots+|i_d|+|I_1|+\dots+|I_N| \leqslant K} \vartheta_{i_1,\dots,i_d,I_1,\dots,I_N}\, \varpi_{i_1\,1} \dots \varpi_{i_d\,d}\, t_1^{i_1} \dots t_d^{i_d}\, \partial_{X_1}^{I_1} \dots \partial_{X_N}^{I_N} \phi(0) \\ &= \sum_{|i|+|I| \leqslant K} \vartheta_{i,I}\, \varpi_i t^i \partial_X^I \phi(0), \end{aligned} \tag{32}$$

where a multi-index notation has been adopted, and if $i = (i_1,\dots,i_d)$, $\varpi_i := \varpi_{i_1\,1} \dots \varpi_{i_d\,d}$. We stress that

$$\varpi_i \neq 0, \tag{33}$$

thanks to (29).

Recalling (22), we write (32) as

$$0 = \sum_{|i|+|I| \leqslant K} \vartheta_{i,I}\, \varpi_i\, t^i \prod_{j=1}^{N} \left(\frac{\lambda_j}{\lambda_{\star,j}}\right)^{\frac{|I_j|}{2s_j}} \partial_{X_j}^{I_j} \tilde{\phi}_{\star,j} \left(\frac{\lambda_j^{1/2s_j}}{\lambda_{\star,j}^{1/2s_j}} (e_j + \varepsilon Y_j)\right).$$

Hence, by Corollary 7, applied to $e := \frac{\lambda_j^{1/2s_j}}{\lambda_{\star,j}^{1/2s_j}} e_j$ and $X := \frac{\lambda_j^{1/2s_j}}{\lambda_{\star,j}^{1/2s_j}} Y_j$, after multiplying by a power of ε and sending $\varepsilon \searrow 0$, we obtain

$$\begin{aligned} 0 &= \sum_{|i|+|I|\leqslant K} \vartheta_{i,I}\, \varpi_i\, t^i \\ &\cdot \prod_{j=1}^{N} \left[\left(\frac{\lambda_j}{\lambda_{\star,j}}\right)^{\frac{|I_j|}{s_j}} (-1)^{|I_j|}\, \kappa_\star\, \lambda_\star\, s\,(s-1)\dots(s-|I_j|+1)\, e_j^{I_j} \left(-\frac{\lambda_j^{\frac{1}{2s_j}}\, e_j}{\lambda_{\star,j}^{\frac{1}{2s_j}}} \cdot \frac{\lambda_j^{\frac{1}{2s_j}}\, Y_j}{\lambda_{\star,j}^{\frac{1}{2s_j}}} \right)_+^{s_j-|I_j|} \right] \\ &= \sum_{|i|+|I|\leqslant K} \vartheta_{i,I}\, \varpi_i\, t^i \prod_{j=1}^{N} \left[\left(\frac{\lambda_j}{\lambda_{\star,j}}\right) (-1)^{|I_j|}\, \kappa_\star\, \lambda_\star\, s\,(s-1)\dots(s-|I_j|+1)\, e_j^{I_j}\; (-e_j\cdot Y_j)_+^{s_j-|I_j|} \right]. \end{aligned}$$

That is, collecting and simplifying some terms, we find that

$$0 = \sum_{|i|+|I|\leqslant K} \tilde\vartheta_{i,I}\, \varpi_i\, t^i\, e^I \prod_{j=1}^{N} (-e_j\cdot Y_j)_+^{-|I_j|}, \tag{34}$$

with

$$\tilde\vartheta_{i,I} := \vartheta_{i,I} \prod_{j=1}^{N} \big(s\,(s-1)\dots(s-|I_j|+1)\big).$$

We remark that, in light of (19),

$$\tilde\vartheta_{i,I} \text{ are not all equal to zero.} \tag{35}$$

Notice that formula (34) is true for any $(t_1,\dots,t_d) \in \mathcal{P}$, any $e_1,\dots,e_N$ satisfying (27), and any $Y_1,\dots,Y_N$, with $Y_j \in \mathbb{R}^{n_j}$ and $e_j \cdot Y_j < 0$.

For this, we take new free parameters $T_1,\dots,T_N$ with $T_j \in \mathbb{R}^{n_j}$ and we choose

$$e_j := \frac{\lambda_{\star,j}^{1/2s_j}}{\lambda_j^{1/2s_j}} \cdot \frac{T_j}{|T_j|} \qquad \text{and} \qquad Y_j := -\frac{T_j}{|T_j|^2}.$$

Then, formula (34) becomes

$$\sum_{|i|+|I|\leqslant K} \tilde\vartheta_{i,I}\, \varpi_i\, t^i\, T^I = 0.$$

By the Identity Principle of Polynomials, this gives that each $\tilde\vartheta_{i,I}\varpi_i$ is equal to zero. Hence, by (33), each $\tilde\vartheta_{i,I}$ is equal to zero. This is in contradiction with (35) and so the desired result is established (when $d \neq 0$).

Now we consider the case in which $d = 0$, i.e. when only “nonlocal variables” are present. For this, we argue recursively on N (i.e. on the number of the “nonlocal variables”). When $N = 1$, that is when there is only one set of “nonlocal variables”, the result is true, thanks to Theorem 3.1 in [4].

Now we suppose that the result is true for $N-1$ and we prove it for N. We set

$$\mathcal{L}' := \sum_{j=1}^{N-1} A_j\,(-\Delta_{X_j})^{s_j} \quad \text{and} \quad \mathcal{L}_N := A_N\,(-\Delta_{X_N})^{s_N}.$$

We denote by $\mathcal{H}'$ the family of all functions $w' \in C(\mathbb{R}^{n_1+\dots+n_{N-1}})$ that are compactly supported and for which there exists a neighborhood of the origin on which w' is smooth and $\mathcal{L}'w' = 0$.

Similarly, we call $\mathcal{H}_N$ the family of all functions $w_N \in C(\mathbb{R}^{n_N})$ that are compactly supported and for which there exists a neighborhood of the origin on which w_N is smooth and $\mathcal{L}_N w_N = 0$.

We also use the notation $X=(X',X_N)\in\mathbb{R}^{n_1+\dots+n_{N-1}}\times\mathbb{R}^{n_N}$ to distinguish the last set of variables. Given any $w'\in\mathcal{H}'$ and any $w_N\in\mathcal{H}_N$, we set

$$W_{w',w_N}(X)=W_{w',w_N}(X',X_N):=w'(X')\,w_N(X_N).$$

Notice that

$$\mathcal{L}W_{w',w_N}(X)=\Big(\mathcal{L}'w'(X')\Big)\,w_N(X_N)+w'(X')\,\Big(\mathcal{L}_N w_N(X_N)\Big).$$

Thus,

$$\text{if } w'\in\mathcal{H}' \text{ and } w_N\in\mathcal{H}_N, \text{ then } W_{w',w_N}\in\mathcal{H}. \tag{36}$$

Again, we argue by contradiction and we suppose that the claim in Lemma 8 is not true, hence there exists a unit vector ϑ such that

$$\mathcal{V}_K \text{ lies in the orthogonal space of } \vartheta. \tag{37}$$

Notice that each component of ϑ can be written as ϑ_I, with $I=(I_1,\dots,I_N)$ and $|I|\leqslant K$. To distinguish the last component we write $I':=(I_1,\dots,I_{N-1})$ and so $\vartheta_I=\vartheta_{(I',I_N)}$ with $|I'|+|I_N|\leqslant K$.

In particular, by (36) and (37), for any $w'\in\mathcal{H}'$ and any $w_N\in\mathcal{H}_N$ we have that

$$\begin{aligned}0&=\sum_{|I|\leqslant K}\vartheta_I\partial_X^I W_{w',w_N}(0)=\sum_{|I'|+|I_N|\leqslant K}\vartheta_{(I',I_N)}\partial_{X'}^{I'}w'(0)\partial_{X_N}^{I_N}w_N(0)\\&=\sum_{|I_N|\leqslant K}\left[\sum_{|I'|\leqslant K-|I_N|}\vartheta_{(I',I_N)}\partial_{X'}^{I'}w'(0)\right]\partial_{X_N}^{I_N}w_N(0)=\sum_{|I_N|\leqslant K}\hat\vartheta_{I_N,w'}\,\partial_{X_N}^{I_N}w_N(0),\end{aligned}$$

where

$$\hat\vartheta_{I_N,w'}:=\sum_{|I'|\leqslant K-|I_N|}\vartheta_{(I',I_N)}\partial_{X'}^{I'}w'(0).$$

That is, all functions in $\mathcal{H}_N$ lie in the orthogonal of the vector with entries $\hat\vartheta_{I_N,w'}$. From Theorem 3.1 in [4] this implies that each $\hat\vartheta_{I_N,w'}$ must vanish, that is

$$\sum_{|I'|\leqslant K-|I_N|}\vartheta_{(I',I_N)}\partial_{X'}^{I'}w'(0)=0$$

for any multi-index I_N with $|I_N|\leqslant K$ and any $w'\in\mathcal{H}'$. Since $\mathcal{H}'$ contains $N-1$ "nonlocal variables", we can now use the inductive hypothesis and conclude that each $\vartheta_{(I',I_N)}$ must vanish. This is a contradiction with the fact that ϑ was supposed to be of unit length and so the proof of Lemma 8 is complete. □

4 Proof of Theorem 2 when f is a monomial

Now we prove Theorem 2 under the additional assumption that f is of monomial type, namely that

$$f(x,X)=\frac{x_1^{i_1}\dots x_d^{i_d}X_1^{I_1}\dots X_N^{I_N}}{\iota!}=\frac{x^i\,X^I}{\iota!}, \tag{38}$$

for some $(i_1,\dots,i_d)\in\mathbb{N}^d$ and $(I_1,\dots,I_N)\in\mathbb{N}^{n_1}\times\mathbb{N}^{n_N}$. Of course, we used here the standard notation for powers of multi-indices: namely if $X_1:=(X_{1,1},\dots,X_{1,n_1})\in\mathbb{R}^{n_1}$ and $I_1:=(I_{1,1},\dots,I_{1,n_1})\in\mathbb{N}^{n_1}$, the notation $X_1^{I_1}$ is short for $X_{1,1}^{I_{1,1}}\dots X_{1,n_1}^{I_{1,n_1}}$. Also, ι is as in (17) and, as customary, we used the multi-index factorial

$$\iota!:=i_1!\,\dots\,i_d!\,I_1!\,\dots\,I_N!,$$

where, once again $I_1!:=I_{1,1}!\,\dots\,I_{1,n_1}!$ and so on.

Then, to prove Theorem 2 in this case, we argue as follows. We define

$$\gamma := \sum_{j=1}^{d} \frac{i_j}{m_j} + \sum_{j=1}^{N} \frac{|I_j|}{2s_j} \tag{39}$$

and $$\mu := \min\left\{ \frac{1}{m_1}, \dots, \frac{1}{m_d}, \frac{1}{2s_1}, \dots, \frac{1}{2s_N} \right\}. \tag{40}$$

We also take $K_o \in \mathbb{N}$ with

$$K_o \geqslant \frac{\gamma+1}{\mu} \tag{41}$$

and we let

$$K := K_o + |i| + |I| + k, \tag{42}$$

where k is the fixed integer given by the statement of Theorem 2. By Lemma 8, there exist a neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^\nu$ and a function $w \in C(\mathbb{R}^\nu)$, compactly supported in $\mathbb{R}^\nu$, such that $w \in C^\infty(\mathcal{N})$, $\Lambda w = 0$ in $\mathcal{N}$, and such that all the derivatives of w in 0 of order up to K vanish, with the exception of $\partial^\iota w(0)$, which is equal to 1. In this way, setting

$$g := w - f, \tag{43}$$

we have that

$$\partial^\alpha g(0) = 0 \text{ for any } \alpha \in \mathbb{N}^\nu \text{ with } |\alpha| \leqslant K.$$

Accordingly, in $\mathcal{N}$ we can write

$$g(x, X) = \sum_{|\tau| \geqslant K+1} x^t \, X^T \, h_\tau(x, X), \tag{44}$$

for functions h_τ that are smooth in $\mathcal{N}$, where the multi-index notation $\tau = (t, T)$ has been used.

Now, we fix $\eta \in (0, 1)$ (to be taken suitably small with respect to the fixed $\varepsilon > 0$ given by the statement of Theorem 2). We define

$$u(x, X) := \frac{1}{\eta^\gamma} \, w\big(\eta^{\frac{1}{m_1}} x_1, \dots, \eta^{\frac{1}{m_d}} x_d, \eta^{\frac{1}{2s_1}} X_1, \dots, \eta^{\frac{1}{2s_N}} X_N\big).$$

Notice that u is compactly supported in $\mathbb{R}^\nu$ and smooth in a neighborhood of the origin (which is large for η small, hence we may suppose that it includes B_1), and in this neighborhood we have

$$\begin{aligned}\eta^\gamma \, \Lambda u(x, X) = \eta \Bigg[&\sum_{j=1}^{d} a_j \partial_{x_j}^{m_j} w\big(\eta^{\frac{1}{m_1}} x_1, \dots, \eta^{\frac{1}{m_d}} x_d, \eta^{\frac{1}{2s_1}} X_1, \dots, \eta^{\frac{1}{2s_N}} X_N\big) \\ &+ \sum_{j=1}^{N} A_j \, (-\Delta_{X_j})^{s_j} w\big(\eta^{\frac{1}{m_1}} x_1, \dots, \eta^{\frac{1}{m_d}} x_d, \eta^{\frac{1}{2s_1}} X_1, \dots, \eta^{\frac{1}{2s_N}} X_N\big) \Bigg] = 0.\end{aligned}$$

These observations establish (4) and (6). Now we prove (5). To this aim, we observe that the monomial structure of f in (38) and the definition of γ in (39) imply that

$$\frac{1}{\eta^\gamma} \, f\big(\eta^{\frac{1}{m_1}} x_1, \dots, \eta^{\frac{1}{m_d}} x_d, \eta^{\frac{1}{2s_1}} X_1, \dots, \eta^{\frac{1}{2s_N}} X_N\big) = f(x, X).$$

Consequently, by (43) and (44),

$$\begin{aligned} u(x, X) - f(x, X) &= \frac{1}{\eta^\gamma} \, g\big(\eta^{\frac{1}{m_1}} x_1, \dots, \eta^{\frac{1}{m_d}} x_d, \eta^{\frac{1}{2s_1}} X_1, \dots, \eta^{\frac{1}{2s_N}} X_N\big) \\ &= \sum_{|\tau| \geqslant K+1} \eta^{\left|\frac{t}{m}\right| + \left|\frac{T}{2s}\right| - \gamma} \, x^t \, X^T \, h_\tau\big(\eta^{\frac{1}{m}} x, \eta^{\frac{1}{2s}} X\big) \end{aligned}$$

where the multi-index notation has been used.

Therefore, for any multi-index $\beta = (b, B)$ with $|\beta| \leqslant k$,

$$\begin{aligned}\partial^\beta \left(u(x,X) - f(x,X)\right) &= \partial_x^b \, \partial_X^B \, \left(u(x,X) - f(x,X)\right) \\ &= \sum_{\substack{|b'|+|b''|=|b| \\ |B'|+|B''|=|B| \\ |\tau|\geqslant K+1}} c_{\tau,\beta} \, \eta^{\left|\frac{t}{m}\right|+\left|\frac{T}{2s}\right|-\gamma+\left|\frac{b''}{m}\right|+\left|\frac{B''}{2s}\right|} \, x^{t-b'} \, X^{T-B'} \, \partial_x^{b''} \, \partial_X^{B''} \, h_\tau\big(\eta^{\frac{1}{m}} x, \eta^{\frac{1}{2s}} X\big),\end{aligned} \tag{45}$$

for suitable coefficients $c_{\tau,\beta}$. Thus, to prove (5), we need to show that this quantity is small if so is η. To this aim, we use (40), (41) and (42) to see that

$$\begin{aligned}\left|\frac{t}{m}\right| + \left|\frac{T}{2s}\right| - \gamma + \left|\frac{b''}{m}\right| + \left|\frac{B''}{2s}\right| &\geqslant \left|\frac{t}{m}\right| + \left|\frac{T}{2s}\right| - \gamma \\ &\geqslant \mu\big(|t| + |T|\big) - \gamma \geqslant K\mu - \gamma \geqslant K_o\mu - \gamma \geqslant 1.\end{aligned}$$

Consequently, we deduce from (45) that $\|u - f\|_{C^k(B_1^\nu)} \leqslant C\eta$, for some $C > 0$. By choosing η sufficiently small with respect to ε, this implies (5). The proof of Theorem 2 when f is a monomial is thus complete.

5 Proof of Theorem 2 when f is a polynomial

If f is a polynomial, we can write f as a finite sum of monomials, say

$$f(x,X) = \sum_{j=1}^{J} c_j f_j(x,X),$$

where each f_j is a monomial as in (38), $c_j \in \mathbb{R}$ and $J \in \mathbb{N}$. Let $c := \max_{j\in J} |c_j|$. Then, we know that Theorem 2 holds for each f_j, in view of the proof given in Section 4, and so we find $u_j \in C^\infty(B_1^\nu) \cap C(\mathbb{R}^\nu)$ and $R_j > 1$ such that $\Lambda u_j = 0$ in B_1^ν, $\|u_j - f_j\|_{C^k(B_1^\nu)} \leqslant \varepsilon$ and $u = 0$ in $\mathbb{R}^\nu \setminus B_{R_j}^\nu$. Hence, we set

$$u(x,X) = \sum_{j=1}^{J} c_j u_j(x,X),$$

and we see that

$$\|u - f\|_{C^k(B_1^\nu)} \leqslant \sum_{j=1}^{J} |c_j| \, \|u_j - f_j\|_{C^k(B_1^\nu)} \leqslant cJ\,\varepsilon.$$

Also, since Λ is linear, we have that $\Lambda u = 0$ in B_1^ν. Finally, u is supported in B_R^ν, being $R := \max_{j\in J} R_j$. This establishes Theorem 2 for polynomials (up to replacing ε with $cJ\,\varepsilon$).

6 Completion of the proof of Theorem 2

Let f be as in the statement of Theorem 2. By a version of the Stone-Weierstraß Theorem (see e.g. Lemma 2.1 in [4]), we know that there exists a polynomial $\tilde f$ such that $\|f - \tilde f\|_{C^k(B_1^\nu)} \leqslant \varepsilon$. Then, we know that Theorem 2 holds for $\tilde f$, in view of the proof given in Section 5, and so we find $u \in C^\infty(B_1^\nu) \cap C(\mathbb{R}^\nu)$ and $R > 1$ such that $\Lambda u = 0$ in B_1^ν, $\|u - \tilde f\|_{C^k(B_1^\nu)} \leqslant \varepsilon$ and $u = 0$ in $\mathbb{R}^\nu \setminus B_R^\nu$. Then, we see that $\|u - f\|_{C^k(B_1^\nu)} \leqslant \|u - \tilde f\|_{C^k(B_1^\nu)} + \|f - \tilde f\|_{C^k(B_1^\nu)} \leqslant 2\varepsilon$, hence Theorem 2 is proved (up to replacing ε with 2ε).

7 Green function computations

In this section, we present the proof of Lemma 4. We recall that such result gives some precise asymptotics on the boundary behavior of the Green function of the fractional Laplacian, which in turn have been exploited in Section 2 to obtain precise boundary information on the solutions of fractional Laplace equations.

Not to interrupt the main arguments of the proofs, we stated Lemma 4 in Section 2 without a proof, and this section is thus devoted to complete this point.

Proof of Lemma 4. We remark that the condition $e + \varepsilon\omega \in B_1$ for all $\varepsilon \in (0, \varepsilon_o]$ says that

$$1 > |e + \varepsilon\omega|^2 = 1 + \varepsilon^2 + 2\varepsilon e \cdot \omega$$

and so in particular

$$-e \cdot \omega > \frac{\varepsilon}{2} > 0. \tag{46}$$

From (11), we have that

$$r_0(e + \varepsilon\omega, z) = \frac{\varepsilon\,(-\varepsilon - 2\,e \cdot \omega)\,(1 - |z|^2)}{|z - e - \varepsilon\omega|^2}. \tag{47}$$

In particular,

$$r_0(e + \varepsilon\omega, z) \leqslant \frac{3\varepsilon}{|z - e + \varepsilon\omega|^2}. \tag{48}$$

Moreover, using a Taylor binomial series,

$$(t+1)^{-\frac{n}{2}} = \sum_{k=0}^{+\infty} \binom{-n/2}{k}\, t^k$$

and therefore

$$\frac{t^{s-1}}{(t+1)^{\frac{n}{2}}} = \sum_{k=0}^{+\infty} \binom{-n/2}{k}\, t^{k+s-1}. \tag{49}$$

Since, by the bounds on the binomial coefficients, we have that

$$\left| \binom{-n/2}{k} \right| \leqslant C\, k^{\frac{n}{2}}, \tag{50}$$

it follows from the root test that the series in (49) is uniformly convergent for any t in a compact subset of $(-1, 1)$. In particular, if we set

$$r_1(x, z) := \min\left\{ r_0(x, z),\ \frac{1}{2} \right\}, \tag{51}$$

we can exchange the integration and summation signs and find that

$$\int_0^{r_1(x,z)} \frac{t^{s-1}\,dt}{(t+1)^{\frac{n}{2}}} = \sum_{k=0}^{+\infty} c_k \left(r_1(x, z)\right)^{k+s},$$

with

$$c_k := \frac{1}{k+s} \binom{-n/2}{k}.$$

Therefore, we have

$$G(x, z) = \mathcal{G}(x, z) + g(x, z), \tag{52}$$

with

$$\mathcal{G}(x,z) := |z-x|^{2s-n} \sum_{k=0}^{+\infty} c_k \big(r_1(x,z)\big)^{k+s}$$
$$\text{and} \qquad g(x,z) := |z-x|^{2s-n} \int_{r_1(x,z)}^{r_0(x,z)} \frac{t^{s-1}\,dt}{(t+1)^{\frac{n}{2}}}.$$

Notice that $g(x,z)=0$ if $r_0(x,z)\leqslant 1/2$. Also, if $r_0(x,z)>1/2$,

$$0\leqslant g(x,z)\leqslant |z-x|^{2s-n}\int_{1/2}^{r_0(x,z)} \frac{t^{s-1}\,dt}{t^{\frac{n}{2}}} \leqslant \begin{cases} C\,|z-x|^{2s-n} & \text{if } n>2s,\\ C\,\log r_0(x,z) & \text{if } n=2s,\\ C\,|z-x|^{2s-n}\big(r_0(x,z)\big)^{s-\frac{n}{2}} & \text{if } n<2s,\end{cases}$$

for some $C>0$. Now we compute this expression in $x:=e+\varepsilon\omega$. Notice that the condition $r_0(e+\varepsilon\omega,z)=r_0(x,z)>1/2$, combined with (48), says that

$$|z-e-\varepsilon\omega|^2\leqslant 9\varepsilon. \tag{53}$$

As a consequence

$$\begin{aligned}\left|\int_{B_1} f(z)\,g(e+\varepsilon\omega,z)\,dz\right| &\leqslant \int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |f(z)|\,|g(e+\varepsilon\omega,z)|\,dz\\ &\leqslant \begin{cases} C\,\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |f(z)|\,|z-e+\varepsilon\omega|^{2s-n}\,dz & \text{if } n>2s,\\ C\,\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |f(z)|\,\log r_0(e+\varepsilon\omega,z)\,dz & \text{if } n=2s,\\ C\,\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |f(z)|\,|z-e+\varepsilon\omega|^{2s-n}\big(r_0(e+\varepsilon\omega,z)\big)^{s-\frac{n}{2}}\,dz & \text{if } n<2s.\end{cases}\end{aligned} \tag{54}$$

Now, if $z\in B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)$, then $|z-e|\leqslant 4\sqrt{\varepsilon}$ and so

$$|f(z)|\leqslant C\varepsilon^{\frac{\alpha}{2}}, \tag{55}$$

with $C>0$ depending on f. Hence recalling (48), after renaming $C>0$, we deduce from (54) that

$$\begin{aligned}\left|\int_{B_1} f(z)\,g(e+\varepsilon\omega,z)\,dz\right| &\leqslant \begin{cases} C\varepsilon^{\frac{\alpha}{2}}\,\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |z-e+\varepsilon\omega|^{2s-n}\,dz & \text{if } n>2s,\\ C\varepsilon^{\frac{\alpha}{2}}\,\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} \log\frac{3\varepsilon}{|z-e+\varepsilon\omega|^2}\,dz & \text{if } n=2s,\\ C\varepsilon^{\frac{\alpha}{2}+s-\frac{n}{2}}\,\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} 1\,dz & \text{if } n<2s,\end{cases}\\ &\leqslant C\varepsilon^{\frac{\alpha}{2}+s}.\end{aligned}$$

This and (52) give that

$$\int_{B_1} f(z)\,G(e+\varepsilon\omega,z)\,dz=\int_{B_1} f(z)\,\mathcal{G}(e+\varepsilon\omega,z)\,dz+o(\varepsilon^s). \tag{56}$$

Now we consider the series defining $\mathcal{G}$ and we split the contribution coming from the index $k=0$ from the ones coming from the indices $k\geqslant 1$, namely we write

$$\begin{aligned} &\mathcal{G}(x,z)=\mathcal{G}_0(x,z)+\mathcal{G}_1(x,z)\\ \text{with}\quad &\mathcal{G}_0(x,z):=\frac{|z-x|^{2s-n}}{s}\big(r_1(x,z)\big)^s\\ \text{and}\quad &\mathcal{G}_1(x,z):=|z-x|^{2s-n}\sum_{k=1}^{+\infty} c_k\big(r_1(x,z)\big)^{k+s}.\end{aligned} \tag{57}$$

So, we use (51) and (55) and obtain that

$$\begin{aligned}
&\left|\int_{B_1\cap B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} f(z)\,\mathcal{G}_1(e+\varepsilon\omega,z)\,dz\right| \leqslant \int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |f(z)|\,\mathcal{G}_1(e+\varepsilon\omega,z)\,dz\\
&\qquad\leqslant C\varepsilon^{\frac{\alpha}{2}}\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |z-e-\varepsilon\omega|^{2s-n}\sum_{k=1}^{+\infty}|c_k|\,\big(r_1(e+\varepsilon\omega,z)\big)^{k+s}\,dz\\
&\qquad\leqslant C\varepsilon^{\frac{\alpha}{2}}\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |z-e-\varepsilon\omega|^{2s-n}\sum_{k=1}^{+\infty}|c_k|\,\big(1/2\big)^{k+s}\,dz\\
&\qquad\leqslant C\varepsilon^{\frac{\alpha}{2}}\int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |z-e-\varepsilon\omega|^{2s-n}\,dz\\
&\qquad\leqslant C\varepsilon^{\frac{\alpha}{2}+s},
\end{aligned}\tag{58}$$

up to renaming $C>0$. On the other hand,

$$|z| = |e+\varepsilon\omega+z-e-\varepsilon\omega| \geqslant |e+\varepsilon\omega|-|z-e-\varepsilon\omega| \geqslant 1-\varepsilon-|z-e-\varepsilon\omega|$$

and therefore

$$|f(z)| \leqslant C\,\big(1-|z|\big)^{\alpha} \leqslant C\,\big(\varepsilon+|z-e-\varepsilon\omega|\big)^{\alpha}.$$

In particular, if $|z-e-\varepsilon\omega|>3\sqrt{\varepsilon}$, then

$$|f(z)| \leqslant C\,|z-e-\varepsilon\omega|^{\alpha}.\tag{59}$$

Also, using (48) and (51), for any $k\geqslant 1$

$$\begin{aligned}
\big(r_1(e+\varepsilon\omega,z)\big)^{k+s} &= \big(r_1(e+\varepsilon\omega,z)\big)^{s+\frac{\alpha}{4}}\,\big(r_1(e+\varepsilon\omega,z)\big)^{k-\frac{\alpha}{4}}\\
&\leqslant \big(r_0(e+\varepsilon\omega,z)\big)^{s+\frac{\alpha}{4}}\left(\frac{1}{2}\right)^{k-\frac{\alpha}{4}} \leqslant \frac{C\,\varepsilon^{s+\frac{\alpha}{4}}}{2^k\,|z-e-\varepsilon\omega|^{2s+\frac{\alpha}{2}}}.
\end{aligned}$$

This and (59) give that, if $z\in B_1\setminus B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)$, then

$$\begin{aligned}
&|f(z)\,\mathcal{G}_1(e+\varepsilon\omega,z)|\\
\leqslant\;& C\,|z-e-\varepsilon\omega|^{\alpha+2s-n}\sum_{k=1}^{+\infty}|c_k|\,\big(r_1(e+\varepsilon\omega,z)\big)^{k+s}\\
\leqslant\;& C\varepsilon^{s+\frac{\alpha}{4}}\,|z-e-\varepsilon\omega|^{\frac{\alpha}{2}-n}\sum_{k=1}^{+\infty}\frac{|c_k|}{2^k},
\end{aligned}$$

and the latter series is convergent, thanks to (50). This implies that

$$\begin{aligned}
&\left|\int_{B_1\setminus B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} f(z)\,\mathcal{G}_1(e+\varepsilon\omega,z)\,dz\right| \leqslant C\varepsilon^{s+\frac{\alpha}{4}}\int_{B_1\setminus B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |z-e-\varepsilon\omega|^{\frac{\alpha}{2}-n}\,dz\\
&\qquad\leqslant C\varepsilon^{s+\frac{\alpha}{4}}\int_{B_1}|z-e-\varepsilon\omega|^{\frac{\alpha}{2}-n}\,dz \leqslant C\varepsilon^{s+\frac{\alpha}{4}}.
\end{aligned}$$

By this and (58), we conclude that

$$\int_{B_1} f(z)\,\mathcal{G}_1(e+\varepsilon\omega,z)\,dz = o(\varepsilon^s).$$

Hence, we insert this information into (56) and, recalling (57), we obtain

$$\int_{B_1} f(z)\,G(e+\varepsilon\omega,z)\,dz = \int_{B_1} f(z)\,\mathcal{G}_0(e+\varepsilon\omega,z)\,dz + o(\varepsilon^s).\tag{60}$$

Now we define

$$\begin{aligned} &\mathcal{D}_1 := \{z \in B_1 \text{ s.t. } r_0(e+\varepsilon\omega, z) > 1/2\} \\ \text{and} \qquad &\mathcal{D}_2 := \{z \in B_1 \text{ s.t. } r_0(e+\varepsilon\omega, z) \leqslant 1/2\}. \end{aligned}$$

If $z \in \mathcal{D}_1$, then (53) holds true, and so we can use (55), to find that

$$\big|f(z)\,\mathcal{G}_0(e+\varepsilon\omega, z)\big| \leqslant C\varepsilon^{\frac{\alpha}{2}}\,|z-e+\varepsilon\omega|^{2s-n}.$$

Consequently, recalling (53),

$$\left|\int_{\mathcal{D}_1} f(z)\,\mathcal{G}_0(e+\varepsilon\omega, z)\,dz\right| \leqslant C\varepsilon^{\frac{\alpha}{2}} \int_{B_{3\sqrt{\varepsilon}}(e+\varepsilon\omega)} |z-e+\varepsilon\omega|^{2s-n}\,dz = C\varepsilon^{\frac{\alpha}{2}+s},$$

up to renaming $C > 0$ once again. In this way, formula (60) reduces to

$$\int_{B_1} f(z)\,G(e+\varepsilon\omega, z)\,dz = \int_{\mathcal{D}_2} f(z)\,\mathcal{G}_0(e+\varepsilon\omega, z)\,dz + o(\varepsilon^s). \tag{61}$$

Now, by (51) and (47), if $z \in \mathcal{D}_2$,

$$\mathcal{G}_0(e+\varepsilon\omega, z) = \frac{|z-e-\varepsilon\omega|^{2s-n}}{s}\big(r_0(e+\varepsilon\omega, z)\big)^s = \frac{\varepsilon^s\,(-\varepsilon-2\,e\cdot\omega)^s\,(1-|z|^2)^s}{s\,|z-e-\varepsilon\omega|^n}.$$

Hence, (61) gives that

$$\begin{aligned} &\lim_{\varepsilon\searrow 0} \varepsilon^{-s}\int_{B_1} f(z)\,G(e+\varepsilon\omega, z)\,dz \\ &= \lim_{\varepsilon\searrow 0}\int_{\{2\varepsilon\,(-\varepsilon-2\,e\cdot\omega)\,(1-|z|^2)\leqslant|z-e-\varepsilon\omega|^2\}} f(z)\,\frac{(-\varepsilon-2\,e\cdot\omega)^s\,(1-|z|^2)^s}{s\,|z-e-\varepsilon\omega|^n}\,dz. \end{aligned} \tag{62}$$

Now, we show the following uniform integrability condition: we set

$$F_\varepsilon(z) := \begin{cases} f(z)\,\dfrac{(-\varepsilon-2\,e\cdot\omega)^s\,(1-|z|^2)^s}{s\,|z-e-\varepsilon\omega|^n} & \text{if } 2\varepsilon\,(-\varepsilon-2\,e\cdot\omega)\,(1-|z|^2)\leqslant|z-e-\varepsilon\omega|^2, \\ 0 & \text{otherwise,} \end{cases}$$

and we prove that for any $\eta > 0$ there exists $\delta > 0$ (depending on η, e and ω, but independent of ε) such that, for any $E \subset \mathbb{R}^d$ with $|E| \leqslant \delta$, we have

$$\int_{B_1\cap E} \big|F_\varepsilon(z)\big|\,dz \leqslant \eta. \tag{63}$$

To this aim, we take E as above and

$$\rho := c_\star\,\varepsilon,$$

with $c_\star \in \big(0, \frac{1}{10}\big)$ to be conveniently chosen in the sequel (also in dependence of ω and e), and we set $E_1 := E \cap B_\rho(e+\varepsilon\omega)$, $E_2 := E \setminus E_1$.

We claim that

$$E_1 \text{ is empty.} \tag{64}$$

For this, we argue by contradiction: if there existed $z \in E_1$, then

$$\varepsilon\,(-\,e\cdot\omega)\,(1-|z|^2) \leqslant 2\varepsilon\,(-\varepsilon-2\,e\cdot\omega)\,(1-|z|^2) \leqslant |z-e-\varepsilon\omega|^2 \leqslant \rho^2,$$

if ε is small enough in dependence of the fixed e and ω (recall (46)), and thus

$$1-|z|^2 \leqslant \frac{C\rho^2}{\varepsilon}, \tag{65}$$

with $C > 0$ also depending on e and ω. On the other hand, we have that $E_1 \subseteq B_\rho(e + \varepsilon\omega)$, therefore

$$|z| \leqslant |e + \varepsilon\omega| + |z - e - \varepsilon\omega| \leqslant \sqrt{1 + \varepsilon^2 + 2\varepsilon e \cdot \omega} + \rho \leqslant 1 - \frac{-\varepsilon e \cdot \omega}{10} + C\varepsilon^2 + \rho,$$

and so

$$|z|^2 \leqslant 1 - \frac{\varepsilon e \cdot \omega}{5} + C\varepsilon^2 + \rho^2.$$

This is a contradiction with (65) if $c_\star$ is appropriately small and so (64) is proved.

So, from now on, $c_\star$ is fixed suitably small. We observe that if $z \in E_2$ then

$$|z - e - \varepsilon\omega| \geqslant \rho = c_\star\varepsilon,$$

and consequently

$$\int_{B_1 \cap E_2} \big|F_\varepsilon(z)\big|\, dz \leqslant \int_{\substack{B_1 \cap E \\ \{|z-e-\varepsilon\omega| \geqslant c_\star\varepsilon\}}} \frac{C\,(1 - |z|)^{s+\alpha}}{|z - e - \varepsilon\omega|^n}\, dz. \tag{66}$$

Now, we distinguish two cases, either $\delta \leqslant \varepsilon^{2n}$ or $\delta > \varepsilon^{2n}$. If $\delta \leqslant \varepsilon^{2n}$, we use (66) to get that

$$\int_{B_1 \cap E_2} \big|F_\varepsilon(z)\big|\, dz \leqslant \int_{\substack{B_1 \cap E \\ \{|z-e-\varepsilon\omega| \geqslant c_\star\varepsilon\}}} \frac{C}{\varepsilon^n}\, dz \leqslant \frac{C\delta}{\varepsilon^n} \leqslant C\sqrt{\delta}. \tag{67}$$

If instead

$$\delta > \varepsilon^{2n}, \tag{68}$$

we observe that

$$|z - e - \varepsilon\omega| \geqslant 1 - |z| - \varepsilon$$

and so we deduce from (66) that

$$\begin{aligned} \int_{B_1 \cap E_2} \big|F_\varepsilon(z)\big|\, dz &\leqslant \int_{\substack{B_1 \cap E \\ \{|z-e-\varepsilon\omega| \geqslant c_\star\varepsilon\}}} \frac{C\,(|z - e - \varepsilon\omega| + \varepsilon)^{s+\alpha}}{|z - e - \varepsilon\omega|^n}\, dz \\ &\leqslant C \int_{\substack{B_1 \cap E \\ \{|z-e-\varepsilon\omega| \geqslant c_\star\varepsilon\}}} \frac{|z - e - \varepsilon\omega|^{s+\alpha}}{|z - e - \varepsilon\omega|^n}\, dz + C \int_{\substack{B_1 \cap E \\ \{|z-e-\varepsilon\omega| \geqslant c_\star\varepsilon\}}} \frac{\varepsilon^{s+\alpha}}{|z - e - \varepsilon\omega|^n}\, dz \\ &=: I_1 + I_2. \end{aligned} \tag{69}$$

To estimate I_1, we split into

$$\begin{aligned} &I_{1,1} := C \int_{\substack{B_1 \cap E \\ \{c_\star\varepsilon \leqslant |z-e-\varepsilon\omega| \leqslant \delta^{1/2n}\}}} |z - e - \varepsilon\omega|^{s+\alpha-n}\, dz \\ \text{and} \qquad &I_{1,2} := C \int_{\substack{B_1 \cap E \\ \{|z-e-\varepsilon\omega| > \delta^{1/2n}\}}} |z - e - \varepsilon\omega|^{s+\alpha-n}\, dz. \end{aligned}$$

Using polar coordinates, we find that

$$I_{1,1} \leqslant C \int_{c_\star\varepsilon}^{\delta^{1/2n}} t^{n-1}\, t^{s+\alpha-n}\, dt \leqslant C\, \left[\left(\delta^{\frac{1}{2n}}\right)^{s+\alpha} - (c_\star\varepsilon)^{s+\alpha} \right] \leqslant C\,\delta^{\frac{s+\alpha}{2n}}. \tag{70}$$

In addition,

$$I_{1,2} \leqslant C \int_{\substack{B_1 \cap E \\ \{|z-e-\varepsilon\omega| > \delta^{1/2n}\}}} |z - e - \varepsilon\omega|^{s-n}\, dz \leqslant C \int_E \delta^{\frac{s-n}{2n}}\, dz \leqslant C\delta^{1 + \frac{s-n}{2n}} = C\delta^{\frac{s+n}{2n}}.$$

This and (70) say that

$$I_1 \leqslant C\,\delta^{\frac{s+\alpha}{2n}} + C\delta^{\frac{s+n}{2n}}. \tag{71}$$

Moreover,

$$I_2 \leqslant C\varepsilon^{s+\alpha} \int_{c_\star\varepsilon}^2 \frac{t^{n-1}}{t^N}\,dt \leqslant C\varepsilon^{s+\alpha}\,|\log\varepsilon| \leqslant C\varepsilon^s \leqslant C\delta^{\frac{s}{2n}},$$

thanks to (68). Hence, using this and (71), and recalling (69), we obtain that

$$\int_{B_1\cap E_2} \big|F_\varepsilon(z)\big|\,dz \leqslant I_1+I_2 \leqslant C\,\delta^{\frac{s+\alpha}{2n}} + C\delta^{\frac{s+n}{2n}} + C\delta^{\frac{s}{2n}}.$$

We now combining this estimate, which is coming from the case in (68), with (67), which was coming from the complementary case, and we see that, in any case,

$$\int_{B_1\cap E_2} \big|F_\varepsilon(z)\big|\,dz \leqslant C\,\delta^\kappa,$$

for some $\kappa > 0$. From this and (64), we obtain that

$$\int_{B_1\cap E} \big|F_\varepsilon(z)\big|\,dz \leqslant C\,\delta^\kappa,$$

Then, choosing δ suitably small with respect to η, we establish (63), as desired.

Notice also that F_ε converges pointwise to $f(z)\,\frac{(-2\,e\cdot\omega)^s\,(1-|z|^2)^s}{s\,|z-e|^n}$. Hence, using (62), (63) and the Vitali Convergence Theorem, we conclude that

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\varepsilon^{-s}\int_{B_1} f(z)\,G(e+\varepsilon\omega,z)\,dz &= \lim_{\varepsilon\searrow 0}\int_{B_1} F_\varepsilon(z)\,dz \\ &= \int_{B_1} f(z)\,\frac{(-2\,e\cdot\omega)^s\,(1-|z|^2)^s}{s\,|z-e|^n}\,dz,\end{aligned}$$

which establishes (12). □

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