Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Time-periodic boundary layer solutions to singularly perturbed parabolic problems

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submitted: September 12, 2016

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No. 2300
Berlin 2016

2010 *Mathematics Subject Classification.* 35B25, 35B10, 35K20, 35K58.

Key words and phrases. Monotone and non-monotone boundary layers, two independent singular perturbation parameters, periodic-parabolic boundary value problem, implicit function theorem.

The authors gratefully acknowledge support from the Russian Foundation of Basic Research (RFBR-DFG 14-01-91333) and from the Deutsche Forschungsgemeinschaft (RE 1336/1-1).

Edited by
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Abstract

In this paper, we present a result of implicit function theorem type, which was designed for applications to singularly perturbed problems. This result is based on fixed point iterations for contractive mappings, in particular, no monotonicity or sign preservation properties are needed. Then we apply our abstract result to time-periodic boundary layer solutions (which are allowed to be non-monotone with respect to the space variable) in semilinear parabolic problems with two independent singular perturbation parameters. We prove existence and local uniqueness of those solutions, and estimate their distance to certain approximate solutions.

1 Introduction

Upper and lower solution techniques and corresponding monotone iterations are classical methods to prove existence of time-periodic solutions to nonlinear parabolic boundary value problems (see, e.g. [2, 11, 12, 20, 22]). In the context of singularly perturbed problems, this approach has allowed to obtain existence and asymptotic expansions of solutions with monotonous boundary [16] and interior [1, 4, 5, 8, 17] layers. However, it turned out to be unsuitable for solutions with more complicated boundary layer structure or interior spikes.

In this paper, we present an alternative approach to singularly perturbed periodic-parabolic boundary value problems which is based on fixed point iterations for contractive mappings, i.e. which is an approach of implicit function theorem type. We apply this approach to time-periodic boundary layer solutions (which are allowed to be non-monotone with respect to the space variable) in problems with two independent singular perturbation parameters. More precisely, we consider semilinear parabolic PDEs of the type

$$\mu\partial_t u(t,x) = \nu^2\partial_x^2 u(t,x) + f(t,x,u(t,x),\mu,\nu), \quad (t,x) \in \mathbb{R} \times (0,1) \tag{1.1}$$

with homogeneous Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t \in \mathbb{R} \tag{1.2}$$

and periodicity condition in time

$$u(t+1,x) = u(t,x), \quad (t,x) \in \mathbb{R} \times [0,1]. \tag{1.3}$$

Here $\mu, \nu > 0$ are two independent small singular perturbation parameters. The right-hand side $f : \mathbb{R} \times [0,1] \times \mathbb{R} \times [0,1]^2 \to \mathbb{R}$ is supposed to be C^3-smooth and 1-periodic with respect to its first argument, i.e. with respect to the time variable t. Moreover, we assume that there exists a continuous 1-periodic function $u^0 : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that

$$f(t,x,u^0(t,x),0,0) = 0, \quad \partial_u f(t,x,u^0(t,x),0,0) < 0, \quad (t,x) \in \mathbb{R} \times [0,1]. \tag{1.4}$$

Our goal is to describe existence, local uniqueness and asymptotic behavior for $\mu, \nu \to 0$ of families (parametrized by μ and ν) $\hat{u}_{\mu,\nu}$ of boundary layer solutions to (1.1)–(1.3), i.e. such that

$$\lim_{(\mu,\nu)\to(0,0)} \hat{u}_{\mu,\nu}(t,x) = u^0(t,x) \quad \text{for all} \quad (t,x) \in \mathbb{R} \times (0,1). \tag{1.5}$$

Such solutions turn out to exist under the following natural assumption: There exist smooth maps $v^0, w^0 : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ such that

$$\left.\begin{array}{ll} \partial_y^2 v^0(t,y) + f(t,0,u^0(t,0) + v^0(t,y),0,0) = 0, & (t,y) \in \mathbb{R} \times (0,\infty), \\ v^0(t,0) + u^0(t,0) = v^0(t,\infty) = 0, & t \in \mathbb{R}, \\ v^0(t+1,y) = v^0(t,y), & (t,y) \in \mathbb{R} \times [0,\infty) \end{array}\right\} \tag{1.6}$$

and

$$\left.\begin{array}{ll} \partial_y^2 w^0(t,y) + f(t,1,u^0(t,1) + w^0(t,y),0,0) = 0, & (t,y) \in \mathbb{R} \times (0,\infty), \\ w^0(t,0) + u^0(t,1) = w^0(t,\infty) = 0, & t \in \mathbb{R}, \\ w^0(t+1,y) = w^0(t,y), & (t,y) \in \mathbb{R} \times [0,\infty). \end{array}\right\} \tag{1.7}$$

Moreover, we suppose

$$\partial_y v^0(t,0) \neq 0 \quad \text{and} \quad \partial_y w^0(t,0) \neq 0 \quad \text{for all} \quad t \in \mathbb{R}. \tag{1.8}$$

The functions v^0 and w^0 will be used to describe the asymptotics of the boundary layers in the vicinity of points $x = 0$ and $x = 1$, respectively. For fixed time variable t such layers can be monotone or non-monotone functions of the space variable. However, due to condition (1.8), the (non-)monotonicity with respect to the space variable of each layer remains unchanged for varying t.

Roughly speaking, we are going to prove that for small μ and ν there exists exactly one solution u to (1.1)–(1.3) which is close to the approximate solution

$$u_\nu(t,x) := u^0(t,x) + v^0\left(t, \frac{x}{\nu}\right) + w^0\left(t, \frac{1-x}{\nu}\right). \tag{1.9}$$

The closeness will be measured with respect to the norms

$$\|u\|_{\mu,\nu} := \left(\int_0^1 \int_0^1 \left(\mu^2 \partial_t u^2 + \nu^4 \partial_x^2 u^2 + \nu^2 \partial_x u^2 + u^2 \right) \frac{dt}{\mu} \frac{dx}{\nu} \right)^{1/2} \tag{1.10}$$

and

$$\|u\|_\infty := \sup\{|u(t,x)| : \ t,x \in [0,1]\}. \tag{1.11}$$

Remark that for all $\mu, \nu \in (0,1]$ and all C^2-functions $u : [0,1]^2 \to \mathbb{R}$, which satisfy the homogeneous Dirichlet boundary conditions (1.2), it holds (cf. (3.4))

$$\|u\|_\infty \leq \sqrt{2}\|u\|_{\mu,\nu}. \tag{1.12}$$

The following theorem is our main result:

Theorem 1.1 *Suppose (1.4) and (1.6)–(1.8). Then the following is true:*

(i) There exist $\varepsilon > 0$ and $c > 0$ such that for all $\mu, \nu \in (0, \varepsilon)$ there exists a solution $u = \hat{u}_{\mu,\nu}$ to (1.1)–(1.3) with

$$\|\hat{u}_{\mu,\nu} - u_\nu\|_\infty \le c(\mu + \nu). \tag{1.13}$$

(ii) There exists $\delta > 0$ such that for all $\mu, \nu \in (0, \varepsilon)$ the following is true: If u is a solution to (1.1)–(1.3) with $\|u - \hat{u}_{\mu,\nu}\|_{\mu,\nu} \le \delta$, then $u = \hat{u}_{\mu,\nu}$.

Remark 1.2 *It is an open problem if the uniqueness assertion (ii) of Theorem 1.1 can be improved to the assertion that there are no solutions $u \neq \hat{u}_{\mu,\nu}$ to (1.1)–(1.3) with $\|u - \hat{u}_{\mu,\nu}\|_\infty \le \delta$, or that there are no solutions $u \neq \hat{u}_{\mu,\nu}$ to (1.1)–(1.3) with $\|u - u_\nu\|_{\mu,\nu} \le \delta$, or, even more, that there are no solutions $u \neq \hat{u}_{\mu,\nu}$ to (1.1)–(1.3) with $\|u - u_\nu\|_\infty \le \delta$.*

Remark 1.3 *Let us denote*

$$\kappa_0 := \min_{t\in\mathbb{R}} \sqrt{|\partial_u f(t, 0, u^0(t, 0), 0, 0)|} \quad \text{and} \quad \kappa_1 := \min_{t\in\mathbb{R}} \sqrt{|\partial_u f(t, 1, u^0(t, 1), 0, 0)|}.$$

Then, assumptions (1.4), (1.6)–(1.8) and smoothness of f imply (see [6, Theorems 4.1 and 4.3]) that there exist $a_0, a_1 > 0$ such that

$$\begin{aligned} |v^0(t, y)| + |\partial_y v^0(t, y)| &\le a_0 e^{-\kappa_0 y} \quad \text{for all} \quad (t, y) \in \mathbb{R} \times [0, \infty), \\ |w^0(t, y)| + |\partial_y w^0(t, y)| &\le a_1 e^{-\kappa_1 y} \quad \text{for all} \quad (t, y) \in \mathbb{R} \times [0, \infty). \end{aligned} \tag{1.14}$$

In particular, the claim (1.5) follows from (1.9), (1.13) and (1.14).

Remark 1.4 *Suppose $u^0(t, 0) < 0$ for all $t \in \mathbb{R}$. Then assumption (1.6) is satisfied if for any fixed t the following is true: The conservative system*

$$v''(y) + f(t, 0, u^0(t, 0) + v(y), 0, 0) = 0 \tag{1.15}$$

has a homoclinic solution $v_* : \mathbb{R} \to \mathbb{R}$ with $v_*(\pm\infty) = 0$ such that there exists y_0 with $v_*(y_0) > -u^0(t, 0)$. Indeed, without loss of generality we can assume $v_*'(0) = 0$. Then there exist $y_1 < 0 < y_2$ such that $v_*(y_1) = v_*(y_2) = -u^0(t, 0)$, $v_*'(y_1) > 0$ and $v_*'(y_2) < 0$, see Fig. 1. Hence, the functions $v^0(t, y) := v_*(y + y_j)$, $j = 1, 2$ satisfy (1.6).

The choice with $j = 1$ leads to a non-monotone function $v^0(t, \cdot)$ and, hence, to a non-monotone boundary layer at $x = 0$ of the solution $\hat{u}_{\mu,\nu}$, produced by Theorem 1.1 (cf. (1.9)). The choice with $j = 2$ leads to a monotone boundary layer.

Similarly one can formulate sufficient conditions for (1.7).

Remark 1.5 *If both boundary layers in the approximate solution (1.9) are monotone, then it turns out that Theorem 1.1 can be proved using upper and lower solutions techniques. However, this is not true anymore, if at least one of the boundary layers v^0 or w^0 is non-monotone.*

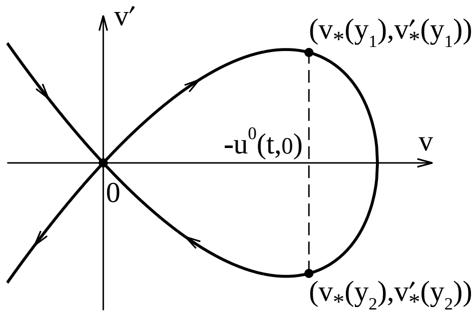

Figure 1: Homoclinic solution v_* to equation (1.15).

The main tool of the proof of Theorem 1.1 is Theorem 2.1 below. Theorem 2.1 is a result of implicit function theorem type, and it was designed for getting existence and local uniqueness of solutions with contrast structures (internal or boundary layers, spikes etc.) to singularly pertubed ODEs and PDEs (cf. [18, 19, 21], see also [23] for a similar approach). In order to prove Theorem 1.1 we will apply Theorem 2.1 to five problems, namely to the main problem (1.1)–(1.3) as well as to the four auxiliary problems (4.36), (4.41), (4.48) and (4.50).

Our paper is organized as follows: In Section 2 we introduce a general abstract setting for singularly perturbed problems and prove an abstract implicit function theorem. In Section 3 we apply this theorem to obtain our main asymptotic result, i.e. Theorem 1.1. Technical results concerned with the construction of an improved approximate solution to problem (1.1)–(1.3) and coercivity estimates for some auxiliary elliptic and parabolic boundary value problems are collected in Sections 4 and 5, respectively.

2 An Implicit Function Theorem for Singularly Perturbed Problems

Let Λ be a set, E be a subset of a normed vector space such that zero belongs to the closure of E, U and V be Banach spaces with norms $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively, and let U^0 be a closed subspace of U. Further, for $\varepsilon \in E$ let be given maps

$$F_\varepsilon : \Lambda \times U \to V \quad \text{and} \quad u^0_\varepsilon : \Lambda \to U.$$

The goal of this section is to state conditions such that for all $\varepsilon \in E$ with $\varepsilon \approx 0$ and for all $\lambda \in \Lambda$ there exists exactly one solution $u \in U^0$ with

$$u \approx u^0_\varepsilon(\lambda) \tag{2.1}$$

to the equation

$$F_\varepsilon(\lambda, u) = 0, \quad u \in U^0. \tag{2.2}$$

We are going to state a result of implicit function theorem type, therefore we suppose that for all $\varepsilon \in E$ and all $\lambda \in \Lambda$ it holds

$$F_\varepsilon(\lambda, \cdot) \in C^1(U; V), \tag{2.3}$$

and

$$\partial_u F_\varepsilon(\lambda, u^0_\varepsilon(\lambda)) \text{ is Fredholm of index zero from } U^0 \text{ into } V. \tag{2.4}$$

In most of our applications it holds $\|u_\varepsilon^0(\lambda)\|_U \to \infty$ for $\varepsilon \to 0$, and the solutions $u \in U^0$ to (2.2), which we will determine, will not be close to $u_\varepsilon^0(\lambda)$ in the sense of the norm $\|\cdot\|_U$. Hence, the closeness (2.1) must be measured by another norm in U, which is weaker than $\|\cdot\|_U$, in general.

Thus, we assume that there is given another norm $\|\cdot\|_\infty$ on U. We use the notation $\|\cdot\|_\infty$ because in most of the applications the elements of U are functions defined on a domain, and $\|\cdot\|_\infty$ is the corresponding L^∞-norm. Remark that in most of the applications U is not complete with respect to $\|\cdot\|_\infty$. We assume that there exists $a > 0$ such that for all $\varepsilon \in E$ and $\lambda \in \Lambda$ we have

$$\|u_\varepsilon^0(\lambda)\|_\infty \le a. \tag{2.5}$$

Theorem 2.1 *Suppose (2.3)–(2.5). Further, suppose that for all $\varepsilon \in E$ and $\lambda \in \Lambda$ there are given norms $\|\cdot\|_\varepsilon$ in U^0 and $|\cdot|_\varepsilon$ in V, which are equivalent to $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively, such that the following is true: There exist $b > 0$ and $c > 0$ such that for all $\varepsilon \in E$ and $\lambda \in \Lambda$ we have*

$$\|u\|_\infty \le b\|u\|_\varepsilon \quad \text{for all} \quad u \in U^0, \tag{2.6}$$

$$\|u\|_\varepsilon \le c|\partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))u|_\varepsilon \quad \text{for all} \quad u \in U^0, \tag{2.7}$$

and that for all $r > 0$ there exists $c_r > 0$ such that for all $\varepsilon \in E$ and $\lambda \in \Lambda$ we have

$$\begin{gathered} |(\partial_u F_\varepsilon(\lambda, u_1) - \partial_u F_\varepsilon(\lambda, u_2))\, u|_\varepsilon \le c_r \|u_1 - u_2\|_\infty \|u\|_\varepsilon \\ \text{for all} \quad u, u_1, u_2 \in U \quad \text{with} \quad \|u_1\|_\infty, \|u_2\|_\infty \le r. \end{gathered} \tag{2.8}$$

Finally, suppose that for all $\varepsilon \in E$ there are given maps $u_\varepsilon^1 : \Lambda \to U^0$ such that

$$\|u_\varepsilon^0(\lambda) - u_\varepsilon^1(\lambda)\|_\infty + |F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))|_\varepsilon \to 0 \quad \text{for} \quad \varepsilon \to 0 \quad \text{uniformly with respect to} \quad \lambda \in \Lambda. \tag{2.9}$$

Then the following is true:

(i) There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_0$ and for all $\lambda \in \Lambda$ there exists a solution $u = \hat{u}_\varepsilon(\lambda)$ to (2.2) with

$$\|\hat{u}_\varepsilon(\lambda) - u_\varepsilon^1(\lambda)\|_\varepsilon \le 4c|F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))|_\varepsilon$$

and, hence, with

$$\|\hat{u}_\varepsilon(\lambda) - u_\varepsilon^0(\lambda)\|_\infty \le \|u_\varepsilon^0(\lambda) - u_\varepsilon^1(\lambda)\|_\infty + 4bc|F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))|_\varepsilon. \tag{2.10}$$

(ii) There exists $\delta > 0$ such that for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_0$ and for all $\lambda \in \Lambda$ there does not exist a solution $u \neq \hat{u}_\varepsilon(\lambda)$ to (2.2) with $\|u - u_\varepsilon^1(\lambda)\|_\varepsilon \le \delta$.

Remark 2.2 *There are two main differences of Theorem 2.1 to the classical implicit function theorem:*

First, the approximate solution $u_\varepsilon^0(\lambda)$ is not defined for $\varepsilon = 0$, in general (like (1.9) does not make sense for $\nu = 0$). Hence, there does not exist a solution to (2.2) with $\varepsilon = 0$ (if equation (2.2) with $\varepsilon = 0$ is defined at all), and one cannot start the iteration procedure for solving (2.2) with $\varepsilon \neq 0$ in a solution to (2.2) with $\varepsilon = 0$, in general.

And second, in (2.2) there appear two parameters ε and λ of quite different nature. The parameter ε is a singular perturbation parameter, and λ is a regular perturbation parameter.

In (1.1)–(1.3) the role of the singular perturbation parameter ε is played by the pair (μ, ν), and there is no regular perturbation parameter λ. In (4.36) the singular perturbation parameter is ν, and t is a regular perturbation parameter. In (4.41) the singular perturbation parameter is μ, and x is a regular perturbation parameter. And finally, in (4.48) and (4.50) the singular perturbation parameter is μ again, and there is no regular perturbation parameter.

Remark 2.3 *For many singularly perturbed boundary value problems approximate solutions u_ε^0 with certain contrast structures and with property (2.5) can be constructed in an ad hoc manner. In those situations Theorem 2.1 provides existence and local uniqueness of exact solutions $\hat{u}_\varepsilon$ close (in the sense of the corresponding L^∞-norm $\|\cdot\|_\infty$) to u_ε^0 and the estimate (2.10) if the following algorithm can be realized:*

First, find Banach spaces U, U^0 and V such that the boundary value problem has an abstract formulation of the type (2.2) with the properties (2.3) and (2.4). Then, find a norm $\|\cdot\|_\varepsilon$ in U which is strong enough such that (2.6) is true. Then, find a norm $|\cdot|_\varepsilon$ in V which is strong enough such that (2.7) is true, but which is, at the same time, weak enough such that (2.8) is true. And finally, find improved approximate solutions u_ε^1 such that (2.9) is true. The better the choice of u_ε^1, the better the a priori estimate (2.10).

Remark 2.4 *In many applications the improved approximate solutions u_ε^1 are known only implicitely. Therefore often the local uniqueness assertion (ii) of Theorem 2.1 is formulated in a slightly weaker form which does not rely on u_ε^1: There are no solutions $u \neq \hat{u}_\varepsilon(\lambda)$ to (2.2) with $\|u - \hat{u}_\varepsilon(\lambda)\|_\varepsilon \leq \delta$. But it turns out that the local uniqueness assertion (ii) of Theorem 2.1 cannot be improved to the assertion that there are no solutions $u \neq \hat{u}_\varepsilon(\lambda)$ to (2.2) with $\|u - u_\varepsilon^0(\lambda)\|_\varepsilon \leq \delta$ or, even more, that there are no solutions $u \neq \hat{u}_\varepsilon(\lambda)$ to (2.2) with $\|u - u_\varepsilon^0(\lambda)\|_\infty \leq \delta$.*

Proof of Theorem 2.1: Let us denote by U_ε^0 and V_ε the spaces U^0 and V equipped with the norms $\|\cdot\|_\varepsilon$ and $|\cdot|_\varepsilon$, respectively. By assumption the spaces U^0 and V are complete with respect to the norms $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively. Moreover, the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_U$ in U^0 are equivalent, as well as the norms $|\cdot|_\varepsilon$ and $\|\cdot\|_V$ in V are equivalent. Hence, the spaces U_ε^0 and V_ε are complete also.

For linear bounded operators $A : U_\varepsilon^0 \to V_\varepsilon$ and $B : V_\varepsilon \to U_\varepsilon^0$ we denote, as usual, by

$$\|A\| := \sup_{\|u\|_\varepsilon = 1} |Au|_\varepsilon \text{ and } \|B\| := \sup_{|v|_\varepsilon = 1} \|Bv\|_\varepsilon$$

their operator norms.

Because of assumptions (2.4) and (2.7) the (restriction to U^0 of the) operator $\partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))$ is bijective from U^0 onto V. We denote by $\partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))^{-1}$ its inverse. Then (2.7) yields

$$\|\partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))^{-1}\| \leq c.$$

Further, because of assumptions (2.5), (2.8) and (2.9) there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_0$ and all $\lambda \in \Lambda$ it holds

$$\|\partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda)) - \partial_u F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))\| \leq c_r \|u_\varepsilon^0(\lambda) - u_\varepsilon^1(\lambda)\|_\infty \leq \frac{1}{2c} < \frac{1}{c} \leq \frac{1}{\|\partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))^{-1}\|}.$$

Here $r > 0$ is chosen such that for all $\varepsilon \in E$ we have $\|u_\varepsilon^0(\lambda)\|_\infty, \|u_\varepsilon^1(\lambda)\|_\infty \le r$ (cf. (2.5) and (2.9)). Hence, for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_0$ and all $\lambda \in \Lambda$ the operator $\partial_u F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))$ is bijective from U^0 onto V, and for all $u \in U^0$ it holds

$$|\partial_u F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))u|_\varepsilon \ge |\partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))u|_\varepsilon - |\left(\partial_u F_\varepsilon(\lambda, u_\varepsilon^1(\lambda)) - \partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))\right) u|_\varepsilon \ge \frac{1}{2c}\|u\|_\varepsilon.$$

Therefore for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_0$ and all $\lambda \in \Lambda$ we have

$$\|\partial_u F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))^{-1}\| \le 2c.$$

Now we are going to solve (2.2). For $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_0$ and $\lambda \in \Lambda$ and $u \in U^0$ we have $F_\varepsilon(\lambda, u) = 0$ if and only if

$$G_\varepsilon(\lambda, u) := u - \partial_u F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))^{-1} F_\varepsilon(\lambda, u) = u. \tag{2.11}$$

Moreover,

$$\begin{aligned} G_\varepsilon(\lambda, u) - G_\varepsilon(\lambda, v) &= \int_0^1 \partial_u G_\varepsilon(\lambda, su + (1-s)v)(u-v)ds = \\ &= \partial_u F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))^{-1} \int_0^1 \left(\partial_u F_\varepsilon(\lambda, u_\varepsilon^1(\lambda)) - \partial_u F_\varepsilon(\lambda, su + (1-s)v)\right)(u-v)ds. \end{aligned}$$

Hence, assumptions (2.6) and (2.8) imply that there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $\delta > 0$ such that for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_1$ and for all $\lambda \in \Lambda$ we have

$$\|G_\varepsilon(\lambda, u) - G_\varepsilon(\lambda, v)\|_\varepsilon \le \frac{1}{2}\|u - v\|_\varepsilon \quad \text{for all} \quad u, v \in K_\varepsilon^\delta(\lambda) := \left\{w \in U^0 : \ \|w - u_\varepsilon^1(\lambda)\|_\varepsilon \le \delta\right\}.$$

Using this, for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_1$ and for all $\lambda \in \Lambda$ and for all $u \in K_\varepsilon^\delta(\lambda)$ we get

$$\begin{aligned} \|G_\varepsilon(\lambda, u) - u_\varepsilon^1(\lambda)\|_\varepsilon &\le \|G_\varepsilon(\lambda, u) - G_\varepsilon(\lambda, u_\varepsilon^1(\lambda))\|_\varepsilon + \|G_\varepsilon(\lambda, u_\varepsilon^1(\lambda)) - u_\varepsilon^1(\lambda)\|_\varepsilon \\ &\le \frac{1}{2}\|u - u_\varepsilon^1(\lambda)\|_\varepsilon + 2c|F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))|_\varepsilon. \end{aligned} \tag{2.12}$$

Hence, assumption (2.9) yields that $G_\varepsilon(\lambda, \cdot)$ maps $K_\varepsilon^\delta(\lambda)$ into $K_\varepsilon^\delta(\lambda)$ for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_1$ and all $\lambda \in \Lambda$, if ε_1 is chosen sufficiently small. Now, Banach's fixed point theorem gives a unique in $K_\varepsilon^\delta(\lambda)$ solution $u = \hat{u}_\varepsilon(\lambda)$ to (2.11) for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_1$ and all $\lambda \in \Lambda$. Moreover, (2.12) yields

$$\|\hat{u}_\varepsilon(\lambda) - u_\varepsilon^1(\lambda)\|_\varepsilon \le \frac{1}{2}\|\hat{u}_\varepsilon(\lambda) - u_\varepsilon^1(\lambda)\|_\varepsilon + 2c|F_\varepsilon(\lambda, u_\varepsilon^1(\lambda))|_\varepsilon,$$

i.e. (2.10). ■

Remark 2.5 *The operator $\partial_u F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda))$ is bijective from U^0 onto V if*

$$\|\partial_u F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda)) - \partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))\| < \frac{1}{c} \le \frac{1}{\|\partial_u F_\varepsilon(\lambda, u_\varepsilon^0(\lambda))^{-1}\|_{\mathcal{L}(V_\varepsilon, U_\varepsilon)}}.$$

But (2.8) and (2.10) imply that this is true for all $\varepsilon \in E$ with $\|\varepsilon\| < \varepsilon_1$ and for all $\lambda \in \Lambda$ if ε_1 is taken sufficiently small. Hence, the classical implicit function theorem yields that the map $\lambda \mapsto \hat{u}_\varepsilon(\lambda)$ is C^1-smooth, if Λ is an open set in a normed vector space and if the maps F_ε are C^1-smooth not only with respect to u, but with respect to the pair (λ, u). Differentiating the identity $F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda)) = 0$ with respect to λ we get

$$\hat{u}_\varepsilon'(\lambda) = -\partial_u F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda))^{-1} \partial_\lambda F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda)).$$

Similarly, if F is C^2-smooth then the map $\lambda \mapsto \hat{u}_\varepsilon(\lambda)$ is C^2-smooth also, and

$$\begin{aligned}\hat{u}_\varepsilon''(\lambda) = -\partial_u F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda))^{-1} \Big(\partial_\lambda^2 F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda)) \quad &+ \quad 2\partial_\lambda \partial_u F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda))\hat{u}_\varepsilon'(\lambda) \\ &+ \quad \partial_u^2 F_\varepsilon(\lambda, \hat{u}_\varepsilon(\lambda))(\hat{u}_\varepsilon'(\lambda), \hat{u}_\varepsilon'(\lambda))\Big).\end{aligned}$$

3 Proof of Theorem 1.1: Verification of (2.3)–(2.8)

In the rest of this paper we will prove Theorem 1.1. Hence, its assumptions (1.4) and (1.6)–(1.8) are always supposed to be satisfied. Note that in this section and later in Propositions 4.4, 4.6 and 4.7 we will apply Theorem 2.1 with different definitions of the set Λ, map F_ε, spaces E, U, U^0, V and their norms. Each time they will be specially specified.

Let us apply Theorem 2.1 to the periodic-boundary value problem (1.1)–(1.3). For that reason we take

$$\left.\begin{aligned} U &:= L^2((0,1); W^{2,2}(0,1)) \cap W^{1,2}((0,1); L^2(0,1)), \\ \|u\|_U &:= \left(\int_0^1 \int_0^1 (\partial_t u^2 + \partial_x^2 u^2 + \partial_x u^2 + u^2) dt dx\right)^{1/2}. \end{aligned}\right\} \tag{3.1}$$

Here, as usual, $L^2((0,1); W^{2,2}(0,1)) \cap W^{1,2}((0,1); L^2(0,1))$ is the space of all (equivalence classes of) measurable functions $u : [0,1]^2 \to \mathbb{R}$ such that u and its distributional derivatives $\partial_t u$, $\partial_x u$ and $\partial_x^2 u$ belong to $L^2((0,1)^2)$. The remaining data for applying Theorem 2.1 to (1.1)–(1.3) are chosen as follows:

$$\left.\begin{aligned} &U^0 := \{u \in U : \ u(t,0) = u(t,1) = 0 \text{ for all } t, \text{ and } u(0,x) = u(1,x) \text{ for all } x\}, \\ &V := L^2((0,1)^2), \quad \|v\|_V := \left(\int_0^1 \int_0^1 v^2 dt dx\right)^{1/2}, \\ &\Lambda := \emptyset, \quad \varepsilon = (\mu, \nu) \in E := (0,1)^2, \quad \|(\mu,\nu)\| := \sqrt{\mu^2 + \nu^2}, \\ &\|u\|_{\mu,\nu} \text{ is defined by (1.10) and } \|u\|_\infty \text{ is defined by (1.11)}, \\ &|v|_{\mu,\nu} := \left(\int_0^1 \int_0^1 v^2 \frac{dt}{\mu}\frac{dx}{\nu}\right)^{1/2}. \end{aligned}\right\} \tag{3.2}$$

Now, we rewrite problem (1.1)–(1.3) in the form

$$[F_{\mu,\nu}(u)](t,x) := \mu \partial_t u(t,x) - \nu^2 \partial_x^2 u(t,x) - f(t,x,u(t,x),\mu,\nu) = 0, \quad u \in U^0,$$

and consider as an approximate solution

$$u_{\mu,\nu}^0 := u_\nu,$$

where u_ν is defined in (1.9).

It is well-known that U is continuously embedded into $C([0,1]^2)$ (cf. [3, Theorem 10.4] or [9, §18.1.3]). Therefore for the C^2-smooth function f we easily verify assumption (2.3) and obtain

$$[\partial_u F_{\mu,\nu}(u)v](t,x) = \mu\partial_t v(t,x) - \nu^2\partial_x^2 v(t,x) - \partial_u f(t,x,u(t,x),\mu,\nu)v(t,x).$$

Moreover, we have

$$[(\partial_u F_{\mu,\nu}(u_1) - \partial_u F_{\mu,\nu}(u_2))u](t,x) = (\partial_u f(t,x,u_2(t,x),\mu,\nu) - \partial_u f(t,x,u_1(t,x),\mu,\nu))u(t,x).$$

Hence, assumption (2.8) is also fulfilled. Assumption (2.4) is fulfilled because of [15, §4]. Assumption (2.5) is fulfilled because the functions u^0, v^0 and w^0 are bounded (cf. (1.4), (1.6) and (1.7)).

The following lemma shows that assumption (2.6) is also fulfilled:

Lemma 3.1 *(i) Let be given $S,Y \geq 1$ and a C^2-function $v : [0,S]\times[0,Y] \to \mathbb{R}$ such that $v(s,0) = v(s,Y) = 0$ for all $s \in [0,S]$. Then for all $s \in [0,S]$ and $y \in [0,Y]$ it holds*

$$|v(s,y)|^2 \leq 2\int_0^Y\int_0^S \left(\partial_s v^2 + \partial_y^2 v^2 + \partial_y v^2 + v^2\right) dsdy \tag{3.3}$$

(ii) Let be given a C^2-function $u : [0,1]^2 \to \mathbb{R}$ such that $u(t,0) = u(t,1) = 0$ for all $t \in [0,1]$. Then for all $\mu,\nu \in (0,1]$ and $t,x \in [0,1]$ it holds

$$|u(t,x)|^2 \leq 2\int_0^1\int_0^1 \left(\mu^2\partial_t u^2 + \nu^4\partial_x^2 u^2 + \nu^2\partial_x u^2 + u^2\right)\frac{dt}{\mu}\frac{dx}{\nu}. \tag{3.4}$$

Proof (i) Let $s \in (0,S]$ and $y \in [0,Y]$ be fixed. Because of $v(s,0)=0$ it holds

$$\begin{aligned} v(s,y)^2 &= \int_0^y \left(\frac{d}{dz}v(s,z)^2\right)dz = 2\int_0^y \partial_y v(s,z)v(s,z)dz \\ &\leq \int_0^Y \left(\partial_y v(s,z)^2 + v(s,z)^2\right)dz. \end{aligned} \tag{3.5}$$

Further, for any $t \in [0,S]$ we have

$$\begin{aligned} &\int_0^Y \left(v(s,z)^2 - v(t,z)^2\right)dz = \int_t^s \left(\frac{d}{dr}\int_0^Y v(r,z)^2 dz\right)dr \\ &= 2\int_t^s\int_0^Y \partial_s v(r,z)v(r,z)dzdr \leq \int_0^S\int_0^Y \left(\partial_s v(r,z)^2 + v(r,z)^2\right)dzdr. \end{aligned}$$

Dividing this by S and integration with respect to t from zero to S we get

$$\int_0^Y v(s,z)^2 dz \leq \int_0^S\int_0^Y \left(\partial_s v(r,z)^2 + \left(1+\frac{1}{S}\right)v(r,z)^2\right)dzdr. \tag{3.6}$$

Similarly,

$$\int_0^Y \left(\partial_y v(s,z)^2 - \partial_y v(t,z)^2\right) dz = \int_t^s \left(\frac{d}{dr}\int_0^Y \partial_y v(r,z)^2 dz\right) dr$$
$$= 2\int_t^s \int_0^Y \partial_s\partial_y v(r,z)\partial_y v(r,z) dz dr = -2\int_t^s \int_0^Y \partial_s v(r,z)\partial_y^2 v(r,z) dz dr$$
$$\leq \int_0^S \int_0^Y \left(\partial_s v(r,z)^2 + \partial_y^2 v(r,z)^2\right) dz dr$$

and, hence,

$$\int_0^Y \partial_y v(s,z)^2 dz \leq \int_0^S \int_0^Y \left(\partial_s v(r,z)^2 + \partial_y^2 v(r,z)^2 + \frac{1}{S}\partial_y v(r,z)^2\right) dz dr. \tag{3.7}$$

Inserting (3.6) and (3.7) into (3.5) we get

$$v(s,y)^2 \leq \int_0^S \int_0^Y \left(2\partial_s v(r,z)^2 + \partial_y^2 v(r,z)^2 + \frac{1}{S}\partial_y v(r,z)^2 + \left(1+\frac{1}{S}\right) v(r,z)^2\right) dz dr.$$

Because of $S \geq 1$ this yields (3.3).

(ii) We get (3.4) by using (3.3) for the function $v(s,y) := u(\mu s, \nu y)$. ■

Assumption (2.7) of Theorem 2.1 in the setting (3.1), (3.2) is satisfied because of Lemma 5.3 and of the density in U^0 of the set of all $u \in C^2([0,1]^2)$ with $u(t,0) = u(t,1) = 0$ and $u(0,x) = u(1,x)$ for all $t,x \in [0,1]$.

For proving Theorem 1.1 it remains to verify assumption (2.9) of Theorem 2.1 in the setting (3.1), (3.2). For sufficiently small μ and ν we have to determine functions $u^1_{\mu,\nu} \in U^0$ such that

$$\|u^1_{\mu,\nu} - u_\nu\|_\infty \leq \text{ const } (\mu + \nu) \tag{3.8}$$

and

$$\int_0^1 \int_0^1 \left(\mu\partial_t u^1_{\mu,\nu}(t,x) - \nu^2\partial_x^2 u^1_{\mu,\nu}(t,x) - f(t,x,u^1_{\mu,\nu}(t,x),\mu,\nu)\right)^2 \frac{dt}{\mu}\frac{dx}{\nu} \leq \text{ const } (\mu+\nu)^2. \tag{3.9}$$

This will be done in the next section.

4 Construction of the improved approximate solution $u^1_{\mu,\nu}$

Following the boundary function method [24] one can construct an approximate solutions $\mathcal{S}_{\mu,\nu}$ to the singularly perturbed problem (1.1)–(1.3) relying on the decomposition

$$\mathcal{S}_{\mu,\nu}(t,x) = \mathcal{U}_{\mu,\nu}(t,x) + \mathcal{V}_{\mu,\nu}\left(t,\frac{x}{\nu}\right) + \mathcal{W}_{\mu,\nu}\left(t,\frac{1-x}{\nu}\right), \tag{4.1}$$

where $\mathcal{U}_{\mu,\nu} : \mathbb{R} \times [0,1] \to \mathbb{R}$ is a function, which approximately satisfies the differential equation (1.1), but not the boundary conditions (1.2), whereas $\mathcal{V}_{\mu,\nu}, \mathcal{W}_{\mu,\nu} : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ are two functions describing the boundary layers at $x = 0$ and $x = 1$, respectively.

The following Lemma shows how to estimate the discrepancy of $\mathcal{S}_{\mu,\nu}$ as an approximate solution to (1.1)–(1.3) by the discrepancies of $\mathcal{U}_{\mu,\nu}$, $\mathcal{V}_{\mu,\nu}$ and $\mathcal{W}_{\mu,\nu}$ as approximate solutions of "their" PDEs and boundary conditions:

Proposition 4.1 *Suppose that for $\mu, \nu \in (0,1]$ are given functions*

$$\mathcal{U}_{\mu,\nu}, \mathcal{R}^u_{\mu,\nu} : \mathbb{R} \times [0,1] \to \mathbb{R},\ \ \mathcal{V}_{\mu,\nu}, \mathcal{W}_{\mu,\nu}, \mathcal{R}^v_{\mu,\nu}, \mathcal{R}^w_{\mu,\nu} : \mathbb{R} \times [0,\infty) \to \mathbb{R}\ \ \text{and}\ \ \mathcal{D}^v_{\mu,\nu}, \mathcal{D}^w_{\mu,\nu} : \mathbb{R} \to \mathbb{R}$$

such that

$$\mu\partial_t \mathcal{U}_{\mu,\nu}(t,x) - \nu^2\partial_x^2 \mathcal{U}_{\mu,\nu}(t,x) = f(t,x,\mathcal{U}_{\mu,\nu}(t,x),\mu,\nu) + \mathcal{R}^u_{\mu,\nu}(t,x), \tag{4.2}$$

$$\begin{aligned}\mu\partial_t \mathcal{V}_{\mu,\nu}(t,y) - \partial_y^2 \mathcal{V}_{\mu,\nu}(t,y) &= f\left(t,\nu y,\mathcal{U}_{\mu,\nu}(t,\nu y) + \mathcal{V}_{\mu,\nu}(t,y),\mu,\nu\right) \\ &\quad - f(t,\nu y,\mathcal{U}_{\mu,\nu}(t,\nu y),\mu,\nu) + \mathcal{R}^v_{\mu,\nu}(t,y),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mu\partial_t \mathcal{W}_{\mu,\nu}(t,y) - \partial_y^2 \mathcal{W}_{\mu,\nu}(t,y) &= f\left(t,1-\nu y,\mathcal{U}_{\mu,\nu}(t,1-\nu y) + \mathcal{W}_{\mu,\nu}(t,y),\mu,\nu\right) \\ &\quad - f(t,1-\nu y,\mathcal{U}_{\mu,\nu}(t,1-\nu y),\mu,\nu) + \mathcal{R}^w_{\mu,\nu}(t,y),\end{aligned} \tag{4.4}$$

$$\mathcal{V}_{\mu,\nu}(t,0) + \mathcal{U}_{\mu,\nu}(t,0) = \mathcal{D}^v_{\mu,\nu}(t), \tag{4.5}$$

$$\mathcal{W}_{\mu,\nu}(t,0) + \mathcal{U}_{\mu,\nu}(t,1) = \mathcal{D}^w_{\mu,\nu}(t). \tag{4.6}$$

Further, suppose that there exists $\kappa > 0$ such that

$$|\mathcal{V}_{\mu,\nu}(t,y)| + |\mathcal{W}_{\mu,\nu}(t,y)| = O(e^{-\kappa y}) \quad \text{for} \quad y \to \infty. \tag{4.7}$$

Then, function $\mathcal{S}_{\mu,\nu}$ defined in (4.1) satisfies

$$\begin{aligned}&\mu\partial_t \mathcal{S}_{\mu,\nu}(t,x) - \nu^2\partial_x^2 \mathcal{S}_{\mu,\nu}(t,x) - f(t,x,\mathcal{S}_{\mu,\nu}(t,x),\mu,\nu) \\ &\quad = \mathcal{R}^u_{\mu,\nu}(t,x) + \mathcal{R}^v_{\mu,\nu}\left(t,\frac{x}{\nu}\right) + \mathcal{R}^w_{\mu,\nu}\left(t,\frac{1-x}{\nu}\right) + O(e^{-\kappa/\nu}) \quad \text{for} \quad \nu \to 0,\end{aligned} \tag{4.8}$$

$$|\mathcal{S}_{\mu,\nu}(t,0) - \mathcal{D}^v_{\mu,\nu}(t)| + |\mathcal{S}_{\mu,\nu}(t,1) - \mathcal{D}^w_{\mu,\nu}(t)| = O(e^{-\kappa/\nu}) \quad \text{for} \quad \nu \to 0. \tag{4.9}$$

Remark 4.2 *The asymptotic estimates $O\left(e^{-\kappa y}\right)$ and $O\left(e^{-\kappa/\nu}\right)$ in (4.7), (4.8) and (4.9) are valid uniformly with respect to all other appearing parameters (i.e. uniformly with respect to $t \in \mathbb{R}$ and $\mu, \nu \in (0,1]$ in (4.7), uniformly with respect to $t \in \mathbb{R}$, $x \in [0,1]$ and $\mu \in (0,1]$ in (4.8) and uniformly with respect to $t \in \mathbb{R}$ and $\mu \in (0,1]$ in (4.9)). Similar convention concerning the meaning of symbol $O(\cdot)$, by default, will be assumed everywhere below.*

Proof of Proposition 4.1: After the inserting $y = x/\nu$ and $y = (1-x)/\nu$ into the equations (4.3) and (4.4), respectively, we sum up equations (4.2)–(4.4). Moreover, we estimate as follows:

$$\Bigg| f(t,x,\mathcal{S}_{\mu,\nu}(t,x),\mu,\nu) - f\left(t,x,\mathcal{U}_{\mu,\nu}(t,x)+\mathcal{V}_{\mu,\nu}\left(t,\frac{x}{\nu}\right),\mu,\nu\right)$$

$$-f\left(t,x,\mathcal{U}_{\mu,\nu}(t,x)+\mathcal{W}_{\mu,\nu}\left(t,\frac{1-x}{\nu}\right),\mu,\nu\right)+f(t,x,\mathcal{U}_{\mu,\nu}(t,x),\mu,\nu)\Bigg|$$

$$=\left|\int_0^1\int_0^1 \partial_u^2 f\left(t,x,\mathcal{U}_{\mu,\nu}(t,x)+s\mathcal{V}_{\mu,\nu}\left(t,\frac{x}{\nu}\right)+r\mathcal{W}_{\mu,\nu}\left(t,\frac{1-x}{\nu}\right),\mu,\nu\right)ds\,dr\right.$$

$$\left.\times\mathcal{V}_{\mu,\nu}\left(t,\frac{x}{\nu}\right)\mathcal{W}_{\mu,\nu}\left(t,\frac{1-x}{\nu}\right)\right| \le \text{const}\left|\mathcal{V}_{\mu,\nu}\left(t,\frac{x}{\nu}\right)\mathcal{W}_{\mu,\nu}\left(t,\frac{1-x}{\nu}\right)\right| = O\left(e^{-\kappa/\nu}\right) \text{ for } \nu\to 0.$$

Using (4.7) we obtain (4.8). Similarly one proves the asymptotic estimate (4.9). ■

The standard algorithm of the boundary function method suggests to use the ansatz

$$\mathcal{U}_{\mu,\nu}(t,x) = u^0(t,x)+\mu u^{10}(t,x)+\nu u^{01}(t,x), \tag{4.10}$$

$$\mathcal{V}_{\mu,\nu}(t,y) = v^0(t,y)+\mu v^{10}(t,y)+\nu v^{01}(t,y), \tag{4.11}$$

$$\mathcal{W}_{\mu,\nu}(t,y) = w^0(t,y)+\mu w^{10}(t,y)+\nu w^{01}(t,y), \tag{4.12}$$

and boundary conditions of the form

$$\mathcal{V}_{\mu,\nu}(t,0)+\mathcal{U}_{\mu,\nu}(t,0) = \mathcal{V}_{\mu,\nu}(t,\infty) = 0, \tag{4.13}$$

$$\mathcal{W}_{\mu,\nu}(t,0)+\mathcal{U}_{\mu,\nu}(t,1) = \mathcal{W}_{\mu,\nu}(t,\infty) = 0. \tag{4.14}$$

More precisely, we insert (4.10)–(4.12) into equations (4.2)–(4.4) with $\mathcal{R}^u_{\mu,\nu} = \mathcal{R}^v_{\mu,\nu} = \mathcal{R}^w_{\mu,\nu} = 0$ and into the boundary conditions (4.5) and (4.6) with $\mathcal{D}^v_{\mu,\nu} = \mathcal{D}^w_{\mu,\nu} = 0$, and perform the Taylor expansion with respect to small parameters μ and ν. Then, collecting separately all the terms proportional to 1, μ and ν (and neglecting all higher order terms with respect to μ and ν) we obtain equations, which have to determine all the components in formulas (4.10)–(4.12). For example, equation (4.2) yields

$$0 = f(t,x,u^0(t,x),0,0), \tag{4.15}$$

$$\partial_t u^0(t,x) = \partial_u f(t,x,u^0(t,x),0,0)u^{10}(t,x)+\partial_\mu f(t,x,u^0(t,x),0,0), \tag{4.16}$$

$$0 = \partial_u f(t,x,u^0(t,x),0,0)u^{01}(t,x)+\partial_\nu f(t,x,u^0(t,x),0,0). \tag{4.17}$$

Remark that equation (4.15) coincides with the definition of u^0 in (1.4), therefore because of the second part of assumption (1.4) we can uniquely solve the linear algebraic equations (4.16) and (4.17), and obtain explicit expressions for the functions u^{10} and u^{01}.

In a similar way, from equations (4.3) and (4.4) with $\mathcal{R}^v_{\mu,\nu} = \mathcal{R}^w_{\mu,\nu} = 0$ we obtain equations determining functions v^0, v^{10}, v^{01}, w^0, w^{10} and w^{01}. In contrast to (4.15)–(4.17), these equations are differential rather than algebraic, therefore we equip them with boundary conditions following from the Taylor expansion of formulas (4.13) and (4.14). For the leading order terms v^0 and w^0 this procedure yields boundary value problems (1.6) and (1.7). For the next terms v^{10}, v^{01}, w^{10} and w^{01} we obtain linear boundary value problems of the form

$$\left.\begin{aligned} &\partial_t v^0 = \partial_y^2 v^{10} + \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)v^{10} \\ &\qquad + \left(\partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0) - \partial_u f(t,0,u^0(t,0),0,0)\right) u^{10}(t,0) \\ &\qquad + \left(\partial_\mu f(t,0,u^0(t,0)+v^0(t,y),0,0) - \partial_\mu f(t,0,u^0(t,0),0,0)\right), \quad y\in(0,\infty), \\ &v^{10}(t,0)+u^{10}(t,0) = v^{10}(t,\infty) = 0, \end{aligned}\right\} \tag{4.18}$$

$$\left.\begin{aligned} &0 = \partial_y^2 v^{01} + \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)v^{01} \\ &\quad + \left(\partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0) - \partial_u f(t,0,u^0(t,0),0,0)\right) u^{01}(t,0) \\ &\quad + \left(\partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0) - \partial_u f(t,0,u^0(t,0),0,0)\right) \partial_x u^0(t,0)y \\ &\quad + \left(\partial_x f(t,0,u^0(t,0)+v^0(t,y),0,0) - \partial_x f(t,0,u^0(t,0),0,0)\right) y \\ &\quad + \left(\partial_\nu f(t,0,u^0(t,0)+v^0(t,y),0,0) - \partial_\nu f(t,0,u^0(t,0),0,0)\right), \quad y\in(0,\infty), \\ &v^{01}(t,0)+u^{01}(t,0) = v^{01}(t,\infty) = 0, \end{aligned}\right\} \tag{4.19}$$

$$\left.\begin{aligned} &\partial_t w^0 = \partial_y^2 w^{10} + \partial_u f(t,1,u^0(t,1)+w^0(t,y),0,0)w^{10} \\ &\qquad + \left(\partial_u f(t,1,u^0(t,1)+w^0(t,y),0,0) - \partial_u f(t,1,u^0(t,1),0,0)\right) u^{10}(t,1) \\ &\qquad + \left(\partial_\mu f(t,1,u^0(t,1)+w^0(t,y),0,0) - \partial_\mu f(t,1,u^0(t,1),0,0)\right), \quad y\in(0,\infty), \\ &w^{10}(t,0)+u^{10}(t,1) = w^{10}(t,\infty) = 0, \end{aligned}\right\} \tag{4.20}$$

$$\left.\begin{aligned} &0 = \partial_y^2 w^{01} + \partial_u f(t,1,u^0(t,1)+w^0(t,y),0,0)w^{01} \\ &\quad + \left(\partial_u f(t,1,u^0(t,1)+w^0(t,y),0,0) - \partial_u f(t,1,u^0(t,1),0,0)\right) u^{01}(t,1) \\ &\quad - \left(\partial_u f(t,1,u^0(t,1)+w^0(t,y),0,0) - \partial_u f(t,1,u^0(t,1),0,0)\right) \partial_x u^0(t,1)y \\ &\quad - \left(\partial_x f(t,1,u^0(t,1)+w^0(t,y),0,0) - \partial_x f(t,1,u^0(t,1),0,0)\right) y \\ &\quad + \left(\partial_\nu f(t,1,u^0(t,1)+w^0(t,y),0,0) - \partial_\nu f(t,1,u^0(t,1),0,0)\right), \quad y\in(0,\infty), \\ &w^{01}(t,0)+u^{01}(t,1) = w^{01}(t,\infty) = 0, \end{aligned}\right\} \tag{4.21}$$

where $t \in \mathbb{R}$ appears as a parameter. Note that because of (1.6) and (1.7) the derivatives $\partial_t v^0$ and $\partial_t w^0$ appearing in (4.18) and (4.20) are determined as solutions of the linear problems

$$\left.\begin{aligned} &\partial_y^2 v(t,y) + \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)v(t,y) + \partial_t f(t,0,u^0(t,0)+v^0(t,y),0,0) \\ &\qquad + \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)\partial_t u^0(t,0) = 0, \quad y\in(0,\infty), \\ &v(t,0)+\partial_t u^0(t,0) = v(t,\infty) = 0, \end{aligned}\right\} \tag{4.22}$$

and

$$\left.\begin{aligned} &\partial_y^2 w(t,y) + \partial_u f(t,1,u^0(t,1)+w^0(t,y),0,0)w(t,y) + \partial_t f(t,1,u^0(t,1)+w^0(t,y),0,0) \\ &\qquad + \partial_u f(t,1,u^0(t,1)+w^0(t,y),0,0)\partial_t u^0(t,1) = 0, \quad y\in(0,\infty), \\ &w(t,0)+\partial_t u^0(t,1) = w(t,\infty) = 0, \end{aligned}\right\} \tag{4.23}$$

respectively.

Proposition 4.3 *Suppose (1.4) and (1.6)–(1.8). Then the problems (4.18)–(4.23) have unique solutions* v^{10}*,* v^{01}*,* w^{10}*,* w^{01}*,* $v = \partial_t v^0$ *and* $w = \partial_t w^0$*. Moreover, for every* $\kappa \in (0, \min(\kappa_0, \kappa_1))$ *we have*

$$\begin{aligned}&\left|v^{10}(t,y)\right| + \left|v^{01}(t,y)\right| + \left|w^{10}(t,y)\right| + \left|w^{01}(t,y)\right| + \left|\partial_t v^0(t,y)\right| + \left|\partial_t w^0(t,y)\right| \\ &+ \left|\partial_t v^{10}(t,y)\right| + \left|\partial_t v^{01}(t,y)\right| + \left|\partial_t w^{10}(t,y)\right| + \left|\partial_t w^{01}(t,y)\right| = O(e^{-\kappa y}) \quad \textit{for} \quad y \to \infty.\end{aligned}$$

Proof: The differential equations of problems (4.18)–(4.23) can be rewritten as ODE systems of the form

$$\frac{d}{dy} z(y) = A_t(y) z(y) + b_t(y), \quad y \geq 0. \tag{4.24}$$

In what follows we will consider (4.22) only, the systems (4.18)–(4.21) and (4.23) can be handled analogously. For (4.22) we have (4.24) with

$$z(y) = \begin{pmatrix} v(y) \\ v'(y) \end{pmatrix}, \quad A_t(y) = \begin{pmatrix} 0 & 1 \\ \partial_u f(t, 0, u^0(t,0) + v^0(t,y), 0, 0) & 0 \end{pmatrix}, \quad b_t(y) = \begin{pmatrix} 0 \\ q(t,y) \end{pmatrix},$$

where

$$q(t,y) = \partial_t f(t, 0, u^0(t,0) + v^0(t,y), 0, 0) + \partial_u f(t, 0, u^0(t,0) + v^0(t,y), 0, 0) \partial_t u^0(t,0). \tag{4.25}$$

Assumption (1.4) and Remark 1.3 imply that the limiting matrix $A_t(\infty)$ has two real eigenvalues

$$\pm\sqrt{|\partial_u f(t, 0, u^0(t,0), 0, 0)|},$$

therefore the homogeneous equation (4.24) (i.e. $b_t = 0$) has an exponential dichotomy on the half-line $y \geq 0$ (see [7, Ch. 6, Prop. 1]). This means that there exists a rank-1 projection operator $P : \mathbb{R}^2 \to \mathbb{R}^2$, and for any $\kappa \in (0, \kappa_0)$ there exists a constant $C > 0$ such that the fundamental matrix $\Phi(y)$ of system (4.24) satisfies

$$\left\|\Phi(y_1) P \Phi^{-1}(y_2)\right\| \leq C e^{-\kappa(y_1 - y_2)} \quad \text{for} \quad 0 \leq y_2 \leq y_1, \tag{4.26}$$

$$\left\|\Phi(y_1)(I - P) \Phi^{-1}(y_2)\right\| \leq C e^{-\kappa(y_2 - y_1)} \quad \text{for} \quad 0 \leq y_1 \leq y_2. \tag{4.27}$$

For any solution to (4.24) with a bounded continuous vector-function b_t there exists $c \in \mathbb{R}$ such that

$$z(y) = c \begin{pmatrix} \partial_y v^0(t,y) \\ \partial_y^2 v^0(t,y) \end{pmatrix} + \int_0^y \Phi(y) P \Phi^{-1}(\xi) b_t(\xi) d\xi - \int_y^\infty \Phi(y)(I - P)\Phi^{-1}(\xi) b_t(\xi) d\xi. \tag{4.28}$$

Moreover, if for a certain $\kappa \in (0, \kappa_0)$ we have $\|b_t(y)\| = O(e^{-\kappa y})$ for $y \to \infty$, then any bounded solution z to (4.24) satisfies $\|z(y)\| = O(e^{-\kappa y})$ for $y \to \infty$. Because of (1.4), (4.25) and of Remark 1.3, for every $\kappa \in (0, \kappa_0)$ it holds

$$\|b_t(y)\| = O(e^{-\kappa y}) \quad \text{for} \quad y \to \infty. \tag{4.29}$$

Therefore solutions to these problems must be of the form (4.28). Then, assumption (1.8) and the Dirichlet boundary condition at $y = 0$ permit us to determine the constant c in (4.28) uniquely. On the other hand, using inequalities (4.26), (4.27), (4.29) and formula (4.28) we obtain the estimates for $\partial_t v^0$. ■

Let $\mathcal{S}_{\mu,\nu}$ be the function given by formulas (4.1) and (4.10)–(4.12), where u^{10}, u^{01}, v^{10}, v^{01}, w^{10} and w^{01} are solutions of the above formulated problems, then the boundary layer functions $\mathcal{V}_{\mu,\nu}$, $\mathcal{W}_{\mu,\nu}$ satisfy the exponential estimates (4.7) and

$$\left\|\mu\partial_t S_{\mu,\nu} - \nu^2\partial_x^2 S_{\mu,\nu} - f(t,x,S_{\mu,\nu}(t,x),\mu,\nu)\right\|_\infty = O\left((\mu+\nu)^2\right) \quad \text{for} \quad \mu,\nu \to 0, \tag{4.30}$$
$$\|S_{\mu,\nu} - u_\nu\|_\infty = O(\mu+\nu) \quad \text{for} \quad \mu,\nu \to 0.$$

Hence, the function $\mathcal{S}_{\mu,\nu}$ seems to be a good candiadate for the improved approximate solution $u^1_{\mu,\nu}$, in particular it satisfies (3.8). But, unfortunately, it does not belong to the subspace U^0 (because it does not satisfy the homogeneous Dirichlet boundary conditions exactly, but only up to an exponentially small error) and it does not satisfy (3.9), in general. Indeed, if we insert $\mathcal{S}_{\mu,\nu}$ instead of $u^1_{\mu,\nu}$ into formula (3.9), then estimate (4.30) yields

$$\left\|\mu\partial_t S_{\mu,\nu} - \nu^2\partial_x^2 S_{\mu,\nu} - f(t,x,S_{\mu,\nu}(t,x),\mu,\nu)\right\|^2_{\mu,\nu} = O\left(\frac{(\mu+\nu)^4}{\mu\nu}\right) \quad \text{for} \quad \mu,\nu \to 0.$$

The ratio $(\mu+\nu)^4/(\mu\nu)$ obviously tends to zero for $\mu = \nu \to 0$, but for μ and ν tending to zero independently, it stays unbounded, therefore we cannot guarantee the smallness of the right-hand side in formula (3.9).

Because of this reason we need to adopt a different strategy. We consider two cases $\mu \le \nu$ and $\mu \ge \nu$ separately. Accordingly, we construct two improved approximate solutions $\mathcal{A}_{\mu,\nu}(x,t)$ and $\mathcal{B}_{\mu,\nu}(x,t)$, which satisfy

$$\left\|\mu\partial_t \mathcal{A}_{\mu,\nu} - \nu^2\partial_x^2 \mathcal{A}_{\mu,\nu} - f(t,x,\mathcal{A}_{\mu,\nu}(t,x),\mu,\nu)\right\|_\infty = O(\mu^2) \quad \text{for} \quad \mu \le \nu \to 0, \tag{4.31}$$
$$\left\|\mu\partial_t \mathcal{B}_{\mu,\nu} - \nu^2\partial_x^2 \mathcal{B}_{\mu,\nu} - f(t,x,\mathcal{B}_{\mu,\nu}(t,x),\mu,\nu)\right\|_\infty = O(\nu^2) \quad \text{for} \quad \nu \le \mu \to 0, \tag{4.32}$$

and

$$\|\mathcal{A}_{\mu,\nu} - u_\nu\|_\infty + \|\mathcal{B}_{\mu,\nu} - u_\nu\|_\infty = O(\mu+\nu) \quad \text{for} \quad \mu,\nu \to 0. \tag{4.33}$$

For $\mu \le \nu$, we apply Theorem 2.1 with $u^1_{\mu,\nu} = \mathcal{A}_{\mu,\nu}$. Then, (4.31) yields

$$\|F_{\mu,\nu}(\mathcal{A}_{\mu,\nu})\|_{\mu,\nu} = O(\sqrt{\mu^3/\nu}) = O(\mu). \tag{4.34}$$

i.e. (3.9).

In the second case $\nu \le \mu$, we apply Theorem 2.1 with $u^1_{\mu,\nu} = \mathcal{B}_{\mu,\nu}$. Then, (4.32) yields

$$\|F_{\mu,\nu}(\mathcal{B}_{\mu,\nu})\|_{\mu,\nu} = O(\sqrt{\nu^3/\mu}) = O(\nu),$$

i.e. (3.9), again. Moreover, (4.33) implies (3.8) in both cases $\mu \le \nu$ and $\nu \le \mu$.

4.1 Case $\mu \le \nu$

We use the following ansatz

$$\mathcal{A}_{\mu,\nu}(t,x) := \mathcal{A}^0_\nu(t,x) + \mu\mathcal{A}^1_\nu(t,x), \tag{4.35}$$

where $\mathcal{A}_\nu^0$ and $\mathcal{A}_\nu^1$ are solutions to the elliptic BVPs

$$\left.\begin{aligned} &0 = \nu^2 \partial_x^2 u + f(t,x,u(t,x),0,\nu), && (t,x) \in \mathbb{R}\times(0,1),\\ &u(t,0) = u(t,1) = 0, && t\in\mathbb{R}, \end{aligned}\right\} \tag{4.36}$$

and

$$\left.\begin{aligned} &\partial_t \mathcal{A}_\nu^0 = \nu^2 \partial_x^2 u + \partial_u f(t,x,\mathcal{A}_\nu^0,0,\nu)u + \partial_\mu f(t,x,\mathcal{A}_\nu^0,0,\nu), && (t,x) \in \mathbb{R}\times(0,1),\\ &u(t,0) = u(t,1) = 0, && t\in\mathbb{R}, \end{aligned}\right\} \tag{4.37}$$

respectively.

Proposition 4.4 *There exist $\nu_0 > 0$ and $c > 0$ such that for all $\nu \in (0,\nu_0)$ and all $t \in [0,1]$ boundary value problems (4.36) and (4.37) have solutions $\mathcal{A}_\nu^0$ and $\mathcal{A}_\nu^1$, respectively, which satisfy*

$$\left|\mathcal{A}_\nu^0(t,x) - u_\nu(t,x)\right| \le c\nu \quad \textit{for all} \quad (t,x) \in [0,1]^2, \tag{4.38}$$

$$\left|\partial_t\mathcal{A}_\nu^0(t,x)\right| + \left|\mathcal{A}_\nu^1(t,x)\right| + \left|\partial_t\mathcal{A}_\nu^1(t,x)\right| \le c \quad \textit{for all} \quad (t,x) \in [0,1]^2. \tag{4.39}$$

Proof: We apply Theorem 2.1 to the boundary value problem (4.36). For that reason we take

$$\begin{aligned} U &:= C^2([0,1]), \quad \|u\|_U := \|u''\|_\infty + \|u'\|_\infty + \|u\|_\infty, \quad \|u\|_\infty := \max\{|u(x)| : x\in[0,1]\},\\ U^0 &:= \{u \in U:\ u(0)=u(1)=0\},\\ V &:= C([0,1]), \quad \|v\|_V := \|v\|_\infty,\\ \lambda &= t \in \Lambda := [0,1], \quad \varepsilon = \nu \in E := (0,1),\\ \|u\|_\nu &:= \nu^2\|u''\|_\infty + \nu\|u'\|_\infty + \|u\|_\infty,\\ |v|_\nu &:= \|v\|_\infty. \end{aligned}$$

Then, problem (4.36) is equivalent to the equation

$$[F_\nu(t,u)](x) := \nu^2\partial_x^2 u(x) + f(t,x,u(x),0,\nu) = 0, \quad u \in U^0.$$

Because the function f is supposed to be C^2-smooth, we have that F_ν is C^2-smooth and

$$\left[\partial_u F_\nu(t,u)v\right](x) = \nu^2\partial_x^2 v(x) + \partial_u f(t,x,u(x),0,\nu)v(x),$$

and

$$\left[(\partial_u F_\nu(t,u_1) - \partial_u F_\nu(t,u_2))\,u\right](x) = \left(\partial_u f(t,x,u_1(x),0,\nu) - \partial_u f(t,x,u_2(x),0,\nu)\right)u(x),$$

therefore we easily verify that the conditions (2.3) and (2.8) in Theorem 2.1 are fulfilled. Moreover, condition (2.6) is fulfilled because $\|u\|_\infty \le \|u\|_\nu$ for all $u \in U^0$.

Let us take

$$u_\nu^0(t) := u_\nu(t,\cdot),$$

where $u_\nu(t,x)$ is defined in (1.9), then condition (2.5) is also fulfilled. The Fredholmness condition (2.4) and the coercivity estimate (2.7) follow from [13, § 4.4] and Lemma 5.4, respectively. However, because of the discrepancy in boundary conditions we have $u_\nu^0 \notin U^0$, therefore we take

$$u_\nu^1(t)(x) := u_\nu^0(t)(x) - u_\nu^0(t)(0) - \Big(u_\nu^0(t)(1) - u_\nu^0(t)(0)\Big)\, x.$$

Now, Remark 1.3 and (4.30) imply

$$\|u_\nu^0(t) - u_\nu^1(t)\|_\infty = O\left(e^{-1/\nu}\right) \quad \text{and} \quad \left|F_\nu(t, u_\nu^1(t))\right|_\nu = O(\nu).$$

Hence, Theorem 2.1 yields the existence of the solution $\mathcal{A}_\nu^0$ to problem (4.36) and estimate (2.10) yields formula (4.38). On the other hand, the smoothness of f and Remark 2.5 imply

$$\left\|\partial_t \mathcal{A}_\nu^0\right\|_\infty + \left\|\partial_t^2 \mathcal{A}_\nu^0\right\|_\infty \le \text{const}.$$

Another corollary of Theorem 2.1 is that for sufficiently small ν, the operator $\partial_u F_\nu(t, \mathcal{A}_\nu^0)$ is bijective from U^0 onto V. Therefore linear boundary value problem (4.37) has a unique solution $\mathcal{A}_\nu^1$, which because of Lemma 5.4 and the smoothness of f satisfies estimates (4.39). ■

Remark 4.5 *Let us insert constructed function $\mathcal{A}_{\mu,\nu}$ into equation (1.1). Then, performing the Taylor expansion with respect to the small parameter μ and taking into account (4.38) and (4.39) we easily obtain (4.31). On the other hand, from formulas (4.35), (4.38) and (4.39) we also obtain*

$$\|\mathcal{A}_{\mu,\nu} - u_\nu\|_\infty = O(\mu + \nu).$$

4.2 Case $\mu \ge \nu$

In this case, we construct an approximate solution of the form

$$\begin{aligned} \mathcal{B}_{\mu,\nu}(t,x) &:= \mathcal{U}_\mu^0(t,x) + \mathcal{V}_\mu^0\left(t, \frac{x}{\nu}\right) + \mathcal{W}_\mu^0\left(t, \frac{1-x}{\nu}\right) \\ &+ \nu\left(\mathcal{U}_\mu^1(t,x) + \mathcal{V}_\mu^1\left(t, \frac{x}{\nu}\right) + \mathcal{W}_\mu^1\left(t, \frac{1-x}{\nu}\right)\right). \end{aligned} \tag{4.40}$$

We use formal decomposition of the original problem (1.1)–(1.3) into equations (4.2)–(4.14) and then perform the Taylor expansion of these equations with respect to the smallest parameter ν only. Thus we obtain two periodic BVPs

$$\left.\begin{aligned} &\mu\partial_t u = f(t,x,u(t,x),\mu,0), && (t,x) \in \mathbb{R}\times(0,1), \\ &u(t+1,x) = u(t,x), && (t,x) \in \mathbb{R}\times(0,1), \end{aligned}\right\} \tag{4.41}$$

and

$$\left.\begin{aligned} &\mu\partial_t u = \partial_u f(t,x,\mathcal{U}_\mu^0(t,x),\mu,0)u + \partial_\nu f(t,x,\mathcal{U}_\mu^0(t,x),\mu,0), && (t,x) \in \mathbb{R}\times(0,1), \\ &u(t+1,x) = u(t,x), && (t,x) \in \mathbb{R}\times(0,1), \end{aligned}\right\} \tag{4.42}$$

which serve to determine the terms $\mathcal{U}_\mu^0$ and $\mathcal{U}_\mu^1$, respectively.

Proposition 4.6 *There exists $\mu_0 > 0$ and $c > 0$ such that for all $\mu \in (0, \mu_0)$ and all $x \in [0,1]$ periodic boundary value problems (4.41) and (4.42) have solutions $\mathcal{U}_\mu^0$ and $\mathcal{U}_\mu^1$, respectively, which satisfy*

$$\left|\mathcal{U}_\mu^0(t,x) - u^0(t,x) - \mu u^{10}(t,x)\right| \le c\mu^2 \quad \textit{for all} \quad (t,x) \in [0,1]^2, \tag{4.43}$$

$$\left|\mathcal{U}_\mu^1(t,x) - u^{01}(t,x)\right| \le c\mu \quad \textit{for all} \quad (t,x) \in [0,1]^2, \tag{4.44}$$

$$\left|\partial_x^2 \mathcal{U}_\mu^0(t,x)\right| + \left|\partial_x^2 \mathcal{U}_\mu^1(t,x)\right| \le c \quad \textit{for all} \quad (t,x) \in [0,1]^2. \tag{4.45}$$

Proof: We apply Theorem 2.1 to the boundary value problem (4.41). For that reason we take

$$\begin{aligned}
U &:= C^1([0,1]), \quad \|u\|_U := \|u'\|_\infty + \|u\|_\infty, \quad \text{where} \quad \|u\|_\infty := \max\{|u(t)| : t \in [0,1]\}, \\
U^0 &:= \{u \in U : \ u(0) = u(1), \ u'(0) = u'(1)\}, \\
V &:= \{v \in C([0,1]) \ : \ v(0) = v(1)\}, \quad \|v\|_V := \|v\|_\infty, \\
\lambda &= x \in \Lambda := [0,1], \quad \varepsilon = \mu \in E := (0,1), \\
\|u\|_\mu &:= \mu\|u'\|_\infty + \|u\|_\infty, \\
|v|_\mu &:= \|v\|_\infty.
\end{aligned}$$

Now, problem (4.41) is equivalent to the equation

$$[F_\mu(x,u)](t) := \mu\partial_t u(t) - f(t,x,u(t),\mu,0) = 0, \quad u \in U^0. \tag{4.46}$$

The C^2-smoothness of function f implies that the derivative

$$[\partial_u F_\mu(x,u)v](t) := \mu\partial_t v(t) - \partial_u f(t,x,u(t),\mu,0)v(t)$$

exists for all $u \in U$ (cf. (2.3)) and satisfies condition (2.8). Obviously, condition (2.6) is also fulfilled.

Let us take

$$u_\mu^0(x) = u_\mu^1(x) := u^0(\cdot,x) + \mu u^{10}(\cdot,x),$$

where $u^0(t,x)$ and $u^{10}(t,x)$ are defined in (1.4) and (4.16), respectively. According to Lemma 5.6, for sufficiently small μ the linear operator $\partial_u F_\mu(x,u_\mu^0(x))$ is bijective from U^0 onto V, hence condition (2.4) is fulfilled. Moreover, inequality (5.32) implies condition (2.7).

Inserting $u_\mu^0(x)$ into equation (4.46) and performing the Taylor expansion with respect to μ, because of (1.4) and (4.16), we obtain

$$\left|F_\mu(x,u_\mu^0(x))\right|_\mu = O(\mu^2).$$

Hence, Theorem 2.1 yields the estimate (4.43).

In contrast to (4.41), problem (4.42) is linear. Using substitution $u(t,x) = u^{01}(t,x) + \tilde{u}(t,x)$ we rewrite it in the form

$$\left.\begin{aligned}
&\mu\partial_t\tilde{u} = \partial_u f(t,x,\mathcal{U}_\mu^0(t,x),\mu,0)\tilde{u} + r_\mu^{\mathrm{u}}(t,x), && (t,x) \in \mathbb{R} \times (0,1), \\
&\tilde{u}(t+1,x) = \tilde{u}(t,x), && (t,x) \in \mathbb{R} \times (0,1),
\end{aligned}\right\} \tag{4.47}$$

where $r_\mu^{\mathrm{u}} = O(\mu)$ because of (4.17) and (4.43). Now, Lemma 5.6 guarantees that problem (4.47) has a unique solution $\tilde{u}$ and estimate (4.44) is fulfilled.

The remaining estimates (4.45) for the derivatives $\partial_x^2\mathcal{U}_\mu^0$ and $\partial_x^2\mathcal{U}_\mu^1$ follow from the C^3-smoothness of nonlinearity f, Remark 2.5 and Lemma 5.6. ■

Functions $\mathcal{U}_\mu^0$ and $\mathcal{U}_\mu^1$, in general, don't satisfy boundary conditions for $x = 0$ and $x = 1$, therefore ansatz (4.40) contains boundary layer functions $\mathcal{V}_\mu^0$, $\mathcal{V}_\mu^1$, $\mathcal{W}_\mu^0$ and $\mathcal{W}_\mu^1$. Two of them $\mathcal{V}_\mu^0$ and $\mathcal{V}_\mu^1$ are determined as solutions to the problems

$$\left.\begin{array}{ll}\mu\partial_t v = \partial_y^2 v + f(t,0,\mathcal{U}_\mu^0(t,0)+v(t,y),\mu,0) - f(t,0,\mathcal{U}_\mu^0(t,0),\mu,0), & (t,y)\in\mathbb{R}\times(0,\infty),\\ v(t,0)+\mathcal{U}_\mu^0(t,0) = v(t,\infty) = 0, & t\in\mathbb{R},\\ v(t+1,y) = v(t,y), & (t,y)\in\mathbb{R}\times(0,\infty),\end{array}\right\}\tag{4.48}$$

and

$$\left.\begin{array}{l}\mu\partial_t v = \partial_y^2 v + \partial_u f(t,0,\mathcal{U}_\mu^0(t,0)+\mathcal{V}_\mu^0(t,y),\mu,0)v\\ \qquad + \big(\partial_u f(t,0,\mathcal{U}_\mu^0(t,0)+\mathcal{V}_\mu^0(t,y),\mu,0) - \partial_u f(t,0,\mathcal{U}_\mu^0(t,0),\mu,0)\big)\,\mathcal{U}_\mu^1(t,0)\\ \qquad + \big(\partial_u f(t,0,\mathcal{U}_\mu^0(t,0)+\mathcal{V}_\mu^0(t,y),\mu,0) - \partial_u f(t,0,\mathcal{U}_\mu^0(t,0),\mu,0)\big)\,\partial_x\mathcal{U}_\mu^0(t,0)y\\ \qquad + \big(\partial_x f(t,0,\mathcal{U}_\mu^0(t,0)+\mathcal{V}_\mu^0(t,y),\mu,0) - \partial_x f(t,0,\mathcal{U}_\mu^0(t,0),\mu,0)\big)\,y\\ \qquad + \big(\partial_\nu f(t,0,\mathcal{U}_\mu^0(t,0)+\mathcal{V}_\mu^0(t,y),\mu,0) - \partial_\nu f(t,0,\mathcal{U}_\mu^0(t,0),\mu,0)\big)\,, \quad (t,y)\in\mathbb{R}\times(0,\infty),\\ v(t,0)+\mathcal{U}_\mu^1(t,0) = v(t,\infty) = 0, \quad t\in\mathbb{R},\\ v(t+1,y) = v(t,y), \quad (t,y)\in\mathbb{R}\times(0,\infty).\end{array}\right\}\tag{4.49}$$

Similarly, for boundary layers $\mathcal{W}_\mu^0$ and $\mathcal{W}_\mu^1$ we write the problems

$$\left.\begin{array}{ll}\mu\partial_t w = \partial_y^2 w + f(t,1,\mathcal{U}_\mu^0(t,1)+w(t,y),\mu,0) - f(t,1,\mathcal{U}_\mu^0(t,1),\mu,0), & (t,y)\in\mathbb{R}\times(0,\infty),\\ w(t,0)+\mathcal{U}_\mu^0(t,1) = w(t,\infty) = 0, & t\in\mathbb{R},\\ w(t+1,y) = w(t,y), & (t,y)\in\mathbb{R}\times(0,\infty),\end{array}\right\}\tag{4.50}$$

and

$$\left.\begin{array}{l}\mu\partial_t w = \partial_y^2 w + \partial_u f(t,1,\mathcal{U}_\mu^0(t,1)+\mathcal{W}_\mu^0(t,y),\mu,0)w\\ \qquad + \big(\partial_u f(t,1,\mathcal{U}_\mu^0(t,1)+\mathcal{W}_\mu^0(t,y),\mu,0) - \partial_u f(t,1,\mathcal{U}_\mu^0(t,1),\mu,0)\big)\,\mathcal{U}_\mu^1(t,1)\\ \qquad - \big(\partial_u f(t,1,\mathcal{U}_\mu^0(t,1)+\mathcal{W}_\mu^0(t,y),\mu,0) - \partial_u f(t,1,\mathcal{U}_\mu^0(t,1),\mu,0)\big)\,\partial_x\mathcal{U}_\mu^0(t,1)y\\ \qquad - \big(\partial_x f(t,1,\mathcal{U}_\mu^0(t,1)+\mathcal{W}_\mu^0(t,y),\mu,0) - \partial_x f(t,1,\mathcal{U}_\mu^0(t,1),\mu,0)\big)\,y\\ \qquad + \big(\partial_\nu f(t,1,\mathcal{U}_\mu^0(t,1)+\mathcal{W}_\mu^0(t,y),\mu,0) - \partial_\nu f(t,1,\mathcal{U}_\mu^0(t,1),\mu,0)\big)\,, \quad (t,y)\in\mathbb{R}\times(0,\infty),\\ w(t,0)+\mathcal{U}_\mu^1(t,1) = w(t,\infty) = 0, \quad t\in\mathbb{R},\\ w(t+1,y) = w(t,y), \quad (t,y)\in\mathbb{R}\times(0,\infty).\end{array}\right\}\tag{4.51}$$

Proposition 4.7 *There exists $\mu_0 > 0$, $c > 0$ and $\kappa > 0$ such that for all $\mu \in (0, \mu_0)$ problems (4.48)–(4.51) have solutions $\mathcal{V}^0_\mu$, $\mathcal{V}^1_\mu$, $\mathcal{W}^0_\mu$ and $\mathcal{W}^1_\mu$, respectively, which satisfy*

$$\left|\mathcal{V}^0_\mu(t,y) - v^0(t,y) - \mu v^{10}(t,y)\right| \le \mu^{3/2} c e^{-\kappa y} \quad \textit{for all} \quad (t,y) \in [0,1] \times [0,\infty), \tag{4.52}$$

$$\left|\mathcal{W}^0_\mu(t,y) - w^0(t,y) - \mu w^{10}(t,y)\right| \le \mu^{3/2} c e^{-\kappa y} \quad \textit{for all} \quad (t,y) \in [0,1] \times [0,\infty), \tag{4.53}$$

$$\left|\mathcal{V}^1_\mu(t,y) - v^{01}(t,y)\right| + \left|\mathcal{W}^1_\mu(t,y) - w^{01}(t,y)\right| \le \sqrt{\mu} c e^{-\kappa y} \quad \textit{for all} \quad (t,y) \in [0,1] \times [0,\infty). \tag{4.54}$$

Proof: We prove only the first part of the assertion concerning functions $\mathcal{V}^0_\mu$ and $\mathcal{V}^1_\mu$. The existence and properties of functions $\mathcal{W}^0_\mu$ and $\mathcal{W}^1_\mu$ can be obtained analogously.

We apply Theorem 2.1 to the boundary value problem (4.48). For that reason we take

$$\begin{aligned}
U &:= L^2\left((0,1); W^{2,2}(0,\infty)\right) \cap W^{1,2}\left((0,1); L^2(0,\infty)\right), \\
\|u\|_U &:= \left(\int_0^1 dt \int_0^\infty \left(\partial_t u^2 + \partial_y^2 u^2 + \partial_y u^2 + u^2\right) dy\right)^{1/2}, \\
U^0 &:= \left\{u \in U : \ u(0,y) = u(1,y) \text{ for all } y, \text{ and } u(t,0) = 0 \text{ for all } t\right\}, \\
V &:= L^2\left((0,1) \times (0,\infty)\right), \quad \|v\|_V := \left(\int_0^1 dt \int_0^\infty v^2 dy\right)^{1/2}, \\
\Lambda &:= \emptyset, \quad \varepsilon = \mu \in E := (0,1), \\
\|u\|_\infty &:= \max\{|u(t,y)| \ : \ (t,y) \in [0,1] \times [0,\infty)\}, \\
\|u\|_\mu &:= \left(\int_0^1 dt \int_0^\infty \left(\mu^2 \partial_t u^2 + \partial_y^2 u^2 + \partial_y u^2 + u^2\right) e^{\kappa y} \frac{dy}{\mu}\right)^{1/2}, \\
|v|_\mu &:= \left(\int_0^1 dt \int_0^\infty v^2 e^{\kappa y} \frac{dy}{\mu}\right)^{1/2}.
\end{aligned}$$

Note, in the definition of the norms $\|\cdot\|_\mu$ and $|\cdot|_\mu$ there appears a coefficient $\kappa > 0$, which will be chosen later independently of μ in accordance with Lemma 5.5.

Given κ, let us choose some $\hat{\kappa} > \kappa/2$ and define an auxiliary function $\tilde{u}_\mu(t,y) = -\mathcal{U}^0_\mu(t,0)e^{-\hat{\kappa}y}$. This function, obviously, satisfies boundary conditions of the problem (4.48) and has finite norm $\|\tilde{u}_\mu\|_\mu < \infty$ for all $\mu \in (0,1]$. Now, problem (4.48) can be rewritten as an abstract equation

$$[F_\mu(u)](t,y) = \tilde{F}_\mu(\tilde{u}_\mu(t,y) + u(t,y)), \quad u \in U^0,$$

where

$$[\tilde{F}_\mu(u)](t,y) = \mu\partial_t u(t,y) - \partial_y^2 u(t,y) - f(t,0,\mathcal{U}^0_\mu(t,0) + u(t,y), \mu, 0) + f(t,0,\mathcal{U}^0_\mu(t,0), \mu, 0)$$

is the differential operator from (4.48).

Using classical embedding theorems for anisotropic Sobolev spaces (cf. [3, Theorem 10.4]) we obtain

$$U \hookrightarrow C((0,1) \times (0,\infty)), \tag{4.55}$$

therefore for a C^2-smooth function f condition (2.3) is fulfilled with

$$[\partial_u F_\mu(u)v](t,y) := \mu\partial_t v(t,y) - \partial_y^2 v(t,y) - \partial_u f(t,0,\mathcal{U}_\mu^0(t,0)(1-e^{-\hat{\kappa}y}) + u(t,y),\mu,0)v(t,y).$$

Moreover, estimate (2.8) is also fulfilled, as follows from the identity

$$\begin{aligned}[(\partial_u F_\mu(u_1) - \partial_u F_\mu(u_2))\, v]\,(t,y) &= \Big(\partial_u f(t,0,\mathcal{U}_\mu^0(t,0)(1-e^{-\hat{\kappa}y}) + u_2(t,y),\mu,0) \\ &- \partial_u f(t,0,\mathcal{U}_\mu^0(t,0)(1-e^{-\hat{\kappa}y}) + u_1(t,y),\mu,0)\Big)\, v(t,y).\end{aligned}$$

However, embedding (4.55) does not yield estimate (2.6), which has to be uniform with respect to $\mu \to 0$. In order to verify it, we show that there exists an extension operator

$$E \,:\, U^0 \to L^2(\mathbb{R}; W^{2,2}(\mathbb{R})) \cap W^{1,2}(\mathbb{R}; L^2(\mathbb{R}))$$

such that

$$\|Eu\|_{L^2(\mathbb{R};W^{2,2}(\mathbb{R}))\cap W^{1,2}(\mathbb{R};L^2(\mathbb{R}))} \le \text{const}\, \|u\|_\mu \quad \text{for all} \quad \mu \in (0,1].$$

Operator E can be constructed as a superposition of the following steps: (i) transform time variable $t \mapsto \tau = t/\mu$, (ii) perform odd extension in y-direction and periodic extension in τ-direction (recall that every $u \in U^0$ vanishes for $y = 0$ and is 1-periodic in t), and finally (iii) multiply the resulting function by a τ-dependent cut-off function, which has bounded derivative and equals to unity on the interval $t \in [0, 1/\mu]$. The existence of E and the continuous embedding $L^2(\mathbb{R}; W^{2,2}(\mathbb{R})) \cap W^{1,2}(\mathbb{R}; L^2(\mathbb{R})) \hookrightarrow C(\mathbb{R}^2)$ (see [14, Ch. 2, Sec. 2, Theorem 6]) yield estimate (2.6).

Now, let us assume

$$u_\mu^0 := v^0(t,y) + \mu v^{10}(t,y) + \mathcal{U}_\mu^0(t,0)e^{-\hat{\kappa}y},$$

where v^0 and v^{10} are bounded functions defined in (1.6) and (4.18), respectively. Then condition (2.5) is obviously fulfilled. Moreover, because of our convention regarding the exponent κ in the definition of norms $\|\cdot\|_\mu$ and $|\cdot|_\mu$, Lemma 5.5 guarantees that condition (2.7) is also fulfilled.

Because of the unbounded spatial domain in problem (4.48), verification of condition (2.4) is less trivial here than it was for problem (1.1)–(1.3) at the beginning of Section 2. We use the decomposition

$$[\partial_u F_\mu(u_\mu^0)v] = (L_1 + L_2)v,$$

where

$$(L_1 v)(t,y) = \mu\partial_t v(t,y) - \partial_y^2 v(t,y) - \partial_u f(t,0,\mathcal{U}_\mu^0(t,0),\mu,0)v(t,y),$$

$$(L_2 v)(t,y) = \Big(\partial_u f(t,0,\mathcal{U}_\mu^0(t,0) + v^0(t,y) + \mu v^{10}(t,y),\mu,0) - \partial_u f(t,0,\mathcal{U}_\mu^0(t,0),\mu,0)\Big)\, v(t,y).$$

From the condition (1.4) and estimate (4.43), it follows that for sufficiently small $\mu > 0$ the operator L_1 is an isomorphism from $L^2((0,1); W^{2,2}(\mathbb{R})) \cap W^{1,2}((0,1); L^2(\mathbb{R}))$ onto $L^2((0,1) \times \mathbb{R})$ (see [25, Ch. 3, Theorem 2.2.2]). Considering its restriction to the subspace of odd functions

$$L^2((0,1); W^{2,2}_{\text{odd}}(\mathbb{R})) \cap W^{1,2}((0,1); L^2_{\text{odd}}(\mathbb{R})),$$

which is isomorphic to U^0, we easily find that L_1 is also an isomorphism from U^0 onto V. Therefore, in order to show that $\partial_u F_\mu(u_\mu^0)$ is Fredholm of index zero it is enough to demonstrate that L_2 is a compact operator.

The latter follows from the following two results. First, because of the exponential decay estimates for v^0 and v^{10} (see Remark 1.3 and Proposition 4.3) we have

$$L_2 = \lim_{R\to\infty} I_{[0,R]} L_2,$$

where $I_{[0,R]}$ is the indicator function of the interval $[0, R]$. Second, the Aubin-Lions lemma yields compact embedding

$$L^2((0,1); W^{2,2}(0,R)) \cap W^{1,2}((0,1); L^2(0,R)) \hookrightarrow L^2((0,1)\times(0,R))$$

for any fixed $R > 0$. Hence, we have verified (2.4).

Comparing boundary conditions of problems (1.6) and (4.18) with the estimate (4.43) we obtain

$$\max_{0\le t\le 1} \left|u^0_\mu(t,0)\right| = \max_{0\le t\le 1} \left|v^0_\mu(t,0) + \mu v^{10}_\mu(t,0) + \mathcal{U}^0_\mu(t,0)\right| = O(\mu^2), \tag{4.56}$$

therefore u^0_μ, in general, produces a small discrepancy in the boundary conditions for $y = 0$. In order to compansate this discrepancy we take

$$u^1_\mu(t,y) := u^0_\mu(t,y) - u^0_\mu(t,0)e^{-\hat\kappa y},$$

where $\hat\kappa > 0$ is the same constant as above. Obviously, because of (4.56) we have

$$\left\|u^0_\mu - u^1_\mu\right\|_\infty = O(\mu^2).$$

Inserting u^1_μ into $F_\mu(u)$ and using the definition of functions v^0, v^{10} (see (1.6), (4.18)), Remark 1.3 and Proposition 4.3 we obtain

$$\left|F_\mu\left(u^1_\mu\right)(t,y)\right| \le \text{const}\, \mu^2 e^{-\kappa y} \quad \text{and} \quad \left|F_\mu\left(u^1_\mu\right)\right|_\mu = O(\mu^{3/2}).$$

Therefore Theorem 2.1 delivers the existence of function $\mathcal{V}^0_\mu$ and estimate (4.52).

We end up the proof by considering the problem (4.49). We use the substitution

$$v(t,y) = v^{01}(t,y) - \left(v^{01}(t,0) + \mathcal{U}^1_\mu(t,0)\right)e^{-\hat\kappa y} + \hat v(t,y),$$

where v^{01} is defined in (4.19). Taking into account Proposition 4.6 and definitions of functions u^0, v^0, u^{10}, u^{01}, v^{10} (see (1.4), (1.6), (4.16)–(4.18)) we transform (4.49) into an equivalent problem for $\hat v$

$$\left.\begin{aligned}
&\mu\partial_t \hat v = \partial_y^2 \hat v + \partial_u f(t,0,\mathcal{U}^0_\mu(t,0) + \mathcal{V}^0_\mu(t,y),\mu,0)\hat v(t,y) + r(t,y), \quad (t,y)\in\mathbb{R}\times(0,\infty),\\
&\hat v(t,0) = \hat v(t,\infty) = 0, \quad t\in\mathbb{R},\\
&\hat v(t+1,y) = \hat v(t,y), \quad (t,y)\in\mathbb{R}\times(0,\infty),
\end{aligned}\right\} \tag{4.57}$$

where

$$|r(t,y)| \le \text{const}\, \mu e^{-\kappa y/2} \quad \text{for} \quad y \ge 0.$$

Preceding application of Theorem 2.1 implies that the linearized operator $\partial_u F_\mu(u^0_\mu)$ is bijective from U^0 onto V. On the other hand, because of the estimate (4.52), for $\mu \to 0$ it is asymptotically close in the operator norm to the differential operator from problem (4.57). Thus, we obtain

$$\|\hat v\|_\mu \le \text{const}\, |r|_\mu = O(\sqrt{\mu}),$$

what yields the first part of the estimate (4.54) concerning $\mathcal{V}^1_\mu$. ■

Remark 4.8 *Function $\mathcal{B}_{\mu,\nu}$ determined by formula (4.40), in general, does not satisfy boundary conditions (1.2) exactly. Therefore, in the proof of Theorem 1.1 we use its modification*

$$\mathcal{B}_{\mu,\nu}(t,x) \mapsto \mathcal{B}_{\mu,\nu}(t,x) - \mathcal{B}_{\mu,\nu}(t,0) - (\mathcal{B}_{\mu,\nu}(t,1) - \mathcal{B}_{\mu,\nu}(t,0))\, x.$$

The modified function $\mathcal{B}_{\mu,\nu}$ satisfies boundary conditions (1.2) automatically. Moreover, Propositions 4.1, 4.6 and 4.7 imply that asymptotic estimates (4.32) and (4.33) are also fulfilled for it.

5 Coercivity estimates

Throughout this section we suppose that u^0, v^0 and w^0 are functions satisfying assumptions (1.4) and (1.6)–(1.8). Below we prove a series of coercivity estimates which are used to justify the construction of the improved approximate solutions $\mathcal{A}_{\mu,\nu}$, $\mathcal{B}_{\mu,\nu}$ and, hence, to prove our main Theorem 1.1.

Lemma 5.1 *There exist $\kappa_* > 0$ and $c > 0$ such that for all $\kappa \in [0, \kappa_*]$ and for all compactly supported functions $u \in C^2([0,\infty))$ with $u(0) = 0$ and for all $t \in [0,1]$ it holds*

$$\begin{aligned} &\int_0^\infty \left(\partial_y^2 u^2 + \partial_y u^2 + u^2\right) e^{\kappa y} dy \\ &\le c \int_0^\infty \left(\partial_y^2 u + \partial_u f(t, 0, u^0(t,0) + v^0(t,y), 0, 0) u\right)^2 e^{\kappa y} dy \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\int_0^\infty \left(\partial_y^2 u^2 + \partial_y u^2 + u^2\right) e^{\kappa y} dy \\ &\le c \int_0^\infty \left(\partial_y^2 u + \partial_u f(t, 1, u^0(t,1) + w^0(t,y), 0, 0) u\right)^2 e^{\kappa y} dy. \end{aligned} \tag{5.2}$$

Proof: Let us consider the inequality (5.1) with $\kappa = 0$. In order to prove it, it is enough to demonstrate that the linear differential operator

$$M_t v = \frac{d^2 v}{dy^2} + \partial_u f(t, 0, u^0(t,0) + v^0(t,y), 0, 0) v$$

is an isomorphism from $W^{2,2}(0,\infty) \cap W_0^{1,2}(0,\infty)$ onto $L^2(0,\infty)$ for all $t \in [0,1]$. This will imply the existence of the inverse operators M_t^{-1}, and hence the inequality (5.1) with $\kappa = 0$ and

$$c = c_0 := \sup_{t \in [0,1]} \left\| M_t^{-1} \right\|^2,$$

where the norms $\left\| M_t^{-1} \right\|$ are uniformly bounded because the operators M_t depend continuously on the parameter t.

Our proof consists of two steps. First we show that M_t is a Fredholm operator of index zero. Then we demonstrate that it is injective. For the first step, we rewrite M_t in the form

$$M_t = M_t^0 + \left(M_t - M_t^0\right), \quad \text{where} \quad M_t^0 := \frac{d^2}{dy^2} + \partial_u f(t,0,u^0(t,0),0,0),$$

and show that M_t^0 is invertible and $M_t - M_t^0$ is a compact operator.

It is well-known that the differential operator M_t^0 with $\partial_u f(t,0,u^0(t,0),0,0) < 0$ (cf. (1.4)) is an isomorphism from $W^{2,2}(\mathbb{R})$ onto $L^2(\mathbb{R})$, see [14]. Taking into account the orthogonal decomposition into subspaces of even and odd functions

$$W^{2,2}(\mathbb{R}) = W^{2,2}_{\text{even}}(\mathbb{R}) \oplus W^{2,2}_{\text{odd}}(\mathbb{R}), \qquad L^2(\mathbb{R}) = L^2_{\text{even}}(\mathbb{R}) \oplus L^2_{\text{odd}}(\mathbb{R}),$$

we easily verify that M_t^0 is also an isomorphism from $W^{2,2}_{\text{odd}}(\mathbb{R})$ onto $L^2_{\text{odd}}(\mathbb{R})$. On the other hand, the restriction of $W^{2,2}_{\text{odd}}(\mathbb{R})$ to the half-line $(0,\infty)$ coincides with the Sobolev space $W^{2,2}(0,\infty) \cap W^{1,2}_0(0,\infty)$, whereas the restriction of $L^2_{\text{odd}}(\mathbb{R})$ to $(0,\infty)$ coincides with $L^2(0,\infty)$. Therefore, due to the local character of differential operator M_t^0, it is an isomorphism from $W^{2,2}(0,\infty) \cap W^{1,2}_0(0,\infty)$ onto $L^2(0,\infty)$.

The difference $M_t - M_t^0$ is a compact multiplication operator from $W^{2,2}(0,\infty)$ to $L^2(0,\infty)$, because of the Kolmogorov-Riesz compactness theorem (cf. [10]) and the estimate

$$\partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0) - \partial_u f(t,0,u^0(t,0),0,0) \to 0 \quad \text{for} \quad y \to \infty,$$

following from Remark 1.3.

We have proved that operator M_t is Fredholm of index zero. Now, let us show that it is injective. Let u be an element of the kernel of operator M_t. The u is C^2-smooth and

$$\frac{d^2u}{dy^2} + \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)u = 0, \quad y \in (0,\infty),$$

and

$$u(0) = 0. \tag{5.3}$$

From (4.28) it follows that u is a scalar multiple of $\partial_y v^0(t,\cdot)$. But $\partial_y v^0(t,0) \neq 0$ (cf. (1.8)), hence (5.3) implies $u = 0$.

We have justified inequality (5.1) for $\kappa = 0$. Let us write it for a function u of the form $u = e^{\kappa y/2} v$ where $v \in C^2([0,\infty))$ has compact support and satisfies $v(0) = 0$, then we obtain

$$\begin{aligned} &\int_0^\infty \left(\left(\partial_y^2 v + \kappa \partial_y v + \frac{\kappa^2}{4} v \right)^2 + \left(\partial_y u + \frac{\kappa}{2} v \right)^2 + v^2 \right) e^{\kappa y} dy \\ &\le c_0 \int_0^\infty \left(\partial_y^2 v + \kappa \partial_y v + \frac{\kappa^2}{4} v + \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0) v \right)^2 e^{\kappa y} dy. \end{aligned} \tag{5.4}$$

Using Cauchy-Schwarz and Young inequalities, it is easy to verify that there exist constants $c_1, c_2 > 0$ depending on the L^∞-estimate of $\partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)$ only, such that for all $|\kappa| \le 1$ we have

$$\begin{aligned} &(1 - c_1|\kappa|) \int_0^\infty \left(\partial_y^2 v^2 + \partial_y u^2 + v^2 \right) e^{\kappa y} dy \\ &\le \int_0^\infty \left(\left(\partial_y^2 v + \kappa \partial_y v + \frac{\kappa^2}{4} v \right)^2 + \left(\partial_y u + \frac{\kappa}{2} v \right)^2 + v^2 \right) e^{\kappa y} dy \end{aligned}$$

and

$$\int_0^\infty \left(\partial_y^2 v + \kappa\partial_y v + \frac{\kappa^2}{4} v + \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)v\right)^2 e^{\kappa y} dy$$
$$\le \int_0^\infty \left(\partial_y^2 v + \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)v\right)^2 e^{\kappa y} dy + c_2|\kappa| \int_0^\infty \left(\partial_y^2 v^2 + \partial_y u^2 + v^2\right) e^{\kappa y} dy.$$

Combining this result with formula (5.4), for sufficiently small κ we obtain inequality (5.1) where c depends on c_0, c_1, c_2 and κ_*.

The inequality (5.2) can be proved analogously. ■

Lemma 5.2 *There exist* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for all* $\mu, \nu \in (0, \varepsilon_0)$*, for all* $u \in C^2([0,1])$ *with* $u(0) = u(1) = 0$ *and for all* $t \in [0,1]$ *it holds*

$$\int_0^1 \left(\nu^4 u''(x)^2 + \nu^2 u'(x)^2 + u(x)^2\right) dx \le c \int_0^1 \left(\nu^2 u''(x) + \partial_u f(t,x,u_\nu(t,x),\mu,\nu)u(x)\right)^2 dx. \quad (5.5)$$

Proof: Suppose the contrary. Then there exist sequences $\mu_n, \nu_n \in (0,1]$, $t_n \in [0,1]$ and $u_n \in C^2([0,1])$, $n = 1, 2, \ldots$, with

$$u_n(0) = u_n(1) = 0 \quad (5.6)$$

such that

$$\int_0^1 \left(\nu_n^4 u_n''(x)^2 + \nu_n^2 u_n'(x)^2 + u_n(x)^2\right) dx = 1 \quad (5.7)$$

and

$$\mu_n + \nu_n + \int_0^1 \left(\nu_n^2 u_n''(x) + \partial_u f(t_n,x,u_{\nu_n}(t_n,x),\mu_n,\nu_n)u_n(x)\right)^2 dx \to 0.$$

Without loss of generality we can assume that $t_n \to t_* \in [0,1]$. Then the smoothness of functions f and u_ν implies

$$\int_0^1 \left(\left(\partial_u f(t_n,x,u_{\nu_n}(t_n,x),\mu_n,\nu_n) - \partial_u f(t_*,x,u_{\nu_n}(t_*,x),0,0)\right) u_n(x)\right)^2 dx \to 0.$$

Hence,

$$\nu_n + \int_0^1 \left(\nu_n^2 u_n''(x) + \partial_u f(t_*,x,u_{\nu_n}(t_*,x),0,0)u_n(x)\right)^2 dx \to 0. \quad (5.8)$$

Our strategy is to show that (5.7) and (5.8) imply a contradiction. This can be done in three steps. In the first step, we will consider two auxiliary sequences

$$v_n(y) := \sqrt{\nu_n}u_n(\nu_n y)\chi(\nu_n y)e^{-\kappa y} \quad \text{and} \quad w_n(y) := \sqrt{\nu_n}u_n(1-\nu_n y)\chi(\nu_n y)e^{-\kappa y}, \quad (5.9)$$

where $\kappa > 0$ is a parameter to be chosen later, and $\chi : [0,\infty) \to [0,\infty)$ is a smooth cut-off function such that

$$\chi(x) = 1 \quad \text{for} \quad 0 \le x \le 1/4, \quad \text{and} \quad \chi(x) = 0 \quad \text{for} \quad x \ge 1/2. \quad (5.10)$$

It will be shown that for every fixed $R > 0$ we have

$$\int_0^R v_n(y)^2 dy + \int_0^R w_n(y)^2 dy \to 0. \tag{5.11}$$

In the second step, we will use this limit to verify that

$$\int_0^1 \left(\left(\partial_u f(t_*, x, u^0_{\nu_n}(t_*, x), 0, 0) - \partial_u f(t_*, x, u^0(t_*, x), 0, 0)\right) u_n(x)\right)^2 dx \to 0. \tag{5.12}$$

From (5.8) and (5.12) follows

$$\int_0^1 \left(\nu_n^2 u_n''(x) + \partial_u f(t_*, x, u^0(t_*, x), 0, 0) u_n(x)\right)^2 dx \to 0 \tag{5.13}$$

with a strictly positive coefficient at u_n (see (1.4)). In the third and last step, we will transform (5.13) into

$$\int_0^1 \left(\nu_n^4 u_n''(x)^2 + \nu_n^2 u_n'(x)^2 + u_n(x)^2\right) dx \to 0. \tag{5.14}$$

This will be a contradiction to the original assumption (5.7).

Step 1. From (5.7) it follows that the functions v_n and w_n defined by (5.9) constitute bounded sequences in the Hilbert space $W^{2,2}(0,\infty)$. Hence, without loss of generality we can assume that there exist $v_*, w_* \in W^{2,2}(0,\infty)$ such that

$$v_n \rightharpoonup v_* \quad \text{and} \quad w_n \rightharpoonup w_* \quad \text{in} \quad W^{2,2}(0,\infty). \tag{5.15}$$

Because of the compact embedding $W^{1,2}(0,R) \hookrightarrow L^2(0,R)$, for proving (5.11) it remains to show that $v_* = w_* = 0$. For the sake of brevity we will prove $v_* = 0$ only. The condition $w_* = 0$ can be verified analogously.

Take a smooth compactly supported test function $\eta : [0,\infty) \to \mathbb{R}$. Take $R > 0$ sufficiently large such that $\operatorname{supp} \eta \subseteq [0, R]$. Then (1.9), (1.14), (5.10), (5.8), (5.9) and (5.15) yield

$$\begin{aligned}
0 &= \lim_{n\to\infty} \int_0^1 \chi(x) \nu_n^{-1/2} e^{-\kappa x/\nu_n} (\nu_n^2 u_n''(x) + \partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) u_n(x)) \eta\left(\frac{x}{\nu_n}\right) dx \\
&= \lim_{n\to\infty} \int_0^{1/4} \nu_n^{-1/2} e^{-\kappa x/\nu_n} (\nu_n^2 u_n''(x) + \partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) u_n(x)) \eta\left(\frac{x}{\nu_n}\right) dx \\
&= \lim_{n\to\infty} \int_0^{1/(4\nu_n)} \left(e^{-\kappa y} \left(v_n(y) e^{\kappa y}\right)'' + \partial_u f(t_*, \nu_n y, u_{\nu_n}(t_*, \nu_n y), 0, 0) v_n(y)\right) \eta(y) dy \\
&= \lim_{n\to\infty} \int_0^R \left(v_n''(y) + 2\kappa v_n'(y) + \kappa^2 v_n(y) + \partial_u f(t_*, \nu_n y, u_{\nu_n}(t_*, \nu_n y), 0, 0) v_n(y)\right) \eta(y) dy \\
&= \lim_{n\to\infty} \int_0^R \left(v_n''(y) + 2\kappa v_n'(y) + \kappa^2 v_n(y) + \partial_u f(t_*, 0, u^0(t_*, 0) + v^0(t_*, y), 0, 0) v_n(y)\right) \eta(y) dy \\
&= \int_0^\infty \left(v_*''(y) + 2\kappa v_*'(y) + \kappa^2 v_*(y) + \partial_u f(t_*, 0, u^0(t_*, 0)) + v^0(t_*, y), 0, 0) v_*(y)\right) \eta(y) dy.
\end{aligned}$$

In other words: v_* is a weak and, hence, classical solution to the linear homogeneous ODE

$$v_*''(y) + 2\kappa v_*'(y) + (\partial_u f(t_*, 0, u^0(t_*, 0) + v^0(t_*, y), 0, 0) + \kappa^2) v_*(y) = 0. \tag{5.16}$$

Moreover, from the compact embedding $W^{1,2}(0,1) \hookrightarrow C([0,1])$, (5.6), (5.9) and (5.15) we have

$$v_*(0) = 0. \tag{5.17}$$

If κ is chosen small enough, then from (4.28) it follows that v_* is a scalar multiple of $\partial_y v^0(t_*, \cdot)$. But $\partial_y v^0(t_*, 0) \neq 0$ (cf. (1.8)), hence (5.17) implies $v_* = 0$.

Step 2. Because of (1.9) and the mean value theorem we have

$$\left|\partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) - \partial_u f(t_*, x, u^0(t_*, x), 0, 0)\right| \le \text{const} \left(\left| v^0 \left(t_*, \frac{x}{\nu_n} \right) \right| + \left| w^0 \left(t_*, \frac{1-x}{\nu_n} \right) \right| \right). \tag{5.18}$$

Hence, (1.14) yields

$$\int_{1/4}^{3/4} \left| \left(\partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) - \partial_u f(t_*, x, u^0(t_*, x), 0, 0) \right) u_n(x) \right|^2 dx \to 0$$

and

$$\begin{aligned}
&\int_0^{1/4} \left| \left(\partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) - \partial_u f(t_*, x, u^0(t_*, x), 0, 0) \right) u_n(x) \right|^2 dx \\
&\le \text{const} \left(\int_0^{1/4} \left| v^0 \left(t_*, \frac{x}{\nu_n} \right) u_n(x) \right|^2 dx + o(1) \right) \\
&= \text{const} \left(\int_0^{1/(4\nu_n)} \left| v^0(t_*, y) e^{\kappa y} v_n(y) \right|^2 dy + o(1) \right) \\
&\le \text{const} \left(\int_0^R |v_n(y)|^2 dy + \int_R^\infty \left| v^0(t_*, y) e^{\kappa y} \right|^2 dy + o(1) \right),
\end{aligned} \tag{5.19}$$

where $R > 0$ is arbitrary. Take κ sufficiently small, i.e. $\kappa \in (0, \kappa_0)$ (cf. (1.14)). Let $\gamma > 0$ be arbitrarily given. Then we always can first take R sufficiently large such that

$$\int_R^\infty \left| v^0(t_*, y) e^{\kappa y} \right|^2 dy < \gamma.$$

Then, fixing this R, we can use limit (5.11) in order to find sufficiently large n such that

$$\int_0^R |v_n(y)|^2 dy < \gamma.$$

Thus we proved that the right-hand side of (5.19) tends to zero for $n \to \infty$. Similarly, using limit (5.11) to control the L^2-norms of functions w_n on bounded intervals, we can prove that

$$\int_{3/4}^1 \left| \left(\partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) - \partial_u f(t_*, x, u^0(t_*, x), 0, 0) \right) u_n(x) \right|^2 dx \to 0.$$

Hence, the limit (5.12) holds true.

Step 3. Because of (1.4) and (5.6)–(5.8) we have

$$\int_0^1 \left(\nu_n^2 u_n'(x)^2 + u_n(x)^2\right) dx \le \text{const} \int_0^1 \left(\nu_n^2 u_n'(x)^2 - \partial_u f(t_*, x, u^0(t_*, x), 0, 0) u_n(x)^2\right) dx$$
$$= \text{const} \int_0^1 \left(-\nu_n^2 u_n''(x) - \partial_u f(t_*, x, u^0(t_*, x), 0, 0) u_n(x)\right) u_n(x) dx \to 0.$$

Using this and (5.8) again we get

$$\lim_{n\to 0} \int_0^1 \nu_n^4 u_n''(x)^2 dx = \lim_{n\to 0} \int_0^1 \left(\nu_n^2 u_n''(x) + \partial_u f(t_*, x, u^0(t_*, x), 0, 0) u_n(x)\right)^2 dx = 0.$$

Thus we have a contradiction with (5.7). ■

Lemma 5.3 *There exist* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for all* $\mu, \nu \in (0, \varepsilon_0)$ *and for all* $u \in C^2(\mathbb{R} \times [0,1])$ *with* $u(t,0) = u(t,1) = 0$ *and* $u(t+1,x) = u(t,x)$ *for all* $t \in \mathbb{R}$ *and* $x \in [0,1]$ *it holds*

$$\int_0^1 \int_0^1 \left(\mu^2 \partial_t u^2 + \nu^4 \partial_x^2 u^2 + \nu^2 \partial_x u^2 + u^2\right) dt dx$$
$$\le c \int_0^1 \int_0^1 \left(\mu \partial_t u - \nu^2 \partial_x^2 u - \partial_u f(t, x, u_\nu(t,x), \mu, \nu) u\right)^2 dt dx. \tag{5.20}$$

Proof: Take $u \in C^2(\mathbb{R} \times [0,1])$ with $u(t,0) = u(t,1) = 0$ and $u(t+1,x) = u(t,x)$ for all $t \in \mathbb{R}$ and $x \in [0,1]$. Then

$$\int_0^1 \int_0^1 \left(\mu \partial_t u - \nu^2 \partial_x^2 u - \partial_u f(t, x, u_\nu(t,x), 0, 0) u\right)^2 dt dx$$
$$= \int_0^1 \int_0^1 \left(\mu^2 \partial_t u^2 + \left(\nu^2 \partial_x^2 u + \partial_u f(t, x, u_\nu(t,x), 0, 0) u\right)^2\right) dt dx$$
$$-2 \int_0^1 \int_0^1 \left(\mu\nu^2 \partial_t u \, \partial_x^2 u + \mu\nu^2 \partial_t u \, \partial_u f(t, x, u_\nu(t,x), 0, 0) u\right) dt dx$$

and

$$\int_0^1 \int_0^1 \partial_t u \, \partial_x^2 u \, dt dx = -\int_0^1 \int_0^1 u \, \partial_x^2 \partial_t u \, dt dx = \int_0^1 \int_0^1 \partial_x u \, \partial_x \partial_t u \, dt dx = 0$$

and

$$\left| \int_0^1 \int_0^1 \partial_t u \, \partial_u f(t, x, u_\nu(t,x), 0, 0) u \, dt dx \right| = \frac{1}{2} \left| \int_0^1 \int_0^1 u^2 \frac{d}{dt} \partial_u f(t, x, u_\nu(t,x), 0, 0) \, dt dx \right|$$
$$\le \text{const} \int_0^1 \int_0^1 u^2 dt dx.$$

Hence, Lemma 5.2 yields

$$\int_0^1\int_0^1 \left(\mu\partial_t u - \nu^2\partial_x^2 u - \partial_u f(t,x,u_\nu(t,x),0,0)u\right)^2 dtdx$$
$$\geq c_1\left(\int_0^1\int_0^1 \left(\mu^2\partial_t u^2 + \nu^4\partial_x^2 u^2 + \nu^2\partial_x u^2 + u^2\right) dtdx - c_2\mu\int_0^1\int_0^1 u^2 dtdx\right)$$

with positive constants c_1 and c_2 which do not depend on μ, ν and u. Taking into account that

$$\int_0^1\int_0^1 \left(\partial_u f(t,x,u_\nu(t,x),\mu,\nu) - \partial_u f(t,x,u_\nu(t,x),0,0)\right)^2 u^2 dtdx \leq \text{const }(\mu+\nu)\int_0^1\int_0^1 u^2 dtdx,$$

we can choose μ and ν sufficiently small such that (5.20) holds true. ■

Lemma 5.4 *There exist $\varepsilon_0 > 0$ and $c > 0$ such that for all $\mu,\nu \in (0,\varepsilon_0)$, for all $u \in C^2([0,1])$ with $u(0) = u(1) = 0$ and for all $t \in [0,1]$ it holds*

$$\nu^2\|u''\|_\infty + \nu\|u'\|_\infty + \|u\|_\infty \leq c \max_{0\leq x\leq 1}\left|\nu^2 u''(x) + \partial_u f(t,x,u_\nu(t,x),\mu,\nu)u(x)\right|. \tag{5.21}$$

Proof: Similar to Lemma 5.2, we use a proof by contradiction. Suppose that (5.21) is not true, then there exist sequences $\mu_n,\nu_n \in (0,1]$, $t_n \in [0,1]$ and $u_n \in C^2([0,1])$, $n = 1,2,\ldots$, with

$$u_n(0) = u_n(1) = 0 \tag{5.22}$$

such that

$$\nu_n^2\|u_n''\|_\infty + \nu_n\|u_n'\|_\infty + \|u_n\|_\infty = 1 \tag{5.23}$$

and

$$\mu_n + \nu_n + \max_{0\leq x\leq 1}\left|\nu_n^2 u_n''(x) + \partial_u f(t_n,x,u_{\nu_n}(t_n,x),\mu_n,\nu_n)u_n(x)\right| \to 0.$$

Without loss of generality we can assume that $t_n \to t_* \in [0,1]$ and

$$\nu_n + \max_{0\leq x\leq 1}\left|\nu_n^2 u_n''(x) + \partial_u f(t_*,x,u_{\nu_n}(t_*,x),0,0)u_n(x)\right| \to 0. \tag{5.24}$$

We aim to show that (5.23) and (5.24) imply a contradiction.

Step 1. Let us consider two auxiliary sequences

$$\tilde{v}_n(y) := u_n(\nu_n y)\chi(\nu_n y)e^{-\kappa y} \quad \text{and} \quad \tilde{w}_n(y) := u_n(1-\nu_n y)\chi(\nu_n y)e^{-\kappa y}, \tag{5.25}$$

where $\kappa > 0$ is a parameter to be chosen later, and χ is a smooth cut-off function satisfying (5.10). From (5.23) it follows that the functions $\tilde{v}_n$ and $\tilde{w}_n$ constitute bounded sequences in the Hilbert space $W^{2,2}(0,\infty)$. Hence, without loss of generality we can assume that there exist $\tilde{v}_*, \tilde{w}_* \in W^{2,2}(0,\infty)$ such that

$$\tilde{v}_n \rightharpoonup \tilde{v}_* \quad \text{and} \quad \tilde{w}_n \rightharpoonup \tilde{w}_* \quad \text{in} \quad W^{2,2}(0,\infty). \tag{5.26}$$

If we show that $\tilde{v}_* = \tilde{w}_* = 0$, then for any fixed $R > 0$, because of the compact embedding $W^{1,2}(0, R) \hookrightarrow C([0, R])$, we obtain

$$\max_{0\le y\le R} |\tilde{v}_n(y)| + \max_{0\le y\le R} |\tilde{w}_n(y)| \to 0. \tag{5.27}$$

For the sake of brevity we will prove $\tilde{v}_* = 0$ only. The condition $\tilde{w}_* = 0$ can be verified analogously.

Take a smooth compactly supported test function $\eta : [0, \infty) \to \mathbb{R}$. Take $R > 0$ sufficiently large such that $\operatorname{supp} \eta \subseteq [0, R]$. Then (1.9), (1.14), (5.10), (5.24)–(5.26) yield

$$\begin{aligned}
0 &= \lim_{n\to\infty} \int_0^1 \chi(x)\nu_n^{-1} e^{-\kappa x/\nu_n} (\nu_n^2 u_n''(x) + \partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0)u_n(x))\eta\left(\frac{x}{\nu_n}\right) dx \\
&= \lim_{n\to\infty} \int_0^{1/4} \nu_n^{-1} e^{-\kappa x/\nu_n} (\nu_n^2 u_n''(x) + \partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0)u_n(x))\eta\left(\frac{x}{\nu_n}\right) dx \\
&= \lim_{n\to\infty} \int_0^{1/(4\nu_n)} \left(e^{-\kappa y} \left(\tilde{v}_n(y) e^{\kappa y}\right)'' + \partial_u f(t_*, \nu_n y, u_{\nu_n}(t_*, \nu_n y), 0, 0)\tilde{v}_n(y)\right) \eta(y) dy \\
&= \lim_{n\to\infty} \int_0^{R} \left(\tilde{v}_n''(y) + 2\kappa \tilde{v}_n'(y) + \kappa^2 \tilde{v}_n(y) + \partial_u f(t_*, \nu_n y, u_{\nu_n}(t_*, \nu_n y), 0, 0)\tilde{v}_n(y)\right) \eta(y) dy \\
&= \lim_{n\to\infty} \int_0^{R} \left(\tilde{v}_n''(y) + 2\kappa \tilde{v}_n'(y) + \kappa^2 \tilde{v}_n(y) + \partial_u f(t_*, 0, u^0(t_*, 0) + v^0(t_*, y), 0, 0)\tilde{v}_n(y)\right) \eta(y) dy \\
&= \int_0^{\infty} \left(\tilde{v}_*''(y) + 2\kappa \tilde{v}_*'(y) + \kappa^2 \tilde{v}_*(y) + \partial_u f(t_*, 0, u^0(t_*, 0)) + v^0(t_*, y), 0, 0)\tilde{v}_*(y)\right) \eta(y) dy.
\end{aligned}$$

In other words: $\tilde{v}_*$ is a weak and, hence, classical solution to the linear homogeneous ODE

$$\tilde{v}_*''(y) + 2\kappa \tilde{v}_*'(y) + (\partial_u f(t_*, 0, u^0(t_*, 0) + v^0(t_*, y), 0, 0) + \kappa^2)\tilde{v}_*(y) = 0.$$

For sufficiently small κ, the latter equation has only trivial solution satisfying boundary conditions $\tilde{v}_*(0) = \tilde{v}_*(\infty) = 0$ (see discussion of the equation (5.16) in the proof of Lemma 5.2), therefore we conclude $\tilde{v}_* = 0$.

Step 2. Next, we show that

$$\max_{0\le x\le 1} \left|\left(\partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) - \partial_u f(t_*, x, u^0(t_*, x), 0, 0)\right) u_n(x)\right| \to 0. \tag{5.28}$$

From (5.25) and (5.27), for any fixed $R > 0$ we have

$$\max_{0\le x\le \nu_n R} |u_n(x)| + \max_{0\le x\le \nu_n R} |u_n(1 - x)| \to 0,$$

therefore estimates (1.14) and assumption (5.23) imply

$$\max_{0\le x\le 1} \left|v^0\left(t_*, \frac{x}{\nu_n}\right) u_n(x)\right| + \max_{0\le x\le 1} \left|w^0\left(t_*, \frac{1-x}{\nu_n}\right) u_n(x)\right| \to 0.$$

Hence, because of the mean value estimate (5.18), we get (5.28).

Step 3. From (5.24) and (5.28) follows

$$\max_{0\le x\le 1} \left|\nu_n^2 u_n''(x) + \partial_u f(t_*, x, u^0(t_*, x), 0, 0)u_n(x)\right| \to 0.$$

Therefore, because of $\partial_u f(t_*, x, u^0(t_*, x), 0, 0) < 0$ (see assumption (1.4)), the strong maximum principle yields

$$\|u_n\|_\infty \to 0.$$

Moreover, using (5.24) we also obtain

$$\begin{aligned}\nu_n^2 \|u_n''\|_\infty &\le \max_{0\le x\le 1} \left|\nu_n^2 u_n'' + \partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) u_n(x)\right| \\ &+ \max_{0\le x\le 1} \left|\partial_u f(t_*, x, u_{\nu_n}(t_*, x), 0, 0) u_n(x)\right| \to 0.\end{aligned}$$

Applying interpolation inequality for C-spaces (see, for example, Lemma 6.3.1 in [14]) we also get

$$\nu_n \|u_n'\|_\infty \to 0,$$

and hence the contradiction with (5.23). ■

Lemma 5.5 *There exist $\kappa_* > 0$, $\mu_0 > 0$ and $c > 0$ such that for all $\kappa \in [0, \kappa_*]$, $\mu \in (0, \mu_0)$ and for all $u \in C^2(\mathbb{R} \times [0,\infty))$ with $u(t, 0) = 0$ and $u(t + 1, y) = u(t, y)$ for all $(t, x) \in \mathbb{R} \times [0,\infty)$ and such that $u(t, \cdot)$ has compact support for all $t \in \mathbb{R}$ it holds*

$$\begin{aligned}&\int_0^1 \int_0^\infty \left(\mu^2 \partial_t u^2 + \partial_y^2 u^2 + \partial_y u^2 + u^2\right) e^{\kappa y} dt dy \\ &\le c \int_0^1 \int_0^\infty \left(\mu \partial_t u - \partial_y^2 u - \partial_u f(t, 0, u^0(t, 0) + v^0(t, y), \mu, \nu) u\right)^2 e^{\kappa y} dt dy. \qquad (5.29)\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 \int_0^\infty \left(\mu^2 \partial_t u^2 + \partial_y^2 u^2 + \partial_y u^2 + u^2\right) e^{\kappa y} dt dy \\ &\le c \int_0^1 \int_0^\infty \left(\mu \partial_t u - \partial_y^2 u - \partial_u f(t, 1, u^0(t, 1) + w^0(t, y), \mu, \nu) u\right)^2 e^{\kappa y} dt dy. \qquad (5.30)\end{aligned}$$

Proof: We will prove inequality (5.29) only. Inequality (5.30) can be considered analogously.

Let us start with the case $\kappa = 0$. Take $u \in C^2(\mathbb{R} \times [0,\infty))$ with $u(t, 0) = 0$ and $u(t + 1, y) = u(t, y)$ for all $t \in \mathbb{R}$ and $y \in [0,\infty)$ and such that $u(t, \cdot)$ has compact support for all $t \in \mathbb{R}$. Then

$$\begin{aligned}&\int_0^1 \int_0^\infty \left(\mu \partial_t u - \partial_y^2 u - \partial_u f(t, 0, u^0(t, 0) + v^0(t, y), 0, 0) u\right)^2 dt dy \\ &= \int_0^1 \int_0^\infty \left(\mu^2 \partial_t u^2 + \left(\partial_y^2 u + \partial_u f(t, 0, u^0(t, 0) + v^0(t, y), 0, 0) u\right)^2\right) dt dy \\ &-2 \int_0^1 \int_0^\infty \left(\mu \partial_t u \, \partial_y^2 u + \mu \partial_t u \, \partial_u f(t, 0, u^0(t, 0) + v^0(t, y), 0, 0) u\right) dt dy\end{aligned}$$

and

$$\int_0^1 \int_0^\infty \partial_t u \, \partial_y^2 u \, dt dy = -\int_0^1 \int_0^\infty u \, \partial_y^2 \partial_t u \, dt dy = \int_0^1 \int_0^\infty \partial_y u \, \partial_y \partial_t u \, dt dy = 0$$

and

$$\left|\int_0^1\int_0^\infty \partial_t u\ \partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)u\ dtdy\right|$$
$$=\frac{1}{2}\left|\int_0^1\int_0^\infty u^2\frac{d}{dt}\partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)\ dtdy\right|\le \text{const}\int_0^1\int_0^\infty u^2dtdy.$$

Hence, Lemma 5.1 yields

$$\int_0^1\int_0^\infty \left(\mu\partial_t u-\partial_y^2 u-\partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)u\right)^2 dtdy$$
$$\ge c_1\left(\int_0^1\int_0^\infty \left(\mu^2\partial_t u^2+\partial_y^2 u^2+\partial_y u^2+u^2\right)dtdy-c_2\mu\int_0^1\int_0^\infty u^2dtdy\right)$$

with positive constants c_1 and c_2 which do not depend on μ, ν and u. Taking into account that

$$\int_0^1\int_0^\infty \left(\partial_u f(t,0,u^0(t,0)+v^0(t,y),\mu,\nu)-\partial_u f(t,0,u^0(t,0)+v^0(t,y),0,0)\right)^2 u^2dtdy$$
$$\le \text{const}\ (\mu+\nu)\int_0^1\int_0^\infty u^2dtdy,$$

we can choose μ and ν sufficiently small such that (5.29) holds true.

Inequality (5.29) for non-zero but sufficiently small κ can be justified if we take $u=e^{\kappa/2}v$ and analyze the resulting expression by analogy with the inequality (5.4) in Lemma 5.1. ■

Lemma 5.6 *There exist $\mu_0>0$ and $c>0$ such that for all $\mu\in(0,\mu_0)$, for all $x\in[0,1]$ and for all $h\in C([0,1])$ with $h(0)=h(1)$, the linear differential equation*

$$\mu u'(t)-\partial_u f(t,x,u^0(t,x),\mu,0)u(t)=h(t) \tag{5.31}$$

has a unique 1-periodic solution $u\in C^1([0,1])$ and it holds

$$\mu\|u'\|_\infty+\|u\|_\infty\le c\|h\|_\infty. \tag{5.32}$$

Proof: Assumption (1.4) implies that the Floquet exponent corresponding to (5.31)

$$Q=\frac{1}{\mu}\int_0^1 \partial_u f(t,x,u^0(t,x),\mu,0)dt$$

is non-degenerate, at least, for sufficiently small $|\mu|\neq 0$. Therefore, one can explicitely verify that in this case equation (5.31) has a unique 1-periodic solution $u\in C^1([0,1])$ determined by the Green's formula

$$u(t)=\int_0^1 G(t,s)h(s)ds$$

with

$$G(t,s) = -\frac{1}{2\mu}\exp\left(\frac{1}{\mu}\int_s^t \partial_u f(t,x,u^0(\xi,x),\mu,0)d\xi - \frac{Q}{2}\,\mathrm{sign}(t-s)\right)/\sinh(Q/2).$$

Function G is sign-preserving and satisfies the identity

$$\int_0^1 G(t,s)\partial_u f(t,x,u^0(s,x),\mu,0)ds = -1,$$

following from the fact that $u(t) = -1$ is a solution to equation (5.31) for $h(t) = \partial_u f(t,x,u^0(t,x),\mu,0)$. Using these properties, we obtain a pointwise estimate

$$|u(t)| = \left|\int_0^1 G(t,s)\partial_u f(t,x,u^0(s,x),\mu,0)\frac{h(s)}{\partial_u f(t,x,u^0(s,x),\mu,0)}ds\right| \le \mathrm{const}\|h\|_\infty \quad \text{for all} \quad t\in[0,1],$$

which together with the inequality

$$|\mu u'(t)| \le |\mu u'(t) - \partial_u f(t,x,u^0(t,x),\mu,0)u(t)| + |\partial_u f(t,x,u^0(t,x),\mu,0)|\,|u(t)|$$

yields the announced coercivity estimate. ■

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