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The Widom-Rowlinson model under spin flip: Immediate loss and sharp recovery of quasilocality

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Abstract

We consider the continuum Widom-Rowlinson model under independent spin-flip dynamics and investigate whether and when the time-evolved point process has an (almost) quasilocal specification (Gibbs-property of the time-evolved measure). Our study provides a first analysis of a Gibbs-non-Gibbs transition for point particles in Euclidean space. We find a picture of loss and recovery, in which even more regularity is lost faster than it is for time-evolved spin models on lattices.

We show immediate loss of quasilocality in the percolation regime, with full measure of discontinuity points for any specification. For the color-asymmetric percolating model, there is a transition from this non-a.s. quasilocal regime back to an everywhere Gibbsian regime. At the sharp reentrance time $t_G > 0$ the model is a.s. quasilocal. For the color-symmetric model there is no reentrance. On the constructive side, for all $t > t_G$, we provide everywhere quasilocal specifications for the time-evolved measures and give precise exponential estimates on the influence of boundary conditions.

1 Introduction

1.1 Gibbsian point particle systems vs. lattice spin systems

The study of spatial point processes has enjoyed considerable attention in the last years. Point processes appear as models for interacting point particles in mathematical statistical mechanics [36, 21, 10, 14] as a description of gases or fluids. Adding to this, there has been a lot of related activity from stochastic geometry [19, 16, 5, 32, 18] and the introduction of Malliavin calculus [28, 31].

The Gibbsian theory of point particles in infinite Euclidean space presents more subtleties than the theory of lattice systems with uniformly convergent Hamiltonians. The issues existence, uniqueness, phase-transitions, variational principle are all more difficult [10, 23, 9, 30, 27, 20, 2]. Loosely speaking Gibbsian point processes are difficult because there is a priori more chance for unboundedness, condensation, divergences for point particles due to spatial degrees of freedom and less uniformity is in the game. Hence not all lattice results have counterparts in the theory of point particles, and for some issues a canonical setup has yet to be found. In the present paper we are contributing to an understanding of Gibbs theory for point processes and its limits by an investigation of the possibility of Gibbs-non-Gibbs transitions.

Parts of the difficulties of systems of point particles are already present in models of unbounded lattice spins which generically have also an unbounded interaction. Here the theory is less complete than the established theory for uniformly convergent Hamiltonians [13, 15]. There are also some links between unbounded lattice spins and point particles: Some proofs

for measures of point particles proceed by reduction to lattice systems via blocking procedures, see[35, 6, 3].

Gibbs-non-Gibbs transitions appear for lattice spin systems with uniformly convergent Hamiltonians where it has been observed that simple stochastic transformations (like spatial block averaging or stochastic time-evolutions) can produce non-localities which lead to a loss of the Gibbs property for the transformed measure [40]. These non-localities appear in the conditional probabilities to see a configuration in a finite volume as a function of the conditioning outside the finite volume. They provide a strong deviation from the spatial Markov property of the image measure and are signs of a lack of regularity of the time-evolved measure. This is remarkable and may sometimes result in serious consequences, like the failure of variational principle, see [24]. For Gibbsian initial measures they are caused by phase transitions of an internal system, conditional to configurations of the image system we want to study. A different source of non-Gibbsian measures of lattice systems are projections of quantum spin chains [7] where a mechanism of quantum entanglement instead of an internal phase transition is responsible for the appearing non-localities.

It is the aim of the present paper to investigate the Widom-Rowlinson model (WRM) [41] as a prototypical system of Gibbsian point particles in all intensity regimes, under a stochastic time-evolution. To our knowledge this is the first study of Gibbsianness (or quasilocality of conditional probabilities) of a transformed system of point particles.

1.2 Results on the WRM under spin flip.

The continuum WRM is a model for point particles in Euclidean space, each carrying one of two colors (or spins). Point configurations are distributed according to Poisson processes with possibly color-dependent intensities, which are conditioned to distances bigger than a given minimal value $2a$, between particles of different spins. The specification kernels obtained by this procedure are clearly local (in particular quasilocal) as a function of the boundary configuration. It is one the first of a class of models of interacting colored point particles which was proved to have a phase transition at large and equal intensities, in spatial dimensions greater or equal to two [4, 35]. We apply a time-evolution which keeps the positions of the particles but randomly changes the colors according to independent Poissonian clocks. Note that only the initial configurations have to obey the color constraint for overlapping discs, see Figure 1 for an illustration.

We prove that the following scenario of Gibbs-non-Gibbs transitions take place. The main features are illustrated in Figure 2. Suppose the model has symmetric and sufficiently high activities, such that there is an infinite cluster. Then there is an immediate loss of quasilocality for any specification (system of conditional probabilities) of the time-evolved measure which persists for all finite times. Moreover, the set of discontinuity points of any specification has measure one w.r.t. the time-evolved measure: There is no a.s. Gibbsianness, but a.s. non-Gibbsianness. The translation-invariant measures μ_t^+ and μ_t^- obtained by time-evolution of the extremal translation invariant Widom-Rowlinson states $\mu^+ > \mu^-$ have each their own specifications which are different for $t < \infty$.

Still in the symmetric high-activity regime, we consider the limiting measure for $t = \infty$, where

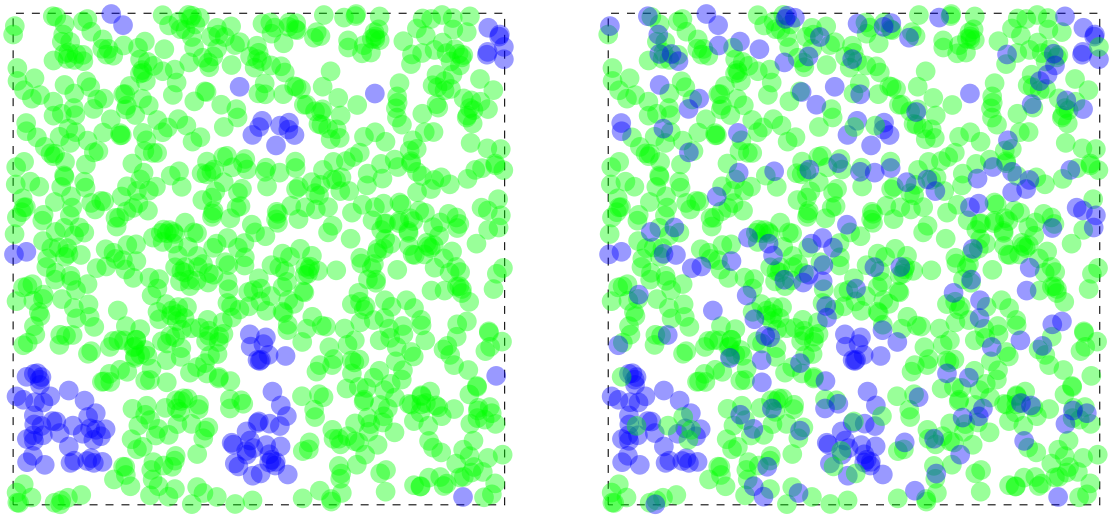

Figure 1: Realization of the WRM in the phase transition regime under independent spin-flip at time zero (left) and for some positive time (right).

we randomly assign colors with equal probability independently of the spatial structure, while keeping the positions fixed. Its internal dependence properties are given by the grey measure which is obtained from the WRM by forgetting the color-assignment and keeping the spatial degrees of freedom only. For this measure we show that it is a.s. non-Gibbs, too. While it is surprising that we even find a full-measure set of bad points, the failure of quasilocality goes in line with examples in which it has been observed that projections (here: to the spatial degrees of freedom) may cause non-localities from Gibbsian measures.

Suppose next that the model has sufficiently high, but different activities, such that there is an infinite cluster. Then we prove that there is a sharp reentrance time $t_G < \infty$ such that the following holds: There is a full-measure set of discontinuity points of any specification for the time-evolved measure for all $t \in (0, t_G)$ and a uniformly quasilocal specification for the time-evolved measure for all $t \in (t_G, \infty]$. For this quasilocal specification we obtain very explicit exponential bounds on the change of the measure in $\Lambda \subset \mathbb{R}^d$ in total variation as a function of the conditioning far away from Λ. At the reentrance time t_G itself, there is non-Gibbsian, but a.s. Gibbsian behavior. We also find a non-percolating small-time regime where almost-sure quasilocality, but not quasilocality everywhere holds, for any specification.

1.3 Lattice spins under time evolution

The time-evolved Ising model in the integer lattice was studied [38]. The authors in particular considered an initial configuration chosen according to a low temperature plus measure of an Ising model in zero magnetic field in the phase transition regime, and considered the symmetric spin-flip dynamics which randomly flips between the two possible spin values plus and minus according to Poissonian clocks, independently over the lattice sites. This paper, in which prototypical behavior of lattice spins under time-evolution was studied for the first time, was very fruitful and stimulating, and it is worth to compare our findings. We see similarities and analogies but also strong differences between point particles and lattice spins like the

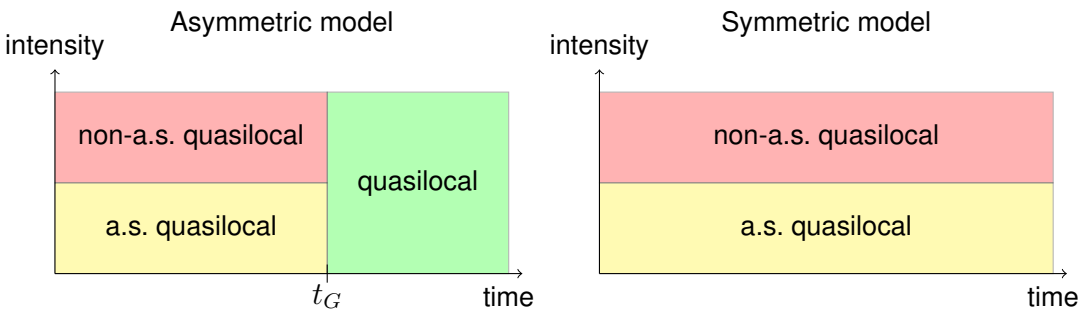

Figure 2: Illustration of Gibbs-non-Gibbs transitions in time and intensity for the WRM under independent spin flip.

Ising model.

Clearly the resulting time-evolved Ising measure μ_t converges locally to the independent symmetric product measure as time goes to infinity. Nevertheless, it was shown that, for large enough t, the conditional probabilities lose the property of quasilocality as a function of the conditioning, at some bad configurations. Different to our findings for the WRM, for small enough t the Gibbs property was proved to be preserved. Indeed, short-time preservation of Gibbsianness is true rather generally [29, 25]. The sharpness of the transition between Gibbs and non-Gibbs at a particular threshold time (excluding multiple ins and outs to Gibbs) was conjectured but not proved.

The hidden phase transitions responsible for the (non-removable) absence of quasilocality of conditional probabilities appear in the infinite system at time zero, conditional on very particular balancing bad configurations which are given at time t. These balancing bad configurations have to be chosen in such a way as to keep the conditional system neutral. An example of such a bad configuration for the time-evolved Ising measure is the plus/minus checkerboard configuration, and the mass of bad configurations w.r.t. the time-evolved measure is zero, so that the time-evolved measure is a.s. Gibbs. In nonzero magnetic field $h > 0$ the situation is different: For large enough t the measure becomes Gibbs again, but sharpness of this reentrance into the Gibbs measures could not be proved.

A similar analysis was carried out for a model of real-valued spins in the phase transition regime under site-wise independent diffusive time-evolution of the spins [26]. We see a picture of short-time preversation of Gibbsianness and loss of the Gibbs property at finite times. As a notable difference to WRM and to the Ising model there is no recovery even in positive magnetic field, which is caused by the unboundedness of spins.

The low-temperature Ising model under spin flip on regular trees was investigated in [37], using entirely different techniques of non-homogeneous tree recursions. As a phenomenon which seems to be possible only for trees the Gibbs-properties depend on the initial Gibbs state: The maximal Gibbs state μ^+ and the Gibbs state obtained with free boundary conditions (which are different on trees) behave very differently under time evolution. The free state has short-time Gibbsianness, but even shows two transitions in time: Non-Gibbsianness with some bad configuration at intermediate time, and full-measure set of discontinuities for large times.

A bulk of related work about Gibb-non-Gibbs transitions under time evolution has appeared [12, 11, 8, 33]. This includes mean-field and Kac-models for which large-deviation techniques lead to variational principles which are more tractable than on the lattice. Compare also the variational approach in path space [39].

1.4 Ideas of Proof.

Our arguments are based on a good understanding of the cluster representation of the conditional probabilities of the time-evolved measure in the form presented in Lemma 3.4. All effects can be seen from here, after suitable limits, where care is needed for the correct treatment of infinite clusters.

We find a number of new physical phenomena due to the spatial degrees of freedom of the colored point cloud which are not present in the Ising model where spatial degrees of freedom are fixed on the lattice. First of all, there is additional complexity due to spatial degrees of freedom: It is possible in the non-quasilocal regime to achieve distortions of the locations of a point cloud in a finite volume w.r.t. the conditional measure by changing only the colors of the conditioning arbitrarily far away and keeping the locations fixed.

The most striking features of our findings about the time-evolved WRM are the immediate loss of quasilocality and the appearance of non almost-sure quasilocality. Both do not appear for the Ising model.

This is best understood on the basis of the cluster representation for conditional probabilities of the time-evolved measure which makes explicit the clusters C of the conditional time-zero model. It allows to see whether transport of information coming from varying boundary conditions far away may (or may not) take place. The perfect color constraint of the WRM keeps a perfectly rigid coupling for the conditional time-zero measure along those clusters. This lossless flow of information is responsible for the immediate loss of Gibbsianness. There are two basic sources for discontinuities of conditional probabilities of the time-evolved measure: These are the color-perturbations far away, keeping cluster structure fixed, and: Spatial perturbations, cutting off an infinite cluster to finite pieces. Both mechanisms assume existence of large clusters, and their absence hence already implies a.s. quasilocality. Color-perturbations in particular allow to show badness in the symmetric high-density regime at any finite time. More than that, they even allow to show badness of any (!) percolating configuration, independently of the coloring. The sharp reentrance time can best be understood in terms of availability of a switch (see Subsection 4.1.2), which describes the interplay between Poisson activities, time, and magnetization at time t on the cluster, and its weight. The form of the switch also explains the immediate loss of quasilocality.

The complete proof of non-existence of an a.s. quasilocal specification in regimes of percolation then also involves a version of conditional probabilities for notably finite clusters (see Proposition 4.11) and a replacement argument for specifications with perturbed conditionings (see Subsection 4.5) which needs a bit more care than for discrete lattice spins. Our proof of existence of a quasilocal specification for $t > t_G$ in arbitrary densities, is constructive (see Propositions 4.4 and 4.5). We define a specification by taking the appropriate formal limit on infinite clusters (see Definition 4.2) and prove specification properties (see Lemma 4.3). The

behavior at the critical times t_G and $t = \infty$ needs modified arguments, in the latter case also involving an argument of cutting off infinite clusters.

1.5 Discussion, generalizations, future research

Summarizing, we have seen that the spin-flip time-evolution of the WRM creates stronger pathologies than it was known from the Ising model on the lattice: It provides the first example of non-a.s. quasilocality created by time evolution (compare however joint measures in [24]). It also provides the first example of immediate loss of quasilocality in non-mean field (for a mean-field example see however the [8]).

How generic are our findings? It would be interesting to change the initial model at time zero to a more general Potts gas model, and see how much of the picture we found in the WRM we can expect to carry over, and what we can expect to be able to prove. We believe that in a finite-range model where the color constraint of the WRM is not strict, there should again be a regime of short-time Gibbsianness w.r.t. τ-topolgy. From a different aspect, working with continuous interactions (as a function of the interparticle distances) would even be nicer, as their corresponding local specifications are Feller for topologies which allow also for spatial variations of points (the vague topology on the positive measures one obtains when one puts a Dirac measure to every particle position) and so there is more regularity in the game. Indeed, the specification of the WRM clearly is non-Feller w.r.t. the vague topology, and the natural topology in which to work for initial measure and also for the time-evolved measure hence is the τ-topology. Next, for models of unbounded range of interactions as starting measures there are new difficulties. For such models it is essential to work with spaces of tempered configurations, with a good definition of temperedness, and a good choice of topology. An analysis would have to start from a generalization of our cluster-representation of the time-evolved conditional probabilities, but this will be more complicated. It could be promising to use continuum percolation tools of [14] in their proof of phase-transitions of general Potts gases in this context, and many interesting challenges and open issues remain.

1.6 Organisation of the manuscript

In Section 2 we present the general framework for Gibbs point processes in Euclidean space and give the definition for Gibbsianness based on the existence of quasilocal specifications. In Section 3 the WRM under independent spin flip is introduced and we give our main results for quasilocal Gibbsianness in time vs. intensity regimes. Section 4 is dedicated to cluster representations of the time-evolved WRM for which we present the properties required for the proofs of the main theorems. All technical proofs are dealt with in Section 5. In the appendix in Section 6 we collect some general results on percolation for the WRM.

2 Gibbs Point Processes

We consider the Euclidean space $\mathbb{R}^d$ with $d \geq 1$ and fix an integer $q \geq 1$. The set $E = \{1, \ldots, q\}$ will play the role of a local state space or in the language of point processes the mark space. Let Ω denote the set of all *locally finite subsets* of $\mathbb{R}^d$, that is, for $\omega \in \Omega$ we have $|\omega_\Lambda| = \#\{\omega \cap \Lambda\} < \infty$ for all bounded sets $\Lambda \subset \mathbb{R}^d$. A configuration of particles with q different colors is given by the vector $\boldsymbol{\omega} = (\omega^{(1)}, \ldots, \omega^{(q)})$ where $\omega^{(i)} \in \Omega$ for all $i \in E$ and $\omega^{(i)} \cap \omega^{(j)} = \emptyset$ for all $i \neq j$. We denote $\boldsymbol{\Omega}$ the set of all *colored configurations*. Let us equip Ω with the σ-algebra $\mathcal{F}$ which is generated by the counting variables $\Omega \ni \omega \mapsto \#(\omega \cap \Lambda)$ for bounded and measurable $\Lambda \Subset \mathbb{R}^d$ and $\boldsymbol{\Omega}$ with the restriction of the product σ-algebra on Ω^q which we denote $\boldsymbol{\mathcal{F}}$. Further we denote by $\boldsymbol{\Omega}_\Lambda$ the set of all colored configurations in the measurable set $\Lambda \subset \mathbb{R}^d$ and equip it with the corresponding σ-algebra $\boldsymbol{\mathcal{F}}_\Lambda$ generated by the counting variables. We write $f \in \boldsymbol{\mathcal{F}}_\Lambda$ if f is measurable w.r.t. $\boldsymbol{\mathcal{F}}_\Lambda$ and $f \in \boldsymbol{\mathcal{F}}_\Lambda^b$ if f is additionally bounded in the supremum norm $\|\cdot\|$. We denote by $\omega = \omega^{(1)} \cup \cdots \cup \omega^{(q)}$ the *grey configuration* of the colored configuration $\boldsymbol{\omega}$. By $\sigma(x) \in E$ we denote the color of the particle $x \in \omega$.

An interaction between particles in $\boldsymbol{\Omega}_\Lambda$ with $\Lambda \Subset \mathbb{R}^d$ and boundary condition $\boldsymbol{\omega}_{\Lambda^c} \in \boldsymbol{\Omega}_{\Lambda^c}$, where $\Lambda^c = \mathbb{R}^d \setminus \Lambda$, is given by the *Hamiltonian*

$$H_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) = \sum_{\eta \Subset \omega_\Lambda \omega_{\Lambda^c} \colon \eta \cap \Lambda \neq \emptyset} \Phi_\eta(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}).$$

where the family of *potentials* Φ_η are measurable functions with values in $\mathbb{R}^d \cup \{\infty\}$, whenever this maybe infinite sum is well defined.

As an example consider the Potts Gas (PG) as presented in [14] where $q \geq 2$ and the potential is given by

$$\Phi_\eta(\boldsymbol{\omega}) = \delta_{\eta=\{x,y\}}[\delta_{\sigma(x)\neq\sigma(y)}\varphi(x-y) + \psi(x-y)] \tag{1}$$

for some measurable and even functions $\varphi, \psi : \mathbb{R}^d \to]-\infty, \infty]$. More precisely, φ is assumed to be positive and finite range and ψ is strongly stable, lower regular and without long-range repulsion, for details see [14, 34]. A special case of the PG for $q = 2$ is the Widom-Rowlinson model (WRM) with $E = \{-1, 1\}$, as presented for example in [4, 41], where $\varphi(x-y) = \infty \times \mathbb{1}_{|x-y|<2a}$ for some parameter $a > 0$ is a hard-core repulsion and $\psi = 0$. The WRM is of finite range with parameter a and satisfies the above mentioned regularity conditions, see [4].

The associated *Gibbsian specification* is given by

$$\gamma_\Lambda(d\boldsymbol{\omega}_\Lambda | \boldsymbol{\omega}_{\Lambda^c}) = \exp(-H_\Lambda(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}))Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c})\boldsymbol{P}_\Lambda(d\boldsymbol{\omega}_\Lambda)$$

where $Z_\Lambda(\boldsymbol{\omega}_{\Lambda^c}) = \int \exp(-H_\Lambda(\tilde{\boldsymbol{\omega}}_\Lambda \boldsymbol{\omega}_{\Lambda^c}))\boldsymbol{P}_\Lambda(d\tilde{\boldsymbol{\omega}}_\Lambda)$ is called the partition function whenever it is well-defined. $\boldsymbol{P}_\Lambda = P_\Lambda^{\lambda_1} \otimes \cdots \otimes P_\Lambda^{\lambda_q}$ here denotes the q-dimensional Poisson point process (PPP) on $\boldsymbol{\Omega}_\Lambda$ with constant intensities $\lambda_1, \ldots, \lambda_q > 0$. That is the measure such that

$$\int dP_\Lambda^\lambda f = e^{-\lambda|\Lambda|} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda^n} dx_1 \cdots dx_n f(\{x_1, \ldots, x_n\})$$

for any bounded and measurable function f on Ω_Λ. In general, a family of proper probability kernels $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{R}^d}$ is called a specification if the following consistency condition is satisfied. For all $\tilde{\boldsymbol{\omega}} \in \boldsymbol{\Omega}$ and measurable $\Lambda \subset \Delta \Subset \mathbb{R}^d$

$$\gamma_\Delta(\gamma_\Lambda(d\boldsymbol{\omega}|\cdot)|\tilde{\boldsymbol{\omega}}) = \gamma_\Delta(d\boldsymbol{\omega}|\tilde{\boldsymbol{\omega}}).$$

In the most general form (see [10]), the set of boundary conditions such that the Hamiltonian H_Λ and Z_Λ are well defined can be characterized as follows. Let $\Phi^- = (-\Phi) \vee 0$, then a configuration $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ is called *admissible* for a region $\Lambda \subset \mathbb{R}^d$, in symbols $\boldsymbol{\omega} \in \boldsymbol{\Omega}^*_\Lambda$, iff

$$H^-_\Lambda(\boldsymbol{\zeta}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) = \sum_{\eta \Subset \zeta_\Lambda \omega_{\Lambda^c} : \eta \cap \Lambda \neq \emptyset} \Phi^-_\eta(\boldsymbol{\zeta}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) < \infty$$

for $\boldsymbol{P}_\Lambda$ almost all $\boldsymbol{\zeta}_\Lambda \in \boldsymbol{\Omega}_\Lambda$ and $0 < Z_\Lambda(\boldsymbol{\omega}_{\Lambda^c}) < \infty$. In particular, the associated Gibbsian kernels γ_Λ are well defined on $\boldsymbol{\Omega}^*_\Lambda$.

Next we give a definition of Gibbs point processes via the DLR equation similar to the one for classical Gibbs measures on deterministic spatial graphs see [13].

Definition 2.1 (Gibbs point processes) *A random field $\mathcal{P}$ is called a Gibbs point process for the specification γ iff for every $\Lambda \Subset \mathbb{R}^d$ and for any $f \in \boldsymbol{\mathcal{F}}^b$,*

$$\int f(\boldsymbol{\omega})\mathcal{P}(d\boldsymbol{\omega}) = \int f(\tilde{\boldsymbol{\omega}}_\Lambda \boldsymbol{\omega}_{\Lambda^c})\gamma_\Lambda(d\tilde{\boldsymbol{\omega}}_\Lambda|\boldsymbol{\omega}_{\Lambda^c})\mathcal{P}(d\boldsymbol{\omega}) \tag{2}$$

and $\mathcal{P}(\boldsymbol{\Omega}^*_\Lambda) = 1$. We denote the set of all such measures $\mathcal{G}(\gamma)$.

For example, for the PG with potential (1), existence of Gibbs measures is proved in [14] where admissibility can be replaced by the notion of temperedness, which is defined without reference to the potential or the volume. Moreover, we note that in the high-intensity regime, phase transitions of multiple Gibbs measures can be observed for the PG.

2.1 Gibbsianness for point processes

In this section we introduce the notion of Gibbsianness for general random fields $\mathcal{P}$ as the existence of a quasilocal specification for $\mathcal{P}$. Similar notions for Gibbsianness in lattice, tree and mean-field situations have been proposed and used to study various statistical mechanics models under transformations, see for example [26]. The criterion for Gibbsianness of continuum random fields presented here is based on the existence of a version of the finite-volume conditional probabilities which constitutes a specification. The additional, and very important, condition is then that the specification is continuous w.r.t. boundary conditions under the τ-topology where $\boldsymbol{\omega}' \Rightarrow \boldsymbol{\omega}$, iff $\boldsymbol{\omega}'(f) \to \boldsymbol{\omega}(f)$ for all $f \in \bigcup_{\Lambda \Subset \mathbb{R}^d} \boldsymbol{\mathcal{F}}^b_\Lambda$ and we used the short-hand notation $\boldsymbol{\omega}(f) = \sum_{(x,\sigma_x)\in\boldsymbol{\omega}} f(x, \sigma_x)$.

Let $B_r(x)$ denote the ball with radius $r > 0$ centered at $x \in \mathbb{R}^d$. We start by labeling points of continuity for a specification and use this to define quasilocality.

Definition 2.2 *Let γ be a specification. A configuration $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ is called* **good** *for γ iff for any $x \in \mathbb{R}^d$ and $0 < r < \infty$ and any observable $f \in \boldsymbol{\mathcal{F}}^b_{B_r(x)}$ we have*

$$\left|\gamma_{B_r(x)}(f|\boldsymbol{\omega}'_{B_r(x)^c}) - \gamma_{B_r(x)}(f|\boldsymbol{\omega}_{B_r(x)^c})\right| \to 0$$

as $\boldsymbol{\omega}' \Rightarrow \boldsymbol{\omega}$. We denote $\boldsymbol{\Omega}(\gamma)$ the set of good configurations. Elements of $\boldsymbol{\Omega} \setminus \boldsymbol{\Omega}(\gamma)$ are called **bad** *for γ and γ is called* **quasilocal** *if $\boldsymbol{\Omega}(\gamma) = \boldsymbol{\Omega}$.*

For example, for the Gibbsian specification of the WRM, any $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ is good since the interaction is of finite range. Even stronger, the WRM is $2a$-Markov in the sense, that $\gamma_{B_r(x)}(d\boldsymbol{\omega}|\cdot)$ is measurable w.r.t. $\boldsymbol{\mathcal{F}}_{B_{r+2a}(x)}$.

It is a subtlety of the theory of Gibbs point processes, that Gibbsian specifications are not always well-defined for all boundary conditions. Even confined to the set of locally finite configurations, the possibility to accumulate arbitrarily many points in finite volumes can lead to blowups in the Hamiltonian if it is of infinite range. This necessitates notions of admissibility or temperedness in the design of the theory which guarantee that the set of configurations where the Gibbsian specification is well-defined, has full mass. In particular, Gibbsian specifications which are not everywhere well defined, can not be quasilocal. Even more dramatically, in the setting of the τ-topology even at a boundary condition $\boldsymbol{\omega}$ where the Gibbsian specification is well-defined one can exhibit a sequence $\boldsymbol{\omega}_n$ of boundary conditions $\boldsymbol{\omega}_n \Rightarrow \boldsymbol{\omega}$ along which the Gibbsian specification is not well-defined. To be more specific, for example for the Gibbsian specification of the PG with infinite range ψ, any $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ would be bad w.r.t. the τ-topology. This can be seen as follows. Away from a large but finite volume, any element of a convergent sequence of configurations can have arbitrarily many more points then $\boldsymbol{\omega}$. Adapting the number of additional points to the, maybe small but non-zero, contribution of ψ leads to the discontinuity.

In the lattice setting, Gibbsianness for a random field $\mathcal{P}$ is defined by the existence of a quasilocal specification for $\mathcal{P}$. As presented in the previous paragraph, this definition can not be directly transferred to the continuum setting since it would, for example, label the infinite-range PG to be non-Gibbs. However, for a random field $\mathcal{P}$ to possess a quasilocal specifications or a specification which is quasilocal away from a set of boundary conditions with zero mass under $\mathcal{P}$ is a way to measure the internal locality structures of $\mathcal{P}$.

Definition 2.3 *We call a random field $\mathcal{P}$* **quasilocally Gibbs** *iff there exists a quasilocal specification γ for $\mathcal{P}$, otherwise we call it* **non-quasilocally Gibbs**. *We call $\mathcal{P}$* **almost-surely quasilocally Gibbs** *iff there exists a specification γ for $\mathcal{P}$ such that $\mathcal{P}(\boldsymbol{\Omega}(\gamma)) = 1$, otherwise we call it* **non-almost-surely quasilocally Gibbs**.

Let us abbreviate quasilocally Gibbsianness with q-Gibbsianness and almost-surely quasilocally Gibbsianness with asq-Gibbsianness. The prime example of random fields $\mathcal{P}$ for which we study Gibbs-non-Gibbs transitions are Gibbs measures under transformations. In the following section, we investigate the WRM under independent spin-flip dynamics and show that it exhibits all the above Gibbsianness properties in certain intensity vs. time regimes.

3 The Widom-Rowlinson model under independent spin-flip dynamics

Let us start by introducing the WRM model on $\mathbb{R}^d$ with $d \geq 2$ and two-dimensional local state space $E = \{-, +\}$. Recall that we write solid $\boldsymbol{\omega}$ for the grey configuration ω colored according to σ_ω, that is, $\boldsymbol{\omega} = \omega^{\sigma_\omega}$. For the WRM the Gibbsian specification is given by $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{R}^d}$ with

$$\gamma_\Lambda(d\boldsymbol{\omega}_\Lambda | \boldsymbol{\omega}_{\Lambda^c}) = \boldsymbol{P}_\Lambda(d\boldsymbol{\omega}_\Lambda)\chi(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c}).$$

Here χ is either one or zero, depending one whether the interspecies distance is bigger or equal than $2a$ for all particles or not. The two-dimensional homogenous PPP $\boldsymbol{P}$ has intensities λ_+ for plus colors and λ_- for minus colors. The usual normalization constant is denoted by Z_Λ. This specification γ is strictly local since it only depends on the boundary condition up to distance $2a$. We may also write this measure on colored particle configurations inside Λ in terms of a two-step procedure by first choosing the particles positions according to a non-colored PPP P with activity $\lambda_+ + \lambda_-$ and afterwards summing over all possible colorings taking into account the compatibility constraints on colors, compare [4, Formula 2.1 and 2.2]. More precisely,

$$\gamma_\Lambda(d\boldsymbol{\omega}_\Lambda | \boldsymbol{\omega}_{\Lambda^c}) = P_\Lambda(d\omega_\Lambda) U(d\sigma_{\omega_\Lambda}) \chi(\boldsymbol{\omega}_\Lambda \boldsymbol{\omega}_{\Lambda^c}) Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c})$$

where U is the Bernoulli measure on the color-space E, independent over the points, which has the probability to see color $+$ given by $\lambda_+/(\lambda_+ + \lambda_-)$.

Note that for $d \geq 2$ the WRM exhibits a phase-transition in the symmetric high-intensity regime, see [4, 35]. More precisely, using the FKG inequality, existence of the limits

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \gamma_\Lambda(d\boldsymbol{\omega}_\Lambda | \pm_{\Lambda^c}) = \mu^\pm(d\boldsymbol{\omega})$$

can be established in all parameter regimes, where $\pm_{\Lambda^c}$ denotes the all plus, respectively all minus boundary condition, see [4, Proposition 2.3.] for the symmetric case. The limiting extremal Gibbs measures $\mu^\pm \in \mathcal{G}(\gamma)$ are invariant under translation and rotation and unequal for sufficiently high intensity. In [4, Corollary] it is shown that existence of percolation in the Random cluster model (sometimes also called the Fortuin-Kasteleyn representation), is a necessary and sufficient condition for symmetry breaking with $\mu^+ \neq \mu^-$.

From now on we call $\lambda_+ = \lambda_-$ the *symmetric regime* and $\lambda_+ > \lambda_-$ the *asymmetric regime*. Let us note that absence of phase-transition for all intensities away from the symmetric high-intensity regime is widely believed to be true but to our knowledge a complete proof is still missing. At low intensities, with possibly different activities, uniqueness can be proved on the lattice in any dimension by cluster expansions. The corresponding result in the continuous setting is standard. Surprisingly, even in the two-dimensional lattice analogue of the WRM, absence of phase-transition in the asymmetric regime is not proven in all parameter regimes, see however [17].

We always start at time zero in some $\mu \in \mathcal{G}(\gamma)$ and apply a rate one Poisson spin-flip dynamics

$$p_t(\sigma_x, \hat{\sigma}_x) = \tfrac{1}{2}(1+e^{-2t})\mathbb{1}_{\sigma_x=\hat{\sigma}_x} + \tfrac{1}{2}(1-e^{-2t})\mathbb{1}_{\sigma_x\neq\hat{\sigma}_x}$$

independent over the sites. We investigate the time-evolved measure $\mu_t = p_t\mu$. In the following subsection, we formulate our main results about Gibbsianness of the time evolved WRM.

3.1 Main results

Let us denote by $\mathcal{G}(\gamma^{\mathrm{sym}})$ the set of Gibbs measures for the symmetric WRM. Moreover we denote by μ^+ is the plus-extremal Gibbs measure. Further, we will refer to the intensity-dependent *critical time* which is given by

$$t_G = \frac{1}{2}\log\frac{\lambda_+ + \lambda_-}{\lambda_+ - \lambda_-}$$

for $\lambda_+ > \lambda_-$. Let us start with our result for the q-Gibbsian regime. For this first observe that for $\lambda_+ > \lambda_-$ and $t_G < t \leq \infty$ we have

$$\frac{1}{R} = \frac{1}{2a}[\log\frac{\lambda_+}{\lambda_-} - \log\frac{1+e^{-2t}}{1-e^{-2t}}] > 0.$$

Let $d(\Delta,\Lambda) = \inf_{x\in\Delta, y\in\Lambda}|x-y|$ denote the set distance between sets $\Delta, \Lambda \subset \mathbb{R}^d$.

Theorem 3.1 *In the asymmetric model let $t_G < t \leq \infty$. Then μ_t^+ is q-Gibbs for a specification $\hat{\gamma}$ with the following exponential decorrelation property. For any $0 < r < \infty$, there exists a finite constant $A = A(\lambda_+, \lambda_-, r)$ such that for all $x \in \mathbb{R}^d$, $\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}$ and observables $f \in \boldsymbol{\mathcal{F}}^b_{B_r(x)}$,*

$$\sup_{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\in\boldsymbol{\Omega}} \left|\hat{\gamma}_{B_r(x)}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B_r(x)}\boldsymbol{\omega}^1_{\Lambda^c}) - \hat{\gamma}_{B_r(x)}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B_r(x)}\boldsymbol{\omega}^2_{\Lambda^c})\right| \leq A\|f\|e^{-d(B_r(x),\Lambda^c)/R}.$$

Next we present our results on asq-Gibbsian regimes. For this we have to collect some information about the support of the transformed measure. Since the time evolution only changes the colors of configurations, all questions concerning grey configurations under the transformed measure can be answered w.r.t. the WRM. Our main concern will be about the existence of infinite clusters in the WRM. A connected component or *cluster* C of points in a grey configuration ω is a subset $C \subset \omega$ where for every $x, y \in C$ there exists a finite set of points $\{x_1, \ldots, x_n\} \subset C$ such that with $x_{n+1} = y$ and $x_0 = x$ we have $|x_i - x_{i-1}| < 2a$ for all $i \in \{1, \ldots, n+1\}$. For any $\Lambda \Subset \mathbb{R}^d$ denote

$$\{\Lambda \leftrightarrow \infty\} = \{\boldsymbol{\omega} \in \boldsymbol{\Omega} : \omega \text{ has an infinite cluster } C \text{ with } C \cap \Lambda \neq \emptyset\}.$$

We will call the parameter regime where $\mu(\{B_r(x) \leftrightarrow \infty\}) > 0$ for some $x \in \mathbb{R}^d$ and $r > 0$ the *high-intensity regime* and the parameter regime where $\mu(\{B_r(x) \leftrightarrow \infty\}) = 0$ for all $x \in \mathbb{R}^d$ and $r > 0$ the *low-intensity regime* of the WRM. We provide proofs of existence of nontrivial high- and low-intensity regimes in the appendix in Section 6.

Discontinuity for small times is based on the existence of infinite clusters. The next results shows that, for low intensities, almost-surely there are no such discontinuities which implies asq-Gibbsianness.

Theorem 3.2 *Consider the symmetric model and let $\mu \in \mathcal{G}(\gamma^{\mathrm{sym}})$ be any starting Gibbs measure. In the low-intensity regime the time-evolved measure μ_t is asq-Gibbs but non-q-Gibbs for all $0 < t \leq \infty$. For the asymmetric model μ_t^+ is asq-Gibbs but non-q-Gibbs for all $0 < t \leq t_G$ in the low-intensity regime.*

Note that the critical time is included in the above result. Further note that in the asq-Gibbsian regimes, μ_t is not Markov in the sense that it depends on the boundary condition only in a finite vicinity. The immediate loss of continuity of any specification, which can usually not be observed in models with fixed geometry, see [25], leading to non-asq-Gibbsianness, is mainly an effect of the hard-core interaction. Note however, that there are very particular examples of mean-field models, see [8], which show immediate loss of Gibbsianness. We do expect short-time preservation of Gibbsianness to be present for instance in models with $\varphi(x-y) = V_0 \times \mathbb{1}_{|x-y|<2a}$ with $V_0 > 0$ large but finite.

For the high-intensity regime and times strictly smaller then the critical one, we show non-asq-Gibbsianness.

Theorem 3.3 *Consider the symmetric model and let $\mu \in \mathcal{G}(\gamma^{\mathrm{sym}})$ be any starting Gibbs measure. Then μ_t is non-asq-Gibbs for all $0 < t < \infty$ in the high-intensity regime. For the asymmetric model μ_t^+ is non-asq-Gibbs for all $0 < t < t_G$ in the high-intensity regime. In both cases if $\mu_t \in \mathcal{G}(\hat{\gamma})$ for some specification $\hat{\gamma}$, then $\mu_t(\mathbf{\Omega}(\hat{\gamma})) = 0$.*

For the symmetric model in the above regimes, we exhibit specifications $\gamma^+ \neq \gamma^-$ for μ_t^+ and μ_t^- which are non-almost-surely quasilocal in the appendix in Section 6.

The method of proof of Theorem 3.3 is based on color perturbations. At the critical time for the symmetric model, $t = \infty$, slightly refined arguments allow us to produce discontinuities with full mass via a different mechanism of spatial perturbations for the Gibbs measure $\mu_\infty^+ = \mu_\infty^-$. The critical time for the asymmetric model shows different behavior. Here the specification $\hat{\gamma}$ that we presented already in Theorem 3.1 is still a specification where discontinuity points now have zero mass. This implies asq-Gibbsianness.

Theorem 3.4 *Consider the symmetric model. Then μ_∞^+ is non-asq-Gibbs in the high-intensity regime. Moreover if $\mu_\infty^+ \in \mathcal{G}(\hat{\gamma})$ for some specification $\hat{\gamma}$, then $\mu_\infty^+(\mathbf{\Omega}(\hat{\gamma})) = 0$. For the asymmetric model $\mu_{t_G}^+$ is asq-Gibbs but non-q-Gibbs in the high-intensity regime.*

Let us note that case $t = \infty$, in the symmetric and in the asymmetric regimes, is equivalent to the case where the colors are simply disregarded. More precisely, the above results imply that $\mu \circ T^{-1}$ with $T : \boldsymbol{\omega} \mapsto \omega$ is non-asq-Gibbs in the symmetric case and asq-Gibbs in the asymmetric case. Table 1 provides an overview for Gibbsiannness transitions in time and intensity for the WRM under independent spin-flip evolution.

Table 1: Gibbsian transitions in time and intensity.

	$\mathcal{G}(\gamma)$	time	high intensity	low intensity
$\lambda_+ > \lambda_-$	μ^+	$0 < t < t_G$	non-asq	asq, non-q
		$t = t_G$	asq, non-q	asq, non-q
		$t_G < t \leq \infty$	q	q
$\lambda_+ = \lambda_-$	μ	$0 < t < \infty$	non-asq	asq, non-q
	μ^+	$t = \infty$	non-asq	asq, non-q

4 Strategy of proofs

As a first step to the proofs, we derive an expression for the conditional expectation of the time-evolved Gibbs measure in a large but finite volume. This expression is based on a reformulation in terms of clusters of the grey configuration. A crucial quantity will be presented which involves the magnetization of the boundary condition. In certain time versus intensity regimes (as in the first and forth line of Table 1) this quantity will act as a 'switch' and infinite clusters can influence the finite-volume conditional probability. In other regimes (as in the second and third line of Table 1), the switch will be inactive and the model will turn out to be asq-Gibbs.

4.1 Notations

Let us introduce the necessary notations. First we write

$$\alpha = \frac{\lambda_+}{\lambda_-}$$

where we always assume $\lambda_+ \geq \lambda_-$ which favors the plus sign. The other case follows by symmetry.

4.1.1 Cluster types

Recall that we write ω for the grey configuration of $\boldsymbol{\omega}$. It will be of central importance to consider the connected components, that is, clusters of the grey configuration ω. We denote by $\mathcal{C}(\omega)$ the set of all clusters of ω respectively $\boldsymbol{\omega}$. Note that ω can be identified with $\mathcal{C}(\omega)$. For some $\Lambda \Subset \mathbb{R}^d$, fix grey configuration $\omega_\Lambda \omega_{\Lambda^c}$. Then we distinguish two types of clusters.

1 $\mathcal{C}_\Lambda(\omega_\Lambda \omega_{\Lambda^c}) = \{C \in \mathcal{C}(\omega_\Lambda \omega_{\Lambda^c}) : C \not\subset \bar{\Lambda}^c\}$,

2 $\mathcal{C}_{\Lambda^c}(\omega_{\Lambda^c}) = \{C \in \mathcal{C}(\omega_\Lambda \omega_{\Lambda^c}) : C \subset \bar{\Lambda}^c\}$

where $\bar{\Lambda} = \bigcup_{x \in \Lambda} B_{2a}(x)$ and the type-two clusters are independent of ω_Λ. In particular $\mathcal{C}(\omega_\Lambda \omega_{\Lambda^c}) = \mathcal{C}_\Lambda(\omega_\Lambda \omega_{\Lambda^c}) \cup \mathcal{C}_{\Lambda^c}(\omega_{\Lambda^c})$ and we will often suppress the dependence on ω_{Λ^c} in both clusters types to ease notation.

4.1.2 Magnetization and the switch

For a given colored configuration $\boldsymbol{\omega}_\Lambda$ we define the *magnetization* as

$$m(\boldsymbol{\omega}_\Lambda) = \frac{1}{|\omega_\Lambda|} \sum_{x \in \omega_\Lambda} \sigma_x \in [-1, 1]$$

where σ is the coloring of ω_Λ. Further we denote by $|\sigma|^\pm$ the number of $\pm$-spins in σ. For magnetization $m \in [-1, 1]$, the sign of the following quantity will be import

$$g(m) = \log \frac{\lambda_+}{\lambda_-} + m \log \frac{1 + e^{-2t}}{1 - e^{-2t}}.$$

Recall our definition $t_G = \frac{1}{2} \log \frac{\lambda_+ + \lambda_-}{\lambda_+ - \lambda_-}$ for $\lambda_+ > \lambda_-$. We can distinguish several regimes:

1 For $\lambda_+ > \lambda_-$: $t_G < t \leq \infty \quad \Rightarrow \quad g(m) > 0$ for all $m \in [-1, 1]$

2 For $\lambda_+ = \lambda_-$: $t = \infty \quad \Rightarrow \quad g(m) = 0$ for all $m \in [-1, 1]$

3 For $\lambda_+ > \lambda_-$: $t_G = t \quad \Rightarrow \quad g(m) > 0$ for all $m \in (-1, 1]$ and $g(-1) = 0$

4 In all other cases $g(m)$ has no definite sign.

In short, the Case (1) implies q-Gibbsianness. The Case (2) although g is fixed, gives rise to non-asq-Gibbsian behavior due to cluster perturbations. The Case (3) implies asq-Gibbsianness since the change of sign is only possible for $m = -1$ which is of zero mass. The Case (4) gives rise to non-asq-Gibbsian behavior due to color perturbations. The quantity g is going to appear in the following form which we will call the *switch*

$$\rho(\boldsymbol{\omega}_C) = \exp\Big(- |\omega_C| g\big(m(\boldsymbol{\omega}_C)\big)\Big).$$

Note that if the cluster C is infinite and the magnetization is well-defined, then $\rho(\boldsymbol{\omega}_C) \in \{0, 1, \infty\}$ depending on the sign of $g\big(m(\boldsymbol{\omega}_C)\big)$.

In the following subsection we give useful representations of the finite-volume versions of the time-evolved Gibbsian specification of the WRM. In particular they will exhibit the switch. To further ease notation, for the rest of the section, we will write $B = B_r(x)$ for some $x \in \mathbb{R}^d$ and $r > 0$ and denote $P^\pm$ the PPP with intensity $\lambda_\pm$.

4.2 Finite-volume conditional probabilities

Let us fix some $0 < t \leq \infty$, $\Lambda \Subset \mathbb{R}^d$ and $\boldsymbol{\omega}_{\Lambda^c} \in \boldsymbol{\Omega}$ a configuration in Λ^c obeying the color constraint. The time-evolved WRM in Λ with (not-time evolved) boundary condition $\boldsymbol{\omega}_{\Lambda^c}$, is given by

$$\mu_{t,\Lambda}^{\boldsymbol{\omega}_{\Lambda^c}}(f) = \int \gamma_\Lambda(d\boldsymbol{\omega}_\Lambda | \boldsymbol{\omega}_{\Lambda^c}) \int p_t(\sigma_{\omega_\Lambda}, d\hat{\sigma}_{\omega_\Lambda}) f(\hat{\boldsymbol{\omega}}_\Lambda).$$

The following cluster representation of this finite-volume time-evolved WRM. Recall that we write σ_ω for the coloring of $\boldsymbol{\omega}$.

Lemma 4.1 *Let $0 < t \leq \infty$, $B \subset \Lambda \Subset \mathbb{R}^d$ and $\boldsymbol{\omega}_{\Lambda^c} \in \boldsymbol{\Omega}$ any configuration in Λ^c obeying the color constraint. Consider the boundary configuration $\hat{\boldsymbol{\omega}}_{\Lambda\setminus B} \in \boldsymbol{\Omega}$, then for any $f \in \boldsymbol{\mathcal{F}}_B^b$, we have $\mu_{t,\Lambda}^{\boldsymbol{\omega}_{\Lambda^c}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) = \gamma_B^{\boldsymbol{\omega}_{\Lambda^c}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B})$ where*

$$\gamma_B^{\boldsymbol{\omega}_{\Lambda^c}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) = \frac{\int P_B^-(d\omega_B) f^\Lambda(\omega_B) \prod_{C\in\mathcal{C}_B(\omega_B)} \left(\alpha^{|C\cap B|}\mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}\right)}{\int P_B^-(d\omega_B) \prod_{C\in\mathcal{C}_B(\omega_B)} \left(\alpha^{|C\cap B|}\mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}\right)}.$$

where $f^\Lambda(\omega_B) = \nu_B^{\boldsymbol{\omega}_{\Lambda^c}}(f(\omega_B,\cdot)|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B},\omega_B)$ with

$$\nu_B^{\boldsymbol{\omega}_{\Lambda^c}}(\hat{\sigma}_{\omega_B}|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B},\omega_B) =$$

$$\frac{\prod_{C\in\mathcal{C}_B(\omega_B)} \left(\alpha^{|C\cap B|}\frac{p_t(+,+)^{|\hat{\sigma}_{C\cap B}|^+}}{p_t(+,-)^{-|\hat{\sigma}_{C\cap B}|^-}}\mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \frac{p_t(-,+)^{|\hat{\sigma}_{C\cap B}|^+}}{p_t(-,-)^{-|\hat{\sigma}_{C\cap B}|^-}}\rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}\right)}{\prod_{C\in\mathcal{C}_B(\omega_B)} \left(\alpha^{|C\cap B|}\mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}\right)}.$$

Some words of explanation. (1) We give a representation of the finite-volume specification γ_B in B within a bigger but still finite-volume Λ in terms of clusters. This has the advantage to well quantify the probabilistic costs of changing color from time zero to time t, dependent on the size of the cluster. More precisely, the indicator functions express the fact that if a cluster is connected to the plus boundary of Λ, then the whole cluster starts to time evolve from the plus color. Moreover, the coloring $\hat{\sigma}_{C\setminus B}$ on a given cluster outside of B, but still inside Λ, at time t, creates an additional weight-factor ρ, the switch. The interpretation is that, according to the coloring in the condition it is more likely (or less likely) for the cluster to start from an all plus or all minus coloring at time zero. Of course, spatial positions of colors in that cluster play no role and thus, the weight factor can be expressed in terms of the magnetization and the size of the cluster. (2) The color-transition probabilities inside B for given grey configurations is given by the measure ν_B. In case f only depends on grey configurations, we have $f^\Lambda = f$. (3) Note the interesting fact that even if f only depends on grey configurations, a perturbation of colors in the boundary condition, that is, a change in the magnetization, leads to a change in the expectation of f w.r.t. the kernel. This is in particular also true in the symmetric case when for example $\omega_{\Lambda^c} = \emptyset_{\Lambda^c}$. Indeed, let $\lambda_+ = \lambda_-$, then

$$\gamma_B^{\emptyset_{\Lambda^c}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) = \frac{\int P_B^-(d\omega_B) f(\omega_B) \prod_{C\in\mathcal{C}_B(\omega_B)} \left(1 + \rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\right)}{\int P_B^-(d\omega_B) \prod_{C\in\mathcal{C}_B(\omega_B)} \left(1 + \rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\right)} \tag{3}$$

where $\rho(\hat{\boldsymbol{\omega}}_{C\setminus B}) = \exp(\sum_{x\in C\setminus B}\hat{\sigma}_x \log\tanh(t))$ and the effect of color perturbations is represented in the exponent. This interesting phenomenon of non-locality is typical for point processes and goes beyond the Ising world.

This finite-volume representation suggests the following heuristics for the infinite volume: Depending on the sign of $g(\boldsymbol{\omega}_C)$, ρ tends to zero, infinity or equals one as $|\omega_C|$ tends to infinity almost surely. In particular, if $t_G < t \leq \infty$ in the asymmetric case or $t = \infty$ in the symmetric case, the switch is inactive, and there is no dependence on the magnetization on these infinite clusters. Thus, in the asymmetric case, ρ becomes small as connected clusters can become large, independently of the size of the magnetization on which we condition and q-Gibbsianness will follow. If the magnetization dependence remains as connected clusters grow large, sensitive dependence on the boundary condition remains and non-q-Gibbssianness will follow. Moreover, note that in the low-intensity regime, the configurations containing an infinite cluster form a nullset, this will lead to asq-Gibbsianness.

4.3 The infinite-volume specification and q-Gibbsianness

First note that for q-Gibbsianness we assume asymmetric parameter regimes where

$$g(m) \geq g(-1) = g_- > 0.$$

In this case, the switch is inactive even on configurations with infinite clusters, where the magnetization can not be changed by local color perturbations.

Inactive switches allow us to build a family of infinite-volume kernels by taking extra care only for the infinite clusters. We denote $\mathcal{C}_B^\infty(\omega_B) \subset \mathcal{C}_B(\omega_B)$ the set of infinity clusters in $\mathcal{C}_B(\omega_B)$ and $\mathcal{C}_B^{\mathrm{f}}(\omega_B) \subset \mathcal{C}_B(\omega_B)$ the set of finite clusters.

Definition 4.2 *We define for* $\Lambda \Subset \mathbb{R}^d$

$$\gamma_\Lambda^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda^c}) = \frac{\int P_\Lambda^-(d\omega_\Lambda) f^\infty(\omega_\Lambda) \prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \left(\alpha^{|C\cap\Lambda|} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right) \prod_{C\in\mathcal{C}_\Lambda^\infty(\omega_\Lambda)} \alpha^{|C\cap\Lambda|}}{\int P_\Lambda^-(d\omega_\Lambda) \prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \left(\alpha^{|C\cap\Lambda|} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right) \prod_{C\in\mathcal{C}_\Lambda^\infty(\omega_\Lambda)} \alpha^{|C\cap\Lambda|}}$$

where $f^\infty(\omega_\Lambda) = \nu_\Lambda^\infty(f(\omega_\Lambda,\cdot)|\hat{\boldsymbol{\omega}}_{\Lambda^c},\omega_\Lambda)$ *with*

$$\nu_\Lambda^\infty(\hat{\sigma}_{\omega_\Lambda}|\hat{\boldsymbol{\omega}}_{\Lambda^c},\omega_\Lambda) = \prod_{C\in\mathcal{C}_\Lambda^\infty(\omega_\Lambda)} p_t(+,+)^{|\hat{\sigma}_{C\cap\Lambda}|^+} p_t(+,-)^{|\hat{\sigma}_{C\cap\Lambda}|^-} \times$$

$$\frac{\prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \left(\alpha^{|C\cap\Lambda|} p_t(+,+)^{|\hat{\sigma}_{C\cap\Lambda}|^+} p_t(+,-)^{|\hat{\sigma}_{C\cap\Lambda}|^-} + p_t(-,+)^{|\hat{\sigma}_{C\cap\Lambda}|^+} p_t(-,-)^{|\hat{\sigma}_{C\cap\Lambda}|^-} \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}{\prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \left(\alpha^{|C\cap\Lambda|} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}.$$

Further we denote $\gamma^\infty = (\gamma_\Lambda^\infty)_{\Lambda\Subset\mathbb{R}^d}$.

Let us first assert properness and consistency of γ^∞.

Lemma 4.3 *For all times and intensities* γ^∞ *is a specification.*

The following proposition shows that in the right parameter regime, where the switch can not be fully used, γ^∞ is indeed a specification for the time-evolved Gibbs measure.

Proposition 4.4 *In the asymmetric regime assume* $t_G \leq t \leq \infty$*, then* $\mu_t^+ \in \mathcal{G}(\gamma^\infty)$.

In the large-time regimes, we can further prove a strong form of quasilocality of γ^∞.

Proposition 4.5 *In the asymmetric regime assume* $t_G < t \leq \infty$*, then* $g_- > 0$ *and there exists finite* $A = A(\lambda_+,\lambda_-,r)$ *such that for all* $\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}$ *and observables* $f \in \boldsymbol{\mathcal{F}}_B^b$*,*

$$\sup_{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\in\boldsymbol{\Omega}} \left|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}_{\Lambda^c}^1) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}_{\Lambda^c}^2)\right| \leq A\|f\|e^{-g_- d(B,\Lambda^c)/(2a)}.$$

Now Theorem 3.1 is a direct application of the preceding results. [Proof of Theorem 3.1] By Lemma 4.3 and Proposition 4.4, γ^∞ is a specification for the time-evolved Gibbs measures. Moreover, γ^∞ is quasilocal by Proposition 4.5 which implies q-Gibbsianness. The more refined exponential locality in the asymmetric case of Proposition 4.5 is simply recorded in Theorem 3.1. □

As mentioned above, γ^∞ can also serve as a specification in other regimes, as long as infinite clusters appear with zero probability. This is the main idea in the next subsection.

4.4 Asq-Gibbsianness

Note that for sufficiently low-intensites, in the symmetric model, the WRM has a unique Gibbs measure [4] while in the asymmetric model this is expected but apparently not proved. The following proposition asserts that for all times in the low-intensity regime, γ^∞ is a specification for the time-evolved measures.

Proposition 4.6 *For the symmetric model $\mu_t \in \mathcal{G}(\gamma^\infty)$ for all $0 < t \leq \infty$ in the low-intensity regime. For the asymmetric model $\mu_t^+ \in \mathcal{G}(\gamma^\infty)$ for all $0 < t \leq t_G$ in the low-intensity regime.*

Moreover, as provided by the following lemma, boundary conditions which do not contain infinite clusters are good points of γ^∞.

Lemma 4.7 *For all $\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}$ which contain no infinite cluster and all $f \in \boldsymbol{\mathcal{F}}_B^b$, there exists $\Delta \Subset \mathbb{R}^d$ such that for all $\Delta \subset \Lambda$*

$$\sup_{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2 \in \boldsymbol{\Omega}} \left|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c}) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c})\right| = 0.$$

The previous results directly imply the asq-Gibbsianness in Theorem 3.2.

[Proof of Theorem 3.2 - asq-Gibbsian part] By Lemma 4.3 and Proposition 4.6 , γ^∞ is a specification for μ_t and $0 < t \leq \infty$ respectively μ_t^+ and $0 < t \leq t_G$ in the low-intensity regime. By Lemma 4.7 we have $\mu_t(\boldsymbol{\Omega}(\gamma^\infty)) = 1$, respectively $\mu_t^+(\boldsymbol{\Omega}(\gamma^\infty)) = 1$, this implies asq-Gibbsianness. □

As for the critical time in the asymmetric regime, first note that on infinite clusters, magnetizations are biased away from minus one. More precisely, let us define $m_t = p_t(+,+) - p_t(+,-) = e^{-2t} > 0$ and for all $\varepsilon > 0$

$$\boldsymbol{\Omega}^\varepsilon = \{\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega} : \liminf_{n\uparrow\infty} m(\hat{\boldsymbol{\omega}}_{C\cap B_n}) \geq \varepsilon \text{ for all infinite clusters } C \text{ of } \hat{\boldsymbol{\omega}}\}.$$

Then we have the following result.

Lemma 4.8 *It is a fact that $\mu_t^+(\boldsymbol{\Omega}^{m_t}) = 1$.*

The next results in particular implies that $\boldsymbol{\Omega}^{m_{t_G}} \subset \boldsymbol{\Omega}(\gamma^\infty)$.

Proposition 4.9 *There exists finite $A = A(\lambda_+, \lambda_-, r)$ such that for all observables $f \in \boldsymbol{\mathcal{F}}_B^b$ and configurations $\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}^{m_{t_G}}$ we have*

$$\sup_{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2 \in \boldsymbol{\Omega}} \left|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c}) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c})\right| \leq A\|f\|e^{-d(B,\Lambda^c)/(2a)}.$$

The previous results directly imply the asq-Gibbsianness in Theorem 3.4.

[Proof of Theorem 3.4 - asq-Gibbsian part] By Proposition 4.4, γ^∞ is a specification for $\mu^+_{t_G}$ which is concentrated on $\mathbf{\Omega}^{m_{t_G}}$ by Lemma 4.8. But by Proposition 4.9, $\mu^+_{t_G}(\mathbf{\Omega}(\gamma^\infty)) = 1$ and thus $\mu^+_{t_G}$ is asq-Gibbs. □

In the next subsections we discuss the non-q-Gibbsian and non-asq-Gibbsian regimes. The main task here is to transfer knowledge of bad points for a given version of finite-volume conditional probabilities on positive-measure subsets of configurations to any other specification.

4.5 Non-asq-Gibbsianness

In this subsection we assume parameter regimes where at least one of two mechanisms is available. The first one involves color perturbations. More precisely, if $g(m)$ can have positive and negative signs as $m \in [-1, 1]$, discontinuities can be produced by changing colors in a large but finite cluster. Existence of infinite clusters can always be assumed when analyzing asq-Gibbsianness and is guaranteed almost surely in the high-intensity regime. The second mechanism works for $t = \infty$ in the symmetric case, where discontinuities can be produced by means of spatial perturbations. Let us start by defining probability kernels similar to γ^∞, but without the infinite components.

Definition 4.10 *We define for* $\Lambda \Subset \mathbb{R}^d$

$$\gamma^{\mathrm{f}}_\Lambda(f|\hat{\boldsymbol{\omega}}_{\Lambda^c}) = \frac{\int P^-_\Lambda(d\omega_\Lambda) f^{\mathrm{f}}(\omega_\Lambda) \prod_{C\in\mathcal{C}^{\mathrm{f}}_\Lambda(\omega_\Lambda)} \left(\alpha^{|C\cap\Lambda|} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}{\int P^-_\Lambda(d\omega_\Lambda) \prod_{C\in\mathcal{C}^{\mathrm{f}}_\Lambda(\omega_\Lambda)} \left(\alpha^{|C\cap\Lambda|} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}$$

where $f^{\mathrm{f}}(\omega_\Lambda) = \nu^{\mathrm{f}}_\Lambda(f(\omega_\Lambda,\cdot)|\hat{\boldsymbol{\omega}}_{\Lambda^c}, \omega_\Lambda)$ *with*

$$\nu^{\mathrm{f}}_\Lambda(\hat{\sigma}_{\omega_\Lambda}|\hat{\boldsymbol{\omega}}_{\Lambda^c}, \omega_\Lambda) = \frac{\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Lambda(\omega_\Lambda)} \left(\alpha^{|C\cap\Lambda|} \frac{p_t(+,+)^{|\hat{\sigma}_{C\cap\Lambda}|^+}}{p_t(+,-)^{-|\hat{\sigma}_{C\cap\Lambda}|^-}} + \frac{p_t(-,+)^{|\hat{\sigma}_{C\cap\Lambda}|^+}}{p_t(-,-)^{-|\hat{\sigma}_{C\cap\Lambda}|^-}} \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}{\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Lambda(\omega_\Lambda)} \left(\alpha^{|C\cap\Lambda|} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}.$$

Further we denote $\gamma^{\mathrm{f}} = (\gamma^{\mathrm{f}}_\Lambda)_{\Lambda\Subset\mathbb{R}^d}$.

Note that we do not claim that γ^{f} is consistent, but we show that γ^{f} is a representation of the conditional probabilities of μ_t away from the infinite components.

Proposition 4.11 *Let* $\mu \in \mathcal{G}(\gamma^{\mathrm{sym}})$ *for the symmetric model or* $\mu = \mu^+$ *for the asymmetric model. Then for all* $0 < t \leq \infty$ *and* μ_t*-almost all* $\hat{\boldsymbol{\omega}} \in \mathbf{\Omega}$ *we have*

$$\mu_t(\cdot|\hat{\boldsymbol{\omega}}_{B^c})\mathbb{1}_{B\not\leftrightarrow\infty}(\omega_{B^c}) = \gamma^{\mathrm{f}}_B(\cdot|\hat{\boldsymbol{\omega}}_{B^c})\mathbb{1}_{B\not\leftrightarrow\infty}(\omega_{B^c}).$$

Note that $\mu_t(B \not\leftrightarrow \infty) = \mu(B \not\leftrightarrow \infty) > 0$ for any $\mu \in \mathcal{G}(\gamma)$. The next proposition asserts that γ^{f}_B is discontinuous at configurations which do have an infinite cluster communicating with B. More precisely, we show that γ^{f}_B is discontinuous even under color perturbation for times smaller then the critical time. In the sequel when we write $\boldsymbol{\omega}^\pm$, we assume $\omega^\pm = \omega$ and $\sigma^\pm_\omega = \pm_\omega$, in words, $\boldsymbol{\omega}^\pm$ is a configuration with only plus or only minus colors.

Proposition 4.12 *Let $0 < t < \infty$ for the symmetric model or $0 < t < t_G$ for the asymmetric model. Then for all $\Lambda \Subset \mathbb{R}^d$ and all $L > 0$, there exists $N \in \mathbb{N}$, $f \in \boldsymbol{\mathcal{F}}_B^b$ and $\delta > 0$ such that for all $n \geq N$,*

$$\inf_{\hat{\boldsymbol{\omega}} \in \{B \leftrightarrow B_n^c\} : |\omega_{\Lambda \setminus B}| < L} \left| \gamma_B^{\mathrm{f}}(f | \hat{\boldsymbol{\omega}}_\Lambda \boldsymbol{\omega}^+_{B_n \setminus \Lambda}) - \gamma_B^{\mathrm{f}}(f | \hat{\boldsymbol{\omega}}_\Lambda \boldsymbol{\omega}^-_{B_n \setminus \Lambda}) \right| > \delta.$$

For the symmetric case at $t = \infty$, the Gibbs measure $\mu_\infty^+ = \mu_\infty^-$ is color-blind and γ^∞ is a specification.

Proposition 4.13 *In the symmetric high-intensity regime $\mu_\infty^+ \in \mathcal{G}(\gamma^\infty)$.*

Moreover, a spatial perturbation can be used to exhibit discontinuities independent of the coloring.

Proposition 4.14 *In the symmetric regime let $t = \infty$, then there exists $f \in \boldsymbol{\mathcal{F}}_B^b$ and $\delta > 0$ such that,*

$$\lim_{\Lambda \uparrow \infty} \inf_{\hat{\boldsymbol{\omega}} \in \{B \leftrightarrow \infty\}} \left| \gamma_B^\infty(f | \hat{\boldsymbol{\omega}}_{B^c}) - \gamma_B^\infty(f | \hat{\boldsymbol{\omega}}_{\Lambda \setminus B}) \right| > \delta.$$

In the high-intensity regime, under μ_t, configurations which have an infinite-cluster connected to B have positive mass. In particular, points of discontinuity for γ^{f} are essential under μ_t and therefore no specification for μ_t can be quasilocal almost surely. This is the main idea for the proof of the non-asq-Gibbsian part of Theorem 3.3 and 3.4.

4.6 Non-q-Gibbsianness

For the asq-Gibbsian regimes, it remains to show non-q-Gibbsianness. For this we use the following argument. We exhibit particular bad boundary conditions which have infinite clusters. They can be approximated by convergent sequences (together with positive mass perturbations, see for example Figure 3) which have growing but finite clusters. In order to show that the jump occurs for any specification we use the positive mass perturbations to replace the unknown specification by γ^{f} for which we know that the jump occurs. This gives the non-q-Gibbsian part in the Theorems 3.2 and 3.4.

5 Proofs

The ordering of the proofs presented in this section is mainly based on the technics required. As a result, they do not follow their numerical ordering. We use some more notation.

[Proof of Lemma 4.1] Let us write very short $U(\sigma_x, \hat{\sigma}_x) = U(\sigma_x) p_t(\sigma_x, \hat{\sigma}_x)$ for the time-dependent double layer single-point measure. We derive the form of the finite-volume specification by introducing a cluster representation, lifting boundary conditions on individual clusters via their magnetizations to the exponential scale and using appropriate normalizations.

Let us start by writing μ_t instead of $\mu_{t,\Lambda}^{\boldsymbol{\omega}_{\Lambda^c}}$. Then we have

$$\mu_t(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) = \frac{\int P_B(d\omega_B)\sum_{\sigma_{\omega_B}}\int U(\sigma_{\omega_B},d\hat{\sigma}_{\omega_B})f(\omega_B,\hat{\sigma}_{\omega_B})\sum_{\sigma_{\omega_{\Lambda\setminus B}}}U(\sigma_{\omega_{\Lambda\setminus B}},\hat{\sigma}_{\omega_{\Lambda\setminus B}})\chi(\boldsymbol{\omega}_\Lambda\boldsymbol{\omega}_{\Lambda^c})}{\int P_B(d\omega_B)\sum_{\sigma_{\omega_B}}U(\sigma_{\omega_B})\sum_{\sigma_{\omega_{\Lambda\setminus B}}}U(\sigma_{\omega_{\Lambda\setminus B}},\hat{\sigma}_{\omega_{\Lambda\setminus B}})\chi(\boldsymbol{\omega}_\Lambda\boldsymbol{\omega}_{\Lambda^c})}$$

where $\boldsymbol{\omega}_\Lambda = \omega_B^{\sigma_{\omega_B}}\omega_{\Lambda\setminus B}^{\sigma_{\omega_{\Lambda\setminus B}}}$. We abbreviate the integration w.r.t. the coloring and write

$$\begin{aligned} U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(f,\omega_B\omega_{\Lambda\setminus B}\big) &= U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}\sigma_{\omega_{\Lambda^c}}}\big(f,\omega_B\omega_{\Lambda\setminus B}\omega_{\Lambda^c}\big) \\ &= \sum_{\sigma_{\omega_B}}\int U(\sigma_{\omega_B},d\hat{\sigma}_{\omega_B})f(\omega_B,\hat{\sigma}_{\omega_B})\sum_{\sigma_{\omega_{\Lambda\setminus B}}}U(\sigma_{\omega_{\Lambda\setminus B}},\hat{\sigma}_{\omega_{\Lambda\setminus B}})\chi(\boldsymbol{\omega}_\Lambda\boldsymbol{\omega}_{\Lambda^c}). \end{aligned}$$

Then, we have the shorthand notation

$$\mu_t(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) = \int P_B(d\omega_B)U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(f,\omega_B\omega_{\Lambda\setminus B}\big)/\int P_B(d\omega_B)U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(1,\omega_B\omega_{\Lambda\setminus B}\big). \tag{4}$$

Due to the color constraint χ, at time zero, there can only be a uniform coloring on every cluster $\mathcal{C}(\omega_B\omega_{\Lambda\setminus B}\omega_{\Lambda^c}) = \mathcal{C}_B(\omega_B)\cup\mathcal{C}_{B^c}$ where the clusters in $\mathcal{C}_{B^c}$ are independent of ω_B. The $\mathcal{C}_B(\omega_B)$ clusters are random variables w.r.t. ω_B. In particular, by the independence of the Bernoulli process, we have

$$U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(f,\omega_B\omega_{\Lambda\setminus B}\big) = U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(f,\mathcal{C}_B(\omega_B)\big)U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(1,\mathcal{C}_{B^c}\big).$$

The last term also appears in the normalization and hence

$$\mu_t(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) = \int P_B(d\omega_B)U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(f,\mathcal{C}_B(\omega_B)\big)/\int P_B(d\omega_B)U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(1,\mathcal{C}_B(\omega_B)\big).$$

Defining a conditional color-expectation of f as

$$f^\Lambda(\omega_B) = U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(f|\mathcal{C}_B(\omega_B)\big) = U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(f,\mathcal{C}_B(\omega_B)\big)/U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(1,\mathcal{C}_B(\omega_B)\big)$$

we arrive at the expression

$$\mu_t(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) = \int P_B(d\omega_B)f^\Lambda(\omega_B)U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(1,\mathcal{C}_B(\omega_B)\big)/\int P_B(d\omega_B)U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(1,\mathcal{C}_B(\omega_B)\big).$$

In words, the conditional probability has been expressed as a conditional Bernoulli average at fixed locations in B which will be averaged over a point measure for colorless point configurations which itself is distorted in a boundary condition-dependent way. Now, the Bernoulli expectations are given by

$$\begin{aligned} U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(1,\mathcal{C}_B(\omega_B)\big) = \prod_{C\in\mathcal{C}_B(\omega_B)} &\big(\hat{\lambda}_+^{|C\cap\Lambda|}p_t(+,+)^{|\hat{\sigma}_{C\cap\Lambda\setminus B}|^+}p_t(+,-)^{|\hat{\sigma}_{C\cap\Lambda\setminus B}|^-}\mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} \\ &+\hat{\lambda}_-^{|C\cap\Lambda|}p_t(-,+)^{|\hat{\sigma}_{C\cap\Lambda\setminus B}|^+}p_t(-,-)^{|\hat{\sigma}_{C\cap\Lambda\setminus B}|^-}\mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}\big). \end{aligned} \tag{5}$$

Note that, all the products over clusters are finite products since, for finite B there is only a finite number $K = K(B)$ of clusters connected to B, but not necessarily a finite number of

points in B. A trivial upper bound for this K would be the volume of B divided by the volume of a ball of radius a. We further note that the expression (5) does not depend on the geometry in a very complicated way. It depends only on the number of points of C in B, the number of points of C in $\Lambda \setminus B$, and the magnetization

$$m_C = \frac{1}{|C \cap \Lambda \setminus B|} \sum_{x \in C \cap \Lambda \setminus B} \hat{\sigma}_x$$

on $C \cap \Lambda \setminus B$. We make further rewritings to make the magnetization of the conditioning explicit. Writing the integers as $|\hat{\sigma}_{C\cap\Lambda\setminus B}|^{\pm} = |C \cap \Lambda \setminus B|(1 \pm m_C)/2$ we obtain

$$\begin{aligned}
&U^{\hat{\sigma}_{\omega_{\Lambda\setminus B}}}\big(1, \mathcal{C}_B(\omega_B)\big) \\
&= \prod_{C \in \mathcal{C}_B(\omega_B)} \Big(\hat{\lambda}_+^{|C\cap\Lambda|} \big(p_t(+,+)p_t(+,-)\big)^{|C\cap\Lambda\setminus B|/2} \big(\tfrac{p_t(+,+)}{p_t(+,-)}\big)^{m_C|C\cap\Lambda\setminus B|/2} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} \\
&\qquad + \hat{\lambda}_-^{|C\cap\Lambda|} \big(p_t(+,+)p_t(+,-)\big)^{|C\cap\Lambda\setminus B|/2} \big(\tfrac{p_t(+,+)}{p_t(+,-)}\big)^{-m_C|C\cap\Lambda\setminus B|/2} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=-} \Big)
\end{aligned}$$

and note that

$$\prod_{C \in \mathcal{C}_B(\omega_B)} \big(p_t(+,+)p_t(+,-)\big)^{|C\cap\Lambda\setminus B|/2} = \big(p_t(+,+)p_t(+,-)\big)^{|\mathcal{C}_B(\omega_B)\cap\Lambda\setminus B|/2}$$

is in fact independent of the configuration ω_B. In particular, it cancels out with the corresponding term in the normalization and we have

$$\begin{aligned}
&\mu_t(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) \\
&= \frac{\int P_B(d\omega_B) f^\Lambda(\omega_B) \prod_{C\in\mathcal{C}_B(\omega_B)} (\hat{\lambda}_+^{|C\cap\Lambda|} q_t^{m_C|C\cap\Lambda\setminus B|/2} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \hat{\lambda}_-^{|C\cap\Lambda|} q_t^{-m_C|C\cap\Lambda\setminus B|/2} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=-})}{\int P_B(d\omega_B) \prod_{C\in\mathcal{C}_B(\omega_B)} (\hat{\lambda}_+^{|C\cap\Lambda|} q_t^{m_C|C\cap\Lambda\setminus B|/2} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \hat{\lambda}_-^{|C\cap\Lambda|} q_t^{-m_C|C\cap\Lambda\setminus B|/2} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=-})}
\end{aligned}$$

where we wrote $q_t = p_t(+,+)/p_t(+,-) = \coth(t)$. Further, for large $|C \setminus B|$ all that matters is the relative size of $\hat{\lambda}_+ q_t^{m_C/2}$ compared to $\hat{\lambda}_- q_t^{-m_C/2}$. For large $|C \setminus B|$ this difference will appear much amplified in the quantities

$$\rho_+^{C\setminus B} = (\hat{\lambda}_+ q_t^{m_C/2})^{|C\cap\Lambda\setminus B|} \qquad \text{and} \qquad \rho_-^{C\setminus B} = (\hat{\lambda}_- q_t^{-m_C/2})^{|C\cap\Lambda\setminus B|}.$$

In particular, using this notation we have

$$\begin{aligned}
&\prod_{C\in\mathcal{C}_B(\omega_B)} (\hat{\lambda}_+^{|C\cap\Lambda|} q_t^{m_C|C\cap\Lambda\setminus B|/2} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \hat{\lambda}_-^{|C\cap\Lambda|} q_t^{-m_C|C\cap\Lambda\setminus B|/2} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}) \\
&\qquad = \prod_{C\in\mathcal{C}_B(\omega_B)} (\hat{\lambda}_+^{|C\cap B|} \rho_+^{C\setminus B} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \hat{\lambda}_-^{|C\cap B|} \rho_-^{C\setminus B} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}) \\
&\qquad = \hat{\lambda}_-^{|\omega_B|} \prod_{C\in\mathcal{C}_B(\omega_B)} \rho_+^{C\setminus B} (\alpha^{|C\cap B|} \mathbb{1}_{\sigma_{C\cap\Lambda^c}=+} + \rho(\hat{\boldsymbol{\omega}}_{C\setminus B}) \mathbb{1}_{\sigma_{C\cap\Lambda^c}=-})
\end{aligned}$$

where $\rho(\hat{\boldsymbol{\omega}}_{C\setminus B}) = \rho_-^{C\setminus B}/\rho_+^{C\setminus B}$. A small inspection yields that $\prod_{C\in\mathcal{C}_B(\omega_B)} \rho_+^{C\setminus B}$ does not depend on ω_B, so we can safely pull it out of the P_B-expectation and it cancels with the

corresponding term in the normalization. Moreover, note that the density $\hat{\lambda}_-^{|\omega_B|}$ can be moved into the intensity of the PPP P_B which gives rise to P_B^- also in the normalization.

Finally, writing $\tilde{\sum}$ for the summation obeying the color constraint, we have

$$\begin{aligned} f^\Lambda(\omega_B) &= \sum_{\hat{\sigma}_{\omega_B}} f(\omega_B, \hat{\sigma}_{\omega_B}) \frac{\prod_{C\in\mathcal{C}(\omega_B)} \tilde{\sum}_{\sigma_{C\cap B}} U(\sigma_{C\cap B}, \hat{\sigma}_{C\cap B}) \sum_{\sigma_{C\setminus B}} U(\sigma_{C\setminus B}, \hat{\sigma}_{C\setminus B})}{\prod_{C\in\mathcal{C}(\omega_B)} \tilde{\sum}_{\sigma_{C\cap B}} U(\sigma_{C\cap B}) \sum_{\sigma_{C\setminus B}} U(\sigma_{C\setminus B}, \hat{\sigma}_{C\setminus B})} \\ &= \sum_{\hat{\sigma}_{\omega_B}} f(\omega_B, \hat{\sigma}_{\omega_B}) \nu_B^{\boldsymbol{\omega}_{\Lambda^c}}(\hat{\sigma}_{\omega_B} | \hat{\boldsymbol{\omega}}_{\Lambda\setminus B}, \omega_B) \end{aligned}$$

and we arrive at the required representation. □

Note that, moving $\alpha^{|\omega_\Lambda|}$ into the Poisson expectation, γ^∞ can also be written in the following shorter but less intuitive form which we will use for the following proofs.

$$\gamma_\Lambda^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda^c}) = \frac{\int P_\Lambda^+(d\omega_\Lambda) f^\infty(\omega_\Lambda) \prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \left(1 + \alpha^{-|C\cap\Lambda|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}{\int P_\Lambda^+(d\omega_\Lambda) \prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \left(1 + \alpha^{-|C\cap\Lambda|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}$$

with $f^\infty(\omega_\Lambda) = \nu_\Lambda^\infty(f(\omega_\Lambda, \cdot)|\hat{\boldsymbol{\omega}}_{\Lambda^c}, \omega_\Lambda)$ where

$$\nu_\Lambda^\infty(\hat{\sigma}_{\omega_\Lambda}|\hat{\boldsymbol{\omega}}_{\Lambda^c}, \omega_\Lambda) = p_t(\hat{\sigma}_{\omega_\Lambda}) \frac{\prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \left(1 + \rho(\hat{\boldsymbol{\omega}}_{C\cap\Lambda})\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}{\prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \left(1 + \alpha^{-|C\cap\Lambda|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\right)}$$

and we abbreviated $p_t(\hat{\sigma}_{\omega_\Lambda}) = p_t(+,+)^{|\hat{\sigma}_{\omega_\Lambda}|^+} p_t(+,-)^{|\hat{\sigma}_{\omega_\Lambda}|^-}$.

[Proof of Proposition 4.3] We first check consistency by direct computation where consistency means, that for all local observable $f \in \boldsymbol{\mathcal{F}}$, $\Lambda \subset \Delta \Subset \mathbb{R}^d$ and boundary conditions $\hat{\boldsymbol{\omega}}$, we have

$$\gamma_\Delta^\infty(\gamma_\Lambda^\infty(f|\cdot)|\hat{\boldsymbol{\omega}}_{\Delta^c}) = \gamma_\Delta^\infty(f|\hat{\boldsymbol{\omega}}_{\Delta^c}). \tag{6}$$

Starting from the l.h.s. of (6), not considering the normalization in Δ, we have the following

equivalencies.

$$\int P^+_\Delta(d\omega_\Delta)\sum_{\hat\sigma_{\omega_\Delta}}\gamma^\infty_\Lambda(f|\boldsymbol{\omega}_{\Delta\setminus\Lambda}\boldsymbol{\omega}_{\Delta^c})p_t(\hat\sigma_{\omega_\Delta})\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Delta(\omega_\Delta\omega_{\Delta^c})}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap\Delta})\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Delta})\big)$$

$$=\int P^+_{\Delta\setminus\Lambda}(d\omega_{\Delta\setminus\Lambda})\sum_{\hat\sigma_{\omega_{\Delta\setminus\Lambda}}}\gamma^\infty_\Lambda(f|\boldsymbol{\omega}_{\Delta\setminus\Lambda}\boldsymbol{\omega}_{\Delta^c})$$

$$\int P^+_\Lambda(d\omega_\Lambda)\sum_{\hat\sigma_{\omega_\Lambda}}p_t(\hat\sigma_{\omega_\Delta})\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Delta(\omega_\Delta\omega_{\Delta^c})}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap\Delta})\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Delta})\big)$$

$$=\int P^+_{\Delta\setminus\Lambda}(d\omega_{\Delta\setminus\Lambda})\sum_{\hat\sigma_{\omega_{\Delta\setminus\Lambda}}}\gamma^\infty_\Lambda(f|\boldsymbol{\omega}_{\Delta\setminus\Lambda}\boldsymbol{\omega}_{\Delta^c})p_t(\hat\sigma_{\omega_{\Delta\setminus\Lambda}})$$

$$\times\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Delta(\omega_\Delta\omega_{\Delta^c})\setminus\mathcal{C}^{\mathrm{f}}_\Lambda(\omega_\Lambda\omega_{\Delta\setminus\Lambda}\omega_{\Delta^c})}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap\Delta})\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Delta})\big)$$

$$\int P^+_\Lambda(d\omega_\Lambda)\sum_{\hat\sigma_{\omega_\Lambda}}p_t(\hat\sigma_{\omega_\Lambda})\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Lambda(\omega_\Lambda\omega_{\Delta\setminus\Lambda}\omega_{\Delta^c})}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap\Delta})\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Delta})\big)$$

$$=\int P^+_{\Delta\setminus\Lambda}(d\omega_{\Delta\setminus\Lambda})\sum_{\hat\sigma_{\omega_{\Delta\setminus\Lambda}}}\int P^+_\Lambda(d\omega_\Lambda)f^\infty(\omega_\Lambda\boldsymbol{\omega}_{\Lambda^c})\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Lambda(\omega_\Lambda\omega_{\Delta\setminus\Lambda}\omega_{\Delta^c})}\big(1+\alpha^{-|C\cap\Lambda|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\big)$$

$$\times p_t(\hat\sigma_{\omega_{\Delta\setminus\Lambda}})\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Delta(\omega_\Delta\omega_{\Delta^c})\setminus\mathcal{C}^{\mathrm{f}}_\Lambda(\omega_\Lambda\omega_{\Delta\setminus\Lambda}\omega_{\Delta^c})}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap\Delta})\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Delta})\big)$$

$$=\int P^+_\Delta(d\omega_\Delta)\sum_{\hat\sigma_{\omega_\Delta}}f(\boldsymbol{\omega}_\Lambda\boldsymbol{\omega}_{\Lambda^c})p_t(\hat\sigma_{\omega_\Delta})\prod_{C\in\mathcal{C}^{\mathrm{f}}_\Delta(\omega_\Delta\omega_{\Delta^c})}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap\Delta})\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Delta})\big)$$

which proves consistency. Since properness is immediate by the definition, we have that γ^∞ is a specification. □

[Proof of Lemma 4.8] First note that for any infinite cluster C of a configuration $\boldsymbol{\omega}$ which is drawn from μ^+ we have $\boldsymbol{\omega}_C=+_C$. In particular, if $\hat{\boldsymbol{\omega}}_C$ is the time evolved configuration $\boldsymbol{\omega}_C$ we have

$$\liminf_{n\uparrow\infty} m(\hat{\boldsymbol{\omega}}_{C\cap B_n})=\liminf_{n\uparrow\infty}|C\cap B_n|^{-1}\sum_{x\in C\cap B_n}\sigma_x(t)$$

where the summation is over independent random variables with distribution $p_t(+,\cdot)$ which has expectation m_t. Thus by the strong law of large numbers $\mu^+_t(\boldsymbol{\Omega}^{m_t})=1$. □

In the sequel we denote by $\mathcal{C}^{\Lambda^c}_B(\omega_B)$ all clusters in $\mathcal{C}_B(\omega_B)$ which are not completely contained in $\Lambda^o=\Lambda\setminus\overline{\Lambda^c}$. Further, $\mathcal{C}^{\Lambda}_B(\omega_B)=\mathcal{C}_B(\omega_B)\setminus\mathcal{C}^{\Lambda^c}_B(\omega_B)$ and $\mathcal{C}^{\mathrm{f},\Lambda^c}_B(\omega_B)\subset\mathcal{C}^{\mathrm{f}}_B(\omega_B)$ is the set of finite clusters not completely contained in Λ^o.

[Proof of Proposition 4.4] The idea for the proof is to use finite-volume approximations. Let $f\in\boldsymbol{\mathcal{F}}^b_B$ where $B=B_r(x)$ for some arbitrary $x\in\mathbb{R}^d$ and $r>0$, then by the FKG inequality, existence of

$$\mu^+_t(f)=\lim_{\Lambda\uparrow\mathbb{R}^d}\mu^{+^{\Lambda^c}}_\Lambda p_t(f)=\lim_{\Lambda\uparrow\mathbb{R}^d}\mu^{+^{\Lambda^c}}_{t,\Lambda}(f)=\lim_{\Lambda\uparrow\mathbb{R}^d}\mu^{+^{\Lambda^c}}_{t,\Lambda}(\gamma^{+^{\Lambda^c}}_B(f|\cdot))$$

is guaranteed, see [4, Proposition 2.3], where $+_{\Lambda^c}$ denotes the all plus boundary condition (at time zero). Then, introducing another volume $\Delta \Subset \mathbb{R}^d$ we can estimate

$$\begin{aligned}
|\mu_t^+(f-\gamma_B^\infty(f|\cdot))| &\le |\mu_t^+(f-\gamma_B^{+_{\Delta^c}}(f|\cdot))| + \|\gamma_B^\infty(f|\cdot)-\gamma_B^{+_{\Delta^c}}(f|\cdot)\| \\
&\le \lim_{\Lambda\uparrow\mathbb{R}^d} |\mu_{t,\Lambda}^{+_{\Lambda^c}}(f-\gamma_B^{+_{\Delta^c}}(f|\cdot))| + \|\gamma_B^\infty(f|\cdot)-\gamma_B^{+_{\Delta^c}}(f|\cdot)\| \\
&\le \limsup_{\Lambda\uparrow\mathbb{R}^d} \|\gamma_B^{+_{\Lambda^c}}(f|\cdot)-\gamma_B^{+_{\Delta^c}}(f|\cdot)\| + \|\gamma_B^\infty(f|\cdot)-\gamma_B^{+_{\Delta^c}}(f|\cdot)\| \\
&\le \limsup_{\Lambda\uparrow\mathbb{R}^d} \|\gamma_B^{+_{\Lambda^c}}(f|\cdot)-\gamma_B^\infty(f|\cdot)\| + 2\|\gamma_B^\infty(f|\cdot)-\gamma_B^{+_{\Delta^c}}(f|\cdot)\|
\end{aligned} \tag{7}$$

where $\|\gamma_B^{+_{\Lambda^c}}(f|\cdot)-\gamma_B^\infty(f|\cdot)\| = \sup_{\hat{\boldsymbol{\omega}}\in\boldsymbol{\Omega}} |\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B^c})-\gamma_B^{+_{\Lambda^c}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B})|$. Hence, it suffices to show that $\|\gamma_B^{+_{\Lambda^c}}(f|\cdot)-\gamma_B^\infty(f|\cdot)\|$ is arbitrarily small for sufficiently large Λ. Let $\hat{\boldsymbol{\omega}}\in\boldsymbol{\Omega}$ then, using Poisson void probabilities to bound denominators away from zero, we have the following estimate

$$\begin{aligned}
&|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B^c}) - \gamma_B^{+_{\Lambda^c}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B})| \\
&\quad\le e^{\lambda_+|B|}\Big[\int P_B^+(d\omega_B)\big|f^\infty(\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\big) \\
&\qquad\qquad - f^\Lambda(\omega_B)\prod_{C\in\mathcal{C}_B^+(\omega_B)}\big(\mathbb{1}_{\sigma_{C\cap\Lambda^c}=+}+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}\big)\big| \\
&\qquad + \|f\|\int P_B^+(d\omega_B)\big|\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\big) \\
&\qquad\qquad - \prod_{C\in\mathcal{C}_B^+(\omega_B)}\big(\mathbb{1}_{\sigma_{C\cap\Lambda^c}=+}+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}\big)\big|\Big]
\end{aligned} \tag{8}$$

where $\mathcal{C}_B^+(\omega_B) = \mathcal{C}_B(\omega_B\omega_{\Lambda\setminus B}+_{\Lambda^c})$. Separating the factors which both products have in common, the last summand in (8) can be bounded from above by

$$\|f\|e^{\lambda_+|B|}\int P_B^{2\lambda_+}(d\omega_B)\Big(\prod_{C\in\mathcal{C}_B^{\mathrm{f},\Lambda^c}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\big)-1\Big). \tag{9}$$

Note, that this is zero if $\mathcal{C}_B^{\mathrm{f},\Lambda^c}(\omega_B)$ is empty for all ω_B. Moreover, for $t > t_G$ we have

$$\rho(\hat{\boldsymbol{\omega}}_{C\setminus B}) \le e^{-|\omega_{C\setminus B}|g_-} \le e^{-g_- d(B,\Lambda^c)/(2a)}.$$

Further, recall that the number of clusters in $|\mathcal{C}_B(\omega_B)| \le K$ is finite where $K = K(r)$. Thus (9) is bounded from above by $2^K e^{-g_- d(B,\Lambda^c)/(2a)}$ which tends to zero as Λ tends to $\mathbb{R}^d$. For $t = t_G$ note that, using Lemma 4.8, instead of $\|\cdot\|$ we can consider

$$\|\gamma_B^{+_{\Lambda^c}}(f|\cdot)-\gamma_B^\infty(f|\cdot)\|_{t_G} = \sup_{\hat{\boldsymbol{\omega}}\in\boldsymbol{\Omega}^{t_G}} |\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B^c}) - \gamma_B^{+_{\Lambda^c}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B})|.$$

In this case $\rho(\hat{\boldsymbol{\omega}}_{C\setminus B}) \le \alpha^{-|\omega_{C\setminus B}|(1+p_{t_G})} \le \alpha^{-d(B,\Lambda^c)/(2a)}$ since $(1+p_{t_G}) > 1$ and thus also in this case (9) tends to zero as Λ tends to $\mathbb{R}^d$.

W.r.t. the first summand in (8) we use very similar arguments. Resolving the color expectation and separating common factors, we have the following upper bound

$$\|f\|e^{3\lambda_+|B|}\int P_B^{4\lambda_+}(d\omega_B)\sup_{\hat{\sigma}_{\omega_B}}\Big(\prod_{C\in\mathcal{C}_B^{\mathrm{f},\Lambda^c}(\omega_B)}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap B})\rho(\hat{\boldsymbol{\omega}}_{C\setminus B})\big)-1\Big).$$

Since $\sup_{\hat{\sigma}_{\omega_B}}\rho(\hat{\boldsymbol{\omega}}_{C\cap B})\leq 1$ we can use the same upper bounds as above for both cases $t>t_G$ and $t=t_G$. □

[Proof of Proposition 4.13] The proof is a simplified version of the proof of Proposition 4.4. Since $t=\infty$ we can assume $f\in\boldsymbol{\mathcal{F}}_B^b$ to be color blind. In particular, we can follow the same steps as above with $f(\boldsymbol{\omega}_B)=f(\omega_B)=f^\infty(\omega_B)=f^\Lambda(\omega_B)$. Then the inequality (8) has the following form,

$$\begin{aligned}&|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B^c})-\gamma_B^{+_{\Lambda^c}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B})|\\&\leq 2\|f\|e^{\lambda_+|B|}\int P_B^+(d\omega_B)\big|2^{|\mathcal{C}_B^{\mathrm{f}}(\omega_B\omega_{\Lambda\setminus B})|}-\prod_{C\in\mathcal{C}_B(\omega_B\omega_{\Lambda\setminus B}+_{\Lambda^c})}\big(\mathbb{1}_{\sigma_{C\cap\Lambda^c}=+}+\mathbb{1}_{\sigma_{C\cap\Lambda^c}=-}\big)\big|.\end{aligned}$$

But the r.h.s. is zero for sufficiently large Λ, which finishes the proof. □

[Proof of Proposition 4.5] The proof is another variation of the proof of Proposition 4.4. Similar to the inequality (8), for boundary conditions $\hat{\boldsymbol{\omega}}^1,\hat{\boldsymbol{\omega}}^2\in\boldsymbol{\Omega}$ with $\hat{\boldsymbol{\omega}}^1_\Lambda=\hat{\boldsymbol{\omega}}^2_\Lambda$ we have

$$\begin{aligned}&|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}^1_{B^c})-\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}^2_{B^c})|\\&\quad\leq e^{\lambda_+|B|}\Big(\int P_B^+(d\omega_B)\big|f_1^\infty(\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f},1}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}^1_{C\setminus B})\big)\\&\qquad\qquad-f_2^\infty(\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f},2}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}^2_{C\setminus B})\big)\big|\\&\qquad+\|f\|\int P_B^+(d\omega_B)\big|\prod_{C\in\mathcal{C}_B^{\mathrm{f},1}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}^1_{C\setminus B})\big)\\&\qquad\qquad-\prod_{C\in\mathcal{C}_B^{\mathrm{f},2}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}^2_{C\setminus B})\big)\big|\Big)\end{aligned}\tag{10}$$

where we indicated the contributions of the different boundary conditions by attaching $1,2$. The second summand in (10), separating again w.r.t. $\mathcal{C}_B^\Lambda(\omega_B)$, can be bounded from above by

$$\begin{aligned}\|f\|e^{\lambda_+|B|}\int P_B^{2\lambda_+}(d\omega_B)\big|&\prod_{C\in\mathcal{C}_B^{\mathrm{f},1,\Lambda^c}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}^1_{C\setminus B})\big)\\&-\prod_{C\in\mathcal{C}_B^{\mathrm{f},2,\Lambda^c}(\omega_B)}\big(1+\alpha^{-|C\cap B|}\rho(\hat{\boldsymbol{\omega}}^2_{C\setminus B})\big)\big|.\end{aligned}\tag{11}$$

Now, if $g_->0$, again $\rho(\hat{\boldsymbol{\omega}}^{1,2}_{C\setminus B})\leq e^{-g_-d(B,\Lambda^c)/(2a)}$ and hence (11) can be bounded from above by $\|f\|e^{\lambda_+|B|}2^Ke^{-g_-d(B,\Lambda^c)/(2a)}$. For the other summand in (10), similar arguments

as above allow the following upper bound

$$\|f\|e^{3\lambda_+|B|}\int P_B^{4\lambda_+}(d\omega_B)\sup_{\hat{\sigma}_{\omega_B}}\Big|\prod_{C\in\mathcal{C}_B^{\mathrm{f},1,\Lambda^c}(\omega_B)}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap B})\rho(\hat{\boldsymbol{\omega}}^1_{C\setminus B})\big) - \prod_{C\in\mathcal{C}_B^{\mathrm{f},2,\Lambda^c}(\omega_B)}\big(1+\rho(\hat{\boldsymbol{\omega}}_{C\cap B})\rho(\hat{\boldsymbol{\omega}}^2_{C\setminus B})\big)\Big|. \tag{12}$$

Again, since $\sup_{\hat{\sigma}_{\omega_B}}\rho(\hat{\boldsymbol{\omega}}_{C\cap B})\leq 1$ we arrive at $\|f\|e^{4\lambda_+|B|}2^K e^{-g_-d(B,\Lambda^c)/(2a)}$ as an upper bound, which gives the desired exponential decay. □

[Proof of Proposition 4.9] Considering the proof of Proposition 4.5, note that the estimates (10), (11) and (12) also hold at the critical time. In particular we still have $\sup_{\hat{\sigma}_{\omega_B}}\rho(\hat{\boldsymbol{\omega}}_{C\cap B})\leq 1$. The difference lies in the fact that at the critical time we have $g_-\geq 0$ and not strictly greater then zero. Observe that $g_-(m)=0$ if and only if $m=-1$ and in particular, under the event $\boldsymbol{\Omega}^{m_{t_G}}$,

$$\rho(\hat{\boldsymbol{\omega}}^{1,2}_{C\setminus B})=\alpha^{-|\omega^{1,2}_{C\setminus B}|(1+m(\hat{\boldsymbol{\omega}}^{1,2}_{C\setminus B}))}\leq\alpha^{-|\omega_{C\cap\Lambda\setminus B}|(1+m(\hat{\boldsymbol{\omega}}_{C\cap\Lambda\setminus B}))}\leq\alpha^{-(1+m_{t_G}/2)d(B,\Lambda^c)/(2a)}$$

for sufficiently large Λ uniformly in all finitely many infinite clusters attached to B. □

[Proof of Proposition 4.6] First note that for $f\in\boldsymbol{\mathcal{F}}^b_\Lambda$, with $\Lambda\Subset\mathbb{R}^d$ and n sufficiently large such that $\Lambda\subset B_n$, we have

$$\begin{aligned}\mu_t(\gamma^\infty_\Lambda(f|\cdot)-f)&=\mu_t((\gamma^\infty_\Lambda(f|\cdot)-f)\mathbb{1}_{\{\Lambda\not\leftrightarrow\infty\}})=\mu_t((\gamma^{\mathrm{f}}_\Lambda(f|\cdot)-f)\mathbb{1}_{\{\Lambda\not\leftrightarrow\infty\}})\\&\leq\mu_t((\gamma^{\mathrm{f}}_\Lambda(f|\cdot)-f)\mathbb{1}_{\{\Lambda\not\leftrightarrow B_n^c\}})+2\|f\|\mu(\mathbb{1}_{\Lambda\not\leftrightarrow\infty}-\mathbb{1}_{\Lambda\not\leftrightarrow B_n^c}).\end{aligned}\tag{13}$$

Further note that by the definition of the low-intensity regime $\lim_{n\uparrow\infty}\mu(\{\Lambda\leftrightarrow B_n^c\})=0$ and hence for the second summand in (13)

$$\mu(\mathbb{1}_{\Lambda\not\leftrightarrow\infty}-\mathbb{1}_{\Lambda\not\leftrightarrow B_n^c})=\mu(\{\Lambda\not\leftrightarrow\infty\}\cap\{\Lambda\leftrightarrow B_n^c\})$$

tends to zero as n tends to infinity. As for the first summand in (13), let $\mu_t=\lim_{\Delta\uparrow\mathbb{R}^d}\mu_{t,\Delta}$ for some suitable boundary condition which we do not make explicit here. Then for $\Delta\supset B_n$, we can estimate

$$\begin{aligned}\mu_t((\gamma^{\mathrm{f}}_\Lambda(f|\cdot)-f)\mathbb{1}_{\Lambda\not\leftrightarrow B_n^c})&\leq\lim_{\Delta\uparrow\mathbb{R}^d}\mu_{t,\Delta}((\gamma^{\mathrm{f}}_\Lambda(f|\cdot)-f)\mathbb{1}_{\Lambda\not\leftrightarrow B_n^c})\\&=\lim_{\Delta\uparrow\mathbb{R}^d}\mu_{t,\Delta}((\gamma^{\cdot^\Delta}_\Lambda(f|\cdot)-f)\mathbb{1}_{\Lambda\not\leftrightarrow B_n^c})=0\end{aligned}$$

where we could replace γ^{f} by $\gamma^{\cdot^\Delta}$ due to the cluster-contraint $\{\Lambda\not\leftrightarrow B_n^c\}$. □

[Proof of Lemma 4.7] Recall that there can only be a finite number of clusters attached to B. For the given configuration $\hat{\boldsymbol{\omega}}$, take $B\subset\Lambda\Subset\mathbb{R}^d$ large enough such that all these clusters are fully contained in Λ, then the result follows. □

[Proof of Lemma 4.9] Again, there can only be a finite number of clusters attached to B and in this case colors of boundary conditions are irrelevant. For the given configuration $\hat{\boldsymbol{\omega}}$, take $B\subset\Lambda\Subset\mathbb{R}^d$ large enough such that all clusters but the unique infinite one are fully contained in Λ, then the result follows. □

[Proof of Proposition 4.11] The proof is analog to the proof of Lemma 4.6. □

[Proof of Proposition 4.12] The main idea for the proof is that any boundary magnetization in a first finite annulus can be uniformly dominated by a large enough but finite second annulus as long as the number of points in the first annulus is uniformly bounded. Indeed, let $\hat{\boldsymbol{\omega}} \in \{B \leftrightarrow B_n^{\mathrm{c}}\}$ with $|\omega_{\Lambda\setminus B}| < L$ and assume for simplicity of the proof, that in $\hat{\boldsymbol{\omega}}$ there is a single cluster C' connected to B from the outside, which then must connect B and B_n^{c}. This is a minor simplification since the number of clusters connected to B can only be finite. Then, by definition,

$$\begin{aligned}
&\big|\gamma_B^{\mathrm{f}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}) - \gamma_B^{\mathrm{f}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda})\big| \\
&= \Big|\frac{\int P_B^-(d\omega_B) f_+^{\mathrm{f}}(\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}\big(\alpha^{|C\cap B|} + \rho(\hat{\boldsymbol{\omega}}_{C\cap\Lambda\setminus B}\boldsymbol{\omega}^+_{C\cap B_n\setminus\Lambda})\big)}{\int P_B^-(d\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}\big(\alpha^{|C\cap B|} + \rho(\hat{\boldsymbol{\omega}}_{C\cap\Lambda\setminus B}\boldsymbol{\omega}^+_{C\cap B_n\setminus\Lambda})\big)} \\
&\qquad - \frac{\int P_B(d\omega_B) f_-^{\mathrm{f}}(\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}\big(\alpha^{|C\cap B|} + \rho(\hat{\boldsymbol{\omega}}_{C\cap\Lambda\setminus B}\boldsymbol{\omega}^-_{C\cap B_n\setminus\Lambda})\big)}{\int P_B^-(d\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}\big(\alpha^{|C\cap B|} + \rho(\hat{\boldsymbol{\omega}}_{C\cap\Lambda\setminus B}\boldsymbol{\omega}^-_{C\cap B_n\setminus\Lambda})\big)}\Big|.
\end{aligned} \tag{14}$$

Recall that the crucial ingredient in the switch ρ is the sign of the quantity $g(m)$ and note that

$$m(\hat{\boldsymbol{\omega}}_{C'\cap\Lambda\setminus B}\boldsymbol{\omega}^+_{C'\cap B_n\setminus\Lambda}) \geq \frac{|\boldsymbol{\omega}^+_{C'\cap B_n\setminus\Lambda}| - |\hat{\boldsymbol{\omega}}_{C'\cap\Lambda\setminus B}|}{|\boldsymbol{\omega}^+_{C'\cap B_n\setminus\Lambda}| + |\hat{\boldsymbol{\omega}}_{C'\cap\Lambda\setminus B}|} \geq \frac{1-2aL/d(B_n^{\mathrm{c}},\Lambda)}{1+2aL/d(B_n^{\mathrm{c}},\Lambda)}$$

which becomes arbitrarily close to 1 for sufficiently large n. On the other hand,

$$m(\hat{\boldsymbol{\omega}}_{C'\cap\Lambda\setminus B}\boldsymbol{\omega}^-_{C'\cap B_n\setminus\Lambda}) \leq \frac{|\hat{\boldsymbol{\omega}}_{C'\cap\Lambda\setminus B}| - |\boldsymbol{\omega}^-_{C'\cap B_n\setminus\Lambda}|}{|\hat{\boldsymbol{\omega}}_{C'\cap\Lambda\setminus B}| + |\boldsymbol{\omega}^-_{C'\cap B_n\setminus\Lambda}|} \leq \frac{2aL/d(B_n^{\mathrm{c}},\Lambda)-1}{2aL/d(B_n^{\mathrm{c}},\Lambda)+1}$$

which becomes arbitrarily close to -1 for sufficiently large n. Now, since the switch can be activated, there exists $n_{\Lambda,K}$ such that for all larger n we have

$$g_- = g(m(\hat{\boldsymbol{\omega}}_{C'\cap\Lambda\setminus B}\boldsymbol{\omega}^-_{C'\cap B_n\setminus\Lambda})) < 0 \text{ and } g_+ = g(m(\hat{\boldsymbol{\omega}}_{C'\cap\Lambda\setminus B}\boldsymbol{\omega}^+_{C'\cap B_n\setminus\Lambda})) > 0.$$

In particular, in the asymmetric case, we have

$$\begin{aligned}
&\big|\gamma_B^{\mathrm{f}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}) - \gamma_B^{\mathrm{f}}(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda})\big| \\
&= \Big|\frac{\int P_B^-(d\omega_B) f_+^{\mathrm{f}}(\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}(\alpha^{|C\cap B|} + e^{-|C\cap B_n\setminus B|g_+})}{\int P_B^-(d\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}(\alpha^{|C\cap B|} + e^{-|C\cap B_n\setminus B|g_+})} \\
&\qquad - \frac{\int P_B^-(d\omega_B) f_-^{\mathrm{f}}(\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}(\alpha^{|C\cap B|}e^{|C\cap B_n\setminus B|g_-} + 1)}{\int P_B^-(d\omega_B)\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)}(\alpha^{|C\cap B|}e^{|C\cap B_n\setminus B|g_-} + 1)}\Big|
\end{aligned}$$

and note that the boundary condition also appears in $f_\pm^{\mathrm{f}}$. Let $f = \mathbb{1}_{\emptyset_B}$, then $f = f_\pm^{\mathrm{f}}$ and the above is bounded from below by

$$\begin{aligned}
e^{-\lambda_-|B|}\Big|\int P_B^-(d\omega_B) &\prod_{C\in\mathcal{C}_B^{\mathrm{f}}(\omega_B)\setminus C'}(\alpha^{|C\cap B|}+1) \\
&\times [\alpha^{|C'\cap B|}(1-e^{|C'\cap B_n\setminus B|g_-}) - (1-e^{-|C'\cap B_n\setminus B|g_+})]\Big|.
\end{aligned} \tag{15}$$

Note that $|C' \cap B_n \setminus B| \geq n/2a$ and thus there exists $\delta > 0$ such that for sufficiently large n we have

$$\alpha^{|C' \cap B|} \geq \alpha > \frac{1 - e^{-|C' \cap B_n \setminus B| g_+}}{1 - e^{|C' \cap B_n \setminus B| g_-}} + \delta$$

and $\exp(|C' \cap B_n \setminus B| g_-) < 1/2$. This implies the following lower bound for (15),

$$\delta e^{-\lambda_- |B|} \int P_B^-(d\omega_B) \prod_{C \in \mathcal{C}_B^{\mathrm{f}}(\omega_B) \setminus C'} (\alpha^{|C \cap B|} + 1) \geq \delta e^{-2\lambda_- |B|}.$$

In the symmetric case we can proceed similar. Using the same notation, we have

$$\begin{aligned} &\left|\gamma_B^{\mathrm{f}}(f|\hat{\boldsymbol{\omega}}_{\Lambda \setminus B} \boldsymbol{\omega}^+_{B_n \setminus \Lambda}) - \gamma_B^{\mathrm{f}}(f|\hat{\boldsymbol{\omega}}_{\Lambda \setminus B} \boldsymbol{\omega}^-_{B_n \setminus \Lambda})\right| \\ &= \Big| \frac{\int P_B^-(d\omega_B) f_+^{\mathrm{f}}(\omega_B) \prod_{C \in \mathcal{C}_B^{\mathrm{f}}(\omega_B)} (1 + e^{-|C \cap B_n \setminus B| g_+})}{\int P_B^-(d\omega_B) \prod_{C \in \mathcal{C}_B^{\mathrm{f}}(\omega_B)} (1 + e^{-|C \cap B_n \setminus B| g_+})} \\ &\qquad - \frac{\int P_B^-(d\omega_B) f_-^{\mathrm{f}}(\omega_B) \prod_{C \in \mathcal{C}_B^{\mathrm{f}}(\omega_B)} (1 + e^{|C \cap B_n \setminus B| g_-})}{\int P_B^-(d\omega_B)(d\omega_B) \prod_{C \in \mathcal{C}_B^{\mathrm{f}}(\omega_B)} (1 + e^{|C \cap B_n \setminus B| g_-})} \Big|. \end{aligned} \tag{16}$$

Now we have to use a color dependent observable f to exhibit lower bounds larger then zero. For example, take $f(\boldsymbol{\omega}_B) = \mathbb{1}_{+_{\omega_B}}$, then we have

$$f_\pm^{\mathrm{f}}(\omega_B) = \nu_B^{\mathrm{f}}(f(\omega_B, \cdot)|\hat{\boldsymbol{\omega}}_{\Lambda \setminus B} \boldsymbol{\omega}^\pm_{B_n \setminus \Lambda}) = \nu_B^{\mathrm{f}}(+_{\omega_B}|\hat{\boldsymbol{\omega}}_{\Lambda \setminus B} \boldsymbol{\omega}^\pm_{B_n \setminus \Lambda})$$

where for $g_- < 0 < g_+$,

$$\nu_B^{\mathrm{f}}(+_{\omega_B}|\hat{\boldsymbol{\omega}}_{\Lambda \setminus B} \boldsymbol{\omega}^\pm_{B_n \setminus \Lambda}, \omega_B) = \frac{\prod_{C \in \mathcal{C}_B^{\mathrm{f}}(\omega_B)} \left(p_t(+,+)^{|C \cap B|} + p_t(-,+)^{|C \cap B|} e^{\mp |C \cap B_n \setminus B| g_\pm}\right)}{\prod_{C \in \mathcal{C}_B^{\mathrm{f}}(\omega_B)} \left(1 + e^{\mp |C \cap B_n \setminus B| g_\pm}\right)}.$$

In particular, inserting this into (16) we can bound (16) from below by

$$\begin{aligned} &\Big| \int P_B^-(d\omega_B) \prod_{C \in \mathcal{C}_B^{\mathrm{f}}(\omega_B) \setminus C'} \left(p_t(+,+)^{|C \cap B|} + p_t(-,+)^{|C \cap B|}\right) \\ &\times \left(p_t(+,+)^{|C' \cap B|} (1 - e^{|C' \cap B_n \setminus B| g_-}) - p_t(-,+)^{|C' \cap B|} (1 - e^{-|C' \cap B_n \setminus B| g_+})\right) \Big|. \end{aligned} \tag{17}$$

Similar to the asymmetric case, there exists $\delta > 0$ such that for sufficiently large n

$$\left(\frac{p_t(+,+)}{p_t(-,+)}\right)^{|C' \cap B|} \geq \frac{p_t(+,+)}{p_t(-,+)} > \frac{1 - e^{-|C' \cap B_n \setminus B| g_+}}{1 - e^{|C' \cap B_n \setminus B| g_-}} + \delta$$

and $\exp(|C' \cap B_n \setminus B| g_-) < 1/2$. For such n we thus get as a lower bound for (17),

$$\delta \int P_B^-(d\omega_B) 2^{-|\omega_B|} p_t(-,+)^{|\omega_B|} \geq \delta e^{-\lambda_- |B|}.$$

This finishes the proof. □

[Proof of Proposition 4.14] Let $t = \infty$ in the symmetric regime, $f = \mathbb{1}_{\emptyset_B}$ and $\hat{\boldsymbol{\omega}} \in \{B \leftrightarrow \infty\}$ then, for sufficiently large Λ we have

$$\begin{aligned}\left|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B^c})\right| &\geq e^{-\lambda_-|B|}\int P_B^-(d\omega_B)(2^{\mathcal{C}_B^{\mathrm{f}}(\omega_B\omega_{\Lambda\setminus B})} - 2^{\mathcal{C}_B^{\mathrm{f}}(\omega_B\omega_{B^c})})\\ &\geq \tfrac{1}{2}e^{-\lambda_-|B|}\int P_B^-(d\omega_B)2^{\mathcal{C}_B^{\mathrm{f}}(\omega_B\omega_{\Lambda\setminus B})} \geq \tfrac{1}{2}e^{-2\lambda_-|B|}\end{aligned}$$

as required. □

[Proof of Theorem 3.3] The idea of the proof is to compare a given μ_t-a.s. continuous specification $\tilde{\gamma}$ to the discontinuous kernel γ^{f} and derive a contradiction. Discontinuities of γ^{f} are based on percolating boundary conditions under a change of coloring. We therefor consider a stochastic kernel, acting only on the colors in a given configuration in the volume Λ, given by

$$\int M_\Lambda(d\tilde{\sigma}_{\omega_\Lambda}|\boldsymbol{\omega})f(\boldsymbol{\omega}_{\Lambda^c},\omega_\Lambda,\tilde{\sigma}_{\omega_\Lambda}) = [\prod_{x\in\omega_\Lambda}\int q(d\tilde{\sigma}_x)]f(\boldsymbol{\omega}_{\Lambda^c},\omega_\Lambda,\tilde{\sigma}_{\omega_\Lambda})$$

where $q(\sigma) = 1/2$. In words, under M_Λ, the color distribution on a given grey configuration ω_Λ is iid equidistributed. We can replace $\tilde{\gamma}$ by γ^{f} under the μ_t-integral only for non-percolating configurations. Hence, consider further the joint distribution $\bar{\mu}$ of the random elements $(\boldsymbol{\omega},\hat{\boldsymbol{\omega}},\boldsymbol{\omega}^1,\boldsymbol{\omega}^2)$, given by

$$\bar{\mu}_t(d\boldsymbol{\omega},d\hat{\boldsymbol{\omega}},d\boldsymbol{\omega}^1_{B_n\setminus\Lambda},d\boldsymbol{\omega}^2_{B_n\setminus\Lambda}) = \mu(d\boldsymbol{\omega})\mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega})M_{B_n\setminus\Lambda}(d\sigma^1_{\omega_{B_n\setminus\Lambda}}|\hat{\boldsymbol{\omega}})M_{B_n\setminus\Lambda}(d\sigma^2_{\omega_{B_n\setminus\Lambda}}|\hat{\boldsymbol{\omega}})$$

for $\Lambda \subset B_n$, where μ the WRM and $\mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega})$ the independent spin-flip transition kernel. Note that for $\int \bar{\mu}_t(d\boldsymbol{\omega},d\hat{\boldsymbol{\omega}},d\boldsymbol{\omega}^1_{B_n\setminus\Lambda},d\boldsymbol{\omega}^2_{B_n\setminus\Lambda})f(\hat{\boldsymbol{\omega}}) = \int \mu_t(d\hat{\boldsymbol{\omega}})f(\hat{\boldsymbol{\omega}})$.

Recall that we write $\boldsymbol{\omega}^\pm$ for configurations where all signs are fixed to be $\pm$. As a first step, we prove that the continuity assumption on $\tilde{\gamma}$ leads to a contradiction. As a second step, we prove that bad points for $\tilde{\gamma}$ have full mass under μ_t. Let us define the integral

$$\begin{aligned}I^\delta_{\Lambda,n} &= \int \bar{\mu}_t(d\boldsymbol{\omega},d\hat{\boldsymbol{\omega}},d\boldsymbol{\omega}^1_{B_n\setminus\Lambda},d\boldsymbol{\omega}^2_{B_n\setminus\Lambda})\gamma^{-1}_{\bar{B}_n\setminus B_n}(\mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}|\boldsymbol{\omega})\mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}(\omega)\mathbb{1}_{+_{\omega_{B_n\setminus\Lambda}}}(\sigma^1_{\omega_{B_n\setminus\Lambda}})\\ &\quad\times \mathbb{1}_{-_{\omega_{B_n\setminus\Lambda}}}(\sigma^2_{\omega_{B_n\setminus\Lambda}})4^{|\omega_{B_n\setminus\Lambda}|}\mathbb{1}_{|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}\hat{\boldsymbol{\omega}}_{B_n^c})-\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda}\hat{\boldsymbol{\omega}}_{B_n^c})|>\delta}\\ &= \int \mu(d\boldsymbol{\omega})\int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega})g_n(\boldsymbol{\omega})\mathbb{1}_{|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}\hat{\boldsymbol{\omega}}_{B_n^c})-\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda}\hat{\boldsymbol{\omega}}_{B_n^c})|>\delta}.\end{aligned}$$

where $g_n(\boldsymbol{\omega}) = \gamma^{-1}_{\bar{B}_n\setminus B_n}(\{\emptyset_{\bar{B}_n\setminus B_n}\}|\boldsymbol{\omega})\mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}(\omega)$ is an integrable density with γ the specification of the WRM. The indicator in g_n, which decouples B_n from B_n^c, will later allow us to replace $\tilde{\gamma}$ by γ^{f}. By the continuity assumption on $\tilde{\gamma}$, we have

$$\begin{aligned}I^\delta_{\Lambda,n} &\leq \int \mu(d\boldsymbol{\omega})g_n(\boldsymbol{\omega})\int \mu_t(d\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}|\boldsymbol{\omega})\mathbb{1}_{\sup_{\boldsymbol{\omega}^{1,2}}|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c})-\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c})|>\delta}\\ &= \int \mu(d\boldsymbol{\omega})\gamma_{\bar{B}_n\setminus B_n}\Big(g_n\int \mu_t(d\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}|\cdot)\mathbb{1}_{\sup_{\boldsymbol{\omega}^{1,2}}|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c})-\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c})|>\delta}|\boldsymbol{\omega}\Big)\\ &= \int \mu(d\boldsymbol{\omega})\gamma_{\bar{B}_n\setminus B_n}(g_n|\boldsymbol{\omega})\int \mu_t(d\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}|\boldsymbol{\omega})\mathbb{1}_{\sup_{\boldsymbol{\omega}^{1,2}}|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c})-\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c})|>\delta}\\ &= \int \mu_t(d\hat{\boldsymbol{\omega}})\mathbb{1}_{\sup_{\boldsymbol{\omega}^{1,2}}|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c})-\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c})|>\delta}\end{aligned}$$

where in the second last step, we pulled out the integral in $\gamma_{\bar{B}_n \setminus B_n}$ using properness. By dominated convergence, using the assumed continuity of $\tilde{\gamma}$, this tends to zero as Λ tends to $\mathbb{R}^d$ for all $\delta > 0$ and $f \in \boldsymbol{\mathcal{F}}^b$.

In order to derive a contradiction, note that since μ_t-a.s. on the decoupling event $\{\emptyset_{\bar{B}_n \setminus B_n}\}$ we have $\tilde{\gamma}_B = \gamma^{\mathrm{f}}_B$. Using Proposition 4.11, we can now replace $\tilde{\gamma}_B$ by the kernel γ^{f}_B in $I_{\Lambda,n}$. Then, by Proposition 4.12, for all $L > 0$, some $f \in \boldsymbol{\mathcal{F}}$, $\delta > 0$ and sufficiently large n we can estimate

$$\begin{aligned}
I^{\delta}_{\Lambda,n} &= \int \mu(d\boldsymbol{\omega}) \int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_n(\boldsymbol{\omega}) \mathbb{1}_{|\gamma^{\mathrm{f}}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}) - \gamma^{\mathrm{f}}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda})|>\delta} \\
&\geq \int \mu(d\boldsymbol{\omega}) \int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_n(\boldsymbol{\omega}) \mathbb{1}_{\{B \leftrightarrow B^{\mathrm{c}}_n\}}(\boldsymbol{\omega}) \mathbb{1}_{\{|\omega_{\Lambda\setminus B}|<K\}}(\boldsymbol{\omega}) \\
&\qquad\qquad \times \mathbb{1}_{|\gamma^{\mathrm{f}}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}) - \gamma^{\mathrm{f}}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda})|>\delta} \\
&\geq \int \mu(d\boldsymbol{\omega}) \mathbb{1}_{\{B \leftrightarrow B^{\mathrm{c}}_n\}}(\boldsymbol{\omega}) \mathbb{1}_{\{|\omega_{\Lambda\setminus B}|<K\}}(\boldsymbol{\omega}) \\
&\geq \int \mu(d\boldsymbol{\omega}) \mathbb{1}_{\{B \leftrightarrow \infty\}}(\boldsymbol{\omega}) \mathbb{1}_{\{|\omega_{\Lambda\setminus B}|<K\}}(\boldsymbol{\omega}).
\end{aligned}$$

Since this is true for all $L > 0$ and by assumption $\mu(\{B \leftrightarrow \infty\}) > 0$, we arrive at the desired contradiction.

As for the almost-sure discontinuity, note that $\lim_{\Delta\uparrow\mathbb{R}^d} \mu(\{\Delta \leftrightarrow \infty\}) = 1$ and thus, for sufficiently large Δ,

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \int \mu_t(d\hat{\boldsymbol{\omega}}) \mathbb{1}_{\sup_{\boldsymbol{\omega}^{1,2}} |\tilde{\gamma}_\Delta(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus\Delta}\boldsymbol{\omega}^1_{\Lambda^{\mathrm{c}}}) - \tilde{\gamma}_\Delta(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus\Delta}\boldsymbol{\omega}^2_{\Lambda^{\mathrm{c}}})|>\delta} > 1 - \varepsilon$$

for any specification $\tilde{\gamma}$ of μ_t. From this we see that the set of bad configurations for $\tilde{\gamma}$ even has full mass under μ_t. □

[Proof of Theorem 3.4 - non-asq-Gibbsian part] Here we consider the symmetric regime with $t = \infty$. First note that, similar to the above for some given specification $\tilde{\gamma}$, using Proposition 4.13 we have

$$\begin{aligned}
&\int \mu^+_\infty(d\hat{\boldsymbol{\omega}}) \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B} \emptyset_{\bar{B}_n\setminus B_n} \hat{\boldsymbol{\omega}}_{(\bar{B}_n)^c}) \\
&\quad = \int \mu^+(d\boldsymbol{\omega}) \gamma^{-1}_{\bar{B}_n\setminus B_n}(\mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}|\boldsymbol{\omega}) \mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}(\omega) \int \mu_\infty(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B^c}) \\
&\quad = \int \mu^+(d\boldsymbol{\omega}) \gamma^{-1}_{\bar{B}_n\setminus B_n}(\mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}|\boldsymbol{\omega}) \mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}(\omega) \int \mu_\infty(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) \gamma^\infty_B(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}) \\
&\quad = \int \mu^+_\infty(d\hat{\boldsymbol{\omega}}) \gamma^\infty_B(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}).
\end{aligned} \tag{18}$$

Hence we have on the one hand,

$$\begin{aligned}
&\int \mu^+_\infty(d\hat{\boldsymbol{\omega}}) \mathbb{1}_{\{|\tilde{\gamma}_B(f|\boldsymbol{\omega}^-_{B_n\setminus B} \emptyset_{\bar{B}_n\setminus B_n} \hat{\boldsymbol{\omega}}_{(\bar{B}_n)^c}) - \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B^c})|>\delta\}} \\
&\quad \leq \int \mu^+_\infty(d\hat{\boldsymbol{\omega}}) \mathbb{1}_{\{\sup_{\boldsymbol{\omega}^{1,2}\in\boldsymbol{\Omega}} |\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}\boldsymbol{\omega}^1_{B^c_n}) - \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}\boldsymbol{\omega}^2_{B^c_n})|>\delta\}}
\end{aligned}$$

which tends to zero as n tends to infinity if we assume $\tilde{\gamma}$ to be almost-surely quasilocal. On the other hand, by Propositions 4.13 and 4.14, there exists $\delta > 0$ and $f \in \boldsymbol{\mathcal{F}}^b$ such that for sufficiently large n we have

$$\begin{aligned}
&\int \mu_\infty^+(d\hat{\boldsymbol{\omega}}) \mathbb{1}_{\{|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}\emptyset_{\bar{B}_n\setminus B_n}\hat{\boldsymbol{\omega}}_{(\bar{B}_n)^c}) - \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B^c})|>\delta\}} \\
&\quad = \int \mu_\infty^+(d\hat{\boldsymbol{\omega}}) \mathbb{1}_{\{|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B^c})|>\delta\}} \\
&\quad \geq \int \mu_\infty^+(d\hat{\boldsymbol{\omega}}) \mathbb{1}_{\{B\leftrightarrow\infty\}}(\hat{\boldsymbol{\omega}}) \mathbb{1}_{\{|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B^c})|>\delta\}} = \mu_t(\{B \leftrightarrow \infty\}) > 0,
\end{aligned}$$

which is a contradiction. As above letting B grow, we see that the set of discontinuity points has full mass. □

[Proof of Theorem 3.2 - non-q-Gibbsian part] The idea of the proof is to exhibit a boundary condition consisting of a unique infinite cluster attached to B. We consider two randomizations of this boundary configurations, first w.r.t. the Lebesgues measures and second w.r.t. μ. This allows us to first replace any given specification $\tilde{\gamma}$ by our known partial specification γ^{f} which is discontinuous at any such boundary condition. Second, using Lebesgue's density theorem we have then deduce that $\tilde{\gamma}$ can not be quasilocal for all such boundary conditions. More precisely, let $\tilde{\gamma}$ be a given specification for μ_t. We show existence of a configuration $\hat{\boldsymbol{\omega}}$ such that

$$\limsup_{\Lambda\uparrow\mathbb{R}^d} \sup_{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2\in\boldsymbol{\Omega}} \left|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c}) - \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c})\right| > 0.$$

Let us define $\boldsymbol{\eta} = (\eta, +_\eta)$ with

$$\eta = \{x \in \mathbb{R}^d : x_1 = na/2 \text{ for some } n \in \mathbb{N}_0 \text{ and } x_i = 0 \text{ for } 2 \leq i \leq d\}.$$

In particular, η consists of a unique cluster in $\{B \leftrightarrow \infty\}$. Define a ε-vicinity of $\boldsymbol{\eta}$ by

$$V_\varepsilon(\eta) = \{\boldsymbol{\omega} \in \boldsymbol{\Omega} : \text{ for all } x \in \eta \text{ there exists exactly one } y \in \omega \text{ such that } |y - x| < \varepsilon\}$$

and note that for $0 < \varepsilon < a/4$, we have $V_\varepsilon(\eta) \subset \{B \leftrightarrow \infty\}$. See Figure 3 for an illustration.

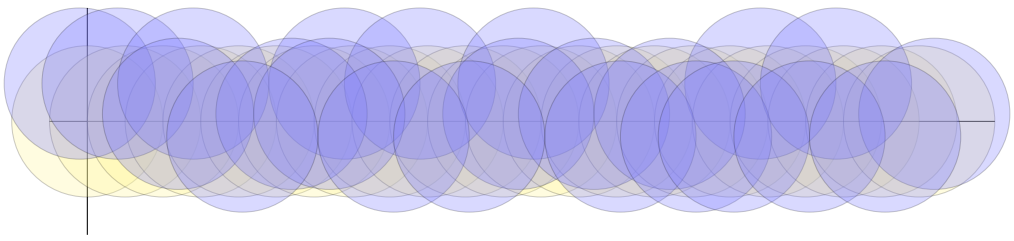

Figure 3: Illustration of the configuration η in yellow and a pertubation in $V_\varepsilon(\eta)$ in blue.

The non-critical case: Let $0 < t < \infty$ for the symmetric case or $0 < t < t_G$ for the asymmetric case and let

$$g_\varepsilon^n[\xi](\boldsymbol{\omega}) = \mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}(\boldsymbol{\omega}) \mathbb{1}_{V_\varepsilon(\xi_{B_n})}(\boldsymbol{\omega}) \gamma_{\bar{B}_n}^{-1}(V_\varepsilon(\xi_{B_n}) \cap \{\emptyset_{\bar{B}_n\setminus B_n}\}|\boldsymbol{\omega})$$

and $V = V_{a/8}$. We consider the integral

$$\begin{aligned} I_{\Lambda,n} &= |V(\eta_{B_n})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_n})}(\xi) \int \mu(d\boldsymbol{\omega}) \int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \\ &\quad \times \left| \tilde{\gamma}_B(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}\hat{\boldsymbol{\omega}}_{B_n^c}) - \tilde{\gamma}_B(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda}\hat{\boldsymbol{\omega}}_{B_n^c}) \right| \\ &= |V(\eta_{B_n})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_n})}(\xi) \int \mu(d\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \\ &\quad \times \left| \gamma_B^{\mathrm{f}}(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}) - \gamma_B^{\mathrm{f}}(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda}) \right| \end{aligned}$$

where we additionally randomize the target configuration η by ξ drawn from the Lebesgue measure on $\mathbb{R}^d$. Then by Proposition 4.12, for all $L > 0$, some $f \in \boldsymbol{\mathcal{F}}$, $\delta > 0$ and sufficiently large n we can estimate

$$I_{\Lambda,n} \geq \delta |V(\eta_{B_n})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_n})}(\xi) \int \mu(d\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \mathbb{1}_{\{|\omega_{\Lambda\setminus B}|<L\}}(\boldsymbol{\omega}).$$

Assuming n to be even larger, also the indicator $\mathbb{1}_{\{|\omega_{\Lambda\setminus B}|<L\}}$ can be dropped, since $V_\varepsilon(\boldsymbol{\eta})$ constrains the number of points $|\omega_{\Lambda\setminus B}|$. This implies $I_{\Lambda,n} \geq \delta$ for all $a/8 > \varepsilon > 0$ and n larger then some $n(\Lambda)$. On the other hand,

$$\begin{aligned} I_{\Lambda,n} &= |V(\eta_{B_n})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_n})}(\xi) \int \mu(d\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) \\ &\quad \times \left| \tilde{\gamma}_B(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^+_{B_n\setminus\Lambda}\hat{\boldsymbol{\omega}}_{B_n^c}) - \tilde{\gamma}_B(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^-_{B_n\setminus\Lambda}\hat{\boldsymbol{\omega}}_{B_n^c}) \right| \\ &\leq |V(\eta_{B_n})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_n})}(\xi) \int \mu(d\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \\ &\quad \times \sup_{\boldsymbol{\omega}^{1,2}} \left| \tilde{\gamma}_B(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c}) - \tilde{\gamma}_B(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c}) \right| \\ &= |V(\eta_{B_n})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_n})}(\xi) \int \mu(d\boldsymbol{\omega}) \gamma_{\bar{B}_n}\big(g_\varepsilon^n[\xi]\tilde{f}\big|\boldsymbol{\omega}\big) \end{aligned}$$

where we wrote $\tilde{f}(\omega) = \sup_{\boldsymbol{\omega}^{1,2}} \left| \tilde{\gamma}_B(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^1_{\Lambda^c}) - \tilde{\gamma}_B(f|\boldsymbol{\omega}^+_{\Lambda\setminus B}\boldsymbol{\omega}^2_{\Lambda^c}) \right|$. Note that $\omega \mapsto \tilde{f}(\omega)$ is $\boldsymbol{\mathcal{F}}_{\Lambda\setminus B}$-measurable, since the integral is w.r.t. the spin flip only. We can further calculate for any $\boldsymbol{\omega}'$

$$\begin{aligned} \gamma_{\bar{B}_n}(g_\varepsilon^n[\xi]\tilde{f}|\boldsymbol{\omega}') &= \frac{\int P_{\bar{B}_n}(d\omega) \mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}(\omega) \mathbb{1}_{V_\varepsilon(\xi_{B_n})}(\omega) \tilde{f}(\omega_{\Lambda\setminus B}) W_{B_n}(\omega)}{\int P_{\bar{B}_n}(d\omega) \mathbb{1}_{\emptyset_{\bar{B}_n\setminus B_n}}(\omega) \mathbb{1}_{V_\varepsilon(\xi_{B_n})}(\omega) W_{B_n}(\omega)} \\ &= |B_\varepsilon|^{-|\xi_{B_n}|} [\prod_{x\in\xi_{B_n}} \int_{B_\varepsilon(x)}] d\omega \tilde{f}(\omega_{\Lambda\setminus B}) \\ &= |B_\varepsilon|^{-|\xi_{\Lambda\setminus B}|} [\prod_{x\in\xi_{\Lambda\setminus B}} \int_{B_\varepsilon(x)}] d\omega \tilde{f}(\omega_{\Lambda\setminus B}) \end{aligned}$$

where we used that the dependents on $\boldsymbol{\omega}'$ can be dropped due to the decoupling event, the measurability of $\tilde{f}$ and the internal color constraint W_{B_n} is constant on $V_\varepsilon(\eta)$. Thus we arrive

at the estimate

$$I_{\Lambda,n} \leq |V(\eta_{\Lambda\setminus B})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{\Lambda\setminus B})}(\xi)|B_\varepsilon|^{-|\xi_{\Lambda\setminus B}|}[\prod_{x\in\xi_{\Lambda\setminus B}} \int_{B_\varepsilon(x)}]d\omega \tilde{f}(\omega_{\Lambda\setminus B}).$$

By Lebesgue's differentiation theorem, the set,

$$\{\xi : \limsup_{\varepsilon\downarrow 0} |B_\varepsilon|^{-|\xi_{\Lambda\setminus B}|}[\prod_{x\in\xi_{\Lambda\setminus B}} \int_{B_\varepsilon(x)}]d\omega \tilde{f}(\omega) \neq \tilde{f}(\xi)\}$$

has Lebesgue measure zero. Hence, using the lower bound, derived above,

$$\delta \leq I_{\Lambda,n} \leq |V(\eta)|^{-1} \int d\xi \mathbb{1}_{V(\eta)}(\xi)\tilde{f}(\xi_{\Lambda\setminus B}).$$

Finally, if $\lim_{\Lambda\uparrow\mathbb{R}^d} \tilde{f}(\xi_{\Lambda\setminus B}) = 0$ for all ξ, by dominated convergence, the r.h.s. would tend to zero, which leads to a contradiction. Hence there exists $\xi \in V(\eta)$ such that $\lim_{\Lambda\uparrow\mathbb{R}^d} \tilde{f}(\xi_{\Lambda\setminus B}) > 0$ as required.

The critical asymmetric case: Using again the kernel M we have with $f = \mathbb{1}_{\emptyset_B}$ that

$$\begin{aligned}
I_{\Lambda,n} &= |V(\eta_{B_{2n}})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_{2n}})}(\xi) \int \mu(d\boldsymbol{\omega}) \int \mu_{t_G}(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_\varepsilon^{2n}[\xi](\boldsymbol{\omega}) \\
&\quad \times \left|\tilde{\gamma}_B(f|\boldsymbol{\omega}^-_{B_{2n}\setminus B}\emptyset_{\bar{B}_{2n}\setminus B_{2n}}\hat{\boldsymbol{\omega}}_{(\bar{B}_{2n})^c}) - \tilde{\gamma}_B(f|\boldsymbol{\omega}^-_{B_n\setminus B}\boldsymbol{\omega}^+_{B_{2n}\setminus B_n}\emptyset_{\bar{B}_{2n}\setminus B_{2n}}\hat{\boldsymbol{\omega}}_{(\bar{B}_{2n})^c})\right| \\
&= |V(\eta_{B_{2n}})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_{2n}})}(\xi) \int \mu(d\boldsymbol{\omega}) g_\varepsilon^{2n}[\xi](\boldsymbol{\omega}) \\
&\quad \times \left|\gamma_B^{\mathrm{f}}(f|\boldsymbol{\omega}^-_{B_{2n}\setminus B}) - \gamma_B^{\mathrm{f}}(f|\boldsymbol{\omega}^-_{B_n\setminus B}\boldsymbol{\omega}^+_{B_{2n}\setminus B_n})\right| \\
&\geq e^{-3\lambda_+|B|}|V(\eta_{B_{2n}})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_{2n}})}(\xi) \int \mu(d\boldsymbol{\omega}) g_\varepsilon^{2n}[\xi](\boldsymbol{\omega}) \int P_B^+(d\omega_B)(1-\alpha^{-2n/a}) \\
&= e^{-3\lambda_+|B|} \int P_B^+(d\omega_B)(1-\alpha^{-2n/a}) \geq e^{-4\lambda_+|B|}
\end{aligned}$$

for sufficiently large n. On the other hand,

$$\begin{aligned}
I_{\Lambda,n} &= |V^-(\eta_{B_{2n}})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_{2n}})}(\xi) \int \mu(d\boldsymbol{\omega}) \int \mu_{t_G}(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_\varepsilon^{2n}[\xi](\boldsymbol{\omega}) \\
&\quad \times \left|\tilde{\gamma}_B(f|\boldsymbol{\omega}^-_{B_{2n}\setminus B}\emptyset_{\bar{B}_{2n}\setminus B_{2n}}\hat{\boldsymbol{\omega}}_{(\bar{B}_{2n})^c}) - \tilde{\gamma}_B(f|\boldsymbol{\omega}^-_{B_n\setminus B}\boldsymbol{\omega}^+_{B_{2n}\setminus B_n}\emptyset_{\bar{B}_{2n}\setminus B_{2n}}\hat{\boldsymbol{\omega}}_{(\bar{B}_{2n})^c})\right| \\
&\leq |V^-(\eta_{B_{2n}})|^{-1} \int d\xi \mathbb{1}_{V(\eta_{B_{2n}})}(\xi) \int \mu(d\boldsymbol{\omega}) g_\varepsilon^{2n}[\xi](\boldsymbol{\omega}) \\
&\quad \times \sup_{\boldsymbol{\omega}^{1,2}} \left|\tilde{\gamma}_B(f|\boldsymbol{\omega}^-_{B_n\setminus B}\boldsymbol{\omega}^1_{B_n^c}) - \tilde{\gamma}_B(f|\boldsymbol{\omega}^-_{B_n\setminus B}\boldsymbol{\omega}^2_{B_n^c})\right|.
\end{aligned}$$

As above, we can further calculate for any $\boldsymbol{\omega}' \in \boldsymbol{\Omega}$,

$$\begin{aligned}
\gamma_{\bar{B}_{2n}}(g_\varepsilon^n[\xi]\tilde{f}|\boldsymbol{\omega}') &= \frac{\int P_{\bar{B}_{2n}}(d\omega)\mathbb{1}_{\emptyset_{\bar{B}_{2n}\setminus B_{2n}}}(\omega)\mathbb{1}_{V_\varepsilon(\xi_{B_{2n}})}(\omega)\tilde{f}(\omega_{B_n\setminus B})W_{B_{2n}}(\omega)}{\int P_{\bar{B}_{2n}}(d\omega)\mathbb{1}_{\emptyset_{\bar{B}_{2n}\setminus B_{2n}}}(\omega)\mathbb{1}_{V_\varepsilon(\xi_{B_{2n}})}(\omega)W_{B_{2n}}(\omega)} \\
&= \frac{\int P_{B_n}(d\omega)\mathbb{1}_{V_\varepsilon(\xi_{B_n})}(\omega)\tilde{f}(\omega_{B_n\setminus B})}{\int P_{B_n}(d\omega)\mathbb{1}_{V_\varepsilon(\xi_{B_n})}(\omega)} = |B_\varepsilon|^{-|\xi_{B_n\setminus B}|}[\prod_{x\in\xi_{B_n\setminus B}} \int_{B_\varepsilon(x)}]d\omega \tilde{f}(\omega_{B_n\setminus B})
\end{aligned}$$

which again leads to the existence of a point of discontinuity of $\tilde{\gamma}$ via Lebesgue's density theorem.

The critical symmetric case: This case is different to the pervious cases since discontinuities can not be produced by color perturbations on finite volumes. Rather discontinuities can for example come from cutting off infinite clusters which form a nullset in the low-intensity regime. But discontinuities can also be produced by glueing together two separate clusters and therefor reduce the number of clusters attached to B. Since this must be possible arbitrarily far away from B, we have to assume that the boundary condition contains two distinguished infinite clusters connectable to B which is of course a nullset as well. One way of marking this precise is the following. Instead of $V_\varepsilon(\boldsymbol{\eta})$ consider the two-arm cluster

$$\bar{\eta} = \{x \in \mathbb{R}^d : \, x_i = na/2 \text{ for some } n \in \mathbb{N}_0 \text{ and } i \in \{1,2\} \text{ and } x_j = 0 \text{ for } 3 \leq i \leq d\}.$$

In particular, $\bar{\eta}$ consists of two clusters in $\{B \leftrightarrow \infty\}$. Note that for $0 < \varepsilon < a/4$, again we have $V_\varepsilon(\bar{\eta}) \subset \{B \leftrightarrow \infty\}$ with two infinite clusters. In this case, the kernel M is not required since we do not need a change of colorings. Instead define

$$\begin{aligned}\zeta_n = \{x \in \mathbb{R}^d : x_1, x_2 > 0, \sqrt{x_1^2 + x_2^2} = n, \arctan \tfrac{x_1}{x_2} = m\tfrac{\pi n}{a} \\ \text{for some } m \in \mathbb{N} \text{ with } 0 \leq m \leq na/2 \text{ and } x_j = 0 \text{ for } 3 \leq i \leq d\},\end{aligned}$$

the gray configuration which has points along the two-dimensional boundary of B_n discretized with mesh size $a/2$. Figure 4 shows an illustration.

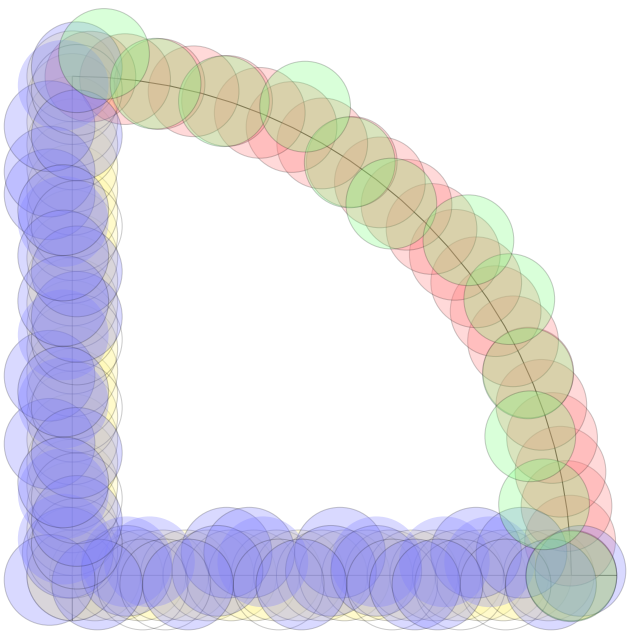

Figure 4: Illustration of the configuration $\bar{\eta}$ in yellow together with ζ_n in red. The purtubation in $V_\varepsilon(\bar{\eta} \cup \zeta_n)$ are indicated in blue and green.

In particular, for all $n \in \mathbb{N}$, ζ_n connects the two clusters in $\bar{\eta}$. Define the density

$$g_\varepsilon^n[\xi](\boldsymbol{\omega}) = \mathbb{1}_{\emptyset_{\bar{B}_n \setminus B_n}}(\boldsymbol{\omega}) \mathbb{1}_{V_\varepsilon(\xi_{B_n})}(\boldsymbol{\omega}) \gamma_{B_n}^{-1}(V_\varepsilon(\xi_{B_n}) \cap \{\emptyset_{\bar{B}_n \setminus B_n}\} | \boldsymbol{\omega})$$

and $V = V_{a/8}$ and consider the integral

$$\begin{aligned} I_{\Lambda,n} &= |V(\bar{\eta}_{B_n\cup\zeta_n})|^{-1} \int d\xi \mathbb{1}_{V(\bar{\eta}_{B_n\cap\zeta_n})}(\xi) \int \mu(d\boldsymbol{\omega}) \int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \\ &\quad \times \left|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}) - \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n^o\setminus B})\right| \\ &= |V(\bar{\eta}_{B_n\cup\zeta_n})|^{-1} \int d\xi \mathbb{1}_{V(\bar{\eta}_{B_n\cap\zeta_n})}(\xi) \int \mu(d\boldsymbol{\omega}) \int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \\ &\quad \times \left|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B_n^o\setminus B})\right| \end{aligned}$$

where we could replace the specifications using a similar argument as in (18). Then, for $f = \mathbb{1}_{\emptyset_B}$ we have

$$\left|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B_n^o\setminus B})\right| \geq e^{-3\lambda_+|B|} \int P_B^+(d\omega_B)(2^{|\mathcal{C}_B^{\mathrm{f}}(\omega_B\omega_{B_n^o\setminus B})|} - 2^{|\mathcal{C}_B^{\mathrm{f}}(\omega_B\omega_{B_n\setminus B})|}).$$

Note that in B_n, the two arms of $\hat{\boldsymbol{\omega}}$ are closed and hence, the number of clusters attached to B is reduced to one. Introducing the indicator, that there is exactly two points in the subregion of B which guarantee connectedness with both infinite components in ω_{B^c} but does not connect them inside B gives the lower bound

$$\left|\gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}) - \gamma_B^\infty(f|\hat{\boldsymbol{\omega}}_{B_n^o\setminus B})\right| \geq \delta^2 e^{-4\lambda_+|B|}\lambda_+^2 > 0.$$

On the other hand,

$$\begin{aligned} I_{\Lambda,n} &= |V(\bar{\eta}_{B_n\cup\zeta_n})|^{-1} \int d\xi \mathbb{1}_{V(\bar{\eta}_{B_n\cap\zeta_n})}(\xi) \int \mu(d\boldsymbol{\omega}) \int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \\ &\quad \times \left|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n\setminus B}) - \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n^o\setminus B})\right| \\ &\leq |V(\bar{\eta}_{B_n\cup\zeta_n})|^{-1} \int d\xi \mathbb{1}_{V(\bar{\eta}_{B_n\cap\zeta_n})}(\xi) \int \mu(d\boldsymbol{\omega}) \int \mu_t(d\hat{\boldsymbol{\omega}}|\boldsymbol{\omega}) g_\varepsilon^n[\xi](\boldsymbol{\omega}) \\ &\quad \times \sup_{\boldsymbol{\omega}^{1,2}} \left|\tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n^o\setminus B}\boldsymbol{\omega}^1_{(B_n^c)^o}) - \tilde{\gamma}_B(f|\hat{\boldsymbol{\omega}}_{B_n^o\setminus B}\boldsymbol{\omega}^2_{(B_n^c)^o})\right| \end{aligned}$$

where the part in $g_\varepsilon^n[\xi]$ involving ζ_n can be integrated out. As above, we can further calculate for any $\boldsymbol{\omega}'$

$$\begin{aligned} \gamma_{B_n^o}(g_\varepsilon^n[\xi]\tilde{f}|\boldsymbol{\omega}') &= \frac{\int P_{B_n^o}(d\omega)\mathbb{1}_{V_\varepsilon(\xi_{B_n^o})}(\omega)\tilde{f}(\omega_{B_n^o\setminus B})W_{B_n^o}(\omega)}{\int P_{B_n^o}(d\omega)\mathbb{1}_{V_\varepsilon(\xi_{B_n^o})}(\omega)W_{B_n^o}(\omega)} \\ &= \frac{\int P_{B_n^o}(d\omega)\mathbb{1}_{V_\varepsilon(\xi_{B_n^o})}(\omega)\tilde{f}(\omega_{B_n^o\setminus B})}{\int P_{B_n^o}(d\omega)\mathbb{1}_{V_\varepsilon(\xi_{B_n^o})}(\omega)} \\ &= |B_\varepsilon|^{-|\xi_{B_n\setminus B}|}[\prod_{x\in\xi_{B_n^o\setminus B}} \int_{B_\varepsilon(x)}]d\omega\tilde{f}(\omega_{B_n^o\setminus B}) \end{aligned}$$

which again leads to the existence of a point of discontinuity of $\tilde{\gamma}$ via Lebesgue's density theorem. □

[Proof of Theorem 3.4 - non-q-Gibbsian part] What remains to be shown is that in the asymmetric high-intensity regime any specification $\tilde{\gamma}$ for $\mu_{t_G}^+$ exhibits discontinuity points. For this note, that the above proof for the critical asymmetric low-intensity regime does not use the fact that we assume low intensity. □

6 Appendix

6.1 Percolation properties of the WRM

In this subsection we derive nontrivial percolation and non-percolation regimes for the WRM. Recall the classical boolean model (or Gilbert disc model) with interaction radius $2a$, see for example [1, Chapter 8.1]. Denote by λ_{c} its critical intensity. The following percolation result is already partially proved in [4].

Lemma 6.1 *(1) Let $\mu \in \mathcal{G}(\gamma)$ with $\lambda_+ \geq \lambda_-$. If $\lambda_+ + \lambda_- < \lambda_{\mathrm{c}}$, then for all $x \in \mathbb{R}^d$ and $0 < r < \infty$ we have*

$$\mu(\{B_r(x) \leftrightarrow \infty\}) = 0.$$

(2) There exists $0 < \zeta < 1$ such that the following holds. Let $\mu \in \mathcal{G}(\gamma^{\mathrm{sym}})$ in the symmetric regime, respectively μ^+ in the asymmetric regime, then if $\lambda_+ + \lambda_- > \lambda_{\mathrm{c}}/\zeta$, respectively $\lambda_+ > \lambda_{\mathrm{c}}/\zeta$, for all $x \in \mathbb{R}^d$ and all $0 < r < \infty$ we have

$$\mu(\{B_r(x) \leftrightarrow \infty\}) > 0 \quad \textit{and} \quad \lim_{r\uparrow\infty} \mu(\{B_r(x) \leftrightarrow \infty\}) = 1.$$

[Proof of Lemma 6.1] The proof uses the FKG-inequality to derive stochastic domination relations between the WRM and the Gilbert disc model. Recall the FKG-inequality for PPP as presented for example in [22, Lemma 2.1]: For a PPP P we have

$$P(fg) \geq P(f)P(g)$$

for measurable functions f, g which are either both increasing or both decreasing. A function f is called increasing if $f(\omega) \geq f(\omega')$ for all $\omega \supset \omega'$ and decreasing if $f(\omega) \leq f(\omega')$ for all $\omega \supset \omega'$.

Note that, for a measurable increasing function f, only depending on the grey configuration and $\Lambda \Subset \mathbb{R}^d$, we have

$$\begin{aligned}\gamma_\Lambda(f|\boldsymbol{\omega}_{\Lambda^c}) &= Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c}) \int P_\Lambda(d\omega_\Lambda) f(\omega_\Lambda \omega_{\Lambda^c}) \int U(d\sigma_{\omega_\Lambda}) \chi(\omega_\Lambda^{\sigma_{\omega_\Lambda}} \boldsymbol{\omega}_{\Lambda^c}) \\ &= \int P_\Lambda(d\omega_\Lambda) f(\omega_\Lambda \omega_{\Lambda^c}) W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda)\end{aligned}$$

where $W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda) = Z_\Lambda^{-1}(\boldsymbol{\omega}_{\Lambda^c}) \sum_{\sigma_{\omega_\Lambda}} U(\sigma_{\omega_\Lambda}) \chi(\omega_\Lambda^{\sigma_{\omega_\Lambda}} \boldsymbol{\omega}_{\Lambda^c})$ is the grey-configuration density of the specification with respect to the underlying PPP. Note that $W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}$ is decreasing and for any $x \in \mathbb{R}^d$ and $0 < r < n < \infty$, the function $\mathbb{1}_{\{B_r(x) \leftrightarrow B_n^c(x)\}}$ is increasing. Thus, by the FKG-inequality,

$$\gamma_{B_n}(\{B_r(x) \leftrightarrow B_n^c(x)\}|\boldsymbol{\omega}_{B_n^c(x)}) \leq \int P_{B_n(x)}(d\omega_{B_n(x)}) \mathbb{1}_{\{B_r(x) \leftrightarrow B_n^c(x)\}}(\omega_{B_n(x)}).$$

Letting n tend to infinity we see that if $\lambda < \lambda_{\mathrm{c}}$, the right hand side converges to zero which proves part (1).

As for part in (2), note that if

$$\zeta = \inf_{\Lambda, \omega_\Lambda \subset \Lambda, y \in \Lambda, \boldsymbol{\omega}_{\Lambda^c}} \frac{W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda \cup \{y\})}{W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda)}$$

exists, then $\hat{W}_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda) = \zeta^{-|\omega_\Lambda|} W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda)$ is increasing since

$$\frac{\zeta^{-(|\omega_\Lambda|+1)} W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda \cup \{y\})}{\zeta^{-|\omega_\Lambda|} W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda)} \geq 1.$$

As shown in [4, Corollary], in the symmetric case, $\zeta = \zeta(d)$ exists with $\zeta(1) = 2^{-2}$, $\zeta(2) = 2^{-6}$ and $\zeta(d) \geq 2^{-3^d}$ for $d \geq 3$. The exponents here correspond to the greatest kissing numbers for d-dimensional spheres. Hence we can rewrite, with $B = B_r(x)$,

$$\begin{aligned}
\gamma_{B_n}(\{B \leftrightarrow B_n^c\} | \boldsymbol{\omega}_{B^c}) &= e^{\lambda |B_n|(\zeta-1)} e^{-\lambda\zeta|B_n|} \sum_{n=0}^{\infty} \frac{(\lambda\zeta)^n}{n!} \int_{B_n^n} d\omega_n \mathbb{1}_{\{B \leftrightarrow B_n^c\}}(\omega_n) \hat{W}_{B_n}^{\boldsymbol{\omega}_{B_n^c}}(\omega_n) \\
&= e^{2\lambda|B_n|(\zeta-1)} \int P_{B_n}^{\lambda\zeta}(d\omega_{B_n}) \mathbb{1}_{\{B \leftrightarrow B_n^c\}}(\omega_{B_n}) \hat{W}_{B_n}^{\boldsymbol{\omega}_{B_n^c}}(\omega_{B_n}) \\
&\geq e^{\lambda|B_n|(\zeta-1)} \int P_{B_n}^{\lambda\zeta}(d\omega_{B_n}) \mathbb{1}_{\{B \leftrightarrow B_n^c\}}(\omega_{B_n}) \int P_{B_n}^{\lambda\zeta}(d\omega_{B_n}) \hat{W}_{B_n}^{\boldsymbol{\omega}_{B_n^c}}(\omega_{B_n}) \\
&= \int P_{B_n}^{\lambda\zeta}(d\omega_{B_n}) \mathbb{1}_{\{B \leftrightarrow B_n^c\}}(\omega_{B_n})
\end{aligned}$$

where $P^{\lambda\zeta}$ is the PPP with intensity $\lambda\zeta$. Consequently

$$\mu(\{B_r(x) \leftrightarrow \infty\}) \geq P^{\lambda\zeta}(\{B_r(x) \leftrightarrow \infty\})$$

and for $\lambda\zeta > \lambda_{\mathrm{c}}$ we have that $P^{\lambda\zeta}(\{B_r(x) \leftrightarrow \infty\}) > 0$ for all $x \in \mathbb{R}^d$ and $0 < r < \infty$.

As for μ^+ consider the boundary condition $+_{\Lambda^c}$ of all plus. In this case, positive lower bounds on

$$\inf_{\Lambda, \omega_\Lambda \subset \Lambda, y \in \Lambda} \frac{W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda y)}{W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_\Lambda)}$$

are slightly more difficult to obtain in comparison to the symmetric case. Indeed, let us exemplify the idea in one spatial dimension. Here the additional particle $y \in \Lambda$ can either be

1 directly attached to the boundary and

  1.1 isolated from any cluster,

  1.2 gluing a cluster to the boundary,

2 not attached to the boundary and

  2.1 isolated from any cluster,

  2.2 attached to one cluster which is attached to the boundary,

2.3 gluing two clusters which are both attached to the boundary,

2.4 gluing two clusters which where both detached from boundary,

2.5 gluing two clusters where only one was attached to the boundary.

To see, that a lower bound is given by $\hat{\lambda}_+\zeta(1)$, where $\zeta(d)$ is defined as in the asymmetric case, we use the cluster representation

$$\frac{W_\Lambda^{+\Lambda^c}(\omega_\Lambda y)}{W_\Lambda^{+\Lambda^c}(\omega_\Lambda)} = \frac{\big[\prod_{C\in\mathcal{C}=\mathcal{C}(\omega_\Lambda y)}\tilde{\sum}_{\sigma_C}U(\sigma_C)\big]\chi\big((\omega_\Lambda y)^{\sigma_\mathcal{C}}+_{\Lambda^c}\big)}{\big[\prod_{C\in\mathcal{C}=\mathcal{C}(\omega_\Lambda)}\tilde{\sum}_{\sigma_C}U(\sigma_C)\big]\chi\big(\omega_\Lambda^{\sigma_\mathcal{C}}+_{\Lambda^c}\big)}$$
$$= \frac{\big[\prod_{C\in\mathcal{C}(\omega_\Lambda y):y\in C}\tilde{\sum}_{\sigma_C}U(\sigma_C)\big]\big[\prod_{C\in\mathcal{C}(\omega_\Lambda y):y\notin C}\tilde{\sum}_{\sigma_C}U(\sigma_C)\big]\chi\big((\omega_\Lambda y)^{\sigma_\mathcal{C}}+_{\Lambda^c}\big)}{\big[\prod_{C\in\mathcal{C}(\omega_\Lambda):B_{2a}(y)\cap C\neq\emptyset}\tilde{\sum}_{\sigma_C}U(\sigma_C)\big]\big[\prod_{C\in\mathcal{C}(\omega_\Lambda):B_{2a}(y)\cap C=\emptyset}\tilde{\sum}_{\sigma_C}U(\sigma_C)\big]\chi\big(\omega_\Lambda^{\sigma_\mathcal{C}}+_{\Lambda^c}\big)}.$$

Now it suffices to consider the clusters which are not affected by the additional particle y. Under the color constraint we find the estimates

$$\frac{\prod_{C\in\mathcal{C}=\mathcal{C}(\omega_\Lambda y):y\in C}\tilde{\sum}_{\sigma_C}U(\sigma_C)}{\prod_{C\in\mathcal{C}=\mathcal{C}(\omega_\Lambda):B_{2a}(y)\cap C\neq\emptyset}\tilde{\sum}_{\sigma_C}U(\sigma_C)} \geq \begin{cases} \hat{\lambda}_+, \text{ in the cases (1a), (2a), (2b), (2c)} \\ \frac{\hat{\lambda}_+^{|C|+1}}{\hat{\lambda}_+^{|C|}+\hat{\lambda}_-^{|C|}} \geq \frac{\hat{\lambda}_+}{2}, \text{ in the cases (1b), (2e)} \\ \frac{\hat{\lambda}_+^{|C_1|+|C_2|+1}+\hat{\lambda}_-^{|C_1|+|C_2|+1}}{(\hat{\lambda}_+^{|C_1|}+\hat{\lambda}_-^{|C_1|})(\hat{\lambda}_+^{|C_2|}+\hat{\lambda}_-^{|C_2|})} \geq \frac{\hat{\lambda}_+}{4}, \text{ in the case (2d).} \end{cases}$$

Similar observations, in view of the dimension-dependent kissing numbers, lead to the following lower bounds in higher dimensions. For $d=2$ we have

$$\inf_{\Lambda,\omega_\Lambda\subset\Lambda,y\in\Lambda}\frac{W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_{n-1}y)}{W_\Lambda^{\boldsymbol{\omega}_{\Lambda^c}}(\omega_{n-1})} \geq \hat{\lambda}_+2^{-6} = \hat{\lambda}_+\zeta(2)$$

and for $d\geq 3$ the bound $\hat{\lambda}_+\zeta(d)$. Using the FKG-inequality as in the symmetric case with $B=B_r(x)$, we get a lower bound

$$\gamma_{B_n(x)}(\{B(x)\leftrightarrow B_n^c(x)\}|\boldsymbol{\omega}_{B(x)^c}) \geq \int P_{B_n(x)}^{\lambda_+\zeta}(d\omega_{B_n(x)})\mathbb{1}_{\{B(x)\leftrightarrow B_n^c(x)\}}(\omega_{B_n(x)})$$

where $P^{\lambda_+\zeta}$ is the PPP with intensity $\lambda_+\zeta$. This concludes the proof. □

6.2 Existence of non-asq-specifications $\gamma^+\neq\gamma^-$ for μ^+ and μ^- in the phase-transition regime

First note that by Lemma 4.8,

$$\mu_t^+(\{\hat{\boldsymbol{\omega}}\in\boldsymbol{\Omega}:\liminf_{n\uparrow\infty}m(\hat{\boldsymbol{\omega}}_{C\cap B_n})>0 \text{ for all infinite clusters } C \text{ of } \hat{\boldsymbol{\omega}}\})=1.$$

In words, under the time evolution a magnetization plus one on an infinite cluster remains positive for all finite times. By symmetry, the same is true for the minus magnetization. In

light of the specification γ^∞ of Section 4.3, and in particular Lemma 4.3, (non-almost-surely quasilocal) specifications for $\mu_t^\pm$ can be defined as

$$\gamma_\Lambda^\pm(f|\hat{\boldsymbol{\omega}}_{\Lambda^c}) = \frac{\int P_\Lambda^-(d\omega_\Lambda) f^\pm(\omega_\Lambda) \prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \big(1+\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\big)}{\int P_\Lambda^-(d\omega_\Lambda) \prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \big(1+\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\big)}$$

where $f^\pm(\omega_\Lambda) = \nu_\Lambda^\pm(f(\omega_\Lambda,\cdot)|\hat{\boldsymbol{\omega}}_{\Lambda^c},\omega_\Lambda)$ with

$$\nu_\Lambda^\pm(\hat{\sigma}_{\omega_\Lambda}|\hat{\boldsymbol{\omega}}_{\Lambda^c},\omega_\Lambda) = \prod_{C\in\mathcal{C}_\Lambda^\infty(\omega_\Lambda)} p_t(\pm,+)^{|\hat{\sigma}_{C\cap\Lambda}|^+} p_t(\pm,-)^{|\hat{\sigma}_{C\cap\Lambda}|^-} \times$$
$$\frac{\prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \big(p_t(+,+)^{|\hat{\sigma}_{C\cap\Lambda}|^+} p_t(+,-)^{|\hat{\sigma}_{C\cap\Lambda}|^-} + p_t(-,+)^{|\hat{\sigma}_{C\cap\Lambda}|^+} p_t(-,-)^{|\hat{\sigma}_{C\cap\Lambda}|^-} \rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\big)}{\prod_{C\in\mathcal{C}_\Lambda^{\mathrm{f}}(\omega_\Lambda)} \big(1+\rho(\hat{\boldsymbol{\omega}}_{C\setminus\Lambda})\big)}.$$

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