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**Mass concentration and aging in the parabolic
Anderson model with doubly-exponential tails**

Marek Biskup¹, Wolfgang König^{2,3}, Renato Soares dos Santos²

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| ¹ University of California, Los Angeles UCLA Department of Mathematics Los Angeles, CA 90095-1555 USA E-Mail: biskup@math.ucla.edu | ² Weierstrass Institute Mohrenstr. 39 10117 Berlin Germany E-Mail: wolfgang.koenig@wias-berlin.de renato.soaresdossantos@wias-berlin.de |
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³ Technische Universität Berlin
Institut für Mathematik
Str. des 17. Juni 136
10623 Berlin
Germany

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: <http://www.wias-berlin.de/>

ABSTRACT. We study the solutions to the Cauchy problem on the with random potential and localised initial data. Here we consider the random Schrödinger operator, i.e., the Laplace operator with random field, whose upper tails are doubly exponentially distributed in our case. We prove that, for large times and with large probability, a majority of the total mass of the solution resides in a bounded neighborhood of a site that achieves an optimal compromise between the local Dirichlet eigenvalue of the Anderson Hamiltonian and the distance to the origin. The processes of mass concentration and the rescaled total mass are shown to converge in distribution under suitable scaling of space and time. Aging results are also established. The proof uses the characterization of eigenvalue order statistics for the random Schrödinger operator in large sets recently proved by the first two authors.

1. INTRODUCTION

Random Schrödinger operators — most notably, the Anderson Hamiltonian $H = \Delta + \xi$ — have been a subject of intense research over several decades. Most of the attention has been paid to the character of the spectrum and the ensuing physical consequences for the *quantum* evolution. However, the associated *parabolic* problem — characterized by the PDE $\partial_t u = \Delta u + \xi u$ — is of as much interest both for theory and applications. Here we study the latter facet of this problem for a specific class of random potentials. Our main result is the proof of localization of the solution to the above PDE for large time in a neighborhood of a process determined solely by the random potential.

A standard way to describe the *parabolic Anderson model* (PAM) is via a solution $u : \mathbb{Z}^d \times [0, \infty) \rightarrow [0, \infty)$ of the Cauchy problem

$$\partial_t u(z, t) = \Delta u(z, t) + \xi(z)u(z, t), \quad z \in \mathbb{Z}^d, t \in (0, \infty), \quad (1.1)$$

$$u(z, 0) = \mathbb{1}_0(z), \quad z \in \mathbb{Z}^d. \quad (1.2)$$

Here ∂_t abbreviates the derivative with respect to t and Δ is the discrete Laplacian acting on $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ as

$$\Delta f(z) := \sum_{y: |y-z|=1} [f(y) - f(z)], \quad (1.3)$$

where $|\cdot|$ denotes the ℓ^1 norm on \mathbb{Z}^d , and $\xi = (\xi(z) : z \in \mathbb{Z}^d)$ is an i.i.d. random potential taking values in $[-\infty, \infty)$.

The interest in (1.1–1.2) for mathematics as well as applications comes from the competing effect of the two terms on the right-hand side of (1.1). Indeed, the Laplacian tends to make the solution smoother over time while the field makes it rougher. The problem (1.1) appears in the studies of chemical kinetics [GM90], hydrodynamics [CM94], and magnetic phenomena [MR94]. We refer to the reviews [M94, CM94] for more background and [GM90] for fundamental mathematical properties of the model. A recent comprehensive survey of mathematical results on the PAM and related models can be found in [K16].

A positive solution to the Cauchy problem (1.1–1.2) exists and is unique as soon as the upper tail of $[\xi(0) / \log \xi(0)]^d$ is integrable [GM90]. Under this condition, there is also a representation in terms of the changed-path measure,

$$Q_t^{(\xi)}(dX) := \frac{1}{U(t)} \exp\left\{\int_0^t \xi(X_s) ds\right\} \mathbb{P}_0(dX), \quad (1.4)$$

on nearest-neighbour paths $X = (X_s)_{s \geq 0}$ on \mathbb{Z}^d , where \mathbb{P}_0 stands for the law of a continuous-time random walk on \mathbb{Z}^d (with generator Δ) started at zero. Indeed, the Feynman-Kac formula shows

$$u(z, t) = U(t) Q_t^{(\zeta)}(X_t = z) = \mathbb{E}_0 \left[e^{\int_0^t \zeta(X_s) ds} \mathbb{1}_{\{X_t = z\}} \right], \quad (1.5)$$

whereby the normalization constant $U(t)$ obtains the meaning

$$U(t) = \sum_{x \in \mathbb{Z}^d} u(x, t) = \mathbb{E}_0 \left(\exp \left\{ \int_0^t \zeta(X_s) ds \right\} \right). \quad (1.6)$$

The aforementioned competition is now obvious probabilistically: the walk would like to maximize the “energy” $\int_0^t \zeta(X_s) ds$, by spending its time at the places where ζ is large, against the “entropy” of such trajectories under the path measure \mathbb{P}_0 .

An alternative and equally useful way to view (1.1) is as the definition of a semigroup $t \mapsto e^{t(\Delta + \zeta)}$ on $\ell^2(\mathbb{Z}^d)$. The solution to (1.1–1.2) is then given by

$$u(x, t) = \langle \delta_x, e^{t(\Delta + \zeta)} \delta_0 \rangle_{\ell^2(\mathbb{Z}^d)}, \quad (1.7)$$

where δ_z is the vector in $\ell^2(\mathbb{Z}^d)$ that is one at z and zero otherwise. This opens up the possibility to control the large- t behavior through spectral analysis of the Anderson Hamiltonian. To this end, it is useful to restrict the problem to a sufficiently large (in t -dependent fashion) finite volume $\Lambda \subset \mathbb{Z}^d$ (with $0 \in \Lambda$) as follows. Denote by H_Λ the Anderson Hamiltonian in Λ with Dirichlet boundary conditions, i.e., for $\phi \in \mathbb{R}^\Lambda$, $H_\Lambda \phi = H\tilde{\phi}$ where $H = \Delta + \zeta$ and $\tilde{\phi}$ is the extension of ϕ to $\mathbb{R}^{\mathbb{Z}^d}$ that is equal to zero on Λ^c . Let u_Λ be the solution to (1.1–1.2) restricted to Λ and with the right-hand side of (1.1) substituted by $H_\Lambda u$. Then the above interpretation yields

$$u_\Lambda(x, t) = \sum_{k=1}^{|\Lambda|} e^{t\lambda_\Lambda^{(k)}} \phi_\Lambda^{(k)}(x) \phi_\Lambda^{(k)}(0), \quad (1.8)$$

where $\lambda_\Lambda^{(k)}$ are the eigenvalues and $\phi_\Lambda^{(k)}$ the corresponding eigenvectors of H_Λ which we assume to be orthonormal in $\ell^2(\mathbb{Z}^d)$.

The competition we described in the context of the changed-path measure (1.4) now reflects itself as follows. The term in the sum in (1.8) that grows the *fastest* in t is that with the largest eigenvalue. However, there is no *a priori* reason for it to be the *dominant* term at a fixed time. Indeed, an eigenvalue will only contribute to (1.8) when its eigenvector puts non-trivial mass on both 0 and x . Since the leading eigenvectors decay exponentially away from their localization centers (Anderson localization), $|\phi_\Lambda^{(k)}(0)|$ will in fact be typically extremely small. It is thus the combined effect of both $e^{t\lambda_\Lambda^{(k)}}$ and $\phi_\Lambda^{(k)}(x) \phi_\Lambda^{(k)}(0)$ that decides which index k will give the main contribution to the sum.

In the present paper, we analyze these competing effects for a class of random potentials with upper tails close to the doubly-exponential distribution, characterized by

$$\text{Prob}(\zeta(0) > r) = \exp\{-e^{r/\rho}\}, \quad r \in \mathbb{R}, \quad (1.9)$$

where $\rho \in (0, \infty)$. (Precise definitions will appear in Section 2.) For these potentials we show that, at all large t , a majority of the total mass $U(t)$ of the solution resides in a bounded neighborhood of a random point Z_t defined entirely by ζ . This point marks a local peak of ζ optimizing the strategy by which the random-walk in (1.4) traverses to Z_t in time $o(t)$ and then “sticks around” Z_t thereafter to enjoy the benefits of a

“strong” local Dirichlet eigenvalue. We also characterize the scaling limits of Z_t and $\frac{1}{t} \log U(t)$, and obtain aging results for both Z_t and $u(x, t)$.

Our results build on a large body of literature on the PAM whose full account here would detract from the main message of the paper. For now let us just say that we extend results from [MOS11, LM12, ST14, FM14], dealing with localization on one lattice site, to a benchmark class of random potentials exemplified by (1.9), where the localization takes place in large domains, albeit not growing with t . An important technical input for us is the recent work [BK16] where eigenvalue order statistics for the Anderson Hamiltonian $H = \Delta + \zeta$ was characterized for this class of ζ . Further connections will be given in Section 3.1.

2. MAIN RESULTS

We now move to the statements of our main results. Throughout the paper, $\ln x$ denotes the natural logarithm of x and $\ln_2 x := \ln \ln x$, $\ln_3 x := \ln \ln \ln x$, etc denote its iterates. We will use “Prob” to denote the probability law of the i.i.d. random field ζ .

2.1 Assumptions.

We begin by identifying the class of potentials to which our results apply. Besides some regularity, the following ensures that the upper tails of $\zeta(0)$ are in the vicinity of the doubly-exponential distribution (1.9).

Assumption 2.1 (Upper tails) *Suppose that $\text{esssup } \zeta(0) = \infty$ and let*

$$F(r) := \ln_2 \frac{1}{\text{Prob}(\zeta(0) > r)}, \quad r > \text{essinf } \zeta(0). \quad (2.1)$$

We assume that F is differentiable on its domain and that

$$\lim_{r \rightarrow \infty} F'(r) = \frac{1}{\rho} \text{ for some } \rho \in (0, \infty). \quad (2.2)$$

The assumption above is exactly as Assumption 1.1 in [BK16], and implies Assumption (F) of [GM98]. While the latter would be enough for most of our needs, the extra requirements of Assumption 2.1 are used in the crucial step, performed in [BK16], of identifying the max-order class of the local principal eigenvalues of the Anderson Hamiltonian. In order to avoid technical inconveniences, we will also assume the following condition on the lower tail of ζ .

Assumption 2.2 (Lower tails) *Let $\zeta^-(x) := \max\{0, -\zeta(x)\}$. We assume that*

$$\int_0^\infty \text{Prob}(\zeta^-(0) > e^s)^{\frac{1}{d}} ds < \infty. \quad (2.3)$$

Assumption 2.2 is only used in the proof of Lemma 8.1, which is used in Proposition 4.4 to give a lower bound for the total mass $U(t)$. Note that (2.3) holds whenever $\ln(1 + \zeta^-(0))$ has a $(d + \varepsilon)$ -th finite moment (cf. [M02]). We believe that, with the use of percolation arguments, this assumption can be relaxed to $\zeta^-(0) > -\infty$ almost surely in $d \geq 2$. In $d = 1$, (2.3) is equivalent to $\ln(1 + \zeta^-(0))$ having the first moment, which is known to be in fact necessary (cf. [BK01b]).

We will assume the validity of Assumptions 2.1–2.2 throughout the rest of the paper without explicitly stating this in each instance.

2.2 Results: Mass concentration.

Recall that $|x|$ denotes the ℓ^1 -norm of x . Our first result concerns the concentration of the total mass of the solution to the Cauchy problem (1.1–1.2):

Theorem 2.3 (Mass concentration) *There is a \mathbb{Z}^d -valued càdlàg stochastic process $(Z_t)_{t>0}$ depending only on ξ such that $t \mapsto |Z_t|$ is non-decreasing and such that the following holds: For each $\delta > 0$, there exists $R \in \mathbb{N}$ such that, for any $l_t > 0$ satisfying $\lim_{t \rightarrow \infty} \frac{1}{t} l_t = 0$,*

$$\lim_{t \rightarrow \infty} \text{Prob} \left(\sup_{s \in [t-l_t, t+l_t]} \sum_{x: |x-Z_t| > R} \frac{u(x, s)}{U(s)} > \delta \right) = 0. \quad (2.4)$$

In words, (2.4) means that the solution at time t is with large probability supported around a single point Z_t , and the control in fact extends to sublinearly-growing intervals of time around t . This cannot be improved to intervals of size growing linearly with t due to the jumps that occur in the process $s \mapsto Z_s$; see Theorem 2.7 below.

Remark 2.4 Note that the asymptotic concentration in one island at time t does not hold almost surely. Indeed, around jump times of $s \mapsto Z_s$, the contributions of *two* islands are equally dominant. Almost-sure concentration in at most two islands for all times, dubbed as a “two-cities theorem”, was shown for the Pareto distribution in [KLMS09], a case in which the islands reduce to single lattice vertices. In order to keep the present paper to a manageable length, we decided not to include almost-sure versions here.

In terms of the path measure $Q_t^{(\xi)}$, Theorem 2.3 can be interpreted as concentration for the law of the position of the path at time t . By letting the radius R grow slowly to infinity, this can be improved to include a majority of the whole random-walk path:

Theorem 2.5 (Path concentration) *For any $\epsilon_t \in (0, 1)$ satisfying $\lim_{t \rightarrow \infty} \epsilon_t \ln_3 t = \infty$,*

$$\lim_{t \rightarrow \infty} Q_t^{(\xi)} \left(\sup_{s \in [\epsilon_t t, t]} |X_s - Z_t| > \epsilon_t \ln t \right) = 0 \quad \text{in probability,} \quad (2.5)$$

where $(Z_t)_{t>0}$ is the stochastic process in Theorem 2.3.

2.3 Results: Scaling limit.

Our next theorem identifies the large- t behavior of the pair of processes $t \mapsto Z_t$ and $t \mapsto \frac{1}{t} \ln U(t)$. While $U(t)$ is continuous, Z_t is only càdlàg and thus it is natural to use the Skorohod topology to discuss distributional convergence. Two relevant scales are

$$d_t := \frac{\rho}{d \ln t} \quad \text{and} \quad r_t := \frac{t d_t}{\ln_3 t} = \frac{\rho}{d \ln t} \frac{t}{\ln_3 t}, \quad (2.6)$$

marking, respectively, the size of fluctuations of $\frac{1}{t} \ln U(t)$ and the typical size of Z_t .

To describe the scaling limit, consider a sample $\{(\lambda_i, z_i) : i \in \mathbb{N}\}$ from the Poisson point process on $\mathbb{R} \times \mathbb{R}^d$ with intensity measure $e^{-\lambda} d\lambda \otimes dz$. For $\theta > 0$ define

$$\psi_\theta(\lambda, z) := \lambda - \frac{|z|}{\theta}. \quad (2.7)$$

It can be checked that, for every $\theta > 0$, the set $\{\psi_\theta(\lambda_i, z_i) : i \in \mathbb{N}\}$ is bounded and locally finite. Moreover, the maximizing point is unique at all but at most a countable set of θ 's and we can thus define $(\bar{\Lambda}_\theta, \bar{Z}_\theta)$ to be the càdlàg maximizer of ψ_θ over the sample points of the process. We set

$$\bar{\Psi}_\theta := \psi_\theta(\bar{\Lambda}_\theta, \bar{Z}_\theta). \quad (2.8)$$

Then we have:

Theorem 2.6 (Scaling limit of the concentration loci and the total mass) *There is a non-decreasing scale function $a_t > 0$ obeying*

$$\lim_{t \rightarrow \infty} \frac{a_t}{\ln_2 t} = \rho \quad (2.9)$$

such that the following holds: The stochastic process $(Z_t)_{t>0}$ in Theorems 2.3 and 2.5 can be chosen such that, for all $s \in (0, \infty)$ and relative to the Skorohod topology on $\mathcal{D}([s, \infty), \mathbb{R} \times \mathbb{R}^d)$,

$$\left(\frac{\frac{1}{\theta t} \ln U(\theta t) - a_{r_t}}{d_t}, \frac{Z_{\theta t}}{r_t} \right)_{\theta \in [s, \infty)} \xrightarrow[t \rightarrow \infty]{\text{law}} (\bar{\Psi}_\theta, \bar{Z}_\theta)_{\theta \in [s, \infty)}. \quad (2.10)$$

In particular, for each $\theta > 0$, the random variable $(\frac{1}{\theta t} \ln U(\theta t) - a_{r_t})/d_t$ converges in law to a Gumbel random variable with scale 1 and location $d \ln(2\theta)$, while $Z_{\theta t}/r_t$ converges in law to a random vector in \mathbb{R}^d with i.i.d. coordinates, each having probability density $(2\theta)^{-1} e^{-|x|/\theta}$ with respect to the Lebesgue measure on \mathbb{R} .

The scaling function a_t characterizes the leading-order scale of the principal Dirichlet eigenvalue of the Anderson Hamiltonian in a box of radius t , as identified in [BK16]. See (7.3) below for a precise definition.

2.4 Results: Aging.

The techniques used to prove the above theorems also permit us to address the phenomenon of *aging* in the problem under consideration. The term ‘‘aging’’ usually refers to the fact that certain decisive changes in the system occur at time scales that increase *proportionally* to the age of the system. Our next result addresses aging in the process $(Z_t)_{t>0}$:

Theorem 2.7 (Aging for the localization process) *For each $s > 0$, and for $(Z_t)_{t>0}$ and $(\bar{Z}_t)_{t>0}$ as in Theorems 2.3, 2.5 and 2.6,*

$$\begin{aligned} & \lim_{t \rightarrow \infty} \text{Prob}(Z_{t+\theta t} = Z_t \ \forall \theta \in [0, s]) \\ &= \lim_{t \rightarrow \infty} \text{Prob}(Z_{t+st} = Z_t) \\ &= \text{Prob}(\bar{Z}_{1+\theta} = \bar{Z}_1 \ \forall \theta \in [0, s]) = \text{Prob}(\Theta > s), \end{aligned} \quad (2.11)$$

where the random variable

$$\Theta := \inf\{\theta > 0 : \bar{Z}_{1+\theta} \neq \bar{Z}_1\} \quad (2.12)$$

is positive and finite almost surely.

In light of Theorem 2.6, Theorem 2.7 can be seen as a reflection of the fact that the functional convergence stated in Theorem 2.6 is not achieved through a large number of microscopic jumps, but rather through sporadic macroscopic jumps.

Our second aging result deals with the jumps in the profile of the normalized solution $u(\cdot, t)/U(t)$. It comes as a consequence of the mass concentration of the normalized solution around Z_t together with Theorem 2.7.

Theorem 2.8 (Aging for the solution) *For any $\varepsilon \in (0, 1)$, the random variable*

$$\frac{1}{t} \inf \left\{ s > 0: \sum_{x \in \mathbb{Z}^d} \left| \frac{u(x, t+s)}{U(t+s)} - \frac{u(x, t)}{U(t)} \right| > \varepsilon \right\} \quad (2.13)$$

converges in distribution as $t \rightarrow \infty$ to the random variable Θ defined in (2.12).

A key point to note about Theorem 2.8 is that the limiting random variable does not depend on ε . The result thus implies that, in fact, the sum in (2.13) jumps from values near 0 to values near 1 as s varies in a time interval of length $o(t)$ centered at Θt .

2.5 Results: Limit profiles.

The localization stated in Theorem 2.3 can be given in a more precise form provided we make an additional uniqueness assumption. To state this assumption, we need further definitions. Given a potential $V: \mathbb{Z}^d \rightarrow \mathbb{R}$, let

$$\mathcal{L}(V) := \sum_{x \in \mathbb{Z}^d} e^{\frac{V(x)}{\rho}}. \quad (2.14)$$

The functional \mathcal{L} plays the role of a large deviation rate function for random potentials ξ with doubly-exponential tails. Whenever $\mathcal{L}(V) < \infty$ (in fact, whenever $V(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$), $\Delta + V$ has a compact resolvent as an operator on $\ell^2(\mathbb{Z}^d)$ and its largest eigenvalue $\lambda^{(1)}(V)$ is well-defined and simple. The constant

$$\chi = \chi(\rho) := -\sup\{\lambda^{(1)}(V): V \in \mathbb{R}^{\mathbb{Z}^d}, \mathcal{L}(V) \leq 1\} \in [0, 2d] \quad (2.15)$$

is important in the description of the asymptotic growth of $U(t)$. The set of centered maximizers

$$\mathcal{M}_\rho^* := \left\{ V \in \mathbb{R}^{\mathbb{Z}^d}: 0 \in \operatorname{argmax}(V), \mathcal{L}(V) \leq 1 \text{ and } \lambda^{(1)}(V) = -\chi \right\} \quad (2.16)$$

is known to be non-empty. The assumption below deals with uniqueness:

Assumption 2.9 (Uniqueness of maximizer) *We assume that $\mathcal{M}_\rho^* = \{V_\rho\}$, i.e., the variational problem (2.15) admits a unique centered solution V_ρ .*

The uniqueness of the centered minimizer is conjectured to hold for all $\rho > 0$, but has so far only been proved for ρ large enough; see [GH99]. In the latter paper it is also shown that, for any $V \in \mathcal{M}_\rho^*$, the non-negative principal eigenfunction of the operator $\Delta + V$ is strictly positive and lies in $\ell^1(\mathbb{Z}^d)$. Under

Assumption (2.9), we will denote henceforth by v_ρ the principal eigenfunction of $\Delta + V_\rho$, normalized so that

$$v_\rho > 0 \quad \text{and} \quad \|v_\rho\|_{\ell^1(\mathbb{Z}^d)} = 1. \quad (2.17)$$

Then we have:

Theorem 2.10 (Limiting profiles) *Suppose that Assumption 2.9 holds and let $(Z_t)_{t>0}$ be the process from Theorems 2.3, 2.5 and 2.6. There exist $\mu_t \in \mathbb{N}$ and $\hat{a}_t > 0$ satisfying $\lim_{t \rightarrow \infty} \mu_t = \infty$ and $\lim_{t \rightarrow \infty} \hat{a}_t / (\rho \ln_2 t) = 1$ such that, for all $\epsilon \in (0, 1)$,*

$$\sup_{s \in [\epsilon t, \epsilon^{-1} t]} \sup_{x \in \mathbb{Z}^d: |x| \leq \mu_t} |\zeta(x + Z_s) - \hat{a}_t - V_\rho(x)| \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in probability.} \quad (2.18)$$

Moreover, for any $l_t > 0$ satisfying $\lim_{t \rightarrow \infty} \frac{1}{t} l_t = 0$,

$$\sup_{s \in [t-l_t, t+l_t]} \sum_{x \in \mathbb{Z}^d} \left| \frac{u(Z_t + x, s)}{U(s)} - v_\rho(x) \right| \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in probability.} \quad (2.19)$$

The scale \hat{a}_t in (2.18) coincides (up to terms that vanish as $t \rightarrow \infty$) with the maximum of ζ inside a box of radius t (cf. Lemma 5.1). Moreover, the scales a_t and \hat{a}_t satisfy $\lim_{t \rightarrow \infty} \hat{a}_t - a_t = \chi$.

The rest of the paper is organized as follows. In Section 3 below we discuss connections to the literature and provide some heuristics. Section 4 contains an extensive overview of our proofs including the definition of the localization process Z_t . The technical core of the paper is formed by Section 5 (properties of the potential and spectral bounds), Section 6 (path expansions) and Section 7 (a point process approach). The bulk of the proofs related to our main results is carried out in Sections 8–11, concerning respectively negligible contributions to the Feynman-Kac formula, localization of relevant eigenfunctions, path localization properties and the analysis of local profiles. The proofs of some technical results are given in Appendices A–C.

3. CONNECTIONS AND HEURISTICS

In this section we make the necessary connections to earlier work on this problem, and also provide a short heuristic argument motivating the definition of the scales in (2.6).

3.1 Relations to earlier work.

Let us give a quick survey on earlier works on the particular question that we consider; we refer to [K16] for a comprehensive account on the parabolic Anderson model and to [M11] for a survey on certain aspects closely related to the present paper.

Much of the effort since 1990 went into developing a characterization of the logarithmic asymptotics of $t \mapsto U(t)$ and its moments, which are all finite if and only if all the positive exponential moments of $\zeta(0)$ are finite. For this case, under a mild regularity assumption, [HKM06] identified *four universality classes* of asymptotic behaviors: the double-exponential tails of the form (1.9) [GM98, GH99, GKM07], the so-called “almost bounded” potentials (corresponding formally to $\rho = 0$) [HKM06], the bounded potentials treated in [BK01a], and potentials with tails heavier than (1.9) (corresponding formally to $\rho = \infty$) [HMS08, KLMS09, LM12, ST14, FM14].

In all of the classes mentioned above, the asymptotics of $U(t)$ is expressed in terms of a variational principle for the local time of the path in $Q_t^{(\xi)}$ and/or the “profile” of ξ that maximizes a local eigenvalue. The picture that emerges is that a typical path sampled from $Q_t^{(\xi)}$ for t large will spend an overwhelming majority of time in a relatively small volume whose location is characterized by a favourable value of the local Dirichlet eigenvalue. Proofs of such statements have first been available for a related version of the model using the method of enlargement of obstacles [S98] and later also for the double-exponential class by probabilistic path expansions [GKM07]. However, neither of these approaches was sharp enough to distinguish among the many “favourable eigenvalues.” In fact, while the expectation was that only a finite number of such eigenvalues needs to be considered, the best available bound on their number was $t^{o(1)}$.

For distributions with tails heavier than (1.9), progress on the path-localization question has been made in [KLMS09] and more recently in [LM12, ST14, FM14]. The distributions considered in these references are, respectively, Pareto, exponential, Weibull with parameter $\gamma \in (0, 2)$ and general Weibull. In these papers it is proven that, with large probability, the solution is asymptotically concentrated on a single lattice point, which is an extremely strong localization property. In the doubly-exponential case considered here, due to less-heavy tails, the localization phenomenon is not so strong; indeed, restricting to any bounded region misses some fraction of the total mass of the solution.

The analysis leading to our result depends crucially on the characterization of the order statistics of local principal eigenvalues for the Anderson Hamiltonian performed in [BK16], which allows us to conveniently represent local eigenvalues through a point process approach. In this aspect, our paper shares similarities with [FM14], which draws heavily upon the analysis of the spectral order statistics in [Ast12, Ast13]. However, our case also harbors many significant differences, caused mainly by the non-degenerate structure of the dominant eigenfunctions.

For the remaining two universality classes of ξ — namely, the bounded and “almost bounded” fields — the path localization question is yet more difficult because the relevant eigenvectors extend over spatial scales that diverge with time. Nevertheless, we expect that our approach provides a correct strategy for tackling these cases as well.

3.2 Some heuristics.

We present next a heuristic calculation based on [BK16] to motivate the appearance of the scale r_t defined in (2.6). We will describe a strategy to obtain a lower bound for the total mass $U(t)$ defined in (1.6). Our actual proof of the corresponding result (cf. Proposition 4.4 below) follows similar but somewhat different steps.

Write $B_t \subset \mathbb{Z}^d$ for the ℓ^1 -ball with radius t , and denote by $\lambda_{B_t}^{(k)}, \phi_{B_t}^{(k)}, 1 \leq k \leq |B_t|$, the eigenvalues (in decreasing order) and corresponding orthonormal eigenfunctions of the Anderson Hamiltonian in B_t with zero Dirichlet boundary conditions. If $Y_{B_t}^{(k)} \in B_t$ are points maximizing $(\phi_{B_t}^{(k)})^2$, it can be shown via spectral methods that

$$\mathbb{E}_{Y_{B_t}^{(k)}} \left[e^{\int_0^t \xi(X_r) dr} \mathbb{1}\{X_r \in B_t \forall r \in [0, t]\} \right] \gtrsim e^{t\lambda_{B_t}^{(k)}}. \quad (3.1)$$

Inserting in (1.6) the event where the random walk X reaches $Y_{B_t}^{(k)}$ at a time $s < t$ and then remains in B_t until time t , using the Markov property we obtain

$$U(t) \geq \mathbb{E}_0 \left[e^{\int_0^t \zeta(X_r) dr} \mathbb{1}\{X_s = Y_{B_t}^{(k)}, X_r \in B_t \forall r \in [s, t]\} \right] \gtrsim \mathbb{P}_0(X_s = Y_{B_t}^{(k)}) e^{(t-s)\lambda_{B_t}^{(k)}} \\ \approx e^{-|Y_{B_t}^{(k)}| \ln(|Y_{B_t}^{(k)}|/s)} e^{(t-s)\lambda_{B_t}^{(k)}}, \quad (3.2)$$

where we assumed $|Y_{B_t}^{(k)}| \gg s$ to approximate the probability $\mathbb{P}_0(X_s = Y_{B_t}^{(k)})$. Optimizing over s gives the candidate $s = |Y_{B_t}^{(k)}|/\lambda_{B_t}^{(k)}$, which we may plug in (3.2) provided that we also assume $|Y_{B_t}^{(k)}|/\lambda_{B_t}^{(k)} < t$. With this choice, (3.2) becomes approximately

$$\exp \left\{ t\lambda_{B_t}^{(k)} - |Y_{B_t}^{(k)}| \ln \lambda_{B_t}^{(k)} \right\} = e^{ta_t} \exp \left\{ td_t \frac{\lambda_{B_t}^{(k)} - a_t}{d_t} - |Y_{B_t}^{(k)}| \ln \lambda_{B_t}^{(k)} \right\}, \quad (3.3)$$

where $a_t \sim \rho \ln_2 t$ is the leading order of the principal Dirichlet eigenvalue of H in a box of radius t as identified in [BK16]. Therein it is shown that the collection of rescaled points $\{(\lambda_{B_t}^{(k)} - a_t)/d_t\}_{1 \leq k \leq |B_t|}$ converges in distribution to (the support of) a Poisson point process. Assuming thus that $(\lambda_{B_t}^{(k)} - a_t)/d_t$ is of finite order, an index k optimizing (3.3) will balance out the two competing terms, implying $|Y_{B_t}^{(k)}| \approx r_t$.

4. MAIN RESULTS FROM KEY PROPOSITIONS

The goal of this section is to give an outline to the proof of Theorems 2.3, 2.5, 2.8 and 2.10. We will achieve this by way of a sequence of propositions that encapsulate the key technical aspects of the whole argument. The proofs of these propositions and of Theorems 2.6–2.7 constitute the remainder of this paper and are the subject of Sections 5–11 as well as the three appendices. Note that Theorem 2.7 will be assumed in Sections 4.5–4.6 below.

Throughout the rest of this work, we set $\mathbb{N} := \{1, 2, \dots\}$, write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and denote by $\text{dist}(\cdot, \cdot)$ the metric derived from the ℓ^1 -norm $|\cdot|$. For a real-valued function f and a positive function g , we write $f(t) = O(g(t))$ as $t \rightarrow \infty$ to denote that there exists $C > 0$ such that $|f(t)| \leq Cg(t)$ for all large enough t , and we write $f(t) = o(g(t))$ in place of $\lim_{t \rightarrow \infty} |f(t)|/g(t) = 0$. In the latter case, we may also alternatively write $|f(t)| \ll g(t)$ or $g(t) \gg |f(t)|$. By $o(\cdot)$ or $O(\cdot)$ we will always mean *deterministic* bounds, i.e., independent of the realization of ζ .

4.1 Definition of the localization process.

In this subsection, we provide the definition of the localization process $(Z_t)_{t>0}$. We start with some necessary notation.

For $\Lambda \subset \mathbb{Z}^d$ finite, we denote by $\lambda_\Lambda^{(1)}$ the largest Dirichlet eigenvalue (i.e., with zero boundary conditions) of $\Delta + \zeta$ in Λ . For $L \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, we let

$$B_L(x) := x + [-L, L]^d \cap \mathbb{Z}^d, \quad (4.1)$$

and when $x = 0$ we write B_L instead of $B_L(0)$.

Fix $\kappa \in (0, 1/d)$. For each $z \in \mathbb{Z}^d$, we define a ζ -dependent radius

$$\varrho_z := \left\lfloor \exp \left\{ \frac{\kappa}{\rho} \zeta(z) \right\} \right\rfloor \quad (4.2)$$

and we let

$$\mathcal{C} := \left\{ z \in \mathbb{Z}^d : \zeta(z) \geq \zeta(y) \forall y \in B_{\varrho_z}(z) \right\} \quad (4.3)$$

denote the set of local maxima of ζ in neighborhoods of radius ϱ_z , which we call *capitals*. Since $\zeta(x)$ has a continuous law, we have $B_{\varrho_z}(z) \cap \mathcal{C} = \{z\}$ for all $z \in \mathcal{C}$ almost surely.

For $z \in \mathcal{C}$, we abbreviate

$$\lambda^{\mathcal{C}}(z) := \lambda_{B_{\varrho_z}(z)}^{(1)}. \quad (4.4)$$

For $t > 0$, we define a *cost functional* over the points $z \in \mathcal{C}$ by setting

$$\Psi_t(z) := \lambda^{\mathcal{C}}(z) - \frac{\ln_3^+ |z|}{t} |z|, \quad \text{where } \ln_3^+ x := \ln_3(x \vee e^e). \quad (4.5)$$

The functional Ψ_t measures the relevance at time t of a capital $z \in \mathcal{C}$ by weighting the principal eigenvalue in $B_{\varrho_z}(z)$ against the ℓ^1 -distance to the origin $|z|$. The next proposition shows that Ψ_t admits a maximizer:

Proposition 4.1 *Almost surely, $|\mathcal{C}| = \infty$ and, for all $t > 0$ and all $\eta \in \mathbb{R}$,*

$$|\{z \in \mathcal{C} : \Psi_t(z) > \eta\}| < \infty. \quad (4.6)$$

The proof of Proposition 4.1 will be given in Section 5. In order to define Z_t as a càdlàg maximizer of Ψ_t , we proceed as follows. Write $(\lambda, z) \succeq (\lambda', z')$ for the usual lexicographical order of $\mathbb{R} \times \mathbb{R}^d$, i.e., $(\lambda, z) \succeq (\lambda', z')$ if either $\lambda > \lambda'$, or $\lambda = \lambda'$ and $z \succeq z'$ according to the usual (non-strict) lexicographical order of \mathbb{R}^d . Now define, recursively for $k \in \mathbb{N}$,

$$\Psi_t^{(k)} := \sup_{z \in \mathcal{C} \setminus \{Z_t^{(1)}, \dots, Z_t^{(k-1)}\}} \Psi_t(z), \quad (4.7)$$

$$\mathfrak{G}_t^{(k)} := \left\{ z \in \mathcal{C} \setminus \{Z_t^{(1)}, \dots, Z_t^{(k-1)}\} : \Psi_t(z) = \Psi_t^{(k)} \right\}, \quad (4.8)$$

and

$$Z_t^{(k)} \in \left\{ z \in \mathfrak{G}_t^{(k)} : (\lambda^{\mathcal{C}}(z), z) \succeq (\lambda^{\mathcal{C}}(\hat{z}), \hat{z}) \forall \hat{z} \in \mathfrak{G}_t^{(k)} \right\}. \quad (4.9)$$

Observe that (4.9) determines $Z_t^{(k)}$ uniquely. Then we set

$$Z_t := Z_t^{(1)}. \quad (4.10)$$

The above definitions ensure that the maps $t \mapsto \Psi_t^{(k)}$ are continuous while $t \mapsto Z_t^{(k)}$ are càdlàg, with $t \mapsto |Z_t|$ non-decreasing (see Lemma 7.5 and (7.40) below).

4.2 Properties of the cost functional.

The technical statements start with a discussion of the properties of the above cost functional Ψ_t and the process Z_t . Recall the definitions of r_t and d_t from (2.6). The various error estimates that are to follow will

require a host of auxiliary scales. First we fix $\epsilon_t \in (0, 1)$, $\epsilon_t \gg (\ln_3 t)^{-1}$ arbitrary as in the statement of Theorem 2.5. Then, similarly to [MP14], we fix e_t, f_t, g_t, h_t and b_t such that

$$e_t, f_t, h_t, b_t \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{and} \quad g_t \xrightarrow[t \rightarrow \infty]{} \infty \quad (4.11)$$

while also

$$\frac{g_t}{\epsilon_t \ln_3 t} \ll b_t \ll f_t h_t \quad \text{and} \quad g_t h_t \ll e_t. \quad (4.12)$$

As an example of scales satisfying (4.11–4.12), one may take suitable powers of $\epsilon_t \ln_3 t$. We then have:

Proposition 4.2 *For all $0 < a \leq b < \infty$ and all $\epsilon \in (0, 1)$,*

$$\left(\Psi_{at}^{(1)} - \Psi_{at}^{(2)} \right) \wedge \left(\Psi_{bt}^{(1)} - \Psi_{bt}^{(2)} \right) > d_t e_t, \quad (4.13)$$

$$r_t f_t < \inf_{s \in [at, bt]} |Z_s| \leq \sup_{s \in [at, bt]} |Z_s| < r_t g_t \quad (4.14)$$

and

$$\rho(1 - \epsilon) \ln_2 t < \inf_{s \in [at, bt]} \zeta(Z_s) \leq \sup_{s \in [at, bt]} \zeta(Z_s) < \rho(1 + \epsilon) \ln_2 t \quad (4.15)$$

hold with probability tending to one as $t \rightarrow \infty$.

Proposition 4.2 is proved in Section 7, together with Theorems 2.6–2.7. The proofs rely strongly on the extreme order statistics of the principal Dirichlet eigenvalue in a box identified in [BK16] and, similarly to the approach of [KLMS09, MOS11, LM12, ST14, FM14, MP14], on a Poisson point process approximation. However, in order to deal with the fact that the local eigenvalues do not depend on bounded regions in space, a coarse-graining scheme taken from [BK16] is required. Our approach provides a quite direct implication of functional convergence and aging for Z_t from the convergence of the underlying point process (in a suitable topology), see in particular Lemmas 7.4, 7.6 and 7.8 below. We believe that this approach could be useful to prove analogous results in other contexts, e.g., the PAM with lighter potential tails.

Notice that in (4.13) we only require a gap between $\Psi_s^{(1)}$ and $\Psi_s^{(2)}$ for $s \in \{at, bt\}$. This is because, while the gap is greater than $d_t e_t$ with large probability at both at and bt , there is by (2.11) a non-zero probability that $s \mapsto Z_s$ jumps in the interval $[at, bt]$, leading to a zero gap at the jump time. Notwithstanding, if no such jump occurs, then the gap remains uniformly positive throughout the interval. Indeed, define

$$\mathcal{G}_{t,s} := \left\{ \Psi_s^{(1)} - \Psi_s^{(2)} \geq d_t e_t \right\}. \quad (4.16)$$

Then we have:

Proposition 4.3 *With probability one, for any $0 < a \leq b < \infty$ and any $t > 0$,*

$$\mathcal{G}_{t,at} \cap \mathcal{G}_{t,bt} \cap \{Z_{at} = Z_{bt}\} = \bigcap_{s \in [at, bt]} (\mathcal{G}_{t,s} \cap \{Z_s = Z_{at}\}). \quad (4.17)$$

The proof of Proposition 4.3 is related to that of Theorem 2.7, and so it is relegated to Section 7 as well.

4.3 Mass decomposition and negligible contributions.

Having dealt with the cost functional and localization process, we proceed by giving estimates on the solution to (1.1–1.2). As noted already earlier, this solution can be written using the Feynman-Kac formula (1.5), which

offers the strategy to control $u(t, x)$ by decomposing the expectation based on various restrictions on the underlying random walk. A starting point is a good lower bound on the total mass $U(t)$:

Proposition 4.4 For any $0 < a \leq b < \infty$,

$$\inf_{s \in [at, bt]} \left\{ \ln U(s) - s\Psi_s^{(1)} \right\} \geq o(td_t b_t \epsilon_t) \quad (4.18)$$

holds with probability tending to 1 as $t \rightarrow \infty$.

For $\Lambda \subset \mathbb{Z}^d$, let

$$\tau_\Lambda := \inf\{s > 0: X_s \in \Lambda\} \quad (4.19)$$

denote the first hitting time of Λ . Our decomposition of (1.5) begins by restricting the expectation to paths that never leave a box of side-length

$$L_t := \lfloor t \ln_2^+ t \rfloor, \quad \text{where } \ln_2^+ t := \ln_2(t \vee e). \quad (4.20)$$

This restriction comes at little loss since we have:

Proposition 4.5 For any $0 < a \leq b < \infty$, there is a $t_0 = t_0(\tilde{\zeta})$ with $t_0 < \infty$ a.s. such that

$$\sup_{s \in [at, bt]} \ln \mathbb{E}_0 \left[e^{\int_0^s \tilde{\zeta}(X_u) du} \mathbb{1}\{\tau_{B_{L_t}^c} \leq s\} \right] \leq -\frac{1}{8}t(\ln_2 t) \ln_3 t \quad (4.21)$$

holds whenever $t > t_0$.

Next we show that the bulk of the contribution to the Feynman-Kac formula comes from the paths that do not even leave the random domain

$$D_{t,s}^\circ := \{x \in \mathbb{Z}^d: |x| \leq |Z_s|(1 + h_t)\}. \quad (4.22)$$

Indeed, the contribution of paths that leave this set is bounded via:

Proposition 4.6 For any $0 < a \leq b < \infty$,

$$\begin{aligned} \sup_{s \in [at, bt]} \left\{ \ln \mathbb{E}_0 \left[e^{\int_0^s \tilde{\zeta}(X_u) du} \mathbb{1}\{\tau_{(D_{t,s}^\circ)^c} \leq s < \tau_{B_{L_t}^c}\} \right] \right. \\ \left. - \max \left\{ s\Psi_s^{(2)}, s\Psi_s^{(1)} - h_t |Z_s| \ln_3 t \right\} \right\} \leq o(td_t b_t) \end{aligned} \quad (4.23)$$

holds with probability tending to 1 as $t \rightarrow \infty$.

Finally, we show that the random walk X enters a fixed-size neighborhood of Z_t by time t with large probability:

Proposition 4.7 For all large enough $v \in \mathbb{N}$ and all $0 < a \leq b < \infty$,

$$\sup_{s \in [at, bt]} \left\{ \ln \mathbb{E}_0 \left[e^{\int_0^s \tilde{\zeta}(X_u) du} \mathbb{1}\{\tau_{B_v(Z_s)} \wedge \tau_{B_{L_t}^c} > s\} \right] - s\Psi_s^{(2)} \right\} \leq o(td_t b_t) \quad (4.24)$$

holds with probability tending to 1 as $t \rightarrow \infty$.

The above propositions will allow us to restrict the Feynman-Kac formula to the event

$$\mathcal{R}_{t,s}^v := \left\{ \tau_{(D_{t,s}^\circ)^c} > s \geq \tau_{B_v(Z_s)} \right\}, \quad (4.25)$$

and proceed to control the result using spectral techniques; see Section 4.4.

Our proofs of Propositions 4.4 and 4.5, given respectively in Sections 8.1 and 8.2, are relatively simple and follow similar results in the literature. Propositions 4.6 and 4.7 are proven in Section 8.3; their main technical point is a path expansion scheme developed in Section 6, based on an approach from [MP14]. Additional difficulties arise in our case due to smaller gaps in the potential, and to the fact that the effective support of the relevant local eigenvalues is unbounded in the limit of large times. This is overcome through a careful analysis of the connectivity properties of the level sets of the potential and their implications for the bounds derived via path expansions.

It is important to note that $\lambda^{\mathcal{C}}(Z_s)$ is the largest possible over all capitals inside $D_{t,s}^\circ$ (cf. Lemma 9.1). This comes as a consequence of the choice of h_t in (4.12), which is of special relevance as it simultaneously allows the proofs of Proposition 4.6 above (for which h_t should be large enough) and Proposition 4.9 below (for which h_t should be small enough). We also note that a complementary bound to (4.18) holds as well (cf. Lemma 8.6), which will be important for the proof of Theorem 2.6 in Section 8.4.

4.4 Localization.

Once the path has been shown to enter a neighborhood of Z_t by time t with large probability, the next item of concern is to show that it will actually not be found far away from Z_t at time t . This will be done by bounding the end-point distribution using the principal eigenfunction $\phi_{t,s}^\circ$ corresponding to the largest Dirichlet eigenvalue of the Anderson Hamiltonian in $D_{t,s}^\circ$, which we assume to be normalised so that

$$\phi_{t,s}^\circ > 0 \text{ on } D_{t,s}^\circ, \quad \phi_{t,s}^\circ = 0 \text{ on } (D_{t,s}^\circ)^c \quad \text{and} \quad \|\phi_{t,s}^\circ\|_{\ell^2(\mathbb{Z}^d)} = 1. \quad (4.26)$$

We have:

Proposition 4.8 *For any $v \in \mathbb{N}$ and $0 < a \leq b < \infty$, the following holds with probability tending to 1 as $t \rightarrow \infty$: For all $s \in [at, bt]$ and all $x \in D_{t,s}^\circ$,*

$$\mathbb{E}_0 \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\mathcal{R}_{t,s}^v \cap \{X_s=x\}} \right] \leq U(s) \sup_{y \in B_v(Z_s)} \phi_{t,s}^\circ(y)^{-3} \phi_{t,s}^\circ(x). \quad (4.27)$$

In order to use the bound in (4.27), we will need an estimate on the decay of $\phi_{t,s}^\circ$ away from Z_s . On the event $\mathcal{G}_{t,s}$ from (4.16), this is the subject of:

Proposition 4.9 *There exist $c_1, c_2 > 0$ and, for all $v \in \mathbb{N}$, also $\varepsilon_v > 0$ such that, for all $0 < a \leq b < \infty$, the following holds on with probability tending to 1 as $t \rightarrow \infty$: For all $s \in [at, bt]$, on $\mathcal{G}_{t,s}$ we have*

$$(i) \quad \phi_{t,s}^\circ(x) \leq c_1 e^{-c_2|x-Z_s|} \quad \forall x \in \mathbb{Z}^d, \quad (4.28)$$

$$(ii) \quad \phi_{t,s}^\circ(y) \geq \varepsilon_v \quad \forall y \in B_v(Z_s). \quad (4.29)$$

Propositions 4.8–4.9 are proven in Section 9. Proposition 4.8 is similar to Proposition 3.11 in [MP14], and its proof is an adaptation of the proof of Theorem 4.1 of [GKM07]. The proof of Proposition 4.9(i) is an

adaptation of the proof of Theorem 1.4 of [BK16], while part (ii) relies on results from [GM98], [GH99] and [GKM07] regarding the optimal shapes of the potential.

4.5 Proof of mass concentration results.

We have now amassed enough information for the proof of Theorem 2.3, assuming Theorem 2.7 and the above propositions:

Proof of Theorem 2.3. Fix $\nu \in \mathbb{N}$ large enough so that Proposition 4.7 is available. Fix $0 < a \leq b < \infty$. We will first show that, for all $\delta > 0$, there exists an $R \in \mathbb{N}$ such that

$$\lim_{t \rightarrow \infty} \text{Prob} \left(\exists s \in [at, bt]: \Psi_s^{(1)} - \Psi_s^{(2)} \geq d_t e_t, Q_s^{(\zeta)}(|X_s - Z_s| > R) > \delta \right) = 0, \quad (4.30)$$

and derive the desired claim from this at the very end.

We begin by noting that Propositions 4.4–4.7 imply that, with probability tending to 1 as $t \rightarrow \infty$,

$$\begin{aligned} & \ln \left(\frac{1}{U(s)} \mathbb{E}_0 \left[e^{\int_0^s \zeta(X_u) du} \mathbb{1}_{(\mathcal{R}_{t,s}^\nu)^c} \right] \right) \\ & \leq -s \min \left\{ \Psi_s^{(1)} - \Psi_s^{(2)}, h_t |Z_s| \ln_3 t, \frac{t \ln_2 t \ln_3 t}{8s} \right\} + o(td_t b_t) \end{aligned} \quad (4.31)$$

holds true for all $s \in [at, bt]$. By Proposition 4.2, on $\mathcal{G}_{t,s} = \{\Psi_s^{(1)} - \Psi_s^{(2)} \geq d_t e_t\}$ we may further bound (4.31) by

$$-at \min \left\{ d_t e_t, h_t r_t f_t \ln_3 t, \frac{1}{8b} \ln_2 t \ln_3 t \right\} + o(td_t b_t) \quad (4.32)$$

which goes to $-\infty$ as $t \rightarrow \infty$ by (2.6) and (4.12) — indeed, (4.12) shows that $e_t \ln_3 t \rightarrow \infty$ (in fact, $e_t \gg g_t / \ln_3 t$ with $g_t \rightarrow \infty$) and so $td_t e_t \gg ct / [(\ln t) \ln_3 t]$ — implying that

$$\lim_{t \rightarrow \infty} \sup_{s \in [at, bt]} \frac{\mathbb{1}_{\mathcal{G}_{t,s}}}{U(s)} \mathbb{E}_0 \left[e^{\int_0^s \zeta(X_u) du} \mathbb{1}_{(\mathcal{R}_{t,s}^\nu)^c} \right] = 0 \quad \text{in probability.} \quad (4.33)$$

Fix now $\delta > 0$ and let $R \in \mathbb{N}$ be large enough such that

$$\varepsilon_\nu^{-3} c_1 \sum_{|x| > R} e^{-c_2 |x|} < \frac{\delta}{2}, \quad (4.34)$$

where c_1, c_2 and ε_ν are as in Proposition 4.9. By Propositions 4.8–4.9,

$$\sup_{s \in [at, bt]} \frac{\mathbb{1}_{\mathcal{G}_{t,s}}}{U(s)} \sum_{x: |x - Z_s| > R} \mathbb{E}_0 \left[e^{\int_0^s \zeta(X_u) du} \mathbb{1}_{\mathcal{R}_{t,s}^\nu \cap \{X_s = x\}} \right] < \frac{\delta}{2} \quad (4.35)$$

with probability tending to 1 as $t \rightarrow \infty$, which together with (4.33) implies (4.30).

To conclude the desired statement from (4.30), fix $l_t > 0$, $l_t = o(t)$ and note that, by Theorem 2.7 and Propositions 4.2–4.3, with probability tending to 1 as $t \rightarrow \infty$,

$$Z_s = Z_t \quad \text{and} \quad \Psi_s^{(1)} - \Psi_s^{(2)} \geq d_t e_t \quad \forall s \in [t - l_t, t + l_t]. \quad (4.36)$$

This together with (4.30) (with $a < 1 < b$) implies (2.4). \square

For the proof of Theorem 2.5, we need two more propositions, which are proved in Section 10. The first one is an improvement of Proposition 4.7:

Proposition 4.10 For $\epsilon_t \in (0, 1)$ satisfying $\epsilon_t \gg (\ln_3 t)^{-1}$ (as in Theorem 2.5), as soon as $\nu \in \mathbb{N}$ is sufficiently large

$$\frac{1}{U(t)} \mathbb{E}_0 \left[e^{\int_0^t \zeta(X_s) ds} \mathbb{1}\{\tau_{(D_{t,t}^\circ)^c} > t \geq \tau_{B_\nu(Z_t)} > \epsilon_t t\} \right] \xrightarrow[t \rightarrow \infty]{} 0 \quad (4.37)$$

in probability.

The second proposition bounds the contribution of paths starting at a point $x \in B_\nu(Z_t)$ and reaching a distance greater than $\frac{1}{2}\epsilon_t \ln t$:

Proposition 4.11 For any $\nu \in \mathbb{N}$, the following holds with probability tending to 1 as $t \rightarrow \infty$: For all $x \in B_\nu(Z_t)$ and all $0 \leq s \leq t$,

$$\mathbb{E}_x \left[e^{\int_0^s \zeta(X_u) du} \mathbb{1}\left\{ \tau_{(D_{t,t}^\circ)^c} > s, \sup_{0 \leq u \leq s} |X_u - x| > \frac{1}{2}\epsilon_t \ln t \right\} \right] \leq t^{-1} \mathbb{E}_x \left[e^{\int_0^s \zeta(X_u) du} \right]. \quad (4.38)$$

Proof of Theorem 2.5. Fix $\nu \in \mathbb{N}$ large enough so that the conclusion of Proposition 4.10 becomes available. Write $\tilde{\tau} := \tau_{B_\nu(Z_t)}$ and note that, when t is large,

$$\left\{ \sup_{s \in [\epsilon_t t, t]} |X_s - Z_t| > \epsilon_t \ln t \right\} \subset (\mathcal{R}_{t,t}^\nu)^c \cup \left\{ \tau_{(D_{t,t}^\circ)^c} > t \geq \tilde{\tau} > \epsilon_t t \right\} \cup A_t, \quad (4.39)$$

where

$$A_t := \left\{ \tau_{(D_{t,t}^\circ)^c} > t, \tilde{\tau} \leq \epsilon_t t, \sup_{s \in [\tilde{\tau}, t]} |X_s - X_{\tilde{\tau}}| > \frac{1}{2}\epsilon_t \ln t \right\}. \quad (4.40)$$

By (4.33), Proposition 4.2 and Proposition 4.10,

$$Q_t^{(\zeta)} \left((\mathcal{R}_{t,t}^\nu)^c \right) \vee Q_t^{(\zeta)} \left(\tau_{(D_{t,t}^\circ)^c} > t \geq \tilde{\tau} > \epsilon_t t \right) \xrightarrow[t \rightarrow \infty]{} 0 \text{ in probability.} \quad (4.41)$$

To control $Q_t^{(\zeta)}(A_t)$, let

$$F_t(x, s) := \mathbb{E}_x \left[e^{\int_0^s \zeta(X_u) du} \mathbb{1}\left\{ \tau_{(D_{t,t}^\circ)^c} > s, \sup_{0 \leq u \leq s} |X_u - x| > \frac{1}{2}\epsilon_t \ln t \right\} \right] \quad (4.42)$$

and use the strong Markov property and Proposition 4.11 to get

$$\begin{aligned} \mathbb{E}_0 \left[e^{\int_0^t \zeta(X_s) ds} \mathbb{1}_{A_t} \right] &= \sum_{x \in B_\nu(Z_t)} \mathbb{E}_0 \left[e^{\int_0^{\tilde{\tau}} \zeta(X_s) ds} \mathbb{1}\left\{ \tau_{(D_{t,t}^\circ)^c} > \tilde{\tau} = \tau_x \leq \epsilon_t t \right\} F_t(x, t - \tilde{\tau}) \right] \\ &\leq t^{-1} U(t) \end{aligned} \quad (4.43)$$

with probability tending to 1 as $t \rightarrow \infty$. The desired claim now readily follows from (4.39), (4.41) and (4.43). \square

4.6 Proof of aging and limit profiles.

The last set of propositions to be introduced here concern the proof of Theorems 2.8 and 2.10. We start with some supporting notation. Given a function $t \mapsto \mu_t$ with $\mu_t \in \mathbb{N}$, let $\phi_{t,s}^\bullet$ denote the eigenfunction

corresponding to the largest Dirichlet eigenvalue of the Anderson operator in $B_{\mu_t}(Z_s)$, normalised so that

$$\phi_{t,s}^\bullet > 0 \text{ on } B_{\mu_t}(Z_s), \quad \phi_{t,s}^\bullet = 0 \text{ on } B_{\mu_t}^c(Z_s) \quad \text{and} \quad \|\phi_{t,s}^\bullet\|_{\ell^1(\mathbb{Z}^d)} = 1. \quad (4.44)$$

(Notice our use of the ℓ^1 -norm here.) When $s = t$ we omit one index from the notation. Recall the choice of $\kappa \in (0, 1/d)$ in (4.2). We then have:

Proposition 4.12 *For any $\mu_t \in \mathbb{N}$ with $1 \ll \mu_t \ll (\ln t)^\kappa$, and any $0 < a \leq b < \infty$,*

$$\lim_{t \rightarrow \infty} \sup_{s \in [at, bt]} \mathbb{1}_{\mathcal{G}_{t,s}} \left\| \frac{u(\cdot, s)}{U(s)} - \phi_{t,s}^\bullet(\cdot) \right\|_{\ell^1(\mathbb{Z}^d)} = 0 \quad \text{in probability.} \quad (4.45)$$

We may thus obtain information about the profile of $u(\cdot, s)$ via that of $\phi_{t,s}^\bullet$. As shown next, this can be achieved under Assumption 2.9, as it uniquely determines the limit profile V_ρ of ξ and the ‘‘shape’’ v_ρ of the principal eigenfunction:

Proposition 4.13 *If Assumption 2.9 holds, then there exists $\mu_t \in \mathbb{N}$ with $1 \ll \mu_t \ll (\ln t)^\kappa$ and a function \hat{a}_t satisfying $\lim_{t \rightarrow \infty} \hat{a}_t / \ln_2 t = \rho$ such that, for any $0 < a \leq b < \infty$, both*

$$\sup_{s \in [at, bt]} \sup_{x \in B_{\mu_t}} |\xi(x + Z_s) - \hat{a}_t - V_\rho(x)| \quad (4.46)$$

and

$$\sup_{s \in [at, bt]} \|\phi_{t,s}^\bullet(Z_s + \cdot) - v_\rho(\cdot)\|_{\ell^1(\mathbb{Z}^d)} \quad (4.47)$$

converge to 0 in probability as $t \rightarrow \infty$.

The proofs of Propositions 4.12–4.13 are based on an approach from [GKM07] and will be given in Section 11 below. Together with Theorem 2.7, they imply Theorem 2.10 as follows.

Proof of Theorem 2.10. Note that (2.18) follows directly from (4.46). For (2.19), use (4.45), (4.47), the triangle inequality for the ℓ^1 -norm and (4.36). \square

Proposition 4.12 (and Theorem 2.7) will also allow us to prove Theorem 2.8.

Proof of Theorem 2.8. We adapt the proof of Theorem 1.1 of [MOS11]. By Theorem 2.7, it is enough to show that, for any $\varepsilon \in (0, 1)$ and $b > 1$,

$$\sup_{s \in [t, bt]} \sum_{z \in \mathbb{Z}^d} \left| \frac{u(z, s)}{U(s)} - \frac{u(z, t)}{U(t)} \right| < \varepsilon \quad \text{if and only if} \quad Z_s = Z_t \quad \forall s \in [t, bt] \quad (4.48)$$

holds with probability tending to 1 as $t \rightarrow \infty$.

Assume first that $Z_s \neq Z_t$ for some $s \in (t, bt]$. By Propositions 4.2 and 4.3, we may assume that $Z_{bt} \neq Z_t$; moreover, $|Z_{bt} - Z_t| > (\ln t)^{\kappa/2}$ by (4.15), the definition of ϱ_z and the fact that $Z_t, Z_{bt} \in \mathcal{C}$. Fixing R so that (4.30) holds with $\delta < \frac{1}{2}(1 - \varepsilon)$, we obtain

$$\sum_{z \in \mathbb{Z}^d} \left| \frac{u(z, bt)}{U(bt)} - \frac{u(z, t)}{U(t)} \right| \geq \sum_{|z - Z_{bt}| \leq R} \left| \frac{u(z, bt)}{U(bt)} \right| - \sum_{|z - Z_t| > R} \left| \frac{u(z, t)}{U(t)} \right| \geq 1 - 2\delta > \varepsilon \quad (4.49)$$

with probability tending to 1 as $t \rightarrow \infty$, proving the ‘‘only if’’ part of (4.48).

Assume now that $Z_s = Z_t \forall s \in [t, bt]$. Then $\phi_{t,s}^\bullet = \phi_t^\bullet$ for all $s \in [t, bt]$, and the “if” part of (4.48) follows by (4.45) with $a = 1 < b$ together with Propositions 4.2–4.3. This finishes the proof. \square

5. PREPARATIONS

In this section we collect auxiliary results that will be used in the remainder of the paper. We start with a few basic properties of the potential field and of the principal Dirichlet eigenvalue of the Anderson Hamiltonian in subdomains of \mathbb{Z}^d , leading to the proof of Proposition 4.1. The two subsequent subsections concern additional properties of the potential field, and the last one contains spectral bounds for the Feynman-Kac formula.

5.1 Potentials and eigenvalues.

First we consider the maximum of the potential in a box. Let \widehat{a}_L be the minimal number satisfying

$$\text{Prob}(\xi(0) > \widehat{a}_L) = L^{-d}, \quad (5.1)$$

which exists since, by Assumption 2.1, $\xi(0)$ has a continuous distribution. Note that, in the notation of [GM98], $\widehat{a}_L = \psi(d \ln L)$. Then we have:

Lemma 5.1 (Maximum of the potential)

$$\lim_{L \rightarrow \infty} \max_{x \in B_L} \xi(x) - \widehat{a}_L = 0 \quad \text{a.s.} \quad (5.2)$$

Proof. See Corollary 2.7 of [GM98]. \square

Let us mention here some properties of \widehat{a}_L . By equation (2.1) of [GM98],

$$\widehat{a}_{k_L} = \widehat{a}_L + o(1) \quad \text{as } L \rightarrow \infty \quad \text{whenever } \ln k_L = \ln L(1 + o(1)) \quad (5.3)$$

and, by Remark 2.1 therein, it is straightforward to verify that $\widehat{a}_L = (\rho + o(1)) \ln_2 L$.

Next we recall the Rayleigh-Ritz formula for the largest eigenvalue of the Anderson Hamiltonian. For $\Lambda \subset \mathbb{Z}^d$ and $V : \mathbb{Z}^d \rightarrow [-\infty, \infty)$, let $\lambda_\Lambda^{(1)}(V)$ denote the largest eigenvalue of the operator $\Delta + V$ in Λ with Dirichlet boundary conditions. Then the Rayleigh-Ritz formula reads

$$\lambda_\Lambda^{(1)}(V) = \sup \left\{ \langle (\Delta + V)\phi, \phi \rangle_{\ell^2(\mathbb{Z}^d)} : \phi \in \mathbb{R}^{\mathbb{Z}^d}, \text{supp } \phi \subset \Lambda, \|\phi\|_{\ell^2(\mathbb{Z}^d)} = 1 \right\}. \quad (5.4)$$

When $V = \xi$ we sometimes write $\lambda_\Lambda^{(1)}$ instead of $\lambda_\Lambda^{(1)}(\xi)$. Here are some straightforward consequences of the Rayleigh-Ritz formula:

1 for any $\Gamma \subsetneq \Lambda$,

$$\max_{z \in \Gamma} V(z) - 2d \leq \lambda_\Gamma^{(1)}(V) \leq \lambda_\Lambda^{(1)}(V) \leq \max_{z \in \Lambda} V(z); \quad (5.5)$$

2 the eigenfunction corresponding to $\lambda_\Lambda^{(1)}(V)$ can be taken non-negative;

3 if V is real-valued and Λ is finite and connected (in the graph-theoretical sense according to the usual nearest-neighbor structure of \mathbb{Z}^d), then the middle inequality in (5.5) is strict and, moreover, the non-negative eigenfunction corresponding to $\lambda_\Lambda^{(1)}(V)$ is strictly positive;

4 for $\Lambda, \Lambda' \subset \mathbb{Z}^d$ such that $\text{dist}(\Lambda, \Lambda') \geq 2$,

$$\lambda_{\Lambda \cup \Lambda'}^{(1)}(V) = \max\{\lambda_{\Lambda}^{(1)}(V), \lambda_{\Lambda'}^{(1)}(V)\}. \quad (5.6)$$

We can now give the proof of Proposition 4.1.

Proof of Proposition 4.1. Note that, for any $R \in \mathbb{N}$ and $z \in \mathbb{Z}^d$,

$$\{z \in \mathcal{C}\} \supseteq \left\{ \zeta(z) \leq \rho \kappa^{-1} \ln R, \zeta(z) = \max_{x \in B_R(z)} \zeta(x) \right\}, \quad (5.7)$$

and the probability of the event on the right-hand side does not depend on z and is positive for some fixed large enough R . As the events on the right of (5.7) depend only on a finite number of coordinates, the second Borel-Cantelli lemma shows $|\mathcal{C}| = \infty$ almost surely. Now, by (5.5), $\lambda^{\mathcal{C}}(z) \leq \zeta(z)$ for any $z \in \mathcal{C}$ while, by Lemma 5.1, almost surely $\zeta(z) \leq 2\rho \ln_2 |z|$ for all $|z|$ large enough. This implies that, almost surely,

$$\limsup_{R \rightarrow \infty} \sup_{z \in \mathcal{C}, |z|=R} \Psi_t(z) \leq \lim_{R \rightarrow \infty} \left(2\rho \ln_2 R - R \frac{\ln_3 R}{t} \right) = -\infty \quad (5.8)$$

for each $t > 0$. This finishes the proof. \square

Next we generalise (2.14–2.15) as follows. For $\Lambda \subset \mathbb{Z}^d$ and $V : \mathbb{Z}^d \rightarrow [-\infty, \infty)$, let

$$\mathcal{L}_{\Lambda}(V) := \sum_{x \in \Lambda} e^{\frac{V(x)}{\rho}} \quad (5.9)$$

with the interpretation $e^{-\infty} := 0$. Then set

$$\chi_{\Lambda} = \chi_{\Lambda}(\rho) := -\sup \left\{ \lambda_{\Lambda}^{(1)}(V) : V \in [-\infty, 0]^{\mathbb{Z}^d}, \mathcal{L}_{\Lambda}(V) \leq 1 \right\}. \quad (5.10)$$

When $\Lambda = \mathbb{Z}^d$ we write just χ . From the definition it follows that, if $\Gamma \subset \Lambda$, then $\chi_{\Gamma} \geq \chi_{\Lambda}$; in particular, $0 \leq \chi \leq \chi_{\Lambda} \leq 2d$ since $\chi_{\{x\}} = 2d$ for any $x \in \mathbb{Z}^d$.

5.2 Islands.

Central to our analysis is a domain truncation method taken from [BK16], which we describe next. Recall the choice of $\kappa \in (0, 1/d)$ in (4.2) and fix an increasing sequence $R_L \in \mathbb{N}$ such that

$$R_L \leq (\ln L) \vee 1 \quad \text{and} \quad R_L \gg (\ln L)^{\beta} \quad \text{as } L \rightarrow \infty \quad \text{for some } \beta \in (\kappa, 1/d). \quad (5.11)$$

This sequence will control the spatial size of the regions in B_L where the field is large, and thus the (principal) local eigenvalue has a chance to be close to maximal. We will often work with R_L satisfying additionally

$$R_L \ll (\ln L)^{\alpha} \quad \text{as } L \rightarrow \infty \quad \text{for some } \alpha \in (\beta, 1/d), \quad (5.12)$$

but for the proof of Proposition 4.11 in Section 10.2 we will need to consider R_L growing as $\ln L$. Unless explicitly mentioned, only (5.11) is assumed in the following. Given $A > 0$ and $L \in \mathbb{N}$, let

$$\Pi_{L,A} := \{z \in B_L : \zeta(z) > \widehat{a}_L - 2A\} \quad (5.13)$$

be the set of high exceedances of the field inside the box B_L , and put

$$D_{L,A} := \bigcup_{z \in \Pi_{L,A}} B_{R_L}(z) \cap B_L. \quad (5.14)$$

The parameter A , providing the cutoff between the “high” and “small” values of the field, will be later fixed to a suitably large value that depends only on the dimension d and the parameter ρ .

Let $\mathfrak{C}_{L,A}$ denote the set of all connected components of $D_{L,A}$, to be called *islands*. For $\mathcal{C} \in \mathfrak{C}_{L,A}$, let

$$z_{\mathcal{C}} := \operatorname{argmax}\{\tilde{\zeta}(z) : z \in \mathcal{C}\} \quad (5.15)$$

be the point of highest potential within \mathcal{C} . Since $\tilde{\zeta}(0)$ has a continuous law, $z_{\mathcal{C}}$ is a.s. well defined for all $\mathcal{C} \in \mathfrak{C}_{L,A}$.

Next we gather useful properties of $\mathfrak{C}_{L,A}$. The first result concerns a uniform bound on the size of the islands. Hereafter we will say that an L -dependent event occurs “almost surely eventually as $L \rightarrow \infty$ ” if there exists a.s. a (random) $L_0 \in \mathbb{N}$ such that the event happens for all $L \geq L_0$. Similar language will be used for events depending on other parameters (e.g. t).

Lemma 5.2 (Maximum size of the islands) *For any $A > 0$, there exists $n_A \in \mathbb{N}$ such that, for any R_L satisfying (5.11), a.s. eventually as $L \rightarrow \infty$, all $\mathcal{C} \in \mathfrak{C}_{L,A}$ satisfy $|\mathcal{C} \cap \Pi_{L,A}| \leq n_A$ and $\operatorname{diam}(\mathcal{C}) \leq n_A R_L$.*

Proof. See the proof of Lemma 6.6 in [BK16]. □

For $\delta > 0$, $A > 0$ and $L \in \mathbb{N}$, let

$$\mathfrak{C}_{L,A}^{\delta} := \{\mathcal{C} \in \mathfrak{C}_{L,A} : \lambda_{\mathcal{C}}^{(1)} > \hat{a}_L - \chi - \delta\} \quad (5.16)$$

denote the set of islands with large principal eigenvalue. We call these *relevant islands*, as their eigenvalue is close to the principal eigenvalue of B_L (cf. Lemma 6.8 of [BK16]).

The following lemma is crucial for the proof of Proposition 7.1 below, from which Proposition 4.2 follows. It allows us to compare the eigenvalues of relevant islands to those of disjoint boxes.

Lemma 5.3 (Coarse-graining for local principal eigenvalues) *Assume R_L satisfies (5.11) and (5.12). Let $N_L \in \mathbb{N}$ satisfy $L^{\beta} \ll N_L \ll L^{\alpha}$ as $L \rightarrow \infty$ for some $0 < \beta < \alpha < 1$. For all $A > 0$ sufficiently large and $\delta > 0$ small enough, the following occurs with probability tending to one as $L \rightarrow \infty$:*

- (i) Each $\mathcal{C} \in \mathfrak{C}_{L,A}^{\delta}$ satisfies $\lambda_{\mathcal{C}}^{(1)} - \lambda_{\mathcal{C}}^{(2)} \geq \frac{1}{2}\rho \ln 2$.
- (ii) For each $\mathcal{C} \in \mathfrak{C}_{L,A}^{\delta}$, there exists $z \in (2N_L + 1)\mathbb{Z}^d$ such that $\mathcal{C} \subset B_{N_L}(z) \subset B_L$.
- (iii) Every two distinct $\mathcal{C}, \mathcal{C}' \in \mathfrak{C}_{L,A}^{\delta}$ satisfy $\operatorname{dist}(\mathcal{C}, \mathcal{C}') > 4dN_L$.
- (iv) Let $\eta_A := \{1 + A/(4d)\}^{-1}$. For any $z \in (2N_L + 1)\mathbb{Z}^d$ such that $B_{N_L}(z) \subset B_L$ and $\lambda_{B_{N_L}(z)}^{(1)} > \hat{a}_L - \chi - \delta + (\eta_A)^{R_L}$, there exists a $\mathcal{C} \in \mathfrak{C}_{L,A}^{\delta}$ satisfying $\mathcal{C} \subset B_{N_L}(z)$ and

$$\lambda_{\mathcal{C}}^{(1)} > \lambda_{B_{N_L}(z)}^{(1)} - (\eta_A)^{R_L}. \quad (5.17)$$

Proof. Let A, δ be as in the statement of Lemma 6.7 of [BK16]; we may assume that $A > \chi + \delta$. Items (i)–(iii) follow from items (1)–(3) in this lemma (the scales there do not match ours exactly, but the proof is the same). For (iv), assume that L is so large that $2d(\eta_A)^{2R_L-1} < (\eta_A)^{R_L}$, and note that $\lambda_{B_{N_L}(z)}^{(1)} - A > \hat{a}_L - 2A$. By Theorem 2.1 of [BK16] applied to $D = B_{N_L}(z)$ and (5.6), there exists $\mathcal{C} \in \mathfrak{C}_{L,A}$, $\mathcal{C} \cap B_{N_L}(z) \neq \emptyset$ such that (5.17) holds. In particular, $\mathcal{C} \in \mathfrak{C}_{L,A}^{\delta}$ so, by item (ii), $\mathcal{C} \subset B_{N_L}(z)$. □

Our next goal is to control the behavior of the potential inside relevant islands. This will be important for the proofs of Propositions 4.7 and 4.9 as well as Lemma 5.8 below. First we will need two lemmas concerning lower and upper bounds for \mathcal{L} .

Lemma 5.4 For any $\Lambda \subset \mathbb{Z}^d$ and any $a \in \mathbb{R}$, if $\lambda_\Lambda^{(1)} \geq a$ then $\mathcal{L}_\Lambda(\xi - a - \chi_\Lambda) \geq 1$.

Proof. This is a consequence of (5.9–5.10) and the fact that $\lambda_\Lambda^{(1)}(V + a) = \lambda_\Lambda^{(1)}(V) + a$. \square

Lemma 5.5 Let R_L satisfy (5.11–5.12). For any $A > 0$,

$$\limsup_{L \rightarrow \infty} \sup_{\mathcal{C} \in \mathfrak{C}_{L,A}} \mathcal{L}_{\mathcal{C}}(\xi - \widehat{a}_L) \leq 1 \quad \text{a.s.} \quad (5.18)$$

Proof. This is a consequence of Lemma 5.2 and a straightforward extension of Corollary 2.12 in [GM98] with R substituted by $n_A R_L$. \square

We will now combine the previous two lemmas with results from [BK16], [GH99] and [GKM07] to obtain upper and lower bounds around \widehat{a}_L for the potential in relevant islands.

Lemma 5.6 (Upper bound for the potential inside relevant islands) Assume (5.11–5.12). For all $\delta \in (0, 1)$ small enough, there exist $A_1 > 4d$ and $\nu_1 \in \mathbb{N}$ such that, for all $A > 0$, a.s. eventually as $L \rightarrow \infty$,

$$\sup_{\mathcal{C} \in \mathfrak{C}_{L,A}^\delta} \sup_{z \in \mathcal{C} \setminus B_{\nu_1}(z_{\mathcal{C}})} \xi(z) \leq \widehat{a}_L - 2A_1. \quad (5.19)$$

Proof. We follow the proof of Lemma 4.8 of [BK16]. Fix $\delta \in (0, 1)$ small enough such that

$$A_1 := -\frac{1}{2}\rho \ln \left(e^{\frac{2\delta}{\rho}} - e^{-\frac{2\delta}{\rho}} \right) > 4d > \chi + \delta, \quad (5.20)$$

and let $r \in \mathbb{N}$ be such that $2d\eta_{A_1}^{2r-1} < \delta$ where $\eta_A := (1 + A/4d)^{-1}$. For $\mathcal{C} \in \mathfrak{C}_{L,A}^\delta$, let

$$S := \{x \in \mathcal{C} : \xi(x) > \widehat{a}_L - 2A_1\}. \quad (5.21)$$

We claim that

$$\text{diam } S \leq 2(r+1)|S|. \quad (5.22)$$

Indeed, suppose by contradiction that (5.22) does not hold. Then $S = S_1 \cup S_2$ with $\text{dist}(S_1, S_2) \geq 2(r+1)$. Let $S_i^r := \{x \in \mathcal{C} : \text{dist}(x, S_i) \leq r\}$, $i = 1, 2$. Then, by (5.6),

$$\lambda_{S_1^r}^{(1)} \vee \lambda_{S_2^r}^{(1)} = \lambda_{S_1^r \cup S_2^r}^{(1)} > \lambda_{\mathcal{C}}^{(1)} - 2d\eta_{A_1}^{2r-1} > \widehat{a}_L - \chi - 2\delta \quad (5.23)$$

where for the first inequality we use Theorem 2.1 of [BK16] applied to $D := \mathcal{C}$ (note that $\lambda_{\mathcal{C}}^{(1)} - A_1 > \widehat{a}_L - 2A_1$ since \mathcal{C} is assumed to be in $\mathfrak{C}_{L,A}^\delta$, i.e., such that $\lambda_{\mathcal{C}}^{(1)} > \widehat{a}_L - \chi - \delta$, and by (5.20)), and the last inequality follows by our choice of r . Supposing without loss of generality that $\lambda_{S_1^r}^{(1)} \geq \lambda_{S_2^r}^{(1)}$, by Lemma 5.4 and (5.23) we have

$$\mathcal{L}_{S_1^r}(\xi - \widehat{a}_L) \geq e^{\frac{\chi_{S_1^r} - \chi - 2\delta}{\rho}} \geq e^{-\frac{2\delta}{\rho}}. \quad (5.24)$$

On the other hand, by Lemma 5.5 we may suppose that $\mathcal{L}_{\mathcal{C}}(\xi - \widehat{a}_L) \leq e^{2\delta/\rho}$. Then, for any $x \in S_2$,

$$\mathcal{L}_{S_1^r}(\xi - \widehat{a}_L) \leq \mathcal{L}_{\mathcal{C}}(\xi - \widehat{a}_L) - e^{\frac{\xi(x) - \widehat{a}_L}{\rho}} \leq e^{\frac{2\delta}{\rho}} - e^{\frac{\xi(x) - \widehat{a}_L}{\rho}}. \quad (5.25)$$

Combining (5.24–5.25) we obtain

$$\tilde{\zeta}(x) - \hat{a}_L \leq \rho \ln \left(e^{\frac{2\delta}{\rho}} - e^{-\frac{2\delta}{\rho}} \right) = -2A_1, \quad (5.26)$$

contradicting $x \in S$. Therefore, (5.22) holds.

To conclude, note that

$$e^{\frac{2\delta}{\rho}} \geq \mathcal{L}_{\mathcal{C}}(\tilde{\zeta} - \hat{a}_L) \geq e^{-\frac{2A_1}{\rho}} |S|. \quad (5.27)$$

Since $z_{\mathcal{C}} \in S$ by (5.5) and (5.20), the inequalities (5.22) and (5.27) now imply (5.19) with $\nu_1 := \lceil 2(r + 1)e^{\frac{2(A_1 + \delta)}{\rho}} \rceil$. \square

Lemma 5.7 (Lower bound for the potential in relevant islands) *Suppose that R_L is such that (5.11–5.12) hold. For any $\nu \in \mathbb{N}$, there exist $A^*, \delta > 0$ such that, for all $A > 0$, the following is true a.s. eventually as $L \rightarrow \infty$:*

$$\inf_{\mathcal{C} \in \mathfrak{C}_{L,A}^{\delta}} \inf_{z \in B_{\nu}(z_{\mathcal{C}})} \tilde{\zeta}(z) \geq \hat{a}_L - 2A^*. \quad (5.28)$$

Proof. Recall the definition of \mathcal{M}_{ρ}^* in (2.16). We note that Lemma 3.2(i) of [GKM07] holds for \mathcal{M}_{ρ}^* in place of \mathcal{M}_{ρ} , as can be inferred from the proof. In particular, $\mathcal{M}_{\rho}^* \neq \emptyset$ and, by Lemma 3.1 therein, all $V \in \mathcal{M}_{\rho}^*$ satisfy $\mathcal{L}(V) = 1$. On the other hand, by (3.21) in [GKM07] together with Theorem 2 and Proposition 3 of [GH99] (see also (5.44) therein),

$$A^* := - \inf_{V \in \mathcal{M}_{\rho}^*} \inf_{x \in B_{\nu}} V(x) < \infty. \quad (5.29)$$

Fix, by (3.6) in [GKM07], $\delta > 0$ small enough such that

$$\left\{ \begin{array}{l} V \in [-\infty, 0]^{\mathbb{Z}^d}, 0 \in \operatorname{argmax}(V), \mathcal{L}(V) \leq 1 \\ \text{and } \inf_{\bar{V} \in \mathcal{M}_{\rho}^*} \sup_{x \in B_{\nu}} |V(x) - \bar{V}(x)| > A^* \end{array} \right\} \Rightarrow \lambda^{(1)}(V) < -\chi - 2\delta. \quad (5.30)$$

Fix $\mathcal{C} \in \mathfrak{C}_{L,A}^{\delta}$ and define

$$V^*(x) := \begin{cases} \tilde{\zeta}(x + z_{\mathcal{C}}) - \hat{a}_L - \delta & \text{if } x + z_{\mathcal{C}} \in \mathcal{C}, \\ -\infty & \text{otherwise.} \end{cases} \quad (5.31)$$

By Lemma 5.1, $V^* \in [-\infty, 0]^{\mathbb{Z}^d}$ a.s. eventually as $L \rightarrow \infty$, and $0 \in \operatorname{argmax}(V^*)$ by the definition of $z_{\mathcal{C}}$. Furthermore, $\mathcal{L}(V^*) = \mathcal{L}_{\mathcal{C}}(\tilde{\zeta} - \hat{a}_L - \delta)$ which is a.s. smaller than 1 for large L by Lemma 5.5. Now, since $\mathcal{C} \in \mathfrak{C}_{L,A}^{\delta}$, we have $\lambda^{(1)}(V^*) = \lambda_{\mathcal{C}}^{(1)} - \hat{a}_L - \delta > -\chi - 2\delta$, and thus the conclusion follows from (5.29–5.30). \square

We end this subsection with a comparison between the islands and capitals with large local eigenvalues, which will be crucial in the proof of Proposition 7.1 below.

Lemma 5.8 *Assume (5.11–5.12). There exists a constant $c_1 > 0$ such that, for all $A > 0$ large enough and $\delta > 0$ small enough, the following occurs with probability tending to one as $L \rightarrow \infty$:*

(i) If $\mathcal{C} \in \mathfrak{C}_{L,A}^\delta$, then $z_{\mathcal{C}} \in \mathcal{C}$, $(\ln L)^{\kappa/2} < \varrho_{z_{\mathcal{C}}} < R_L$ and

$$0 \leq \lambda_{\mathcal{C}}^{(1)} - \lambda^{\mathcal{C}}(z_{\mathcal{C}}) \leq e^{-c_1(\ln L)^{\kappa/2}}. \quad (5.32)$$

(ii) For all $z \in \mathcal{C}$ such that $B_{\varrho_z}(z) \subset B_L$ and $\lambda^{\mathcal{C}}(z) > \widehat{a}_L - \chi - \delta$, there exists $\mathcal{C} \in \mathfrak{C}_{L,A}^\delta$ such that $z = z_{\mathcal{C}}$ and (5.32) holds.

Proof. Let $A, \delta > 0$ satisfy the hypotheses of Lemmas 5.3 and 5.6, and let $A_1 > 0$, $\nu_1 \in \mathbb{N}$ as in Lemma 5.6. We may assume that $2A > A_1$. For (i), note that, if $\mathcal{C} \in \mathfrak{C}_{L,A}^\delta$, then $(\ln L)^{\kappa/2} + \nu_1 < \varrho_{z_{\mathcal{C}}} \leq \max_{z \in B_L} \varrho_z < R_L$ for all L large enough by (4.2), (5.2), (5.5) and (5.11), and thus $z_{\mathcal{C}} \in \mathcal{C}$. By Lemma 5.6, the set $\{x \in \mathcal{C} : \text{dist}(x, \Pi_{L,A_1}) \leq (\ln L)^{\kappa/2}\}$ is contained in $B_{\varrho_{z_{\mathcal{C}}}}(z_{\mathcal{C}})$ and thus (5.32) follows by Theorem 2.1 of [BK16] with $c_1 := \ln(1 + A_1/(4d))$. For (ii), note that, again by (5.5), $\zeta(z) > \widehat{a}_L - A_1$ and thus $z \in \Pi_{L,A}$. Letting $\mathcal{C} \in \mathfrak{C}_{L,A}$ such that $z \in \mathcal{C}$, note that $B_{\varrho_z}(z) \subset \mathcal{C}$ since $\varrho_z < R_L$, and thus $\mathcal{C} \in \mathfrak{C}_{L,A}^\delta$. Since $\varrho_z > \nu_1$, $z = z_{\mathcal{C}}$ by Lemma 5.6, and (5.32) follows by item (i). \square

5.3 Connectivity properties of the potential field.

In this section, we provide bounds on the number of points in which the potential achieves high values inside connected sets of the lattice. These will be important in the proof of Proposition 6.1. We will use the following concentration inequality for Binomial random variables.

Lemma 5.9 *Let $\text{Bin}(p, n)$ denote a Binomial random variable with parameters p and n . Then, for all $u > 0$,*

$$P(\text{Bin}(p, n) > u) \leq \exp \left\{ -u \left(\ln \frac{u}{np} - 1 \right) \right\}. \quad (5.33)$$

Proof. Since, for any $\alpha > 0$,

$$E[e^{\alpha \text{Bin}(p, n)}] = \{1 + p(e^\alpha - 1)\}^n \leq e^{npe^\alpha}, \quad (5.34)$$

(5.33) follows by applying Markov's inequality and maximizing over $\alpha > 0$. \square

Our first lemma reads as follows.

Lemma 5.10 (Number of intermediate peaks of the potential) *For each $\beta \in (0, 1)$ there is $\varepsilon \in (0, \beta/2)$ such that, a.s. eventually as $L \rightarrow \infty$, for all finite connected subsets $\Lambda \subset \mathbb{Z}^d$ with $\Lambda \cap B_L \neq \emptyset$ and $|\Lambda| \geq (\ln L)^\beta$,*

$$N_\Lambda := |\{z \in \Lambda : \zeta(z) > (1 - \varepsilon)\widehat{a}_L\}| \leq \frac{|\Lambda|}{(\ln L)^\varepsilon}. \quad (5.35)$$

Proof. Let $\varepsilon \in (0, \delta/2)$ be small enough so that, for all L large enough,

$$p_L := \text{Prob}(\zeta(0) > (1 - \varepsilon)\widehat{a}_L) \leq \exp \left\{ -(\ln L)^{1 - \frac{\delta}{2}} \right\}. \quad (5.36)$$

This is possible by e.g. Lemma 6.1 in [BK16]. Now fix a point $x \in B_L$ and $n \in \mathbb{N}$. The number of connected subsets $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| = n$ and $x \in \Lambda$ is at most $e^{c_0 n}$ for some $c_0 > 0$ independent of x (see e.g.

[G99], Section 4.2). For such a Λ , the random variable N_Λ has a $\text{Bin}(p_L, n)$ -distribution. Using (5.33) and a union bound, we obtain

$$\begin{aligned} \text{Prob}\left(\exists \text{ connected } \Lambda \ni x, |\Lambda| = n \text{ and } N_\Lambda > n/(\ln L)^\varepsilon\right) \\ \leq \exp\left\{-n\left((\ln L)^{1-\frac{\delta}{2}-\varepsilon} - c_0 - \frac{1+\varepsilon \ln_2 L}{(\ln L)^\varepsilon}\right)\right\}. \end{aligned} \quad (5.37)$$

When L is large enough, the expression in the parentheses above is at least $\frac{1}{2}(\ln L)^{1-\frac{\delta}{2}-\varepsilon}$. Summing over $n \geq (\ln L)^\delta$ and $x \in B_L$, we get

$$\text{Prob}\left(\begin{array}{l} \exists \text{ connected } \Lambda \text{ such that } \Lambda \cap B_L \neq \emptyset, \\ |\Lambda| \geq (\ln L)^\varepsilon \text{ and (5.35) does not hold} \end{array}\right) \leq c_1 L^d \exp\left\{-c_2 (\ln L)^{1+\frac{\delta}{2}-\varepsilon}\right\} \quad (5.38)$$

for some positive constants c_1, c_2 . By our choice of ε , (5.38) is summable on L , so the conclusion follows from the Borel-Cantelli lemma. \square

A similar computation allows us to bound the number of high exceedances of the potential.

Lemma 5.11 (Number of high exceedances of the potential) *For each $A > 0$, there is a constant $C \geq 1$ such that, for all $\delta \in (0, 1)$, the following holds a.s. eventually as $L \rightarrow \infty$: For all finite connected subsets $\Lambda \subset \mathbb{Z}^d$ with $\Lambda \cap B_L \neq \emptyset$ and $|\Lambda| \geq C(\ln L)^\delta$ it holds that*

$$|\Lambda \cap \Pi_{L,A}| \leq \frac{|\Lambda|}{(\ln L)^\delta}. \quad (5.39)$$

Proof. The proof proceeds exactly as for Lemma 5.10 by noting that, by Lemma 6.1 in [BK16],

$$p_L := \text{Prob}(0 \in \Pi_{L,A}) \leq L^{-\varepsilon} \quad (5.40)$$

for some $\varepsilon \in (0, 1)$ and all large enough L , and then taking $C > 2(d+1)/\varepsilon$. \square

5.4 Spectral bounds.

Here we state some spectral bounds for the Feynman-Kac formula. The results in this section are deterministic, i.e., they hold for any fixed choice of potential $\zeta \in \mathbb{R}^{\mathbb{Z}^d}$.

Fix a finite connected subset $\Lambda \subset \mathbb{Z}^d$, and let H_Λ denote the restriction of the Anderson Hamiltonian to Λ with Dirichlet boundary conditions. For $z \in \Lambda$, let u_Λ^z be the positive solution of

$$\begin{aligned} \partial_t u(x, t) &= H_\Lambda u(x, t), & x \in \Lambda, t > 0, \\ u(x, 0) &= \mathbb{1}_z(x), & x \in \Lambda, \end{aligned} \quad (5.41)$$

and set $U_\Lambda^z(t) := \sum_{x \in \Lambda} u_\Lambda^z(x, t)$. The solution admits the Feynman-Kac representation

$$u_\Lambda^z(x, t) = \mathbb{E}_z \left[\exp \left\{ \int_0^t \zeta(X_s) ds \right\} \mathbb{1}\{\tau_{\Lambda^c} > t, X_t = x\} \right], \quad (5.42)$$

where τ_{Λ^c} is as in (4.19). It also admits the spectral representation

$$u_{\Lambda}^z(x, t) = \sum_{k=1}^{|\Lambda|} e^{t\lambda_{\Lambda}^{(k)}} \phi_{\Lambda}^{(k)}(z) \phi_{\Lambda}^{(k)}(x), \quad (5.43)$$

where $\lambda_{\Lambda}^{(1)} > \lambda_{\Lambda}^{(2)} \geq \dots \geq \lambda_{\Lambda}^{(|\Lambda|)}$ and $\phi_{\Lambda}^{(1)}, \phi_{\Lambda}^{(2)}, \dots, \phi_{\Lambda}^{(|\Lambda|)}$ are respectively the eigenvalues and corresponding orthonormal eigenfunctions of H_{Λ} . One may exploit these representations to obtain bounds for one in terms of the other, as shown by the following lemma.

Lemma 5.12 (Bounds on the solution) *For any finite $\Lambda \subset \mathbb{Z}^d$, any $z \in \Lambda$ and any $t > 0$,*

$$\begin{aligned} e^{t\lambda_{\Lambda}^{(1)}} \phi_{\Lambda}^{(1)}(z)^2 &\leq \mathbb{E}_z \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{\tau_{\Lambda^c} > t, X_t = z\}} \right] \\ &\leq \mathbb{E}_z \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{\tau_{\Lambda^c} > t\}} \right] \leq e^{t\lambda_{\Lambda}^{(1)}} |\Lambda|^{3/2}. \end{aligned} \quad (5.44)$$

Proof. The first and last inequalities follow directly from (5.42–5.43); the middle inequality is elementary. \square

The second lemma bounds the Feynman-Kac formula integrated up to an exit time.

Lemma 5.13 (Mass up to an exit time) *For any $z \in \Lambda$ and $\gamma > \lambda_{\Lambda}^{(1)}$,*

$$\mathbb{E}_z \left[\exp \left\{ \int_0^{\tau_{\Lambda^c}} (\xi(X_s) - \gamma) ds \right\} \right] \leq 1 + \frac{2d|\Lambda|}{\gamma - \lambda_{\Lambda}^{(1)}}. \quad (5.45)$$

Proof. See Lemma 4.2 in [GKM07]. \square

The next lemma is a well-known representation for the principal eigenfunction.

Lemma 5.14 *For any $x, y \in \Lambda$,*

$$\frac{\phi_{\Lambda}^{(1)}(x)}{\phi_{\Lambda}^{(1)}(y)} = \mathbb{E}_x \left[\exp \left\{ \int_0^{\tau_y} (\xi(X_u) - \lambda_{\Lambda}^{(1)}) du \right\} \mathbb{1}_{\{\tau_y < \tau_{\Lambda^c}\}} \right]. \quad (5.46)$$

Proof. See e.g. Proposition 3.3 in [MP14]. \square

Our last lemma bounds the Feynman-Kac formula when the random walk is restricted to hit a subset, and is the principal ingredient in the proof of Proposition 4.8.

Lemma 5.15 (Bound by principal eigenfunction) *For all $t > 0, z, x \in \Lambda$ and $\Gamma \subset \Lambda$,*

$$\mathbb{E}_z \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{X_t = x, \tau_{\Lambda^c} > t \geq \tau_{\Gamma}\}} \right] \leq U_{\Lambda}^z(t) \phi_{\Lambda}^{(1)}(x) \sup_{y \in \Gamma} |\phi_{\Lambda}^{(1)}(y)|^{-3}. \quad (5.47)$$

Proof. We adapt the proof of Theorem 4.1 of [GKM07]. Fix $z \in \mathbb{Z}^d$ and, for $x \in \mathbb{Z}^d$ and $t > 0$, denote

$$w(x, t) := \mathbb{E}_x \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{X_t = z, \tau_{\Lambda^c} > t \geq \tau_{\Gamma}\}} \right]. \quad (5.48)$$

Note that, by invariance under time reversal, (5.48) is equal to the left-hand side of (5.47). It will suffice to show that, for any $0 < s \leq t$ and $y \in \Gamma$,

$$\mathbb{E}_y \left[e^{\int_0^{t-s} \xi(X_u) du} \mathbb{1}_{\{X_{t-s} = z, \tau_{\Lambda^c} > t-s\}} \right] \leq e^{-s\lambda_{\Lambda}^{(1)}} |\phi_{\Lambda}^{(1)}(y)|^{-2} w(y, t). \quad (5.49)$$

Indeed, by the strong Markov property, $w(x, t)$ equals

$$\begin{aligned} & \sum_{y \in \Gamma} \mathbb{E}_x \left[e^{\int_0^{\tau_y} \xi(X_u) du} \mathbb{1}_{\{\tau_{\Lambda^c} > \tau_y = \tau_T \leq t\}} \left(\mathbb{E}_y \left[e^{\int_0^{t-s} \xi(X_u) du} \mathbb{1}_{\{X_{t-s}=z, \tau_{\Lambda^c} > t-s\}} \right] \right)_{s=\tau_y} \right] \\ & \leq \sum_{y \in \Gamma} |\phi_{\Lambda}^{(1)}(y)|^{-2} w(y, t) \mathbb{E}_x \left[e^{\int_0^{\tau_y} (\xi(X_u) - \lambda_{\Lambda}^{(1)}) du} \mathbb{1}_{\{\tau_{\Lambda^c} > \tau_y\}} \right] \\ & = \phi_{\Lambda}^{(1)}(x) \sum_{y \in \Gamma} |\phi_{\Lambda}^{(1)}(y)|^{-3} w(y, t) \leq \phi_{\Lambda}^{(1)}(x) \sup_{y \in \Gamma} |\phi_{\Lambda}^{(1)}(y)|^{-3} U_{\Lambda}^z(t), \end{aligned} \quad (5.50)$$

where for the second line we used (5.49) and, for the last one, we invoked (5.46) and one more time applied the invariance under time reversal.

In order to prove (5.49), we restrict to $X_s = y$ inside the expectation defining $w(y, t)$ to obtain

$$w(y, t) \geq \mathbb{E}_y \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{X_s=y, \tau_{\Lambda^c} > s\}} \right] \mathbb{E}_y \left[e^{\int_0^{t-s} \xi(X_u) du} \mathbb{1}_{\{X_{t-s}=z, \tau_{\Lambda^c} > t-s\}} \right]. \quad (5.51)$$

By Lemma 5.12,

$$\mathbb{E}_y \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{X_s=y, \tau_{\Lambda^c} > s\}} \right] \geq e^{s\lambda_{\Lambda}^{(1)}} |\phi_{\Lambda}^{(1)}(y)|^2, \quad (5.52)$$

implying (5.49) as desired. \square

6. PATH EXPANSIONS

In this section, we develop a setup to bound the contribution of certain specific classes of random-walk paths to the Feynman-Kac formula. This leads to Propositions 6.1–6.2 below, which are the key to the proof of Propositions 4.6–4.7 in Section 8 and Propositions 4.10–4.11 in Section 10.

6.1 Key propositions.

To start, we define various sets of nearest-neighbour paths in \mathbb{Z}^d as follows. For $\ell \in \mathbb{N}_0$ and subsets $\Lambda, \Lambda' \subset \mathbb{Z}^d$, define

$$\mathcal{P}_{\ell}(\Lambda, \Lambda') := \left\{ (\pi_0, \dots, \pi_{\ell}) \in (\mathbb{Z}^d)^{\ell+1} : \begin{array}{l} \pi_0 \in \Lambda, \pi_{\ell} \in \Lambda' \\ |\pi_i - \pi_{i-1}| = 1 \quad i = 1, \dots, \ell \end{array} \right\} \quad (6.1)$$

and set

$$\begin{aligned} \mathcal{P}(\Lambda, \Lambda') &:= \bigcup_{\ell \in \mathbb{N}_0} \mathcal{P}_{\ell}(\Lambda, \Lambda'), \\ \mathcal{P}_{\ell} &:= \mathcal{P}_{\ell}(\mathbb{Z}^d, \mathbb{Z}^d), \\ \mathcal{P} &:= \mathcal{P}(\mathbb{Z}^d, \mathbb{Z}^d) = \bigcup_{\ell \in \mathbb{N}_0} \mathcal{P}_{\ell}. \end{aligned} \quad (6.2)$$

When Λ or Λ' consists of a single point, we write x instead of $\{x\}$. If $\pi \in \mathcal{P}_{\ell}$, we set $|\pi| := \ell$. We write $\text{supp}(\pi) := \{\pi_0, \dots, \pi_{|\pi|}\}$ to denote the set of points visited by π .

Let $X = (X_t)_{t \geq 0}$ be a continuous-time simple symmetric random walk with total jump rate $2d$; this is the process that “drives” the Feynman-Kac formula. We denote by $(T_n)_{n \in \mathbb{N}_0}$ the sequence of its jump times (with $T_0 := 0$). For $\ell \in \mathbb{N}_0$, let $\pi^{(\ell)}(X) := (X_0, \dots, X_{T_{\ell}})$ be the path in \mathcal{P}_{ℓ} consisting of the first ℓ steps

of X and, for $t \geq 0$, $\pi(X_{0,t})$ the path in \mathcal{P} consisting of all the steps taken by X between the times 0 and t . Recall the definition (4.19) of the hitting times τ_Λ .

For $\pi \in \mathcal{P}$, $L \in \mathbb{N}$ and $A > 0$, we define

$$\lambda_{L,A}(\pi) := \sup \left\{ \lambda_{\mathcal{C}}^{(1)} : \mathcal{C} \in \mathfrak{C}_{L,A}, \text{supp}(\pi) \cap \mathcal{C} \cap \Pi_{L,A} \neq \emptyset \right\}, \quad (6.3)$$

with the convention $\sup \emptyset = -\infty$. This is the largest principal eigenvalue among the components of $\mathfrak{C}_{L,A}$ encountered by the path.

The main results of this section are the following two propositions.

Proposition 6.1 *Let R_L satisfy (5.11–5.12). For any $A > 0$, there exists a constant $c_A > 0$ such that the following holds a.s. eventually as $L \rightarrow \infty$: For each $x \in B_L$, each $t > 0$, each $\mathcal{N} \subset \mathcal{P}(x, \mathbb{Z}^d)$ satisfying $\text{supp}(\pi) \subset B_L$ and $\max_{1 \leq \ell \leq |\pi|} |\pi_\ell - x| \geq \ln L$ for all $\pi \in \mathcal{N}$, and each assignment $\pi \mapsto (\gamma_\pi, z_\pi) \in \mathbb{R} \times \mathbb{Z}^d$ such that*

$$\gamma_\pi \geq \lambda_{L,A}(\pi) \vee (\widehat{a}_L - A) + e^{-R_L} \quad (6.4)$$

and

$$z_\pi \in \text{supp}(\pi) \cup \bigcup_{\substack{\mathcal{C} \in \mathfrak{C}_{L,A}: \\ \text{supp}(\pi) \cap \mathcal{C} \cap \Pi_{L,A} \neq \emptyset}} \mathcal{C} \quad (6.5)$$

are true for all $\pi \in \mathcal{N}$, we have

$$\ln \mathbb{E}_x \left[e^{\int_0^t \zeta(X_s) ds} \mathbb{1}\{\pi(X_{0,t}) \in \mathcal{N}\} \right] \leq \sup_{\pi \in \mathcal{N}} \left\{ t \gamma_\pi - (\ln_3(4dL) - c_A) |z_\pi - x| \right\}. \quad (6.6)$$

While we assume (5.11–5.12) in most of the paper, the proof of Proposition 4.11 will require us to work without (5.12). In this setting, we have the following:

Proposition 6.2 *For $A > 0$, let $n_A \in \mathbb{N}$ as in Lemma 5.2. For any $R_L \in \mathbb{N}$ that obeys (5.11) and any $\vartheta_L \in \mathbb{N}$ satisfying $\vartheta_L \ll \ln_3 L$ as $L \rightarrow \infty$, the following holds a.s. eventually as $L \rightarrow \infty$: For each $x \in B_L$, each $t > 0$, each $\mathcal{N} \subset \mathcal{P}(x, \mathbb{Z}^d)$ satisfying $\text{supp}(\pi) \subset B_L$ and $\max_{1 \leq \ell \leq |\pi|} |\pi_\ell - x| \geq (n_A + 1)R_L$ for all $\pi \in \mathcal{N}$, and each $\pi \mapsto \gamma_\pi \in \mathbb{R}$ that obeys*

$$\gamma_\pi \geq \lambda_{L,A}(\pi) \vee (\widehat{a}_L - A) + e^{-\vartheta_L R_L}, \quad \pi \in \mathcal{N}, \quad (6.7)$$

$$\ln \mathbb{E}_x \left[e^{\int_0^t \zeta(X_s) ds} \mathbb{1}\{\pi(X_{0,t}) \in \mathcal{N}\} \right] \leq t \sup_{\pi \in \mathcal{N}} \gamma_\pi - \frac{1}{2} R_L \ln_3 L. \quad (6.8)$$

To prove Propositions 6.1–6.2, we will need a key lemma (Lemma 6.5 below), whose proof in turn depends on intermediate results obtained in the following two subsections. We emphasize that all of these results are deterministic, i.e., they hold for any fixed potential $\zeta \in \mathbb{R}^{\mathbb{Z}^d}$.

6.2 Mass of the solution along excursions.

In order to control the contribution to the mass given by a path, it will be important for us to control the contribution of its excursions outside of $\Pi_{L,A}$ (recall (5.13)). A useful result is the following:

Lemma 6.3 (Path evaluation) *For any $\ell \in \mathbb{N}_0$, any path $\pi \in \mathcal{P}_\ell$ and any γ such that $\gamma > \max_{i < |\pi|} \zeta(\pi_i) - 2d$,*

$$\mathbb{E}_{\pi_0} \left[\exp \left\{ \int_0^{T_\ell} (\zeta(X_s) - \gamma) ds \right\} \middle| \pi^{(\ell)}(X) = \pi \right] = \prod_{i=0}^{\ell-1} \frac{2d}{2d + \gamma - \zeta(\pi_i)}. \quad (6.9)$$

Proof. The left-hand side of (6.9) can be directly evaluated using the fact that T_ℓ is the sum of ℓ i.i.d. $\text{Exp}(2d)$ random variables that are independent of $\pi^{(\ell)}(X)$. The condition on γ ensures that all integrals are finite. \square

For a path $\pi \in \mathcal{P}$, any $L \in \mathbb{N}$ and any $\varepsilon \in (0, 1)$, we denote

$$M_\pi^{L,\varepsilon} := \left| \{x \in \text{supp}(\pi) \setminus \{\pi_{|\pi|}\} : \zeta(x) \leq (1 - \varepsilon)\widehat{a}_L\} \right|. \quad (6.10)$$

Then we have:

Lemma 6.4 (Mass of excursions) *For any $A, \varepsilon > 0$, there exist $c > 0$ and $L_0 \in \mathbb{N}$ such that, for all $L \geq L_0$, all $\gamma > \widehat{a}_L - A$ and all $\pi \in \mathcal{P}$ satisfying $\pi_i \notin \Pi_{L,A}$ for all $i < \ell := |\pi|$,*

$$\mathbb{E}_{\pi_0} \left[\exp \left\{ \int_0^{T_\ell} (\zeta(X_t) - \gamma) ds \right\} \middle| \pi^{(\ell)}(X) = \pi \right] \leq q_A^\ell e^{(c - \ln_3 L) M_\pi^{L,\varepsilon}}, \quad (6.11)$$

where $q_A := (1 + A/2d)^{-1}$.

Note that the statement of Lemma 6.4 allows for $\pi_\ell \in \Pi_{L,A}$.

Proof. By our assumptions on π and γ , we can use Lemma 6.3. Splitting the product on the right-hand side of (6.9) according to whether $\zeta(\pi_i)$ is larger than $(1 - \varepsilon)\widehat{a}_L$ or not, and using that $\zeta(\pi_i) \leq \widehat{a}_L - 2A$ for all $i < |\pi|$, we obtain that (6.11) is at most

$$q_A^\ell \left[q_A \frac{\varepsilon \widehat{a}_L - A}{2d} \right]^{-|\{i < \ell : \zeta(\pi_i) \leq (1 - \varepsilon)\widehat{a}_L\}|}. \quad (6.12)$$

For large L , $\widehat{a}_L \geq \frac{1}{2}\rho \ln_2 L$ and the number within square brackets in (6.12) exceeds $q_A \varepsilon \rho (\ln_2 L) / 5d > 1$. Since $|\{i < |\pi| : \zeta(\pi_i) \leq (1 - \varepsilon)\widehat{a}_L\}| \geq M_\pi^{L,\varepsilon}$, (6.11) holds with $c := \ln(1 \vee 5d(q_A \varepsilon \rho)^{-1})$. \square

6.3 Equivalence classes of paths.

Here we develop a setup similar to Section 6.2 of [MP14]. The idea is to categorize paths $\pi \in \mathcal{P}$ according to their excursions between $\Pi_{L,A}$ and $D_{L,A}^c$ (cf. (5.13–5.14)) and then apply the results from Sections 5.4 and 6.2. Note that $\text{dist}(\Pi_{L,A}, D_{L,A}^c) \geq R_L$.

First we discuss the concatenation of paths. If π and π' are two paths in \mathcal{P} such that $\pi_{|\pi|} = \pi'_0$, we define their concatenation as

$$\pi \circ \pi' := (\pi_0, \dots, \pi_{|\pi|}, \pi'_1, \dots, \pi'_{|\pi'|}) \in \mathcal{P}. \quad (6.13)$$

Note that $|\pi \circ \pi'| = |\pi| + |\pi'|$.

If a path $\pi \in \mathcal{P}$ is contained in B_L and intersects $\Pi_{L,A}$, then it can be decomposed into an initial path, a sequence of excursions between $\Pi_{L,A}$ and $D_{L,A}^c$, and a terminal path. Explicitly, there exists $m_\pi \in \mathbb{N}$ such that

$$\pi = \check{\pi}^{(1)} \circ \hat{\pi}^{(1)} \circ \dots \circ \check{\pi}^{(m_\pi)} \circ \hat{\pi}^{(m_\pi)} \circ \bar{\pi}, \quad (6.14)$$

where the paths in (6.14) are contained in B_L and satisfy

$$\begin{aligned} \check{\pi}^{(1)} &\in \mathcal{P}(\mathbb{Z}^d, \Pi_{L,A}) & \text{and} & & \check{\pi}_i^{(1)} &\notin \Pi_{L,A}, & 0 \leq i < |\check{\pi}^{(1)}|, \\ \check{\pi}^{(k)} &\in \mathcal{P}(D_{L,A}^c, \Pi_{L,A}) & \text{and} & & \check{\pi}_i^{(k)} &\notin \Pi_{L,A}, & 0 \leq i < |\check{\pi}^{(k)}|, & 2 \leq k \leq m_\pi, \\ \hat{\pi}^{(k)} &\in \mathcal{P}(\Pi_{L,A}, D_{L,A}^c) & \text{and} & & \hat{\pi}_i^{(k)} &\in D_{L,A}, & 0 \leq i < |\hat{\pi}^{(k)}|, & 1 \leq k \leq m_\pi - 1, \\ \hat{\pi}^{(m_\pi)} &\in \mathcal{P}(\Pi_{L,A}, \mathbb{Z}^d) & \text{and} & & \hat{\pi}_i^{(m_\pi)} &\in D_{L,A}, & 0 \leq i < |\hat{\pi}^{(m_\pi)}|, \end{aligned} \quad (6.15)$$

while

$$\begin{aligned} \bar{\pi} &\in \mathcal{P}(D_{L,A}^c, \mathbb{Z}^d), \bar{\pi}_i \notin \Pi_{L,A} \quad \forall i \geq 0 & \text{if } \hat{\pi}^{(m_\pi)} \in \mathcal{P}(\Pi_{L,A}, D_{L,A}^c), \\ \bar{\pi}_0 &\in D_{L,A}, |\bar{\pi}| = 0 & \text{otherwise.} \end{aligned} \quad (6.16)$$

Note that the decomposition (6.14–6.16) is unique, and that the paths $\check{\pi}^{(1)}$, $\hat{\pi}^{(m_\pi)}$ and $\bar{\pi}$ can have zero length.

For $L \in \mathbb{N}$ and $\varepsilon > 0$, whenever $\text{supp}(\pi) \cap \Pi_{L,A} \neq \emptyset$, we define

$$n_\pi := \sum_{i=1}^{m_\pi} |\check{\pi}^{(i)}| + |\bar{\pi}| \quad \text{and} \quad k_\pi^{L,\varepsilon} := \sum_{i=1}^{m_\pi} M_{\check{\pi}^{(i)}}^{L,\varepsilon} + M_{\bar{\pi}}^{L,\varepsilon} \quad (6.17)$$

to be respectively the total time spent in exterior excursions and the sum of the numbers of moderately low points of the potential visited by exterior excursions (excluding their last point). In the case when $\text{supp}(\pi) \cap \Pi_{L,A} = \emptyset$, we set $m_\pi := 0$, $n_\pi := |\pi|$ and $k_\pi^{L,\varepsilon} := M_\pi^{L,\varepsilon}$. Recall from (6.3) that, in this case, $\lambda_{L,A}(\pi) = -\infty$.

We say that $\pi, \pi' \in \mathcal{P}$ are *equivalent*, written $\pi' \sim \pi$, if $m_\pi = m_{\pi'}$, $\check{\pi}'^{(i)} = \check{\pi}^{(i)}$ for all $i = 1, \dots, m_\pi$ and $\bar{\pi}' = \bar{\pi}$ if $\bar{\pi}_0 \in D_{L,A}^c$. If $\pi' \sim \pi$, then $n_{\pi'}$, $k_{\pi'}^{L,\varepsilon}$ and $\lambda_{L,A}(\pi')$ are all equal to the counterparts for π .

To state our next lemma, we define, for $m, n \in \mathbb{N}_0$,

$$\mathcal{P}^{(m,n)} = \{\pi \in \mathcal{P} : m_\pi = m, n_\pi = n\}, \quad (6.18)$$

and we denote by

$$C_{L,A} := \max\{|\mathcal{C}| : \mathcal{C} \in \mathfrak{C}_{L,A}\} \quad (6.19)$$

the maximal size of the islands in $\mathfrak{C}_{L,A}$.

Lemma 6.5 *For any $A, \varepsilon > 0$, there exist $c > 0$ and $L_0 \in \mathbb{N}$ such that, for all $L \geq L_0$, all $m, n \in \mathbb{N}_0$, all $\pi \in \mathcal{P}^{(m,n)}$ with $\text{supp}(\pi) \subset B_L$, all $\gamma > \lambda_{L,A}(\pi) \vee (\hat{a}_L - A)$ and all $t \geq 0$,*

$$\begin{aligned} \mathbb{E}_{\pi_0} \left[e^{\int_0^t (\xi(X_s) - \gamma) ds} \mathbb{1}\{\pi(X_{0,t}) \sim \pi\} \right] \\ \leq \left(C_{L,A}^{3/2} \right)^{\mathbb{1}\{m>0\}} \left(1 + \frac{2d C_{L,A}}{\gamma - \lambda_{L,A}(\pi)} \right)^m \left(\frac{qA}{2d} \right)^n e^{(c - \ln_3 L) k_\pi^{L,\varepsilon}}. \end{aligned} \quad (6.20)$$

Proof. Fix $A, \varepsilon > 0$ and let $c > 0$, $L_0 \in \mathbb{N}$ be as given by Lemma 6.4. For $0 \leq s \leq t < \infty$, set $I_s^t := e^{\int_s^t (\xi(X_u) - \gamma) du}$. Our strategy is to prove the claim by induction on m .

Suppose first that $m = 1$, let $\ell := |\check{\pi}^{(1)}|$ and set $z := \check{\pi}_\ell^{(1)}$. There are two possibilities: either $\bar{\pi}_0$ belongs to $D_{L,A}$ or not. Focussing first on the case $\bar{\pi}_0 \in D_{L,A}$, which in particular implies $|\bar{\pi}| = 0$, the strong Markov property yields %

$$\begin{aligned} \mathbb{E}_{\pi_0} \left[I_0^t \mathbb{1}_{\{\pi(X_{0,t}) \sim \pi\}} \right] &= \mathbb{E}_{\pi_0} \left[I_0^{T_\ell} I_{T_\ell}^t \mathbb{1}_{\{\pi^{(\ell)}(X) = \check{\pi}^{(1)}\}} \mathbb{1}_{\{T_\ell < t\}} \mathbb{1}_{\{X_{s+T_\ell} \in D_{L,A} \forall s \in [0, t - T_\ell]\}} \right] \\ &= \mathbb{E}_{\pi_0} \left[I_0^{T_\ell} \mathbb{1}_{\{\pi^{(\ell)}(X) = \check{\pi}^{(1)}\}} \mathbb{1}_{\{T_\ell < t\}} \left(\mathbb{E}_z \left[I_0^{t-s} \mathbb{1}_{\{\tau_{D_{L,A}^c} > t-s\}} \right] \right)_{s=T_\ell} \right]. \end{aligned} \quad (6.21)$$

Since $z \in \Pi_{L,A}$, we may write \mathcal{C}_z to denote the island in $\mathfrak{C}_{L,A}$ containing z . As $\tau_{D_{L,A}^c} = \tau_{\mathcal{C}_z^c}$ \mathbb{P}_z -a.s., Lemma 5.12 and our hypothesis on γ bound the inner expectation in (6.21) by $|\mathcal{C}_z|^{3/2}$. Applying Lemmas 5.2 and 6.4, we further bound (6.21) by

$$|\mathcal{C}_z|^{3/2} \mathbb{E}_{\pi_0} \left[I_0^{T_\ell} \mathbb{1}_{\{\pi^{(\ell)}(X) = \check{\pi}^{(1)}\}} \right] \leq C_{L,A}^{3/2} \left(\frac{q_A}{2d} \right)^\ell e^{(c - \ln_3 L) M_{\check{\pi}^{(1)}}^{L,\varepsilon}}, \quad (6.22)$$

thus proving (6.20) in the case $m = 1$, $\bar{\pi}_0 \in D_{L,A}$.

Next let us assume $x := \bar{\pi}_0 \in D_{L,A}^c$. Abbreviating $\sigma := \inf\{s > T_\ell : X_s \notin D_{L,A}\}$, we may write

$$\mathbb{E}_{\pi_0} \left[I_0^t \mathbb{1}_{\{\pi(X_{0,t}) \sim \pi\}} \right] \leq \mathbb{E}_{\pi_0} \left[I_0^\sigma \mathbb{1}_{\{\pi^{(\ell)}(X) = \check{\pi}^{(1)}, \sigma < t\}} \left(\mathbb{E}_x \left[I_0^{t-s} \mathbb{1}_{\{\pi(X_{0,t-s}) = \bar{\pi}\}} \right] \right)_{s=\sigma} \right]. \quad (6.23)$$

Let $\ell_* := |\bar{\pi}|$ and note that, since $\bar{\pi}_{\ell_*} \notin \Pi_{L,A}$, by the hypothesis on γ we have

$$\mathbb{E}_x \left[I_0^{t-s} \mathbb{1}_{\{\pi(X_{0,t-s}) = \bar{\pi}\}} \right] \leq \mathbb{E}_x \left[I_0^{T_{\ell_*}} \mathbb{1}_{\{\pi^{(\ell_*)}(X) = \bar{\pi}\}} \right] \leq \left(\frac{q_A}{2d} \right)^{\ell_*} e^{(c - \ln_3 L) M_{\bar{\pi}}^{L,\varepsilon}} \quad (6.24)$$

by Lemma 6.4. On the other hand, by Lemmas 5.13 and 6.4,

$$\begin{aligned} \mathbb{E}_{\pi_0} \left[I_0^\sigma \mathbb{1}_{\{\pi^{(\ell)}(X) = \check{\pi}^{(1)}\}} \right] &= \mathbb{E}_{\pi_0} \left[I_0^{T_\ell} \mathbb{1}_{\{\pi^{(\ell)}(X) = \check{\pi}^{(1)}\}} \right] \mathbb{E}_z \left[I_0^{\tau_{\mathcal{C}_z^c}} \right] \\ &\leq \left(1 + \frac{2d C_{L,A}}{\gamma - \lambda_{L,A}(\pi)} \right) \left(\frac{q_A}{2d} \right)^\ell e^{(c - \ln_3 L) M_{\check{\pi}^{(1)}}^{L,\varepsilon}}. \end{aligned} \quad (6.25)$$

Putting together (6.23)–(6.25), we finish the proof of the case $m = 1$.

By induction, assume now that the statement is proven for some fixed $m \geq 1$, and let $\pi \in \mathcal{D}^{(m+1,n)}$. Define $\pi' := \check{\pi}^{(2)} \circ \hat{\pi}^{(2)} \circ \dots \circ \check{\pi}^{(m+1)} \circ \hat{\pi}^{(m+1)} \circ \bar{\pi}$. Then $\pi' \in \mathcal{D}^{(m,n')}$ where $n = |\check{\pi}^{(1)}| + n'$, and $k_{\bar{\pi}}^{L,\varepsilon} = k_{\pi'}^{L,\varepsilon} + M_{\check{\pi}^{(1)}}^{L,\varepsilon}$. Setting $\ell := |\check{\pi}^{(1)}|$, $x := \check{\pi}_0^{(2)}$ and $\sigma := \inf\{s > T_\ell : X_s \notin D_{L,A}\}$, we get

$$\mathbb{E}_{\pi_0} \left[I_0^t \mathbb{1}_{\{\pi(X_{0,t}) \sim \pi\}} \right] \leq \mathbb{E}_{\pi_0} \left[I_0^\sigma \mathbb{1}_{\{\pi^{(\ell)}(X) = \check{\pi}^{(1)}, \sigma < t\}} \left(\mathbb{E}_x \left[I_0^{t-s} \mathbb{1}_{\{\pi(X_{0,t-s}) \sim \pi'\}} \right] \right)_{s=\sigma} \right], \quad (6.26)$$

from which (6.20) follows using the induction hypothesis and (6.25). The case $m = 0$ follows from equation (6.24) after substituting $\bar{\pi}$ by π and $t - s$ by t . \square

6.4 Proof of Propositions 6.1–6.2.

We are now ready to present the proofs of the above key propositions.

Proof of Proposition 6.2. The proof is based on Lemma 6.5 and results from Sections 5.2–5.3. Fix $A > 0$ and, for β as in (5.11), take $\varepsilon \in (0, 1/2)$ as in Lemma 5.10. Let $L_0 \in \mathbb{N}$ be as given by Lemma 6.5 and take $L \geq L_0$ so large that the conclusions of Lemmas 5.10 and 5.2 hold. Fix $x \in B_L$. Recall the definition of $\mathcal{P}^{(m,n)}$. Noting that the relation \sim is an equivalence relation in $\mathcal{P}^{(m,n)}$, define

$$\widetilde{\mathcal{P}}_x^{(m,n)} := \{\text{equivalence classes of the paths in } \mathcal{P}(x, \mathbb{Z}^d) \cap \mathcal{P}^{(m,n)}\}. \quad (6.27)$$

We first claim that, for a constant $c_1 \in \mathbb{N}$, a.s. eventually as $L \rightarrow \infty$,

$$|\widetilde{\mathcal{P}}_x^{(m,n)}| \leq (c_1 R_L^d)^m (2d)^n \quad \forall m, n \in \mathbb{N}_0. \quad (6.28)$$

Indeed, (6.28) is clear if $m = 0$. To prove it in the case $m \geq 1$, write, for $\Lambda \subset \mathbb{Z}^d$, $\partial\Lambda := \{z \notin \Lambda : \text{dist}(z, \Lambda) = 1\}$. By Lemma 5.2, there is a $c_0 \in \mathbb{N}$ such that

$$|\partial\mathcal{C}| \leq 2d|\mathcal{C}| \leq c_0 R_L^d \quad \forall \mathcal{C} \in \mathfrak{C}_{L,A} \quad \text{a.s. eventually as } L \rightarrow \infty. \quad (6.29)$$

We then define a map $\Phi: \widetilde{\mathcal{P}}_x^{(m,n)} \rightarrow \mathcal{P}_n(x, \mathbb{Z}^d) \times \{1, \dots, c_0 R_L^d + 1\}^m$ as follows: For each $\Lambda \subset \mathbb{Z}^d$ with $1 \leq |\Lambda| \leq c_0 R_L^d$, fix an injection $f_\Lambda: \Lambda \rightarrow \{1, \dots, c_0 R_L^d\}$. Given a path $\pi \in \mathcal{P}^{(m,n)} \cap \mathcal{P}(x, \mathbb{Z}^d)$, decompose π as in (6.13) and, abusing notation slightly, write $\check{\pi}^{(m+1)}$ for $\bar{\pi}$. Now let $\tilde{\pi}$ be the path obtained from $\check{\pi}^{(k)}$, $1 \leq k \leq m+1$, by progressively shifting, for $2 \leq k \leq m+1$, the starting point of each $\check{\pi}^{(k)}$ to the terminal point of $\check{\pi}^{(k-1)}$ and concatenating these shifted paths together. Note that, for each $2 \leq k \leq m$, the starting point $\check{\pi}_0^{(k)}$ lies in $\partial\mathcal{C}_k$ for some $\mathcal{C}_k \in \mathfrak{C}_{L,A}$, while $\check{\pi}_0^{(m+1)} = \bar{\pi}_0 \in \partial\bar{\mathcal{C}} \cup \bar{\mathcal{C}}$ for some $\bar{\mathcal{C}} \in \mathfrak{C}_{L,A}$. Thus we may set

$$\Phi(\pi) := \begin{cases} (\tilde{\pi}, f_{\partial\mathcal{C}_2}(\check{\pi}_0^{(2)}), \dots, f_{\partial\mathcal{C}_m}(\check{\pi}_0^{(m)}), c_0 R_L^d + 1) & \text{if } \bar{\pi}_0 \in \bar{\mathcal{C}} \subset D_{L,A}, \\ (\tilde{\pi}, f_{\partial\mathcal{C}_2}(\check{\pi}_0^{(2)}), \dots, f_{\partial\mathcal{C}_m}(\check{\pi}_0^{(m)}), f_{\partial\bar{\mathcal{C}}}(\bar{\pi}_0)) & \text{if } \bar{\pi}_0 \in \partial\bar{\mathcal{C}} \subset D_{L,A}^c. \end{cases} \quad (6.30)$$

As is readily checked, $\Phi(\pi)$ depends only on the equivalence class of π and, when restricted to equivalence classes, Φ is injective. Thus (6.28) follows with e.g. $c_1 := 2c_0$.

Take now $\mathcal{N} \subset \mathcal{P}(x, \mathbb{Z}^d)$ as in the statement, and set

$$\widetilde{\mathcal{N}}^{(m,n)} := \{\text{equivalence classes of paths in } \mathcal{N} \cap \mathcal{P}^{(m,n)}\} \subset \widetilde{\mathcal{P}}_x^{(m,n)}. \quad (6.31)$$

(6.28) permits us to write

$$\begin{aligned} \mathbb{E}_x \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{\pi(X_{0,t}) \in \mathcal{N}\}} \right] &= \sum_{m,n \in \mathbb{N}_0} \sum_{\pi \in \widetilde{\mathcal{N}}^{(m,n)}} \mathbb{E}_x \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{\pi(X_{0,t}) \sim \pi\}} \right] \\ &\leq \sum_{m,n \in \mathbb{N}_0} (c_1 R_L^d)^m (2d)^n \sup_{\pi \in \mathcal{N}^{(m,n)}} \mathbb{E}_x \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{\pi(X_{0,t}) \sim \pi\}} \right], \end{aligned} \quad (6.32)$$

where we use the convention $\sup \emptyset = 0$. For fixed $\pi \in \mathcal{N}^{(m,n)}$, by the hypothesis on γ_π we may apply (6.20), (6.4–6.5), Lemma 5.2 and (5.11) to obtain, for all L large enough,

$$(c_1 R_L^d)^m (2d)^n \mathbb{E}_x \left[e^{\int_0^t \xi(X_s) ds} \mathbb{1}_{\{\pi(X_{0,t}) \sim \pi\}} \right] \leq e^{t\gamma_\pi} \left(R_L^{4d} e^{\theta_L R_L} \right)^m q_A^n e^{(c - \ln_3 L) k_\pi^{L,\varepsilon}} \quad (6.33)$$

for some constant $c > 0$. We now claim that, for large enough L ,

$$k_\pi^{L,\varepsilon} \geq \{(m-1) \vee 1\} R_L \{1 - 2(\ln L)^{-\varepsilon}\}. \quad (6.34)$$

Indeed, when $m = 0$, $|\text{supp}(\pi)| \geq (n_A + 1)R_L$. When $m \geq 2$, $|\text{supp}(\check{\pi}^{(i)})| \geq R_L$ for all $2 \leq i \leq m$. When $m = 1$, there are two cases: if $\text{supp}(\check{\pi}^{(1)}) \cap D_{L,A}^c \neq \emptyset$, then $|\text{supp}(\check{\pi}^{(1)})| \geq R_L$ while, if

$\text{supp}(\tilde{\pi}^{(1)}) \subset D_{L,A}$, then $|\text{supp}(\tilde{\pi})| \geq R_L$ by the assumption $\max_{1 \leq \ell \leq |\pi|} |\pi_\ell - x| > (n_A + 1)R_L$ together with (5.11) and Lemma 5.2. Thus (6.34) holds by (6.17), (6.10), (5.11) and Lemma 5.10.

Using (6.34), $\vartheta_L \ll \ln_3 L$ and $n \geq k_\pi^{L,\varepsilon}$, for large L we may further bound (6.33) by

$$\begin{aligned} & \left[R_L^{8d} e^{2\vartheta_L R_L} e^{-(2\vartheta_L + \frac{1}{2})R_L} \right]^m q_A^n e^{t\gamma_\pi} e^{(c+1+2\vartheta_L - \ln_3 L)k_\pi^{L,\varepsilon}} \\ & \leq q_A^{R_L/3} \left[R_L^{8d} e^{-\frac{R_L}{2}} \right]^m q_A^{n/2} e^{t\gamma_\pi} e^{(c+1+2\vartheta_L - \ln_3 L)k_\pi^{L,\varepsilon}}. \end{aligned} \quad (6.35)$$

Inserting this back into (6.32), we obtain

$$\mathbb{E}_x \left[e^{\int_0^t \tilde{\xi}(X_s) ds} \mathbb{1}_{\{\pi(X_{0,t}) \in \mathcal{N}\}} \right] \leq \sup_{\pi \in \mathcal{N}} \exp \left\{ t\gamma_\pi + (c+1+2\vartheta_L - \ln_3 L)k_\pi^{L,\varepsilon} \right\}. \quad (6.36)$$

Now (6.8) follows from (6.36), (6.34) and $\vartheta_L \ll \ln_3 L$. \square

Proof of Proposition 6.1. Note that, for large L , the assumptions of Proposition 6.1 imply those of Proposition 6.2 with $\vartheta_L \equiv 1$, and thus we may use (6.36). We proceed to bound $k_\pi^{L,\varepsilon}$ using assumption (5.12). For $\alpha \in (0, 1/d)$ as in (5.12), let $C \geq 1$ be as in Lemma 5.11 with $\delta \in (\alpha d, 1)$ and set $\varepsilon' := \delta - \alpha d > 0$. Assume that L is so large that the conclusion of Lemma 5.11 is in place.

Note that, by Lemma 5.2, there exists a constant $c_2 > 0$ such that

$$k_\pi^{L,\varepsilon} \geq M_\pi^{L,\varepsilon} - |\text{supp}(\pi) \cap \Pi_{L,A}| c_2 R_L^d. \quad (6.37)$$

By (5.11) and our hypothesis on \mathcal{N} , $|\text{supp}(\pi)| \geq \ln L \geq C(\ln L)^\delta$ for large L . Hence, by Lemma 5.11,

$$|\text{supp}(\pi) \cap \Pi_{L,A}| \leq \frac{|\text{supp}(\pi)|}{(\ln L)^\delta} \leq \frac{|\text{supp}(\pi)|}{R_L^d (\ln L)^{\varepsilon'}} \quad (6.38)$$

by (5.12) and our choice of δ, ε' . By Lemma 5.10, $M_\pi^{L,\varepsilon} + 1 \geq |\text{supp}(\pi)| \{1 - (\ln L)^{-\varepsilon}\}$. Thus

$$k_\pi^{L,\varepsilon} \geq |\text{supp}(\pi)| \left\{ 1 - (\ln L)^{-1} - (\ln L)^{-\varepsilon} - c_2 (\ln L)^{-\varepsilon'} \right\}. \quad (6.39)$$

Now, by Lemma 5.2 and (6.4–6.5), $|\text{supp}(\pi)| \geq |z_\pi| - n_A R_L$; this in conjunction with $|\text{supp}(\pi)| \geq \ln L$ implies

$$|\text{supp}(\pi)| \geq |z_\pi - x| \left(1 - \frac{n_A R_L}{\ln L} \right). \quad (6.40)$$

From (6.39–6.40) and (5.11) we obtain $(c+3 - \ln_3 L)k_\pi^{L,\varepsilon} \leq (c+4 - \ln_3(4dL))|z_\pi - x|$ for large enough L , which together with (6.36) implies (6.6). \square

7. ANALYSIS OF THE COST FUNCTIONAL

In this section, we identify the order statistics of Ψ_t and give the proofs of Theorem 2.7 and Propositions 4.2–4.3. Motivated by Proposition 6.1 and Lemma 5.8, we define the following generalization of the cost functional: For $t > 0$ and $c \in \mathbb{R}$, let

$$\Psi_{t,c}(z) := \lambda^{\mathcal{C}}(z) - (\ln_3^+ |z| - c)^+ \frac{|z|}{t}, \quad z \in \mathcal{C}, \quad (7.1)$$

where $\lambda^{\mathcal{C}}(z)$ is as in (4.4). Arguing as for (4.6), we can see that, almost surely,

$$|\{z \in \mathcal{C} : \Psi_{t,c}(z) > \eta\}| < \infty \quad \text{for all } t > 0, \eta \in \mathbb{R}, \quad (7.2)$$

and thus we may define $\Psi_{t,c}^{(k)}$ and $Z_{t,c}^{(k)}$ analogously to the corresponding objects for Ψ_t .

Fix $N_t \in \mathbb{N}$ such that $t^\beta \ll N_t \ll t^\alpha$ for some $0 < \beta < \alpha < 1$. Noting that r_t is strictly increasing for large enough t , we may take $t \mapsto L_t^* \in \mathbb{N}$ such that $L_{r_t}^* = L_t$. Set $\widehat{N}_t := N_{L_t^*}$ and define a_t to be the smallest positive number such that

$$\text{Prob} \left(\lambda_{B_{\widehat{N}_t}}^{(1)} > a_t \right) = \frac{(2\widehat{N}_t)^d}{t^d}. \quad (7.3)$$

Such an a_t exists (for t large enough) since $\lambda_{B_{\widehat{N}_t}}^{(1)}$ is continuously distributed. Note that

$$L_t^* \sim \frac{d}{\rho} t (\ln t) (\ln_2 t) \ln_3 t \quad \text{as } t \rightarrow \infty, \quad (7.4)$$

and thus also $t^\beta \ll \widehat{N}_t \ll t^{\alpha'}$ for some $0 < \beta < \alpha' < 1$. An important result of [BK16] (Theorem 2.4 therein) is that, for any $\theta \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \frac{t^d}{(2\widehat{N}_t)^d} \text{Prob} \left(\lambda_{B_{\widehat{N}_t}}^{(1)} > a_t + \theta d_t \right) = e^{-\theta}, \quad (7.5)$$

where d_t is as in (2.6). A strengthened version of this statement (see (7.19) below) will allow us to identify the order statistics of $\Psi_{t,c}$. Together with Theorem 2.3 and Lemma 6.8 in [BK16], (7.5) implies that $a_t = \widehat{a}_t - \chi + o(1)$. In particular, $a_t = (\rho + o(1)) \ln_2 t$.

For $0 < a \leq b < \infty$, $c \in \mathbb{R}$ and $k \in \mathbb{N}$, we define the events

$$\begin{aligned} \mathcal{E}_{t,a,b,c}^{(k)} := & \left\{ \min_{i=1,\dots,k} \left(\Psi_{at,c}^{(i)} - \Psi_{at,c}^{(i+1)} \right) \wedge \left(\Psi_{bt,c}^{(i)} - \Psi_{bt,c}^{(i+1)} \right) > d_t e_t \right\} \\ & \cap \bigcap_{s \in [at, bt]} \left\{ a_{r_t} + d_t g_t > \Psi_{s,c}^{(1)} \geq \Psi_{s,c}^{(k)} > a_{r_t} - d_t g_t \right\} \\ & \cap \bigcap_{s \in [at, bt]} \left\{ r_t f_t < \min_{1 \leq i \leq k} |Z_{s,c}^{(i)}| \leq \max_{1 \leq i \leq k} |Z_{s,c}^{(i)}| < r_t g_t \right\}. \end{aligned} \quad (7.6)$$

When $c = 0$ and/or $k = 1$, we omit them in the notation.

For $a \in (0, \infty)$, let $\mathcal{C}([a, \infty), \mathbb{R}^n)$, resp. $\mathcal{D}([a, \infty), \mathbb{R}^n)$, denote the set of continuous, resp. càdlàg, functions from $[a, \infty)$ to \mathbb{R}^n , both equipped with the Skorohod topology. The following result is the main objective of this section.

Proposition 7.1 *For all $c \in \mathbb{R}$, $k \in \mathbb{N}$ and $a > 0$, the stochastic process*

$$\left(\left(\frac{\Psi_{\theta t,c}^{(1)} - a_{r_t}}{d_{r_t}}, \frac{\lambda^{\mathcal{C}}(Z_{\theta t,c}^{(1)}) - a_{r_t}}{d_{r_t}}, \frac{Z_{\theta t,c}^{(1)}}{r_t} \right), \dots, \left(\frac{\Psi_{\theta t,c}^{(k)} - a_{r_t}}{d_{r_t}}, \frac{\lambda^{\mathcal{C}}(Z_{\theta t,c}^{(k)}) - a_{r_t}}{d_{r_t}}, \frac{Z_{\theta t,c}^{(k)}}{r_t} \right) \right)_{\theta \in [a, \infty)} \quad (7.7)$$

belongs to $(\mathcal{C}([a, \infty), \mathbb{R}) \times \mathcal{D}([a, \infty), \mathbb{R}) \times \mathcal{D}([a, \infty), \mathbb{R}^d))^k$ and converges in distribution as $t \rightarrow \infty$ with respect to the Skorohod topology of $\mathcal{D}([a, \infty), (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d)^k)$ to the process

$$\left(\left(\bar{\Psi}_\theta^{(1)}, \bar{\Lambda}_\theta^{(1)}, \bar{Z}_\theta^{(1)} \right), \dots, \left(\bar{\Psi}_\theta^{(k)}, \bar{\Lambda}_\theta^{(k)}, \bar{Z}_\theta^{(k)} \right) \right)_{\theta \in [a, \infty)} \quad (7.8)$$

where $\bar{\Psi}_\theta^{(i)} := \bar{\Lambda}_\theta^{(i)} - \frac{1}{\theta} \bar{Z}_\theta^{(i)}$ and $(\bar{\Lambda}_\theta^{(i)}, \bar{Z}_\theta^{(i)})_{i=1}^k$ are the k first ordered maximizers of the functional $\psi_\theta(\lambda, z) = \lambda - \frac{|z|}{\theta}$ over the points (λ, z) of a Poisson point process on $\mathbb{R} \times \mathbb{R}^d$ with intensity $e^{-\lambda} d\lambda \otimes dz$, chosen in such a way that $\bar{\Psi}_\theta^{(i)}$ is continuous and $\bar{\Lambda}_\theta^{(i)}, \bar{Z}_\theta^{(i)}$ càdlàg.

In particular, the probability of the event $\mathcal{E}_{t,a,b,c}^{(k)}$ defined in (7.6) converges to 1 as $t \rightarrow \infty$ and, for any fixed $\theta \in (0, \infty)$, the random vector

$$\left(\frac{\Psi_{\theta t, c}^{(1)} - a_{r_t}}{d_{r_t}}, \frac{Z_{\theta t, c}^{(1)}}{r_t} \right), \dots, \left(\frac{\Psi_{\theta t, c}^{(k)} - a_{r_t}}{d_{r_t}}, \frac{Z_{\theta t, c}^{(k)}}{r_t} \right) \quad (7.9)$$

converges in law to a random vector in $(\mathbb{R} \times \mathbb{R}^d)^k$ with distribution given by

$$\mathbb{1}\{\psi_1 > \dots > \psi_k\} e^{-\left(\frac{1}{\theta}|z_1| + \dots + \frac{1}{\theta}|z_k| + \psi_1 + \dots + \psi_k + (2\theta)^d e^{-\psi_k}\right)} \prod_{i=1}^k d\psi_i \otimes dz_i. \quad (7.10)$$

From this we immediately obtain:

Proof of Proposition 4.2. (4.13–4.14) follow directly from Proposition 7.1 and (2.6), while for (4.15) we use additionally Lemma 5.1 and $\xi(z) \geq \lambda^{\mathcal{C}}(z) \geq \Psi_t(z)$ for $z \in \mathcal{C}$, as implied by (5.5). \square

Note that the part of Theorem 2.6 concerning $(Z_t)_{t>0}$ already follows from Proposition 7.1. Another useful consequence is the following comparison between $\Psi_{t,c}$ and Ψ_t .

Lemma 7.2 For any $c \in \mathbb{R}$ and any $0 < a \leq b < \infty$, on the event $\mathcal{E}_{t,a,b,c}^{(2)}$ the following holds for all $s \in [at, bt]$:

$$\left| \sup_{z \neq Z_s} \Psi_{s,c}(z) - \Psi_s^{(2)} \right| \leq o(d_t b_t), \quad (7.11)$$

and

$$\left| \Psi_{s,c}(Z_s) - \Psi_s^{(1)} \right| \leq o(d_t b_t). \quad (7.12)$$

Proof. The supremum in (7.11) is attained at $Z_{s,c}^{(1)}$ if $Z_{s,c}^{(1)} \neq Z_s$, or $Z_{s,c}^{(2)}$ if $Z_{s,c}^{(1)} = Z_s$. Since $|Z_{s,c}^{(1)}| \vee |Z_{s,c}^{(2)}| \leq r_t g_t$ on $\mathcal{E}_{t,a,b,c}^{(2)}$, in either case we have

$$\left| \sup_{z \neq Z_s} \Psi_{s,c}(z) - \Psi_s^{(2)} \right| \leq |c| \frac{r_t g_t}{at} = o(d_t b_t) \quad (7.13)$$

since $r_t g_t / t = o(d_t b_t)$ by (4.12). The bound (7.12) is obtained analogously. \square

The proof of Proposition 7.1 is based on a point process approach, which we describe next. This approach will also allow us to prove Proposition 4.3 and Theorem 2.7.

7.1 A point process approach.

The key to the proofs of Proposition 7.1 and Theorem 2.7 is the convergence of suitably rescaled set $\{(\lambda^{\mathcal{C}}(z), z) : z \in \mathcal{C}\}$ to (the support of) a Poisson point process. We follow the setup and notation of [R87] for point processes; some arguments are for brevity relegated to the appendices.

Since we will need to apply the stated Poisson convergence to infer convergence of certain non-local minimizing functions, we will need to compactify some sets of $\mathbb{R} \times \mathbb{R}^d$ as follows. Embed $\mathbb{R} \times \mathbb{R}^d$ in a locally compact Polish space \mathfrak{E} such that the set

$$\mathcal{H}_\eta^\theta := \left\{ (\lambda, z) \in \mathbb{R} \times \mathbb{R}^d : \lambda > \frac{|z|}{\theta} + \eta \right\} \subset \mathfrak{E} \quad (7.14)$$

is relatively compact for any $\eta \in \mathbb{R}$ and $\theta \in (0, \infty)$ and, for each compact $K \subset \mathfrak{E}$, there exist $\theta > 0, \eta \in \mathbb{R}$ such that $K \subset \overline{\mathcal{H}_\eta^\theta}$. A suitable choice of \mathfrak{E} is given in Appendix B. Note that a Poisson point process in $\mathbb{R} \times \mathbb{R}^d$ with intensity $e^{-\lambda} d\lambda \otimes dz$ can be extended to \mathfrak{E} as the latter measure is a Radon measure on \mathfrak{E} . Denote by $\mathcal{M}_{\mathbb{P}} = \mathcal{M}_{\mathbb{P}}(\mathfrak{E})$ the set of point measures (i.e., \mathbb{N}_0 -valued Radon measures) on \mathfrak{E} . We equip $\mathcal{M}_{\mathbb{P}}$ with the topology of vague convergence, and let $\text{supp}(\mathcal{P})$ denote the support of $\mathcal{P} \in \mathcal{M}_{\mathbb{P}}$.

Let us denote

$$\mathcal{P}_t := \sum_{z \in \mathcal{C}} \delta_{(Y_t(z), z/t)} \quad \text{where} \quad Y_t(z) := \frac{\lambda^{\mathcal{C}}(z) - a_t}{d_t}. \quad (7.15)$$

Then we have:

Proposition 7.3 *The point process \mathcal{P}_t defined in (7.15) belongs almost surely to $\mathcal{M}_{\mathbb{P}}$ and converges in distribution as $t \rightarrow \infty$ with respect to the vague topology of $\mathcal{M}_{\mathbb{P}}$ to a Poisson point process supported in $\mathbb{R} \times \mathbb{R}^d \subset \mathfrak{E}$ with intensity measure $e^{-\lambda} d\lambda \otimes dz$.*

The proof of the Proposition 7.3 relies on the following lemma.

Lemma 7.4 *Let μ be a Radon measure on \mathbb{R} such that $\mu \otimes dz$ is a Radon measure on \mathfrak{E} . Let $\widehat{N}_t \in \mathbb{N}_0$ such that $\widehat{N}_t \ll t$ as $t \rightarrow \infty$ and assume that, for each $t > 0$, $(\widehat{Y}_t(z))_{z \in (2\widehat{N}_t+1)\mathbb{Z}^d}$ is a sequence of i.i.d. real-valued random variables satisfying the following two conditions:*

(i) For each $s \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \frac{t^d}{(2\widehat{N}_t^d + 1)} \text{Prob} \left(\widehat{Y}_t(0) > s \right) = \mu(s, \infty). \quad (7.16)$$

(ii) For each $\theta > 0, \eta \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} \sum_{x \in (2\widehat{N}_t+1)\mathbb{Z}^d : |x| \geq tn} \text{Prob} \left(\widehat{Y}_t(0) > \frac{|x|}{\theta t} + \eta \right) = 0. \quad (7.17)$$

Then, for each $t > 0$ large enough, the point process

$$\widehat{\mathcal{P}}_t := \sum_{x \in (2\widehat{N}_t+1)\mathbb{Z}^d} \delta_{(\widehat{Y}_t(x), x/t)} \quad (7.18)$$

belongs almost surely to $\mathcal{M}_{\mathbb{P}}$, and converges in distribution as $t \rightarrow \infty$ with respect to the vague topology of $\mathcal{M}_{\mathbb{P}}$ to a Poisson point process in $\mathbb{R} \times \mathbb{R}^d \subset \mathfrak{E}$ with intensity measure $\mu \otimes dz$.

Proof. Note first that, by (7.17), the expected value of $\widehat{\mathcal{P}}_t(\mathcal{H}_\eta^\theta)$ is finite for all $\theta > 0, \eta \in \mathbb{R}$ when t is large enough, and hence $\widehat{\mathcal{P}}_t \in \mathcal{M}_P$. The claimed convergence may be proved by a straightforward generalization of Proposition 3.21 of [R87], with $[0, \infty)$ therein substituted by \mathbb{R}^d and E therein substituted by \mathbb{R} (see also [HMS08, Lemma 2.4]). Indeed, we only need to verify (3.20) and (3.21) in [R87]. For (3.21), we note that, for any compact $K \subset \mathfrak{E}$, there exists $\eta \in \mathbb{R}$ such that $K \cap (\mathbb{R} \times \mathbb{R}^d) \subset [\eta, \infty) \times \mathbb{R}^d$, and thus (3.21) follows from (7.16). For (3.20), it suffices to prove that

$$\sum_{x \in (2\widehat{N}_t + 1)\mathbb{Z}^d} \text{Prob} \left(\widehat{Y}_t(0) \in \cdot \right) \otimes \delta_{x/t}(\mathrm{d}z) \xrightarrow{t \rightarrow \infty} \mu \otimes \mathrm{d}z \text{ vaguely in } \mathcal{M}_P. \quad (7.19)$$

Indeed, by (7.16), the convergence in (7.19) holds when evaluated on functions with support contained in the closure of a set of the form $[-n, \infty) \times [-n, n]^d \subset \mathfrak{E}$ with $n \in \mathbb{N}$. This is extended to functions compactly supported in \mathfrak{E} by applying (7.17) and the fact that, for any compact $K \subset \mathfrak{E}$, there exists $\theta > 0, \eta \in \mathbb{R}$ such that $K \subset \mathcal{H}_\eta^\theta$. \square

We can now proceed to:

Proof of Proposition 7.3. We will first use Lemma 7.4 to obtain convergence of an auxiliary process. Define

$$\widehat{Y}_t(x) := \frac{\lambda_{B_{\widehat{N}_t}}^{(1)}(x) - a_t}{d_t}, \quad x \in (2\widehat{N}_t + 1)\mathbb{Z}^d, \quad (7.20)$$

and let $\widehat{\mathcal{P}}_t$ be defined as in (7.18). We claim the following:

$$\text{The statement of Proposition 7.3 holds for } \widehat{\mathcal{P}}_t \text{ in place of } \mathcal{P}_t. \quad (7.21)$$

Indeed, condition (7.16) follows from (7.5), while (7.17) is proved in Appendix A.

Arguing as in the proof of Proposition 4.1, we see that, almost surely, $\mathcal{P}_t \in \mathcal{M}_P$ for all t large enough. By (7.21) and since \mathcal{P}_t and $\widehat{\mathcal{P}}_t$ are simple, it suffices to show that, for any $\theta \in (0, \infty)$ and $\eta \in \mathbb{R}$, with probability tending to 1 as $t \rightarrow \infty$ there exists a bijective map

$$T_t: \text{supp}(\widehat{\mathcal{P}}_t) \cap \mathcal{H}_\eta^\theta \rightarrow \text{supp}(\mathcal{P}_t) \cap \mathcal{H}_\eta^\theta \quad (7.22)$$

such that

$$\sup_{\Xi \in \text{supp}(\widehat{\mathcal{P}}_t) \cap \mathcal{H}_\eta^\theta} \text{dist}(T_t(\Xi), \Xi) \xrightarrow{t \rightarrow \infty} 0 \text{ in probability.} \quad (7.23)$$

To this end, pick $x \in (2\widehat{N}_t + 1)\mathbb{Z}^d$ such that $(\widehat{Y}_t(x), x/t) \in \mathcal{H}_\eta^\theta$. We first claim that, a.s. eventually as $t \rightarrow \infty$, all such x satisfy

$$B_{\widehat{N}_t}(x) \subset B_{L_t^*} \quad \text{and} \quad \lambda_{B_{\widehat{N}_t}}^{(1)}(x) > \widehat{a}_{L_t^*} - \chi + o(1). \quad (7.24)$$

Indeed, the second claim above follows from (5.3). If the first were violated, then by (5.5), Lemma 5.1 and the fact that $s \mapsto 2\rho(d_t)^{-1} \ln_2 s - s/(\theta t)$ is decreasing for $s \geq 2d\theta t \ln t$, we would have, a.s. eventually as $t \rightarrow \infty$,

$$\frac{\lambda_{B_{\widehat{N}_t}}^{(1)}(x) - a_t}{d_t} - \frac{|x|}{\theta t} \leq \frac{2\rho \ln_2 |x|}{d_t} - \frac{|x|}{\theta t} \leq \frac{2\rho \ln_2 L_t^*}{d_t} - \frac{L_t^* - 2d\widehat{N}_t}{\theta t} \xrightarrow{t \rightarrow \infty} -\infty \quad (7.25)$$

by (7.4), contradicting $(\widehat{Y}_t(x), x/t) \in \mathcal{H}_\eta^\theta$. This finishes the proof of (7.24). Now, since $\widehat{N}_t = N_{L_t^*}$, by Lemmas 5.3 and 5.8 there exists, with probability tending to 1 as $t \rightarrow \infty$, a unique $z \in \mathcal{C}$ satisfying

$$B_{Q_z}(z) \subset B_{\widehat{N}_t}(x) \quad \text{and} \quad \lambda_{B_{\widehat{N}_t}(x)}^{(1)} - \lambda^{\mathcal{C}}(z) \leq 2e^{-c_1(\ln L_t^*)^{\kappa/2}}, \quad (7.26)$$

which allows us to define an injective map

$$T_t \left(\widehat{Y}_t(x), \frac{x}{t} \right) := \left(Y_t(z), \frac{z}{t} \right) \in \text{supp}(\mathcal{P}_t). \quad (7.27)$$

Let us verify that T_t satisfies the desired properties. Indeed, (7.23) follows since

$$\left| \widehat{Y}_t(x) - Y_t(z) \right| + \left| \frac{z - x}{\theta t} \right| \leq \frac{2e^{-c_1(\ln L_t^*)^{\kappa/2}}}{d_t} + 2d \frac{\widehat{N}_t}{\theta t} =: \varepsilon_t \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (7.28)$$

and thus we only need to show that, with probability tending to 1 as $t \rightarrow \infty$, (7.27) is in \mathcal{H}_η^θ and T_t is surjective. Indeed, by (7.21), with probability tending to 1 as $t \rightarrow \infty$,

$$\widehat{\mathcal{P}}_t \left(\mathcal{H}_{\eta - \varepsilon_t}^\theta \setminus \mathcal{H}_{\eta + \varepsilon_t}^\theta \right) = 0, \quad (7.29)$$

implying by (7.28) that (7.27) is in \mathcal{H}_η^θ . Moreover, if $(Y_t(z), z/t) \in \mathcal{H}_\eta^\theta$ for some $z \in \mathcal{C}$, then as before $\lambda^{\mathcal{C}}(z) > \widehat{a}_{L_t^*} - \chi + o(1)$ and $B_{Q_z}(z) \subset B_{L_t^*}$. Thus, by Lemmas 5.8 and 5.3, there exists $x \in (2\widehat{N}_t + 1)\mathbb{Z}^d$ such that (7.26) and (7.28) hold, implying by (7.29) that $(Y_t(z), z/t)$ is the image by T_t of a point in $\text{supp}(\widehat{\mathcal{P}}_t) \cap \mathcal{H}_\eta^\theta$. This finishes the proof. \square

7.2 Order statistics: proof of Propositions 7.1 and 4.3 and Theorem 2.7.

Our next task is to translate (4.7–4.9) (and generalizations thereof) in terms of maps defined on point measures. We start with some necessary notation.

Denote by $\widehat{\mathcal{M}}_{\mathbb{P}}$ the set of positive measures \mathcal{P} on $\mathbb{R} \times \mathbb{R}^d$ that can be represented as

$$\mathcal{P} = \sum_{i \in \mathbb{N}} \delta_{(\lambda_i, z_i)} \quad \text{for some } (\lambda_i, z_i) \in \mathbb{R} \times \mathbb{R}^d, \quad (7.30)$$

i.e., $\widehat{\mathcal{M}}_{\mathbb{P}}$ is the set of integer-valued σ -finite Borel measures on $\mathbb{R} \times \mathbb{R}^d$.

Fix a measurable map $\vartheta: \mathbb{R}^d \rightarrow \mathbb{R}^d$. For a measure $\mathcal{P} \in \widehat{\mathcal{M}}_{\mathbb{P}}$ as in (7.30), we define

$$\mathcal{P}^\vartheta := \sum_{i \in \mathbb{N}} \delta_{(\lambda_i, \vartheta(z_i))}, \quad (7.31)$$

and we set

$$\mathcal{M}_{\mathbb{P}, \vartheta} := \{ \mathcal{P} \in \widehat{\mathcal{M}}_{\mathbb{P}} : \mathcal{P}^\vartheta \in \mathcal{M}_{\mathbb{P}} \}. \quad (7.32)$$

Finally, we generalise (2.7) by setting, for $\theta > 0$,

$$\psi_\theta^\vartheta(\lambda, z) := \lambda - \frac{|\vartheta(z)|}{\theta}, \quad (\lambda, z) \in \mathbb{R} \times \mathbb{R}^d. \quad (7.33)$$

Now, for $\mathcal{P} \in \mathcal{M}_{\mathbb{P}, \vartheta}$ and $\theta > 0$, we set, recursively for $i \in \mathbb{N}$, $i \leq |\text{supp}(\mathcal{P})|$,

$$\begin{aligned} \Psi_\vartheta^{(i)}(\mathcal{P})(\theta) &:= \\ &\sup \left\{ \psi_\theta^\vartheta(\lambda, z) : (\lambda, z) \in \text{supp}(\mathcal{P}) \setminus \left\{ \Xi_\vartheta^{(1)}(\mathcal{P})(\theta), \dots, \Xi_\vartheta^{(i-1)}(\mathcal{P})(\theta) \right\} \right\}, \end{aligned} \quad (7.34)$$

$$\mathfrak{S}_\vartheta^{(i)}(\mathcal{P})(\theta) := \left\{ (\lambda, z) \in \text{supp}(\mathcal{P}) \setminus \left\{ \Xi_\vartheta^{(1)}(\mathcal{P})(\theta), \dots, \Xi_\vartheta^{(i-1)}(\mathcal{P})(\theta) \right\} : \psi_\vartheta^\theta(\lambda, z) = \Psi_\vartheta^{(i)}(\mathcal{P})(\theta) \right\} \quad (7.35)$$

and

$$\Xi_\vartheta^{(i)}(\mathcal{P})(\theta) \in \left\{ (\lambda, z) \in \mathfrak{S}_\vartheta^{(i)}(\mathcal{P})(\theta) : (\lambda, z) \succeq (\lambda', z') \forall (\lambda', z') \in \mathfrak{S}_\vartheta^{(i)}(\mathcal{P})(\theta) \right\}, \quad (7.36)$$

where \succeq is the usual lexicographical order of $\mathbb{R} \times \mathbb{R}^d$ as introduced right before (4.7). Note that $\Xi_\vartheta^{(i)}(\mathcal{P})$ is well defined since the set in (7.36) has cardinality 1. Writing

$$\Xi_\vartheta^{(i)}(\mathcal{P}) =: \left(\Lambda_\vartheta^{(i)}(\mathcal{P}), Z_\vartheta^{(i)}(\mathcal{P}) \right), \quad (7.37)$$

we put

$$\Phi_\vartheta^{(i)}(\mathcal{P}) := \left(\Psi_\vartheta^{(i)}(\mathcal{P}), \Lambda_\vartheta^{(i)}(\mathcal{P}), Z_\vartheta^{(i)}(\mathcal{P}) \right). \quad (7.38)$$

When ϑ is the identity, i.e., $\vartheta(z) = z$ for all $z \in \mathbb{R}^d$, we omit it from the notation.

The functions defined above enjoy the following properties.

Lemma 7.5 *For any $\vartheta : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and any $\mathcal{P} \in \mathcal{M}_{\mathbb{P}, \vartheta}$, the following hold:*

- (i) $\Psi_\vartheta^{(1)}(\mathcal{P})$, $\Lambda_\vartheta^{(1)}(\mathcal{P})$ and $|\vartheta(Z_\vartheta^{(1)}(\mathcal{P}))|$ are non-decreasing and, if $\Xi_\vartheta^{(1)}(\mathcal{P})(\theta_0) \neq \Xi_\vartheta^{(1)}(\mathcal{P})(\theta_1)$ for some $\theta_0 < \theta_1$, then they are strictly smaller at θ_0 than at θ_1 .
- (ii) For any $a \in (0, \infty)$ and any $i \in \mathbb{N}$, $i \leq |\text{supp}(\mathcal{P})|$,

$$\Psi_\vartheta^{(i)}(\mathcal{P}) \in \mathcal{C}([a, \infty), \mathbb{R}) \quad \text{and} \quad \Xi_\vartheta^{(i)}(\mathcal{P}) \in \mathcal{D}([a, \infty), \mathbb{R} \times \mathbb{R}^d). \quad (7.39)$$

The set of discontinuities of $\Xi_\vartheta^{(i)}(\mathcal{P})$ is discrete and, if $\text{supp}(\mathcal{P}^\vartheta) \cap (\mathbb{R} \times \{0\}) = \emptyset$, then $\Psi_\vartheta^{(1)}(\mathcal{P})$ is strictly increasing.

The proof of Lemma 7.5 is postponed to Appendix C. It already implies the properties claimed for $\Psi_t^{(k)}, Z_t^{(k)}$ at the end of Section 4.1: indeed, they follow from the representation

$$\left(\Psi_t^{(k)}, \lambda^{\mathcal{C}}(Z_t^{(k)}), Z_t^{(k)} \right) = \Phi_\vartheta^{(k)}(\mathcal{P}_{\mathcal{C}})(t) \quad \text{with} \quad \vartheta(z) := z \ln_3^+ |z|, \quad \mathcal{P}_{\mathcal{C}} := \sum_{z \in \mathcal{C}} \delta_{(\lambda^{\mathcal{C}}(z), z)}. \quad (7.40)$$

Note that $\mathcal{P}_{\mathcal{C}} \in \mathcal{M}_{\mathbb{P}, \vartheta}$ almost surely by (4.6), and that $|\vartheta(z_1)| > |\vartheta(z_0)|$ implies $|z_1| > |z_0|$.

Next we consider continuity of $\mathcal{P} \mapsto \Phi^{(i)}(\mathcal{P})$ with respect to the Skorohod topology, i.e., specializing to the case where ϑ is the identity. To this end, we define the following subsets of $\mathcal{M}_{\mathbb{P}}$, indexed by $a \in (0, \infty)$:

$$\begin{aligned} \widetilde{\mathcal{M}}_{\mathbb{P}}^a := & \left\{ \mathcal{P} \in \mathcal{M}_{\mathbb{P}} : \text{supp}(\mathcal{P}) \subset \mathbb{R} \times \mathbb{R}^d \setminus (\mathbb{R} \times \{0\}), \right. \\ & (\lambda, z) \mapsto \lambda \text{ is injective over } \text{supp}(\mathcal{P}), \\ & \mathcal{P}(\partial \mathcal{H}_\eta^\theta) \leq 1 \quad \forall \theta \in \{a\} \cup (0, \infty) \cap \mathbb{Q}, \eta \in \mathbb{R}, \\ & \mathcal{P}(\partial \mathcal{H}_\eta^\theta) \leq 2 \quad \forall \theta \in (0, \infty), \eta \in \mathbb{R}, \\ & \left. |\{\eta \in \mathbb{R} : \mathcal{P}(\partial \mathcal{H}_\eta^\theta) = 2\}| \leq 1 \quad \forall \theta \in (0, \infty) \right\}. \end{aligned} \quad (7.41)$$

Then we have:

Lemma 7.6 Fix $a \in (0, \infty)$ and $\mathcal{P} \in \widetilde{\mathcal{M}}_{\mathbb{P}}^a$. Let $\vartheta_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $t > 0$, satisfy

$$(i) \vartheta_t(z) \xrightarrow[t \rightarrow \infty]{} z \text{ locally uniformly for } z \in \mathbb{R}^d \setminus \{0\}, \text{ and} \quad (7.42)$$

$$(ii) \text{ there exists a } c_* \in (0, 1] \text{ such that, for all } \delta > 0, \liminf_{t \rightarrow \infty} \inf_{|z| \geq \delta} \frac{|\vartheta_t(z)|}{|z|} \geq c_*. \quad (7.43)$$

Let $\mathcal{P}_t \in \mathcal{M}_{\mathbb{P}} \cap \mathcal{M}_{\mathbb{P}, \vartheta_t}$ such that $\mathcal{P}_t \xrightarrow[t \rightarrow \infty]{} \mathcal{P}$ vaguely in $\mathcal{M}_{\mathbb{P}}$. Then also $\mathcal{P}_t^{\vartheta_t} \rightarrow \mathcal{P}$ vaguely and, for all $k \in \mathbb{N}$, $k \leq |\text{supp}(\mathcal{P})|$,

$$\left(\Phi_{\vartheta_t}^{(i)}(\mathcal{P}_t) \right)_{1 \leq i \leq k} \xrightarrow[t \rightarrow \infty]{} \left(\Phi^{(i)}(\mathcal{P}) \right)_{1 \leq i \leq k} \quad (7.44)$$

in the Skorohod topology of $\mathcal{D}([a, \infty), (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d)^k)$. In particular, $(\Phi^{(i)})_{1 \leq i \leq k}$ is continuous at \mathcal{P} with respect to the Skorohod topology.

Lemma 7.6 will be also proved in Appendix C. We note that it may be used to study the continuity of $\mathcal{P} \mapsto \Phi_{\vartheta}^{(i)}(\mathcal{P})$ when ϑ is a homeomorphism by using the representation $\Lambda^{(i)}(\mathcal{P}^{\vartheta}) = \Lambda_{\vartheta}^{(i)}(\mathcal{P})$, $Z^{(i)}(\mathcal{P}^{\vartheta}) = \vartheta(Z_{\vartheta}^{(i)}(\mathcal{P}))$, which is valid e.g. whenever $\mathcal{P}^{\vartheta} \in \widetilde{\mathcal{M}}_{\mathbb{P}}^a$.

We now use Lemma 7.6 to finish the:

Proof of Proposition 7.1. By Lemma 7.5, we may realise the processes in (7.8) as

$$\left(\overline{\Psi}_{\theta}^{(i)}, \overline{\Lambda}_{\theta}^{(i)}, \overline{Z}_{\theta}^{(i)} \right) = \Phi^{(i)}(\mathcal{P}_{\infty})(\theta) \quad (7.45)$$

where \mathcal{P}_{∞} is a Poisson point process on $\mathbb{R} \times \mathbb{R}^d$ with intensity $e^{-\lambda} d\lambda \otimes dz$. Note that, for each $a > 0$, $\mathcal{P}_{\infty} \in \widetilde{\mathcal{M}}_{\mathbb{P}}^a$ almost surely. On the other hand, we also have the representation

$$\left(\frac{\Psi_{\theta t, c}^{(i)} - a_{r_t}}{d_{r_t}}, \frac{\lambda^{\mathcal{C}}(Z_{\theta t, c}^{(i)}) - a_{r_t}}{d_{r_t}}, \frac{Z_{\theta t, c}^{(i)}}{r_t} \right) = \Phi_{\vartheta_t}^{(i)}(\mathcal{P}_{r_t})(\theta) \quad (7.46)$$

where \mathcal{P}_t is as in (7.15) and

$$\vartheta_t(z) := z \left(\frac{\ln_3^+ |r_t z| - c}{\ln_3 t} \right)^+ \frac{d_t}{d_{r_t}}. \quad (7.47)$$

Note that, by (7.2), $\mathcal{P}_{r_t} \in \mathcal{M}_{\mathbb{P}, \vartheta_t}$ almost surely for all t large enough. The convergence claimed in Proposition 7.1 now follows by Proposition 7.3 and Lemma 7.6 together with (7.45), (7.46)–(7.47) and the Skorohod representation theorem; in fact,

$$\left(\mathcal{P}_{r_t}, \left(\Phi_{\vartheta_t}^{(i)}(\mathcal{P}_{r_t})(\theta) \right)_{\theta \in [a, \infty), 1 \leq i \leq k} \right) \xrightarrow[t \rightarrow \infty]{\text{law}} \left(\mathcal{P}_{\infty}, \left(\Phi^{(i)}(\mathcal{P}_{\infty})(\theta) \right)_{\theta \in [a, \infty), 1 \leq i \leq k} \right). \quad (7.48)$$

The statement regarding $\mathcal{E}_{a, b, c}^{(k)}$ follows from the distributional convergence since $d_{r_t} = d_t(1 + o(1))$ and, by the continuity properties of $\overline{\Psi}_{\theta}^{(i)}$ and $\overline{Z}_{\theta}^{(i)}$,

$$\begin{aligned} -\infty < \inf_{\theta \in [a, b]} \overline{\Psi}_{\theta}^{(i)} \leq \sup_{\theta \in [a, b]} \overline{\Psi}_{\theta}^{(i)} < \infty, \quad 0 < \inf_{\theta \in [a, b]} |\overline{Z}_{\theta}^{(i)}| \leq \sup_{\theta \in [a, b]} |\overline{Z}_{\theta}^{(i)}| < \infty \\ \text{and } \left(\overline{\Psi}_a^{(i)} - \overline{\Psi}_a^{(i+1)} \right) \wedge \left(\overline{\Psi}_b^{(i)} - \overline{\Psi}_b^{(i+1)} \right) > 0 \end{aligned} \quad (7.49)$$

hold almost surely for each $i \in \mathbb{N}$. The expression for the density in (7.10) follows from an analogous calculation as performed in the proof of Proposition 3.2 in [ST14]. \square

Next we interpret the event in Theorem 2.7 in terms of the underlying point measure, which is still kept rather general:

Lemma 7.7 *For any $\vartheta : \mathbb{R}^d \rightarrow \mathbb{R}^d$, any $\mathcal{P} \in \mathcal{M}_{\mathcal{P},\vartheta}$ and any $0 < a < b < \infty$, the following statements are equivalent:*

$$\begin{aligned}
(1) \quad & Z_{\vartheta}^{(1)}(\mathcal{P})(a) = Z_{\vartheta}^{(1)}(\mathcal{P})(b); \\
(2) \quad & \Lambda_{\vartheta}^{(1)}(\mathcal{P})(a) = \Lambda_{\vartheta}^{(1)}(\mathcal{P})(b); \\
(3) \quad & \Xi_{\vartheta}^{(1)}(\mathcal{P})(\theta) = \Xi_{\vartheta}^{(1)}(\mathcal{P})(a) \text{ for all } \theta \in [a, b]; \\
(4) \quad & \mathcal{P} \left\{ (\lambda, z) : \begin{array}{l} \psi_b^{\vartheta}(\lambda, z) > \psi_b^{\vartheta}(\Xi_{\vartheta}^{(1)}(\mathcal{P})(a)), \text{ or} \\ \psi_b^{\vartheta}(\lambda, z) = \psi_b^{\vartheta}(\Xi_{\vartheta}^{(1)}(\mathcal{P})(a)) \text{ and } \lambda > \Lambda_{\vartheta}^{(1)}(\mathcal{P})(a) \end{array} \right\} = 0.
\end{aligned} \tag{7.50}$$

Proof. The equivalence between (1) and (2) follows from Lemma 7.5(i), and thus either of them implies (3) since $\Lambda_{\vartheta}^{(1)}(\mathcal{P})$ is non-decreasing. The implications (3) \Rightarrow (4) and (4) \Rightarrow (2) are then easily verified using the definition of $\Xi_{\vartheta}^{(i)}$. \square

We study next continuity properties of the event in item (4) above. To this end, we define, for $\vartheta : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\mathcal{P} \in \mathcal{M}_{\mathcal{P},\vartheta}$, $(\lambda, z) \in \mathbb{R} \times \mathbb{R}^d$ and $\theta > 0$,

$$\mathcal{F}_{\vartheta}^{\theta}(\mathcal{P}, \lambda, z) := \mathcal{P} \left\{ (\lambda', z') : \begin{array}{l} \psi_{\theta}^{\vartheta}(\lambda', z') > \psi_{\theta}^{\vartheta}(\lambda, z), \text{ or} \\ \psi_{\theta}^{\vartheta}(\lambda', z') = \psi_{\theta}^{\vartheta}(\lambda, z) \text{ and } \lambda' > \lambda \end{array} \right\} \in \mathbb{N}_0. \tag{7.51}$$

When ϑ is the identity, we again omit it from the notation. Then we have:

Lemma 7.8 *Fix $b \in (0, \infty)$, $\mathcal{P} \in \widetilde{\mathcal{M}}_{\mathcal{P}}^b$ and take $\vartheta_t, \mathcal{P}_t$ as in the statement of Lemma 7.6. Assume that $(\lambda_*, z_*) \in \text{supp}(\mathcal{P})$, $(\lambda_t, z_t) \in \text{supp}(\mathcal{P}_t)$ are such that $(\lambda_t, z_t) \rightarrow (\lambda_*, z_*)$ as $t \rightarrow \infty$. Then*

$$\mathcal{F}_b^{\vartheta_t}(\mathcal{P}_t, \lambda_t, z_t) \xrightarrow[t \rightarrow \infty]{} \mathcal{F}_b(\mathcal{P}, \lambda_*, z_*). \tag{7.52}$$

The proof of Lemma 7.8 is again deferred to Appendix C. Together with Lemma 7.7, it permits us to give the:

Proof of Theorem 2.7. Fix $0 < a < b < \infty$ and use the representation (7.46–7.47) (with $c = 0$), Lemma 7.7 and (7.51) to write

$$\begin{aligned}
Z_{at} = Z_{bt} &\Leftrightarrow Z_{\vartheta_t} = Z_{at} \quad \forall \theta \in [a, b] \\
&\Leftrightarrow \mathcal{F}_b^{\vartheta_t}(\mathcal{P}_{r_t}, \Lambda_{\vartheta_t}^{(1)}(\mathcal{P}_{r_t})(a), Z_{\vartheta_t}^{(1)}(\mathcal{P}_{r_t})(a)) = 0.
\end{aligned} \tag{7.53}$$

Since $\mathcal{P}_{\infty} \in \widetilde{\mathcal{M}}_{\mathcal{P}}^a \cap \widetilde{\mathcal{M}}_{\mathcal{P}}^b$ a.s., the result follows from Lemma 7.8, (7.48) and (7.45). \square

The last objective of the section is to prove Proposition 4.3. Our next lemma shows that its statement holds in fact more generally:

Lemma 7.9 *For any $\vartheta : \mathbb{R}^d \rightarrow \mathbb{R}^d$, any $\mathcal{P} \in \mathcal{M}_{\mathcal{P},\vartheta}$ and any $0 < a < b < \infty$, if*

$$\Xi_{\vartheta}^{(1)}(\mathcal{P})(\theta) = \Xi_{\vartheta}^{(2)}(\mathcal{P})(a) \quad \forall \theta \in [a, b] \tag{7.54}$$

then

$$\inf_{\theta \in [a, b]} \left\{ \Psi_{\vartheta}^{(1)}(\mathcal{P})(\theta) - \Psi_{\vartheta}^{(2)}(\mathcal{P})(\theta) \right\} = \min_{\theta \in \{a, b\}} \left\{ \Psi_{\vartheta}^{(1)}(\mathcal{P})(\theta) - \Psi_{\vartheta}^{(2)}(\mathcal{P})(\theta) \right\}. \quad (7.55)$$

Proof. For $\theta \in [a, b]$ and $i \in \{1, 2\}$, put $(\hat{\lambda}_{\theta}^{(i)}, \hat{z}_{\theta}^{(i)}) := \Xi_{\vartheta}^{(i)}(\mathcal{P})(\theta)$ and write

$$\Psi_{\vartheta}^{(1)}(\mathcal{P})(\theta) - \Psi_{\vartheta}^{(2)}(\mathcal{P})(\theta) = \hat{\lambda}_{\theta}^{(1)} - \hat{\lambda}_{\theta}^{(2)} - \frac{|\vartheta(\hat{z}_{\theta}^{(1)})| - |\vartheta(\hat{z}_{\theta}^{(2)})|}{\theta}. \quad (7.56)$$

If $|\vartheta(\hat{z}_{\theta}^{(1)})| \geq |\vartheta(\hat{z}_{\theta}^{(2)})|$, substitute $\theta = a$ in the denominator above and use (7.54) to obtain

$$\begin{aligned} \Psi_{\vartheta}^{(1)}(\mathcal{P})(\theta) - \Psi_{\vartheta}^{(2)}(\mathcal{P})(\theta) &\geq \Psi_{\vartheta}^{(1)}(\mathcal{P})(a) - \psi_a^{\vartheta}(\hat{\lambda}_{\theta}^{(2)}, \hat{z}_{\theta}^{(2)}) \\ &\geq \Psi_{\vartheta}^{(2)}(\mathcal{P})(a) - \Psi_{\vartheta}^{(2)}(\mathcal{P})(a). \end{aligned} \quad (7.57)$$

If $|\vartheta(\hat{z}_{\theta}^{(1)})| < |\vartheta(\hat{z}_{\theta}^{(2)})|$, substituting $\theta = b$ instead we analogously get

$$\Psi_{\vartheta}^{(1)}(\mathcal{P})(\theta) - \Psi_{\vartheta}^{(2)}(\mathcal{P})(\theta) \geq \Psi_{\vartheta}^{(2)}(\mathcal{P})(b) - \Psi_{\vartheta}^{(2)}(\mathcal{P})(b). \quad (7.58)$$

In either case, (7.55) follows. \square

We can finally conclude the:

Proof of Proposition 4.3. Follows from Lemmas 7.7 and 7.9 together with (7.40). \square

8. MASS DECOMPOSITION

In this section, we prove Proposition 4.4 in Subsection 8.1, Proposition 4.5 in Subsection 8.2, Propositions 4.6–4.7 in Subsection 8.3 and Theorem 2.6 in Subsection 8.4.

8.1 Lower bound for the total mass.

We begin with a lower bound for the mass up to the hitting time of a point.

Lemma 8.1 *Under Assumption 2.2, there exists a constant $K > 1$ such that, a.s. eventually as $T \rightarrow \infty$, for all $\theta \geq T$ and all $x \in \mathbb{Z}^d$ with $|x| > 4d\theta$,*

$$\mathbb{E}_0 \left[e^{\int_0^{\tau_x} \xi(X_u) du} \mathbb{1}_{\{\tau_x \leq \theta\}} \right] \geq \exp \left\{ -|x| \ln \frac{K|x|}{\theta} \right\}. \quad (8.1)$$

Proof. We follow the proof of Lemma 4.3 of [GM90] (case of $d = 1$ therein). Fix a path π from 0 to x such that $|\pi| = |x|$. Then the left-hand side of (8.1) is at least

$$(2d)^{-|x|} \mathbb{E}_0 \left[\exp \left\{ - \sum_{i=0}^{|x|-1} \sigma_i \xi^-(\pi_i) \right\} \mathbb{1}_{\{\sum_{i=0}^{|x|-1} \sigma_i \leq \theta\}} \right] \quad (8.2)$$

where $(\sigma_i)_{i=0}^\infty$ are i.i.d. exponential random variables with parameter $2d$. We can further bound (8.2) from below by

$$(2d)^{-|x|} e^{-\theta} \mathbb{P}_0 \left(\sigma_i \leq \frac{\theta}{|x|} \frac{1}{1 + \xi^-(\pi_i)} \quad \forall i = 0, \dots, |x| - 1 \right) \quad (8.3)$$

$$\geq (2d)^{-|x|} e^{-\theta} \prod_{i=0}^{|x|-1} \left(\frac{\theta}{|x|} \frac{d}{1 + \xi^-(\pi_i)} \right) = \exp \left\{ -|x| \ln \frac{2d|x|}{\theta d} - \theta - \sum_{i=0}^{|x|-1} \ln(1 + \xi^-(\pi_i)) \right\} \quad (8.4)$$

where we used $1 - e^{-y} \geq \frac{1}{2}y$ when $0 < y < \frac{1}{2}$. By Theorem 1.1 of [M02] and Assumption 2.2, there exists a constant $c_0 > 0$ such that, a.s. eventually as $|x| \rightarrow \infty$,

$$\sum_{i=0}^{|x|-1} \ln(1 + \xi^-(\pi_i)) \leq c_0|x|. \quad (8.5)$$

Now (8.1) follows from (8.3–8.5) and $\theta < |x|/(4d)$. \square

We can now prove Proposition 4.4.

Proof of Proposition 4.4. For a finite connected subset $\Lambda \subset \mathbb{Z}^d$, let $\phi_\Lambda^{(1)}$ be the normalised eigenfunction of H_Λ corresponding to its largest eigenvalue $\lambda_\Lambda^{(1)}$ as in Section 5.4. Let $x_0 \in \Lambda$ be a point where $\phi_\Lambda^{(1)}$ attains its maximum, and note that, since $\|\phi_\Lambda^{(1)}\|_{\ell^2(\mathbb{Z}^d)} = 1$, $|\phi_\Lambda^{(1)}(x_0)|^2 \geq |\Lambda|^{-1}$. By Lemma 5.12,

$$\mathbb{E}_{x_0} \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{\tau_{\Lambda^c} > s\}} \right] \geq e^{s\lambda_\Lambda^{(1)}} |\phi_\Lambda^{(1)}(x_0)|^2 \geq e^{s\lambda_\Lambda^{(1)} - \ln|\Lambda|}. \quad (8.6)$$

Using the Feynman-Kac formula, the strong Markov property and (8.6), we obtain, for any $\theta < s$,

$$\begin{aligned} U(s) &\geq \mathbb{E}_0 \left[\exp \left\{ \int_0^{\tau_{x_0}} \xi(X_u) du \right\} \mathbb{1}_{\{\tau_{x_0} \leq \theta\}} \mathbb{E}_{x_0} \left[e^{\int_0^{s-r} \xi(X_u) du} \mathbb{1}_{\{\tau_{\Lambda^c} > s-r\}} \right]_{r=\tau_{x_0}} \right] \\ &\geq e^{s\lambda_\Lambda^{(1)} - \ln|\Lambda| - \theta|\lambda_\Lambda^{(1)}|} \mathbb{E}_0 \left[\exp \left\{ \int_0^{\tau_{x_0}} \xi(X_u) du \right\} \mathbb{1}_{\{\tau_{x_0} \leq \theta\}} \right]. \end{aligned} \quad (8.7)$$

Specializing now to $\Lambda := B_{\varrho_{Z_s}}(Z_s)$, let $K > 1$ as in Lemma 8.1 and set $\theta := K|x_0|/\lambda^\mathcal{C}(Z_s)$. By Lemma 5.1 and Proposition 4.2, we may assume that $\varrho_{Z_s} \leq \ln t$. Thus on $\mathcal{E}_{t,a,b}$ we have

$$\frac{|x_0|}{s} \leq \frac{|Z_s| + |x_0 - Z_s|}{at} \leq \frac{r_t g_t + 2d \ln t}{at} = o(d_t b_t \epsilon_t), \quad (8.8)$$

while $\lambda^\mathcal{C}(Z_s) \geq \Psi_s^{(1)} \geq a_{r_t} - d_t g_t \rightarrow \infty$ as $t \rightarrow \infty$ since $d_t g_t = o(1)$. Therefore, $\theta < |x_0|/(4d) < s$ for large enough t . On the other hand, by Lemma 5.1, on $\mathcal{E}_{t,a,b}$ we have

$$\lambda^\mathcal{C}(Z_s) \leq \zeta(Z_s) \leq 2\rho \ln_2 |Z_s| \leq 2\rho \ln_2 t \quad (8.9)$$

for large enough t since $r_t g_t = o(t)$. Hence

$$\theta \geq \frac{r_t f_t - 2d \ln t}{2\rho \ln_2 t} \rightarrow \infty \text{ as } t \rightarrow \infty, \quad (8.10)$$

and so we may apply Lemma 8.1 to (8.7) obtaining

$$\frac{\ln U(s)}{s} \geq \lambda^{\mathcal{C}}(Z_s) - \frac{|x_0|}{s} \ln \lambda^{\mathcal{C}}(Z_s) - K \frac{|x_0|}{s} + o(d_t b_t \epsilon_t). \quad (8.11)$$

Now, by (8.9),

$$\frac{\ln U(s)}{s} \geq \Psi_s^{(1)} - \frac{|x_0 - Z_s| \ln_3^+ |Z_s|}{s} - (|\ln 2\rho| + K) \frac{|x_0|}{s} + o(d_t b_t \epsilon_t), \quad (8.12)$$

and to conclude we note that the second and third terms in (8.12) are also $o(d_t b_t \epsilon_t)$. \square

8.2 Macrobox truncation.

Next we prove Proposition 4.5, ensuring that the Feynman-Kac formula is not affected by restricting to random-walk paths that do not leave a box of side $L_t = \lfloor t \ln_2^+ t \rfloor$ around the starting point.

Proof of Proposition 4.5. We follow the proof of Proposition 2.1 in [FM14]. First write

$$\mathbb{E}_0 \left[e^{\int_0^s \zeta(X_u) du} \mathbb{1}_{\{\sup_{\theta \in [0,s]} |X_\theta| \geq L_t\}} \right] \leq \sum_{n=L_t}^{\infty} \exp \left\{ s \max_{x \in B_n} \zeta(x) \right\} \mathbb{P}_0 \left(\sup_{\theta \in [0,s]} |X_\theta| = n \right). \quad (8.13)$$

Denoting by J_s the number of jumps of X up to time s , the fact that J_s is a Poisson random variable with parameter $2ds$ gives

$$\mathbb{P}_0 \left(\sup_{\theta \in [0,s]} |X_\theta| = n \right) \leq \mathbb{P}_0 (J_s \geq n) \leq \frac{(2ds)^n}{n!}. \quad (8.14)$$

By Lemma 5.1, we have that $\max_{x \in B_n} \zeta(x) \leq 2\rho \ln_2 n$ a.s. for all n large enough. Using Stirling's formula, we note that, since $s \in [at, bt]$, the term corresponding to n in the sum in (8.13) is at most

$$\exp \left\{ 2\rho b t \ln_2 n - n(\ln n - \ln t - c) \right\} \quad (8.15)$$

for some deterministic constant $c > 0$. Now, when $n \geq L_t$ and t is large enough, $\ln n - \ln t - c \geq \frac{1}{2} \ln_3 t$. Since the function $x \mapsto 2\rho b t \ln_2 x - \frac{x}{4} \ln_3 t$ is strictly decreasing on $[L_t, \infty)$ and negative at $x = L_t$, a.s. for all t large enough, (8.13) is smaller than

$$\sum_{n=L_t}^{\infty} e^{-\frac{n}{4} \ln_3 t} \leq 2e^{-\frac{L_t}{4} \ln_3 t}. \quad (8.16)$$

Plugging in the definition of L_t now yields (4.21). \square

8.3 Negligible contributions.

In this subsection we prove Propositions 4.6 and 4.7. Here and in the next subsection we will work with R_L satisfying (5.11–5.12). It will be useful to introduce yet another family of auxiliary cost functionals $\tilde{\Psi}_{t,s,c}$, indexed by $t, s \geq 0$, $c \in \mathbb{R}$, and defined on the elements of $\mathfrak{C}_{L_t, A}$ as follows:

$$\tilde{\Psi}_{t,s,c}(\mathcal{C}) := \lambda_{\mathcal{C}}^{(1)} - \frac{(\ln_3^+ |z_{\mathcal{C}}| - c)^+}{s} |z_{\mathcal{C}}|, \quad \mathcal{C} \in \mathfrak{C}_{L_t, A}. \quad (8.17)$$

These functionals will be convenient to express bounds to the Feynman-Kac formula obtained via Proposition 6.1. In order to compare $\tilde{\Psi}_{t,s,c}$ and Ψ_t , we will need the following.

Lemma 8.2 *Almost surely for all $t, s > 0$, there exists a component $\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}$ such that, for all $0 < a \leq b < \infty$, the following holds with probability tending to 1 as $t \rightarrow \infty$:*

$$z_{\mathcal{C}_{t,s}} = Z_s \quad \forall s \in [at, bt]. \quad (8.18)$$

Proof. By Lemma 5.8, there exists a $\delta > 0$ such that, with probability tending to 1 as $t \rightarrow \infty$, whenever $|Z_s| + 2d_Q Z_s < L_t$ and $\lambda^{\mathcal{C}}(Z_s) > \hat{a}_{L_t} - \chi - \delta$ we can find a unique $\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}$ with $z_{\mathcal{C}_{t,s}} = Z_s$. Fixing $\mathcal{C}_t^* \in \mathfrak{C}_{L_t, A}$ in an arbitrary (measurable) fashion, we define $\mathcal{C}_{t,s} = \mathcal{C}_t^*$ when either the conclusion of Lemma 5.8 does not hold, or when Z_s does not satisfy the properties above. By Proposition 4.2, $\mathcal{C}_{t,s}$ satisfies (8.18) with probability tending to 1 as $t \rightarrow \infty$. \square

When $t = s$ we write \mathcal{C}_t instead of $\mathcal{C}_{t,s}$.

The following lemma relates $\tilde{\Psi}_{t,s,c}$ to Ψ_t .

Lemma 8.3 *For all $A > 0$ large enough and any $0 < a \leq b < \infty$, $\delta > 0$ and $c \in \mathbb{R}$,*

$$\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}^\delta, \quad \left| \tilde{\Psi}_{t,s,c}(\mathcal{C}_{t,s}) - \Psi_s^{(1)} \right| \leq o(dtbt) \quad \text{and} \quad \left| \max_{\mathcal{C} \neq \mathcal{C}_{t,s}} \tilde{\Psi}_{t,s,c}(\mathcal{C}) - \Psi_s^{(2)} \right| \leq o(dtbt) \quad (8.19)$$

hold for all $s \in [at, bt]$ with probability tending to 1 as $t \rightarrow \infty$.

Proof. Fix $A, \delta > 0$ as in Lemma 5.8 and let $\mathcal{C} \in \mathfrak{C}_{L_t, A}$. By this lemma and Proposition 7.1, we may assume that, if $\mathcal{C} \notin \mathfrak{C}_{L_t, A}^\delta$, then $\tilde{\Psi}_{t,s,c}(\mathcal{C}) \leq \lambda_{\mathcal{C}}^{(1)} \leq \hat{a}_{L_t} - \chi - \delta < \Psi_s^{(2)}$ while, if $\mathcal{C} \in \mathfrak{C}_{L_t, A}^\delta$, $z_{\mathcal{C}} \in \mathcal{C}$ and

$$\tilde{\Psi}_{t,s,c}(\mathcal{C}) = \Psi_{s,c}(z_{\mathcal{C}}) + o(dtbt). \quad (8.20)$$

(8.19) follows by considering in (8.20) the cases $z_{\mathcal{C}} = Z_s$ and $z_{\mathcal{C}} \neq Z_s$ and applying Lemma 7.2. \square

Recall (6.3) and consider the following classes of paths: First set

$$\mathcal{N}_{t,s}^{(0)} := \{ \pi \in \mathcal{P}(0, \mathbb{Z}^d) : \text{supp}(\pi) \subset B_{L_t}, \text{supp}(\pi) \cap (D_{t,s}^\circ)^c \neq \emptyset \} \quad (8.21)$$

and then let

$$\mathcal{N}_{t,s}^{(1)} := \{ \pi \in \mathcal{N}_{t,s}^{(0)} : \lambda_{L_t, A}(\pi) \leq \lambda_{\mathcal{C}_{t,s}}^{(1)} \} \quad \text{and} \quad \mathcal{N}_{t,s}^{(2)} := \mathcal{N}_{t,s}^{(0)} \setminus \mathcal{N}_{t,s}^{(1)}, \quad (8.22)$$

where $\mathcal{C}_{t,s}$ is as in Lemma 8.2. Note that, if $\tau_{(D_{t,s}^\circ)^c} \leq s < \tau_{B_{L_t}^c}$, then $\pi(X_{0,s}) \in \mathcal{N}_{t,s}^{(1)} \cup \mathcal{N}_{t,s}^{(2)}$ and hence we may bound the contribution of each class of paths separately. This is carried out in the following lemma, using Proposition 6.1.

Lemma 8.4 *For all $A > 0$ large enough, there exists $c > 0$ such that, for all $0 < a \leq b < \infty$,*

$$\begin{aligned} \ln \mathbb{E}_0 \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{\pi(X_{0,s}) \in \mathcal{N}_{t,s}^{(1)}\}} \right] \\ \leq s \tilde{\Psi}_{t,s,c}(\mathcal{C}_{t,s}) - (\ln_3(4dL_t) - c)(1 + h_t)|Z_s| + o(td_t b_t), \end{aligned} \quad (8.23)$$

and

$$\ln \mathbb{E}_0 \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{\pi(X_{0,s}) \in \mathcal{N}_{t,s}^{(2)}\}} \right] \leq s \max_{\mathcal{C} \neq \mathcal{C}_{t,s}} \tilde{\Psi}_{t,s,c}(\mathcal{C}) + o(td_t b_t) \quad (8.24)$$

hold for all $s \in [at, bt]$ with probability tending to 1 as $t \rightarrow \infty$.

Proof. On $\mathcal{E}_{t,a,b}$ (cf. (7.6)), $\inf_{s \in [at, bt]} |Z_s| \gg \ln L_t$ and so we may apply Proposition 6.1 to $\mathcal{N}_{t,s}^{(1)}$ and $\mathcal{N}_{t,s}^{(2)}$. Choose γ_π, z_π as follows. For $\pi \in \mathcal{N}_{t,s}^{(1)}$, let $\gamma_\pi = \lambda_{\mathcal{C}_{t,s}}^{(1)} + d_t / \ln_3 t$ and take z_π arbitrarily in $\text{supp}(\pi) \cap (D_{t,s}^\circ)^c \neq \emptyset$. If $\pi \in \mathcal{N}_{t,s}^{(2)}$, then $\text{supp}(\pi) \cap \Pi_{L_t, A} \neq \emptyset$ and we may set $\gamma_\pi = \lambda_{L_t, A}(\pi) + d_t / \ln_3 t$, $z_\pi = z_{\mathcal{C}_\pi}$ where $\mathcal{C}_\pi \in \mathfrak{C}_{L_t, A}$ is such that $\lambda_{L_t, A}(\pi) = \lambda_{\mathcal{C}_\pi}^{(1)}$. Note that, by Lemma 8.3, we may assume that $\lambda_{\mathcal{C}_{t,s}}^{(1)} > \widehat{a}_{L_t} - A$. Then (8.23–8.24) follow by substituting our choice of γ_π, z_π in (6.6), using the definition of $\widetilde{\Psi}_{t,s,c}$, the fact that $|z_\pi| > |Z_s|(1 + h_t)$ for $\pi \in \mathcal{N}_{t,s}^{(1)}$ and noting that $d_t / \ln_3 t = o(d_t b_t)$ by (4.12). \square

Proof of Proposition 4.6. This now follows from Lemmas 8.3–8.4, Proposition 4.2, the definition of d_t and r_t in (2.6) and the relations between the various error scales in (4.12). \square

Next we turn to Proposition 4.7. Note that paths avoiding $B_\nu(Z_s)$ do not necessarily exit an ℓ^1 -ball of radius $\ln L_t$, so we may not directly use Proposition 6.1. As the points in $\Pi_{L_t, A}$ are typically far away from the origin, this can be remedied by considering

$$\begin{aligned} \mathcal{N}_t^{(3)} &:= \left\{ \pi \in \mathcal{P}(0, \mathbb{Z}^d) : \text{supp}(\pi) \subset B_{L_t} \setminus \Pi_{L_t, A} \right\}, \\ \mathcal{N}_{t,s}^{(4)} &:= \left\{ \pi \in \mathcal{P}(0, \mathbb{Z}^d) : \text{supp}(\pi) \subset B_{L_t} \setminus B_\nu(Z_s), \text{supp}(\pi) \cap \Pi_{L_t, A} \neq \emptyset \right\}. \end{aligned} \quad (8.25)$$

Since $\tau_{B_\nu(Z_s)} \wedge \tau_{B_{L_t}^c} > s$ implies $\pi(X_{0,s}) \in \mathcal{N}_t^{(3)} \cup \mathcal{N}_{t,s}^{(4)}$, we may again control the contribution of each set separately. For $\mathcal{N}_t^{(3)}$ this is an easy task since, for all $s \in [at, bt]$,

$$\ln \mathbb{E}_0 \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{\pi_{0,s}(X) \in \mathcal{N}_t^{(3)}\}} \right] \leq s(\widehat{a}_{L_t} - 2A) \quad (8.26)$$

by the definition of $\Pi_{L_t, A}$. For $\mathcal{N}_{t,s}^{(4)}$, we may again apply Proposition 6.1:

Lemma 8.5 *For all $A > 0$ large enough, there exists $\nu_1 \in \mathbb{N}$ and $c > 0$ such that, for all $0 < a \leq b < \infty$, the following holds with probability tending to 1 as $t \rightarrow \infty$. For all $\nu \geq \nu_1$, $s \in [at, bt]$ and $\theta > 0$,*

$$\ln \mathbb{E}_0 \left[e^{\int_0^\theta \xi(X_u) du} \mathbb{1}_{\{\pi(X_{0,\theta}) \in \mathcal{N}_{t,s}^{(4)}\}} \right] \leq \theta \left(\max_{\mathcal{C} \in \mathfrak{C}_{t,s}} \widetilde{\Psi}_{t,\theta,c}(\mathcal{C}) \vee (\widehat{a}_{L_t} - 4d) + o(d_t b_t) \right) \quad (8.27)$$

where $o(d_t b_t)$ does not depend on θ .

Proof. Let $\delta, A_1 > 4d$ and ν_1 be as in Lemma 5.6, and assume that t is large enough for the conclusions of this lemma to hold with $L = L_t$. We may assume $A > A_1$.

We will apply Proposition 6.1 using the islands of \mathfrak{C}_{L_t, A_1} . We may do so as, by Lemma 5.1, $\Pi_{L_t, A} \cap B_{\ln L_t} = \emptyset$ almost surely when t is large, and thus all $\pi \in \mathcal{N}_{t,s}^{(4)}$ exit a box of radius $\ln L_t$. Let $c = c_{A_1}$ be as in (6.6). Since $A > A_1$,

$$\forall \mathcal{C} \in \mathfrak{C}_{L_t, A_1}, \exists \mathcal{C}' \in \mathfrak{C}_{L_t, A} \text{ s.t. } \mathcal{C} \subset \mathcal{C}'. \quad (8.28)$$

Recall the definition of $\lambda_{L_t, A}(\pi)$ in (6.3). For $\pi \in \mathcal{N}_{t,s}^{(4)}$, let $z_\pi := z_{\mathcal{C}_\pi}$ where $\mathcal{C}_\pi \in \mathfrak{C}_{L_t, A_1}$ is such that $\pi \cap \mathcal{C} \cap \Pi_{L_t, A_1} \neq \emptyset$ and $\lambda_{L_t, A_1}(\pi) = \lambda_{\mathcal{C}_\pi}^{(1)}$. Note that $z_\pi = z_{\mathcal{C}'_\pi}$ where $\mathcal{C}'_\pi \in \mathfrak{C}_{L_t, A}$. When t is large enough, $\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}^\delta$ by Lemma 8.3; hence, by Lemma 5.6 and the definition of $\mathcal{N}_{t,s}^{(4)}$,

$$\mathcal{C}_\pi \cap \mathcal{C}_{t,s} = \emptyset. \quad (8.29)$$

From (8.28–8.29), we conclude that

$$\begin{aligned}
& \theta \lambda_{L_t, A_1}(\pi) - (\ln_3(4dL_t) - c) |z_\pi| \\
&= \theta \lambda_{\mathcal{C}_\pi}^{(1)} - (\ln_3(4dL_t) - c) |z_{\mathcal{C}_\pi}| \\
&\leq \theta \sup \left\{ \lambda_{\mathcal{C}'}^{(1)} - (\ln_3^+ |z_{\mathcal{C}'}| - c) + \frac{|z_{\mathcal{C}'}|}{\theta} : \mathcal{C}' \in \mathfrak{C}_{L_t, A} \setminus \{\mathcal{C}_{t, s}\} \right\}.
\end{aligned} \tag{8.30}$$

Choosing now $\gamma_\pi = \lambda_{L_t, A_1}(\pi) \vee (\widehat{a}_{L_t} - 4d) + d_t / \ln_3 t$, (8.27) follows from (6.6), (8.30) and (4.12). \square

Proof of Proposition 4.7. Proposition 4.7 now follows from (8.26) together with Lemma 8.5 applied to $\theta = s$, Lemma 8.3 and the fact that, by Proposition 7.1 and the properties of a_L, \widehat{a}_L and χ , $\Psi_s^{(2)} > (\widehat{a}_{L_t} - 4d) \vee (\widehat{a}_{L_t} - 2A)$ for all $s \in [at, bt]$ with probability tending to 1 as $t \rightarrow \infty$. \square

8.4 Upper bound for the total mass and proof of Theorem 2.6.

We will prove Theorem 2.6 by comparing $\frac{1}{t} \ln U(t)$ to $\Psi_t^{(1)}$ and then applying Proposition 7.1. The last missing ingredient is the following upper bound for $U(t)$. Recall that we assume (5.11–5.12).

Lemma 8.6 (Upper bound for the total mass) *For any $0 < a \leq b < \infty$,*

$$\sup_{s \in [at, bt]} \left\{ \ln U(s) - s \Psi_s^{(1)} \right\} \leq o(td_t b_t) \tag{8.31}$$

holds with probability tending to 1 as $t \rightarrow \infty$.

Proof. Applying Proposition 6.1 to the set of paths

$$\mathcal{N}_t^{(5)} := \left\{ \pi \in \mathcal{P}(0, \mathbb{Z}^d) : \text{supp}(\pi) \subset B_{L_t}, \text{supp}(\pi) \cap \Pi_{L_t, A} \neq \emptyset \right\} \tag{8.32}$$

with $\gamma_\pi := \lambda_{L_t, A}(\pi) \vee (\widehat{a}_{L_t} - A) + d_t / \ln_3 t$ and $z_\pi := z_{\mathcal{C}_\pi}$ where $\mathcal{C}_\pi \in \mathfrak{C}_{L_t, A}$ satisfies $\lambda_{L_t, A}(\pi) = \lambda_{\mathcal{C}_\pi}^{(1)}$, we obtain

$$\begin{aligned}
\ln \mathbb{E}_0 \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}\{\pi_{0, s}(X) \in \mathcal{N}_t^{(5)}\} \right] &\leq s \max_{\mathcal{C} \in \mathfrak{C}_{L_t, A}} \widetilde{\Psi}_{t, s, \mathcal{C}}(\mathcal{C}) + o(td_t b_t) \\
&\leq s \Psi_s^{(1)} + o(td_t b_t)
\end{aligned} \tag{8.33}$$

with probability tending to 1 as $t \rightarrow \infty$ by (2.6), (4.12), (6.6), (8.17) and Lemma 8.3. Then (8.31) follows since, by (8.26) and Propositions 4.2, 4.4 and 4.5, the difference between $\ln U(s)$ and the left-hand side of (8.33) is bounded by $o(1)$ uniformly on $s \in [at, bt]$ with probability tending to 1 as $t \rightarrow \infty$. \square

Proof of Theorem 2.6. Proposition 4.4 and Lemma 8.6 imply that, for any $0 < a \leq b < \infty$,

$$\lim_{t \rightarrow \infty} \sup_{s \in [at, bt]} \frac{\left| \frac{1}{s} \ln U(s) - \Psi_s^{(1)} \right|}{d_t} = 0 \quad \text{in probability,} \tag{8.34}$$

and thus the theorem follows from Proposition 7.1 and $d_{r_t} = d_t(1 + o(1))$. \square

9. LOCALIZATION

In this section we prove Propositions 4.8–4.9, dealing with localization of the solution to the PAM as well as the eigenfunction $\phi_{t,s}^\circ$. The proof of the former proposition is actually quite short:

Proof of Proposition 4.8. By (4.12) and Proposition 4.2, $B_V(Z_s) \subset D_{t,s}^\circ$ for all $s \in [at, bt]$ with probability tending to 1 as $t \rightarrow \infty$, and thus we may apply Lemma 5.15 to $\Lambda = D_{t,s}^\circ$, $z = 0$, $\Gamma = B_V(Z_s)$. \square

We now turn to the proof of Proposition 4.9. The first step is to obtain a spectral gap in the inner domain $D_{t,s}^\circ$, which is a consequence of our choice of the scale h_t in (4.12). Recall the following useful formulas for the second largest eigenvalue of the Anderson Hamiltonian in a subset of \mathbb{Z}^d : For $\Lambda \subset \mathbb{Z}^d$, let $\lambda_\Lambda^{(k)}$, $\phi_\Lambda^{(k)}$ be the eigenvalues and eigenvectors of H_Λ as in Section 5.4. Then we may write

$$\lambda_\Lambda^{(2)} = \sup \left\{ \langle (\Delta + \xi)\phi, \phi \rangle : \phi \in \mathbb{R}^{\mathbb{Z}^d}, \text{supp } \phi \subset \Lambda, \|\phi\|_{\ell^2(\mathbb{Z}^d)} = 1, \phi \perp \phi_\Lambda^{(1)} \right\}. \quad (9.1)$$

A consequence of (9.1) and (5.4) is that, if $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ satisfy $\text{dist}(\Lambda_1, \Lambda_2) \geq 2$, then

$$\lambda_{\Lambda_1}^{(1)} \geq \lambda_{\Lambda_2}^{(1)} \quad \Rightarrow \quad \lambda_{\Lambda_1 \cup \Lambda_2}^{(2)} = \max \left\{ \lambda_{\Lambda_1}^{(2)}, \lambda_{\Lambda_2}^{(1)} \right\}. \quad (9.2)$$

In the following, we assume that the scale sequence R_L obeys (5.11–5.12). Recall the component $\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}$ from Lemma 8.2, and the notation $\mathcal{G}_{t,s} := \{\Psi_s^{(1)} - \Psi_s^{(2)} > e_t d_t\}$. We then have:

Lemma 9.1 (Spectral gap) *For any $A > 0$ large enough and any $0 < a \leq b < \infty$, it holds with probability tending to 1 as $t \rightarrow \infty$ that, for all $s \in [at, bt]$, on $\mathcal{G}_{t,s}$,*

$$\lambda_{\mathcal{C}_{t,s}}^{(1)} > \sup_{\substack{\mathcal{C} \in \mathfrak{C}_{L_t, A} \setminus \{\mathcal{C}_{t,s}\} \\ \text{dist}(\mathcal{C}, D_{t,s}^\circ) \leq (\ln t)^2}} \lambda_{\mathcal{C}}^{(1)} + d_t e_t + o(d_t e_t) \quad (9.3)$$

and

$$\lambda_{D_{t,s}^\circ}^{(1)} > \lambda_{D_{t,s}^\circ}^{(2)} + d_t e_t + o(d_t e_t). \quad (9.4)$$

Proof. Let t be large enough such that the conclusion of Lemma 5.2 is in place. Then, for any $\mathcal{C} \in \mathfrak{C}_{L_t, A} \setminus \{\mathcal{C}_{t,s}\}$, by (8.17) and Lemma 8.3, on $\mathcal{G}_{t,s}$ we have

$$\lambda_{\mathcal{C}_{t,s}}^{(1)} - \lambda_{\mathcal{C}}^{(1)} \geq d_t e_t + o(d_t b_t) - \frac{|z_{\mathcal{C}}| \ln_3^+ |z_{\mathcal{C}}| - |Z_s| \ln_3^+ |Z_s|}{s} \quad (9.5)$$

with probability tending to 1 as $t \rightarrow \infty$. By Proposition 7.1 and Lemma 5.2, we may assume that $|Z_s| \geq t^{1/2}$ and that, for all $\mathcal{C} \in \mathfrak{C}_{L_t, A}$ such that $\text{dist}(\mathcal{C}, D_{t,s}^\circ) \leq (\ln t)^2$, $|z_{\mathcal{C}}| \leq |Z_s|(1 + h_t) + (\ln t)^2 + n_A R_{L_t} < t$. With the help of (2.6), (4.12) and (5.11), we can see that the right-hand side of (9.5) is at least

$$\begin{aligned} d_t e_t + o(d_t b_t) - 2(\ln_3 t) \frac{|Z_s| h_t + (\ln t)^3}{s} &\geq d_t e_t + o(d_t b_t) - 2(\ln_3 t) \frac{r_t g_t h_t + (\ln t)^3}{at} \\ &= d_t e_t + o(d_t e_t), \end{aligned} \quad (9.6)$$

thus proving (9.3).

To show (9.4), we may assume $\lambda_{D_{t,s}^\circ}^{(2)} > \lambda_{D_{t,s}^\circ}^{(1)} - A/4$ since otherwise (9.4) is trivially satisfied. For $A > \chi + 1$ large enough, take $\delta \in (0, 1)$ as in Lemma 5.3. By Lemma 5.2, Proposition 4.2 and Lemma 8.3,

we may assume that $\mathcal{C}_{t,s} \subset D_{t,s}^\circ$ and $\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}^\delta$. Thus, by (9.3), $\lambda_{D_{t,s}^\circ}^{(1)} - A \geq \lambda_{\mathcal{C}_{t,s}}^{(1)} - A \geq \widehat{a}_{L_t} - 2A$. Applying Theorem 2.1 of [BK16] to $D := D_{t,s}^\circ$ together with (5.6) and (9.2), we obtain

$$\lambda_{D_{t,s}^\circ}^{(2)} < \left(\sup_{\mathcal{C} \neq \mathcal{C}_{t,s}: \mathcal{C} \cap D_{t,s}^\circ \neq \emptyset} \lambda_{\mathcal{C}}^{(1)} \right) \vee \lambda_{\mathcal{C}_{t,s}}^{(2)} + 2d(\eta_A)^{R_{L_t}}, \quad \text{where } \eta_A := \left(1 + \frac{A}{4d}\right)^{-1}. \quad (9.7)$$

Now, by Lemma 5.3(i), (9.3) and (9.7),

$$\lambda_{D_{t,s}^\circ}^{(1)} - \lambda_{D_{t,s}^\circ}^{(2)} > \{d_t e_t + o(d_t e_t)\} \wedge \frac{1}{2} \rho \ln 2 - 2d(\eta_A)^{R_{L_t}}, \quad (9.8)$$

which proves (9.4) since $(\eta_A)^{R_{L_t}} = o(d_t e_t)$ by (2.6), (4.12) and (5.11). \square

We are now in position to finish the proof.

Proof of Proposition 4.9(i). We can use the proof of Theorem 1.4 in [BK16] with the following three main modifications:

- 1 In the part of the proof dealing with large distances, Theorem 2.5 of [BK16] is invoked, with the generic component \mathcal{C} appearing in its statement now set to $\mathcal{C}_{t,s}$ (which we may and do assume to be contained in $D_{t,s}^\circ$ by Lemma 8.2). For that we need to show that, with probability tending to 1 as $t \rightarrow \infty$,

$$\|\phi_{t,s}^\circ \mathbb{1}_{\mathcal{C}_{t,s}}\|_2 > \frac{1}{2} \quad \forall s \in [at, bt]. \quad (9.9)$$

The proof of Theorem 2.5 then shows that this inequality characterizes \mathcal{C} .

- 2 Still in the part dealing with large distances, we use (9.4) instead of Lemma 8.1 of [BK16].
- 3 In the second part of the proof dealing with short distances, use (5.19) instead of Lemma 4.8 of [BK16].

With these modifications, the proof goes through in our case.

In order to complete the proof, it thus remains establish (9.9). Let $D := D_{t,s}^\circ \setminus \mathcal{C}_{t,s}$. We first claim that, with probability tending to 1 as $t \rightarrow \infty$,

$$\lambda_D^{(1)} \leq \lambda_{\mathcal{C}_{t,s}}^{(1)} - d_t e_t + o(d_t e_t). \quad (9.10)$$

Indeed, take $A > \chi + \delta$. By Lemma 8.3, we may assume that $\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}^\delta$, and thus we may also assume that $\lambda_D^{(1)} > \widehat{a}_{L_t} - A$ since otherwise (9.10) is satisfied. By Theorem 2.1 of [BK16] and (5.6),

$$\lambda_D^{(1)} \leq \sup\{\lambda_{\mathcal{C}}^{(1)} : \mathcal{C} \in \mathfrak{C}_{L_t, A} \setminus \{\mathcal{C}_{t,s}\}, \mathcal{C} \cap D_{t,s}^\circ \neq \emptyset\} + 2d(\eta_A)^{R_{L_t}} \quad (9.11)$$

where $\eta_A := (1 + A/(4d))^{-1}$, so (9.10) follows by Lemma 9.1, (2.6), (4.12) and (5.11). Now, for $x \in D$, the eigenfunction $\phi_{t,s}^\circ$ satisfies the equation

$$\left(-H_D - \lambda_{D_{t,s}^\circ}^{(1)}\right) \phi_{t,s}^\circ(x) = \sum_{y \in \partial D, |y-x|=1} \phi_{t,s}^\circ(y) \quad (9.12)$$

where H_D is the Anderson operator in D with Dirichlet boundary conditions and $\partial D := \{x \in D_{t,s}^\circ \setminus D : \exists y \in D, |y-x|=1\}$. By Lemma 4.2 of [BK16],

$$\|\phi_{t,s}^\circ \mathbb{1}_{\partial D}\|_{\ell^2(\mathbb{Z}^d)} \leq \{1 + A/(2d)\}^{-2R_{L_t}} \leq (\eta_A)^{R_{L_t}}. \quad (9.13)$$

Using (9.12–9.13) together with the operator norm of the resolvent of $-H_D$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\phi_{t,s}^\circ \mathbb{1}_D\|_{\ell^2(\mathbb{Z}^d)} &\leq \text{dist}(\lambda_{D_{t,s}^\circ}^{(1)}, \text{Spec}(-H_D))^{-1} 2d(\eta_A)^{R_{L_t}} \\ &\leq (\ln t)^2 (\eta_A)^{R_{L_t}} = o(1), \end{aligned} \quad (9.14)$$

where the last line holds by (9.10), $\lambda_{D_{t,s}^\circ}^{(1)} \geq \lambda_{\mathcal{C}_{t,s}}^{(1)}$, (2.6), (4.12) and (5.11). As $\|\phi_{t,s}^\circ\|_{\ell^2(\mathbb{Z}^d)} = 1$, this implies (9.9) as desired. \square

Proof of Proposition 4.9(ii). To prove (4.29), we use (4.28), the representation (5.46) and Lemma 5.7. Let c_1, c_2 as in (4.28). Since $\phi_{t,s}^\circ$ is normalized in $\ell^2(\mathbb{Z}^d)$, there exists $\nu_0 = \nu_0(c_1, c_2)$ such that, for all $\nu \geq \nu_0$,

$$\max_{y \in B_\nu(Z_s)} \phi_{t,s}^\circ(y) \geq \max_{y \in B_{\nu_0}(Z_s)} \phi_{t,s}^\circ(y) \geq \frac{1}{2} |B_{\nu_0}|^{-\frac{1}{2}} =: \varepsilon_0 > 0. \quad (9.15)$$

Fix $\nu \geq \nu_0$ and let A^*, δ and A be as in Lemma 5.7. When t is large, the conclusion of this lemma holds with $L := L_t$. By Lemma 8.3, we may assume that $\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}^\delta$, and thus (5.28) holds for $\mathcal{C}_{t,s}$. On the other hand, by (5.5) we have, with probability tending to 1 as $t \rightarrow \infty$,

$$\lambda_{D_{t,s}^\circ}^{(1)} \leq \max_{x \in D_{t,s}^\circ} \zeta(x) \leq \max_{x \in B_{L_t}} \zeta(x) \leq \widehat{a}_{L_t} + 1, \quad (9.16)$$

by Proposition 4.2 and Lemma 5.1. Since $Z_s = z_{\mathcal{C}_{t,s}}$, for any $z \in B_\nu(Z_s)$,

$$\lambda_{D_{t,s}^\circ}^{(1)} - \zeta(z) \leq 2A^* + 1 =: A'. \quad (9.17)$$

Let $\bar{x} \in B_\nu(Z_s)$ with $\phi_{t,s}^\circ(\bar{x}) = \max_{y \in B_\nu(Z_s)} \phi_{t,s}^\circ(y)$. For $y \in B_\nu(Z_s)$, fix a shortest-distance path π from y to \bar{x} inside $B_\nu(Z_s)$. Then

$$\begin{aligned} \mathbb{E}_y \left[\exp \left\{ \int_0^{\tau_{\bar{x}}} (\zeta(X_s) - \lambda_{D_{t,s}^\circ}^{(1)}) ds \right\} \mathbb{1}_{\{\tau_{\bar{x}} < \tau_{(D_{t,s}^\circ)^c}\}} \right] \\ \geq \mathbb{E}_y \left[\exp \left\{ \int_0^{T_{|\pi|}} (\zeta(X_s) - \lambda_{D_{t,s}^\circ}^{(1)}) ds \right\} \mathbb{1}_{\{\pi^{(|\pi|)}(X) = \pi\}} \right] \\ = \prod_{i=0}^{|\pi|-1} \frac{1}{2d + \lambda_{D_{t,s}^\circ}^{(1)} - \zeta(\pi_i)} \geq (2d + A')^{-2d\nu} =: \varepsilon_1 > 0 \end{aligned} \quad (9.18)$$

by Lemma 6.3 and (9.17). To conclude, invoke (5.46) to write

$$\phi_{t,s}^\circ(y) = \phi_{t,s}^\circ(\bar{x}) \mathbb{E}_y \left[\exp \left\{ \int_0^{\tau_{\bar{x}}} (\zeta(X_s) - \lambda_{D_{t,s}^\circ}^{(1)}) ds \right\} \mathbb{1}_{\{\tau_{\bar{x}} < \tau_{(D_{t,s}^\circ)^c}\}} \right] \geq \varepsilon_0 \varepsilon_1 \quad (9.19)$$

by (9.15) and (9.18). The claim follows with $\varepsilon_\nu := \varepsilon_0 \varepsilon_1 > 0$. \square

10. PATH CONCENTRATION

In this section, we prove Propositions 4.10 and 4.11; these proofs come in Sections 10.1 and 10.2, respectively. We assume throughout that $A > 0$ and $\nu \in \mathbb{N}$ have been fixed at sufficiently large values to

satisfy the hypotheses of all previous results. We also assume that R_L obeys (5.11–5.12). In order to avoid repetition, statements inside proofs are tacitly assumed to hold with probability tending to 1 as $t \rightarrow \infty$.

10.1 Fast approach to the localization center.

Recall the component $\mathcal{C}_t = \mathcal{C}_{t,t} \in \mathfrak{C}_{L_t, A}$ from Lemma 8.2. We first show that, under $Q_t^{(\xi)}$, the random walk exits a box of radius $\ln L_t$ by time $\epsilon_t t$, at least on the event that a neighborhood of the localization center Z_t is hit by time t .

Lemma 10.1 *In probability under the law of ξ ,*

$$\frac{1}{U(t)} \mathbb{E}_0 \left[e^{\int_0^t \xi(X_u) du} \mathbb{1}_{\{\tau_{(D_{t,t}^\circ)^c} > t \geq \tau_{B_V(Z_t)}, \tau_{B_{[\ln L_t]}^c} > \epsilon_t t\}} \right] \xrightarrow[t \rightarrow \infty]{} 0. \quad (10.1)$$

Proof. Note that $\tau_{B_V(Z_t)} > \tau_{B_{[\ln L_t]}^c}$. For $x \in B_{[\ln L_t]}$, we may apply Proposition 6.1 to the set of paths

$$\mathcal{N}_{t,x}^{(6)} := \left\{ \pi \in \mathcal{P}(x, \mathbb{Z}^d) : \text{supp}(\pi) \subset D_{t,t}^\circ, \text{supp}(\pi) \cap B_V(Z_t) \neq \emptyset \right\} \quad (10.2)$$

with $\gamma_\pi = \lambda_{\mathcal{C}_t}^{(1)} + d_t / \ln_3 t$ and $z_\pi \in B_V(Z_t)$ arbitrary, which is justified by Lemma 8.3, Lemma 9.1 and Proposition 4.2. Since $|z_\pi - x| \geq |Z_t| - 2d_V - 2d_{[\ln L_t]}$, we obtain

$$\begin{aligned} \ln \mathbb{E}_x \left[e^{\int_0^{(1-\epsilon_t)t} \xi(X_u) du} \mathbb{1}_{\{\tau_{(D_{t,t}^\circ)^c} > (1-\epsilon_t)t \geq \tau_{B_V(Z_t)}\}} \right] \\ \leq (1-\epsilon_t)t \lambda_{\mathcal{C}_t}^{(1)} - |Z_t| \ln_3 |Z_t| + o(td_t b_t). \end{aligned} \quad (10.3)$$

On the other hand, by Lemma 5.1, a.s. eventually as $t \rightarrow \infty$,

$$\ln \mathbb{E}_0 \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{\tau_{B_{[\ln L_t]}^c} > s\}} \right] \leq s \max_{x \in B_{[\ln L_t]}} \xi(x) \leq s 2\rho \ln_3 t \quad \forall s \geq 0. \quad (10.4)$$

Now use the Markov property at time $\epsilon_t t$ together with (10.3–10.4) and Proposition 4.4 to obtain

$$\begin{aligned} \frac{1}{U(t)} \mathbb{E}_0 \left[e^{\int_0^t \xi(X_u) du} \mathbb{1}_{\{\tau_{(D_{t,t}^\circ)^c} > t \geq \tau_{B_V(Z_t)}, \tau_{B_{[\ln L_t]}^c} > \epsilon_t t\}} \right] \\ \leq \exp \left\{ -\epsilon_t t (\lambda_{\mathcal{C}_t}^{(1)} - 2\rho \ln_3 t) + o(td_t b_t) \right\} \end{aligned} \quad (10.5)$$

which goes to 0 as $t \rightarrow \infty$ by Lemma 8.3, (4.11) and $\epsilon_t \gg (\ln_3 t)^{-1}$. \square

The following result can be seen as an alternative version of Lemma 8.5.

Lemma 10.2 *There exists a constant $c > 0$ such that, with probability tending to 1 as $t \rightarrow \infty$,*

$$\begin{aligned} \ln \mathbb{E}_0 \left[e^{\int_0^{\epsilon_t t} \xi(X_u) du} \mathbb{1}_{\{\tau_{B_V(Z_t)} \wedge \tau_{(D_{t,t}^\circ)^c} > \epsilon_t t \geq \tau_{B_{[\ln L_t]}^c}, X_{\epsilon_t t} = x\}} \right] \\ \leq \epsilon_t t \max_{\mathcal{C} \neq \mathcal{C}_t} \lambda_{\mathcal{C}}^{(1)} - (\ln_3(4dL_t) - c) |x| + o(\epsilon_t t d_t b_t) \end{aligned} \quad (10.6)$$

for all $x \in \mathbb{Z}^d$, and $o(\epsilon_t t d_t b_t)$ in (10.6) does not depend on x .

Proof. Let $A > A_1$ where $A_1 > 4d$ is as in Lemma 5.6, and define the set of paths

$$\mathcal{N}_{t,x}^{(7)} := \left\{ \pi \in \mathcal{P}(0, x) : D_{t,t}^\circ \supset \text{supp}(\pi) \not\subset B_{\lfloor \ln L_t \rfloor}, \text{supp}(\pi) \cap B_\nu(Z_t) = \emptyset \right\}. \quad (10.7)$$

We wish to apply Proposition 6.1 to $\mathcal{N}_{t,x}^{(7)}$ using the islands of \mathfrak{C}_{L_t, A_1} (i.e., with $L = L_t$, $A = A_1$ therein), similarly as in the proof of Lemma 8.5. To that end we take, for all $\pi \in \mathcal{N}_{t,x}^{(7)}$, $\gamma_\pi := \max_{\mathcal{C} \neq \mathcal{C}_t} \lambda_{\mathcal{C}}^{(1)} + d_t / \ln_3 t$ (where the supremum is taken over $\mathcal{C} \in \mathfrak{C}_{L_t, A} \setminus \mathcal{C}_t$), and $z_\pi := x$. Let us check that γ_π satisfies (6.4). Indeed, by Lemma 8.3 and Proposition 7.1, we may assume that $\sup_{\mathcal{C} \neq \mathcal{C}_t} \lambda_{\mathcal{C}}^{(1)} > \widehat{a}_{L_t} - A_1$. Moreover, reasoning as in the arguments leading to (8.28–8.29), we obtain $\lambda_{L_t, A_1}(\pi) \leq \sup_{\mathcal{C} \neq \mathcal{C}_t} \lambda_{\mathcal{C}}^{(1)}$ for all $\pi \in \mathcal{N}_{t,x}^{(7)}$, so (6.4) follows. Inserting our choice of γ_π, z_π in (6.6) and using (4.12), we obtain (10.6) with $c = c_{A_1}$. \square

We can now finish the proof of Proposition 4.10.

Proof of Proposition 4.10. The key point is to show that, for some constant $c > 0$ and uniformly in $x \in \mathbb{Z}^d$,

$$\begin{aligned} & \mathbb{E}_0 \left[e^{\int_0^t \xi(X_u) du} \mathbb{1} \left\{ \tau_{(D_{t,t}^\circ)^c} > t \geq \tau_{B_\nu(Z_t)} > \epsilon_t t \geq \tau_{B_{\lfloor \ln L_t \rfloor}^c}, X_{\epsilon_t t} = x \right\} \right] \\ & \leq \exp \left\{ \epsilon_t t \sup_{\mathcal{C} \neq \mathcal{C}_t} \lambda_{\mathcal{C}}^{(1)} + (1 - \epsilon_t) t \lambda_{\mathcal{C}_t}^{(1)} - (\ln_3(4dL_t) - c) |Z_t| + o(\epsilon_t t d_t b_t) \right\}. \end{aligned} \quad (10.8)$$

Indeed, assuming (10.8), Propositions 4.2 and 4.4 allow us to write

$$\begin{aligned} & \frac{1}{U(t)} \mathbb{E}_0 \left[e^{\int_0^t \xi(X_u) du} \mathbb{1}_{\{\tau_{(D_{t,t}^\circ)^c} > t \geq \tau_{B_\nu(Z_t)} > \epsilon_t t \geq \tau_{B_{\lfloor \ln L_t \rfloor}^c}\}} \right] \\ & \leq \frac{|D_{t,t}^\circ|}{U(t)} \sup_{x \in \mathbb{Z}^d} \mathbb{E}_0 \left[e^{\int_0^t \xi(X_u) du} \mathbb{1}_{\{\tau_{(D_{t,t}^\circ)^c} > t \geq \tau_{B_\nu(Z_t)} > \epsilon_t t \geq \tau_{B_{\lfloor \ln L_t \rfloor}^c}, X_{\epsilon_t t} = x\}} \right] \\ & \leq \exp \left\{ -\epsilon_t t (\lambda_{\mathcal{C}_t}^{(1)} - \max_{\mathcal{C} \neq \mathcal{C}_t} \lambda_{\mathcal{C}}^{(1)}) + o(\epsilon_t t d_t b_t) \right\} \xrightarrow[t \rightarrow \infty]{} 0 \quad \text{in probability} \end{aligned} \quad (10.9)$$

by Lemma 9.1 and (4.12). This and Lemma 10.1 yield (4.37).

In order to prove (10.8), suppose first that $\text{dist}(x, B_\nu(Z_t)) \geq \ln L_t$. Then we may apply Proposition 6.1 to the set of paths

$$\mathcal{N}_{t,x}^{(8)} := \left\{ \pi \in \mathcal{P}(x, \mathbb{Z}^d) : \text{supp}(\pi) \subset D_{t,t}^\circ, \text{supp}(\pi) \cap B_\nu(Z_t) \neq \emptyset \right\} \quad (10.10)$$

with $\gamma_\pi = \lambda_{\mathcal{C}_t}^{(1)} + d_t / \ln_3 t$ and $z_\pi \in B_\nu(Z_t) \cap \text{supp}(\pi)$ arbitrary, obtaining

$$\begin{aligned} & \ln \mathbb{E}_x \left[e^{\int_0^{(1-\epsilon_t)t} \xi(X_u) du} \mathbb{1}_{\{\tau_{(D_{t,t}^\circ)^c} > (1-\epsilon_t)t \geq \tau_{B_\nu(Z_t)}\}} \right] \\ & \leq (1 - \epsilon_t) t \lambda_{\mathcal{C}_t}^{(1)} - (\ln_3(4dL_t) - c_A) |Z_t - x| + o(\epsilon_t t d_t b_t) \end{aligned} \quad (10.11)$$

since $|z_\pi - x| \geq |Z_t - x| - 2d\nu$. Noting that both (10.11) and (10.6) remain true if we substitute c and c_A by $c \vee c_A$, (10.8) follows by applying the Markov property at time $\epsilon_t t$ and then using (10.11), Lemma 10.2 and the triangle inequality.

If instead $\text{dist}(x, B_\nu(Z_t)) < \ln L_t$, we may bound

$$\begin{aligned} \mathbb{E}_x \left[e^{\int_0^{(1-\epsilon_t)t} \xi(X_u) du} \mathbb{1}_{\{\tau_{(D_{t,t}^\circ)^c} > (1-\epsilon_t)t \geq \tau_{B_\nu(Z_t)}\}} \right] &\leq e^{(1-\epsilon_t)t \lambda_{D_{t,t}^\circ}^{(1)} |D_{t,t}^\circ|^{\frac{3}{2}}} \\ &\leq \exp \left\{ (1-\epsilon_t)t \lambda_{D_{t,t}^\circ}^{(1)} + o(\epsilon_t t d_t b_t) \right\} \end{aligned} \quad (10.12)$$

by Lemma 5.12. By Theorem 2.1 of [BK16] together with Lemma 9.1 and (5.6),

$$\lambda_{D_{t,t}^\circ}^{(1)} < \lambda_{\mathcal{C}_t}^{(1)} + o(\epsilon_t t d_t b_t). \quad (10.13)$$

Since $|x| > |Z_t| - 2d\nu - \ln L_t$, (10.8) again follows using the Markov property together with (10.12–10.13) and Lemma 10.2. \square

10.2 Path concentration.

In this section, we address the principal ingredient needed for the proof of path localization, culminating in the proof of Proposition 4.11.

For $L \in \mathbb{N}$, we define $\tilde{\epsilon}_L := \inf\{\epsilon_s : s > 0, L_s = L\}$ and put

$$\tilde{R}_L := \left\lfloor \frac{\tilde{\epsilon}_L \ln L}{2(n_A + 1)} \right\rfloor. \quad (10.14)$$

Note that \tilde{R}_L satisfies (5.11) but *not* (5.12). Furthermore, $(n_A + 1)\tilde{R}_L \leq \frac{1}{2}\epsilon_t \ln t$.

Let $\tilde{\mathfrak{C}}_{L,A}$ be the analogue of $\mathfrak{C}_{L,A}$ using the radius \tilde{R}_L , and let $\tilde{\mathcal{C}}_t \in \tilde{\mathfrak{C}}_{L,A}$ such that $Z_t \in \tilde{\mathcal{C}}_t \cap \Pi_{L_t,A}$. This is well-defined with probability tending to 1 as $t \rightarrow \infty$ since, by (5.5) and Proposition 7.1, we may assume that $Z_t \in \Pi_{L_t,A}$. Note that, without assuming (5.12), we cannot use Lemma 5.8; in particular, it may be that $Z_t \neq z_{\tilde{\mathcal{C}}_t}$. Nonetheless, we still have the following.

Lemma 10.3 *With probability tending to 1 as $t \rightarrow \infty$,*

$$\lambda_{\tilde{\mathcal{C}}_t}^{(1)} > \sup_{\tilde{\mathcal{C}} \in \tilde{\mathfrak{C}}_{L_t,A} \setminus \{\tilde{\mathcal{C}}_t\} : \tilde{\mathcal{C}} \cap D_{t,t}^\circ \neq \emptyset} \lambda_{\tilde{\mathcal{C}}}^{(1)} + d_t \epsilon_t + o(d_t \epsilon_t). \quad (10.15)$$

In particular, $\lambda_{\tilde{\mathcal{C}}_t}^{(1)} = \max\{\lambda_{\tilde{\mathcal{C}}}^{(1)} : \tilde{\mathcal{C}} \in \tilde{\mathfrak{C}}_{L_t,A}, \tilde{\mathcal{C}} \cap D_{t,t}^\circ \neq \emptyset\}$.

Proof. Fix $R_L \leq \tilde{R}_L$ satisfying (5.11–5.12) and let $\mathcal{C}_t = \mathcal{C}_{t,t} \in \mathfrak{C}_{L_t,A}$ as in Lemma 8.2. Then $\mathcal{C}_t \subset \tilde{\mathcal{C}}_t$ and thus $\lambda_{\tilde{\mathcal{C}}_t}^{(1)} \geq \lambda_{\mathcal{C}_t}^{(1)}$. Now fix $\tilde{\mathcal{C}} \in \tilde{\mathfrak{C}}_{L_t,A} \setminus \{\tilde{\mathcal{C}}_t\}$, $\tilde{\mathcal{C}} \cap D_{t,t}^\circ \neq \emptyset$. Applying Theorem 2.1 of [BK16] to $D := \tilde{\mathcal{C}}$ and then (5.6) and Lemma 5.2, we get

$$\lambda_{\tilde{\mathcal{C}}}^{(1)} \leq \sup_{\mathcal{C} \in \mathfrak{C}_{L_t,A} : \mathcal{C} \cap \tilde{\mathcal{C}} \neq \emptyset} \lambda_{\mathcal{C}}^{(1)} + 2d(\eta_A)^{R_{L_t}} \leq \sup_{\substack{\mathcal{C} \in \mathfrak{C}_{L_t,A} \setminus \{\mathcal{C}_t\} : \\ \text{dist}(\mathcal{C}, D_{t,t}^\circ) \leq (\ln t)^2}} \lambda_{\mathcal{C}}^{(1)} + 2d(\eta_A)^{R_{L_t}} \quad (10.16)$$

where $\eta_A := (1 + A/(4d))^{-1}$. Hence (10.15) follows from Lemma 9.1. \square

We can now give the proof of Proposition 4.11.

Proof of Proposition 4.11. Let $n_A \in \mathbb{N}$ be as in Lemma 5.2. Fix $x \in B_\nu(Z_t)$ and define the set of paths

$$\mathcal{N}_{t,x}^{(9)} := \left\{ \pi \in \mathcal{P}(x, \mathbb{Z}^d) : \text{supp}(\pi) \subset D_{t,t}^\circ, \max_{1 \leq \ell \leq |\pi|} |\pi_\ell - x| > (n_A + 1) \tilde{R}_{L_t} \right\}. \quad (10.17)$$

Let $\vartheta_L := 3(n_A + 1) \lfloor \tilde{\epsilon}_L^{-1} \rfloor$ and note that

$$\vartheta_L \ll \ln_3 L \text{ as } L \rightarrow \infty \quad \text{and} \quad \vartheta_L \tilde{R}_L \geq \ln L \text{ for all } L \text{ large enough.} \quad (10.18)$$

Choosing $\gamma_\pi := \lambda_{\tilde{\mathcal{C}}_t}^{(1)} + 2/t$, by Lemma 10.3, Proposition 7.1 and (10.18), we may apply Proposition 6.2 (using the islands of $\mathfrak{C}_{L_t, A}$) to $\mathcal{N}_{t,x}^{(9)}$, obtaining, for all $0 \leq s \leq t$,

$$\mathbb{E}_x \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{\tau_{(D_{t,t}^\circ)^c} > s, \sup_{0 \leq u \leq s} |X_u - x| > \frac{1}{2} \epsilon_t \ln t\}} \right] \leq e^2 \exp \left\{ s \lambda_{\tilde{\mathcal{C}}_t}^{(1)} - \frac{1}{2} \tilde{R}_{L_t} \ln_3 L_t \right\} \quad (10.19)$$

since $\frac{1}{2} \epsilon_t \ln t \geq (n_A + 1) \tilde{R}_{L_t}$. Now note that, by Lemma 5.12 and Proposition 4.9(ii),

$$\mathbb{E}_x \left[e^{\int_0^s \xi(X_u) du} \right] \geq \mathbb{E}_x \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{\tau_{D_{t,t}^\circ} > s, X_s = x\}} \right] \geq \epsilon_\nu^2 \exp \left\{ s \lambda_{D_{t,t}^\circ}^{(1)} \right\}. \quad (10.20)$$

Then (4.38) follows from (10.19–10.20) and $\tilde{\mathcal{C}}_t \subset D_{t,t}^\circ$. \square

11. LOCAL PROFILES

In this section we prove Propositions 4.12 and 4.13 dealing with the local “shapes” of the solution to the PAM and of the potential configuration in the vicinity of the localization center. In the following we will always assume that $A > 0$ and $\nu \in \mathbb{N}$ have been taken large enough so as to satisfy the hypotheses of all previous results. We start with Proposition 4.13.

Proof of Proposition 4.13. Fix $0 < a \leq b < \infty$. Let $d(\cdot, \cdot)$ be a metric under which $[-\infty, 0]^{\mathbb{Z}^d}$ is compact and has the topology of pointwise convergence. Since for each $R \in \mathbb{N}$ the principal Dirichlet eigenvalue of $\Delta + V_\rho$ in B_R is simple, there exists $\epsilon_R > 0$ such that

$$d(V, V_\rho) < \epsilon_R \Rightarrow \sup_{x \in B_R} |V(x) - V_\rho(x)| \vee \left\| v_V^R - v_\rho^R \right\|_{\ell^1} < \frac{1}{R}, \quad (11.1)$$

where v_V^R , resp., v_ρ^R are the principal Dirichlet eigenfunctions of $\Delta + V$, resp., $\Delta + V_\rho$ in B_R , both normalised in ℓ^1 . Under Assumption 2.9, Lemma 3.2(i) in [GKM07] shows that the quantity

$$\mathcal{F}(\epsilon) := -\chi - \sup \left\{ \lambda^{(1)}(V) : V \in [-\infty, 0]^{\mathbb{Z}^d}, \mathcal{L}(V) \leq 1, 0 \in \text{argmax}(V), d(V, V_\rho) \geq \epsilon \right\} \quad (11.2)$$

is strictly positive for $\epsilon > 0$. By Lemmas 5.1, 5.5 and 8.3, Proposition 4.2 and the properties of a_L, \hat{a}_L , there exists a deterministic non-increasing function $\delta_t > 0$ such that $\delta_t \rightarrow 0$ as $t \rightarrow \infty$ and the following holds with probability tending to 1 as $t \rightarrow \infty$:

$$\max_{x \in B_{L_t}} \tilde{\zeta}(x) < \hat{a}_{L_t} + \delta_t, \quad \inf_{s \in [at, bt]} \lambda_{\mathcal{C}_{t,s}}^{(1)} > \hat{a}_{L_t} - \chi - \delta_t \quad (11.3)$$

and

$$\sup_{s \in [at, bt]} \mathcal{L}_{\mathcal{C}_{t,s}}(\tilde{\zeta} - \hat{a}_{L_t} - \delta_t) \leq 1. \quad (11.4)$$

Letting $t_R > 0$ with $t_R \rightarrow \infty$ be such that $\delta_t < \frac{1}{2}\mathcal{F}(\varepsilon_R)$ for all $t \geq t_R$, we define

$$\mu_t := \inf\{R \in \mathbb{N} : t_R \leq t < t_{R+1}\}. \quad (11.5)$$

Note that $\mu_t \rightarrow \infty$, and we may suppose that $\mu_t \ll (\ln t)^\kappa$ by making t_R grow sufficiently fast with R . Then, defining

$$V^*(x) := \begin{cases} \xi(x + Z_s) - \widehat{a}_{L_t} - \delta_t & \text{if } x + Z_s \in \mathcal{C}_{t,s}, \\ -\infty & \text{otherwise,} \end{cases} \quad (11.6)$$

we have $V^* \in [-\infty, 0]^{\mathbb{Z}^d}$, $\mathcal{L}(V^*) = \mathcal{L}_{\mathcal{C}_{t,s}}(\xi - \widehat{a}_{L_t} - \delta_t) \leq 1$ and $0 \in \operatorname{argmax}(V^*)$. Furthermore, $\lambda^{(1)}(V^*) = \lambda_{\mathcal{C}_{t,s}}^{(1)} - \widehat{a}_{L_t} - \delta_t > -\chi - \mathcal{F}(\varepsilon_{\mu_t})$. Since $v_{V^*}^{\mu_t}(\cdot) = \phi_{t,s}^\bullet(\cdot + Z_s)$,

$$\sup_{x \in \mu_t} |\xi(x + Z_s) - \widehat{a}_{L_t} - V_\rho(x)| \vee \|\phi_{t,s}^\bullet(Z_s + \cdot) - v_\rho^{\mu_t}(\cdot)\|_{\ell^1} < \frac{1}{\mu_t} + \delta_t \quad (11.7)$$

by (11.1) and the definition of $\mathcal{F}(\varepsilon)$. To conclude, we observe that $\widehat{a}_{L_t} = \widehat{a}_t + o(1)$ and that, by Lemma 3.3(iii) of [GKM07], $\lim_{t \rightarrow \infty} \|v_\rho^{\mu_t} - v_\rho\|_{\ell^1} = 0$. \square

Next we prove Proposition 4.12 by adapting the strategy of Section 8.2 of [GKM07]. The proof is based on two lemmas whose proofs will be postponed to subsequent subsections. Fix $\mu_t \in \mathbb{N}$, $1 \ll \mu_t \ll R_t$, which is enough by (5.11). We will again decompose the solution with the help of the Feynman-Kac representation, which states that, for a function $f : \mathbb{Z}^d \rightarrow [0, \infty)$, $f \not\equiv 0$, the function

$$(x, t) \mapsto \mathbb{E}_x \left[e^{\int_0^t \xi(X_s) ds} f(X_t) \right] \quad (11.8)$$

is the unique positive solution of the equation (1.1) with initial condition f .

Fix an auxiliary function $t \mapsto T_t \in \mathbb{N}$ such that $\sqrt{\mu_t} \ll T_t \ll \mu_t$. For notational convenience we set $B_{t,s} := B_{\mu_t}(Z_s)$. Using (11.8), we may write $u(x, s) = u^{(1)}(x, s; t) + u^{(2)}(x, s; t)$ where

$$u^{(1)}(x, s; t) := \mathbb{E}_x \left[e^{\int_0^s \xi(X_u) du} \mathbb{1}_{\{X_s=0, \tau_{B_{t,s}^c} > T_t\}} \right] \quad (11.9)$$

and $u^{(2)}$ is defined by replacing $\tau_{B_{t,s}^c} > T_t$ by the complementary inequality. The first lemma shows that the contribution of $u^{(2)}$ is negligible.

Lemma 11.1 *For any $0 < a \leq b < \infty$,*

$$\lim_{t \rightarrow \infty} \sup_{s \in [at, bt]} \mathbb{1}_{\mathcal{G}_{t,s}} \sum_{x \in \mathbb{Z}^d} \frac{u^{(2)}(x, s; t)}{U(s)} = 0 \quad \text{in probability.} \quad (11.10)$$

Finally, the second lemma controls the distance between $u^{(1)}$ and $\phi_{t,s}^\bullet$.

Lemma 11.2 *For any $0 < a \leq b < \infty$,*

$$\lim_{t \rightarrow \infty} \sup_{s \in [at, bt]} \mathbb{1}_{\mathcal{G}_{t,s}} \sum_{x \in \mathbb{Z}^d} \left| \frac{u^{(1)}(x, s; t)}{U(s)} - \phi_{t,s}^\bullet(x) \right| = 0 \quad \text{in probability.} \quad (11.11)$$

Proof of Proposition 4.12. Follows directly from Lemmas 11.1–11.2. \square

The remainder of this section is devoted to the proofs of Lemmas 11.1–11.2. In order to avoid repetition, we fix here $0 < a \leq b < \infty$, and all statements made in what follows are assumed to hold for all $s \in [at, bt]$ with probability tending to 1 as $t \rightarrow \infty$.

11.1 Contribution of $u^{(2)}$.

Proof of Lemma 11.1. Recall that $B_{t,s} = B_{\mu_t}(Z_s)$ and note that, since $u^{(2)}(x, s; t) \leq u(x, s)$, (4.30) implies

$$\lim_{t \rightarrow \infty} \sup_{s \in [at, bt]} \mathbb{1}_{\mathcal{G}_{t,s}} \sum_{x \notin B_{t,s}} \frac{u^{(2)}(x, s; t)}{U(s)} = 0 \quad \text{in probability.} \quad (11.12)$$

We thus only need to consider the sum over $x \in B_{t,s}$. Using the strong Markov property for X , we may write

$$u^{(2)}(x, s; t) = \mathbb{E}_x \left[\exp \left\{ \int_0^{\tau_{B_{t,s}^c}} \zeta(X_\theta) d\theta \right\} u(X_{\tau_{B_{t,s}^c}}, s - \tau_{B_{t,s}^c}) \mathbb{1}_{\{X_s=0, \tau_{B_{t,s}^c} \leq T_t\}} \right]. \quad (11.13)$$

Consider the event

$$\mathcal{R}_{t,s,\theta}^v := \left\{ \tau_{(D_{t,s}^\circ)^c} > \theta \geq \tau_{B_v(Z_s)} \right\}, \quad (11.14)$$

introduce the functions

$$u_1(x, \theta) := \mathbb{E}_x \left[e^{\int_0^\theta \zeta(X_u) du} \mathbb{1}_{\{X_\theta=0\} \cap \mathcal{R}_{t,s,\theta}^v} \right] \quad (11.15)$$

and

$$u_2(x, \theta) := \mathbb{E}_x \left[e^{\int_0^\theta \zeta(X_u) du} \mathbb{1}_{\{X_\theta=0\} \cap (\mathcal{R}_{t,s,\theta}^v)^c} \right] \quad (11.16)$$

and define $u_i^{(2)}(x, s; t)$, $i = 1, 2$, by substituting u_i for u in (11.13). Then, clearly, we have $u^{(2)}(x, s; t) = u_1^{(2)}(x, s; t) + u_2^{(2)}(x, s; t)$. Our strategy is to separately estimate the contribution of $u_1^{(2)}$ and $u_2^{(2)}$.

Starting with $u_2^{(2)}$, we claim that, for all $\theta < s$,

$$u_2(x, s - \theta) \leq e^{\theta(2d - \bar{\zeta}(0))} u_2(x, s). \quad (11.17)$$

Indeed, (11.17) can be obtained from (11.16) with $\theta = s$ by intersecting with the event $(R_{t,s,s-\theta}^v)^c \cap \{X_u = 0 \forall u \in [s - \theta, s]\}$ and applying the Markov property. The inequality (11.17) in turn shows

$$\sum_{x \in B_{t,s}} \frac{u_2^{(2)}(x, s; t)}{U(s)} \leq |B_{\mu_t}| e^{T_t(2d + |\bar{\zeta}(0)| + 2\rho \ln_2 t)} \sum_{x \in \mathbb{Z}^d} \frac{u_2(x, s)}{U(s)}, \quad (11.18)$$

where we bound $\bar{\zeta}(X_\theta) \leq 2\rho \ln_2 t$ by Lemma 5.1 noting that $B_{t,s} \subset B_t$. By (4.31–4.32) (and invariance under time-reversal of the law of X), on $\mathcal{G}_{t,s}$ we can bound (11.18) by

$$|B_{\mu_t}| \exp \left\{ -t(\ln t)^{-2} + T_t(2d + |\bar{\zeta}(0)| + 2\rho \ln_2 t) \right\}, \quad (11.19)$$

which tends to 0 as $t \rightarrow \infty$.

Thus we are left with controlling $u_1^{(2)}$. To this end, recall the setup of Lemma 5.15 and set $\Lambda := D_{t,s}^\circ$ and $\Gamma := B_v(Z_s)$. Applying (5.50) with t substituted by $t - s$ and then (5.49) to u_1 , we obtain, on $\mathcal{G}_{t,s}$,

$$u_1(x, s - \theta) \leq e^{-\theta \lambda_{t,s}^\circ} \left(\inf_{y \in \Gamma} \phi_{t,s}^\circ(y) \right)^{-5} \phi_{t,s}^\circ(x) \sum_{y \in \Gamma} u_1(y, s) \leq e^{-\theta \lambda_{t,s}^\circ} \varepsilon_v^{-5} \phi_{t,s}^\circ(x) U(s), \quad (11.20)$$

where $\lambda_{t,s}^\circ$ is the largest Dirichlet eigenvalue of $H_{D_{t,s}^\circ}$ and ε_V is as in Proposition 4.9(ii). Inserting (11.20) in the definition of $u_1^{(2)}$, we obtain, for some constant $c_0 > 0$,

$$\sum_{x \in B_{t,s}} \frac{u_1^{(2)}(x, s; t)}{U(s)} \leq c_0 \mu_t^d \sup_{x \notin B_{t,s}} \phi_{t,s}^\circ(x) \sup_{x \in B_{t,s}} \mathbb{E}_x \left[e^{\int_0^{\tau_{B_{t,s}}^c} (\xi(X_u) - \lambda_{t,s}^\circ) du} \mathbb{1}_{\{\tau_{B_{t,s}}^c \leq T_t\}} \right]. \quad (11.21)$$

Since $B_{t,s} \subset D_{t,s}^\circ$, (5.5) shows that $\max_{x \in B_{t,s}} \xi(x) - \lambda_{t,s}^\circ \leq 2d$. Applying Proposition 4.9(i), on $\mathcal{G}_{t,s}$ we may further bound (11.21) by

$$c_0 c_1 \mu_t^d e^{-c_2 \mu_t + 2d T_t}. \quad (11.22)$$

Since (11.22) tends to 0 as $t \rightarrow \infty$, the proof of Lemma 11.1 is concluded. \square

11.2 Contribution of $u^{(1)}$.

Let $\lambda_{t,s}^{(k)}, \phi_{t,s}^{(k)}$ be the ordered Dirichlet eigenvalues and respective orthonormal eigenfunctions of the Anderson operator in $B_{t,s}$. We extend the eigenfunctions to be 0 outside of $B_{t,s} = B_{\mu_t}(Z_s)$. In our previous notation, $\lambda_{t,s}^\bullet = \lambda_{t,s}^{(1)}$ and $\phi_{t,s}^\bullet = \phi_{t,s}^{(1)} / \|\phi_{t,s}^{(1)}\|_{\ell^1(\mathbb{Z}^d)}$. We start with the following important fact.

Lemma 11.3 *For any $0 < a \leq b < \infty$, with probability tending to 1 as $t \rightarrow \infty$,*

$$\inf_{s \in [at, bt]} \lambda_{t,s}^{(1)} > \widehat{a}_{L_t} - \chi + o(1), \quad (11.23)$$

and

$$\inf_{s \in [at, bt]} \lambda_{t,s}^{(1)} - \lambda_{t,s}^{(2)} \geq \frac{1}{3} \rho \ln 2. \quad (11.24)$$

Proof. By Proposition 4.2, Lemma 8.3 and $\widehat{a}_{L_t} = \widehat{a}_{r_t} + o(1)$, may assume that $\lambda_{\mathcal{C}_{t,s}}^{(1)} > \widehat{a}_{L_t} - \chi + o(1)$. In particular, $\mathcal{C}_{t,s} \in \mathfrak{C}_{L_t, A}^\delta$ for any $\delta \in (0, 1)$, so by Lemma 5.3(i),

$$\lambda_{\mathcal{C}_{t,s}}^{(1)} - \lambda_{\mathcal{C}_{t,s}}^{(2)} > \frac{1}{2} \rho \ln 2. \quad (11.25)$$

Since $B_{t,s} \subset \mathcal{C}_{t,s}$, $\lambda_{t,s}^{(2)} \leq \lambda_{\mathcal{C}_{t,s}}^{(2)}$ by the minimax formula (see e.g. the proof of Lemma 4.3 in [BK16]). Furthermore, by Lemma 5.6 together with Theorem 2.1 of [BK16] (note that $\lambda_{\mathcal{C}_{t,s}}^{(1)} - A_1 > \widehat{a}_{L_t} - 2A_1$),

$$\lambda_{t,s}^{(1)} > \lambda_{\mathcal{C}_{t,s}}^{(1)} - 2d \left(1 + \frac{A_1}{4d} \right)^{1-2(\mu_t - \nu_1)}. \quad (11.26)$$

Now (11.23–11.24) follows from (11.25–11.26). \square

Lemma 11.3 will allow us to prove the following localization property for $\phi_{t,s}^{(1)}$.

Lemma 11.4 *There exist $c_1, c_2 \in (0, \infty)$ and, for fixed $R \in \mathbb{N}$, a constant $\varepsilon_R^\bullet > 0$ such that, for all $0 < a \leq b < \infty$, the following holds with probability tending to 1 as $t \rightarrow \infty$: For all $s \in [at, bt]$,*

$$\phi_{t,s}^{(1)}(x) \leq c_1 e^{-c_2 |x - Z_s|} \quad \forall x \in \mathbb{Z}^d, \quad (11.27)$$

and

$$\phi_{t,s}^{(1)}(y) \geq \varepsilon_R^\bullet \quad \forall y \in B_R(Z_s). \quad (11.28)$$

Proof. Fix A_1, ν_1 as in Lemma 5.6 and take $r > \nu_1$. By Lemma 4.2 of [BK16] and (11.23),

$$\sum_{x \in B_{t,s} \setminus B_r(Z_s)} |\phi_{t,s}^{(1)}(x)|^2 \leq \left(1 + \frac{A_1}{2d}\right)^{-2(r-\nu_1)}, \quad (11.29)$$

proving (11.27). The bound (11.28) is obtained using (11.27) and Lemma 5.7 as in the proof of Proposition 4.9(ii). \square

We can now finish the proof of Lemma 11.2.

Proof of Lemma 11.2. Using the Markov property, we can write

$$u^{(1)}(x, s; t) = \mathbb{E}_x \left[e^{\int_0^{T_t} \xi(X_u) du} u(X_{T_t}, s - T_t) \mathbb{1}_{\{\tau_{B_{t,s}^c} > T_t\}} \right]. \quad (11.30)$$

Since

$$(x, T) \mapsto \mathbb{E}_x \left[e^{\int_0^T \xi(X_u) du} u(X_T, s - T) \mathbb{1}_{\{\tau_{B_{t,s}^c} > T\}} \right] \quad (11.31)$$

solves the parabolic equation (5.41) with $\Lambda := B_{t,s}$ and initial condition $u(\cdot, s - T_t)$, an eigenvalue expansion as (5.43) gives

$$u^{(1)}(x, s; t) = \sum_{k=1}^{|B_{t,s}|} e^{T_t \lambda_{t,s}^{(k)}} \phi_{t,s}^{(k)}(x) \langle \phi_{t,s}^{(k)}, u(\cdot, s - T_t) \rangle, \quad (11.32)$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product in $\ell^2(\mathbb{Z}^d)$.

Set $U^{(1)}(s; t) := \sum_{x \in \mathbb{Z}^d} u^{(1)}(x, s; t)$ and note that, by Lemma 11.1,

$$\lim_{t \rightarrow \infty} \sup_{s \in [at, bt]} \mathbb{1}_{\mathcal{G}_{t,s}} \left| \frac{u^{(1)}(s; t)}{U(s)} - 1 \right| = 0 \quad \text{in probability.} \quad (11.33)$$

Hence it is enough to prove (11.11) with $U(s)$ substituted by $U^{(1)}(s; t)$. Using (11.32) we may write

$$\frac{U^{(1)}(x, s; t)}{U^{(1)}(s; t)} = \frac{\phi_{t,s}^{(1)}(x) + E_{t,s}(x)}{\|\phi_{t,s}^{(1)}\|_{\ell^1(\mathbb{Z}^d)} + \sum_{x \in \mathbb{Z}^d} E_{t,s}(x)} \quad (11.34)$$

where

$$E_{t,s}(x) := \sum_{k=2}^{|B_{t,s}|} e^{-T_t(\lambda_{t,s}^{(1)} - \lambda_{t,s}^{(k)})} \phi_{t,s}^{(k)}(x) \frac{\langle \phi_{t,s}^{(k)}, u(\cdot, s - T_t) \rangle}{\langle \phi_{t,s}^{(1)}, u(\cdot, s - T_t) \rangle}. \quad (11.35)$$

Noting that $\|\phi_{t,s}^{(1)}\|_{\ell^1(\mathbb{Z}^d)} \geq \|\phi_{t,s}^{(1)}\|_{\ell^2(\mathbb{Z}^d)} = 1$, we obtain

$$\left\| \frac{u^{(1)}(\cdot, s; t)}{U^{(1)}(s; t)} - \phi_{t,s}^{(1)}(\cdot) \right\|_{\ell^1(\mathbb{Z}^d)} \leq \|E_{t,s}\|_{\ell^1(\mathbb{Z}^d)} \left(1 + \frac{1}{U^{(1)}(s; t)} + \frac{\|E_{t,s}\|_{\ell^1(\mathbb{Z}^d)}}{U^{(1)}(s; t)} \right) \quad (11.36)$$

and our problem is reduced to showing that

$$\lim_{t \rightarrow \infty} \sup_{s \in [at, bt]} \mathbb{1}_{\mathcal{G}_{t,s}} \|E_{t,s}\|_{\ell^1(\mathbb{Z}^d)} = 0 \quad \text{in probability.} \quad (11.37)$$

To this end, we first use the Cauchy-Schwarz inequality and Parseval's identity to obtain

$$\begin{aligned} |E_{t,s}(x)| &\leq \frac{e^{-T_t(\lambda_{t,s}^{(1)} - \lambda_{t,s}^{(2)})}}{\langle \phi_{t,s}^{(1)}, u(\cdot, s - T_t) \rangle} \left(\sum_{k=1}^{|B_{t,s}|} \langle \phi_{t,s}^{(k)}, \mathbb{1}_x \rangle^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{|B_{t,s}|} \langle \phi_{t,s}^{(k)}, u(\cdot, s - T_t) \rangle^2 \right)^{\frac{1}{2}} \\ &= e^{-T_t(\lambda_{t,s}^{(1)} - \lambda_{t,s}^{(2)})} \frac{\|u(\cdot, s - T_t)\|_{\ell^2(\mathbb{Z}^d)}}{\langle \phi_{t,s}^{(1)}, u(\cdot, s - T_t) \rangle} \mathbb{1}_{B_{t,s}}(x). \end{aligned} \quad (11.38)$$

Once we show that, for some positive constants c_0, c_1 , on $\mathcal{G}_{t,s}$

$$\|u(\cdot, s - T_t)\|_{\ell^2(\mathbb{Z}^d)} \leq c_0 e^{-T_t \lambda_{t,s}^\bullet} U(s), \quad (11.39)$$

and

$$\langle \phi_{t,s}^{(1)}, u(\cdot, s - T_t) \rangle \geq c_1 e^{-T_t \lambda_{t,s}^\bullet} U(s), \quad (11.40)$$

then using (11.38–11.40) and (11.24) we will be able to bound

$$\sup_{s \in [at, bt]} \mathbb{1}_{\mathcal{G}_{t,s}} \|E_{t,s}\|_{\ell^1(\mathbb{Z}^d)} \leq \frac{c_0}{c_1} (3\mu_t)^d e^{-\frac{\rho \ln 2}{3} T_t} \quad (11.41)$$

which tends to 0 as $t \rightarrow \infty$ by our choice of T_t . Thus it only remains to prove (11.39–11.40).

We start with (11.39). By the triangle inequality,

$$\|u(\cdot, s - T_t)\|_{\ell^2(\mathbb{Z}^d)} \leq \|u_1(\cdot, s - T_t)\|_{\ell^2(\mathbb{Z}^d)} + \|u_2(\cdot, s - T_t)\|_{\ell^2(\mathbb{Z}^d)} \quad (11.42)$$

where u_1, u_2 are defined as in (11.15–11.16). Reasoning as in (11.17–11.19), we can see that, on $\mathcal{G}_{t,s}$,

$$\begin{aligned} \frac{\|u_2(\cdot, s - T_t)\|_{\ell^2(\mathbb{Z}^d)}}{U(s)} &\leq \frac{\|u_2(\cdot, s - T_t)\|_{\ell^1(\mathbb{Z}^d)}}{U(s)} \\ &\leq \exp \left\{ T_t(2d + |\zeta(0)|) - t(\ln t)^{-2} \right\} \ll e^{-T_t \lambda_{t,s}^\bullet} \end{aligned} \quad (11.43)$$

since $\lambda_{t,s}^\bullet \leq \max_{x \in B_{t,s}} \zeta(x) \leq 2\rho \ln_2 t$ by Lemma 5.1. On the other hand, using (11.20) we get, on $\mathcal{G}_{t,s}$,

$$\frac{\|u_1(\cdot, s - T_t)\|_{\ell^2(\mathbb{Z}^d)}}{U(s)} \leq \varepsilon_v^{-5} e^{-T_t \lambda_{t,s}^\circ} \leq \varepsilon_v^{-5} e^{-T_t \lambda_{t,s}^\bullet} \quad (11.44)$$

since $\lambda_{t,s}^\circ \geq \lambda_{t,s}^\bullet$. This shows (11.39).

For (11.40), let $u^{(1)}, u^{(2)}$ be as in (11.9) and write

$$\begin{aligned} \langle u(\cdot, s), \phi_{t,s}^{(1)} \rangle &= \langle u^{(1)}(\cdot, s; t), \phi_{t,s}^{(1)} \rangle + \langle u^{(2)}(\cdot, s; t), \phi_{t,s}^{(1)} \rangle \\ &= e^{T_t \lambda_{t,s}^\bullet} \langle u(\cdot, s - T_t), \phi_{t,s}^{(1)} \rangle + \langle u^{(2)}(\cdot, s; t), \phi_{t,s}^{(1)} \rangle \end{aligned} \quad (11.45)$$

to obtain

$$\langle u(\cdot, s - T_t), \phi_{t,s}^{(1)} \rangle = e^{-T_t \lambda_{t,s}^\bullet} \left\{ \langle u(\cdot, s), \phi_{t,s}^{(1)} \rangle - \langle u^{(2)}(\cdot, s; t), \phi_{t,s}^{(1)} \rangle \right\}. \quad (11.46)$$

Fix $R \in \mathbb{N}$ such that (4.30) holds with $\delta < \frac{1}{2}$ and, for this R , take $\varepsilon_R^\bullet > 0$ as in (11.28). Then on $\mathcal{G}_{t,s}$ we can estimate

$$\langle u(\cdot, s), \phi_{t,s}^{(1)} \rangle \geq \sum_{x \in B_R(Z_s)} \phi_{t,s}^{(1)}(x) u(x, s) \geq \varepsilon_R^\bullet (1 - \delta) U(s) > \frac{1}{2} \varepsilon_R^\bullet U(s). \quad (11.47)$$

On the other hand, by Lemma 11.1, the second term inside the brackets in (11.46) multiplied by $\mathbb{1}_{\mathcal{G}_{t,s}}$ is smaller than $\varepsilon_R^\bullet U(s)/4$ with probability tending to 1, proving (11.40) with $c_1 = \frac{1}{4}\varepsilon_R^\bullet$. This concludes the proof of Lemma 11.2. \square

A. A TAIL ESTIMATE

In this section we prove (7.16) for \widehat{Y}_t given by (7.20) using an approach from [BK16]. We will strongly rely on Assumption 2.1. The first step concerns the tail of ζ .

Lemma A.1 *For any $\varepsilon > 0$, there exists $t_0 > 0$ such that, for all $t \geq t_0$,*

$$t^d \text{Prob}(\zeta(0) > \widehat{a}_t + sd_t) \leq e^{-s(1-\varepsilon)} \quad \forall s \geq 0. \quad (\text{A.1})$$

Proof. Recall the definition of F in (2.1). Note that $t^d = \exp(e^{F(\widehat{a}_t)})$ to write

$$\begin{aligned} & -\ln \left\{ t^d \text{Prob}(\zeta(0) > \widehat{a}_t + sd_t) \right\} \\ &= e^{F(\widehat{a}_t)} \left(e^{F(\widehat{a}_t + sd_t) - F(\widehat{a}_t)} - 1 \right) \geq e^{F(\widehat{a}_t)} \{F(\widehat{a}_t + sd_t) - F(\widehat{a}_t)\} \end{aligned} \quad (\text{A.2})$$

where in the last inequality we used $e^x - 1 \geq x$. Using (2.2) and the Mean Value Theorem, we obtain $F(\widehat{a}_t + sd_t) - F(\widehat{a}_t) \geq sd_t(1 - \varepsilon)/\rho$ for all $s \geq 0$ if t is large enough. Since $d_t = \rho e^{-F(\widehat{a}_t)}$, (A.1) follows from (A.2). \square

Lemma A.1 will allow us to reduce the sum in (7.16) to $|x| \leq 6d\theta t/d_t$.

Corollary A.2 *For any $\eta \in \mathbb{R}$, $\theta \in (0, \infty)$,*

$$\lim_{t \rightarrow \infty} \sum_{\substack{x \in (2\widehat{N}_t + 1)\mathbb{Z}^d \\ |x| > 6d\theta t/d_t}} \text{Prob} \left(\widehat{Y}_t(0) > \frac{|x|}{\theta t} + \eta \right) = 0. \quad (\text{A.3})$$

Proof. Recall that $\max_{x \in B_{\widehat{N}_t}} \zeta(x) \geq \lambda_{B_{\widehat{N}_t}}^{(1)}$ by (5.5). Using $a_t = \widehat{a}_t - \chi + o(1)$ and $\chi \leq 2d$, we obtain, for each $L \in \mathbb{N}$,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \sum_{\substack{x \in (2\widehat{N}_t + 1)\mathbb{Z}^d \\ |x| > 6d\theta t/d_t}} \text{Prob} \left(\widehat{Y}_t(0) > \frac{|x|}{\theta t} + \eta \right) \\ & \leq \limsup_{t \rightarrow \infty} \sum_{\substack{x \in (2\widehat{N}_t + 1)\mathbb{Z}^d \\ |x| > 6d\theta t/d_t}} |B_{\widehat{N}_t}| \text{Prob} \left(\zeta(0) > \widehat{a}_t + \frac{d_t}{2} \left(\frac{|x|}{\theta t} + 2\eta \right) \right) \\ & \leq \limsup_{t \rightarrow \infty} \sum_{\substack{x \in \mathbb{Z}^d \\ |x| > Lt/(2\widehat{N}_t + 1)}} \frac{|B_{\widehat{N}_t}|}{t^d} \exp \left\{ -\frac{1}{4} \left(\frac{|x|(2\widehat{N}_t + 1)}{\theta t} + 2\eta \right) \right\} \\ & = \int_{|z| \geq L} e^{-\frac{1}{4} \left(\frac{|z|}{\theta} + 2\eta \right)} dz \end{aligned} \quad (\text{A.4})$$

by Lemma A.1 and (2.6). Since the integral converges to 0 as $L \rightarrow \infty$, (A.3) follows. \square

To control the sum in (7.16) with $|x| \leq t6d\theta/d_t$, we will use the following lemma.

Lemma A.3 *There exist $c_0, \varepsilon > 0$ such that, for all large enough t and $s \geq 0$,*

$$\frac{t^d}{(2\widehat{N}_t)^d} \text{Prob} \left(\widehat{Y}_t(0) > s \right) \leq 4e^{-c_0 s} + t^{-\varepsilon}. \quad (\text{A.5})$$

Before we prove Lemma A.3, let us finish the proof of (7.16).

Proof of (7.16). By Corollary A.2, we only need to control the sum for $|x| \leq t6d\theta/d_t$. Fix $\eta \in \mathbb{R}$. Letting $n \geq \theta|\eta|$ and $|x| \geq nt$, we have $0 \leq |x|/(\theta t) + \eta \leq 6d/d_t$. Thus we may bound, by Lemma A.3,

$$\begin{aligned} & \sum_{\substack{x \in (2\widehat{N}_t+1)\mathbb{Z}^d \\ nt \leq |x| \leq t6d\theta/d_t}} \text{Prob} \left(\widehat{Y}_t(0) > \frac{|x|}{\theta t} + \eta \right) \\ & \leq \frac{c_2(\ln t)^d}{t^\varepsilon} + \sum_{\substack{x \in \mathbb{Z}^d \\ nt \leq |x| (2\widehat{N}_t+1) \leq t6d\theta/d_t}} \frac{(2\widehat{N}_t)^d}{t^d} 2 \exp \left\{ -c_0 \left(\frac{|x|(2\widehat{N}_t+1)}{\theta t} + \eta \right) \right\} \end{aligned} \quad (\text{A.6})$$

for a constant $c_2 > 0$ and large enough t . To conclude (7.16), note that the right-hand side of (A.6) converges as $t \rightarrow \infty$ to

$$2 \int_{|z| \geq n} e^{-c_0 \left(\frac{|z|}{\theta} + \eta \right)} dz, \quad (\text{A.7})$$

which converges itself to 0 as $n \rightarrow \infty$. \square

The remainder of this section is dedicated to the proof of Lemma A.3. Note that Assumption 2.1 implies that $\zeta(0)$ has a density f with respect to Lebesgue measure given by

$$f(r) = \begin{cases} F'(r) \exp \left\{ F(r) - e^{F(r)} \right\}, & r > \text{essinf } \zeta(0), \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.8})$$

The following bound holds for f .

Lemma A.4 *Fix a finite $\Lambda \subset \mathbb{Z}^d$ and two functions $\alpha, \varphi: \Lambda \rightarrow \mathbb{R}$. Then, as $t \rightarrow \infty$,*

$$\prod_{x \in \Lambda} \frac{f(\widehat{a}_t + \varphi(x) + \alpha(x)d_t)}{f(\widehat{a}_t + \varphi(x))} \leq \exp \left\{ -(1 + o(1)) \sum_{x \in \Lambda} \alpha(x) e^{\frac{\varphi(x)}{\rho}} + o(1) \mathcal{L}_\Lambda(\varphi) \right\} \quad (\text{A.9})$$

where $\mathcal{L}_\Lambda(\varphi)$ is as in (5.9) and $o(1)$ is uniform on Λ and on α, φ whenever $\alpha(x) \geq 0$ and $|\varphi(x)|$ is uniformly bounded. If additionally $\alpha(x)$ is uniformly bounded, then equality holds in (A.9).

Proof. One can follow the reasoning leading to the proof of Lemma 7.5 in [BK16]. \square

Fix now $c_0 := \frac{1}{2}e^{-3(d+1)/\rho}$; this will be the constant appearing in (A.3). The following corollary is a convenient rephrasing of (A.9).

Corollary A.5 *There exists $t_0 > 0$ such that, for all $t \geq t_0$, $s \geq 0$, $\Lambda \subset \mathbb{Z}^d$ and all $\alpha, \varphi: \Lambda \rightarrow \mathbb{R}$ with $\alpha(x) \geq 0$, $-2(d+1) \leq \varphi(x) \leq 1$,*

$$\prod_{x \in \Lambda} \frac{f(\widehat{a}_t + \varphi(x) + s\alpha(x)d_t)}{f(\widehat{a}_t + \varphi(x))} \leq \exp \left\{ -2c_0s \sum_{x \in \mathbb{Z}^d} \alpha(x) + \mathcal{L}_\Lambda(\varphi) \right\}. \quad (\text{A.10})$$

We can now prove Lemma A.3.

Proof of Lemma A.3. For $t > 0$ such that $a_t > \text{essinf } \zeta(0) + 1$, define the continuous map

$$\mathcal{F}_{t,s}(r) := \begin{cases} r & \text{if } r \leq a_t - 1, \\ r - sd_t & \text{if } r \geq a_t + sd_t, \\ \text{linear,} & \text{otherwise.} \end{cases} \quad (\text{A.11})$$

Then $\mathcal{F}_{t,s}$ is bijective with the inverse given by

$$\mathcal{F}_{t,s}^{-1}(r) := \begin{cases} r & \text{if } r \leq a_t - 1, \\ r + sd_t & \text{if } r \geq a_t, \\ \text{linear,} & \text{otherwise.} \end{cases} \quad (\text{A.12})$$

Let $\tilde{\zeta}_{t,s}(x) := \mathcal{F}_{t,s}(\zeta(x))$. Then $\tilde{\zeta}_{t,s}(x)$ is absolutely continuous with respect to $\zeta(x)$ with density

$$\frac{d\tilde{\zeta}_{t,s}(x)}{d\zeta(x)}(r) = \begin{cases} 1 & \text{if } r \leq a_t - 1, \\ (1 + sd_t)^{\mathbb{1}_{\{r < a_t\}}} \frac{f(\mathcal{F}_{t,s}^{-1}(r))}{f(r)} & \text{otherwise.} \end{cases} \quad (\text{A.13})$$

Recalling that $\lambda_{B_R}^{(1)}(\zeta)$ denotes the principal Dirichlet eigenvalue of $\Delta + \zeta$ in B_R , define

$$G_{t,s} := \left\{ \zeta: \lambda_{B_{R_t}}^{(1)}(\zeta) > a_t + sd_t, \mathcal{L}_{B_{R_t}}(\zeta - \widehat{a}_t) \leq \ln 2, \max_{x \in B_{R_t}} \zeta(x) \leq \widehat{a}_t + 1 \right\}. \quad (\text{A.14})$$

Since $\zeta(x) - 2d_t \leq \tilde{\zeta}_{t,s}(x) \leq \zeta(x)$, $\zeta \in G_{t,s}$ implies $\tilde{\zeta}_{t,s} \in G_{t,0}$. Write

$$\text{Prob}(\tilde{\zeta}_{t,s} \in G_{t,0}) = E \left[\mathbb{1}_{G_{t,0}}(\tilde{\zeta}) (1 + sd_t)^{|\{x \in B_{R_t}: a_t - 1 < \zeta(x) < a_t\}|} \prod_{\substack{x \in B_{R_t} \\ \zeta(x) > a_t - 1}} \frac{f(\mathcal{F}_{t,s}^{-1}(\zeta(x)))}{f(\zeta(x))} \right] \quad (\text{A.15})$$

where E denotes expectation with respect to Prob . Bound the middle term in (A.15) by

$$(1 + sd_t)^{|B_{R_t}|} \leq e^{sd_t(2R_t+1)^d} \leq e^{sc_0} \quad (\text{A.16})$$

for large t by (5.11). For the product term, define $\varphi(x) := \zeta(x) - \widehat{a}_t$ and $\alpha(x) \geq 0$ by the equation $\zeta(x) + sd_t\alpha(x) = \mathcal{F}_{t,s}^{-1}(\zeta(x))$. Noting that, if $\alpha(x) \neq 0$ then $-2(d+1) \leq \varphi(x) \leq 1$, by Corollary A.5

$$\prod_{x \in B_{R_t}: \zeta(x) > a_t - 1} \frac{f(\mathcal{F}_{t,s}^{-1}(\zeta(x)))}{f(\zeta(x))} \leq 2 \exp \left\{ -2c_0s \sum_{x \in B_{R_t}: \zeta(x) > a_t - 1} \alpha(x) \right\} \quad (\text{A.17})$$

since $\mathcal{L}_{B_{R_t}}(\varphi) \leq \ln 2$ on $G_{t,0}$. Moreover, on this event we have $\xi(x) > a_t$ and thus $\alpha(x) = 1$ for some $x \in B_{R_t}$ since $\max_{x \in B_{R_t}} \xi(x) \geq \lambda_{B_{R_t}}^{(1)}(\xi)$. Noting now that, by (A.1) and Lemma 6.4 of [BK16],

$$\text{Prob}\left(\lambda_{B_{R_t}}^{(1)}(\xi) > a_t + sd_t\right) \leq \text{Prob}(\xi \in G_{t,s}) + o(t^{-(d+\varepsilon_0)}) \quad (\text{A.18})$$

for some $\varepsilon_0 > 0$, we obtain by (A.14–A.18)

$$\text{Prob}\left(\lambda_{B_{R_t}}^{(1)}(\xi) \geq a_t + sd_t\right) \leq 2e^{-c_0s} \text{Prob}\left(\lambda_{B_{R_t}}^{(1)}(\xi) \geq a_t\right) + o(t^{-(d+\varepsilon_0)}). \quad (\text{A.19})$$

To pass the estimate to $\lambda_{B_{\widehat{N}_t}}^{(1)}(\xi)$, note first that, by Lemma 7.6 of [BK16],

$$\limsup_{t \rightarrow \infty} \frac{t^d}{(2R_t)^d} \text{Prob}\left(\lambda_{B_{R_t}}^{(1)}(\xi) \geq a_t\right) \leq 1, \quad (\text{A.20})$$

and thus for large t the right-hand side of (A.19) is at most $3e^{-c_0s}(2R_t/t)^d + o(t^{-(d+\varepsilon_0)})$. Moreover, by Lemma 7.7 of [BK16] applied to $t_L := a_L - \widehat{a}_L + sd_L$ and $R'_L := (\ln_2 L)^2$,

$$\frac{t^d}{(2\widehat{N}_t)^d} \text{Prob}\left(\lambda_{B_{\widehat{N}_t}}^{(1)}(\xi) \geq a_t + sd_t\right) \leq \widehat{N}_t^{-d} + 4e^{-c_0s} + o(t^{-\varepsilon_0}) \quad (\text{A.21})$$

for t large enough, noting that $o(L^{-d})$ and $o(1)$ in equation (7.27) of [BK16] are uniform on the sequence t_L . Note that the factor 2 multiplying R_t and \widehat{N}_t here and not in [BK16] appears since our boxes have side-length $2R + 1$ while theirs R . Recalling that $\widehat{N}_t \gg t^\beta$ for some $\beta > 0$ and taking $\varepsilon := \varepsilon_0 \wedge (\beta d)$, the lemma is proved. \square

B. COMPACTIFICATION

Let $\mathfrak{E} := (\mathbb{R} \times \mathbb{R}^d) \cup [0, \infty)$ be equipped with a metric \mathbf{d} defined by setting, for $\theta, \theta' \in [0, \infty)$ and $(\lambda, z), (\lambda', z') \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} \mathbf{d}(\theta, \theta') &:= |\theta - \theta'|, & \mathbf{d}(\theta, (\lambda, z)) &:= e^{-\lambda} + \left| \frac{|z|}{1 \vee \lambda} - \theta \right|, \\ \mathbf{d}((\lambda, z), (\lambda', z')) &:= e^{-\lambda \wedge \lambda'} \left(1 - e^{-|\lambda - \lambda'| - |z - z'|} \right) + \left| \frac{|z|}{1 \vee \lambda} - \frac{|z'|}{1 \vee \lambda'} \right|. \end{aligned} \quad (\text{B.1})$$

One may verify that \mathbf{d} is indeed a metric under which \mathfrak{E} is separable, complete and locally compact. Moreover:

Lemma B.1 *For any $(\theta, \eta) \in (0, \infty) \times \mathbb{R}$, the set $\mathcal{H}_\eta^\theta \subset \mathfrak{E}$ defined in (7.14) is relatively compact.*

Proof. Note that the closure of \mathcal{H}_η^θ in \mathfrak{E} is given by

$$\overline{\mathcal{H}_\eta^\theta} = \left\{ (\lambda, z) \in \mathbb{R} \times \mathbb{R}^d : \lambda - \frac{|z|}{\theta} \geq \eta \right\} \cup [0, \theta]. \quad (\text{B.2})$$

Fix a sequence $(\Xi_n)_{n \in \mathbb{N}}$ in $\overline{\mathcal{H}_\eta^\theta}$ and consider the following three cases:

- 1 $\Xi_n \in [0, \theta]$ for infinitely many n ;

- 2 $\Xi_n = (\lambda_n, z_n) \in \mathbb{R} \times \mathbb{R}^d$ for all but a finite number of n and $(\lambda_n)_{n \in \mathbb{N}}$ is bounded, implying that $\{\Xi_n : n \in \mathbb{N}\}$ is contained in a compact subset of $\mathbb{R} \times \mathbb{R}^d$;
- 3 $\Xi_n = (\lambda_n, z_n) \in \mathbb{R} \times \mathbb{R}^d$ for all but a finite number of n and $\lim_{n \rightarrow \infty} \lambda_n = \infty$. Note that $\limsup_{n \rightarrow \infty} |z_n|/\lambda_n \leq \theta$.

As is directly checked, in each case there exists a subsequence converging in \mathfrak{E} to a point of $\overline{\mathcal{H}_\eta^\theta}$, thus proving the claim. \square

We finish the section with the following important property of \mathfrak{E} .

Lemma B.2 *For any compact set $K \subset \mathfrak{E}$, there exist $\theta \in (0, \infty)$ and $\eta \in \mathbb{R}$ such that $K \cap (\mathbb{R} \times \mathbb{R}^d) \subset \mathcal{H}_\eta^\theta$.*

Proof. Cover each $x \in K$ with an open set $\mathcal{H}_{\eta_x}^{\theta_x} \cup [0, \theta_x)$ for some $\theta_x > 0, \eta_x \in \mathbb{R}$. Use compactness to extract a finite subcover corresponding to x_1, \dots, x_N and set $\theta := \max_{i=1}^N \theta_{x_i}, \eta := \min_{i=1}^N \eta_{x_i}$ to obtain the result. \square

C. PROPERTIES OF THE COST FUNCTIONAL

In this section we prove Lemmas 7.5, 7.6 and 7.8.

Proof of Lemma 7.5(i). Fix $\theta_0 < \theta_1$ and set $(\lambda_i, z_i) = \Xi_\theta^{(1)}(\mathcal{P})(\theta_i), i = 0, 1$. Then

$$\theta_0(\lambda_1 - \lambda_0) \leq |\vartheta(z_1)| - |\vartheta(z_0)| \leq \theta_1(\lambda_1 - \lambda_0) \quad (\text{C.1})$$

by the definition of $\Psi_\theta^{(1)}(\mathcal{P})$, so that all three functions are non-decreasing. Now, if $(\lambda_0, z_0) \neq (\lambda_1, z_1)$, then one of the inequalities above is strict, since otherwise $\lambda_1 = \lambda_0, |\vartheta(z_1)| = |\vartheta(z_0)|$ and we would have $(\lambda_i, z_i) \in \mathfrak{S}_\theta^{(1)}(\mathcal{P})(\theta_j)$ with $i \neq j \in \{1, 2\}$, implying that $(\lambda_1, z_1) = (\lambda_0, z_0)$ by the definition of $\Xi_\theta^{(1)}(\mathcal{P})$. This concludes the proof. \square

Proof of Lemma 7.5(ii). We will first consider the case $|\text{supp}(\mathcal{P})| < \infty$. We may assume $|\text{supp}(\mathcal{P})| \geq 2$ since otherwise there is nothing to prove.

Consider first the case $i = 1$. $\Psi_\theta^{(1)}(\mathcal{P})$ is continuous as the pointwise maximum of finitely many continuous functions. Lemma 7.5(i) implies that $\Xi_\theta^{(1)}(\mathcal{P})$ jumps finitely many times, and thus has left limits; let us to show that it is càdlàg. Fix $\theta_0 > 0$ and let $(\lambda_0, z_0) := \Xi_\theta^{(1)}(\mathcal{P})(\theta_0)$. Note first that, if $(\lambda, z) \in \mathfrak{S}_\theta^{(1)}(\mathcal{P})(\theta_0)$, then $\psi_\theta^\vartheta(\lambda, z) \leq \psi_\theta^\vartheta(\lambda_0, z_0)$ for all $\theta \geq \theta_0$ because $\lambda \leq \lambda_0$ by definition. On the other hand, if $(\lambda, z) \notin \mathfrak{S}_\theta^{(1)}(\mathcal{P})(\theta_0)$, then there exists $\delta_{\lambda, z} > 0$ such that $\psi_\theta^\vartheta(\lambda, z) < \psi_\theta^\vartheta(\lambda_0, z_0)$ for all $\theta \in [\theta_0, \theta_0 + \delta_{\lambda, z}]$. Setting $\delta > 0$ to be the smallest among these, we can see that

$$(\lambda_0, z_0) \in \mathfrak{S}_\theta^{(1)}(\mathcal{P})(\theta) \subset \mathfrak{S}_\theta^{(1)}(\mathcal{P})(\theta_0) \quad \forall \theta \in [\theta_0, \theta_0 + \delta] \quad (\text{C.2})$$

implying $\Xi_\theta^{(1)}(\mathcal{P})(\theta) = \Xi_\theta^{(1)}(\mathcal{P})(\theta_0)$ for all $\theta \in [\theta_0, \theta_0 + \delta]$, i.e., $\Xi_\theta^{(1)}(\mathcal{P})$ is right-continuous.

Assume now by induction that the statement of Lemma 7.5(ii) has been proved in the case $|\text{supp}(\mathcal{P})| < \infty$ for all $i \leq k-1$, $k \geq 2$. Note that, by the definition of $\Phi_\vartheta^{(k)}$,

$$\Phi_\vartheta^{(k)}(\mathcal{P})(\theta) = \sum_{\Xi \in \text{supp}(\mathcal{P})} \mathbb{1}_{\{\Xi_\vartheta^{(1)}(\mathcal{P})(\theta) = \Xi\}} \Phi_\vartheta^{(k-1)}(\mathcal{P}_\Xi)(\theta) \quad (\text{C.3})$$

where $\mathcal{P}_\Xi(\cdot) := \mathcal{P}(\cdot \setminus \{\Xi\})$. Since $\Xi_\vartheta^{(1)}(\mathcal{P})$ is càdlàg, it follows from the induction hypothesis that $\Phi_\vartheta^{(k)}(\mathcal{P})$ is also càdlàg. To prove in addition that $\Psi_\vartheta^{(k)}(\mathcal{P})$ is continuous, we only need to show that, if $\Xi_0 := \Xi_\vartheta^{(1)}(\mathcal{P})(\theta-) \neq \Xi_\vartheta^{(1)}(\mathcal{P})(\theta) =: \Xi$, then $\Psi_\vartheta^{(k-1)}(\mathcal{P}_{\Xi_0})(\theta) = \Psi_\vartheta^{(k-1)}(\mathcal{P}_\Xi)(\theta)$; but this follows from the definition of $\Psi_\vartheta^{(k-1)}$ since, by the continuity of $\Psi_\vartheta^{(1)}(\mathcal{P})$, $\psi_\vartheta^\theta(\Xi_0) = \psi_\vartheta^\theta(\Xi)$. This finishes the proof in the case $|\text{supp}(\mathcal{P})| < \infty$.

The case $|\text{supp}(\mathcal{P})| = \infty$ can be reduced to the previous one as follows. First note that we may substitute $(0, \infty)$ by $[a, b]$ with $0 < a < b < \infty$ arbitrary. Fix $i \in \mathbb{N}$. Since $\mathcal{H}_\eta^a \uparrow \mathbb{R} \times \mathbb{R}^d$ as $\eta \rightarrow -\infty$, \mathcal{H}_η^b is relatively compact and $\mathcal{P}^\vartheta \in \mathcal{M}_P$, there exists an $\eta \in \mathbb{R}$ such that $i \leq |\text{supp}(\mathcal{P}^\vartheta) \cap \mathcal{H}_\eta^a| \leq \mathcal{P}^\vartheta(\mathcal{H}_\eta^b) < \infty$. Noting that, on $[a, b]$, $\Phi_\vartheta^{(i)}(\mathcal{P}) = \Phi_\vartheta^{(i)}(\mathcal{P}')$ where $\mathcal{P}'(\cdot) := \mathcal{P}(\cdot \cap \{(\lambda, z) : (\lambda, \vartheta(z)) \in \mathcal{H}_\eta^b\})$, we fall into the previous case.

For the last statements, note that the proof above shows that $\Xi_\vartheta^{(i)}(\mathcal{P})$ jumps finitely many times in each compact interval $[\theta_1, \theta_2] \subset (0, \infty)$. Moreover, if $\vartheta(Z_\vartheta^{(1)}(\mathcal{P})(\theta_1)) \neq 0$ and $\Xi_\vartheta^{(1)}(\mathcal{P})$ is constant in $[\theta_1, \theta_2]$, then $\Psi_\vartheta^{(1)}(\mathcal{P})$ is strictly increasing in $[\theta_1, \theta_2]$. \square

Proof of Lemma 7.6. We first consider the case $1 \leq |\text{supp}(\mathcal{P})| < \infty$. By Proposition 3.13 of [R87], for t large enough there exist bijections $T_t : \text{supp}(\mathcal{P}) \rightarrow \text{supp}(\mathcal{P}_t)$ such that

$$\lim_{t \rightarrow \infty} \sup_{\Xi \in \text{supp}(\mathcal{P})} \text{dist}(T_t(\Xi), \Xi) = 0. \quad (\text{C.4})$$

Moreover, letting $\mathcal{T}_t(\lambda, z) := (\lambda, \vartheta_t(z))$, by (7.42) and $\text{supp}(\mathcal{P}) \cap \mathbb{R} \times \{0\} = \emptyset$ we have

$$\lim_{t \rightarrow \infty} \sup_{\Xi \in \text{supp}(\mathcal{P})} \text{dist}(\mathcal{T}_t \circ T_t(\Xi), \Xi) = 0, \quad (\text{C.5})$$

and $\mathcal{T}_t \circ T_t$ is a bijection onto $\text{supp}(\mathcal{P}_t^{\vartheta_t})$. In particular, $\mathcal{P}_t^{\vartheta_t} \rightarrow \mathcal{P}$.

Let $a_0 := a$ and, recursively for $\ell \in \mathbb{N}$,

$$a_\ell := \inf\{\theta > a_{\ell-1} : \exists 1 \leq i \leq |\text{supp}(\mathcal{P})|, \Xi_\vartheta^{(i)}(\mathcal{P})(\theta) \neq \Xi_\vartheta^{(i)}(\mathcal{P})(a_{\ell-1})\}. \quad (\text{C.6})$$

Note that $\Xi_\vartheta^{(i)}(\mathcal{P})$ jumps finitely many times: for $i = 1$ this follows by Lemma 7.5(i), and for $i \geq 2$, by induction using (C.3). Thus $\ell_* = \ell_*(a, \mathcal{P}) := \inf\{\ell \geq 0 : a_{\ell+1} = \infty\} < \infty$.

We proceed by induction on ℓ_* , starting with $\ell_* = 0$. Since $\mathcal{P} \in \widetilde{\mathcal{M}}_P^a$, the values $i \mapsto \psi_a(\Xi_\vartheta^{(i)}(\mathcal{P})(a))$ are all distinct, which together with (C.4)–(C.5) implies that $\Xi_\vartheta^{(i)}(\mathcal{P}_t)(a) = T_t(\Xi_\vartheta^{(i)}(\mathcal{P})(a))$ for all i when t is large enough. In particular, (C.4) implies the result in the case $\ell_* = 0$. Assume by induction that, for some $L \in \mathbb{N}$, the statement has been proved for all $a' \in (0, \infty)$ and $\mathcal{P}' \in \widetilde{\mathcal{M}}_P^{a'}$ satisfying $|\text{supp}(\mathcal{P}')| < \infty$ and $\ell_*(a', \mathcal{P}') \leq L-1$, and suppose that $\ell_* = \ell_*(a, \mathcal{P}) = L$ (in which case necessarily $|\text{supp}(\mathcal{P})| \geq 2$).

Note now that, because $\mathcal{P} \in \widetilde{\mathcal{M}}_P^a$, there exists a unique i_1 such that both $\Xi^{(i_1)}(\mathcal{P})$ and $\Xi^{(i_1+1)}(\mathcal{P})$ jump at a_1 while $\Xi^{(i)}(\mathcal{P})$ is continuous at a_1 for all $i \notin \{i_1, i_1 + 1\}$. Furthermore, $\Xi^{(i_1)}(\mathcal{P})(a_1)$ is the point $\Xi \in \text{supp}(\mathcal{P})$ minimizing $\mathcal{F}_a(\Xi, \Xi^{(i_1)}(\mathcal{P})(a))$ where

$$\mathcal{F}_\theta((\lambda_1, z_1), (\lambda_2, z_2)) := \begin{cases} \frac{|z_1| - |z_2|}{\lambda_1 - \lambda_2} & \text{if } \lambda_1 > \lambda_2 \text{ and } \psi_\theta(\lambda_1, z_1) < \psi_\theta(\lambda_2, z_2), \\ \infty & \text{otherwise,} \end{cases} \quad (\text{C.7})$$

and also $a_1 - a = \mathcal{F}_a(\Xi^{(i_1)}(\mathcal{P})(a_1), \Xi^{(i_1)}(\mathcal{P})(a)), \Xi^{(i_1+1)}(\mathcal{P})(a_1) = \Xi^{(i_1)}(\mathcal{P})(a)$.

Let now a_ℓ^t, ℓ_*^t be the analogous of a_ℓ, ℓ_* for $\Xi_{\theta_t}^{(i)}(\mathcal{P}_t)$ and fix $a' \in (a_1, a_2) \cap \mathbb{Q}$. By (C.4)–(C.5) and the previous discussion, when t is large enough, $\Xi_{\theta_t}^{(i)}(\mathcal{P}_t)$ does not jump in $[a, a']$ for all $i \notin \{i_1, i_1 + 1\}$. Moreover, $\Xi_{\theta_t}^{(i_1)}(\mathcal{P}_t)(a_1^t) = T_t(\Xi^{(i_1)}(\mathcal{P})(a_1)), \Xi_{\theta_t}^{(i_1+1)}(\mathcal{P}_t)(a_1^t) = \Xi_{\theta_t}^{(i_1)}(\mathcal{P}_t)(a) = T_t(\Xi^{(i_1)}(\mathcal{P})(a)), a_1^t < a' < a_2^t$ and

$$\begin{aligned} |a_1 - a_1^t| &= \left| \mathcal{F}_a(\Xi^{(1)}(\mathcal{P})(a_1), \Xi^{(1)}(\mathcal{P})(a)) - \mathcal{F}_a(\Xi_{\theta_t}^{(1)}(\mathcal{P}_t)(a_1^t), \Xi_{\theta_t}^{(1)}(\mathcal{P}_t)(a)) \right| \\ &\leq \max_{\substack{\Xi_1, \Xi_2 \in \text{supp}(\mathcal{P}) \\ \mathcal{F}_a(\Xi_1, \Xi_2) < \infty}} |\mathcal{F}_a(\Xi_1, \Xi_2) - \mathcal{F}_a(T_t(\Xi_1), T_t(\Xi_2))| \xrightarrow{t \rightarrow \infty} 0 \end{aligned} \quad (\text{C.8})$$

by (C.4). Define now a time change $\sigma_t : [a, a'] \rightarrow [a, a']$ by setting

$$\sigma_t(a) = a, \quad \sigma_t(a_1) = a_1^t, \quad \sigma_t(a') = a' \quad \text{and linear otherwise.} \quad (\text{C.9})$$

Then, by (C.4) and (C.8),

$$\lim_{t \rightarrow \infty} \sup_{1 \leq i \leq |\text{supp}(\mathcal{P})|} \sup_{\theta \in [a, a']} |\sigma_t(\theta) - \theta| \vee \left| \Phi_{\theta_t}^{(i)}(\mathcal{P}_t)(\sigma_t(\theta)) - \Phi^{(i)}(\mathcal{P})(\theta) \right| = 0. \quad (\text{C.10})$$

Since $\ell_*(a', \mathcal{P}) = L - 1$ and $\mathcal{P} \in \widetilde{\mathcal{M}}_P^a$, by the induction hypothesis we can extend σ_t to $[a, \infty)$ in such a way that (C.10) holds with $[a, a']$ substituted by $[a, \infty)$, finishing the proof in the case $|\text{supp}(\mathcal{P})| < \infty$.

Consider now the case $|\text{supp}(\mathcal{P})| = \infty$. Let us first show (7.44). Fix $k \in \mathbb{N}$ and a point $b \in (a, \infty) \cap \mathbb{Q}$. Note that, since $\mathcal{P} \in \widetilde{\mathcal{M}}_P^a$, b is a continuity point of $\Phi^{(i)}(\mathcal{P})$ for all $1 \leq i \leq k$. Let $\eta \in \mathbb{R}$ be negative enough such that, for all t large enough,

$$k \leq |\text{supp}(\mathcal{P}) \cap \mathcal{H}_\eta^a| = |\text{supp}(\mathcal{P}_t) \cap \mathcal{H}_\eta^a| \leq \mathcal{P}_t(\mathcal{H}_\eta^{2b/c_*}) = \mathcal{P}(\mathcal{H}_\eta^{2b/c_*}) < \infty, \quad (\text{C.11})$$

where $c_* \in (0, 1]$ is as in (7.43); this is possible because $\mathcal{P} \in \mathcal{M}_P$ and $\mathcal{P}_t \rightarrow \mathcal{P}$. Moreover, since $\text{supp}(\mathcal{P}) \cap \mathbb{R} \times \{0\} = \emptyset$, by (7.42)–(7.43) we may also assume that

$$k \leq |\text{supp}(\mathcal{P}_t^{\theta_t}) \cap \mathcal{H}_\eta^a| \quad \text{and} \quad \text{supp}(\mathcal{P}_t^{\theta_t}) \cap \mathcal{H}_\eta^b \subset \mathcal{T}_t \left(\text{supp}(\mathcal{P}_t) \cap \mathcal{H}_\eta^{2b/c_*} \right), \quad (\text{C.12})$$

where \mathcal{T}_t is defined right before (C.5). Now (C.11)–(C.12) imply that, on $[a, b]$, $\Phi^{(i)}(\mathcal{P}) = \Phi^{(i)}(\mathcal{P}')$ and $\Phi_{\theta_t}^{(i)}(\mathcal{P}_t) = \Phi_{\theta_t}^{(i)}(\mathcal{P}_t')$ for all $1 \leq i \leq k$, where $\mathcal{P}'(\cdot) := \mathcal{P}(\cdot \cap \mathcal{H}_\eta^{2b/c})$ and analogously for \mathcal{P}_t' . Since $\mathcal{P}_t' \rightarrow \mathcal{P}'$, (7.44) follows by the previous case and Theorem 16.2 of [B99]. The convergence $\mathcal{P}_t^{\theta_t} \rightarrow \mathcal{P}$ follows from (C.12), (7.42) and $\mathcal{P}_t \rightarrow \mathcal{P}$ (note that b, η above can be taken arbitrarily large, respect. negative). \square

Proof of Lemma 7.8. For $(\lambda, z) \in \mathbb{R} \times (\mathbb{R}^d \setminus \{0\})$, let

$$\mathcal{A}(\lambda, z) := \left\{ (\lambda', z') \in \mathbb{R} \times \mathbb{R}^d : \begin{array}{l} \psi_b(\lambda', z') > \psi_b(\lambda, z) \text{ or} \\ \psi_b(\lambda', z') = \psi_b(\lambda, z) \text{ and } \lambda' > \lambda \end{array} \right\}. \quad (\text{C.13})$$

Note that, by the definition of \mathcal{P}^ϑ , $\mathcal{F}_b^\vartheta(\mathcal{P}, \lambda, z) = \mathcal{P}^\vartheta \{ \mathcal{A}(\lambda, \vartheta(z)) \}$. Since $\vartheta_t(z_t) \rightarrow z_*$ by (7.42) and $\mathcal{P}_t^{\vartheta_t} \rightarrow \mathcal{P}$ by Lemma 7.6, we may assume that $\vartheta_t(z) = z$ for all $z \in \mathbb{R}^d$.

Now, since $\mathcal{P} \in \widetilde{\mathcal{M}}_P^b$, $\mathcal{F}(\mathcal{P}, \lambda_*, z_*) = \mathcal{P} \left\{ \overline{\mathcal{H}_{\psi_b(\lambda_*, z_*)}^b} \right\}$ and there exists a $\delta > 0$ such that

$$\mathcal{P} \left\{ \overline{\mathcal{H}_{\psi_b(\lambda_*, z_*) - \delta}^b} \right\} = 1 + \mathcal{P} \left\{ \overline{\mathcal{H}_{\psi_b(\lambda_*, z_*) + \delta}^b} \right\}. \quad (\text{C.14})$$

On the other hand, since $\mathcal{P}_t \rightarrow \mathcal{P}$ and $(\lambda_t, z_t) \rightarrow (\lambda_*, z_*)$, when t is large we also have

$$\mathcal{P}_t \left\{ \overline{\mathcal{H}_{\psi_b(\lambda_*, z_*) \pm \delta}^b} \right\} = \mathcal{P} \left\{ \overline{\mathcal{H}_{\psi_b(\lambda_*, z_*) \pm \delta}^b} \right\} \quad \text{and} \quad (\lambda_t, z_t) \in \mathcal{H}_{\psi_b(\lambda_*, z_*) - \delta}^b \setminus \mathcal{H}_{\psi_b(\lambda_*, z_*) + \delta}^b. \quad (\text{C.15})$$

In particular, for all t large enough,

$$\mathcal{P}_t \{ \mathcal{A}(\lambda_t, z_t) \} = \mathcal{P}_t \left\{ \overline{\mathcal{H}_{\psi_b(\lambda_*, z_*) + \delta}^b} \right\} = \mathcal{P} \left\{ \overline{\mathcal{H}_{\psi_b(\lambda_*, z_*) + \delta}^b} \right\} = \mathcal{P} \left\{ \mathcal{H}_{\psi_b(\lambda_*, z_*)}^b \right\}, \quad (\text{C.16})$$

concluding the proof. \square

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