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Localization of the principal Dirichlet eigenvector in the heavy-tailed random conductance model

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We study the asymptotic behavior of the principal eigenvector and eigenvalue of the random conductance Laplacian in a large domain of $\mathbb{Z}^d$ ($d \geq 2$) with zero Dirichlet condition. We assume that the conductances w are positive i.i.d. random variables, which fulfill certain regularity assumptions near zero. If $\gamma = \sup\{q \geq 0 \colon \mathbb{E}[w^{-q}] < \infty\} < 1/4$, then we show that for almost every environment the principal Dirichlet eigenvector asymptotically concentrates in a single site and the corresponding eigenvalue scales subdiffusively. The threshold $\gamma_{\mathrm{c}} = 1/4$ is sharp. Indeed, other recent results imply that for $\gamma > 1/4$ the top of the Dirichlet spectrum homogenizes. Our proofs are based on a spatial extreme value analysis of the local speed measure, Borel-Cantelli arguments, the Rayleigh-Ritz formula, results from percolation theory, and path arguments.

Contents

1 Introduction **1**
1.1 Model and main objects . . . 2
1.2 Main results . . . 3
1.3 Comparison with former results . . . 6
1.4 Survey on proofs for upper bounds . . . 7
1.5 Heuristics for critical moments and relation to heat-kernel upper bounds . . . 8
1.6 Survey on proofs for lower bounds . . . 9

2 Borel-Cantelli arguments **10**

3 Percolation results **15**

4 Path argument **19**
4.1 Asymptotics of the principal Dirichlet eigenvalue . . . 20
4.2 Proof of Propositions 4.6 and 4.7 . . . 21

5 Localization of the principal eigenvector **22**
5.1 Auxiliary lemmas . . . 23
5.2 Proof of Theorem 1.8 . . . 25

1 Introduction

In dimensions greater than one, the spectrum of the i.i.d. random conductance Laplacian displays a sharp transition between complete localization and complete homogenization. This is the result of the present paper in combination with recent papers from Flegel, Heida, and Slowik [FHS] and Neukamm, Schäffner, and Schlömerkemper [NSS16]. While the other two papers cover spectral homogenization, we investigate the localization phase. A simple moment condition distinguishes between the two phases.

More precisely, we investigate the spectrum of the Laplacian $\mathcal{L}_{\boldsymbol{w}}$ associated with the random conductance model on the euclidean lattice $\mathbb{Z}^d$. The Laplacian acts on real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ as

$$(\mathcal{L}_{\boldsymbol{w}} f)(x) = \sum_{y \colon |x-y|_1 = 1} w_{xy}(f(y) - f(x)) \qquad (x \in \mathbb{Z}^d)\,. \tag{1.1}$$

We assume that the conductances w are positive, independent and identically distributed (i.i.d.) random variables. We describe the almost-sure behavior of the principal eigenvector with zero Dirichlet conditions outside a growing centered ball B. It turns out that its behavior strongly depends on the lower tails of the conductances. To be more precise, let us denote the expectation with respect to the conductances by $\mathbb{E}$ and define $\gamma = \sup\{q \geq 0\colon \mathbb{E}[w^{-q}] < \infty\}$. Then we show that, under some further regularity assumptions,

$$\gamma < 1/4 \Rightarrow \begin{cases} \text{a.s. complete localization of principal Dirichlet eigenvector and} \\ \text{a.s. subdiffusive scaling of principal Dirichlet eigenvalue.} \end{cases} \tag{1.2}$$

On the other hand, as a special case the results in [FHS] and [NSS16] imply that

$$\gamma > 1/4 \Rightarrow \begin{cases} \text{a.s. complete homogenization of first Dirichlet eigenvectors and} \\ \text{a.s. convergence of diffusively rescaled first Dirichlet eigenvalues.} \end{cases} \tag{1.3}$$

We comment on this in Section 1.3. Together, (1.2) and (1.3) imply that in the i.i.d. random conductance model there is a dichotomy between a completely homogenized and a completely localized phase.

Moreover, it is remarkable that the critical exponent $\gamma_c = 1/4$ coincides with the critical exponent for the validity of a local central limit theorem (LCLT) of the corresponding random walk (see e.g. [BKM15]):

$$\gamma > 1/4 \Rightarrow \text{LCLT holds} \qquad \text{and} \qquad \gamma < 1/4 \Rightarrow \text{LCLT does not hold.}$$

The validity of a local CLT is a very strong kind of heat-kernel homogenization. But in contrast to the principal Dirichlet eigenvector, the heat kernel does not display such a completely different behavior for $\gamma < 1/4$. Although the heat kernel decays anomalously for γ small enough [FM06, BBHK04, Bou10, BB12], a quenched functional CLT (QFCLT) still holds under minimal assumptions on the i.i.d. environment [ABDH12]. This is indeed *not* a contradiction since the QFCLT associates with macroscopic properties of the random walk, whereas the anomalous heat-kernel bounds as well as the local CLT and the principal Dirichlet eigenvector are all sensitive to microscopic trapping structures.

This paper is organized as follows: In Section 1.1 we define the model and our main objects. We present our main results in Section 1.2. In Section 1.3 we compare our results to former results. Section 1.5 contains a heuristic explanation why the critical exponent $\gamma_c = 1/4$ decides between spectral homogenization and localization. In the same section we comment on how a subdiffusive upper bound for the principal Dirichlet eigenvalue contradicts diffusive heat-kernel upper bounds. We survey the proofs concerning the eigenvalues in Sections 1.4 and 1.6 where we rely on technical results from the subsequent sections. Section 2 contains Borel-Cantelli arguments, which extend results from [CD81], [Kes03] and [BKM15]. In Section 3 we adapt some standard results on percolation theory from [Bar04], [MR04] and [BKM15] to our needs. Section 4 contains a path argument similar to the one in [BKM15]. Finally, we prove the localization of the principal eigenvector in Section 5.

1.1 Model and main objects

We consider the lattice with vertex set $\mathbb{Z}^d$ ($d \geq 2$) and edge set $\mathfrak{E}_d = \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y|_1 = 1\}$. If two sites $x, y \in \mathbb{Z}^d$ are neighbors according to $\mathfrak{E}_d$, we also write $x \sim y$. To each edge $e \in \mathfrak{E}_d$ we assign a positive random variable w_e. In analogy to a d-dimensional resistor network, we call the random weights w_e *conductances*. We take $(\Omega, \mathcal{F}) = \big((0, \infty)^{\mathfrak{E}_d}, \mathcal{B}((0, \infty))^{\otimes \mathfrak{E}_d}\big)$ as the underlying measurable space and assume that an environment $\boldsymbol{w} = (w_e)_{e \in \mathfrak{E}_d} \in \Omega$ is a family of i.i.d. positive random variables with law $\mathbb{P}$. We denote the expectation w.r.t. to $\mathbb{P}$ by $\mathbb{E}$.

If e is the edge between the sites $x, y \in \mathbb{Z}^d$, we will also write w_{xy} or $w_{x,y}$ instead of w_e. Note that by definition of the edge set $\mathfrak{E}_d$, the edges are undirected, whence $w_{xy} = w_{yx}$. If we want to refer to an arbitrary copy of the conductances in general, we simply write w, i.e., for a set $A \in \mathcal{F}$, the expression $\mathbb{P}[w \in A]$ equals $\mathbb{P}[w_e \in A]$ for an arbitrary edge e.

We call

$$F\colon [0, \infty) \to [0, 1]\colon u \mapsto \mathbb{P}[w \leq u] \tag{1.4}$$

the distribution function of the conductances.

Given a realization $(w_{xy})_{\{x,y\}\in\mathfrak{E}_d}$ of the environment, we consider the Markov chain on $\mathbb{Z}^d$ with transitions rates given by the conductances w_{xy}. Its generator $\mathcal{L}_{\boldsymbol{w}}$ is defined as in (1.1). This Markov chain is known as the variable-speed random walk among random conductances. During the last decades both physicists and mathematicians have analyzed the random conductance model extensively and many questions regarding central limit theorems and heat-kernel behavior have been answered (for reviews see [BG90] and [Bis11], respectively). The model became popular for the description of materials where the transition rates between different states are independent of the states' energy levels. This is the case e.g. for the spectral transport of optical excitations among impurity ions [Lyo79], or charge transport in the one-dimensional ionic conductor hollandite [BBSA79].

Our goal is to study the behavior of the principal eigenvalue $\lambda_1^{(n)}$ and eigenvector $\psi_1^{(n)}$ of the sign-inverted generator $-\mathcal{L}_{\boldsymbol{w}}$ in the ball

$$B_n := \{x \in \mathbb{Z}^d \colon |x|_\infty \le n\} = [-n,n]^d \cap \mathbb{Z}^d \tag{1.5}$$

with zero Dirichlet conditions at the boundary. For a real-valued function $f \in \ell^2(\mathbb{Z}^d)$ let us define the Dirichlet energy $\mathcal{E}^{\boldsymbol{w}}(f)$ w.r.t. to the operator $-\mathcal{L}_{\boldsymbol{w}}$ by

$$\mathcal{E}^{\boldsymbol{w}}(f) = \langle f, -\mathcal{L}_{\boldsymbol{w}} f\rangle\,, \tag{1.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product. Then, according to the Courant-Fischer theorem, the principal Dirichlet eigenvalue is given by the variational formula

$$\lambda_1^{(n)} = \inf\{\mathcal{E}^{\boldsymbol{w}}(f) \colon f \in \ell^2(\mathbb{Z}^d),\, \mathrm{supp} f \subseteq B_n,\, \|f\|_2 = 1\}\,. \tag{1.7}$$

The function f that minimizes the RHS of (1.7) is the principal Dirichlet eigenfunction $\psi_1^{(n)}$, whence $\lambda_1^{(n)} = \mathcal{E}^{\boldsymbol{w}}(\psi_1^{(n)})$.

Remark 1.1 (Perron-Frobenius)**.** *For a given box B_n the operator $\mathcal{L}_{\boldsymbol{w}}$ together with the zero Dirichlet boundary conditions can be written as a $|B_n| \times |B_n|$-matrix with non-negative entries everywhere except on the diagonal. Since the matrix is finite-dimensional, we can add a multiple of the identity to obtain a non-negative primitive matrix without changing the matrix' spectrum. By the Perron-Frobenius theorem (see e.g. [Sen81, Ch. 1]) it follows that its principal eigenvalue is unique and we can assume without loss of generality that its principal eigenvector is positive.*

In this paper we are especially interested in the behavior of the principal Dirichlet eigenvalue and eigenfunction for dimensions $d \ge 2$ and for conductances with a very heavy tail near zero. More precisely, we consider those cases where the conductances are distributed such that a local central limit theorem is not valid (cf. [BKM15, Remark 1.10]). Under different circumstances, the principal Dirichlet eigenvalue and eigenfunctions were studied before: Boivin and Depauw [BD03] proved that the top of the spectrum homogenizes. Recent results from Neukamm, Schäffner, and Schlömerkemper [NSS16] and Flegel, Heida, and Slowik [FHS] imply that the uniform ellipticity condition can be weakened to suitable moment conditions. The one-dimensional case was thoroughly covered by Faggionato [Fag12]. In Section 1.3 we comment on this background and how our results relate to this previous work.

1.2 Main results

First we give asymptotic lower and upper bounds for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$. How can we determine whether a function $g\colon (0,\infty) \to (0,\infty)$ that decreases monotonically to zero, is such an asymptotic lower or upper bound? To settle this, we will see that it is crucial to determine whether the box B_n contains a site such that all the $2d$ incident conductances are less than or equal to $g(n)$. We call such a site a $g(n)$-trap. A function $\Lambda_g\colon (0,\infty) \to (0,\infty)$ that carries the information about how many $g(n)$-traps we can expect in the box B_n, is defined by

$$\Lambda_g(n) = n^d \mathbb{P}[w \le g(n)]^{2d}\,. \tag{1.8}$$

Note that the factor n^d scales like the number of sites in the box B_n and the factor $\mathbb{P}[w \le g(n)]^{2d}$ relates to the probability that for a given site all the $2d$ incident links carry a conductance less than or equal to $g(n)$. We will see in Lemma 2.6 that if Λ_g diverges fast enough, then $\mathbb{P}$-a.s. for n large enough the box B_n contains at least one $g(n)$-trap. On the other hand, if Λ_g decreases fast enough to zero, then B_n does $\mathbb{P}$-a.s. not contain a $g(n)$-trap for n large enough, see Lemma 2.1.

In our results we often require recurring conditions on the function $g\colon (0,\infty) \to (0,\infty)$.

Assumption 1.2. *Let $g\colon (0,\infty) \to (0,\infty)$.*

(a) The function g varies regularly at infinity with index less than -2.

(b) The function $u \mapsto u^2 g(u)$ is monotone and has a finite limit as u tends to infinity.

(b') The function $u \mapsto u^2 g(u)$ converges monotonically to zero as u tends to infinity.

Our first theorem gives a sufficient and a necessary condition for when the function g is an asymptotic upper bound for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$. Note that, given one of the Assumptions 1.2 (a) or (b') is true, then the sufficient and necessary conditions coincide up to the case where Λ_g scales exactly like $\log\log n$. We summarize all the conditions of the following two theorems in a graphical overview (see Fig. 1.1).

Theorem 1.3 (Upper bound)**.** *Let $g : (0,\infty) \to (0,\infty)$ be a function that converges monotonically to zero and let Λ_g be as in (1.8). Then the following statements are true:*

(i) If there exists $\epsilon > 0$ such that for all n large enough

$$\frac{\Lambda_g(n)}{\log\log n} \ge 2 + \epsilon\,, \tag{1.9}$$

then there exists a constant $C < \infty$ such that $\mathbb{P}$-a.s. for n large enough $\lambda_1^{(n)} \le C g(n)$.

(ii) On the other hand, if

$$\lim_{u\to\infty} \frac{\Lambda_g(u)}{\log\log u} = 0\,, \tag{1.10}$$

and one of the Assumptions 1.2 (a) or (b') is true, then $\mathbb{P}$-a.s. $\limsup_{n\to\infty} \frac{\lambda_1^{(n)}}{g(n)} = \infty$.

We prove part (i) of the theorem in Section 1.4 and part (ii) in Section 1.6. Note that in (ii) the Assumptions 1.2 (a) and (b') correspond to the fact that we can only deduce that the limit superior diverges if we assume that g is in $o(n^{-2})$. This is because in the diffusive regime $\lambda_1^{(n)}$ scales like n^{-2}.

In the case where the distribution function $F(a)$ scales like a^γ with $\gamma > 0$, Theorem 1.3 (i) implies the following corollary.

Corollary 1.4. *Let $\delta > 0$. If F varies regularly at zero with index $\gamma > 0$, then $\mathbb{P}$-a.s. for n large enough the function $g(n) = n^{-\frac{1}{2\gamma}+\delta}$ is an asymptotic upper bound for $\lambda_1^{(n)}$.*

Note that if F varies regularly at zero with index $\gamma > 0$, then $\gamma = \sup\{q \ge 0\colon \mathbb{E}[w^{-q}] < \infty\}$, as defined in the introduction.

The second theorem gives conditions for when the function g is an asymptotic lower bound of the principal Dirichlet eigenvalue $\lambda_1^{(n)}$. Note that, given one of the Assumptions 1.2 (a) or (b) is true and Λ_g is bounded, then the condition in (1.11) is sharp. We further comment on these conditions in Section 1.6. As with the conditions of Theorem 1.3, we summarize them in the graphical overview Fig. 1.1.

Theorem 1.5 (Lower Bound)**.** *Let $g : (0,\infty) \to (0,\infty)$ be a decreasing function that fulfills one of the Assumptions 1.2 (a) or (b). Let Λ_g be as in (1.8). Then the following statements are true: If*

$$\int_0^\infty u^{-1}\Lambda_g(u)\,\mathrm{d}u < \infty \tag{1.11}$$

and Λ_g is bounded from above, then there exists a constant $c > 0$ such that $\mathbb{P}$-a.s. for n large enough $\lambda_1^{(n)} \geq cg(n)$. If, on the other hand, Condition (1.11) does not hold, then $\mathbb{P}$-a.s. $\liminf_{n\to\infty} \frac{\lambda_1^{(n)}}{g(n)} = 0$.

We prove the first part of this theorem in Section 1.6. The second part, i.e., where Condition (1.11) does not hold, is covered in Section 1.4.

Similarly as for Theorem 1.3, we obtain the following corollary.

Corollary 1.6. *Let $\delta > 0$. If F varies regularly at zero with index $\gamma \in (0, 1/4]$, then $\mathbb{P}$-a.s. for n large enough the function $g(n) = n^{-\frac{1}{2\gamma}-\delta}$ is an asymptotic lower bound for $\lambda_1^{(n)}$. Furthermore, if F varies regularly at zero with index $\gamma > 1/4$, then there exists $c > 0$ such that cn^{-2} is an asymptotic lower bound for $\lambda_1^{(n)}$. Then $\mathcal{E}^{\boldsymbol{w}}(f) \geq cn^2 \|f\|_2$ for all $f \in \ell^2(B_n)$, which is a Poincaré inequality for functions with bounded support.*

When we set $g(u) = F^{-1}(u^{-1/2})$, then Theorems 1.3 (ii) and 1.5 directly imply the following corollary.

Corollary 1.7. *Assume that there exists $v > 0$ such that $F|_{[0,v)}$ is invertible and that the function $u \mapsto u^2 F^{-1}\big(u^{-\frac{1}{2}}\big)$ converges monotonically to zero. Then*

$$\liminf_{n\to\infty} \frac{\lambda_1^{(n)}}{F^{-1}\Big(n^{-\frac{1}{2}}\Big)} = 0 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\lambda_1^{(n)}}{F^{-1}\Big(n^{-\frac{1}{2}}\Big)} = \infty \qquad \mathbb{P}\textit{-a.s.} \tag{1.12}$$

We comment on this behavior in Remark 1.12 in Section 1.4.

Note that in the special case where there exists $\gamma > 0$ such that the law $\mathbb{P}$ of the conductances fulfills $\mathbb{P}[w \leq a] = a^\gamma$ for $a \in [0, 1]$, Corollary 1.7 implies that

$$\liminf_{n\to\infty} n^{\frac{1}{2\gamma}} \lambda_1^{(n)} = 0 \quad \text{and} \quad \limsup_{n\to\infty} n^{\frac{1}{2\gamma}} \lambda_1^{(n)} = \infty \qquad \mathbb{P}\text{-a.s.} \tag{1.13}$$

Finally, we show that if we define the local speed measure $\pi\colon \mathbb{Z}^d \to [0, \infty)$ by

$$\pi_x = \sum_{y\colon y\sim x} w_{xy}\,, \quad x \in \mathbb{Z}^d\,, \tag{1.14}$$

then $\mathbb{P}$-a.s. as n tends to infinity, the first Dirichlet eigenvector $\psi_1^{(n)}$ localizes in the sequence of sites $(z_n)_{n\in\mathbb{N}}$ that minimize π over B_n.

Theorem 1.8 (Localization of the principal Dirichlet eigenvector)**.** *Let F vary regularly at zero with index $\gamma \in [0, 1/4)$. Further, assume that there exists $v > 0$ such that $F|_{[0,v)}$ is invertible. In the case where $\gamma = 0$, assume additionally that there exists $\epsilon_1 \in (0, 1)$ such that the product $n^{2+\epsilon_1} F^{-1}\big(n^{-1/2}\big)$ converges monotonically to zero as n grows to infinity. For $n \in \mathbb{N}$ let z_n be the site that minimizes π over B_n. Then $\mathbb{P}$-a.s. the mass of the principal Dirichlet eigenvector $\psi_1^{(n)}$ with zero Dirichlet conditions outside the box B_n increasingly concentrates in the site z_n, i.e.,*

$$\psi_1^{(n)}(z_n) \to 1 \quad \mathbb{P}\textit{-a.s.}$$

More precisely, $\mathbb{P}$-a.s. for n large enough

$$\psi_1^{(n)}(z_n)^2 \geq 1 - n^{-\epsilon_1/4}\,, \tag{1.15}$$

where for $\gamma > 0$ the value of $\epsilon_1 \in (0, 1)$ is chosen such that $1/(2\gamma) > 2 + \epsilon_1$. As a consequence $\lambda_1^{(n)}$ $\mathbb{P}$-a.s. behaves like $\min_{x\in B_n} \pi_x$ for large n, i.e.,

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{\lambda_1^{(n)}}{\min_{x\in B_n} \pi_x} = 1\right] = 1\,. \tag{1.16}$$

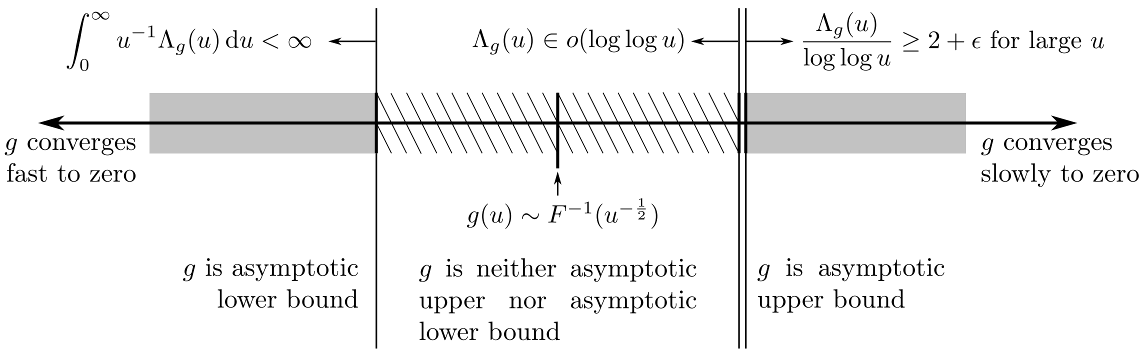

Figure 1.1: Visualization of our results from Theorems 1.3 and 1.5 for a fixed distribution function F. The figure shows the space of functions $g : (0,\infty) \to (0,\infty)$ that decrease to zero. The space is depicted such that if $f \in o(g)$, then f appears left of g. For simplicity we assume that g fulfills one of the Assumptions 1.2 (a) or (b'). If $F^{-1}(u^{-1/2}) \in o(g(u))$, then $\Lambda_g(u)$ diverges. If g even decays slowly enough such that condition (1.9) is fulfilled, then there exists $C < \infty$ such that $\mathbb{P}$-a.s. for n large enough $\lambda_1^{(n)} \leq Cg(n)$. On the other hand, if $g(u) \in o(F^{-1}(u^{-1/2}))$, then $\Lambda_g(u)$ converges to zero. If g even decays fast enough such that (1.11) is fulfilled, then there exists $c > 0$ such that $\mathbb{P}$-a.s. for n large enough $\lambda_1^{(n)} \geq cg(n)$. The figure also shows that around $g(u) \sim F^{-1}(u^{-1/2})$ there is an interval where g is definitely neither an a.s. asymptotic upper nor an a.s. asymptotic lower bound.

We prove this theorem in Section 5.

Remark 1.9. *If F varies regularly at zero with index $\gamma \in [0, 1/4)$, then there exists $\eta > 0$ such that the expectation $\mathbb{E}\big[w^{-1/4+\eta}\big]$ diverges.*

Remark 1.10 (Dimension one)**.** *Note that we cannot expect that a result like Theorem 1.8 holds in dimension one. This is because in dimension one, the probabilistic cost to generate a hardly reachable area is independent of the area's diameter.*

Remark 1.11 (Constant speed)**.** *If the conductances are bounded from above, we conjecture that, qualitatively, the above results should also hold for the constant-speed random conductance model, i.e., where the Laplacian is given by*

$$(\mathcal{L}_{\boldsymbol{w}} f)(x) = \pi_x^{-1} \sum_{y:\,|x-y|_1=1} w_{xy}(f(y) - f(x)) \qquad (x \in \mathbb{Z}^d, f \in \ell^2(\mathbb{Z}^d)) \,.$$

In this case, the critical exponent $\gamma_c^\pi = \frac{1}{8}\frac{d}{d-1/2}$ (cf. [BKM15, Theorem 1.8 (1)]). Further, the typical trapping structures are not single sites but pairs of sites (cf. [ADS16, Figure 1]). In a similar way as we adapt the proof techniques of [BKM15] for the variable-speed case, this should be possible for the constant-speed model. However, the proofs become much more technical.

1.3 Comparison with former results

Our investigation on the spectral behavior of the random conductance generator supplements the results of former research.

Boivin and Depauw [BD03] proved spectral homogenization for stationary and ergodic conductances that fulfill the uniform ellipticity condition, i.e., where there exist positive and finite constants a, b that uniformly bound the conductances from above and below. As a special case they showed that for uniformly elliptic i.i.d. conductances there exists a constant $c > 0$ such that if $\lambda_1^{(n)} < \lambda_2^{(n)} \leq \ldots \leq \lambda_k^{(n)}$ are the first k Dirichlet eigenvalues of $-\mathcal{L}_{\boldsymbol{w}}$, then for almost every realization of the conductance landscape

$$\lim_{n\to\infty} n^2 \lambda_k^{(n)} = c\lambda_k \,,$$

where λ_k is the kth eigenvalue of the operator $-\Delta$ in $(-1,1)^d$ with zero Dirichlet conditions. Additionally, the principal Dirichlet eigenfunction of $-\mathcal{L}_{\boldsymbol{w}}$ converges, properly rescaled, to the principal Dirichlet eigenfunction of the operator $-\Delta$ in $(-1,1)^d$ with zero Dirichlet conditions [BD03, Theorem 1, Corollary 1].

In the special case of dimension one, the uniform ellipticity condition was already weakened by Faggionato [Fag12]: She showed that for $d=1$ a finite inverse moment of w is sufficient for spectral homogenization [Fag12, Proposition 2.6]. Further, if the inverse conductances w^{-1} are i.i.d. and in the domain of attraction of an α-stable law with $0<\alpha<1$, then Faggionato showed that the vector of the first k Dirichlet eigenvalues rescaled by $n^{1+1/\alpha}$ times a slowly varying function converges in distribution to the vector of the first k Dirichlet eigenvalues of a random generalized differential operator [Fag12, Theorem 2.5].

Recent results from two teams of authors imply that the uniform ellipticity condition can also be weakened in higher dimensions: Neukamm, Schäffner, and Schlömerkemper [NSS16, Corollary 3.4, Proposition 3.18] proved amongst other results that for $\gamma > 1/4$ the Dirichlet energy of $-\mathcal{L}_{\boldsymbol{w}}$ Γ-converges to a deterministic, homogeneous integral. This together with their compactness result [NSS16, Lemma 3.9] and [Mas93, Theorem 13.5] implies that Conditions I–IV of [JKO94, Chapter 11] are fulfilled and spectral convergence follows. On the other hand, Flegel, Heida, and Slowik [FHS] use the method of stochastic two-scale convergence by Zhikov and Pyatniskii [ZP06] to show that the Poisson equation homogenizes. Their approach is similar to the one of Faggionato [Fag08] who already employed two-scale convergence in order to show homogenization for a Laplacian with shifted spectrum and bounded conductances. From the homogenization of the Poisson equation, the spectral homogenization follows again by [JKO94, Chapter 11]. Furthermore, Flegel, Heida, and Slowik identify the corresponding limit operator.

The basis for both [NSS16] and [FHS] are Poincaré and Sobolev inequalities that were already used by Andres, Deuschel, and Slowik [ADS16] to prove a quenched local CLT under suitable moment conditions.

1.4 Survey on proofs for upper bounds

Let us consider the variational formula (1.7). The equation implies that for any real-valued test function $f \in \ell^2(\mathbb{Z}^d)$ with supp $f \subseteq B_n$ and $\|f\|_2 = 1$ we can estimate

$$\lambda_1^{(n)} \leq \langle f, -\mathcal{L}_{\boldsymbol{w}} f\rangle = \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{y\colon y\sim x} w_{xy}(f(x)-f(y))^2\,.$$

Suppose that z_n is a random site that minimizes π (see (1.14)) in B_n. Now we choose the function f such that its whole mass is concentrated in the site $z_n \in B_n$, i.e., $f = \delta_{z_n}$. When we insert this into the variational formula (1.7), then we obtain that

$$\lambda_1^{(n)} \leq \min_{x\in B_n} \pi_x \leq 2d \min_{x\in B_n} \max_{y\colon x\sim y} w_{xy}\,. \tag{1.17}$$

It remains to find conditions under which the above RHS can be bounded from above by a decreasing function $g(n)$. As we have already mentioned before, a quantity which carries this information, is the function Λ_g defined in (1.8), as we see in the two following proofs.

Proof of Theorem 1.3 (i). Condition (1.9) together with Lemma 2.6 implies that $\mathbb{P}$-a.s. for n large enough there exists a site $z_n \in B_n$ such that $\max_{y\colon y\sim z_n} w_{z_n y} \leq g(n)$. Choose the test function $f_n = \delta_{z_n}$ and insert it into the variational formula (1.7). The claim follows. □

Proof of Theorem 1.5 if Condition (1.11) *fails.* If Condition (1.11) fails, then

$$c\int_0^\infty u^{-1}\Lambda_g(u)\,\mathrm{d}u = \infty \quad \text{for any } c>0. \tag{1.18}$$

Let $\mathfrak{N} = \{e \in \mathfrak{E}_d\colon 0 \in e\}$ be the set of edges incident to the origin and note that $|\mathfrak{N}| = 2d$. A substitution of variables and Lemma 2.1 imply that for any $c > 0$ the following event occurs $\mathbb{P}$-a.s. infinitely often as $n \to \infty$:

There exists a site $z_n \in B_{n+1}$ such that all edges in $\tau_{z_n} \circ \mathfrak{N}$ have conductance smaller than or equal to $g(cn)$. Here, τ_z ($z \in \mathbb{Z}^d$) denotes the spatial shift operator.

Every time this event occurs, we choose the test function $f_n = \delta_{z_n}$ (as in the proof of Theorem 1.3 (i)), insert it into the variational formula (1.7) and immediately obtain that $\mathbb{P}$-a.s.

$$\liminf_{n\to\infty} \frac{\lambda_1^{(n)}}{g(cn)} \le 2d \quad \text{for any } c > 0\,.$$

We now show that this implies the claim. Suppose that $c \in (0,1)$. We have assumed that one of the Assumptions 1.2 (a) or (b) is true. In any case it follows that there exists $c_1 > 0$ independent of c such that eventually $g(cn) \le c_1 c^{-2} g(n)$. It follows that $\mathbb{P}$-a.s. $\liminf_{n\to\infty} \frac{\lambda_1^{(n)}}{g(n)} \le 2dc_1c^2$. This holds for any $c \in (0,1)$, implying that $\mathbb{P}$-a.s. $\liminf_{n\to\infty} \lambda_1^{(n)}/g(n) = 0$. □

Remark 1.12. *Now we can intuitively understand the result of Corollary 1.7: For the choice $g(u) = F^{-1}(u^{-1/2})$, the function Λ_g is constant one. Therefore for every $c > 0$ $\mathbb{P}$-a.s. there exists an infinite subsequence n_k where the box B_{n_k} contains a $(cg(n_k))$-trap. However, as we will see in Section 1.6, $\mathbb{P}$-a.s. there also exists an infinite subsequence n'_k where the box $B_{n'_k}$ does not contain a sufficiently good trap. It follows that the asymptotics of $\lambda_1^{(n)}$ fluctuate around the asymptotics of $F^{-1}(u^{-1/2})$.*

1.5 Heuristics for critical moments and relation to heat-kernel upper bounds

Our first aim in this section is to give a heuristic argument why the $1/4$-moment of the inverse conductance w^{-1} decides between spectral homogenization and localization. Our second goal is to explain why the subdiffusive scaling of the principal Dirichlet eigenvalue contradicts the validity of a local central limit theorem.

Let us first consider a realization of the environment in which all conductances are equal to one. In this situation the operator $\mathcal{L}_{\boldsymbol{w}}$ generates a simple random walk on $\mathbb{Z}^d$. It is known that in this case the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ of $-\mathcal{L}_{\boldsymbol{w}}$ in B_n with zero Dirichlet conditions scales like n^{-2} (see e.g. [BD03, Eq. (7)]). We call this diffusive scaling. In contrast, if $\lambda_1^{(n)} \in o(n^{-2})$, we say that $\lambda_1^{(n)}$ scales subdiffusively.

We will now investigate under what circumstances we can expect subdiffusive behavior. Let us recall (1.17), i.e., that $\lambda_1^{(n)} \le 2d \min_{x\in B_n} \max_{y:\, x\sim y} w_{xy}$.

For simplicity we assume that there exists $\gamma > 0$ such that the law $\mathbb{P}$ of the conductances fulfills

$$\mathbb{P}[w \le a] = a^\gamma \qquad \text{for } a \in [0,1]\,, \tag{1.19}$$

and we choose $g(n) = n^{-\frac{1}{2\gamma}+\delta}$ with $\delta \in \mathbb{R}$. Note that $\gamma = \sup\{q \ge 0\colon \mathbb{E}[w^{-q}] < \infty\}$ as defined in the introduction. We recall that a $g(n)$-trap is a site where all the $2d$ incident conductances are less than or equal to $g(n)$. We observe that

$$\Lambda_g(n) = n^{2d\gamma\delta}\,,$$

which diverges if $\delta > 0$ and converges to zero if $\delta < 0$. Correspondingly, the Borel-Cantelli arguments of Lemmas 2.1 and 2.6 imply that if we fix an environment $\boldsymbol{w} \in \Omega$, then $\mathbb{P}$-a.s. for n large enough there exists a $g(n)$-trap in B_n if $\delta > 0$. On the other hand, there does not exist a $g(n)$-trap in B_n if $\delta < 0$. This was already pointed out in [BKM15, Remark 1.10].

Now we simply note that if and only if $\gamma < 1/4$, then there exists $\delta > 0$ such that $g(n) = n^{-\frac{1}{2\gamma}+\delta}$ is in $o(n^{-2})$. Hence the principal Dirichlet eigenvalue scales subdiffusively if $\gamma < 1/4$. It is remarkable that this condition is also sufficient to prove a complete asymptotic localization of the principal Dirichlet eigenvector, see Theorem 1.8.

On the other hand, if $\gamma \ge 1/4$, then these considerations do not provide us with a sub-diffusive upper bound for $\lambda_1^{(n)}$. To the contrary: For $\gamma > 1/4$ Theorem 1.5 implies that there exists a constant $c > 0$ such that $\mathbb{P}$-a.s. for n large enough $\lambda_1^{(n)} \ge cn^{-2}$.

Note that this is equivalent to a Poincaré inequality for functions with bounded support, which states that there exists $c > 0$ such that $\mathbb{P}$-a.s. for n large enough

$$\langle f, -\mathcal{L}_{\boldsymbol{w}} f\rangle \geq cn^{-2}\|f\|_2 \quad \text{(for } f \in \ell^2(\mathbb{Z}^d) \text{ with supp } f \subseteq B_n\text{)},$$

which for i.i.d. conductances with finite expectation of $w^{-1/4}$ is a consequence of [ADS16, Proposition 2.4] (with $q = d/2$, η a step function and ν_ω replaced a $\tilde{\nu}_\omega$ which for each neighbor sums over the optimal detour from the $2d$ independent paths in Figure 2 of [ADS16]).

Furthermore, $\lambda_1^{(n)} \geq cn^{-2}$ is also a necessary consequence of a diffusive scaling of quenched heat-kernel upper bounds and thus the validity of a quenched local central limit theorem. The local CLT was established in 2015 by the two teams of authors Andres, Deuschel, Slowik [ADS16] and Boukhadra, Kumagai, Mathieu [BKM15] who require that for i.i.d. conductances there exists $\epsilon > 0$ such that $\mathbb{E}[w]$ and $\mathbb{E}[w^{-1/4-\epsilon}]$ are finite. Let us briefly comment on this.

First we define the random walk among random conductances. This is the Markov chain X_t that is generated by the operator $\mathcal{L}_{\boldsymbol{w}}$. Its behavior is as follows: When the walker is at a site $x \in \mathbb{Z}^d$ it waits for an exponential time with expectation π_x^{-1} (see (1.14)) and then jumps to one of the neighboring sites. This is why we call π_x the local speed of the random walk at the site x. To which neighbor the random walker jumps is random with probabilities proportional to the corresponding conductances: If z is a specific neighbor of x, the random walker jumps to z with probability w_{xz}/π_x. We call $P_x^{\boldsymbol{w}}$ the probability w.r.t. to the random walk where the superscript $\boldsymbol{w}$ refers to a fixed environment (quenched probability) and the subscript x refers to the starting point of the random walker: $P_x^{\boldsymbol{w}}[X_0 = x] = 1$. The corresponding expectation is called $E_x^{\boldsymbol{w}}$.

Let τ_A be the escape time from a set $A \subset \mathbb{Z}^d$, i.e., $\tau_A = \inf\{t \geq 0 \colon X_t \notin A\}$. There exists a natural relation between the principal Dirichlet eigenvalue of the operator $-\mathcal{L}_{\boldsymbol{w}}$ and the expected escape time $E_x^{\boldsymbol{w}}[\tau_{B_n}]$ from the box B_n, see [BdH15, Section 8.4.1]:

$$\lambda_1^{(n)} \geq \left(\max_{z\in B_n} E_z^{\boldsymbol{w}}\left[\tau_{B_n}\right]\right)^{-1}.$$

Thus, an upper bound for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ implies a lower bound on the maximal expected escape time from the box B_n.

Now Lemma 2.1(i) of [BKM15] implies that if the heat kernel

$$p_t(x,y) := P_x^{\boldsymbol{w}}[X_t = y] \qquad (x, y \in \mathbb{Z}^d\,,\ t \geq 0)\,,$$

has a diffusive on-diagonal upper bound, i.e. there exists $c \in (0,\infty)$ and a random $n_0 \in \mathbb{N}$ such that

$$p_{n^2}(x,y) \leq cn^{-d} \qquad \forall x, y \in B_n\,,\ n \geq n_0\,,$$

then $\max_{z\in B_n} E_z^{\boldsymbol{w}}\left[\tau_{B_n}\right] \sim n^{-2}$. Diffusive heat-kernel upper bounds are a necessary condition for the validity of a local CLT.

But if we assume that the principal Dirichlet eigenvalue scales subdiffusively, i.e., $\lambda_1^{(n)} \in o(n^{-2})$, then eventually $\max_{z\in B_n} E_z^{\boldsymbol{w}}\left[\tau_{B_n}\right] \gg n^{-2}$ and therefore a subdiffusively scaling principal Dirichlet eigenvalue contradicts the validity of a local CLT.

We can explain the exploding escape times by showing that a large box contains some sites where the expected time to even leave the initial position is anomalously long. Although this effect is related to the one responsible for the anomalous heat-kernel decay observed in [BBHK04], it is still a different one. In [BBHK04], the dominating effect is that a random walk finds a trap elsewhere and then returns to its initial position. This behavior, however, has a more complex dependence on the Laplacian's eigenvalues.

1.6 Survey on proofs for lower bounds

For the lower bound of the principal Dirichlet eigenvalue we have to put in significantly more work than for the upper bound. The key idea, however, is linked to the considerations for the shape theorem of first-passage percolation, see e.g. Cox and Durrett [CD81] for the sample case $d = 2$. The philosophy is that we have to show that each site in the box B_n is sufficiently well reachable by conductances that are significantly greater than the lower bound candidate g. Note that this is similar to the idea of Lemma 4.6 in [BKM15] where the authors proved this for a polynomial tail of the conductances with parameter γ and the candidate $g(n) = n^{-\alpha}$ with $\alpha > 1/(2\gamma)$. It turns out that a crucial element of the proof is to give a condition that implies that $\mathbb{P}$-a.s. for n large enough all sites in the box B_n have at least one link with conductance greater than $g(n)$, similar to [CD81, p. 585] and [Kes86, p. 127]. In general, if g is monotonically decreasing and $w_1 \ldots, w_{2d}$ are $2d$ independent copies of the conductance w, and

$$\mathbb{E}\Big[g^{-1}(\max\{w_1,\ldots,w_{2d}\})^d\Big] = d\int_0^\infty u^{-1}\Lambda_g(u)\mathrm{d}u < \infty\,, \tag{1.20}$$

then $\mathbb{P}$-a.s. for n large enough, all sites in the box B_n have at least one link with conductance greater than $g(n)$. This together with a path argument, which we adapt from [BKM15] gives the $\mathbb{P}$-a.s. lower bound for the principal Dirichlet eigenvalue $\lambda_1^{(n)}$ (given that $g(n)$ is not asymptotically larger than n^{-2}). On the other hand, if Condition (1.20) is violated, then the same arguments as in Cox and Durrett [CD81, p. 585] yield that $\mathbb{P}$-a.s. as n tends to infinity, the box B_n contains a $g(n)$-trap infinitely often. We have already dealt with this case at the end of Section 1.4.

In what follows we give a survey on the proofs of Theorem 1.3 (ii) as well as Theorem 1.5 if Condition (1.11) (or equivalently (1.20)) holds. The arguments described above are made rigorous in several auxiliary lemmas, which we present in the subsequent sections.

Proof of Theorem 1.5 if Condition (1.11) holds. By virtue of Lemma 2.1 with $\mathfrak{A} = \{e \in \mathfrak{E}_d : 0 \in e\}$ it follows that $\mathbb{P}$-a.s. there exists $n_1^* \in \mathbb{N}$ such that for all $n \geq n_1^*$ all sites $z \in B_n$ have an incident edge with conductance greater than $g(n)$. Further, let $b \in \mathbb{N}$ be such that $b \gg 4d$. Since we assumed that Λ_g is bounded from above, it follows by virtue of Corollary 2.4 (with $m = 2d$ and $k = d$) that there exists $\epsilon > 0$ such that $\mathbb{P}$-a.s. there exists $n_2^* \in \mathbb{N}$ such that for all $n \geq n_2^*$ and for all $z \in B_{n+b}$ the box $B_b(z)$ contains at most $3d - 1$ links with conductance less than or equal to $g(n^{1-\epsilon})$. Set $n_k = k + \max(n_1^*, n_2^*)$. The claim follows by virtue of Proposition 4.7. □

Proof of Theorem 1.3 (ii). Condition (1.10) together with the Borel-Cantelli argument of Lemma 2.7 implies that for any $\bar{c} > 0$, $\mathbb{P}$-a.s. as the box size n grows to infinity, there exists a random subsequence $n' = n'(\omega)$ along which each site $z \in B_{n'}$ has at least one incident link e such that $w_e > \bar{c}g(n')$. Further, by Corollary 2.5 we know the following: For a fixed $b \in \mathbb{N}$ there exists $\epsilon > 0$ such that $\mathbb{P}$-a.s. for n large enough, there are at most $2d$ edges in any subbox $B_b(z) \subset B_{n+b}$ with conductance smaller than or equal to $g(n^{1-\epsilon})$. It follows that we can apply Proposition 4.7 with $\bar{c}g$ instead of g and obtain that there exists $C > 0$ (independent of $\bar{c}$) such that $\mathbb{P}$-a.s. along the subsequence n'_k and for k large enough the following holds: For any $f\colon \mathbb{Z}^d \to \mathbb{R}$ with supp $f \subseteq B_{n'_k}$ it is

$$\mathcal{E}^{\boldsymbol{w}}(f) \geq C\bar{c}g(n'_k)\|f\|_2^2\,.$$

Since this holds for any $\bar{c} > 0$, this implies the claim. □

2 Borel-Cantelli arguments

In this section we always assume that the dimension $d \geq 2$ and that the conductances are i.i.d. with law $\mathbb{P}$. We further let $g\colon (0,\infty) \to (0,\infty)$ be a function that decreases monotonically to zero. Moreover, we use the following abbreviations: For $\alpha > 0$ and an edge set $\mathfrak{A} \subseteq \mathfrak{E}_d$ we define the event

$$J_\alpha(\mathfrak{A}) = \{\exists e \in \mathfrak{A} : w_e > \alpha\}\,.$$

For a set $A \subset \mathbb{Z}^d$ we define $\mathfrak{E}(A)$ to be the set of edges that connect a site in A with a neighbor in positive axes direction (i.e., right, above, in front, etc.), i.e.,

$$\mathfrak{E}(A) = \{\{x,y\} \in \mathfrak{E}_d : x \in A \text{ and } \exists j \in \{1,\ldots,d\} \text{ such that } y = x + \boldsymbol{e}_j\}\,,$$

where $\{\boldsymbol{e}_j\}$ is the set of unit base vectors of $\mathbb{Z}^d$. For $\mathfrak{A} \subseteq \mathfrak{E}_d$ we write $\tau_z \circ \mathfrak{A}$ for the translation of $\mathfrak{A}$ by $z \in \mathbb{Z}^d$.

Lemma 2.1. *If $b \in \mathbb{N}$ and $\mathfrak{A} \subseteq \mathfrak{E}(B_b)$ is an edge set with $|\mathfrak{A}| = m$, then*

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{z\in B_{n+b}} J_{g(n)}(\tau_z \circ \mathfrak{A})\right] = \begin{cases} 1\,, & \text{if } \int_0^\infty u^{d-1}\mathbb{P}[w \le g(u)]^m \mathrm{d}u < \infty\,, \quad (2.1a) \\ 0\,, & \text{otherwise.} \quad (2.1b) \end{cases}$$

I.e., if and only if the integral $\int_0^\infty u^{d-1}\mathbb{P}[w \le g(u)]^m \mathrm{d}u$ is finite, then $\mathbb{P}$-a.s. for n large enough for all sites $z \in B_{n+b}$ the edge set $\tau_z \circ \mathfrak{A}$ contains a conductance greater than $g(n)$. Otherwise the counter event occurs infinitely often.

Remark 2.2. *The result of Lemma 2.1 as well as the proof are generalizations of the considerations of Cox and Durrett [CD81] and Kesten [Kes03, p. 108] (there, $m = 2d$ and $g(n) = n^{-1}$). For the sake of completeness, we included the proofs here.*

Proof of Lemma 2.1. For (2.1a): We first show that

$$0 = 1 - \mathbb{P}\left[\liminf_{|z|_\infty\to\infty} J_{g(|z|_\infty - b)}(\tau_z \circ \mathfrak{A})\right] = \mathbb{P}\left[\limsup_{|z|_\infty\to\infty} \left(J_{g(|z|_\infty - b)}(\tau_z \circ \mathfrak{A})\right)^{\mathrm{c}}\right]. \tag{2.2}$$

We achieve this by applying the first Borel-Cantelli lemma, i.e., we have to estimate

$$\begin{aligned} \sum_{z\in\mathbb{Z}^d\setminus B_b} \mathbb{P}\left[\left(J_{g(|z|_\infty - b)}(\tau_z \circ \mathfrak{A})\right)^{\mathrm{c}}\right] &= \sum_{k=b+1}^{\infty} \sum_{|z|_\infty = k} \mathbb{P}[w \le g(|z|_\infty - b)]^m \\ &\le 2d \sum_{k=b+1}^{\infty} (2k+1)^{d-1}\mathbb{P}[w \le g(k-b)]^m \,. \end{aligned}$$

Since $g(\cdot)$ is monotonically decreasing, $\mathbb{P}[w \le g(\cdot)]$ is monotonically decreasing as well. Further, there exists an index k_b such that $k - b \ge 2^{-1}(k+1)$ for all $k \ge k_b$. It follows that there exists $C < \infty$ such that

$$\sum_{z\in\mathbb{Z}^d\setminus B_b} \mathbb{P}\left[\left(J_{g(|z|_\infty - b)}(\tau_z \circ \mathfrak{A})\right)^{\mathrm{c}}\right] \le C + 4^d d \sum_{k=b+1}^{\infty} (2^{-1}(k+1))^{d-1}\mathbb{P}\left[w \le g\left(2^{-1}(k+1)\right)\right]^m .$$

Thus, there exists $c < \infty$ such that the LHS is bounded from above by $C + c\int_0^\infty u^{d-1}\mathbb{P}[w \le g(u)]^m \mathrm{d}u$, which is finite by assumption. The claim (2.2) follows from the first Borel-Cantelli lemma.

To arrive at the claim of the lemma, we observe that (2.2) implies that there $\mathbb{P}$-a.s. exists $n^* \in \mathbb{N}$ such that for all $|z|_\infty \ge n^*$ the set $\tau_z \circ \mathfrak{A}$ contains at least one conductance with $w > g(|z|_\infty - b)$. If $n > n^*$ and $z \in B_{n+b}\setminus B_{n^*}$, i.e., $|z|_\infty \in (n^*, n+b]$, this means that $\tau_z \circ \mathfrak{A}$ contains at least one conductance with $w > g(n)$ (recall that g is monotonically decreasing). Since n^* is finite and g decreases monotonically to zero, it also follows that there exists a finite $n' \ge n^*$ such that for all edges $e \in \mathfrak{E}(B_{n^*+1})$ we have $g(n') < w_e$. Thus, $\bigcap_{z\in B_{n+b}} J_{g(n)}(\tau_z \circ \mathfrak{A})$ is true $\mathbb{P}$-a.s. for n large enough.

For (2.1b): Let $\int_0^\infty u^{d-1}\mathbb{P}[w \le g(u)]^m \mathrm{d}u = \infty$. We want to show that this implies

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{z\in B_{n+b}} J_{g(n)}(\tau_z \circ \mathfrak{A})\right] = 0\,. \tag{2.3}$$

Let us define the set $A_b = (2b+1)\mathbb{Z}^d$. It suffices to prove the claim (2.3) for the intersection over $z \in B_{n+b} \cap A_b$, which in turn follows by the second Borel-Cantelli lemma if

$$\sum_{z\in A_b} \mathbb{P}\left[\left(J_{g(|z|+b)}(\tau_z \circ \mathfrak{A})\right)^{\mathrm{c}}\right] = \infty\,,$$

since the events $\left\{J_{g(|z|+b)}(\tau_z \circ \mathfrak{A})\right\}_{z\in A_b}$ are independent. To prove that the above sum diverges, we observe that there exists a constant $C > 0$ such that

$$\sum_{z\in A_b} \mathbb{P}\big[\big(J_{g(|z|+b)}(\tau_z \circ \mathfrak{A})\big)^{c}\big] \geq 2d(2b+1)^d \sum_{k=1}^{\infty}(2k-1)^{d-1}\mathbb{P}[w \leq g((2b+1)k+b)]^m$$

$$\geq C \int_0^{\infty} u^{d-1}\mathbb{P}[w \leq g(u)]^m \mathrm{d}u\,.$$

By the assumption that $\int_0^\infty u^{d-1}\mathbb{P}[w \leq g(u)]^m \mathrm{d}u = \infty$, the sum diverges. □

Corollary 2.3 (of Lemma 2.1)**.** *Let $b \in \mathbb{N}$ and $m \leq |\mathfrak{E}(B_b)|$. Then the following equivalence holds:*

$$\int_0^{\infty} u^{d-1}\mathbb{P}[w \leq g(u)]^m \mathrm{d}u < \infty \;\Leftrightarrow\; \mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{\substack{\mathfrak{A}\subseteq\mathfrak{E}(B_b),\\ |\mathfrak{A}|\geq m}} \bigcap_{z\in B_{n+b}} J_{g(n)}(\tau_z \circ \mathfrak{A})\right] = 1\,. \tag{2.4}$$

Proof of Corollary 2.3. For "⇐", we apply Lemma 2.1 for an arbitrary $\mathfrak{A} \subseteq \mathfrak{E}(B_b)$ with $|\mathfrak{A}| = m$. For "⇒", note that since B_b is finite, the intersection over the edge sets $\mathfrak{A}$ on the RHS of (2.4) runs over finitely many events. By virtue of Lemma 2.1 the claim holds for each of these events and therefore also for the finite intersection. □

Corollary 2.4 (of Corollary 2.3)**.** *Let $b \in \mathbb{N}$, $m, k \in \mathbb{N}$ with $m < |\mathfrak{E}(B_b)|$. If $u^d\mathbb{P}[w \leq g(u)]^m$ is bounded from above, then*

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{\substack{\mathfrak{A}\subseteq\mathfrak{E}(B_b),\\ |\mathfrak{A}|\geq m+k}} \bigcap_{z\in B_{n+b}} J_{g(n^{1-\epsilon})}(\tau_z \circ \mathfrak{A})\right] = 1 \qquad \textit{for all } \epsilon \in [0, k(m+k)^{-1})\,. \tag{2.5}$$

Proof. We show that the integral $\int_0^\infty v^{d-1}\mathbb{P}\big[w \leq g\big(v^{1-\epsilon}\big)\big]^{m+k}\mathrm{d}v$ is finite and then we apply Corollary 2.3. The change of variable $v^{1-\epsilon} = u$ yields

$$\int_0^{\infty} v^{d-1}\mathbb{P}\big[w \leq g\big(v^{1-\epsilon}\big)\big]^{m+k}\mathrm{d}v = (1-\epsilon)^{-1}\int_0^{\infty} u^{d(1-\epsilon)^{-1}-1}\mathbb{P}[w \leq g(u)]^{m+k}\,\mathrm{d}u\,.$$

Now we consider that

$$u^{d(1-\epsilon)^{-1}-1}\mathbb{P}[w \leq g(u)]^{m+k} = u^{d\left((1-\epsilon)^{-1}-1-\frac{k}{m}\right)-1}\big(u^d\mathbb{P}[w \leq g(u)]^m\big)^{1+\frac{k}{m}}\,.$$

Since both $u^d\mathbb{P}[w \leq g(u)]^m$ and $\mathbb{P}[w \leq g(u)]^m$ are bounded from above, we obtain that

$$\int_0^{\infty} u^{d(1-\epsilon)^{-1}-1}\mathbb{P}[w \leq g(u)]^{m+k}\,\mathrm{d}u \leq \int_0^{u_1} u^{d(1-\epsilon)^{-1}-1}\,\mathrm{d}u + C\int_{u_1}^{\infty} u^{d\left((1-\epsilon)^{-1}-1-\frac{k}{m}\right)-1}\,\mathrm{d}u < \infty$$

for any $u_1 \in (0,\infty)$ and a suitable $C < \infty$. Since $\epsilon \in [0,1)$ and $d \geq 2$, the first integral on the RHS is finite. Further, since $\epsilon < k(m+k)^{-1}$, the second integral on the RHS is finite as well. □

For the next three results, we define Λ_g as in (1.8).

Corollary 2.5 (of Corollary 2.3)**.** *Let $b \geq 2$ and assume that $\Lambda_g(u)/\log\log u$ is bounded from above. Then there exists $\epsilon > 0$ such that*

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{\substack{\mathfrak{A}\subseteq\mathfrak{E}(B_b),\\ |\mathfrak{A}|\geq 2d+1}} \bigcap_{z\in B_{n+b}} J_{g(n^{1-\epsilon})}(\tau_z \circ \mathfrak{A})\right] = 1\,. \tag{2.6}$$

Proof. We show that the integral in (2.4) is finite for $m = 2d+1$ and $g(u^{1-\epsilon})$ instead of $g(u)$. The assumption on the function Λ_g implies that there exists $C < \infty$ such that

$$\frac{u^{(d+\frac{1}{2})(1-\epsilon)}\mathbb{P}\big[w \le g(u^{1-\epsilon})\big]^{2d+1}}{(\log\log u^{1-\epsilon})^{1+\frac{1}{2d}}} < C \quad \text{for all } u > 0 \text{ and } \epsilon \in (0,1).$$

It follows that

$$u^d\mathbb{P}\big[w \le g(u^{1-\epsilon})\big]^{2d+1} \le Cu^{-\frac{1}{2}+\epsilon(d+\frac{1}{2})}\big(\log\log u^{1-\epsilon}\big)^{1+\frac{1}{2d}} \quad \text{for all } u > 0.$$

For all $\epsilon < (2d+1)^{-1}$, this implies that the integral $\int_0^\infty u^{d-1}\mathbb{P}\big[w \le g(u^{1-\epsilon})\big]^{2d+1}\,\mathrm{d}u$ is finite. The claim follows by virtue of Corollary 2.3. □

For the next lemma we need the following definition: The even lattice is the set

$$A_{\mathrm{e}} = \big\{z \in \mathbb{Z}^d\colon |z|_1 \equiv 0 \text{ mod } 2\big\}\,. \tag{2.7}$$

Accordingly, the odd lattice is $A_{\mathrm{o}} = \mathbb{Z}^d \backslash A_{\mathrm{e}}$, see Fig. 2.1 for a sketch. Now, the next lemma states that if a certain condition is fulfilled, then $\mathbb{P}$-a.s. for n large enough there exist sites $z_n^{\mathrm{e}} \in B_n \cap A_{\mathrm{e}}$ and $z^{\mathrm{o}} \in B_n \cap A_{\mathrm{o}}$ such that all incident links to these sites have conductance less than or equal to $g(n)$. This implies that there are at least two sites in B_n whose incident links are all less than or equal to $g(n)$.

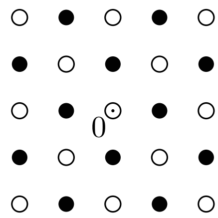

Figure 2.1: Even (white circles) and odd lattice (black circles) as defined in (2.7) and below.

Lemma 2.6. *Let $\mathfrak{N} = \{e \in \mathfrak{E}_d\colon 0 \in e\}$ and $A \in \{A_{\mathrm{e}}, A_{\mathrm{o}}\}$. Then the following implication is true: If there exists $\epsilon > 0$ and $n^* \in \mathbb{N}$ such that*

$$\frac{\Lambda_g(n)}{\log\log n} \ge 2+\epsilon \text{ for all } n \ge n^*, \text{ then } \mathbb{P}\left[\limsup_{n\to\infty}\left(\bigcap_{z\in B_n\cap A} J_{g(n)}(\tau_z \circ \mathfrak{N})\right)\right] = 0\,,$$

i.e., $\mathbb{P}$-a.s. for n large enough there exists a site $z_n \in B_n \cap A$ that is completely surrounded by edges with conductances less than or equal to $g(n)$.

Proof of Lemma 2.6. We first prove the claim for the subsequence $n_j = 2^j$ with $j \in \mathbb{N}$ and with $g(2n)$ instead of $g(n)$. Then we show how to infer the claim along the whole sequence $n \in \mathbb{N}$.

For the first part, note that the sets $A_{\mathrm{e}} \cap B_n$ and $A_{\mathrm{o}} \cap B_n$ have cardinality greater than $2^{d-1}n^d$. Further, for any $\alpha > 0$ the events $\{J_\alpha(\tau_z \circ \mathfrak{N})\}_{z\in A_{\mathrm{e}}}$ are independent, as well as are the events $\{J_\alpha(\tau_z \circ \mathfrak{N})\}_{z\in A_{\mathrm{o}}}$. Let $w_1, \dots, w_{2d}$ be $2d$ independent copies of w. Then for both $A \in \{A_{\mathrm{e}}, A_{\mathrm{o}}\}$ we can estimate

$$\begin{aligned}\mathbb{P}\left[\bigcap_{z\in B_{n_j}\cap A} J_{g(2n_j)}(\tau_z \circ \mathfrak{N})\right] &= \mathbb{P}[\max\{w_1,\dots,w_{2d}\} > g(2n_j)]^{|B_{n_j}\cap A|}\\ &\le \Big(1 - \mathbb{P}[w \le g(2n_j)]^{2d}\Big)^{2^{d-1}n_j^d} \le \exp\Big(-\frac{1}{2}(2n_j)^d\mathbb{P}[w \le g(2n_j)]^{2d}\Big)\,.\end{aligned}$$

The assumption on Λ_g implies that the RHS is summable along the sequence $n_j = 2^j$. Thus, it follows directly by the Borel-Cantelli lemma that the statement of this lemma holds along the subsequence n_j and with $g(2n_j)$ instead of $g(n_j)$.

To infer the claim of the lemma along the entire sequence, we define

$$M_n^{\mathrm{e,o}} := \inf_{x\in B_n\cap A_{\mathrm{e,o}}}\ \sup_{e\in\tau_x\circ\mathfrak{N}} w_e.$$

Note that both $M_n^{\mathrm{e,o}}$ are monotonically decreasing in n. By the first part of the proof we know that

$$\mathbb{P}\left[\liminf_{j\to\infty}\frac{M_{n_j}^{\mathrm{e,o}}}{g(2n_j)} \le 1\right] = 1\,. \tag{2.8}$$

Now for $n \in \mathbb{N}$ we choose j_n such that

$$2^{j_n} \leq n \leq 2^{j_n+1}.$$

Since g^{-1} and $M^{\mathrm{e,o}}_{(\,\cdot\,)}$ are all monotonically decreasing, this implies that

$$M^{\mathrm{e,o}}_{2^{j_n}} \geq M^{\mathrm{e,o}}_{n} \geq M^{\mathrm{e,o}}_{2^{j_n+1}} \quad \text{and} \quad g\big(2^{j_n}\big) \geq g(n) \geq g\big(2^{j_n+1}\big)$$

whence especially

$$\frac{M^{\mathrm{e,o}}_{n}}{g(n)} \leq \frac{M^{\mathrm{e,o}}_{2^{j_n}}}{g(2^{j_n+1})} = \frac{M^{\mathrm{e,o}}_{2^{j_n}}}{g(2\cdot 2^{j_n})}.$$

Thus, the claim follows by (2.8). $\square$

Lemma 2.7. *Let $\mathfrak{N}$ be as in Lemma 2.6. If the function $u \mapsto ug(u)$ decreases monotonically to zero and*

$$\lim_{u\to\infty} \frac{\Lambda_g(u)}{\log\log u} = 0, \quad \textit{then} \quad \mathbb{P}\left[\limsup_{n\to\infty}\left(\bigcap_{z\in B_n} J_{cg(n)}(\tau_z \circ \mathfrak{N})\right)\right] = 1 \quad \forall c > 0. \tag{2.9}$$

Proof. For $A \in \mathbb{Z}^d$, a fixed $c > 0$, and a fixed function g let us abbreviate

$$H^n_A = \bigcap_{z\in A} J_{cg(n)}(\tau_z \circ \mathfrak{N}).$$

Let us briefly outline the idea of the proof: It is sufficient to show that the claim is true along the subsequence $n_j = j^j$. First we show that

$$\sum_{j=1}^{\infty} \mathbb{P}\Big[H^{n_j}_{B_{n_j}}\Big] = \infty \tag{2.10}$$

which, since $H^{n_j}_{B_{n_j}} \subset H^{n_j}_{B_{n_j}\setminus B_{n_{j-1}+1}}$, implies that $\sum_{j=1}^{\infty} \mathbb{P}\Big[H^{n_j}_{B_{n_j}\setminus B_{n_{j-1}+1}}\Big] = \infty$. Note that since for $i, j \in \mathbb{N}$ with $i \neq j$ the intersection

$$\left(\bigcup_{z\in B_{n_j}\setminus B_{n_{j-1}+1}} \tau_z \circ \mathfrak{N}\right) \cap \left(\bigcup_{z\in B_{n_i}\setminus B_{n_{i-1}+1}} \tau_z \circ \mathfrak{N}\right) = \emptyset,$$

the events $\Big\{H^{n_j}_{B_{n_j}\setminus B_{n_{j-1}+1}}\Big\}_{j\geq 2}$ are independent. Thus, we can infer by the second Borel-Cantelli lemma that

$$\mathbb{P}\Big[\limsup_{j\to\infty} H^{n_j}_{B_{n_j}\setminus B_{n_{j-1}+1}}\Big] = 1. \tag{2.11}$$

Then we show that

$$\mathbb{P}\Big[\liminf_{j\to\infty} H^{n_j}_{B_{n_{j-1}+1}}\Big] = 1. \tag{2.12}$$

Since by definition

$$H^{n_j}_{B_{n_j}} = H^{n_j}_{B_{n_j}\setminus B_{n_{j-1}+1}} \cap H^{n_j}_{B_{n_{j-1}+1}},$$

(2.12) together with (2.11) implies the claim of the lemma.

Let us start with the proof of (2.10). We note that for A_{e} and A_{o} as defined in (2.7) the FKG-inequality implies that

$$\mathbb{P}\Big[H^{n_j}_{B_{n_j}}\Big] = \mathbb{P}\Big[H^{n_j}_{A_{\mathrm{e}}\cap B_{n_j}} \cap H^{n_j}_{A_{\mathrm{o}}\cap B_{n_j}}\Big] \geq \mathbb{P}\Big[H^{n_j}_{A_{\mathrm{e}}\cap B_{n_j}}\Big]^2$$

Then we recall that A_{e} was constructed such that $H^{n_j}_{A_{\mathrm{e}}\cap B_{n_j}}$ is the intersection of less than $(2n+1)^d$ i.i.d. subevents $\{J_{cg(n_j)}(\tau_z \circ \mathfrak{N})\}_{z\in A_{\mathrm{e}}\cap B_{n_j}}$, each with probability

$$\mathbb{P}\big[J_{cg(n_j)}(\mathfrak{N})\big] = 1 - \mathbb{P}[w \le cg(n_j)]^{2d}.$$

Thus for j large enough, there exists $C < \infty$ such that

$$\begin{aligned}
\mathbb{P}\Big[H^{n_j}_{B_{n_j}}\Big] &\ge \Big(1 - \mathbb{P}[w \le cg(n_j)]^{2d}\Big)^{2(2n_j+1)^d} \\
&= \left(\Big(1 - \mathbb{P}[w \le cg(n_j)]^{2d}\Big)^{\mathbb{P}[w\le cg(n_j)]^{-2d}}\right)^{2(2n_j+1)^d\mathbb{P}[w\le cg(n_j)]^{2d}} \\
&\ge \exp\Big(-Cn_j^d\mathbb{P}[w \le cg(n_j)]^{2d}\Big) = \exp(-C\Lambda_{cg}(n_j)). \qquad (2.13)
\end{aligned}$$

Now we observe that the assumptions on g and Λ_g imply that the RHS of (2.13) is not summable for any $c > 0$. This concludes the argument for (2.10).

Let us proceed with the proof of (2.12). Note that for any $\epsilon > 0$ we have $n_{j+1} \ge h(n_j)$ with $h(u) = u(\log u)^{1-\epsilon}$. This is because for j large enough and any $\epsilon > 0$ we have

$$n_{j+1} = (j+1)\left(1 + \frac{1}{j}\right)^j j^j \ge (j+1)n_j \ge n_j(\log n_j)^{1-\epsilon}.$$

Thus, (2.12) is a consequence of

$$\mathbb{P}\left[\liminf_{n\to\infty} \bigcap_{z\in B_{n+1}} J_{cg(h(n))}(\tau_z \circ \mathfrak{A})\right] = 1 \quad \forall c > 0.$$

By virtue of Lemma 2.1 we can thus verify (2.12) by showing that for all $c > 0$ the integral

$$\int_0^\infty u^{d-1}\mathbb{P}[w \le cg(h(u))]^{2d}\mathrm{d}u < \infty.$$

To see that the integral is indeed finite, we consider the following: There exists a constant $C < \infty$ such that

$$\begin{aligned}
\int_0^\infty u^{d-1}\mathbb{P}[w \le cg(h(u))]^{2d}\mathrm{d}u &\le C + \int_2^\infty u^{-1}\left(\frac{u}{h(u)}\right)^d \Big(h(u)^d\,\mathbb{P}[w \le cg(h(u))]^{2d}\Big)\,\mathrm{d}u \\
&= C + \int_2^\infty u^{-1}(\log u)^{-d(1-\epsilon)}\Lambda_{cg}(h(u))\,\mathrm{d}u.
\end{aligned}$$

By the condition on Λ_g and the monotonicity condition on g follows that the above RHS is finite. □

3 Percolation results

In this section we adapt three standard percolation results that we need for the path arguments of the next section in order to establish the lower bound for the principal Dirichlet eigenvalue.

We consider the standard Bernoulli bond percolation on the graph $(\mathbb{Z}^d, \mathfrak{E}_d)$, i.e., we assume that the conductances are independent random variables with common law $\mathbb{P}$ such that an individual conductance is 1 with probability p and 0 otherwise. For an introduction to percolation we refer the reader to [Gri99]. As in the previous section, we call $\boldsymbol{w} = (w_e)_{e\in\mathfrak{E}_d} \in \{0,1\}^{\mathfrak{E}_d}$ an environment and we denote the law of the environment by $\mathbb{P}$. If the conductance

w_e of an edge e is equal to 1, then we call e an open edge. Otherwise we call the edge e closed. Given a realization $\boldsymbol{w}$ of the environment, we denote the set of open edges by $\mathfrak{E}_{\mathrm{O}} \subset \mathfrak{E}_d$.

Consider the random graph $(\mathbb{Z}^d, \mathfrak{E}_{\mathrm{O}})$. Following the terminology of Grimmet [Gri99], we call the connected subgraphs of this graph *open clusters* and, for $x \in \mathbb{Z}^d$, we write $\mathscr{C}(x)$ for the open cluster that contains the site x. Note that $\mathscr{C}(x) \subset (\mathbb{Z}^d, \mathfrak{E}_{\mathrm{O}})$ is a graph. We define the clusters this way in order to make sense of Dirichlet forms defined as in (3.2) below. However, when we write $|\mathscr{C}(x)|$, we refer to the number of sites in $\mathscr{C}(x)$. Furthermore, when $\mathscr{C}$ is a cluster and y is a site in the vertex set of $\mathscr{C}$, then we use the shorthand notation $y \in \mathscr{C}$. Similarly, if e is in the edge set of $\mathscr{C}$, then we write $e \in \mathscr{C}$.

We say that a path $l = (x_0, \ldots, x_m)$ is open if and only if $\{x_{i-1}, x_i\} \in \mathfrak{E}_{\mathrm{O}}$ for all $i \in \{1, \ldots, m\}$. Given a realization $\boldsymbol{w}$ of the environment, we write $d_{\boldsymbol{w}}(x, y)$ for the length of the shortest open path between the sites x and y. We say that $d_{\boldsymbol{w}}(x, y) = \infty$ whenever x and y are not connected through an open path.

Let $p_c(d)$ be the critical probability such that $\mathbb{P}$-a.s. there exists an infinite open cluster $\mathscr{C}_\infty$. This cluster is $\mathbb{P}$-a.s. unique. We assume that $p_c(d) < p < 1$. Note that $\mathscr{C}_\infty$ contains all sites x that are connected to infinity through an open path *as well as* all open edges that are incident to a site in $\mathscr{C}_\infty$. We further define $\mathscr{H}$ as the complement of $\mathscr{C}_\infty$ in $\mathbb{Z}^d$, i.e., we regard $\mathscr{H}$ as a set of sites.

The main object of this section is to collect results from the literature and adapt the details such they exactly fit our needs.

Lemma 3.1. *Let $\eta \in (0,1)$. Then for p sufficiently close to one, there exist constants $C < \infty$ and $c > 0$ such that*

$$\mathbb{P}[|B_n \cap \mathscr{C}_\infty| \le \eta |B_n|] \le C\mathrm{e}^{-cn} \quad \text{for all } n \ge 1. \tag{3.1}$$

For the proof of this lemma we refer the reader to [BKM15, Lemma 4.2]. The second lemma is an implication of Lemma 3.1 above.

Lemma 3.2. *For p sufficiently close to one and $\mathbb{P}$-a.s. for n large enough there exists an injective map $\varphi_1 : \mathscr{H} \cap B_n \to \mathscr{C}_\infty$ such that for any site $x \in \mathscr{H} \cap B_n$ the distance $|x - \varphi_1(x)|_1 \le 2d(\log n)^{(d+1)}$.*

Proof of Lemma 3.2. The proof of this lemma follows the lines of the first paragraph of the proof of [BKM15, Lemma 4.7] but we included the proof here for completeness. For $z \in \mathbb{Z}^d$ and $m \ge 0$, we denote $B_m(z) = \{x \in \mathbb{Z}^d \colon |x - z|_\infty \le m\}$. Choose the percolation parameter p such that (3.1) is fulfilled with $\eta > \frac{1}{2}$. Let $m = \lfloor (\log n)^{d+1} \rfloor$ and consider the disjoint partition $\mathcal{P}_m := \{B_m((2m+1)z)\}_{z \in \mathbb{Z}^d}$ of $\mathbb{Z}^d$. Then Lemma 3.1 implies that there exist $c, C \in (0, \infty)$ such that

$$\begin{aligned}\mathbb{P}\Bigg[\bigcup_{\substack{B \in \mathcal{P}_m,\\ B \cap B_n \neq \emptyset}} \{|B \cap \mathscr{C}_\infty| \le \eta |B|\}\Bigg] &\le \sum_{\substack{B \in \mathcal{P}_m,\\ B \cap B_n \neq \emptyset}} \mathbb{P}[|B \cap \mathscr{C}_\infty| \le \eta |B|] \\ &\le C(2n+1)^d \exp\Big(-c(\log n)^{d+1}\Big),\end{aligned}$$

which is summable. By the Borel-Cantelli lemma it follows that $\mathbb{P}$-a.s. for n large enough we have $|B \cap \mathscr{C}_\infty| > |B|/2$ in any $B \in \mathcal{P}_m$ with $B \cap B_n \neq \emptyset$.

Now we construct φ_1 as follows: For $x \in \mathscr{H} \cap B_n$ choose $B \in \mathcal{P}_m$ (unique) such that $x \in B$. Choose $\varphi_1(x) \in B \cap \mathscr{C}_\infty$ in an injective way - this is possible since $|\mathscr{H} \cap B| < |\mathscr{C}_\infty \cap B|$. The ℓ_1-distance between x and $\varphi_1(x)$ is thus smaller than or equal to $2d(\log n)^{(d+1)}$. □

For $f : \mathbb{Z}^d \to \mathbb{R}$ with $\|f\|_2^2 < \infty$ we define the Dirichlet-form $\mathcal{E}_{\mathscr{C}_\infty}(f)$:

$$\mathcal{E}_{\mathscr{C}_\infty}(f) = \sum_{\{x,y\} \in \mathscr{C}_\infty} (f(x) - f(y))^2 , \tag{3.2}$$

as well as the norm $\|f\|_{\ell^2(\mathscr{C}_\infty)} = \sum_{x \in \mathscr{C}_\infty} f^2(x)$.

In the following lemma we give a lower bound for the principal Dirichlet eigenvalue on $B_n \cap \mathscr{C}_\infty$. The lemma is similar to Theorem 1.3 from [MR04] with the difference that $B_n \cap \mathscr{C}_\infty$ is in general not connected and we do not include the condition that $0 \in \mathscr{C}_\infty$.

Lemma 3.3. *Let $d \geq 2$ and choose p such that Lemma 3.1 holds with $\eta > \frac{1}{2}$. Then there exists a deterministic constant $c > 0$ such that $\mathbb{P}$-a.s. for n large enough and all real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ with supp $f \subseteq B_n$ we have*

$$\|f\|^2_{\ell^2(\mathscr{C}_\infty)} \leq cn^2 \mathcal{E}_{\mathscr{C}_\infty}(f)\,. \tag{3.3}$$

We prove this lemma at the end of the section. It is rather standard given the relative isoperimetric inequality in Lemma 3.4 below (see e.g. [SC97, p. 83]) but since the details are slightly different, we include the proof for the convenience of the reader. For the proof we need some additional definitions.

Let $n \in \mathbb{N}$. If $x, y \in B_n = [-n, n]^d \cap \mathbb{Z}^d$, we write $d_{\omega,n}(x, y)$ for the shortest open path between x and y that is contained in B_n. We further say that $\mathscr{B}$ is an open cluster in B_n if and only if $\mathscr{B}$ is an open cluster and $d_{\omega,n}(x, y) < \infty$ for all $x, y \in \mathscr{B}$. The largest open cluster in B_n is denoted by $\mathscr{C}_n^\vee$. Note that it is *not* necessarily $\mathscr{C}_n^\vee \subset \mathscr{C}_\infty$.

Let $A \subseteq \mathscr{C}_n^\vee$ be a set of sites. We define the relative edge boundary of A with respect to $\mathscr{C}_n^\vee$ as the edge set

$$\partial_{\mathrm{E}}(A|\mathscr{C}_n^\vee) = \{\{x, y\} \in \mathscr{C}_n^\vee : x \in A \text{ and } y \in \mathscr{C}_n^\vee \backslash A\}\,.$$

Further we need the following auxiliary lemma about properties of $\mathscr{C}_n^\vee$ and a relative isoperimetric inequality. In this lemma we collect well-known results from [Bar04] and [MR04] and adapt them to our needs.

Lemma 3.4. *Let $d \geq 2$ and choose p such that Lemma 3.1 holds with $\eta > \frac{1}{2}$. Then there exists $\rho \in [1, \infty)$ such that $\mathbb{P}$-a.s. exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ the following statements are true: i) $\mathscr{C}_n^\vee \subset \mathscr{C}_\infty$, ii) $\mathscr{C}_\infty \cap B_{\frac{n}{2\rho}} \subset \mathscr{C}_n^\vee$, iii) $|B_n| \leq 2|B_n \cap \mathscr{C}_\infty|$, and iv) the relative isoperimetric inequality holds:*

$$\inf_{\substack{A \subset \mathscr{C}_n^\vee, \\ |A| \leq \frac{1}{2}|\mathscr{C}_n^\vee|}} \frac{|\partial_{\mathrm{E}}(A|\mathscr{C}_n^\vee)|}{|A|} \geq \frac{c}{n}\,. \tag{3.4}$$

Proof. Since $p > p_c$, item (i) follows from Lemma 2.19 and Remark 2 on page 3049 of Barlow [Bar04]. Item (iii) follows from Lemma 3.1 together with a Borel-Cantelli argument.

Item (iv) follows by virtue of [Bar04, Proposition 2.11, Proposition 2.12(a)] and the Borel-Cantelli lemma, i.e., there exists $c > 0$ such that $\mathbb{P}$-a.s. there exists $N_1(\omega) < \infty$ such that for all $n \geq N_1$ the following statement is true: For all connected open subsets $A \subset \mathscr{C}_n^\vee$ with $|A| \leq \frac{1}{2}|\mathscr{C}_n^\vee|$ and such that $\mathscr{C}_n^\vee \backslash A$ is connected, the isoperimetric inequality

$$\frac{|\partial_{\mathrm{E}}(A|\mathscr{C}_n^\vee)|}{|A|} \geq \frac{c}{n}$$

holds. By the same arguments as in [MR04, Sec. 3.1], this implies that (3.4) also holds for arbitrary $A \subset \mathscr{C}_n^\vee$ with $|A| \leq \frac{1}{2}|\mathscr{C}_n^\vee|$.

Finally we show (ii), i.e., that there exists $\rho \in [1, \infty)$ such that $\mathbb{P}$-a.s. there exists $N_2(\omega) < \infty$ such that $\mathscr{C}_\infty \cap B_{\frac{n}{2\rho}} \subset \mathscr{C}_n^\vee$ for all $n \geq N_2$. The proof is similar to the one in Appendix B of [MR04] but we present it here for the convenience of the reader. Assume that there exists a site $x \in \mathscr{C}_\infty \cap B_{\frac{n}{2\rho}} \backslash \mathscr{C}_n^\vee$. We estimate

$$\mathbb{P}\Big[\exists x \in B_{\frac{n}{2\rho}} \cap \mathscr{C}_\infty \backslash \mathscr{C}_n^\vee\Big] \leq \mathbb{P}\Big[\Big\{\exists x \in B_{\frac{n}{2\rho}} \cap \mathscr{C}_\infty \backslash \mathscr{C}_n^\vee\Big\} \cap \{\mathscr{C}_n^\vee \subset \mathscr{C}_\infty\}\Big] + \mathbb{P}[\mathscr{C}_n^\vee \not\subset \mathscr{C}_\infty]$$

From (i) it follows that the second term on the RHS is negligible and we therefore only consider the first term. If $\mathscr{C}_n^\vee \subset \mathscr{C}_\infty$, then $\mathbb{P}$-a.s. for n large enough there exists a site $y \in \mathscr{C}_n^\vee$ with norm $|y|_\infty = \lfloor n/\rho \rfloor$, for a sketch see Fig 3.1. But this implies that the ℓ^1-distance between x and y can be estimated from above and below by

$$\frac{n}{2\rho} - 1 \leq \left\lfloor \frac{n}{\rho} \right\rfloor - \frac{n}{2\rho} \leq |x - y|_\infty \leq |x - y|_1 \leq 2d|x - y|_\infty \leq \frac{3dn}{\rho}\,. \tag{3.5}$$

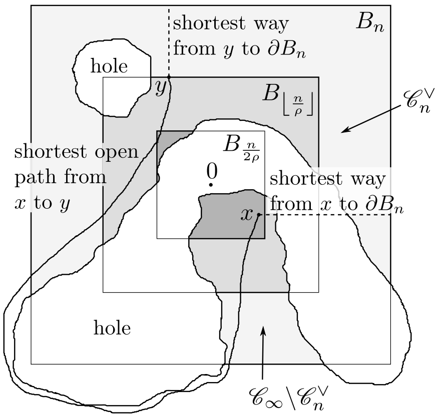

Figure 3.1: Sketch for the estimation (3.5) and (3.6). We assume that there exists a site $x \in B_{\frac{n}{2\rho}} \cap \mathscr{C}_\infty \backslash \mathscr{C}_n^\vee$. Further, we can show that $\mathbb{P}$-a.s. for n large enough there exists an element $y \in \mathscr{C}_n^\vee$ such that $|y|_\infty = \lfloor n/\rho \rfloor$.

Further, since $x \notin \mathscr{C}_n^\vee$ but $y \in \mathscr{C}_n^\vee$, it follows that the shortest open path between x and y must leave the box B_n, see e.g. Fig 3.1. Thus, the chemical distance between the sites x and y is bounded from below by the ℓ^∞-distances of x and y to the boundary ∂B_n, i.e.

$$d_{\boldsymbol{w}}(x,y) \geq n - \frac{n}{2\rho} + n - \frac{n}{\rho} \geq \left(\frac{2\rho}{3d} - \frac{1}{2d}\right)|x-y|_1 \,. \tag{3.6}$$

By virtue of [AP96, Theorem 1.1] there exists $\rho^* = \rho^*(p,d) \in [1,\infty)$ and $\beta = \beta(p,d) > 0$ such that

$$\mathbb{P}[x,y \in \mathscr{C}_\infty \text{ and } d_{\boldsymbol{w}}(x,y) > \rho^*|x-y|_1] \leq \mathrm{e}^{-\beta|x-y|_1} \,.$$

Let us choose ρ such that $\frac{2\rho}{3d} - \frac{1}{2d} \geq \rho^*$. Then it follows that

$$\begin{aligned}
&\mathbb{P}\Big[\Big\{\exists x \in B_{\frac{n}{2\rho}} \cap \mathscr{C}_\infty \backslash \mathscr{C}_n^\vee\Big\} \cap \{\mathscr{C}_n^\vee \subseteq \mathscr{C}_\infty\}\Big] \\
&\qquad \leq \sum_{x:\ |x|_\infty \leq \frac{n}{2\rho}} \ \sum_{y:\ |y|_\infty = \frac{n}{\rho}} \mathrm{e}^{-\beta|x-y|_1} \leq (2n+1)^{2d} \mathrm{e}^{-\beta\left(\frac{n}{2\rho}-1\right)} \,,
\end{aligned}$$

which is summable. It follows that $\mathbb{P}$-a.s. there exists $N_2 < \infty$ such that $\mathscr{C}_\infty \cap B_{\frac{n}{2\rho}} \subset \mathscr{C}_n^\vee$ for all $n \geq N_2$. $\square$

Proof of Lemma 3.3. Let n^* be as in Lemma 3.4 and let $n \geq n^*$. Further let $f\colon \mathbb{Z}^d \to \mathbb{R}$ such that $\operatorname{supp} f \subseteq B_n$. It is sufficient to show that (3.3) holds if f is the principal Dirichlet eigenvector of the generator belonging to the Dirichlet form $\mathcal{E}_{\mathscr{C}_\infty}$. Similarly to Remark 1.1, the Perron-Frobenius theorem implies that we can assume w.l.o.g. that $f \geq 0$. We apply Hölder's inequality to obtain

$$\|f\|_{\ell^2(\mathscr{C}_\infty)} \sqrt{\mathcal{E}_{\mathscr{C}_\infty}(f)} \geq \frac{1}{4d} \sum_{\{x,y\} \in \mathscr{C}_\infty} |f(x) - f(y)|(f(x)+f(y)) = \frac{1}{4d} \sum_{\{x,y\} \in \mathscr{C}_\infty} \left|f^2(x) - f^2(y)\right| . \tag{3.7}$$

Now we use a standard approach which is known as the co-area formula (see e.g. [SC97, p. 83]):

$$\sum_{\{x,y\} \in \mathscr{C}_\infty} \left|f^2(x) - f^2(y)\right| = \sum_{x \in \mathscr{C}_\infty} \ \sum_{\substack{y:\ \{x,y\} \in \mathscr{C}_\infty, \\ f(x) \geq f(y)}} \int_0^\infty \mathbb{1}_{\{f^2(x) > t \geq f^2(y)\}} \,\mathrm{d}t \,,$$

where we can interchange the double sum and the integral since f has bounded support. If we define the set of sites $A_t = \left\{x \in \mathscr{C}_\infty \colon f^2(x) > t\right\}$, then we see that

$$\sum_{x \in \mathscr{C}_\infty} \ \sum_{\substack{y:\ \{x,y\} \in \mathscr{C}_\infty, \\ f(x) \geq f(y)}} \mathbb{1}_{\{f(x)^2 > t \geq f^2(y)\}} = |\partial_{\mathrm{E}}(A_t | \mathscr{C}_\infty)| \,.$$

We aim to apply the relative isoperimetric inequality (3.4). Let us choose ρ as in Lemma 3.4. Since supp $f \subseteq B_n$ and $n \geq n^*$, we have $A_t \subseteq B_n \cap \mathscr{C}_\infty \subseteq \mathscr{C}^\vee_{8\rho n}$ for all $t > 0$ by Lemma 3.4 (ii). This implies $\partial_{\mathrm{E}}(A_t|\mathscr{C}_\infty) = \partial_{\mathrm{E}}\left(A_t|\mathscr{C}^\vee_{8\rho n}\right)$ as well. We further have $|A_t| \leq |B_n| \leq \frac{1}{4}|B_{4n}| \leq \frac{1}{2}|\mathscr{C}^\vee_{8\rho n}|$ by Lemma 3.4 (ii) and (iii). By the relative isoperimetric inequality (3.4) we therefore obtain

$$\sum_{\{x,y\}\in\mathscr{C}_\infty} \left|f^2(x) - f^2(y)\right| \geq \frac{c}{8\rho n}\int_0^\infty |A_t|\,\mathrm{d}t = \frac{c}{8\rho n}\sum_{x\in\mathscr{C}_\infty} f^2(x)\,.$$

Together with (3.7) this implies that

$$\sqrt{\mathcal{E}_{\mathscr{C}_\infty}(f)} \geq \frac{c}{32 d\rho n}\|f\|_{\ell^2(\mathscr{C}_\infty)}\,.$$

□

4 Path argument

In this section we give the two Propositions 4.6 and 4.7, which transfer the knowledge we obtained by the Borel-Cantelli arguments in Section 2 to lower bounds of Dirichlet forms. In order to achieve this, Lemma 4.3 generalizes and modifies a path argument similar to the one in [BKM15, Lemma 4.7]. Before we start, we give a definition which is crucial for the remaining part of the paper.

Definition 4.1. *Let $\mathscr{G} = (V, \mathfrak{E})$ be an undirected graph and $\boldsymbol{w} = (w_e)_{e\in\mathfrak{E}}$. For $f\colon V \to \mathbb{R}$, we define the Dirichlet energy on $\mathscr{G}$:*

$$\mathcal{E}^{\boldsymbol{w}}_{\mathscr{G}}(f) = \frac{1}{2}\sum_{x\in V}\sum_{\substack{y\in V,\\ \{x,y\}\in\mathfrak{E}}} w_{xy}(f(x) - f(y))^2\,. \tag{4.1}$$

Remark 4.2. *For $\xi > 0$ let us define $a_e = \mathbb{1}_{\{w_e \geq \xi\}}$ ($e \in \mathfrak{E}_d$). Let us call an edge e open if and only if $a_e = 1$ and let $\mathscr{C}$ be an open cluster in the environment $\boldsymbol{a} = (a_e)_{e\in\mathfrak{E}_d}$. Then, with reference to (3.2), we obtain that $\xi\mathcal{E}_{\mathscr{C}}(f) \leq \mathcal{E}^{\boldsymbol{w}}_{\mathscr{C}}(f)$ for all real-valued functions $f \in \ell^2(\mathbb{Z}^d)$.*

Since we apply a similar argument for two slightly different situations (i.e., once for the proofs of Theorems 1.3 and 1.5, see Proposition 4.7, and once for the proof of Theorem 1.8, see Proposition 4.6), we kept the conditions of the following lemma as general as necessary.

Lemma 4.3. *Let $\mathscr{G} = (V, \mathfrak{E})$ be an undirected graph and let $\mathscr{C} = (V_{\mathscr{C}}, \mathfrak{E}_{\mathscr{C}})$ be a subgraph of $\mathscr{G}$. Assume that $v, L \in (0,\infty)$ and $B \subseteq V$ are such that the following conditions are fulfilled:*

(i) There exists a constant $\mu > 0$ such that for all $f\colon V \to \mathbb{R}$ with supp $f \subseteq B$ the following inequality holds:

$$\mathcal{E}^{\boldsymbol{w}}_{\mathscr{C}}(f) \geq \mu\|f\|^2_{\ell^2(\mathscr{C})}. \tag{4.2}$$

(ii) There exists an injective map $\varphi : B\backslash V_{\mathscr{C}} \to V_{\mathscr{C}}$ such that the following holds: From any $x \in B\backslash V_{\mathscr{C}}$ there exists a (self-avoiding) directed path $l(x,\varphi(x))$ to $\varphi(x)$ in $\mathscr{G}$ such that

(ii).1 all $e \in l(x,\varphi(x))$ fulfill $w_e > v$,

(ii).2 $|l(x,\varphi(x))| \leq L$.

Then for all $f\colon V \to \mathbb{R}$ with supp $f \subseteq B$ the following holds:

$$\mathcal{E}^{\boldsymbol{w}}_{\mathscr{G}}(f) \geq \left((2L)^{d+1}v^{-1} + 3\mu^{-1}\right)^{-1}\|f\|^2_{\ell^2(\mathscr{G})}\,. \tag{4.3}$$

Proof of Lemma 4.3. We generalize the proof of [BKM15, Lemma 4.7]. Let $f\colon V \to \mathbb{R}$ with $\operatorname{supp} f \subseteq B$. For the following calculation we abbreviate $f(y) - f(z) = \mathrm{d}f((y,z))$ where (y,z) is the (directed) link from site y to its neighbor z. For $x \in B \backslash V_{\mathscr{C}}$ we write $f(x)$ as a telescopic sum

$$f(x) = \sum_{b \in l(x,\varphi(x))} \mathrm{d}f(b) + f(\varphi(x)) .$$

We apply the Cauchy-Schwarz inequality and expand the terms on the RHS by the conductances:

$$f^2(x) \leq \frac{2|l(x,\varphi(x))|}{v} \sum_{b \in l(x,\varphi(x))} w_b \, (\mathrm{d}f(b))^2 + 2f^2(\varphi(x)) .$$

Now we sum over all $x \in B \backslash V_{\mathscr{C}}$ and use the upper bound for $|l(x,\varphi(x))|$ according to Condition (ii)b:

$$\sum_{x \in B \backslash V_{\mathscr{C}}} f^2(x) \leq \frac{2L}{v} \sum_{x \in B \backslash V_{\mathscr{C}}} \sum_{b \in l(x,\varphi(x))} w_b \, (\mathrm{d}f(b))^2 + 2 \sum_{x \in B \backslash V_{\mathscr{C}}} f^2(\varphi(x)) . \tag{4.4}$$

Let us look at the last term on the RHS: By definition φ is injective and its image is in $V_{\mathscr{C}}$. This means that

$$\sum_{x \in B \backslash V_{\mathscr{C}}} f^2(\varphi(x)) \leq \sum_{x \in V_{\mathscr{C}}} f^2(x) .$$

Due to the limited length of the path $l(x,\varphi(x))$ the sum over $b \in l(x,\varphi(x))$ on the RHS in (4.4) uses each edge at most $(2L)^d$ times, whence

$$\sum_{x \in B \backslash V_{\mathscr{C}}} \sum_{b \in l(x,\varphi(x))} w_b \, (\mathrm{d}f(b))^2 \leq (2L)^d \mathcal{E}^{\boldsymbol{w}}_{\mathscr{G}}(f) .$$

Completing the sum to all sites $x \in \mathscr{G}$ and using the comparability between $\mathcal{E}^{\boldsymbol{w}}_{\mathscr{G}}(f)$ and $\mathcal{E}^{\boldsymbol{w}}_{\mathscr{C}}(f)$, we obtain by virtue of Condition (i):

$$\sum_{x \in V} f^2(x) \leq \frac{(2L)^{d+1}}{v} \mathcal{E}^{\boldsymbol{w}}_{\mathscr{G}}(f) + 3 \sum_{x \in \mathscr{C}} f^2(x) \leq \left(\frac{(2L)^{d+1}}{v} + \frac{3}{\mu} \right) \mathcal{E}^{\boldsymbol{w}}_{\mathscr{G}}(f) .$$

□

4.1 Asymptotics of the principal Dirichlet eigenvalue

From the path argument in Lemma 4.3 we can use our observations from Section 2 to obtain lower bounds of the Dirichlet forms. We use similar arguments as in [BKM15, Lemma 5.1].

We fix $\xi > 0$ such that

$$\mathbb{P}[w > \xi] > p_c(d) . \tag{4.5}$$

Moreover, we fix an environment $\boldsymbol{w}$ and define a new environment $\boldsymbol{a}$ by setting

$$a_e = \mathbb{1}_{\{w_e > \xi\}} \qquad (e \in \mathfrak{E}_d) , \tag{4.6}$$

similarly as in Remark 4.2. We denote the unique infinite cluster of the environment $\boldsymbol{a}$ by $\mathscr{C}^\xi$ and we use the same shorthand notations as explained at the beginning of Section 3. Further we define $\mathscr{C}^\xi_n$ as the restriction of $\mathscr{C}^\xi$ to the box B_n and and similarly the holes $\mathscr{H}^\xi_n$.

Additionally, we define a second percolation environment $\tilde{\boldsymbol{w}}_{g(n)}$ for $g : (0,\infty) \to (0,\infty)$ by setting

$$\tilde{w}_{g(n)}(e) = w_e \mathbb{1}_{\{w_e > g(n)\}} \qquad (e \in \mathfrak{E}_d) . \tag{4.7}$$

Thus, links with conductance less than or equal to $g(n)$ are considered to be closed and all others keep their original conductance. With this terminology we can now define the following clusters.

Definition 4.4. *Let $\mathscr{D}_{g(n)}$ be the unique infinite open cluster of $\tilde{\boldsymbol{w}}_{g(n)}$. Regarding this cluster, we use the same shorthand notations as introduced at the beginning of Section 3. Furthermore, let $\mathscr{I}_{g(n)} = B_n \backslash \mathscr{D}_{g(n)}$ be the set of holes.*

For a fixed function g and a fixed $\epsilon > 0$ we often abbreviate $\mathscr{D}_n = \mathscr{D}_{g(n^{1-\epsilon})}$ and $\mathscr{I}_n = \mathscr{I}_{g(n^{1-\epsilon})}$.

Definition 4.5. *We say a set $\mathscr{I} \subset \mathbb{Z}^d$ is* **sparse** *if the set $\mathscr{I}$ does not contain any neighboring sites.*

Proposition 4.6. *Let $g : [0, \infty) \to [0, \infty)$ be a decreasing function. Let $b \in \mathbb{N}$ be such that $b \gg 4d$. Further, assume that there exists $\epsilon > 0$ such that along the subsequence $(n_k)_{k\in\mathbb{N}}$ for all $z \in B_{n+b}$ the box $B_b(z)$ contains at most $3d-1$ links with conductance less than or equal to $g(n^{1-\epsilon})$. Define $\mathscr{D}_n = \mathscr{D}_{g(n^{1-\epsilon})}$. Then there exists $c \in (0, \infty)$ (independent of g) such that $\mathbb{P}$-a.s. for k large enough, and all real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ with supp $f \subseteq B_{n_k}$ we obtain*

$$\mathcal{E}^{\boldsymbol{w}}_{\mathscr{D}_{n_k}}(f) \geq \left(2^{d+1}(\log n_k)^{4d^2} g\left(n_k^{1-\epsilon}\right)^{-1} + cn_k^2\right)^{-1} \|f\|^2_{\ell^2(\mathscr{D}_{n_k})},$$

with $\mathcal{E}^{\boldsymbol{w}}_{\mathscr{D}_{n_k}}$ as in Definition 4.1.

Proposition 4.7. *Let the assumptions of Proposition 4.6 be true as well as one of Assumptions 1.2 (a) or (b). Further, assume that there exists an infinite subsequence $(n_k)_{k\in\mathbb{N}}$ such that for all $z \in B_{n_k}$ there exists an incident edge with conductance greater than $g(n_k)$. Then there exists $c > 0$ such that $\mathbb{P}$-a.s. for k large enough, and all real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ with supp $f \subseteq B_{n_k}$ we obtain*

$$\mathcal{E}^{\boldsymbol{w}}(f) \geq cg(n_k)\|f\|_2^2 .$$

If one of the Assumptions 1.2 (a) or (b') is fulfilled, then the constant c can be chosen independently of g.

We prove these propositions in the next section.

4.2 Proof of Propositions 4.6 and 4.7

Proof of Proposition 4.6. We apply Lemma 4.3. Let $\mathscr{G}_n = \mathscr{D}_n$. Further, let $v_n = g\left(n^{1-\epsilon}\right)$.

In order to define $\mathscr{C} \subset \mathscr{G}_n$ as required by Lemma 4.3, we fix $\xi > 0$ according to (4.5) and let $\mathscr{C} = \mathscr{C}^\xi$. Since g is decreasing to zero, $\mathscr{C}^\xi \subset \mathscr{G}_n$ for n large enough.

By Lemma 3.3 we know that there exists $c_1 > 0$ such that $\mathbb{P}$-a.s. for n large enough and all real-valued functions $f \in \ell^2(\mathbb{Z}^d)$ with supp $f \subseteq B_n$ we have

$$\|f\|_2^2 \leq c_1 n^2 \mathcal{E}_{\mathscr{C}}(f) \leq \xi^{-1} c_1 n^2 \mathcal{E}^{\boldsymbol{w}}_{\mathscr{C}}(f) . \tag{4.8}$$

Thus, if we choose $\mu_n = \frac{\xi}{c_1 n^2}$ in the place of μ in (4.2), then Condition (i) of Lemma 4.3 is fulfilled.

We are now going to construct the map $\varphi\colon\ B_n \cap \mathscr{D}_n \cap \mathscr{H}^\xi \to \mathscr{C}^\xi$ and the path $l(x, \varphi(x))$. For the next paragraph we say that a conductance is "bad" if it is smaller than or equal to $g(n^{1-\epsilon})$. By Lemma 3.2 there exists an injective map $\varphi_1\colon\ \mathscr{H}^\xi \cap B_n \to \mathscr{C}^\xi$ such that there exists a directed path $l_1(x, \varphi_1(x))$ in $(\mathbb{Z}^d, \mathfrak{E}_d)$ from x to $\varphi_1(x)$ of length $|l_1(x, \varphi_1(x))| \leq 2d(\log n)^{(d+1)}$. Let $\varphi = \varphi_1|_{\mathscr{H}^\xi \cap B_n \cap \mathscr{D}_n}$. By assumption, along the subsequence n_k, each subbox $B_b(z)$ with $z \in B_{n+b}$ contains at most $3d - 1$ bad conductances. In addition, by the construction of $\mathscr{D}_n$, each site $x \in \mathscr{D}_n$ is connected to infinity by a path that contains only good conductances. Thus, we construct the path $l(x, \varphi(x))$ by the following algorithm: The path $l(x, \varphi(x))$ follows $l_1(x, \varphi(x))$ until it hits an edge with a bad conductance. In this case it takes the shortest detour around that edge to meet $l_1(x, \varphi(x))$ again without using any edge with a bad conductance. Since each $B_b(z)$ with $z \in B_{n+b}$ contains at most $3d - 1$ bad conductances, this detour contains less than $6d$ edges. After the detour, $l(x, \varphi(x))$ continues again on $l_1(x, \varphi(x))$ until it hits the next bad conductance and so on. By this construction there exists a $C < \infty$, such that for all $x \in B_n \cap \mathscr{D}_n \cap \mathscr{H}^\xi$ and for n large enough the length of the resulting path $|l(x, \varphi(x))| \leq C(\log n)^{(d+1)} < (\log n)^{2d}$.

We can now apply Lemma 4.3 and obtain that along the subsequence n_k $\mathbb{P}$-a.s. for k large enough for all $f\colon \mathscr{D}_{n_k} \to \mathbb{R}$ with supp $f \subseteq B_{n_k}$ the following holds:

$$\mathcal{E}^{\boldsymbol{w}}_{\mathscr{D}_{n_k}}(f) \geq \Big(2^{d+1}(\log n_k)^{4d^2} g\big(n_k^{1-\epsilon}\big)^{-1} + 3c_1 n_k^2 \xi^{-1}\Big)^{-1} \|f\|^2_{\ell^2(\mathscr{D}_{n_k})}\,.$$

Since both c_1 and ξ are independent of g, the claim follows. □

Proof of Proposition 4.7. Again, we apply Lemma 4.3. Let $\mathscr{G} = \big(\mathbb{Z}^d, \mathfrak{E}_d\big)$ and $v_n = g(n)$. Further, let $\mathscr{D}_n$ be as in Proposition 4.6 and let $\mathscr{C}_n = \mathscr{D}_n$. Then Condition (i) of Lemma 4.3 is fulfilled with

$$\mu_n = \Big(2^{d+1}(\log n)^{4d^2} g\big(n^{1-\epsilon}\big)^{-1} + 3c_1 n^2 \xi^{-1}\Big)^{-1}\,.$$

By assumption, along the subsequence n_k, each site $x \in \mathscr{I}_n := B_n \backslash \mathscr{D}_n$ has only neighbors in $\mathscr{D}$ and there exists a a neighbor $\varphi(x)$ such that the conductance $w_{x,\varphi(x)} > g(n)$. Since each box $B_b(z)$ for $z \in B_{n+b}$ contains at most $3d-1$ links with conductance less than or equal to $g(n)$ and $b \gg 1$, the map $\varphi\colon \mathscr{I}_n \to \mathscr{D}_n$ is injective. By the choice of φ, there exists a path $l(x,\varphi(x))$ of length one that fulfills the requirements of Lemma 4.3.

It follows that $\mathbb{P}$-a.s. for k large enough the following holds for all $f\colon \mathbb{Z}^d \to \mathbb{R}$ with supp $f \subseteq B_{n_k}$:

$$\mathcal{E}^{\boldsymbol{w}}(f) \geq \Big(2^{d+1} g(n_k)^{-1} + 2^{d+3}(\log n_k)^{4d^2} g\big(n_k^{1-\epsilon}\big)^{-1} + 9c_1 n_k^2 \xi^{-1}\Big)^{-1} \|f\|_2^2\,.$$

We have assumed that one of Assumptions 1.2 (a) or (b) is fulfilled. Let us first assume that Assumption 1.2 (b) is true and that the limit of $u^2 g(u)$ is smaller than $c_2 \in (0,\infty)$. It follows that eventually

$$9c_1\xi^{-1} n_k^2 g(n_k) < 9c_1c_2\xi^{-1} \quad \text{and } 2^{d+3}(\log n_k)^{4d^2} \frac{g(n_k)}{g\big(n_k^{1-\epsilon}\big)} < 2^{d+5}(\log n_k)^{4d^2} n_k^{-2\epsilon} < 1\,,$$

and therefore $\mathbb{P}$-a.s. for k large enough

$$\mathcal{E}^{\boldsymbol{w}}(f) \geq \frac{g(n_k)}{1 + 2^{d+1} + 9c_1c_2\xi^{-1}} \|f\|_2^2 \qquad (\text{supp} f \subseteq B_{n_k})\,.$$

If we assume that Assumption 1.2 (b') is fulfilled, then eventually even $9c_1\xi^{-1}n_k^2 g(n_k) < 1$ and thus the lower bound becomes independent of c_1, c_2, and ξ.

Let us now assume that Condition (a) is true. Then there exists $\rho < -2$ such that we can write $g(n) = n^\rho L(n)$ where L varies slowly at infinity. It follows that eventually

$$9c_1\xi^{-1} n_k^2 g(n_k) < 1 \quad \text{and} \quad 2^{d+3} C g(n_k)(\log n_k)^{4d^2} g\big(n_k^{1-\epsilon}\big)^{-1} = n_k^{\rho\epsilon} \frac{2^{d+3}C(\log n_k)^{4d^2} L(n_k)}{L(n_k^{1-\epsilon})} < 1\,.$$

It follows that in this case $\mathbb{P}$-a.s. for k large enough

$$\mathcal{E}^{\boldsymbol{w}}(f_k) \geq \frac{1}{2^{d+1}+2} g(n_k) \|f_k\|_2^2\,.$$

□

5 Localization of the principal eigenvector

We start with a collection of auxiliary lemmas and give the proof of Theorem 1.8 in Section 5.2.

5.1 Auxiliary lemmas

Let v be as in Theorem 1.8 and let n^* be the smallest integer greater than or equal to $\big(F^{-1}(v)\big)^{-2}$. For $n \geq n^*$ we abbreviate

$$g(n) := F^{-1}(n^{-1/2})\,. \tag{5.1}$$

Note that $\Lambda_g(n) = 1$ for $n \geq n^*$. In Lemma 5.3 we are going to see that the principal eigenvector $\psi_1^{(n)}$ tends to concentrate in the site $z_n \in B_n$ that minimizes the measure π_z. In addition to z_n we are also interested in properties of the site $z_{(2,n)}$ which is the location of the second-smallest value of π_z for $z \in B_n$.

The first lemma states some structural properties of the environment.

Lemma 5.1. *Let g be as in (5.1) and $\epsilon_2 \in (0, 1/3)$. Let $b \in \mathbb{N}$ such that $b \gg 4d$. Then $\mathbb{P}$-a.s. for n large enough and for all $z \in B_{n+b}$ the box $B_b(z)$ contains at most $3d-1$ links with conductance less than or equal to $g(n^{1-\epsilon_2})$. Furthermore, if $\mathscr{D}_n$ is as in Definition 4.4 with $\epsilon = \epsilon_2$, then $\mathbb{P}$-a.s. for n large enough the set $\mathscr{I}_n = B_n \backslash \mathscr{D}_n$ is sparse and both $z_n, z_{(2,n)} \in \mathscr{I}_n$.*

Proof. Since Λ_g is constant for $n \geq n^*$ and therefore bounded, the first claim follows by virtue of Corollary 2.4 (with $m = 2d$ and $k = d$).

To show that $\mathscr{I}_n$ is $\mathbb{P}$-a.s. sparse, we have to show that no two sites in $\mathscr{I}_n$ are neighbors. Let us assume that for infinitely many n there exists a pair of neighbors $x, y \in \mathscr{I}_n$. We note that at least $4d-2$ closed links are necessary to separate two neighboring sites from an infinite open cluster in $\mathbb{Z}^d$. But $4d-2 > 3d-1$ in any dimension $d \geq 2$. This is a contradiction. Therefore the above considerations imply that $\mathbb{P}$-a.s. for n large enough the set $\mathscr{I}_n$ is sparse.

For the last statement consider the following: Since the quotient $\Lambda_{g((\cdot)^{1-\epsilon_2/2})}/\log\log n$ diverges for n growing to infinity, Lemma 2.6 implies that $\mathbb{P}$-a.s. for n large enough $\pi_{z_n} \leq \pi_{z_{(2,n)}} < 2dg(n^{1-\epsilon_2/2})$. But this already implies that eventually $z_n, z_{(2,n)} \in \mathscr{I}_n$. $\square$

Let $\psi_1^{(n)}$ be the principal Dirichlet eigenvector of the operator $-\mathcal{L}_{\boldsymbol{w}}$ in the box B_n with zero Dirichlet conditions.

Lemma 5.2. *Let the function g be as in (5.1). Assume that there exists $\epsilon_1 \in (0, 1)$ such that one of the two cases occurs: g varies regularly at infinity with index $\rho < -(2+\epsilon_1)$ or the product $n^{2+\epsilon_1}g(n)$ converges monotonically to zero as n grows to infinity. Further, let $\epsilon = \epsilon_2 = \frac{7\epsilon_1}{8(2+\epsilon_1)}$ and $\mathscr{D}_n$ be as in Definition 4.4. Then $\mathbb{P}$-a.s. for n large enough*

$$\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)} \leq n^{-\epsilon_1/2}\,. \tag{5.2}$$

Proof. By Lemma 5.1, we can apply Proposition 4.6 to the set $\mathscr{D}_n$, i.e., there exists $c > 0$ such that $\mathbb{P}$-a.s. for n large enough

$$\mathcal{E}^{\boldsymbol{w}}_{\mathscr{D}_n}(f) \geq \left(2^{d+1}(\log n)^{4d^2} g(n^{1-\epsilon_2})^{-1} + cn^2\right)^{-1} \|f\|^2_{\ell^2(\mathscr{D}_n)}\,, \tag{5.3}$$

for any function $f\colon \mathbb{Z}^d \to \mathbb{R}$ with supp $f \subseteq B_n$. In any case, the assumptions imply that the product $n^{2+\epsilon_1}g(n)$ converges to zero as n grows to infinity. It follows that $n^2 g(n^{1-\epsilon_2})/(\log n)^{4d^2}$ converges to zero as well. Therefore, if $C = 2^{d+1} + 1$(5.3) implies that $\mathbb{P}$-a.s. for n large enough

$$\mathcal{E}^{\boldsymbol{w}}_{\mathscr{D}_n}(f) \geq \frac{1}{C}\frac{g(n^{1-\epsilon_2})}{(\log n)^{4d^2}} \|f\|^2_{\ell^2(\mathscr{D}_n)}\,.$$

On the other hand, we know that for any $\epsilon_3 > 0$ the term $\Lambda_{g((\cdot)^{1-\epsilon_3})}(n)/\log\log n$ diverges. Let us specifically choose $\epsilon_3 = \epsilon_1(8(2+\epsilon_1))^{-1}$. Now we use Theorem 1.3 (i) and the fact that the Dirichlet form $\mathcal{E}^{\boldsymbol{w}}$ majorizes $\mathcal{E}^{\boldsymbol{w}}_{\mathscr{D}_n}$ to infer that there exists $c_1 < \infty$ such that $\mathbb{P}$-a.s. for n large enough

$$c_1 g(n^{1-\epsilon_3}) \geq \lambda_1^{(n)} = \mathcal{E}^{\boldsymbol{w}}\left(\psi_1^{(n)}\right) \geq \mathcal{E}^{\boldsymbol{w}}_{\mathscr{D}_n}(\psi_1^{(n)}) \geq \frac{1}{C}\frac{g(n^{1-\epsilon_2})}{(\log n)^{4d^2}} \|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)}\,. \tag{5.4}$$

When we solve this inequality for $\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)}$, we obtain that

$$\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)} \le c_1 C\, \frac{g(n^{1-\epsilon_3})(\log n)^{4d^2}}{g(n^{1-\epsilon_2})}\,.$$

To finish the proof we use one of the additional assumptions about g: If g varies regularly at infinity with index $\rho < -(2+\epsilon_1)$, then we can write $g(n) = n^\rho L(n)$ where L varies slowly at infinity. In this case we observe that eventually

$$c_1 C\, \frac{g(n^{1-\epsilon_3})(\log n)^{4d^2}}{g(n^{1-\epsilon_2})} = c_1 C n^{\frac{3\rho\epsilon_1}{4(2+\epsilon_1)}}\, \frac{(\log n)^{4d^2} L(n^{1-\epsilon_3})}{L(n^{1-\epsilon_2})} \le n^{-\epsilon_1/2}\,,$$

which implies the claim. In the other case, i.e., if the product $n^{2+\epsilon_1} g(n)$ converges monotonically to zero as n tends to infinity, we observe that eventually

$$c_1 C\, \frac{g(n^{1-\epsilon_3})(\log n)^{4d^2}}{g(n^{1-\epsilon_2})} = c_1 C n^{-(2+\epsilon_1)(\epsilon_2-\epsilon_3)} (\log n)^{4d^2} \le n^{-\epsilon_1/2}\,,$$

which implies the claim as well. □

Lemma 5.3. *Let $y, z \in B_n$ with $\pi_z < \pi_y$ and $y \nsim z$. Assume that $\psi_1^{(n)}$ is nonnegative. Further, define $m_y = 2\max_{x:\, x\sim y} \psi_1^{(n)}(x)$. Then the mass $\psi_1^{(n)}(y)$ is bounded from above by*

$$\psi_1^{(n)}(y) \le \frac{m_y}{1 - \frac{\pi_z}{\pi_y}}\,. \tag{5.5}$$

Proof of Lemma 5.3. We assume the contrary, i.e., we assume that

$$m_y \pi_y + \psi_1^{(n)}(y)(\pi_z - \pi_y) < 0\,. \tag{5.6}$$

Then we define a new function $\phi : \mathbb{Z}^d \to \mathbb{R}_+$ by setting

$$\phi(x) = \begin{cases} \psi_1^{(n)}(x), & \text{for } x \notin \{y, z\}, \\ m_y, & \text{for } x = y, \\ \sqrt{\psi_1^{(n)}(y)^2 + \psi_1^{(n)}(z)^2 - m_y^2}, & \text{for } x = z. \end{cases} \tag{5.7}$$

Note that since (5.6) implies that $\psi_1^{(n)}(y) > m_y$, we observe that $\phi(z) > \psi_1^{(n)}(z)$. Obviously, supp $\phi \subseteq B_n$ and $\|\phi\|_2 = 1$. Therefore, by the variational formula (1.7) and Remark 1.1, the Dirichlet energy $\langle \phi, -\mathcal{L}_{\boldsymbol{w}}\phi\rangle$ is larger than the principal Dirichlet eigenvalue $\lambda_1^{(n)}$.

However, the Dirichlet energy of ϕ is given by

$$\begin{aligned}\langle \phi, -\mathcal{L}_{\boldsymbol{w}}\phi\rangle = \lambda_1^{(n)} &+ \Bigg[\sum_{x:x\sim y} w_{xy}\Big(\psi_1^{(n)}(x) - m_y\Big)^2 - \sum_{x:x\sim y} w_{xy}\Big(\psi_1^{(n)}(x) - \psi_1^{(n)}(y)\Big)^2\Bigg] \\ &+ \Bigg[\sum_{x:x\sim z} w_{xz}\Big(\psi_1^{(n)}(x) - \phi(z)\Big)^2 - \sum_{x:x\sim z} w_{xz}\Big(\psi_1^{(n)}(x) - \psi_1^{(n)}(z)\Big)^2\Bigg]. \end{aligned} \tag{5.8}$$

Evaluation of the first bracketed summand on the RHS gives:

$$\begin{aligned}&\sum_{x:x\sim y} w_{xy}\Big(\psi_1^{(n)}(x) - m_y\Big)^2 - \sum_{x:x\sim y} w_{xy}\Big(\psi_1^{(n)}(x) - \psi_1^{(n)}(y)\Big)^2 \\ &\qquad = \sum_{x:x\sim y} w_{xy}\Big(\psi_1^{(n)}(y) - m_y\Big)\Big(2\psi_1^{(n)}(x) - m_y - \psi_1^{(n)}(y)\Big)\end{aligned}$$

$$\leq -\psi_1^{(n)}(y) \sum_{x:x\sim y} w_{xy} \Big(\psi_1^{(n)}(y) - m_y\Big)\,, \tag{5.9}$$

where the last inequality follows by the definition of m_y and since the Assumption (5.6) implies that $\psi_1^{(n)}(y) > m_y$. Further, we evaluate the second bracketed summand on the RHS of (5.8) as

$$\sum_{x:x\sim z} w_{xz} \Big(\psi_1^{(n)}(x) - \phi(z)\Big)^2 - \sum_{x:x\sim z} w_{xz} \Big(\psi_1^{(n)}(x) - \psi_1^{(n)}(z)\Big)^2$$
$$= \sum_{x:x\sim z} w_{xz} \Big(\phi(z) - \psi_1^{(n)}(z)\Big)\Big(\phi(z) + \psi_1^{(n)}(z) - 2\psi_1^{(n)}(x)\Big)\,.$$

Since we assumed that $\psi_1^{(n)}$ is nonnegative and since the Assumption (5.6) implies that $\phi(z) > \psi_1^{(n)}(z)$, we conclude that

$$\sum_{x:x\sim z} w_{xz} \Big(\psi_1^{(n)}(x) - \phi(z)\Big)^2 - \sum_{x:x\sim z} w_{xz} \Big(\psi_1^{(n)}(x) - \psi_1^{(n)}(z)\Big)^2$$
$$\leq \sum_{x:x\sim z} w_{xz} \Big(\phi(z)^2 - \psi_1^{(n)}(z)^2\Big) = \sum_{x:x\sim z} w_{xz} \Big(\psi_1^{(n)}(y)^2 - m_y^2\Big)\,, \tag{5.10}$$

where the last equality follows by the definition of $\phi(z)$. When we insert (5.9) and (5.10) into (5.8), then we obtain

$$\langle \phi, -\mathcal{L}_{\boldsymbol{w}}\phi\rangle \leq \lambda_1^{(n)} - \psi_1^{(n)}(y) \sum_{x:x\sim y} w_{xy} \Big(\psi_1^{(n)}(y) - m_y\Big) + \sum_{x:x\sim z} w_{xz} \Big(\psi_1^{(n)}(y)^2 - m_y^2\Big)$$
$$= \lambda_1^{(n)} + \psi_1^{(n)}(z)^2(\pi_z - \pi_y) + m_y\Big(\psi_1^{(n)}(y)\pi_y - m_y\pi_z\Big)$$
$$\leq \lambda_1^{(n)} + \psi_1^{(n)}(y)\Big[m_y\pi_y + \psi_1^{(n)}(y)(\pi_z - \pi_y)\Big]\,. \tag{5.11}$$

Under Assumption (5.6) and the assumption that $\psi_1^{(n)}(y)$ is nonnegative it follows that the Dirichlet energy of ϕ is not larger than $\lambda_1^{(n)}$. This is a contradiction to the Perron-Frobenius theorem, see Remark 1.1. □

Let F_π be the distribution function of π, i.e., the distribution function of the sum of $2d$ independent copies of the conductance w. Then we have the following lemma.

Lemma 5.4. *If there exists $\gamma \in [0, 1/4)$ such that F varies regularly at zero with index γ, then F_π varies regularly near zero with index $2d\gamma$.*

Proof. Let $\mathscr{L}[F]$ be the Laplace transform of F. Then the Laplace transform of F_π fulfills

$$\mathscr{L}[F_\pi] = (\mathscr{L}[F])^{2d}\,.$$

By virtue of the Tauberian theorems, more precisely by virtue of Theorem 3 in [Fel71, XIII.5] (or, equivalently Theorem 1.7.1' of [BGT89]), $\mathscr{L}[F]$ varies regularly at infinity with index $-\gamma$. It follows that $\mathscr{L}[F_\pi]$ varies regularly at infinity with index $-2d\gamma$. Hence, by another application of Theorem 3 in [Fel71, XIII.5] we obtain that F_π varies regularly at zero with index $2d\gamma$. □

5.2 Proof of Theorem 1.8

Proof of Theorem 1.8. Recall the definition of the local speed measure π, i.e., $\pi(x) = \pi_x = \sum_{y:\, x\sim y} w_{xy}$ for $x \in \mathbb{Z}^d$. If π_z is very small, then z is a potential trap. The smaller π_z, the better the trap z. We now show that the random sequence $(z_n)_{n\in\mathbb{N}}$ of sites defined by $z_n = \arg\min_{x\in B_n} \pi_x$ is the sequence that we are looking for. Note that $\mathbb{P}$-a.s. the site, where the minimum is attained, is unique. Let $z_{(2,n)}$ be the site in B_n with second-smallest measure π.

Let $n^* = \left\lceil \left(F^{-1}(v)\right)^{-2} \right\rceil$ and abbreviate $g(n) = F^{-1}(n^{-1/2})$ for $n \geq n^*$. Let us recall that we assumed that one of the two following cases occurs: $\gamma \in (0, 1/4)$ or there exists $\epsilon_1 \in (0,1)$ such that the product $n^{2+\epsilon_1} g(n)$ converges monotonically to zero as n grows to infinity.

In the case where $\gamma > 0$ the inverse F^{-1} varies regularly at zero with index $1/\gamma$ (see e.g. [Sen76, p. 21]). It follows that g varies regularly at infinity with index $-1/(2\gamma)$. Since in addition $\gamma < 1/4$, there exists $\epsilon_1 \in (0,1)$ such that $-1/(2\gamma) < -(2+\epsilon_1)$.

In both cases we define $\mathscr{D}_n$ as in Definition 4.4 with $\epsilon = \frac{7\epsilon_1}{8(2+\epsilon_1)}$. Let $\mathscr{I}_n = B_n \backslash \mathscr{D}_n$. By virtue of Lemma 5.1 we know that $\mathbb{P}$-a.s. for n large enough the set $\mathscr{I}_n$ is sparse in the sense of Definition 4.5 and $z_n, z_{(2,n)} \in \mathscr{I}_n$.

Now, let $\alpha_n = n^{-\epsilon_1/8}$ and note that

$$\left\{ \|\psi_1^{(n)}\|^2_{\ell^2(B_n\backslash\{z_n\})} > \alpha_n^2 \right\} \subseteq \left\{ \|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)} > \frac{\alpha_n^2}{2} \right\} \cup \left\{ \|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{I}_n\backslash\{z_n\})} > \frac{\alpha_n^2}{2} \right\}. \tag{5.12}$$

However, by virtue of Lemma 5.2 we know that $\mathbb{P}$-a.s. for n large enough

$$\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)} \leq \alpha_n^4 \,, \tag{5.13}$$

and thus $\mathbb{P}$-a.s. the limit superior of the first event on the RHS vanishes.

In order to estimate the probability of the second event on the RHS of (5.12), we now estimate $\|\psi_1^{(n)}\|_{\ell^2(\mathscr{I}_n\backslash\{z_n\})}$ in terms of $\|\psi_1^{(n)}\|_{\ell^2(\mathscr{D}_n)}$. By virtue of Remark 1.1, we can assume without loss of generality that $\psi_1^{(n)}$ nonnegative. Let $y \in \mathscr{I}_n\backslash\{z_n\}$ and define $m_y = 2\max_{x: x\sim y} \psi_1^{(n)}(x)$. On the event where $\mathscr{I}_n$ is sparse, it is further $y \nsim z_n$. Therefore we know by virtue of Lemma 5.3 that $\psi_1^{(n)}(y) \leq m_y \left(1 - \frac{\pi_{z_n}}{\pi_y}\right)^{-1}$. By definition $\pi_y \geq \pi_{z_{(n,2)}}$ and thus it follows that

$$\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{I}_n\backslash\{z_n\})} \leq \left(1 - \frac{\pi_{z_n}}{\pi_{z_{(n,2)}}}\right)^{-2} \sum_{y\in\mathscr{I}_n\backslash\{z_n\}} m_y^2 \quad \text{(given } \mathscr{I}_n \text{ is sparse)}.$$

Moreover, on the event where $\mathscr{I}_n$ is sparse, any neighbor of $y \in \mathscr{I}_n$ is in $\mathscr{D}_n$ and therefore

$$\|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{I}_n\backslash\{z_n\})} \leq 8d \left(1 - \frac{\pi_{z_n}}{\pi_{z_{(n,2)}}}\right)^{-2} \|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{D}_n)} \quad \text{(given } \mathscr{I}_n \text{ is sparse)}. \tag{5.14}$$

On the event where (5.13) is true and $\mathscr{I}_n$ is sparse, we hence infer that

$$\left\{ \|\psi_1^{(n)}\|^2_{\ell^2(\mathscr{I}_n\backslash\{z_n\})} > \frac{\alpha_n^2}{2} \right\} \subseteq \left\{ 4\sqrt{d}\,\alpha_n > 1 - \frac{\pi_{z_n}}{\pi_{z_{(n,2)}}} \right\}.$$

It remains to show that $\mathbb{P}$-a.s. for n large enough $4\sqrt{d}\,\alpha_n \leq 1 - \pi_{z_n}/\pi_{z_{(n,2)}}$. We achieve this by an investigation of the extreme value statistics of the local speed measure $\{\pi_z\}_{z\in B_n}$, which follows below. For $n \in \mathbb{N}$ let $\pi_{1,|B_n|} \leq \pi_{2,|B_n|} \leq \ldots \leq \pi_{|B_n|,|B_n|}$ be an ordering of the set $\{\pi_z\}_{z\in B_n}$. Note that this means that $\pi_{1,|B_n|} = \pi_{z_n}$ and $\pi_{2,|B_n|} = \pi_{z_{(n,2)}}$. We abbreviate $E_n = \left\{ 4\sqrt{d}\,\alpha_n > 1 - \frac{\pi_{1,|B_n|}}{\pi_{2,|B_n|}} \right\}$ and $G_n = \{z_n, z_{(n,2)} \in \mathscr{I}_n\} \cap \{\mathscr{I}_n \text{ sparse}\}$. From the previous considerations we already know that

$$\begin{aligned}\mathbb{P}\left[\limsup_{n\to\infty} \left\{ \|\psi_1^{(n)}\|^2_{\ell^2(B_n\backslash\{z_n\})} > \alpha_n^2 \right\}\right] &\leq \mathbb{P}\left[\limsup_{n\to\infty} (E_n \cap G_n)\right] \\ &\leq \lim_{n\to\infty} \mathbb{P}[E_n \cap G_n] + \lim_{n\to\infty} \mathbb{P}\left[\bigcup_{k=n+1}^{\infty} (E_k \cap G_k \cap E_{k-1}^{c})\right]\end{aligned}$$

We now split the event

$$E_n = \left(E_n \cap \left\{\pi_{2,|B_n|} < \pi_{2,|B_{n-1}|}\right\}\right) \cup \left(E_n \cap \left\{\pi_{2,|B_n|} = \pi_{2,|B_{n-1}|}\right\}\right)$$

and observe that $E_n \cap \left\{\pi_{2,|B_n|} = \pi_{2,|B_{n-1}|}\right\} \cap E_{n-1}^{\mathrm{c}} = \emptyset$ and thus

$$\begin{aligned}
&\mathbb{P}\left[\limsup_{n\to\infty}\left\{\|\psi_1^{(n)}\|^2_{\ell^2(B_n\setminus\{z_n\})} > \alpha_n^2\right\}\right] \\
&\quad\le \lim_{n\to\infty}\mathbb{P}[E_n\cap G_n] + \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\mathbb{P}\left[E_k\cap G_k \middle| \pi_{2,|B_n|} < \pi_{2,|B_{n-1}|}\right]\mathbb{P}\left[\pi_{2,|B_n|} < \pi_{2,|B_{n-1}|}\right].
\end{aligned}$$

First, we estimate the factor $\mathbb{P}\left[\pi_{2,|B_n|} < \pi_{2,|B_{n-1}|}\right]$. To this end we recall that the set $\{\pi_z\}_{z\in B_n}$ consists of dependent random variables. However π_x is independent from π_y if the sites x and y are not neighbors. Let us recall the definition of the even lattice $A_{\mathrm{e}} \subset \mathbb{Z}^d$ and the odd lattice $A_{\mathrm{o}} \subset \mathbb{Z}^d$ as in (2.7) and below. With these definitions the set $\{\pi_z\}_{z\in B_n\cap A_{\mathrm{e}}}$ is a set of i.i.d. random variables, as well as is $\{\pi_z\}_{z\in B_n\cap A_{\mathrm{o}}}$. Let $\pi^{\mathrm{e}}_{1,|B_n|} \le \pi^{\mathrm{e}}_{2,|B_n|} \le \ldots$ be the ordering of $\{\pi_z\}_{z\in B_n\cap A_{\mathrm{e}}}$ and $\pi^{\mathrm{o}}_{1,|B_n|} \le \pi^{\mathrm{o}}_{2,|B_n|} \le \ldots$ be the ordering of $\{\pi_z\}_{z\in B_n\cap A_{\mathrm{o}}}$. Now we observe that for $n \ge 2$ the statement $\pi_{2,|B_n|} < \pi_{2,|B_{n-1}|}$ implies that $\pi^{\mathrm{e}}_{2,|B_n|} < \pi^{\mathrm{e}}_{2,|B_{n-1}|}$ or $\pi^{\mathrm{o}}_{2,|B_n|} < \pi^{\mathrm{o}}_{2,|B_{n-1}|}$. Thus, by virtue of [Res87, Proposition 4.3], there exists a constant $C < \infty$ such that for $n \ge 2$ we have:

$$\mathbb{P}\left[\pi_{2,|B_n|} < \pi_{2,|B_{n-1}|}\right] \le \mathbb{P}\left[\left\{\pi^{\mathrm{e}}_{2,|B_n|} < \pi^{\mathrm{e}}_{2,|B_{n-1}|}\right\} \cup \left\{\pi^{\mathrm{o}}_{2,|B_n|} < \pi^{\mathrm{o}}_{2,|B_{n-1}|}\right\}\right] \le Cn^{-1}.$$

We now define F_π as the distribution function of π. By definition, $F_\pi^{\frac{1}{2d}}$ is an increasing function and thus

$$E_n \subseteq \left\{F_\pi^{\frac{1}{2d}}\left(\pi_{1,|B_n|}\right) \ge F_\pi^{\frac{1}{2d}}\left((1-4\sqrt{d}\,\alpha_n)\pi_{2,|B_n|}\right)\right\}.$$

Our next aim is to extract the factor $1-4\sqrt{d}\,\alpha_n$ from inside the function argument. To this end we observe that by virtue of Lemma 5.4 and the assumption that F varies regularly at zero with index $\gamma \in [0, 1/4)$, we know that F_π varies regularly at zero with index $2d\gamma$. For any $\delta \in (0,1)$ it follows that

$$\lim_{a\to 0}\frac{F_\pi^{\frac{1}{2d}}((1-\delta)a)}{F_\pi^{\frac{1}{2d}}(a)} = (1-\delta)^\gamma > 1-\delta.$$

Thus, there exists $a^* > 0$ such that for all $a \in (0, a^*)$ we have $F_\pi^{\frac{1}{2d}}((1-\delta)a) \ge (1-\delta)F_\pi^{\frac{1}{2d}}(a)$.

Since both α_n and $\pi_{2,|B_n|}$ are converging to zero as a function of n, we obtain that for n large enough

$$\begin{aligned}
E_n &\subseteq \left\{\frac{F_\pi^{\frac{1}{2d}}\left(\pi_{1,|B_n|}\right)}{F_\pi^{\frac{1}{2d}}\left(\pi_{2,|B_n|}\right)} \ge 1-4\sqrt{d}\,\alpha_n\right\} \\
&= \left\{\log F_\pi^{\frac{1}{2d}}\left(\pi_{2,|B_n|}\right) - \log F_\pi^{\frac{1}{2d}}\left(\pi_{1,|B_n|}\right) \le -\log(1-4\sqrt{d}\,\alpha_n)\right\}
\end{aligned}$$

We observe that by definition the random variable $F_\pi^{\frac{1}{2d}}(\pi_x)$ $(x\in\mathbb{Z}^d)$ is an inverse Pareto variable with parameter $\lambda = 2d$. It follows that the random variable $\sigma_x := -\log F_\pi^{\frac{1}{2d}}(\pi_x)$ is exponentially distributed with parameter $2d$. In analogy to the definitions above, we define $\sigma_{1,|B_n|} \ge \sigma_{2,|B_n|} \ge \ldots \ge \sigma_{|B_n|,|B_n|}$ as an ordering of the set $\{\sigma_z\}_{z\in B_n}$.

Finally, we notice that the event G_n implies that the sites z_n and $z_{(n,2)}$ are not neighbors. Thus, when we apply Bayes' theorem to express the probability $\mathbb{P}[E_n \cap G_n]$ as a conditional probability, we notice that

$$\mathbb{P}[E_n\cap G_n] \le \mathbb{P}\left[\sigma_{1,|B_n|} - \sigma_{2,|B_n|} \le -\log\left(1-4\sqrt{d}\,\alpha_n\right) \,\middle|\, z_n \not\sim z_{(n,2)}\right].$$

By the condition $z_n \not\sim z_{(n,2)}$, the random variables $\sigma_{1,|B_n|}$ and $\sigma_{2,|B_n|}$ become independent except for the condition that $\sigma_{1,|B_n|} \leq \sigma_{2,|B_n|}$. But since the exponential distribution is memoryless, this condition simply implies that $\sigma_{1,|B_n|} - \sigma_{2,|B_n|}$ is again a nonnegative exponential random variable with parameter $2d$. It follows that

$$\mathbb{P}[E_n \cap G_n] \leq 1 - \mathrm{e}^{2d\log(1-4\sqrt{d}\,\alpha_n)} = 1 - \left(1 - 4\sqrt{d}\,\alpha_n\right)^{2d} \leq 8d^{\frac{3}{2}}\alpha_n\,.$$

With the same argument (i.e., that the exponential distribution is memoryless) we infer that

$$\mathbb{P}\big[E_k \cap G_k \big| \pi_{2,|B_n|} < \pi_{2,|B_{n-1}|}\big] \leq 8d^{\frac{3}{2}}\alpha_k\,,$$

and thus

$$\mathbb{P}\left[\limsup_{n\to\infty}\Big\{\big\|\psi_1^{(n)}\big\|^2_{\ell^2(B_n\setminus\{z_n\})} > \alpha_n^2\Big\}\right] \leq 8d^{\frac{3}{2}} \lim_{n\to\infty}\left(\alpha_n + C\sum_{k=n+1}^{\infty}\alpha_k k^{-1}\right),$$

which is zero since α_k decreases as $k^{-\epsilon_1/8}$.

In remains to prove (1.16). We already know that there exists $\epsilon_1 > 0$ such that $\mathbb{P}$-a.s. for n large enough

$$\psi_1^{(n)}(z_n)^2 \geq 1 - n^{-\epsilon_1/4}\,.$$

Further, by virtue of the variational formula (1.7) the event $\{\psi_1^{(n)}(z_n)^2 \geq 1 - n^{-\epsilon_1/4}\}$ implies that

$$\lambda_1^{(n)} \in \left[(\sqrt{1-n^{-\epsilon_1/4}} - n^{-\epsilon_1/8})^2 \min_{z\in B_n}\pi_z, (1-n^{-\epsilon_1/4})\min_{z\in B_n}\pi_z + n^{-\epsilon_1/4}\lambda_1^{(n)}\right].$$

The claim (1.16) follows. □

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