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# Distributed optimal control of a nonstandard nonlocal phase field system

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Abstract

We investigate a distributed optimal control problem for a nonlocal phase field model of viscous Cahn–Hilliard type. The model constitutes a nonlocal version of a model for two-species phase segregation on an atomic lattice under the presence of diffusion that has been studied in a series of papers by P. Podio-Guidugli and the present authors. The model consists of a highly nonlinear parabolic equation coupled to an ordinary differential equation. The latter equation contains both nonlocal and singular terms that render the analysis difficult. Standard arguments of optimal control theory do not apply directly, although the control constraints and the cost functional are of standard type. We show that the problem admits a solution, and we derive the first-order necessary conditions of optimality.

# 1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote an open and bounded domain whose smooth boundary $\Gamma$ has the outward unit normal $\mathbf{n}$; let $T > 0$ be a given final time, and set $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$. We study in this paper distributed optimal control problems of the following form:

**(CP)** Minimize the cost functional

$$J(u,\rho,\mu) = \frac{\beta_1}{2}\int_0^T\!\!\int_\Omega |\rho-\rho_Q|^2\,dx\,dt + \frac{\beta_2}{2}\int_0^T\!\!\int_\Omega |\mu-\mu_Q|^2\,dx\,dt + \frac{\beta_3}{2}\int_0^T\!\!\int_\Omega |u|^2\,dx\,dt \tag{1.1}$$

subject to the state system

$$(1+2\,g(\rho))\,\partial_t\mu + \mu\,g'(\rho)\,\partial_t\rho - \Delta\mu = u \quad \text{a.e. in } Q, \tag{1.2}$$

$$\partial_t\rho + B[\rho] + F'(\rho) = \mu\,g'(\rho) \quad \text{a.e. in } Q, \tag{1.3}$$

$$\partial_{\mathbf{n}}\mu = 0 \quad \text{a.e. on } \Sigma, \tag{1.4}$$

$$\rho(\cdot,0) = \rho_0\,, \quad \mu(\cdot,0) = \mu_0, \quad \text{a.e. in } \Omega, \tag{1.5}$$

and to the control constraints

$$u \in \mathcal{U}_{\rm ad} := \big\{u \in H^1(0,T;L^2(\Omega)) \,:\, 0 \le u \le u_{\max} \ \text{a.e. in } Q \ \text{and } \|u\|_{H^1(0,T;L^2(\Omega))} \le R\big\}. \tag{1.6}$$

Here, $\beta_1,\beta_2,\beta_3 \ge 0$ and $R > 0$ are given constants, with $\beta_1+\beta_2+\beta_3 > 0$, and the threshold function $u_{\max} \in L^\infty(Q)$ is nonnegative. Moreover, $\rho_Q, \mu_Q \in L^2(Q)$ represent prescribed target functions of the tracking-type functional $J$. Although more general cost functionals could be admitted for large parts of the subsequent analysis, we restrict ourselves to the above situation for the sake of a simpler exposition.

The state system (1.2)–(1.5) constitutes a *nonlocal* version of a phase field model of Cahn–Hilliard type describing phase segregation of two species (atoms and vacancies, say) on a lattice, which was recently studied in [18]. In the (simpler) original *local* model, which was introduced in [25], the nonlocal term $B[\rho]$ is a replaced by the diffusive term $-\Delta\rho$. The local model has been the subject of intensive research in the past years; in this connection, we refer the reader to [4–7, 9–12]. In particular, in [8] the analogue of the control problem **(CP)** for the local case was investigated for the special situation $g(\rho) = \rho$, for which the optimal boundary control problems was studied in [14].

The state variables of the model are the *order parameter* $\rho$, interpreted as a volumetric density, and the *chemical potential* $\mu$; for physical reasons, we must have $0 \le \rho \le 1$ and $\mu > 0$ almost everywhere in $Q$. The control function $u$ on the right-hand side of (1.2) plays the role of a *microenergy source*. We remark at this place that the requirement encoded in the definition of $\mathcal{U}_{\rm ad}$, namely that $u$ be nonnegative, is indispensable for the analysis of the forthcoming sections. Indeed, it is needed to guarantee the nonnegativity of the chemical potential $\mu$.

The nonlinearity $F$ is a double-well potential defined in the interval $(0,1)$, whose derivative $F'$ is singular at the endpoints $\rho = 0$ and $\rho = 1$: e.g., $F = F_1 + F_2$, where $F_2$ is smooth and

$$F_1(\rho) = \hat{c}\,(\rho\,\log(\rho) + (1-\rho)\,\log(1-\rho)), \quad \text{with a constant } \hat{c} > 0. \tag{1.7}$$

The presence of the nonlocal term $B[\rho]$ in (1.3) constitutes the main difference to the local model. Simple examples are given by integral operators of the form

$$B[\rho](x,t) = \int_0^t \int_\Omega k(t,s,x,y)\,\rho(y,s)\,ds\,dy \tag{1.8}$$

and purely spatial convolutions like

$$B[\rho](x,t) = \int_\Omega k(|y-x|)\,\rho(y,t)\,dy, \tag{1.9}$$

with sufficiently regular kernels.

Optimal control problems of the above type often occur in industrial production processes. For instance, consider a metallic workpiece consisting of two different component materials that tend to separate. Then a typical goal would be to monitor the production process in such a way that a desired distribution of the two materials (represented by the function $\rho_Q$) is realized during the time evolution in order to guarantee a wanted behavior of the workpiece; the deviation from the desired phase distribution is measured by the first summand in the cost $J$. The third summand of $J$ represents the costs due to the control action $u$; the size of the factors $\beta_i \ge 0$ then reflects the relative importance that the two conflicting interests "realize the desired phase distribution as closely as possible" and "minimize the cost of the control action" have for the manufacturer.

The state system (1.2)–(1.5) is singular, with highly nonlinear and nonstandard coupling. In particular, unpleasant nonlinear terms involving time derivatives occur in (1.2), and the expression $F'(\rho)$ in (1.3) may become singular. Moreover, the nonlocal term $B[\rho]$ is a source for possible analytical difficulties, and the absence of the Laplacian in (1.3) may cause a low regularity of the order parameter $\rho$. We remark that the state system (1.2)–(1.5) was recently analyzed in [18] for the case $u = 0$ (no control); results concerning well-posedness and regularity were established.

The mathematical literature on control problems for phase field systems involving equations of viscous or nonviscous Cahn–Hilliard type is still scarce and quite recent. We refer in this connection to the works [2, 3, 16, 17, 21, 28]. Control problems for convective Cahn–Hilliard systems were studied in [29, 30], and a few analytical contributions were made to the coupled Cahn–Hilliard/Navier–Stokes system (cf. [19, 20, 22, 23]). The very recent contribution [13] deals with the optimal control of a Cahn–Hilliard type system arising in the modeling of solid tumor growth.

The paper is organized as follows: in Section 2, we state the general assumptions and derive new regularity and stability results for the state system. In Section 3, we establish the directional differentiability of the control-to-state operator, and the final Section 4 brings the main results of this paper, namely, the derivation of the first-order necessary conditions of optimality.

Throughout this paper, we will use the following notation: we denote for a (real) Banach space $X$ by $\|\cdot\|_X$ its norm and the norm of $X \times X \times X$, by $X'$ its dual space, and by $\langle\cdot,\cdot\rangle_X$ the dual pairing between $X'$ and $X$. If $X$ is an inner product space, then the inner product is denoted by $(\cdot,\cdot)_X$. The only exception from this convention is given by the $L^p$ spaces, $1 \le p \le \infty$, for which we use the abbreviating notation $\|\cdot\|_p$ for the norms. Furthermore, we put

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{w \in H^2(\Omega) : \partial_{\mathbf{n}} w = 0 \ \text{ a.e. on } \Gamma\}.$$

We have the dense and continuous embeddings $W \subset V \subset H \cong H' \subset V' \subset W'$, where $\langle u, v\rangle_V = (u, v)_H$ and $\langle u, w\rangle_W = (u, w)_H$ for all $u \in H$, $v \in V$, and $w \in W$.

In the following, we will make repeated use of Young's inequality

$$a\,b \le \delta\, a^2 + \frac{1}{4\delta} b^2 \quad \text{for all } a, b \in \mathbb{R} \quad \text{and } \delta > 0, \tag{1.10}$$

as well as of the fact that for three dimensions of space and smooth domains the embeddings $V \subset L^p(\Omega)$, $1 \le p \le 6$, and $H^2(\Omega) \subset C^0(\overline{\Omega})$ are continuous and (in the first case only for $1 \le p < 6$) compact. In particular, there are positive constants $\widetilde{K}_i$, $i = 1, 2, 3$, which depend only on the domain $\Omega$, such that

$$\|v\|_6 \ \le\ \widetilde{K}_1 \|v\|_V \quad \forall\, v \in V, \tag{1.11}$$

$$\|v\, w\|_H \ \le\ \|v\|_6\, \|w\|_3 \ \le\ \widetilde{K}_2 \|v\|_V\, \|w\|_V \quad \forall\, v, w \in V, \tag{1.12}$$

$$\|v\|_\infty \ \le\ \widetilde{K}_3 \|v\|_{H^2(\Omega)} \quad \forall\, v \in H^2(\Omega). \tag{1.13}$$

We also set for convenience

$$Q_t := \Omega \times (0, t) \quad \text{and} \quad Q^t := \Omega \times (t, T), \quad \text{for } t \in (0, T). \tag{1.14}$$

Please note the difference between the subscript and the superscript in the above notation.

About time derivatives of a time-dependent function $v$, we point out that we will use both the notations $\partial_t v$, $\partial_t^2 v$ and the shorter ones $v_t$, $v_{tt}$.

## 2 Problem statement and results for the state system

Consider the optimal control problem (1.1)–(1.6). We make the following assumptions on the data:

**(A1)** $F = F_1 + F_2$, where $F_1 \in C^3(0,1)$ is convex, $F_2 \in C^3[0,1]$, and

$$\lim_{r \searrow 0} F_1'(r) = -\infty, \quad \lim_{r \nearrow 1} F_1'(r) = +\infty. \tag{2.1}$$

**(A2)** $\rho_0 \in V$, $F'(\rho_0) \in H$, $\mu_0 \in W$, where $\mu_0 \geq 0$ a. e. in $\Omega$,

$$\inf \{\rho_0(x) : x \in \Omega\} > 0, \quad \sup \{\rho_0(x) : x \in \Omega\} < 1\,. \tag{2.2}$$

**(A3)** $g \in C^3[0,1]$ satisfies $g(\rho) \geq 0$ and $g''(\rho) \leq 0$ for all $\rho \in [0,1]$.

**(A4)** The nonlocal operator $B\colon L^1(Q) \to L^1(Q)$ satisfies the following conditions:
**(i)** For every $t \in (0,T]$, we have

$$B[v]|_{Q_t} = B[w]|_{Q_t} \quad \text{whenever } v|_{Q_t} = w|_{Q_t}. \tag{2.3}$$

**(ii)** For all $p \in [2,+\infty]$, we have $B(L^p(Q_t)) \subset L^p(Q_t)$ and

$$\|B[v]\|_{L^p(Q_t)} \leq C_{B,p} \left(1 + \|v\|_{L^p(Q_t)}\right) \tag{2.4}$$

for every $v \in L^p(Q)$ and $t \in (0,T]$.
**(iii)** For every $v, w \in L^1(0,T;H)$ and $t \in (0,T]$, it holds that

$$\int_0^t \|B[v](s) - B[w](s)\|_6 \, ds \leq C_B \int_0^t \|v(s) - w(s)\|_H \, ds\,. \tag{2.5}$$

**(iv)** It holds, for every $v \in L^2(0,T;V)$ and $t \in (0,T]$, that

$$\|\nabla B[v]\|_{L^2(0,t;H)} \leq C_B\big(1 + \|v\|_{L^2(0,t;V)}\big). \tag{2.6}$$

**(v)** For every $v \in H^1(0,T;H)$, we have $\partial_t B[v] \in L^2(Q)$ and

$$\|\partial_t B[v]\|_{L^2(Q)} \leq C_B\big(1 + \|\partial_t v\|_{L^2(Q)}\big). \tag{2.7}$$

**(vi)** $B$ is continuously Fréchet differentiable as a mapping from $L^2(Q)$ into $L^2(Q)$, and the Fréchet derivative $DB[\overline{v}] \in \mathcal{L}(L^2(Q), L^2(Q))$ of $B$ at $\overline{v}$ has for every $\overline{v} \in L^2(Q)$ and $t \in (0,T]$ the following properties:

$$\|DB[\overline{v}](w)\|_{L^p(Q_t)} \leq C_B \|w\|_{L^p(Q_t)} \quad \forall\, w \in L^p(Q), \quad \forall\, p \in [2,6], \tag{2.8}$$

$$\|\nabla(DB[\overline{v}](w))\|_{L^2(Q_t)} \leq C_B \|w\|_{L^2(0,t;V)} \quad \forall\, w \in L^2(0,T;V)\,, \tag{2.9}$$

$$\Big| \int_0^t\!\!\int_\Omega \nabla(DB[\overline{v}](w)) \cdot \nabla w \, dx \, ds \Big| \leq C_B \|w\|^2_{L^2(0,t;V)} \quad \forall\, w \in L^2(0,T;V). \tag{2.10}$$

In the above formulas, $C_{B,p}$ and $C_B$ denote given positive structural constants. We also notice that (2.8) implicitely requires that $DB[\overline{v}](w)|_{Q_t}$ depends only on $w|_{Q_t}$. However, this is a consequence of (2.3).

The statements related to the control problem **(CP)** depend on the assumptions made in the Introduction. We recall them here.

**(A5)** $J$ and $\mathcal{U}_{\rm ad}$ are defined by (1.1) and (1.6), respectively, where

$$\beta_1\,,\ \beta_2\,,\ \beta_3 \geq 0, \quad \beta_1+\beta_2+\beta_3>0, \quad \text{and} \quad R>0. \tag{2.11}$$

$$\rho_Q, \mu_Q \in L^2(Q), \quad u_{\max}\in L^\infty(Q) \quad \text{and} \quad u_{\max}\geq 0 \quad \text{a.e. in } Q. \tag{2.12}$$

Remark 1: In view of (2.8), for every $t\in[0,T]$ it holds that

$$\|B[v]-B[w]\|_{L^2(Q_t)} \leq C_B\,\|v-w\|_{L^2(Q_t)} \quad \forall\, v,w\in L^2(Q)\,, \tag{2.13}$$

that is, the condition (2.9) in [18] is fulfilled. Moreover, (2.4) and (2.6) imply that $B$ maps $L^2(0,T;V)$ into itself and that, for all $t\in(0,T]$ and $v\in L^2(0,T;V)$,

$$\Big|\int_0^t\!\!\int_\Omega \nabla B[v]\cdot\nabla v\,dx\,ds\Big| \leq C_B\left(1+\|v\|^2_{L^2(0,t;V)}\right),$$

which means that also the condition (2.10) in [18] is satisfied. Moreover, thanks to (2.8) and (2.9), there is some constant $\widetilde{C}_B>0$ such that

$$\|DB[\overline{v}](w)\|_{L^2(0,t;V)} \leq \widetilde{C}_B\,\|w\|_{L^2(0,t;V)} \quad \forall\,\overline{v}\in L^2(Q),\ \forall\, w\in L^2(0,T;V)\,. \tag{2.14}$$

Remark 2: We recall (cf. [18]) that the integral operator (1.9) satisfies the conditions (2.3) and (2.4), provided that the integral kernel $k$ belongs to $C^1(0,+\infty)$ and fulfills, with suitable constants $C_1>0$, $C_2>0$, $0<\alpha<\frac{3}{2}$, $0<\beta<\frac{5}{2}$, the growth conditions

$$|k(r)|\leq C_1\,r^{-\alpha}, \quad |k'(r)|\leq C_2\,r^{-\beta}, \quad \forall\, r>0\,.$$

In this case, we have $2\alpha<3$ and thus, for all $v,w\in L^1(0,T;H)$ and $t\in(0,T]$,

$$\begin{aligned}
&\int_0^t \|B[v](s)-B[w](s)\|_6\,ds \\
&\leq C_1\int_0^t\Big(\int_\Omega\Big|\int_\Omega |y-x|^{-\alpha}|v(y,s)-w(y,s)|\,dy\Big|^6\,dx\Big)^{1/6}ds \\
&\leq C_3\int_0^t\Big(\int_\Omega\Big|\Big(\int_\Omega |y-x|^{-2\alpha}\,dy\Big)^{1/2}\|v(s)-w(s)\|_H\Big|^6\,dx\Big)^{1/6}ds \\
&\leq C_4\int_0^t \|v(s)-w(s)\|_H\,ds,
\end{aligned}$$

with global constants $C_i$, $3 \le i \le 4$; the condition (2.5) is thus satisfied. Also condition (2.6) holds true in this case: indeed, for every $t \in (0,T]$ and $v \in L^2(0,T;V)$, we find, since $\frac{6\beta}{5} < 3$, that

$$\begin{aligned}
\|\nabla B[v]\|^2_{L^2(0,t;H)} &\le C_2 \int_0^t\!\!\int_\Omega \Big| \int_\Omega |y-x|^{-\beta} |v(y,s)|\, dy \Big|^2 dx\, ds \\
&\le C_5 \int_0^t\!\!\int_\Omega \Big( \int_\Omega |y-x|^{-6\beta/5} dy \Big)^{5/3} \|v(s)\|_6^2\, dx\, ds \\
&\le C_6 \int_0^t \|v(s)\|_V^2\, ds\,.
\end{aligned}$$

Finally, since the operator $B$ is linear in this case, we have $DB[\overline{v}] = B$ for every $\overline{v} \in L^2(Q)$, and thus also **(A4)(v)** and (2.8)–(2.14) are fulfilled. Notice that the above growth conditions are met by, e.g., the three-dimensional Newtonian potential, where $k(r) = \frac{c}{r}$ with some $c \neq 0$.

We also note that **(A2)** implies $\mu_0 \in C(\overline{\Omega})$, and **(A1)** and (2.2) ensure that both $F(\rho_0)$ and $F'(\rho_0)$ are in $L^\infty(\Omega)$, whence in $H$. Moreover, the logarithmic potential (1.7) obviously fulfills the condition (2.1) in **(A1)**.

We have the following existence and regularity result for the state system.

THEOREM 2.1: *Suppose that* **(A1)**–**(A5)** *are satisfied. Then the state system* (1.2)–(1.5) *has for every* $u \in \mathcal{U}_{\rm ad}$ *a unique solution* $(\rho,\mu)$ *such that*

$$\rho \in H^2(0,T;H) \cap W^{1,\infty}(0,T;L^\infty(\Omega)) \cap H^1(0,T;V), \tag{2.15}$$

$$\mu \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \subset L^\infty(Q). \tag{2.16}$$

*Moreover, there are constants* $0 < \rho_* < \rho^* < 1$, $\mu^* > 0$, *and* $K_1^* > 0$, *which depend only on the given data, such that for every* $u \in \mathcal{U}_{\rm ad}$ *the corresponding solution* $(\rho,\mu)$ *satisfies*

$$0 < \rho_* \le \rho \le \rho^* < 1, \quad 0 \le \mu \le \mu^*, \quad \text{a.e. in } Q, \tag{2.17}$$

$$\begin{aligned}
&\|\mu\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)\cap L^\infty(Q)} \\
&+ \|\rho\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;L^\infty(\Omega))\cap H^1(0,T;V)} \le K_1^*.
\end{aligned} \tag{2.18}$$

PROOF: In the following, we denote by $C_i > 0$, $i \in \mathbb{N}$, constants which may depend on the data of the control problem **(CP)** but not on the special choice of $u \in \mathcal{U}_{\rm ad}$. First, we note that in [18, Thms. 2.1, 2.2] it has been shown that under the given assumptions there exists for $u \equiv 0$ a unique solution $(\rho,\mu)$ with the properties

$$0 < \rho < 1, \quad \mu \ge 0, \quad \text{a.e. in } Q, \tag{2.19}$$

$$\rho \in L^\infty(0,T;V), \quad \partial_t \rho \in L^6(Q), \tag{2.20}$$

$$\mu \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^\infty(Q) \cap L^2(0,T;W^{2,3/2}(\Omega)). \tag{2.21}$$

A closer inspection of the proofs in [18] reveals that the line of argumentation used there (in particular, the proof that $\mu$ is nonnegative) carries over with only minor modifications to general right-hand sides $u \in \mathcal{U}_{\rm ad}$. We thus infer that (1.2)–(1.5) enjoys for every $u \in \mathcal{U}_{\rm ad}$ a unique solution satisfying (2.19)–(2.21); more precisely, there is some $C_1 > 0$ such that

$$\|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^\infty(Q)\cap L^2(0,T;W^{2,3/2}(\Omega))} \\ + \|\rho\|_{L^\infty(0,T;V)} + \|\partial_t\rho\|_{L^6(Q)} \le C_1 \quad \forall\, u \in \mathcal{U}_{\rm ad}. \tag{2.22}$$

Moreover, invoking (2.19), and (2.4) for $p = +\infty$, we find that

$$\|B[\rho]\|_{L^\infty(Q)} \le C_2 \quad \forall\, u \in \mathcal{U}_{\rm ad},$$

and it follows from (2.22) and the general assumptions on $\rho_0$, $g$, and $F$, that there are constants $\rho_*, \rho^*$ such that, for every $u \in \mathcal{U}_{\rm ad}$,

$$\begin{aligned} &0 < \rho_* \le \inf\{\rho_0(x) : x \in \Omega\} \le \sup\{\rho_0(x) : x \in \Omega\} \le \rho^* < 1, \\ &F'(\rho) + B[\rho] - \mu\, g'(\rho) \le 0 \ \text{ if } 0 < \rho \le \rho_*, \\ &F'(\rho) + B[\rho] - \mu\, g'(\rho) \ge 0 \ \text{ if } \rho^* \le \rho < 1. \end{aligned}$$

Therefore, multiplying (1.3) by the positive part $(\rho-\rho^*)^+$ of $\rho - \rho^*$, and integrating over $Q$, we find that

$$\begin{aligned} 0 &= \int_0^T\!\!\int_\Omega \partial_t\rho\,(\rho-\rho^*)^+\,dx\,dt + \int_0^T\!\!\int_\Omega (F'(\rho) + B[\rho] - \mu\, g'(\rho))(\rho-\rho^*)^+\,dx\,dt \\ &\ge \frac{1}{2}\int_\Omega \left|(\rho(t)-\rho^*)^+\right|^2 dx, \end{aligned}$$

whence we conclude that $(\rho-\rho^*)^+ = 0$, and thus $\rho \le \rho^*$, almost everywhere in $Q$. The other bound for $\rho$ in (2.17) is proved similarly.

It remains to show the missing bounds in (2.18) (which then also imply the missing regularity claimed in (2.15)–(2.16)). To this end, we employ a bootstrapping argument. First, notice that **(A3)** and the already proved bounds (2.22) and (2.17) imply that the expressions $\mu\, g'(\rho)\,\partial_t\rho$ and $(1+2g(\rho))\,\partial_t\mu$ are bounded in $L^2(Q)$. Hence, by comparison in (1.2), the same holds true for $\Delta\mu$, and thus standard elliptic estimates yield that

$$\|\mu\|_{L^2(0,T;W)} \le C_3 \quad \forall\, u \in \mathcal{U}_{\rm ad}. \tag{2.23}$$

Next, observe that **(A1)** and (2.17) imply that $\|F'(\rho)\|_{L^\infty(Q)} \le C_4$, and comparison in (1.3), using **(A3)**, yields that

$$\|\partial_t\rho\|_{L^\infty(Q)} \le C_5 \quad \forall\, u \in \mathcal{U}_{\rm ad}. \tag{2.24}$$

In addition, we infer from the estimates shown above, and using (2.6), that the right-hand side of the identity

$$\nabla\rho_t = -F''(\rho)\,\nabla\rho - \nabla B[\rho] + g'(\rho)\,\nabla\mu + \mu\, g''(\rho)\,\nabla\rho \tag{2.25}$$

is bounded in $L^2(Q)$, so that

$$\|\partial_t\rho\|_{L^2(0,T;V)} \le C_6 \quad \forall\, u \in \mathcal{U}_{\rm ad}. \tag{2.26}$$

We also note that the time derivative $\partial_t(-F'(\rho) - B[\rho] + \mu g'(\rho))$ exists and is bounded in $L^2(Q)$ (cf. (2.7)). We thus have

$$\|\rho_{tt}\|_{L^2(Q)} \le C_7 \quad \forall\, u \in \mathcal{U}_{\rm ad}. \tag{2.27}$$

At this point, we observe that Eq. (1.2) can be written in the form

$$a\,\partial_t\mu + \mu\,\partial_t a - \Delta\mu = b, \quad \text{with } a := 1 + 2g(\rho), \quad b := u + \mu\, g'(\rho)\,\partial_t\rho,$$

where, thanks to the above estimates, we have, for every $u \in \mathcal{U}_{\rm ad}$,

$$\|a\|_{L^\infty(Q)} + \|\partial_t a\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)} \le C_8, \tag{2.28}$$

$$\left\|\partial_t^2 a\right\|_{L^2(Q)} = 2\,\|g''(\rho)\rho_t^2 + g'(\rho)\rho_{tt}\|_{L^2(Q)} \le C_9, \tag{2.29}$$

$$\|\partial_t b\|_{L^2(Q)} = \|u_t + \mu_t g'(\rho)\rho_t + \mu g''(\rho)\rho_t^2 + \mu g'(\rho)\rho_{tt}\|_{L^2(Q)} \le C_{10}. \tag{2.30}$$

Since also $\mu_0 \in W$, we are thus in the situation of [15, Thm. 3.4], whence we obtain that $\partial_t\mu \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and $\mu \in L^\infty(0,T;W)$. Moreover, a closer look at the proof of [15, Thm. 3.4] reveals that we also have the estimates

$$\|\partial_t\mu\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\mu\|_{L^\infty(0,T;W)} \le C_{11}. \tag{2.31}$$

This concludes the proof of the assertion. □

Remark 3: From the estimates (2.17) and (2.18), and using the continuity of the embedding $V \subset L^6(\Omega)$, we can without loss of generality (by possibly choosing a larger $K_1^*$) assume that also

$$\begin{aligned} &\max_{0\le i\le 3} \|F^{(i)}(\rho)\|_{L^\infty(Q)} + \max_{0\le i\le 3} \|g^{(i)}(\rho)\|_{L^\infty(Q)} \\ &+ \|\nabla\mu\|_{L^\infty(0,T;L^6(\Omega)^3)} + \|\partial_t\mu\|_{L^2(0,T;V)} \\ &+ \|B[\rho]\|_{H^1(0,T;L^2(\Omega))\cap L^\infty(Q)\cap L^2(0,T;V)} \le K_1^* \quad \forall\, u \in \mathcal{U}_{\rm ad}. \end{aligned} \tag{2.32}$$

According to Theorem 2.1, the control-to-state mapping $\mathcal{S} : \mathcal{U}_{\rm ad} \ni u \mapsto (\rho,\mu)$ is well defined. We now study its stability properties. We have the following result.

Theorem 2.2: *Suppose that* **(A1)**–**(A5)** *are fulfilled, and let* $u_i \in \mathcal{U}_{\rm ad}$, $i = 1,2$, *be given and* $(\rho_i,\mu_i) = \mathcal{S}(u_i)$, $i = 1,2$, *be the associated solutions to the state system* (1.2)–(1.5). *Then there exists a contant* $K_2^* > 0$, *which depends only on the data of the problem, such that, for every* $t \in (0,T]$,

$$\begin{aligned} &\|\rho_1 - \rho_2\|_{H^1(0,t;H)\cap L^\infty(0,t;L^6(\Omega))} + \|\mu_1 - \mu_2\|_{H^1(0,t;H)\cap L^\infty(0,t;V)\cap L^2(0,t;W)} \\ &\le K_2^*\,\|u_1 - u_2\|_{L^2(0,t;H)}. \end{aligned} \tag{2.33}$$

Proof: Taking the difference of the equations satisfied by $(\rho_i,\mu_i)$, $i = 1,2$, and setting $u := u_1 - u_2$, $\rho := \rho_1 - \rho_2$, $\mu := \mu_1 - \mu_2$, we first observe that we have almost everywhere in $Q$ the identities

$$(1+2g(\rho_1))\,\partial_t\mu \,+\, g'(\rho_1)\,\partial_t\rho_1\,\mu - \Delta\mu \,+\, 2(g(\rho_1)-g(\rho_2))\,\partial_t\mu_2$$
$$= u \,-\, (g'(\rho_1)-g'(\rho_2))\,\partial_t\rho_1\,\mu_2 \,-\, g'(\rho_2)\,\mu_2\,\partial_t\rho\,, \tag{2.34}$$

$$\partial_t\rho \,+\, F'(\rho_1) \,-\, F'(\rho_2) \,+\, B[\rho_1]-B[\rho_2]$$
$$= g'(\rho_1)\,\mu \,+\, (g'(\rho_1)-g'(\rho_2))\,\mu_2\,, \tag{2.35}$$

as well as

$$\partial_{\mathbf{n}}\mu = 0 \quad \text{a. e. on } \Sigma, \quad \mu(\cdot,0)=\rho(\cdot,0)=0 \quad \text{a. e. in } \Omega. \tag{2.36}$$

Let $t \in (0,T]$ be arbitrary. In the following, we repeatedly use the global estimates (2.17), (2.18), and (2.32), without further reference. Moreover, we denote by $c>0$ constants that may depend on the given data of the state system, but not on the choice of $u_1,u_2 \in \mathcal{U}_{\rm ad}$; the meaning of $c$ may change between and even within lines. We establish the validity of (2.33) in a series of steps.

Step 1: To begin with, we first observe that

$$(1+2g(\rho_1))\mu\partial_t\mu + g'(\rho_1)\partial_t\rho_1\mu^2 = \partial_t\Bigl(\Bigl(\frac{1}{2}+g(\rho_1)\Bigr)\mu^2\Bigr).$$

Hence, multiplying (2.34) by $\mu$ and integrating over $Q_t$ and by parts, we obtain that

$$\int_\Omega \Bigl(\frac{1}{2}+g(\rho_1(t))\Bigr)\mu^2(t)\,dx \,+\, \int_0^t\!\!\int_\Omega |\nabla\mu|^2\,dx\,ds \,\le\, \sum_{j=1}^3 |I_j|\,, \tag{2.37}$$

where the expressions $I_j$, $j=1,2,3$, defined below, are estimated as follows: first, we apply **(A3)**, the mean value theorem, and Hölder's and Young's inequalities, to find, for every $\gamma>0$ (to be chosen later), that

$$I_1 := -2\int_0^t\!\!\int_\Omega (g(\rho_1)-g(\rho_2))\,\partial_t\mu_2\,\mu\,dx\,ds \le c\int_0^t \|\partial_t\mu_2(s)\|_6\,\|\mu(s)\|_3\,\|\rho(s)\|_2\,ds$$
$$\le \gamma\int_0^t \|\mu(s)\|_V^2\,ds \,+\, \frac{c}{\gamma}\int_0^t \|\partial_t\mu_2(s)\|_V^2\,\|\rho(s)\|_H^2\,ds, \tag{2.38}$$

where it follows from (2.32) that the mapping $s\mapsto \|\partial_t\mu_2(s)\|_V^2$ belongs to $L^1(0,T)$. Next, we see that

$$I_2 := \int_0^t\!\!\int_\Omega \Bigl(u-(g'(\rho_1)-g'(\rho_2))\partial_t\rho_1\,\mu_2\Bigr)\,\mu\,dx\,ds$$
$$\le c\int_0^t\!\!\int_\Omega (|u|+|\rho|)|\mu|\,dx\,ds \,\le\, c\int_0^t\!\!\int_\Omega (u^2+\rho^2+\mu^2)\,dx\,ds\,. \tag{2.39}$$

Finally, Young's inequality yields that

$$I_3 := -\int_0^t\!\!\int_\Omega g'(\rho_2)\,\mu_2\,\rho_t\,\mu\,dx\,ds \,\le\, \gamma\int_0^t\!\!\int_\Omega \rho_t^2\,dx\,ds \,+\, \frac{c}{\gamma}\int_0^t\!\!\int_\Omega \mu^2\,dx\,ds\,. \tag{2.40}$$

Combining (2.37)–(2.40), and recalling that $g(\rho_1)$ is nonnegative, we have thus shown the estimate

$$\frac{1}{2}\,\|\mu(t)\|_H^2 \,+\, (1-\gamma)\int_0^t \|\mu(s)\|_V^2\,ds \,\le\, \gamma\int_0^t\!\!\int_\Omega \rho_t^2\,dx\,ds \,+\, c\int_0^t\!\!\int_\Omega u^2\,dx\,ds$$
$$+\, c\left(1+\gamma^{-1}\right)\int_0^t \left(\|\mu(s)\|_H^2 \,+\, (1+\|\partial_t\mu_2(s)\|_V^2)\,\|\rho(s)\|_H^2\right)ds\,. \tag{2.41}$$

Next, we add $\rho$ on both sides of (2.35) and multiply the result by $\rho_t$. Integrating over $Q_t$, using the Lipschitz continuity of $F'$ (when restricted to $[\rho_*,\rho^*]$), (2.13) and Young's inequality, we easily find the estimate

$$(1-\gamma)\int_0^t\!\!\int_\Omega \rho_t^2\,dx\,ds \,+\, \frac{1}{2}\,\|\rho(t)\|_H^2 \,\le\, \frac{c}{\gamma}\int_0^t\!\!\int_\Omega (\rho^2+\mu^2)\,dx\,ds\,. \tag{2.42}$$

Therefore, combining (2.41) with (2.42), choosing $\gamma > 0$ small enough, and invoking Gronwall's lemma, we have shown that

$$\|\mu\|_{L^\infty(0,t;H)\cap L^2(0,t;V)} \,+\, \|\rho\|_{H^1(0,t;H)} \,\le\, c\|u\|_{L^2(0,t;H)} \quad \forall\, t\in(0,T]. \tag{2.43}$$

<u>STEP 2:</u> Next, we multiply (2.35) by $\rho|\rho|$ and integrate over $Q_t$. We obtain

$$\frac{1}{3}\,\|\rho(t)\|_3^3 \,\le\, \sum_{j=1}^3 |J_j|, \tag{2.44}$$

where the expressions $J_j$, $1\le j\le 3$, are estimated as follows: at first, we simply have

$$J_1 := \int_0^t\!\!\int_\Omega (-F'(\rho_1)+F'(\rho_2)+\mu_2(g'(\rho_1)-g'(\rho_2)))\,\rho\,|\rho|\,dx\,ds$$
$$\le c\int_0^t \|\rho(s)\|_3^3\,ds. \tag{2.45}$$

Moreover, invoking (2.43), Hölder's inequality, as well as the global bounds once more,

$$J_2 := \int_0^t\!\!\int_\Omega \mu\,g'(\rho_1)\,\rho\,|\rho|\,dx\,ds \,\le\, c\int_0^t \|\mu(s)\|_6\,\|\rho(s)\|_2\,\|\rho(s)\|_3\,ds$$
$$\le \int_0^t \|\rho(s)\|_3^3\,ds \,+\, c\int_0^t \|\mu(s)\|_V^{3/2}\,\|\rho(s)\|_H^{3/2}\,ds$$
$$\le \int_0^t \|\rho(s)\|_3^3\,ds \,+\, c\,\|\rho\|_{L^\infty(0,t;H)}^{3/2}\,\|\mu\|_{L^{3/2}(0,t;V)}^{3/2}$$
$$\le \int_0^t \|\rho(s)\|_3^3\,ds \,+\, c\,\|\rho\|_{L^\infty(0,t;H)}^{3/2}\,\|\mu\|_{L^2(0,t;V)}^{3/2}$$
$$\le \int_0^t \|\rho(s)\|^3\,ds \,+\, c\,\|u\|_{L^2(0,t;H)}^3\,. \tag{2.46}$$

In addition, condition (2.5), Hölder's inequality, and (2.43), yield that

$$\begin{aligned} J_3 : &= -\int_0^t\!\!\int_\Omega (B[\rho_1]-B[\rho_2])\,\rho\,|\rho|\,dx\,ds \\ &\leq c\int_0^t \|\rho(s)\|_3\,\|\rho(s)\|_2\,\|B[\rho_1](s)-B[\rho_2](s)\|_6\,ds \\ &\leq c\sup_{0\leq s\leq t}\|\rho(s)\|_3\,\|\rho\|_{L^\infty(0,t;H)}\int_0^t\|\rho(s)\|_H\,ds \\ &\leq \frac{1}{6}\sup_{0\leq s\leq t}\|\rho(s)\|_3^3\,+\,c\,\|u\|^3_{L^2(0,t;H)}\,. \end{aligned} \tag{2.47}$$

Combining the estimates (2.44)–(2.47), and invoking Gronwall's lemma, we can easily infer that

$$\|\rho\|_{L^\infty(0,t;L^3(\Omega))}\,\leq\,c\,\|u\|_{L^2(0,t;H)}\quad\text{for all }\ t\in(0,T]. \tag{2.48}$$

Step 3: With the above estimates proved, the road is paved for multiplying (2.34) by $\mu_t$. Integration over $Q_t$ yields that

$$\int_0^t\!\!\int_\Omega (1+2g(\rho_1))\,\mu_t^2\,dx\,ds\,+\,\frac{1}{2}\,\|\nabla\mu(t)\|_H^2\,\leq\,\sum_{j=1}^5|K_j|, \tag{2.49}$$

where the expressions $K_j$, $1\leq j\leq 5$, are estimated as follows: at first, using the global bounds and Young's inequality, we have for every $\gamma>0$ (to be specified later) the bound

$$\begin{aligned} K_1 : &= -\int_0^t\!\!\int_\Omega g'(\rho_1)\,\partial_t\rho_1\,\mu\,\mu_t\,dx\,ds\,\leq\,\gamma\int_0^t\!\!\int_\Omega\mu_t^2\,dx\,ds\,+\,\frac{c}{\gamma}\int_0^t\!\!\int_\Omega\mu^2\,dx\,ds \\ &\leq\gamma\int_0^t\!\!\int_\Omega\mu_t^2\,dx\,ds\,+\,\frac{c}{\gamma}\,\|u\|^2_{L^2(0,t;H)}\,. \end{aligned} \tag{2.50}$$

Next, thanks to the mean value theorem, and employing (2.32) and (2.48), we find that

$$\begin{aligned} K_2 : &= -2\int_0^t\!\!\int_\Omega (g(\rho_1)-g(\rho_2))\,\partial_t\mu_2\,\mu_t\,dx\,ds\,\leq\,c\int_0^t\!\!\int_\Omega|\rho|\,|\partial_t\mu_2|\,|\mu_t|\,dx\,ds \\ &\leq c\int_0^t\|\rho(s)\|_3\,\|\partial_t\mu_2(s)\|_6\,\|\mu_t(s)\|_2\,ds \\ &\leq\gamma\int_0^t\!\!\int_\Omega\mu_t^2\,dx\,ds\,+\,\frac{c}{\gamma}\,\|\rho\|^2_{L^\infty(0,t;L^3(\Omega))}\int_0^t\|\partial_t\mu_2(s)\|_V^2\,ds \\ &\leq\gamma\int_0^t\!\!\int_\Omega\mu_t^2\,ds\,+\,\frac{c}{\gamma}\,\|u\|^2_{L^2(0,t;H)}\,. \end{aligned} \tag{2.51}$$

Moreover, we infer that

$$K_3 := \int_0^t\!\!\int_\Omega u\,\mu_t\,dx\,ds\,\leq\,\gamma\int_0^t\!\!\int_\Omega\mu_t^2\,dx\,ds\,+\,\frac{c}{\gamma}\,\|u\|^2_{L^2(0,t;H)}\,, \tag{2.52}$$

as well as, invoking the mean value theorem once more,

$$K_4 := -\int_0^t\!\!\int_\Omega (g'(\rho_1) - g'(\rho_2))\,\partial_t\rho_1\,\mu_2\,\mu_t\,dx\,ds \,\le\, c\int_0^t\!\!\int_\Omega |\rho|\,|\mu_t|\,dx\,ds$$
$$\le\, \gamma\int_0^t\!\!\int_\Omega \mu_t^2\,dx\,ds \,+\, \frac{c}{\gamma}\,\|u\|^2_{L^2(0,t;H)}\,, \tag{2.53}$$

and, finally, using (2.43) and Young's inequality,

$$K_5 := -\int_0^t\!\!\int_\Omega g'(\rho_2)\,\mu_2\,\rho_t\,\mu_t\,dx\,ds \,\le\, c\int_0^t\!\!\int_\Omega |\rho_t|\,|\mu_t|\,dx\,ds$$
$$\le\, \gamma\int_0^t\!\!\int_\Omega \mu_t^2\,dx\,ds \,+\, \frac{c}{\gamma}\,\|\rho\|^2_{H^1(0,t;H)}$$
$$\le\, \gamma\int_0^t\!\!\int_\Omega \mu_t^2\,dx\,ds \,+\, \frac{c}{\gamma}\,\|u\|^2_{L^2(0,t;H)}\,. \tag{2.54}$$

Now we combine the estimates (2.49)–(2.54) and choose $\gamma > 0$ appropriately small. It then follows that

$$\|\mu\|_{H^1(0,t;H)\cap L^\infty(0,t;V)} \,\le\, c\,\|u\|_{L^2(0,t;H)}\,. \tag{2.55}$$

Finally, we come back to (2.34) and employ the global bounds (2.17), (2.18), (2.32), and the estimates shown above, to conclude that

$$\|\Delta\mu\|_{L^2(0,t;H)} \,\le\, c\,\big(\|\mu_t\|_{L^2(0,t;H)} \,+\, \|\mu\|_{L^2(0,t;H)} \,+\, \|\rho_t\|_{L^2(0,t;H)}$$
$$+\, \|\rho\|_{L^2(0,t;H)} \,+\, \|u\|_{L^2(0,t;H)}\big) \,+\, c\,\|\rho\,\partial_t\mu_2\|_{L^2(0,t;H)}$$
$$\le\, c\,\|u\|_{L^2(0,t;H)}\,, \tag{2.56}$$

where the last summand on the right-hand side was estimated as follows:

$$\int_0^t\!\!\int_\Omega |\rho|^2\,|\partial_t\mu_2|^2\,dx\,ds \,\le\, c\int_0^t \|\partial_t\mu_2(s)\|_6^2\,\|\rho(s)\|_3^2\,ds$$
$$\le\, c\,\|\rho\|^2_{L^\infty(0,t;L^3(\Omega))}\int_0^t \|\partial_t\mu_2(s)\|_V^2\,ds \,\le\, c\,\|u\|^2_{L^2(0,t;H)}\,.$$

Invoking standard elliptic estimates, we have thus shown that

$$\|\mu\|_{L^2(0,t;W)} \,\le\, c\,\|u\|_{L^2(0,t;H)}\,. \tag{2.57}$$

<u>Step 4:</u> It remains to show the $L^\infty(0,t;L^6(\Omega))$–estimate for $\rho$. To this end, we multiply (2.35) by $\rho|\rho|^4$ and integrate over $Q_t$. It follows that

$$\frac{1}{6}\,\|\rho(t)\|_6^6 \,\le\, \sum_{j=1}^3 |L_j|, \tag{2.58}$$

where quantities $L_j$, $1 \le j \le 3$, are estimated as follows: at first, we obtain from the global estimates (2.18) and (2.32), that

$$L_1 := \int_0^t\!\!\int_\Omega \bigl(-F'(\rho_1) + F'(\rho_2) + \mu_2(g'(\rho_1) - g'(\rho_2))\bigr)\,\rho\,|\rho|^4\,dx\,ds$$
$$\le c\int_0^t \|\rho(s)\|_6^6\,ds\,. \tag{2.59}$$

Moreover, from (2.55) and Hölder's and Young's inequalities we conclude that

$$L_2 := \int_0^t\!\!\int_\Omega g'(\rho_1)\,\mu\,\rho\,|\rho|^4\,dx\,ds \;\le\; c\int_0^t \|\mu(s)\|_6\,\|\rho(s)\|_6^5\,ds$$
$$\le c\,\|\mu\|_{L^\infty(0,t;V)}\;\|\rho\|^5_{L^5(0,t;L^6(\Omega))} \le c\,\|\mu\|^6_{L^\infty(0,t;V)} + c\,\|\rho\|^6_{L^5(0,t;L^6(\Omega))}$$
$$\le c\,\|u\|^6_{L^2(0,t;H)} + c\int_0^t \|\rho(s)\|_6^6\,ds\,. \tag{2.60}$$

Finally, we employ (2.5) and (2.43) to infer that

$$L_3 := -\int_0^t\!\!\int_\Omega (B[\rho_1] - B[\rho_2])\,\rho\,|\rho|^4\,dx\,ds$$
$$\le c\int_0^t \|B(\rho_1](s) - B[\rho_2](s)\|_6\,\|\rho(s)\|_6^5\,ds$$
$$\le c\sup_{0\le s\le t}\|\rho(s)\|_6^5\int_0^t \|\rho(s)\|_H\,ds$$
$$\le \frac{1}{12}\sup_{0\le s\le t}\|\rho(s)\|_6^6 + c\,\|u\|^6_{L^2(0,t;H)}\,. \tag{2.61}$$

Combining the estimates (2.58)–(2.61), and invoking Gronwall's lemma, then we readily find the estimate

$$\|\rho\|_{L^\infty(0,t;L^6(\Omega))} \le c\,\|u\|_{L^2(0,t;H)}\,,$$

which concludes the proof of the assertion. □

# 3 Directional differentiability of the control-to-state mapping

In this section, we prove the relevant differentiability properties of the solution operator $\mathcal{S}$. To this end, we introduce the spaces

$$\mathcal{X} := H^1(0,T;H) \cap L^\infty(Q),$$
$$\mathcal{Y} := H^1(0,T;H) \times \bigl(L^\infty(0,T;H) \cap L^2(0,T;V)\bigr)\,,$$

endowed with their natural norms

$$\|u\|_{\mathcal{X}} := \|u\|_{H^1(0,T;H)} + \|u\|_{L^\infty(Q)} \quad \forall\, u \in \mathcal{X},$$
$$\|(\rho,\mu)\|_{\mathcal{Y}} := \|\rho\|_{H^1(0,T;H)} + \|\mu\|_{L^\infty(0,T;H)} + \|\mu\|_{L^2(0,T;V)} \quad \forall\,(\rho,\mu) \in \mathcal{Y}\,,$$

and consider the control-to-state operator $\mathcal{S}$ as a mapping between $\mathcal{U}_{\rm ad} \subset \mathcal{X}$ and $\mathcal{Y}$. Now let $\overline{u} \in \mathcal{U}_{\rm ad}$ be fixed and put $(\overline{\rho}, \overline{\mu}) := \mathcal{S}(\overline{u})$. We then study the linearization of the state system (1.2)–(1.5) at $\overline{u}$, which is given by:

$$(1+2g(\overline{\rho}))\,\eta_t \,+\, 2g'(\overline{\rho})\,\overline{\mu}_t\,\xi \,+\, g'(\overline{\rho})\,\overline{\rho}_t\,\eta \,+\, \overline{\mu}\,g''(\overline{\rho})\,\overline{\rho}_t\,\xi \,+\, \overline{\mu}g'(\overline{\rho})\,\xi_t \\ -\Delta\eta = h \quad \text{a.e. in } Q, \tag{3.1}$$

$$\xi_t + F''(\overline{\rho})\,\xi + DB[\overline{\rho}](\xi) = \overline{\mu}\,g''(\overline{\rho})\,\xi \,+\, g'(\overline{\rho})\,\eta \quad \text{a.e. in } Q, \tag{3.2}$$

$$\partial_{\mathbf{n}}\eta = 0 \quad \text{a.e. on } \Sigma, \tag{3.3}$$

$$\eta(0) = \xi(0) = 0 \quad \text{a.e. in } \Omega. \tag{3.4}$$

Here, $h \in \mathcal{X}$ must satisfy $\overline{u} + \overline{\lambda}h \in \mathcal{U}_{\rm ad}$ for some $\overline{\lambda} > 0$. Provided that the system (3.1)–(3.4) has for any such $h$ a unique solution pair $(\xi, \eta)$, we expect that the directional derivative $\delta\mathcal{S}(\overline{u}; h)$ of $\mathcal{S}$ at $\overline{u}$ in the direction $h$ (if it exists) ought to be given by $(\xi, \eta)$. In fact, the above problem makes sense for every $h \in L^2(Q)$, and it is uniquely solvable under this weaker assumption.

THEOREM 3.1: *Suppose that the general hypotheses* **(A1)**–**(A5)** *are satisfied and let* $h \in L^2(Q)$. *Then, the linearized problem* (3.1)–(3.4) *has a unique solution* $(\xi, \eta)$ *satisfying*

$$\xi \in H^1(0,T;H) \cap L^\infty(0,T;L^6(\Omega)), \tag{3.5}$$

$$\eta \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W). \tag{3.6}$$

PROOF: We first prove uniqueness. Since the problem is linear, we take $h = 0$ and show that $(\xi, \eta) = (0, 0)$. We add $\eta$ and $\xi$ to both sides of equations (3.1) and (3.2), respectively, then multiply by $\eta$ and $\xi_t$, integrate over $Q_t$, and sum up. By observing that

$$(1+2g(\overline{\rho}))\eta_t\eta + g'(\overline{\rho})\overline{\rho}_t|\eta|^2 = \partial_t\big[\big(\tfrac{1}{2} + g(\overline{\rho})\big)|\eta|^2\big],$$

and recalling that $g \geq 0$, we obtain

$$\frac{1}{2}\int_\Omega |\eta(t)|^2\,dx + \int_0^t \|\eta(s)\|_V^2\,ds + \frac{1}{2}\int_\Omega |\xi(t)|^2\,dx + \int_0^t\!\!\int_\Omega |\xi_t|^2\,dx\,ds \leq \sum_{j=1}^3 H_j,$$

where the terms $H_j$ are defined and estimated as follows. We have

$$H_1 := -\int_0^t\!\!\int_\Omega 2\,g'(\overline{\rho})\,\overline{\mu}_t\,\xi\,\eta\,dx\,ds \leq c\int_0^t \|\overline{\mu}_t(s)\|_3\,\|\xi(s)\|_2\,\|\eta(s)\|_6\,ds \\ \leq \frac{1}{2}\int_0^t \|\eta(s)\|_V^2\,ds + c\int_0^t \|\overline{\mu}_t(s)\|_V^2\,\|\xi(s)\|_2^2\,ds\,,$$

and we notice that the function $s \mapsto \|\overline{\mu}_t(s)\|_V^2$ belongs to $L^1(0,T)$, by (2.31) for $\overline{\mu}$. Next, we easily have the estimate

$$H_2 := \int_0^t\!\!\int_\Omega \big(\eta - \overline{\mu}\,g''(\overline{\rho})\,\overline{\rho}_t\,\xi - \overline{\mu}\,g'(\overline{\rho})\,\xi_t\big)\,\eta\,dx\,ds \\ \leq \frac{1}{4}\int_0^t\!\!\int_\Omega |\xi_t|^2\,dx\,ds + c\int_0^t\!\!\int_\Omega (|\xi|^2 + |\eta|^2)\,dx\,ds\,.$$

Finally, recalling (2.8), it is clear that

$$\begin{aligned} H_3 &:= \int_0^t\!\!\int_\Omega \big((\xi + \overline{\mu}\, g''(\overline{\rho}) - F''(\overline{\rho}))\,\xi - DB[\overline{\rho}](\xi) + g'(\overline{\rho})\,\eta\big)\,\xi_t\,dx\,ds \\ &\leq \frac{1}{4}\int_0^t\!\!\int_\Omega |\xi_t|^2\,dx\,ds + c\int_0^t\!\!\int_\Omega (|\xi|^2 + |\eta|^2)\,dx\,ds\,. \end{aligned} \tag{3.7}$$

Therefore, it suffices to collect these inequalities and apply Gronwall's lemma in order to conclude that $\xi = 0$ and $\eta = 0$.

The existence of a solution is proved in several steps. First, we introduce an approximating problem depending on the parameter $\varepsilon \in (0,1)$. Then, we show well-posedness for this problem and perform suitable a priori estimates. Finally, we construct a solution to problem (3.1)–(3.4) by letting $\varepsilon$ tend to zero. For the sake of simplicity, in performing the uniform a priori estimates, we denote by $c > 0$ different constants that may depend on the data of the system but not on $\varepsilon \in (0,1)$; the actual value of $c$ may change within formulas and lines. On the contrary, the symbol $c_\varepsilon$ stands for (different) constants that can depend also on $\varepsilon$. In particular, $c_\varepsilon$ is independent of the parameter $\delta$ that enters an auxiliary problem we introduce later on.

<u>STEP 1:</u> We approximate $\overline{\rho}$ and $\overline{\mu}$ by suitable $\rho^\varepsilon, \mu^\varepsilon \in C^\infty(\overline{Q})$ as specified below. For every $\varepsilon \in (0,1)$, it holds that

$$\rho_{**} \leq \rho^\varepsilon \leq \rho^{**} \text{ in } \overline{Q} \quad \text{and} \quad \|\rho^\varepsilon_t\|_{L^\infty(Q)} + \|\mu^\varepsilon\|_{H^1(0,T;L^3(\Omega))\cap L^\infty(Q)} \leq C^*, \tag{3.8}$$

for some constants $\rho_{**}, \rho^{**} \in (0,1)$ and $C^* > 0$; as $\varepsilon \searrow 0$, we have

$$\begin{gathered} \rho^\varepsilon \to \overline{\rho}, \quad \rho^\varepsilon_t \to \overline{\rho}_t, \quad \mu^\varepsilon \to \overline{\mu}, \quad \text{in } L^p(Q), \text{ for every } p < +\infty \text{ and a.e. in } Q, \\ \text{and} \quad \mu^\varepsilon_t \to \overline{\mu}_t \quad \text{in } L^2(0,T;L^3(\Omega)). \end{gathered} \tag{3.9}$$

In order to construct regularizing families as above, we can use, for instance, extension outside $Q$ and convolution with mollifiers.
Next, we introduce the approximating problem of finding $(\xi^\varepsilon, \eta^\varepsilon)$ satisfying

$$\xi^\varepsilon_t + F''(\overline{\rho})\,\xi^\varepsilon + DB[\overline{\rho}](\xi^\varepsilon) = \overline{\mu}\,g''(\overline{\rho})\,\xi^\varepsilon + g'(\overline{\rho})\,\eta^\varepsilon \quad \text{a.e. in } Q, \tag{3.10}$$

$$\begin{aligned} &(1 + 2g(\rho^\varepsilon))\,\eta^\varepsilon_t + g'(\rho^\varepsilon)\,\rho^\varepsilon_t\,\eta^\varepsilon \\ &\quad + 2g'(\overline{\rho})\,\mu^\varepsilon_t\,\xi^\varepsilon + \overline{\mu} g''(\overline{\rho})\,\overline{\rho}_t\,\xi^\varepsilon + \overline{\mu} g'(\overline{\rho})\,\xi^\varepsilon_t - \Delta\eta^\varepsilon = h \quad \text{a.e. in } Q, \end{aligned} \tag{3.11}$$

$$\partial_{\mathbf{n}}\eta^\varepsilon = 0 \quad \text{a.e. on } \Sigma, \tag{3.12}$$

$$\eta^\varepsilon(0) = \xi^\varepsilon(0) = 0 \quad \text{a.e. in } \Omega. \tag{3.13}$$

In order to solve (3.10)–(3.13), we introduce the spaces

$$\mathcal{V} := H \times V \quad \text{and} \quad \mathcal{H} := H \times H,$$

and present our problem in the form

$$\frac{d}{dt}\,(\xi,\eta) + \mathcal{A}^\varepsilon(\xi,\eta) = f \quad \text{and} \quad (\xi,\eta)(0) = (0,0),$$

in the framework of the Hilbert triplet $(\mathcal{V},\mathcal{H},\mathcal{V}')$. We look for a weak solution and aim at applying [1, Thm. 3.2, p. 256]. To this end, we have to split $\mathcal{A}^\varepsilon$ in the form $\mathcal{Q}^\varepsilon+\mathcal{R}^\varepsilon$, where $\mathcal{Q}^\varepsilon$ is the uniformly elliptic principal part and the remainder $\mathcal{R}^\varepsilon$ satisfies the requirements [1, (4.4)–(4.5), p. 259]. We notice at once that these conditions are trivially fulfilled whenever

$$\mathcal{R}^\varepsilon=(\mathcal{R}_1^\varepsilon,\mathcal{R}_2^\varepsilon)\in\mathcal{L}(L^2(0,T;\mathcal{H}),L^2(0,T;\mathcal{H}))\,, \tag{3.14}$$

$$\left|\int_0^t\!\!\int_\Omega\bigl(\mathcal{R}_1^\varepsilon(v,w)\,v+\mathcal{R}_2^\varepsilon(v,w)\,w\bigr)\,dx\,ds\right|\le C_{\mathcal{R}^\varepsilon}\int_0^t\!\!\int_\Omega\bigl(|v|^2+|w|^2\bigr)\,ds\,, \tag{3.15}$$

for some constant $C_{\mathcal{R}^\varepsilon}$, and every $v,w\in L^2(0,T;H)$ and $t\in[0,T]$. In order to present (3.10)–(3.13) in the desired form, we multiply (3.11) by $a^\varepsilon:=1/(1+2g(\rho^\varepsilon))$ and notice that

$$-a^\varepsilon\Delta\eta^\varepsilon=-\operatorname{div}(a^\varepsilon\nabla\eta^\varepsilon)+\nabla a^\varepsilon\cdot\nabla\eta^\varepsilon.$$

As $a^\varepsilon\ge\alpha:=1/(1+2\sup g)$ and $\nabla a^\varepsilon$ is bounded, we can fix a real number $\lambda^\varepsilon>0$ such that

$$\int_\Omega\bigl(a^\varepsilon(t)|\nabla w|^2+(\nabla a^\varepsilon(t)\cdot\nabla w)w+\lambda^\varepsilon|w|^2\bigr)\,dx\;\ge\;\frac{\alpha}{2}\,\|w\|_V^2 \tag{3.16}$$

for every $w\in V$ and $t\in[0,T]$. Next, we replace $\xi_t^\varepsilon$ in (3.11) by using (3.10). Therefore, we see that a possible weak formulation of (3.10)–(3.12) is given by

$$\begin{aligned}&\int_\Omega\xi_t^\varepsilon(t)v\,dx+{}_{V'}\langle\eta_t^\varepsilon(t),w\rangle_V+{}_{\mathcal{V}'}\langle\mathcal{Q}^\varepsilon(t)(\xi^\varepsilon,\eta^\varepsilon)(t),(v,w)\rangle_{\mathcal{V}}\\&\quad+\int_\Omega\bigl(\mathcal{R}_1^\varepsilon(\xi^\varepsilon,\eta^\varepsilon)(t)v+\mathcal{R}_2^\varepsilon(\xi^\varepsilon,\eta^\varepsilon)(t)\,w\bigr)\,dx=\int_\Omega a^\varepsilon(t)\,h(t)\,w\,dx\\&\text{for a. a. } t\in(0,T)\text{ and every }(v,w)\in\mathcal{V},\end{aligned} \tag{3.17}$$

where the symbols $\langle\cdot,\cdot\rangle$ stand for the duality pairings and $\mathcal{Q}^\varepsilon$ and $\mathcal{R}_i^\varepsilon$ have the meaning explained below. The time-dependent operator $\mathcal{Q}^\varepsilon(t)$ from $\mathcal{V}$ into $\mathcal{V}'$ is defined by

$$\begin{aligned}&{}_{\mathcal{V}'}\langle\mathcal{Q}^\varepsilon(t)(\hat v,\hat w),(v,w)\rangle_{\mathcal{V}}\\&=\int_\Omega\bigl(\hat v\,v+a^\varepsilon(t)\,\nabla\hat w\cdot\nabla w+(\nabla a^\varepsilon(t)\cdot\nabla\hat w)\,w+\lambda^\varepsilon\,\hat w\,w\bigr)\,dx\end{aligned} \tag{3.18}$$

for every $(\hat v,\hat w),(v,w)\in\mathcal{V}$ and $t\in[0,T]$. By construction, the bilinear form given by the right-hand side of (3.18) is continuous on $\mathcal{V}\times\mathcal{V}$, depends smoothly on time, and is $\mathcal{V}$-coercive uniformly with respect to $t$ (see (3.16)). The operators

$$\mathcal{R}_i^\varepsilon\in\mathcal{L}(L^2(0,T;\mathcal{H}),L^2(0,T;H))$$

account for the term $\lambda^\varepsilon\eta^\varepsilon$ that has to be added also to the right-hand side of (3.11) and for all the contributions to the equations that have not been considered in the principal part. They have the form

$$(\mathcal{R}_i^\varepsilon(v,w))(t)=a_{i1}^\varepsilon(t)v+a_{i2}^\varepsilon(t)w+a_{i3}^\varepsilon(t)\bigl(DB[\overline{\rho}](v)\bigr)(t) \tag{3.19}$$

for $(v,w) \in L^2(0,T;\mathcal{H})$, with some coefficients $a^\varepsilon_{ij} \in L^\infty(Q)$. Therefore, by virtue of (2.8), we see that

$$\begin{aligned}
&\int_0^t\!\!\int_\Omega \bigl(\mathcal{R}^\varepsilon_1(v,w)\,v + \mathcal{R}^\varepsilon_1(v,w)\,w\bigr)\,dx\,ds \\
&\le c\int_0^t\!\!\int_\Omega (|v|^2+|w|^2)\,dx\,ds + c\,\|DB[\overline{\rho}](v)\|^2_{L^2(Q_t)} \\
&\le c\int_0^t\!\!\int_\Omega (|v|^2+|w|^2)\,dx\,ds\,,
\end{aligned}$$

for every $(v,w) \in L^2(0,T;\mathcal{H})$ and every $t \in [0,T]$. Thus, the conditions (3.14)–(3.15) are fulfilled, and the result of [1] mentioned above can be applied. We conclude that the Cauchy problem for (3.17) has a unique solution $(\xi^\varepsilon,\eta^\varepsilon)$ satisfying

$$\begin{aligned}
&(\xi^\varepsilon,\eta^\varepsilon) \in H^1(0,T;\mathcal{V}') \cap L^2(0,T;\mathcal{V}), \quad \text{i.e.,} \\
&\xi^\varepsilon \in H^1(0,T;H) \quad \text{and} \quad \eta^\varepsilon \in H^1(0,T;V') \cap L^2(0,T;V).
\end{aligned}$$

On the other hand, this solution has to satisfy

$$\langle \partial_t\eta^\varepsilon, w\rangle + \int_\Omega a^\varepsilon\,\nabla\eta^\varepsilon\cdot\nabla w\,dx = \int_\Omega \varphi_\varepsilon\,w\,dx \quad \text{a.e. in (0,T), for every } w\in V\,,$$

for some $\varphi_\varepsilon \in L^2(Q)$. From standard elliptic regularity, it follows that $\eta^\varepsilon \in H^1(0,T;H) \cap L^2(0,T;W)$.

In the next steps, besides of Young's inequality, we make repeated use of the global estimates (2.17), (2.18), and (2.32), for $\overline{\rho}$ and $\overline{\mu}$, without further reference.

<u>Step 2:</u> For convenience, we refer to Eqs. (3.10)–(3.12) (using the language that is proper for strong solutions), but it is understood that they are meant in the variational sense (3.17). We add $\xi^\varepsilon$ and $\eta^\varepsilon$ to both sides of (3.10) and (3.11), respectively; then, we multiply the resulting equalities by $\xi^\varepsilon_t$ and $\eta^\varepsilon$, integrate over $Q_t$, and sum up. By observing that

$$(1+2g(\rho^\varepsilon))\,\eta^\varepsilon_t\,\eta^\varepsilon + g'(\rho^\varepsilon)\,\rho^\varepsilon_t\,|\eta^\varepsilon|^2 = \partial_t\bigl[\bigl(\tfrac12 + g(\rho^\varepsilon)\bigr)\,|\eta^\varepsilon|^2\bigr],$$

and recalling that $g \ge 0$, we obtain

$$\frac12\int_\Omega |\xi^\varepsilon(t)|^2\,dx + \int_0^t\!\!\int_\Omega |\xi^\varepsilon_t|^2\,dx\,ds + \frac12\int_\Omega |\eta^\varepsilon(t)|^2\,dx + \int_0^t \|\eta^\varepsilon(s)\|^2_V\,ds \le \sum_{j=1}^3 I_j,$$

where the terms $I_j$ are defined and estimated as follows. In view of (2.8), we first infer that

$$\begin{aligned}
I_1 :&= \int_0^t\!\!\int_\Omega \bigl(\xi^\varepsilon - F''(\overline{\rho})\,\xi^\varepsilon - DB[\overline{\rho}](\xi^\varepsilon) + \overline{\mu}\,g''(\overline{\rho})\,\xi^\varepsilon + g'(\overline{\rho})\,\eta^\varepsilon\bigr)\,\xi^\varepsilon_t\,dx\,ds \\
&\le \frac14\int_0^t\!\!\int_\Omega |\xi^\varepsilon_t|^2\,dx\,ds + c\int_0^t\!\!\int_\Omega \bigl(|\xi^\varepsilon|^2 + |\eta^\varepsilon|^2\bigr)\,dx\,ds\,.
\end{aligned}$$

Next, we have

$$
\begin{aligned}
I_2 :&= \int_0^t\!\!\int_\Omega \big(\eta^\varepsilon - \overline{\mu}\, g''(\overline{\rho})\, \overline{\rho}_t\, \xi^\varepsilon - \overline{\mu} g'(\overline{\rho})\, \xi_t^\varepsilon + h\big)\, \eta^\varepsilon \, dx\, ds \\
&\le \frac{1}{4}\int_0^t\!\!\int_\Omega |\xi_t^\varepsilon|^2 \, dx\, ds + c\int_0^t\!\!\int_\Omega \big(|\xi^\varepsilon|^2 + |\eta^\varepsilon|^2\big)\, dx\, ds + c\,.
\end{aligned}
$$

Finally, by virtue of the Hölder and Sobolev inequalities, we have

$$
\begin{aligned}
I_3 :&= -\int_0^t\!\!\int_\Omega 2g'(\overline{\rho})\, \mu_t^\varepsilon\, \xi^\varepsilon\, \eta^\varepsilon \, dx\, ds \le c\int_0^t \|\mu_t^\varepsilon(s)\|_3\, \|\xi^\varepsilon(s)\|_2\, \|\eta^\varepsilon(s)\|_6\, ds \\
&\le \frac{1}{2}\int_0^t \|\eta^\varepsilon(s)\|_V^2\, ds + c\int_0^t \|\mu_t^\varepsilon(s)\|_3^2\, \|\xi^\varepsilon(s)\|_2^2\, ds\,.
\end{aligned}
$$

At this point, we recall all the inequalities we have proved, notice that (3.8) implies that the function $s \mapsto \|\mu_t^\varepsilon(s)\|_3^2$ is bounded in $L^1(0,T)$, and apply the Gronwall lemma. We obtain

$$
\|\xi^\varepsilon\|_{H^1(0,T;H)} + \|\eta^\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\,. \tag{3.20}
$$

<u>STEP 3:</u> We would now like to test (3.10) by $(\xi^\varepsilon)^5$. However, this function is not admissible, unfortunately. Therefore, we introduce a suitable approximation. To start with, we consider the Cauchy problem

$$
\hat{\xi}_t + b\,\hat{\xi} + L(\hat{\xi}) = f^\varepsilon \quad \text{and} \quad \hat{\xi}(0) = 0, \tag{3.21}
$$

where we have set, for brevity,

$$
b := F''(\overline{\rho}) - \overline{\mu}\, g''(\overline{\rho}), \quad L := DB[\overline{\rho}], \quad \text{and} \quad f^\varepsilon := g'(\overline{\rho})\, \eta^\varepsilon. \tag{3.22}
$$

By (3.10), $\hat{\xi} := \xi^\varepsilon$ is a solution belonging to $H^1(0,T;H)$. On the other hand, such a solution is unique. Indeed, multiplying by $\hat{\xi}$ the corresponding homogeneous equation (i.e., $f^\varepsilon$ is replaced by $0$), and invoking (2.8) and Gronwall's lemma, one immediately obtains that $\hat{\xi} = 0$. We conclude that $\hat{\xi} := \xi^\varepsilon$ is the unique solution to (3.21).

At this point, we approximate $\xi^\varepsilon$ by the solution to a problem depending on the parameter $\delta \in (0,1)$, in addition. Namely, we look for $\xi^{\varepsilon\delta}$ satisfying the parabolic-like equation

$$
\xi_t^{\varepsilon\delta} - \delta\, \Delta \xi^{\varepsilon\delta} + b^\delta\, \xi^{\varepsilon\delta} + L(\xi^{\varepsilon\delta}) = f^\varepsilon, \tag{3.23}
$$

complemented with the Neumann boundary condition $\partial_{\mathbf{n}} \xi^{\varepsilon\delta} = 0$ and the initial condition $\xi^{\varepsilon\delta}(0) = 0$. In (3.23), $b^\delta$ is an approximation of $b$ belonging to $C^\infty(\overline{Q})$ that satisfies

$$
\|b^\delta\|_{L^\infty(Q)} \le c, \quad \text{and} \quad b^\delta \to b \quad \text{a.e. in } Q \text{ as } \delta \searrow 0. \tag{3.24}
$$

This problem has a unique weak solution $\xi^{\varepsilon\delta} \in H^1(0,T;V') \cap L^2(0,T;V)$, as one easily sees by arguing as we did for the more complicated system (3.10)–(3.13) and applying [1, Thm. 3.2, p. 256].

We now aim to show that $\xi^{\varepsilon\delta}$ is bounded. To this end, we introduce the operator $A_\delta \in \mathcal{L}(V,V')$ defined by

$$\langle A_\delta v, w\rangle := \int_\Omega (\delta\,\nabla v\cdot\nabla w + v\,w)\,dx \quad \text{for every } v,w\in V\,,$$

and observe that $A_\delta$ is an isomorphism. Moreover, Eq. (3.23), complemented with the boundary and initial conditions, can be written as

$$\xi^{\varepsilon\delta}_t + A_\delta \xi^{\varepsilon\delta} = f^{\varepsilon\delta} := f^\varepsilon - (1+b^\delta)\,\xi^{\varepsilon\delta} + L(\xi^{\varepsilon\delta}) \quad \text{and} \quad \xi^{\varepsilon\delta}(0)=0\,. \tag{3.25}$$

Now, by also accounting for (2.9), we notice that $f^\varepsilon$, $\xi^{\varepsilon\delta}$, $b^\delta\xi^{\varepsilon\delta}$, and $L(\xi^{\varepsilon\delta})$, all belong to $L^2(0,T;V)$. Hence, $f^{\varepsilon\delta}\in L^2(0,T;V)$, so that $A_\delta f^{\varepsilon\delta}\in L^2(0,T;V')$, and we can consider the unique solution $\zeta^{\varepsilon\delta}\in H^1(0,T;V')\cap L^2(0,T;V)$ to the problem

$$\zeta^{\varepsilon\delta}_t + A_\delta\zeta^{\varepsilon\delta} = A_\delta f^{\varepsilon\delta} \quad\text{and}\quad \zeta^{\varepsilon\delta}(0)=0\,.$$

Now, $A_\delta^{-1}\zeta^{\varepsilon\delta}$ satisfies

$$(A_\delta^{-1}\zeta^{\varepsilon\delta})_t + A_\delta(A_\delta^{-1}\zeta^{\varepsilon\delta}) = A_\delta^{-1}A_\delta f^{\varepsilon\delta} = f^{\varepsilon\delta} \quad\text{and}\quad (A_\delta^{-1}\zeta^{\varepsilon\delta})(0)=0,$$

so that a comparison with (3.25) shows that $\xi^{\varepsilon\delta} = A_\delta^{-1}\zeta^{\varepsilon\delta}$, by uniqueness. Since $\zeta^{\varepsilon\delta}\in L^\infty(0,T;H)$, and $A_\delta^{-1}(H)=W$ by elliptic regularity, we deduce that $\xi^{\varepsilon\delta}\in L^\infty(0,T;W)$. Therefore, $\xi^{\varepsilon\delta}$ is bounded, as claimed.

Consequently, $(\xi^{\varepsilon\delta})^5$ is an admissible test function, since it clearly belongs to the space $L^2(0,T;V)$. By multiplying (3.23) by $(\xi^{\varepsilon\delta})^5$ and integrating over $Q_t$, we obtain that

$$\frac16\int_\Omega |\xi^{\varepsilon\delta}(t)|^6\,dx \,+\, 5\,\delta\int_0^t\!\!\int_\Omega |\xi^{\varepsilon\delta}|^4\,|\nabla\xi^{\varepsilon\delta}|^2\,dx\,ds = \sum_{j=1}^3 K_j,$$

where the terms $K_j$ are defined and estimated as follows. First, recalling (3.24), we deduce that

$$K_1 := -\int_0^t\!\!\int_\Omega b^\delta\,\xi^{\varepsilon\delta}\,(\xi^{\varepsilon\delta})^5\,dx\,ds \,\le\, c\int_0^t\!\!\int_\Omega |\xi^\varepsilon|^6\,dx\,ds\,.$$

On the other hand, Hölder's inequality, and assumption (2.8) with $p=6$, imply that

$$\begin{aligned} K_2 :&= -\int_0^t\!\!\int_\Omega L(\xi^{\varepsilon\delta})\,(\xi^{\varepsilon\delta})^5\,dx\,ds \,\le\, c\,\|L\xi^{\varepsilon\delta}\|_{L^6(Q_t)}\,\|(\xi^{\varepsilon\delta})^5\|_{L^{6/5}(Q_t)} \\ &\le c\,\|\xi^{\varepsilon\delta}\|_{L^6(Q_t)}\,\|\xi^{\varepsilon\delta}\|^5_{L^6(Q_t)} \,=\, c\int_0^t\!\!\int_\Omega |\xi^{\varepsilon\delta}|^6\,dx\,ds. \end{aligned}$$

Finally, also invoking Sobolev's inequality, we see that

$$\begin{aligned} K_3 :&= \int_0^t\!\!\int_\Omega f^\varepsilon\,(\xi^{\varepsilon\delta})^5\,dx\,ds \,\le\, c\int_0^t \|\eta^\varepsilon(s)\|_6\,\|(\xi^{\varepsilon\delta}(s))^5\|_{6/5}\,ds \\ &\le c\int_0^t \|\eta^\varepsilon(s)\|_6\,\|\xi^{\varepsilon\delta}(s)\|_6^5\,ds \le c\int_0^t \|\eta^\varepsilon(s)\|_V\,\bigl(1+\|\xi^{\varepsilon\delta}(s)\|_6^6\bigr)\,ds\,. \end{aligned}$$

Collecting the above estimates, and noting that the function $s \mapsto \|\eta^\varepsilon(s)\|_V$ is bounded in $L^1(0,T)$ by (3.20), we can apply the Gronwall lemma to conclude that

$$\|\xi^{\varepsilon\delta}\|_{L^\infty(0,T;L^6(\Omega))} \le c\,. \tag{3.26}$$

At this point, we quickly show that $\xi^{\varepsilon\delta}$ converges to $\xi^\varepsilon$ as $\delta \searrow 0$, at least for a subsequence. Indeed, one multiplies (3.23) first by $\xi^{\varepsilon\delta}$, and then by $\xi_t^{\varepsilon\delta}$, and proves that

$$\|\xi^{\varepsilon\delta}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c_\varepsilon,$$

uniformly with respect to $\delta$. Then, by weak compactness and (3.24) (which implies convergence of $b^\delta$ to $b$ in $L^p(Q)$ for every $p < +\infty$), it is straightforward to see that $\xi^{\varepsilon\delta}$ converges to a solution $\hat{\xi}$ to the problem associated with (3.21). As $\hat{\xi} = \xi^\varepsilon$, we have proved what we have claimed. This, and (3.26), yield that

$$\|\xi^\varepsilon\|_{L^\infty(0,T;L^6(\Omega))} \le c\,. \tag{3.27}$$

<u>STEP 4:</u> At this point, we can multiply (3.11) by $\eta_t^\varepsilon$ and integrate over $Q_t$. By recalling that $g \ge 0$, we obtain

$$\int_0^t\!\!\int_\Omega |\eta_t^\varepsilon|^2\,dx\,ds + \frac{1}{2}\int_\Omega |\nabla\eta^\varepsilon(t)|^2\,dx \le \sum_{j=1}^3 L_j,$$

where each term $L_j$ is defined and estimated below. First, by taking advantage of (3.27) and (3.8) for $\mu_t^\varepsilon$, we have

$$\begin{aligned} L_1 :&= -\int_0^t\!\!\int_\Omega 2g'(\overline{\rho})\,\mu_t^\varepsilon\,\xi^\varepsilon\,\eta_t^\varepsilon\,dx\,ds \le c\int_0^t \|\mu_t^\varepsilon(s)\|_3\,\|\xi^\varepsilon(s)\|_6\,\|\eta_t^\varepsilon(s)\|_2\,ds \\ &\le c\int_0^t \|\mu_t^\varepsilon(s)\|_3\,\|\eta_t^\varepsilon(s)\|_2\,ds \le \frac{1}{4}\int_0^t\!\!\int_\Omega |\eta_t^\varepsilon|^2\,dx\,ds + c\int_0^T \|\mu_t^\varepsilon(s)\|_3^2\,ds \\ &\le \frac{1}{4}\int_0^t\!\!\int_\Omega |\eta_t^\varepsilon|^2\,dx\,ds + c\,. \end{aligned}$$

Next, using (3.8) for $\rho_t^\varepsilon$ and (3.20), we obtain that

$$\begin{aligned} L_2 := -\int_0^t\!\!\int_\Omega g'(\rho^\varepsilon)\rho_t^\varepsilon\eta^\varepsilon\eta_t^\varepsilon\,dx\,ds &\le \frac{1}{4}\int_0^t\!\!\int_\Omega |\eta_t^\varepsilon|^2\,dx\,ds + c\int_0^t\!\!\int_\Omega |\eta^\varepsilon|^2\,dx\,ds \\ &\le \frac{1}{4}\int_0^t\!\!\int_\Omega |\eta_t^\varepsilon|^2\,dx\,ds + c\,. \end{aligned}$$

Finally, in view of (3.20), we have

$$\begin{aligned} L_3 :&= \int_0^t\!\!\int_\Omega \big(-\overline{\mu}\,g''(\overline{\rho})\,\overline{\rho}_t\,\xi^\varepsilon - \overline{\mu}\,g'(\overline{\rho})\,\xi_t^\varepsilon + h\big)\,\eta_t^\varepsilon\,dx\,ds \\ &\le \frac{1}{4}\int_0^t\!\!\int_\Omega |\eta_t^\varepsilon|^2\,dx\,ds + c\int_0^t\!\!\int_\Omega \big(|\xi^\varepsilon|^2 + |\xi_t^\varepsilon|^2 + 1\big)\,dx\,ds \\ &\le \frac{1}{4}\int_0^t\!\!\int_\Omega |\eta_t^\varepsilon|^2\,dx\,ds + c\,. \end{aligned}$$

By collecting the above estimates, we conclude that

$$\|\eta^\varepsilon_t\|_{L^2(0,T;H)} + \|\eta^\varepsilon\|_{L^2(0,T;V)} \le c\,. \tag{3.28}$$

As a consequence, we can estimate $\Delta\eta^\varepsilon$ in $L^2(Q)$, just by comparison in (3.11). Using standard elliptic regularity, we deduce that

$$\|\eta^\varepsilon\|_{L^2(0,T;W)} \le c\,. \tag{3.29}$$

STEP 5: At this point, we are ready to prove the existence part of the statement. Indeed, the estimates (3.20) and (3.27)–(3.29) yield that

$$\xi^\varepsilon \to \xi \qquad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;L^6(\Omega)),$$
$$\eta^\varepsilon \to \eta \qquad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W),$$

as $\varepsilon \searrow 0$, at least for a subsequence. By accounting for (3.9) and the Lipschitz continuity of $g$ and $g'$, it is straightforward to see that $(\xi,\eta)$ is a solution to problem (3.1)–(3.4). This completes the proof. □

We are now prepared to show that $\mathcal{S}$ is directionally differentiable. We have the following result:

THEOREM 3.2: *Suppose that the general hypotheses* **(A1)–(A5)** *are satisfied, and let $\overline{u} \in \mathcal{U}_{\rm ad}$ be given and $(\overline{\rho},\overline{\mu}) = \mathcal{S}(\overline{u})$. Moreover, let $h \in \mathcal{X}$ be a function such that $\overline{u}+\overline{\lambda}h \in \mathcal{U}_{\rm ad}$ for some $\overline{\lambda} > 0$. Then the directional derivative $\delta\mathcal{S}(\overline{u};h)$ of $\mathcal{S}$ at $\overline{u}$ in the direction $h$ exists in the space $(\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$, and we have $\delta\mathcal{S}(\overline{u};h) = (\xi,\eta)$, where $(\xi,\eta)$ is the unique solution to the linearized system* (3.1)–(3.4).

PROOF: We have $\overline{u}+\lambda h \in \mathcal{U}_{\rm ad}$ for $0<\lambda\le\overline{\lambda}$, since $\mathcal{U}_{\rm ad}$ is convex. We put, for any such $\lambda$,

$$u^\lambda := \overline{u}+\lambda h, \quad (\rho^\lambda,\mu^\lambda) := \mathcal{S}(u^\lambda), \quad y^\lambda := \rho^\lambda - \overline{\rho} - \lambda\xi, \quad z^\lambda := \mu^\lambda - \overline{\mu} - \lambda\eta.$$

Notice that $(\rho^\lambda,\mu^\lambda)$ and $(\overline{\rho},\overline{\mu})$ fulfill the global bounds (2.17), (2.18), and (2.32), and that $(y^\lambda,z^\lambda) \in \mathcal{Y}$ for all $\lambda \in [0,\overline{\lambda}]$. Moreover, by virtue of Theorem 2.2, we have the estimate

$$\|\rho^\lambda - \overline{\rho}\|_{H^1(0,t;H)\cap L^\infty(0,t;L^6(\Omega))} + \|\mu^\lambda - \overline{\mu}\|_{H^1(0,t;H)\cap L^\infty(0,t;V)\cap L^2(0,t;W)}$$
$$\le K_2^*\,\lambda\|h\|_{L^2(0,t;H)}\,. \tag{3.30}$$

We also notice that, owing to (2.17) and the assumptions on $F$ and $g$, it follows from Taylor's theorem that

$$\left|F'(\rho^\lambda) - F'(\overline{\rho}) - \lambda F''(\overline{\rho})\xi\right| \le c\left|y^\lambda\right| + c\left|\rho^\lambda - \overline{\rho}\right|^2 \quad \text{a.e. in } Q, \tag{3.31}$$
$$\left|g(\rho^\lambda) - g(\overline{\rho}) - \lambda g'(\overline{\rho})\xi\right| \le c\left|y^\lambda\right| + c\left|\rho^\lambda - \overline{\rho}\right|^2 \quad \text{a.e. in } Q, \tag{3.32}$$
$$\left|g'(\rho^\lambda) - g'(\overline{\rho}) - \lambda g''(\overline{\rho})\xi\right| \le c\left|y^\lambda\right| + c\left|\rho^\lambda - \overline{\rho}\right|^2 \quad \text{a.e. in } Q, \tag{3.33}$$

where, here and in the remainder of the proof, we denote by $c$ constants that may depend on the data of the system but not on $\lambda \in [0,\overline{\lambda}]$; the actual value of $c$ may change within formulas and lines. Moreover, by the Fréchet differentiability of $B$ (recall assumption **(A4)(vi)** and the fact that, for $\overline{v}, v \in L^2(Q)$, the restrictions of $B[v]$ and $DB[\overline{v}](v)$ to $Q_t$ depend only on $v|_{Q_t}$), we have (cf. (3.30))

$$\|B[\rho^\lambda] - B[\overline{\rho}] - \lambda\, DB[\overline{\rho}](\xi)\|_{L^2(Q_t)} \,\le\, c\,\|y^\lambda\|_{L^2(Q_t)} + R\left(\lambda\,\|h\|_{L^2(Q_t)}\right), \tag{3.34}$$

with a function $R:(0,+\infty)\to(0,+\infty)$ satisfying $\lim_{\sigma\searrow 0} R(\sigma)/\sigma = 0$. As we want to prove that $\delta\mathcal{S}(\overline{u};h)=(\xi,\eta)$, according to the definition of directional differentiability, we need to show that

$$\begin{aligned} 0 \,&=\, \lim_{\lambda\searrow 0} \frac{\|\mathcal{S}(\overline{u}+\lambda h) - \mathcal{S}(\overline{u}) - \lambda\,(\xi,\eta)\|_{\mathcal{Y}}}{\lambda} \\ &=\, \lim_{\lambda\searrow 0} \frac{\left\|y^\lambda\right\|_{H^1(0,T;H)} \,+\, \left\|z^\lambda\right\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}}{\lambda}\,. \end{aligned} \tag{3.35}$$

To begin with, using the state system (1.2)–(1.5) and the linearized system (3.1)–(3.4), we easily verify that for $0<\lambda\le\overline{\lambda}$ the pair $(z^\lambda, y^\lambda)$ is a strong solution to the system

$$\begin{aligned} &(1+2g(\overline{\rho}))z_t^\lambda \,+\, g'(\overline{\rho})\overline{\rho}_t z^\lambda \,+\, \overline{\mu}\,g'(\overline{\rho})y_t^\lambda - \Delta z^\lambda \\ &= -\,2\left(g(\rho^\lambda)-g(\overline{\rho})\right)\left(\mu_t^\lambda - \overline{\mu}_t\right) \,-\, 2\,\overline{\mu}_t\left(g(\rho^\lambda)-g(\overline{\rho})-\lambda g'(\overline{\rho})\xi\right) \\ &\quad -\,\overline{\mu}\,\overline{\rho}_t\left(g'(\rho^\lambda)-g'(\overline{\rho})-\lambda g''(\overline{\rho})\xi\right) \,-\, \overline{\mu}\left(g'(\rho^\lambda)-g'(\overline{\rho})\right)\left(\rho_t^\lambda-\overline{\rho}_t\right) \\ &\quad -\left(\mu^\lambda-\overline{\mu}\right)\left[\left(g'(\rho^\lambda)-g'(\overline{\rho})\right)\overline{\rho}_t + g'(\rho^\lambda)\left(\rho_t^\lambda-\overline{\rho}_t\right)\right] \quad \text{a.e. in } Q, \end{aligned} \tag{3.36}$$

$$\begin{aligned} y_t^\lambda &= -\left(F'(\rho^\lambda)-F'(\overline{\rho})-\lambda F''(\overline{\rho})\xi\right) - \left(B[\rho^\lambda]-B[\overline{\rho}]-\lambda DB[\overline{\rho}](\xi)\right) \\ &\quad +\, g'(\overline{\rho})z^\lambda \,+\, \overline{\mu}\left(g'(\rho^\lambda)-g'(\overline{\rho})-\lambda g''(\overline{\rho})\xi\right) \\ &\quad +\left(\mu^\lambda-\overline{\mu}\right)\left(g'(\rho^\lambda)-g'(\overline{\rho})\right) \quad \text{a.e. in}\, Q, \end{aligned} \tag{3.37}$$

$$\partial_{\mathbf{n}} z^\lambda = 0 \quad \text{a.e. on } \Sigma, \tag{3.38}$$

$$z^\lambda(0) = y^\lambda(0) = 0 \quad \text{a.e. in } \Omega. \tag{3.39}$$

In the following, we make repeated use of the mean value theorem and of the global estimates (2.17), (2.18), (2.32), and (3.30), without further reference. For the sake of a better readability, we will omit the superscript $\lambda$ of the quantities $y^\lambda$, $z^\lambda$ during the estimations, writing it only at the end of the respective estimates.

STEP 1: Let $t\in(0,T]$ be fixed. First, observe that

$$\partial_t\Big(\Big(\frac{1}{2}+g(\overline{\rho})\Big)z^2\Big) \,=\, (1+2\,g(\overline{\rho}))\,z\,z_t \,+\, g'(\overline{\rho})\,\overline{\rho}_t\,z^2\,.$$

Hence, multiplication of (3.36) by $z$ and integration over $Q_t$ yields the estimate

$$\int_\Omega\Big(\frac{1}{2}+g(\overline{\rho}(t))\Big)z^2(t)\,dx \,+\, \int_0^t\!\!\int_\Omega |\nabla z|^2\,dx\,ds \,\le\, c\sum_{j=1}^7 |I_j|\,, \tag{3.40}$$

where the quantities $I_j$, $1 \le j \le 7$, are specified and estimated as follows: at first, Young's inequality shows that, for every $\gamma > 0$ (to be chosen later),

$$I_1 := -\int_0^t\!\!\int_\Omega \overline{\mu}\, g'(\overline{\rho})\, y_t\, z\, dx\, ds \;\le\; \gamma \int_0^t\!\!\int_\Omega y_t^2\, dx\, ds \;+\; \frac{c}{\gamma}\int_0^t\!\!\int_\Omega z^2\, dx\, ds\,. \tag{3.41}$$

Moreover, we have, by Hölder's and Young's inequalities and (3.30),

$$\begin{aligned} I_2 :&= -2\int_0^t\!\!\int_\Omega \bigl(g(\rho^\lambda) - g(\overline{\rho})\bigr)\bigl(\mu_t^\lambda - \overline{\mu}_t\bigr)\, z\, dx\, ds \\ &\le c\int_0^t \|\rho^\lambda(s) - \overline{\rho}(s)\|_6\, \|\mu_t^\lambda(s) - \overline{\mu}_t(s)\|_2\, \|z(s)\|_3\, ds \\ &\le c\, \|\rho^\lambda - \overline{\rho}\|_{L^\infty(0,t;L^6(\Omega))}\, \|\mu^\lambda - \overline{\mu}\|_{H^1(0,t;H)}\, \|z\|_{L^2(0,t;V)} \\ &\le \gamma\, \|z\|^2_{L^2(0,t;V)} \;+\; \frac{c}{\gamma}\, \lambda^4\,. \end{aligned} \tag{3.42}$$

Next, we employ (3.32), the Hölder and Young inequalities, and (3.30), to infer that

$$\begin{aligned} I_3 :&= -2\int_0^t\!\!\int_\Omega \overline{\mu}_t\, \bigl(g(\rho^\lambda) - g(\overline{\rho}) - \lambda g'(\overline{\rho})\xi\bigr)\, z\, dx\, ds \\ &\le c\int_0^t\!\!\int_\Omega |\overline{\mu}_t|\, \bigl(|y| \;+\; |\rho^\lambda - \overline{\rho}|^2\bigr)\; |z|\, dx\, ds \\ &\le c\int_0^t \|\overline{\mu}_t(s)\|_6\, \bigl(\|y(s)\|_2\, \|z(s)\|_3 \;+\; \|\rho^\lambda(s) - \overline{\rho}(s)\|_6^2\, \|z(s)\|_2\bigr)\; ds \\ &\le \gamma\int_0^t \|z(s)\|_V^2\, ds \;+\; \frac{c}{\gamma}\int_0^t \|\overline{\mu}_t(s)\|_V^2\, \|y(s)\|_H^2\, ds \\ &\quad + c\int_0^t \|\overline{\mu}_t(s)\|_V^2\, \|z(s)\|_H^2\, ds \;+\; c\, \|\rho^\lambda - \overline{\rho}\|^4_{L^\infty(0,t;V)} \\ &\le \gamma\int_0^t \|z(s)\|_V^2\, ds \;+\; \Bigl(1 + \frac{c}{\gamma}\Bigr)\int_0^t \|\overline{\mu}_t(s)\|_V^2\bigl(\|y(s)\|_H^2 \;+\; \|z(s)\|_H^2\bigr)\, ds + c\,\lambda^4, \end{aligned} \tag{3.43}$$

where we observe that, in view of (2.18), the mapping $s \mapsto \|\overline{\mu}_t(s)\|_V^2$ belongs to $L^1(0,T)$. Likewise, utilizing (2.18), (3.33), (3.30), and the Hölder and Young inequalities, it is straightforward to deduce that

$$\begin{aligned} I_4 :&= -\int_0^t\!\!\int_\Omega \overline{\mu}\, \overline{\rho}_t\, \bigl(g'(\rho^\lambda) - g'(\overline{\rho}) - \lambda g''(\overline{\rho})\xi\bigr)\, z\, dx\, ds \\ &\le c\int_0^t\!\!\int_\Omega \bigl(|y| + |\rho^\lambda - \overline{\rho}|^2\bigr)\, |z|\, dx\, ds \\ &\le c\int_0^t\!\!\int_\Omega \bigl(y^2 + z^2\bigr)\, dx\, ds \;+\; c\int_0^t \|\rho^\lambda(s) - \overline{\rho}(s)\|_4^2\, \|z(s)\|_2\, ds \\ &\le c\int_0^t\!\!\int_\Omega \bigl(y^2 + z^2\bigr)\, dx\, ds \;+\; c\,\lambda^4\,. \end{aligned} \tag{3.44}$$

In addition, arguing similarly, we have

$$
\begin{aligned}
I_5 :=& -\int_0^t\!\!\int_\Omega \overline{\mu}\left(g'(\rho^\lambda)-g'(\overline{\rho})\right)\left(\rho_t^\lambda-\overline{\rho}_t\right) z\,dx\,ds \\
\le&\ c\int_0^t \|\rho^\lambda(s)-\overline{\rho}(s)\|_6\,\|\rho_t^\lambda(s)-\overline{\rho}_t(s)\|_2\,\|z(s)\|_3\,ds \\
\le&\ c\,\|\rho^\lambda-\overline{\rho}\|_{L^\infty(0,t;L^6(\Omega))}\,\|\rho^\lambda-\overline{\rho}\|_{H^1(0,t;H)}\,\|z\|_{L^2(0,t;V)} \\
\le&\ \gamma\int_0^t \|z(s)\|_V^2\,ds\,+\,\frac{c}{\gamma}\,\lambda^4\,,
\end{aligned}
\tag{3.45}
$$

as well as

$$
\begin{aligned}
I_6 :=& -\int_0^t\!\!\int_\Omega \overline{\rho}_t\left(\mu^\lambda-\overline{\mu}\right)\left(g'(\rho^\lambda)-g'(\overline{\rho})\right) z\,dx\,ds \\
\le&\ c\int_0^t\!\!\int_\Omega \left|\mu^\lambda-\overline{\mu}\right|\left|\rho^\lambda-\overline{\rho}\right||z|\,dx\,ds \\
\le&\ c\int_0^t \|\rho^\lambda(s)-\overline{\rho}(s)\|_6\,\|\mu^\lambda(s)-\overline{\mu}(s)\|_3\,\|z(s)\|_2\,ds \\
\le&\ c\int_0^t\!\!\int_\Omega z^2\,dx\,ds\,+\,c\,\lambda^4\,.
\end{aligned}
\tag{3.46}
$$

Finally, we find that

$$
\begin{aligned}
I_7 :=& -\int_0^t\!\!\int_\Omega \left(\mu^\lambda-\overline{\mu}\right) g'(\rho^\lambda)\left(\rho_t^\lambda-\overline{\rho}_t\right) z\,dx\,ds \\
\le&\ c\int_0^t \|\mu^\lambda(s)-\overline{\mu}(s)\|_6\,\|\rho_t^\lambda(s)-\overline{\rho}_t(s)\|_2\,\|z(s)\|_3\,ds \\
\le&\ c\,\|\mu^\lambda-\overline{\mu}\|_{L^\infty(0,t;V)}\,\|\rho^\lambda-\overline{\rho}\|_{H^1(0,t;H)}\,\|z\|_{L^2(0,t;V)} \\
\le&\ \gamma\int_0^t \|z(s)\|_V^2\,ds\,+\,\frac{c}{\gamma}\,\lambda^4\,.
\end{aligned}
\tag{3.47}
$$

In conclusion, combining the estimates (3.40)–(3.47), and choosing $\gamma=\frac{1}{8}$, we have shown that

$$
\begin{aligned}
\frac{1}{2}\,\left\|z^\lambda(t)\right\|_H^2\,+\,\frac{1}{2}\int_0^t \left\|z^\lambda(s)\right\|_V^2 ds\,\le\,\frac{1}{8}\int_0^t\!\!\int_\Omega \left|y_t^\lambda\right|^2 dx\,ds\,+\,c\,\lambda^4 \\
+\,c\int_0^t \left(1+\|\overline{\mu}_t(s)\|_V^2\right)\left(\left\|y^\lambda(s)\right\|_H^2+\left\|z^\lambda(s)\right\|_H^2\right) ds\,.
\end{aligned}
\tag{3.48}
$$

<u>STEP 2:</u> Let $t\in(0,T]$ be fixed. We add $y$ to both sides of (3.37), multiply the resulting identity by $y_t$, and integrate over $Q_t$ to obtain

$$
\int_0^t\!\!\int_\Omega y_t^2\,dx\,ds\,+\,\frac{1}{2}\|y(t)\|_H^2\,\le\,\sum_{j=1}^{6}|J_j|\,,
\tag{3.49}
$$

where the terms $J_j$, $1 \le j \le 6$, are specified and estimated as follows: at first, we have, for every $\gamma > 0$ (to be specified later),

$$J_1 := \int_0^t\!\!\int_\Omega y\, y_t \, dx\, ds \;\le\; \gamma \int_0^t\!\!\int_\Omega y_t^2 \, dx\, ds \;+\; \frac{c}{\gamma} \int_0^t\!\!\int_\Omega y^2 \, dx\, ds\,. \tag{3.50}$$

Then, we employ (2.18), (2.32), (3.30), and (3.31), as well as Hölder's and Young's inequalities, to obtain the estimate

$$\begin{aligned}
J_2 :&= -\int_0^t\!\!\int_\Omega \bigl(F'(\rho^\lambda) - F'(\overline{\rho}) - \lambda F''(\overline{\rho})\xi\bigr)\, y_t \, dx\, ds \\
&\le c \int_0^t\!\!\int_\Omega \bigl(|y| + |\rho^\lambda - \overline{\rho}|^2\bigr)\, |y_t| \, dx\, ds \\
&\le c \int_0^t \bigl(\|y(s)\|_2 \,+\, \|\rho^\lambda(s) - \overline{\rho}(s)\|_4^2\bigr)\, \|y_t(s)\|_2 \, ds \\
&\le \gamma \int_0^t\!\!\int_\Omega y_t^2 \, dx\, ds \,+\, \frac{c}{\gamma} \int_0^t\!\!\int_\Omega y^2 \, dx\, ds \,+\, \frac{c}{\gamma}\, \lambda^4\,.
\end{aligned} \tag{3.51}$$

By the same token, this time invoking (3.33), we find that

$$\begin{aligned}
J_3 :&= \int_0^t\!\!\int_\Omega \overline{\mu}\, \bigl(g'(\rho^\lambda) - g'(\overline{\rho}) - \lambda g''(\overline{\rho})\xi\bigr)\, y_t \, dx\, ds \\
&\le \gamma \int_0^t\!\!\int_\Omega y_t^2 \, dx\, ds \,+\, \frac{c}{\gamma} \int_0^t\!\!\int_\Omega y^2 \, dx\, ds \,+\, \frac{c}{\gamma}\, \lambda^4\,.
\end{aligned} \tag{3.52}$$

Moreover, we obviously have

$$J_4 := \int_0^t\!\!\int_\Omega g'(\overline{\rho})\, z\, y_t \, dx\, ds \;\le\; \gamma \int_0^t\!\!\int_\Omega y_t^2 \, dx\, ds \;+\; \frac{c}{\gamma} \int_0^t\!\!\int_\Omega z^2 \, dx\, ds\,. \tag{3.53}$$

Also, using (3.30) and the global bounds once more, we obtain that

$$\begin{aligned}
J_5 :&= \int_0^t\!\!\int_\Omega (\mu^\lambda - \overline{\mu})\, (g'(\rho^\lambda) - g'(\overline{\rho}))\, y_t \, dx\, ds \\
&\le c \int_0^t \|\mu^\lambda(s) - \overline{\mu}(s)\|_6 \, \|\rho^\lambda(s) - \overline{\rho}(s)\|_3 \, \|y_t(s)\|_2 \; ds \\
&\le \gamma \int_0^t\!\!\int_\Omega y_t^2 \, dx\, dx \,+\, \frac{c}{\gamma}\, \lambda^4\,.
\end{aligned} \tag{3.54}$$

Finally, invoking (3.34) and Young's inequality, we have the estimate

$$\begin{aligned}
J_6 :&= -\int_0^t\!\!\int_\Omega \bigl(B[\rho^\lambda] - B[\overline{\rho}] - \lambda DB[\overline{\rho}](\xi)\bigr)\, y_t \, dx\, ds \\
&\le \bigl\|B[\rho^\lambda] - B[\overline{\rho}] - \lambda DB[\overline{\rho}](\xi)\bigr\|_{L^2(Q_t)} \, \|y_t\|_{L^2(Q_t)} \\
&\le \gamma \int_0^t\!\!\int_\Omega y_t^2 \, dx\, ds \,+\, \frac{c}{\gamma}\, \|y\|_{L^2(Q_t)}^2 + \frac{c}{\gamma}\, \bigl(R\bigl(\lambda\, \|h\|_{L^2(Q)}\bigr)\bigr)^2\,.
\end{aligned} \tag{3.55}$$

Thus, combining the estimates (3.49)–(3.55), and choosing $\gamma = \frac{1}{8}$, we have shown that, for every $t \in (0,T]$, we have the estimate

$$\frac{1}{4}\int_0^t\!\!\int_\Omega |y_t^\lambda|^2\,dx\,ds \,+\, \frac{1}{2}\,\|y^\lambda(t)\|_H^2 \\ \le\, c\left(\int_0^t \|y^\lambda(s)\|_H^2\,ds + \lambda^4 \,+ \big(R\left(\lambda\,\|h\|_{L^2(Q)}\right)\big)^2\right). \tag{3.56}$$

<u>STEP 3:</u> We now add the estimates (3.48) and (3.56). It follows that, with suitable global constants $c_1 > 0$ and $c_2 > 0$, we have for every $t \in (0,T]$ the estimate

$$\left\|z^\lambda(t)\right\|_H^2 \,+\, \left\|z^\lambda\right\|_{L^2(0,t;V)}^2 \,+\, \left\|y^\lambda(t)\right\|_H^2 \,+\, \left\|y_t^\lambda\right\|_{L^2(0,t;H)}^2 \\ \le\, c_1\,Z(\lambda) \,+\, c_2\int_0^t \left(1 + \|\overline{\mu}_t(s)\|_V^2\right)\left(\left\|y^\lambda(s)\right\|_H^2 \,+\, \left\|z^\lambda(s)\right\|_H^2\right)\,ds\,, \tag{3.57}$$

where we have defined, for $\lambda > 0$, the function $Z$ by

$$Z(\lambda) := \,\lambda^4 \,+\, (R(\lambda\,\|h\|_{L^2(Q)}))^2\,. \tag{3.58}$$

Recalling that the mapping $s \mapsto \|\overline{\mu}_t(s)\|_V^2$ belongs to $L^1(0,T)$, we conclude from Gronwall's lemma that, for every $t \in (0,T]$,

$$\left\|y^\lambda\right\|_{H^1(0,t;H)}^2 \,+\, \left\|z^\lambda\right\|_{L^\infty(0,t;H)\cap L^2(0,t;V)}^2 \\ \le\, c_1\,Z(\lambda)\exp\left(c_2\int_0^T \left(1 + \|\overline{\mu}_t(s)\|_V^2\right)\,ds\right) \,\le\, c\,Z\,(\lambda)\,. \tag{3.59}$$

Since $\lim_{\lambda\searrow 0} Z(\lambda)/\lambda^2 = 0$ (recall (3.34)), we have finally shown the validity of (3.35). This concludes the proof of the assertion. □

We are now in the position to derive the following result.

COROLLARY 3.3: *Let the general hypotheses* **(A1)**–**(A5)** *be fulfilled and assume that* $\overline{u} \in \mathcal{U}_{\rm ad}$ *is a solution to the control problem* **(CP)** *with associated state* $(\overline{\rho},\overline{\mu}) = \mathcal{S}(\overline{u})$*. Then we have, for every* $v \in \mathcal{U}_{\rm ad}$*,*

$$\beta_1\int_0^T\!\!\int_\Omega (\overline{\rho} - \rho_Q)\,\xi\,dx\,dt \,+\, \beta_2\int_0^T\!\!\int_\Omega (\overline{\mu} - \mu_Q)\,\eta\,dx\,dt \\ +\, \beta_3\int_0^T\!\!\int_\Omega \overline{u}\,(v - \overline{u})\,dx\,dt \,\ge\, 0\,, \tag{3.60}$$

*where* $(\xi,\eta)$ *denotes the (unique) solution to the linearized system* (3.1)–(3.4) *for* $h = v - \overline{u}$*.*

Proof: Let $v \in \mathcal{U}_{\rm ad}$ be arbitrary. Then $h = v - \overline{u}$ is an admissible direction, since $\overline{u} + \lambda h \in \mathcal{U}_{\rm ad}$ for $0 < \lambda \le 1$. For any such $\lambda$, we have

$$\begin{aligned} 0 \le\ & \frac{J(\overline{u}+\lambda h, \mathcal{S}(\overline{u}+\lambda h)) - J(\overline{u}, \mathcal{S}(\overline{u}))}{\lambda} \\ \le\ & \frac{J(\overline{u}+\lambda h, \mathcal{S}(\overline{u}+\lambda h)) - J(\overline{u}, \mathcal{S}(\overline{u}+\lambda h))}{\lambda} \\ & + \frac{J(\overline{u}, \mathcal{S}(\overline{u}+\lambda h) - J(\overline{u}, \mathcal{S}(\overline{u}))}{\lambda}\,. \end{aligned}$$

It follows immediately from the definition of the cost functional $J$ that the first summand on the right-hand side of this inequality converges to $\int_0^T\!\!\int_\Omega \beta_3\, \overline{u}\, h\, dx\, dt$ as $\lambda \searrow 0$. For the second summand, we obtain from Theorem 3.2 that

$$\begin{aligned} & \lim_{\lambda\searrow 0} \frac{J(\overline{u}, \mathcal{S}(\overline{u}+\lambda h) - J(\overline{u}, \mathcal{S}(\overline{u}))}{\lambda} \\ & = \beta_1 \int_0^T\!\!\int_\Omega (\overline{\rho} - \rho_Q)\xi\, dx\, dt \,+\, \beta_2 \int_0^T\!\!\int_\Omega (\overline{\mu} - \mu_Q)\eta\, dx\, dt\,, \end{aligned}$$

whence the assertion follows. □

# 4 Existence and first-order necessary conditions of optimality

In this section, we derive the first-order necessary conditions of optimality for problem **(CP)**. We begin with an existence result.

Theorem 4.1: *Suppose that the conditions* **(A1)**–**(A5)** *are satisfied. Then the problem* **(CP)** *has a solution* $\overline{u} \in \mathcal{U}_{\rm ad}$.

Proof: Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}_{\rm ad}$ be a minimizing sequence for **(CP)**, and let $\{(\rho_n, \mu_n)\}_{n\in\mathbb{N}}$ be the sequence of the associated solutions to (1.2)–(1.5). We then can infer from the global estimate (2.18) the existence of a triple $(\overline{u}, \overline{\rho}, \overline{\mu})$ such that, for a suitable subsequence again indexed by $n$,

$$\begin{aligned} u_n &\to \bar{u} \quad \text{weakly star in } H^1(0,T;H)\cap L^\infty(Q), \\ \rho_n &\to \bar{\rho} \quad \text{weakly star in } H^2(0,T;H)\cap W^{1,\infty}(0,T;L^\infty(\Omega))\cap H^1(0,T;V), \\ \mu_n &\to \bar{\mu} \quad \text{weakly star in } W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W). \end{aligned}$$

Clearly, we have that $\overline{u} \in \mathcal{U}_{\rm ad}$ and, by virtue of the Aubin-Lions lemma (cf. [24, Thm. 5.1, p. 58]) and similar compactness results (cf. [27, Sect. 8, Cor. 4]),

$$\rho_n \to \overline{\rho} \quad \text{strongly in } L^2(Q), \tag{4.1}$$

whence also $\rho_* \le \overline{\rho} \le \rho^*$ a. e. in $Q$ and

$$B[\rho_n] \to B[\overline{\rho}] \quad \text{strongly in } L^2(Q),$$
$$\Phi(\rho_n) \to \Phi(\overline{\rho}) \quad \text{strongly in } L^2(Q), \quad \text{for } \Phi \in \{F', g, g'\},$$

thanks to the general assumptions on $B$, $F$ and $g$, as well as the strong convergence

$$\mu_n \to \overline{\mu} \quad \text{strongly in } C^0([0,T]; C^0(\overline{\Omega})) = C^0(\overline{Q}). \tag{4.2}$$

From this, we easily deduce that

$$g(\rho_n)\,\partial_t\mu_n \to g(\overline{\rho})\,\partial_t\overline{\mu} \quad \text{weakly in } L^1(Q),$$
$$\mu_n\, g'(\rho_n)\,\partial_t\rho_n \to \overline{\mu}\, g'(\overline{\rho})\,\partial_t\overline{\rho} \quad \text{weakly in } L^1(Q)\,.$$

In summary, if we pass to the limit as $n \to \infty$ in the state equations (1.2)–(1.3), written for the triple $(u_n, \rho_n, \mu_n)$, we find that $(\overline{\rho}, \overline{\mu})$ satisfies (1.2) and (1.3). Moreover, $\overline{\mu} \in L^\infty(0,T;W)$ satisfies the boundary condition (1.4), and it is easily seen that also the initial conditions (1.5) hold true. In other words, we have $(\overline{\rho}, \overline{\mu}) = \mathcal{S}(\overline{u})$, that is, the triple $(\overline{u}, \overline{\rho}, \overline{\mu})$ is admissible for the control problem **(CP)**. From the weak sequential lower semicontinuity of the cost functional $J$ it finally follows that $\bar{u}$, together with $(\overline{\rho}, \overline{\mu}) = \mathcal{S}(\overline{u})$, is a solution to **(CP)**. This concludes the proof. □

We now turn our interest to the derivation of first-order necessary optimality conditions for problem **(CP)**. To this end, we generally assume in the following that the hypotheses **(A1)**–**(A5)** are satisfied and that $\overline{u} \in \mathcal{U}_{\rm ad}$ is an optimal control with associated state $(\overline{\rho}, \overline{\mu})$, which has the properties (2.17)–(2.32). We now aim to eliminate $\xi$ and $\eta$ from the variational inequality (3.60). To this end, we employ the adjoint state system associated with (1.2)–(1.5) for $\overline{u}$, which is formally given by:

$$-\,(1 + 2g(\overline{\rho}))\,p_t - g'(\overline{\rho})\,\overline{\rho}_t\,p - \Delta p - g'(\overline{\rho})\,q$$
$$= \beta_2(\overline{\mu} - \mu_Q) \quad \text{in } Q, \tag{4.3}$$
$$-\,q_t + F''(\overline{\rho})\,q - \overline{\mu}\,g''(\overline{\rho})\,q + g'(\overline{\rho})\,(\overline{\mu}_t\,p - \overline{\mu}\,p_t) + DB[\overline{\rho}]^*(q)$$
$$= \beta_1(\overline{\rho} - \rho_Q) \quad \text{in } Q, \tag{4.4}$$
$$\partial_{\mathbf{n}} p = 0 \quad \text{on } \Sigma, \tag{4.5}$$
$$p(T) = q(T) = 0 \quad \text{in } \Omega\,. \tag{4.6}$$

In (4.4), $DB[\overline{\rho}]^* \in \mathcal{L}(L^2(Q), L^2(Q))$ denotes the adjoint operator associated with the operator $DB[\overline{\rho}] \in \mathcal{L}(L^2(Q), L^2(Q))$, thus defined by the identity

$$\int_0^T\!\!\int_\Omega DB[\overline{\rho}]^*(v)\,w\,dx\,dt \;=\; \int_0^T\!\!\int_\Omega v\,DB[\overline{\rho}](w)\,dx\,dt \quad \forall\, v, w \in L^2(Q)\,. \tag{4.7}$$

As, for every $v \in L^2(Q)$, the restriction of $DB[\overline{\rho}](v)$ to $Q_t$ depends only on $v|_{Q_t}$, it follows that, for every $w \in L^2(Q)$, the restriction of $DB[\overline{\rho}]^*(w)$ to $Q^t = \Omega \times (t,T)$ (see (1.14)) depends only on $w|_{Q^t}$. Moreover, (2.8) implies that

$$\|DB[\overline{\rho}]^*(w)\|_{L^2(Q^t)} \le C_B \|w\|_{L^2(Q^t)} \quad \forall\, w \in L^2(Q). \tag{4.8}$$

We also note that in the case of the integral operator (1.9) it follows from Fubini's theorem that $DB[\overline{\rho}]^* = DB[\overline{\rho}] = B$.

We have the following existence and uniqueness result for the adjoint system.

Theorem 4.2: *Suppose that* **(A1)**–**(A5)** *are fulfilled, and assume that* $\overline{u} \in \mathcal{U}_{\rm ad}$ *is a solution to the control problem* **(CP)** *with associated state* $(\overline{\rho},\overline{\mu}) = \mathcal{S}(\overline{u})$. *Then the adjoint system* (4.3)–(4.6) *has a unique solution* $(p,q)$ *satisfying*

$$p \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \quad \textit{and} \quad q \in H^1(0,T;H)\,. \tag{4.9}$$

Proof: Besides of Young's inequality, we make repeated use of the global estimates (2.17)–(2.18) and (2.32) for $\overline{\rho}$ and $\overline{\mu}$, without further reference. Moreover, we denote by $c$ different positive constants that may depend on the given data of the state system and of the control problem; the meaning of $c$ may change between and even within lines.

We first prove uniqueness. Thus, we replace the right-hand sides of (4.3) and (4.4) by 0 and prove that $(p,q) = (0,0)$. We add $p$ to both sides of (4.3) and multiply by $-p_t$. At the same time, we multiply (4.4) by $q$. Then we add the resulting equalities and integrate over $Q^t = \Omega \times (t,T)$. As $g$ is nonnegative, and thanks to (2.8), we obtain that

$$\begin{aligned}
&\int_t^T\!\!\int_\Omega |p_t|^2\,dx\,ds + \frac12\,\|p(t)\|_V^2 + \frac12 \int_\Omega |q(t)|^2\,dx \\
&\le \int_t^T\!\!\int_\Omega \bigl(-p - g'(\overline{\rho})\,\overline{\rho}_t\,p - g'(\overline{\rho})\,q\bigr)\,p_t\,dx\,ds \\
&\quad + \int_t^T\!\!\int_\Omega \bigl((\overline{\mu}\,g''(\overline{\rho}) - F''(\overline{\rho}))\,q + \overline{\mu}\,g'(\overline{\rho})\,p_t - DB[\overline{\rho}](q)\bigr)\,q\,dx\,ds \\
&\quad - \int_t^T\!\!\int_\Omega g'(\overline{\rho})\,\overline{\mu}_t\,p\,q\,dx\,ds \\
&\le \frac12 \int_t^T\!\!\int_\Omega |p_t|^2\,dx\,ds + c\int_t^T\!\!\int_\Omega (p^2+q^2)\,dx\,ds + c\int_t^T\!\!\int_\Omega |\overline{\mu}_t|\,|p|\,|q|\,dx\,ds\,.
\end{aligned}$$

The last integral is estimated as follows: employing the Hölder, Sobolev and Young inequalities, we have

$$\begin{aligned}
&\int_t^T\!\!\int_\Omega |\overline{\mu}_t|\,|p|\,|q|\,dx\,ds \le \int_t^T \|\overline{\mu}_t(s)\|_3\,\|p(s)\|_6\,\|q(s)\|_2\,ds \\
&\le c\int_t^T \bigl(\|\overline{\mu}_t(s)\|_V^2\,\|p(s)\|_V^2 + \|q(s)\|_H^2\bigr)\,ds\,.
\end{aligned}$$

As the function $s \mapsto \|\overline{\mu}_t(s)\|_V^2$ belongs to $L^1(0,T)$, we can apply the backward version of Gronwall's lemma to conclude that $(p,q) = (0,0)$.

The existence of a solution to (4.3)–(4.6) is proved in several steps.

<u>Step 1:</u> We approximate $\overline{\rho}$ and $\overline{\mu}$ by functions $\rho^\varepsilon, \mu^\varepsilon \in C^\infty(\overline{Q})$ satisfying (3.8)–(3.9)

and look for a solution $(p^\varepsilon, q^\varepsilon)$ to the following problem:

$$
\begin{aligned}
&-(1+2g(\rho^\varepsilon))\, p_t^\varepsilon - g'(\overline{\rho})\, \overline{\rho}_t\, p^\varepsilon - \Delta p^\varepsilon - g'(\overline{\rho})\, q^\varepsilon \\
&\quad = \beta_2(\overline{\mu} - \mu_Q) \quad \text{in } Q, \qquad (4.10)\\
&-q_t^\varepsilon - \varepsilon\, \Delta q^\varepsilon + F''(\overline{\rho})\, q^\varepsilon - \overline{\mu}\, g''(\overline{\rho})\, q^\varepsilon + g'(\rho^\varepsilon)\, (\mu_t^\varepsilon\, p^\varepsilon - \mu^\varepsilon\, p_t^\varepsilon) \\
&\quad + DB[\overline{\rho}]^*(q^\varepsilon) = \beta_1(\overline{\rho} - \rho_Q) \quad \text{in } Q, \qquad (4.11)\\
&\quad \partial_{\mathbf{n}} p^\varepsilon = \partial_{\mathbf{n}} q^\varepsilon = 0 \quad \text{on } \Sigma, \qquad (4.12)\\
&\quad p^\varepsilon(T) = q^\varepsilon(T) = 0 \quad \text{in } \Omega\,. \qquad (4.13)
\end{aligned}
$$

We prove that this problem has a unique solution satisfying

$$
p^\varepsilon, q^\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\,. \qquad (4.14)
$$

To this end, we present (4.10)–(4.12) as an abstract backward equation, namely,

$$
-\frac{d}{dt}\,(p^\varepsilon, q^\varepsilon)(t) + \mathcal{A}^\varepsilon(t)\,(p^\varepsilon, q^\varepsilon)(t) + (\mathcal{R}^\varepsilon(p^\varepsilon, q^\varepsilon))(t) = f^\varepsilon(t), \qquad (4.15)
$$

in the framework of the Hilbert triplet $(\mathcal{V}, \mathcal{H}, \mathcal{V}')$, where

$$
\mathcal{V} := V \times V \quad \text{and} \quad \mathcal{H} := H \times H\,.
$$

Notice that (4.15), together with the regularity (4.14), means that

$$
\begin{aligned}
&-\big((p_t^\varepsilon, q_t^\varepsilon)(t), (v,w)\big)_{\mathcal{H}} + a^\varepsilon\big(t; (p^\varepsilon, q^\varepsilon)(t), (v,w)\big) \\
&\quad + \big((\mathcal{R}^\varepsilon(p^\varepsilon, q^\varepsilon))(t), (v,w)\big)_{\mathcal{H}} = \big(f^\varepsilon(t), (v,w)\big)_{\mathcal{H}} \\
&\text{for every } (v,w) \in \mathcal{V} \text{ and a. a. } t \in (0,T), \qquad (4.16)
\end{aligned}
$$

where $a^\varepsilon(t;\cdot,\cdot)$ is the bilinear form associated with the operator $\mathcal{A}^\varepsilon(t) : \mathcal{V} \to \mathcal{V}'$; moreover, $(\cdot,\cdot)_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$ (equivalent to the usual one) that one has chosen, the embedding $\mathcal{H} \subset \mathcal{V}'$ being dependent on such a choice. In fact, we will not use the standard inner product of $\mathcal{H}$, which will lead to a nonstandard embedding $\mathcal{H} \subset \mathcal{V}'$. We aim at applying first [1, Thm. 3.2, p. 256], in order to find a unique weak solution, as we did for the linearized problem; then, we derive the full regularity required in (4.14). We set, for convenience,

$$
\varphi_\varepsilon := \frac{1}{1+2g(\rho^\varepsilon)} \quad \text{and} \quad \psi_\varepsilon := \frac{\mu^\varepsilon\, g'(\rho^\varepsilon)}{1+2g(\rho^\varepsilon)} = \varphi_\varepsilon\, \mu^\varepsilon\, g'(\rho^\varepsilon),
$$

and choose a constant $M_\varepsilon$ such that

$$
\varphi_\varepsilon \le M_\varepsilon, \quad |\psi_\varepsilon| \le M_\varepsilon, \quad |\nabla\varphi_\varepsilon| \le M_\varepsilon, \quad \text{and} \quad |\nabla\psi_\varepsilon| \le M_\varepsilon, \quad \text{a. e. in } Q\,.
$$

Moreover, we introduce three parameters $\lambda^\varepsilon, \lambda_1^\varepsilon, \lambda_2^\varepsilon$, whose values will be specified later on. In order to transform our problem, we compute $p_t^\varepsilon$ from (4.10) and substitute in (4.11).

Moreover, we multiply (4.10) by $\varphi_\varepsilon$. Finally, we add and subtract the same terms for convenience. Then (4.10)–(4.11) is equivalent to the system

$$
\begin{aligned}
&-p^\varepsilon_t - \varphi_\varepsilon\,\Delta p^\varepsilon + \lambda^\varepsilon_1\,p^\varepsilon \\
&\quad - \lambda^\varepsilon_1\,p^\varepsilon - \varphi_\varepsilon\,g'(\overline{\rho})\,\overline{\rho}_t\,p^\varepsilon - \varphi_\varepsilon\,g'(\overline{\rho})\,q^\varepsilon \;=\; \varphi_\varepsilon\,\beta_2(\overline{\mu}-\mu_Q), \\
&-q^\varepsilon_t - \varepsilon\,\Delta q^\varepsilon + \psi_\varepsilon\,\Delta p^\varepsilon + \lambda^\varepsilon_2\,q^\varepsilon \\
&\quad - \lambda^\varepsilon_2\,q^\varepsilon + F''(\overline{\rho})\,q^\varepsilon - \overline{\mu}\,g''(\overline{\rho})\,q^\varepsilon + g'(\rho^\varepsilon)\,\mu^\varepsilon_t\,p^\varepsilon \\
&\quad + \psi_\varepsilon\,\big(g'(\overline{\rho})\,\overline{\rho}_t\,p^\varepsilon + g'(\overline{\rho})\,q^\varepsilon + \beta_2(\overline{\mu}-\mu_Q)\big) \\
&\quad + DB[\overline{\rho}]^*(q^\varepsilon) = \beta_1(\overline{\rho}-\rho_Q)\,.
\end{aligned}
$$

By observing that

$$
-\varphi_\varepsilon\,\Delta p^\varepsilon = -\operatorname{div}(\varphi_\varepsilon\nabla p^\varepsilon) + \nabla\varphi_\varepsilon\cdot\nabla p^\varepsilon\,,
$$

and that the same holds true with $\psi_\varepsilon$ in place of $\varphi_\varepsilon$, we see that the latter system, complemented with the boundary condition (4.12), is equivalent to

$$
\begin{aligned}
&-\int_\Omega p^\varepsilon_t(t)\,v\,dx + a^\varepsilon_1(t;p^\varepsilon(t),v) + \int_\Omega (\mathcal{R}^\varepsilon_1(p^\varepsilon,q^\varepsilon))(t)\,v\,dx \\
&\quad = \int_\Omega \varphi_\varepsilon(t)\,\beta_2(\overline{\mu}-\mu_Q)(t)\,v\,dx \\
&-\int_\Omega q^\varepsilon_t(t)\,w\,dx + a^\varepsilon_2(t;(p^\varepsilon(t),q^\varepsilon(t)),w) + \int_\Omega (\mathcal{R}^\varepsilon_2(p^\varepsilon,q^\varepsilon))(t)\,w\,dx \\
&\quad = -\int_\Omega \psi_\varepsilon(t)\,\beta_2(\overline{\mu}-\mu_Q)(t)\,w\,dx + \int_\Omega \beta_1(\overline{\rho}-\rho_Q)(t)\,w\,dx
\end{aligned}
$$

for every $(v,w)\in\mathcal{V}$ and a. a. $t\in(0,T)$, where the forms $a^\varepsilon_i$ are defined below and the operators $\mathcal{R}^\varepsilon_i$ account for all the other terms on the left-hand sides of the equations. We set, for every $t\in[0,T]$ and $\hat v,\hat w,v,w\in V$,

$$
\begin{aligned}
&a^\varepsilon_1(t;\hat v,v) := \int_\Omega \big(\varphi_\varepsilon(t)\,\nabla\hat v\cdot\nabla v + (\nabla\varphi_\varepsilon(t)\cdot\nabla\hat v)\,v + \lambda^\varepsilon_1\,\hat v\,v\big)\,dx\,, \\
&a^\varepsilon_2(t;(\hat v,\hat w),w) \\
&\quad := \int_\Omega \big(\varepsilon\,\nabla\hat w\cdot\nabla w - \psi_\varepsilon(t)\,\nabla\hat v\cdot\nabla w - (\nabla\psi_\varepsilon(t)\cdot\nabla\hat v)\,w + \lambda^\varepsilon_2\,\hat w\,w\big)\,dx\,.
\end{aligned}
$$

Now, we choose the values of $\lambda^\varepsilon_i$ and of the further parameter $\lambda^\varepsilon$ in such a way as to guarantee some coerciveness. Putting $\alpha:=1/(1+2\sup g)$, we have that

$$
\begin{aligned}
a^\varepsilon_1(t;v,v) &\ge \int_\Omega \big(\alpha\,|\nabla v|^2 - M_\varepsilon\,|\nabla v|\,|v| + \lambda^\varepsilon_1\,v^2\big)\,dx \\
&\ge \int_\Omega \big(\alpha\,|\nabla v|^2 - \tfrac{\alpha}{2}\,|\nabla v|^2 - \tfrac{M_\varepsilon^2}{2\alpha}\,v^2 + \lambda^\varepsilon_1\,v^2\big)\,dx\,.
\end{aligned}
$$

Therefore, the choice $\lambda^\varepsilon_1 := \frac{\alpha}{2} + \frac{M_\varepsilon^2}{2\alpha}$ yields

$$
a^\varepsilon_1(t;v,v) \;\ge\; \frac{\alpha}{2}\,\|v\|_V^2 \quad \text{for every } v\in V \text{ and } t\in[0,T]\,.
$$

Next, we deal with $a_2^\varepsilon$. We have, for every $v,w\in V$ and $t\in[0,T]$,

$$a_2^\varepsilon(t;(v,w),w) \geq \int_\Omega \bigl(\varepsilon|\nabla w|^2 - M_\varepsilon|\nabla v|\,|\nabla w| - M_\varepsilon|\nabla v|\,|w| + \lambda_2^\varepsilon w^2\bigr)\,dx$$
$$\geq \int_\Omega \bigl(\varepsilon|\nabla w|^2 - \tfrac{\varepsilon}{2}|\nabla w|^2 - \tfrac{M_\varepsilon^2}{2\varepsilon}\,|\nabla v|^2 - \tfrac{M_\varepsilon^2}{2\varepsilon}\,|\nabla v|^2 - \tfrac{\varepsilon}{2}\,|w|^2 + \lambda_2^\varepsilon w^2\bigr)\,dx$$
$$= \int_\Omega \bigl(\tfrac{\varepsilon}{2}|\nabla w|^2 + (\lambda_2^\varepsilon - \tfrac{\varepsilon}{2})w^2 - \tfrac{M_\varepsilon^2}{\varepsilon}\,|\nabla v|^2\bigr)\,dx\,,$$

and the choice $\lambda_2^\varepsilon := \varepsilon$ leads to

$$a_2^\varepsilon(t;(v,w),w) \;\geq\; \frac{\varepsilon}{2}\,\|w\|_V^2 - \frac{M_\varepsilon^2}{\varepsilon}\,\|v\|_V^2\,.$$

Therefore, if we choose $\lambda^\varepsilon$ such that $\lambda^\varepsilon\,\frac{\alpha}{2} - \frac{M_\varepsilon^2}{\varepsilon} \geq \frac{\varepsilon}{2}$, then we obtain

$$\lambda^\varepsilon a_1^\varepsilon(t;v,v) + a_2^\varepsilon(t;(v,w),w) \geq \frac{\varepsilon}{2}\bigl(\|v\|_V^2 + \|w\|_V^2\bigr)$$

for every $(v,w)\in\mathcal{V}$ and $t\in[0,T]$. Hence, if we define $a^\varepsilon : [0,T]\times\mathcal{V}\times\mathcal{V}\to\mathbb{R}$ by setting

$$a^\varepsilon(t;(\hat v,\hat w),(v,w)) := \lambda_1^\varepsilon a_1^\varepsilon(\hat v,v) + a_2^\varepsilon(t;(\hat v,\hat w),w)\,,$$

then we obtain a time-dependent continuous bilinear form that is coercive on $\mathcal{V}$ (endowed whith its standard norm), uniformly with respect to $t\in[0,T]$. Moreover, $a^\varepsilon$ depends smoothly on $t$, and (4.10)–(4.12) is equivalent to

$$-\int_\Omega \bigl(\lambda^\varepsilon p_t^\varepsilon(t)\,v + q_t^\varepsilon(t)\,w\bigr)\,dx + a^\varepsilon\bigl(t;(p^\varepsilon(t),q^\varepsilon(t)),(v,w)\bigr)$$
$$+\int_\Omega \bigl\{\lambda^\varepsilon\,(\mathcal{R}_1^\varepsilon(p^\varepsilon,q^\varepsilon))(t)\,v + (\mathcal{R}_2^\varepsilon(p^\varepsilon,q^\varepsilon))(t)\,w\bigr\}\,dx$$
$$= \int_\Omega \bigl((\lambda^\varepsilon\,\varphi_\varepsilon - \psi_\varepsilon)(t)\,\beta_2(\overline{\mu} - \mu_Q)(t)\,v + \beta_1\,(\overline{\rho} - \rho_Q)(t)\,w\bigr)\,dx$$

for every $(v,w)\in\mathcal{V}$ and a. a. $t\in(0,T)$. Therefore, the desired form (4.16) is achieved if we choose the scalar product in $\mathcal{H}$ as follows:

$$\bigl((\hat v,\hat w),(v,w)\bigr)_{\mathcal{H}} := \int_\Omega (\lambda^\varepsilon\,\hat v\,v + \hat w\,w)\,dx \quad \text{for every } (\hat v,\hat w),(v,w)\in\mathcal{H}.$$

Notice that this leads to the following nonstandard embedding $\mathcal{H}\subset\mathcal{V}'$:

$${}_{\mathcal{V}'}\langle(\hat v,\hat w),(v,w)\rangle_{\mathcal{V}} = \bigl((\hat v,\hat w),(v,w)\bigr)_{\mathcal{H}} = \lambda^\varepsilon\,{}_{V'}\langle\hat v,v\rangle_V + {}_{V'}\langle\hat w,w\rangle_V$$

for every $(\hat v,\hat w)\in\mathcal{H}$ and $(v,w)\in\mathcal{V}$, provided that the embedding $H\subset V'$ is the usual one, i. e., corresponds to the standard inner product of $H$. As the remainder, given by the terms $\mathcal{R}_1^\varepsilon$ and $\mathcal{R}_2^\varepsilon$, satisfies the backward analogue of (3.14)–(3.15) (see also (4.8)), the quoted result of [1] can be applied, and problem (4.10)–(4.13) has a unique solution satisfying

$$(p^\varepsilon,q^\varepsilon)\in H^1(0,T;\mathcal{V}')\cap L^2(0,T;\mathcal{V}).$$

Moreover, if we move the remainder of (4.15) to the right-hand side, we see that

$$-\frac{d}{dt}(p^\varepsilon, q^\varepsilon) + \mathcal{A}^\varepsilon(p^\varepsilon, q^\varepsilon) \in L^2(0,T;\mathcal{H}).$$

Therefore, by also accounting for (4.13), we deduce that $(p^\varepsilon, q^\varepsilon) \in H^1(0,T;\mathcal{H})$ as well as $\mathcal{A}^\varepsilon(p^\varepsilon, q^\varepsilon) \in L^2(0,T;\mathcal{H})$. Hence, we have that $p^\varepsilon, q^\varepsilon \in L^2(0,T;W)$, by standard elliptic regularity.

<u>Step 2:</u> We add $p^\varepsilon$ to both sides of (4.10) and multiply by $-p^\varepsilon_t$. At the same time, we multiply (4.11) by $q^\varepsilon$. Then, we sum up and integrate over $Q^t$. As $g \geq 0$, we easily obtain that

$$\begin{aligned}
&\frac{1}{2}\,\|p^\varepsilon(t)\|_V^2 + \int_t^T\!\!\int_\Omega |p^\varepsilon_t|^2\,dx\,ds + \frac{1}{2}\int_\Omega |q^\varepsilon(t)|^2\,dx + \varepsilon\int_t^T\!\!\int_\Omega |\nabla q^\varepsilon|^2\,dx\,ds \\
&\leq c\int_t^T\!\!\int_\Omega |p^\varepsilon|\,|p^\varepsilon_t|\,dx\,ds + c\int_t^T\!\!\int_\Omega |q^\varepsilon|\,|p^\varepsilon_t|\,dx\,ds + c\int_t^T\!\!\int_\Omega |q^\varepsilon|^2\,dx\,ds \\
&\quad + c\int_t^T\!\!\int_\Omega |\mu^\varepsilon_t|\,|p^\varepsilon|\,|q^\varepsilon|\,dx\,ds + \int_t^T\!\!\int_\Omega |DB[\overline{\rho}]^*(q^\varepsilon)|\,|q^\varepsilon|\,dx\,ds + c\,\|p^\varepsilon\|^2_{L^2(Q^t)} + c\,.
\end{aligned}$$

Just two of the terms on the right-hand side need some treatment. We have

$$\begin{aligned}
&\int_t^T\!\!\int_\Omega |\mu^\varepsilon_t|\,|p^\varepsilon|\,|q^\varepsilon|\,dx\,ds \leq \int_t^T \|\mu^\varepsilon_t(s)\|_3\,\|p^\varepsilon(s)\|_6\,\|q^\varepsilon(s)\|_2\,ds \\
&\leq c\int_t^T \|p^\varepsilon(s)\|_V^2\,ds + c\int_t^T \|\mu^\varepsilon_t(s)\|_3^2\|q^\varepsilon(s)\|_2^2\,ds\,,
\end{aligned}$$

and we observe that the function $s \mapsto \|\mu^\varepsilon_t(s)\|_3^2$ belongs to $L^1(0,T)$, by (3.8). Moreover, the Schwarz inequality and (4.8) immediately yield that

$$\int_t^T\!\!\int_\Omega |DB[\overline{\rho}]^*(q^\varepsilon)|\,|q^\varepsilon|\,dx\,ds \leq C_B\,\|q^\varepsilon\|^2_{L^2(Q^t)}\,.$$

Therefore, we can apply the backward version of Gronwall's lemma to obtain that

$$\|p^\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|q^\varepsilon\|_{H^1(0,T;H)} + \varepsilon^{1/2}\|q^\varepsilon\|_{L^2(0,T;V)} \leq c\,. \tag{4.17}$$

By comparison in (4.10), we see that $\Delta p^\varepsilon$ is bounded in $L^2(Q)$. Hence,

$$\|p^\varepsilon\|_{L^2(0,T;W)} \leq c\,. \tag{4.18}$$

<u>Step 3:</u> We multiply (4.11) by $-q^\varepsilon_t$ and integrate over $Q^t$. We obtain

$$\begin{aligned}
&\int_t^T\!\!\int_\Omega |q^\varepsilon_t|^2\,dx\,ds + \frac{\varepsilon}{2}\int_\Omega |\nabla q^\varepsilon(t)|^2\,dx \\
&\leq c\int_t^T\!\!\int_\Omega |q^\varepsilon|\,|q^\varepsilon_t|\,dx\,ds + c\int_t^T \|\mu^\varepsilon_t(s)\|_3\,\|p^\varepsilon(s)\|_6\,\|q^\varepsilon_t(s)\|_2\,ds \\
&\quad + c\int_t^T\!\!\int_\Omega |p^\varepsilon_t|\,|q^\varepsilon_t|\,dx\,ds + \int_t^T\!\!\int_\Omega |DB[\overline{\rho}]^*(q^\varepsilon)|\,|q^\varepsilon_t|\,dx\,ds\,.
\end{aligned}$$

Thanks to (4.8) once more, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\int_t^T\!\!\int_\Omega |q_t^\varepsilon|^2\,dx\,ds + \frac{\varepsilon}{2}\int_\Omega |\nabla q^\varepsilon(t)|^2\,dx \\
&\le c\int_t^T\!\!\int_\Omega |q^\varepsilon|^2\,dx\,ds + c\int_t^T \|\mu_t^\varepsilon(s)\|_3^2\,\|p^\varepsilon(s)\|_V^2\,ds \\
&\quad + c\int_t^T\!\!\int_\Omega |p_t^\varepsilon|^2\,dx\,ds\,.
\end{aligned}
$$

Thus, (3.8) and (4.17) imply that

$$
\|q_t^\varepsilon\|_{L^2(0,T;H)} + \varepsilon^{1/2}\|q^\varepsilon\|_{L^\infty(0,T;V)} \le c\,. \tag{4.19}
$$

STEP 4: Now, we let $\varepsilon$ tend to zero and construct a solution to (4.3)–(4.6). By (4.17)–(4.19) we have, at least for a subsequence,

$$
\begin{aligned}
p^\varepsilon \to p &\qquad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\,, \\
q^\varepsilon \to q &\qquad \text{weakly in } H^1(0,T;H)\,, \\
\varepsilon q^\varepsilon \to 0 &\qquad \text{strongly in } L^\infty(0,T;V)\,,
\end{aligned}
$$

for some pair $(p,q)$ satisfying the regularity requirements (4.9). By accounting for (3.9) and the Lipschitz continuity of $g$ and $g'$, it is straightforward to see that $(p,q)$ is a solution to problem (4.3)–(4.6). This completes the proof. □

COROLLARY 4.3: *Suppose that* **(A1)**–**(A5)** *are fulfilled, and assume that* $\overline{u}\in\mathcal{U}_{\mathrm{ad}}$ *is an optimal control of* **(CP)** *with associated state* $(\overline{\rho},\overline{\mu})=\mathcal{S}(\overline{u})$ *and adjoint state* $(p,q)$. *Then it holds the variational inequality*

$$
\int_0^T\!\!\int_\Omega (p+\beta_3\,\overline{u})(v-\overline{u})\,dx\,dt \;\ge\; 0 \quad \forall\, v\in\mathcal{U}_{\mathrm{ad}}\,. \tag{4.20}
$$

PROOF: We fix $v\in\mathcal{U}_{\mathrm{ad}}$ and choose $h=v-\overline{u}$. Then, we write the linearized system (3.1)–(3.4) and multiply the equations (3.1) and (3.2) by $p$ and $q$, respectively. At the same time, we consider the adjoint system and multiply the equations (4.3) and (4.4) by $-\eta$ and $-\xi$, respectively. Then, we add all the equalities obtained in this way and integrate over $Q$. Many terms cancel out. In particular, this happens for the contributions given by the Laplace operators, due to the boundary conditions (3.3) and (4.5), as well as for the terms involving $DB[\overline{\rho}]$ and $DB[\overline{\rho}]^*$, by the definition of adjoint operator (see (4.7)). Thus, it remains

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \big(2g'(\overline{\rho})\,\overline{\rho}_t\,\eta\,p + (1+2g(\overline{\rho}))\,\eta_t\,p + (1+2g(\overline{\rho}))\,\eta\,p_t\big)\,dx\,dt \\
&\quad + \int_0^T\!\!\int_\Omega \big(\overline{\mu}_t\,g'(\overline{\rho})\,\xi\,p + \overline{\mu}\,g''(\overline{\rho})\,\overline{\rho}_t\,\xi\,p + \overline{\mu}\,g'(\overline{\rho})\,\xi_t\,p + \overline{\mu}\,g'(\overline{\rho})\,\xi\,p_t\big)\,dx\,dt \\
&\quad + \int_0^T\!\!\int_\Omega (\xi_t\,q + \xi\,q_t)\,dx\,dt \\
&= \int_0^T\!\!\int_\Omega \big((v-\overline{u})\,p - \beta_2(\overline{\mu}-\mu_Q)\,\eta - \beta_1(\overline{\rho}-\rho_Q)\,\xi\big)\,dx\,dt
\end{aligned}
$$

Now, we observe that the expression on the left-hand side coincides with

$$\int_0^T\!\!\int_\Omega \partial_t\bigl\{(1+2g(\overline{\rho}))\,\eta\,p+\overline{\mu}\,g'(\overline{\rho})\,\xi\,p+\xi q\bigr\}\,dx\,dt\,.$$

Thus, it vanishes, due to the initial and final conditions (3.4) and (4.6). This implies that

$$\int_0^T\!\!\int_\Omega \bigl(\beta_1(\overline{\rho}-\rho_Q)\,\eta+\beta_2(\overline{\mu}-\mu_Q)\,\xi\bigr)\,dx\,dt=\int_0^T\!\!\int_\Omega (v-\overline{u})\,p\,dx\,dt\,.$$

Therefore, (4.20) follows from (3.60). □

REMARK 4: The variational inequality (4.20) forms together with the state system (1.2)–(1.5) and the adjoint system (4.3)–(4.6) the system of first-order necessary optimality conditions for the control problem **(CP)**. Notice that in the case $\beta_3>0$ the function $-\beta_3^{-1}p$ is nothing but the $L^2(Q)$ orthogonal projection of $\overline{u}$ onto $\mathcal{U}_{\rm ad}$.

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