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Stochastic homogenization of rate-independent systems

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Abstract

We study the stochastic and periodic homogenization 1-homogeneous convex functionals. We proof some convergence results with respect to stochastic two-scale convergence, which are related to classical Γ-convergence results. The main result is a general lim inf-estimate for a sequence of 1-homogeneous functionals and a two-scale stability result for sequences of convex sets. We apply our results to the homogenization of rate-independent systems with 1-homogeneous dissipation potentials and quadratic energies. In these applications, both the energy and the dissipation potential have an underlying stochastic microscopic structure. We study the particular homogenization problems of Prandlt-Reuss plasticity, Coulomb friction on a macroscopic surface and Coulomb friction on microscopic fissures.

1 Introduction

We study (stochastic) homogenization problems of the form

$$0 \in \partial\Psi_\varepsilon(\partial_t u^\varepsilon) + D\mathcal{E}_\varepsilon(t, u^\varepsilon)\,, \tag{1.1}$$

where $\mathcal{E}_\varepsilon : [0,T] \times \mathscr{B}_\varepsilon \to \overline{\mathbb{R}}$ is a proper, quadratic functional and $\Psi_\varepsilon : \mathscr{B}_\varepsilon \to \overline{\mathbb{R}}$ is proper and 1-homogeneous and $\mathscr{B}_\varepsilon$ is an ε-dependent Banach space. As usual in homogenization, the index $\varepsilon > 0$ is a smallnes parameter and (in general) relates to the scale of the underlying geometry of the pysical system, such as crystaline structure, microscopic cracks etc.. We work with quadratic energies on Hilbert spaces although our ideas also apply to more general settings.

Systems of the form (1.1) arise in various applications, among which we focus on Prandtl-Reuss plasticity and Coulomb-friction. The concept of rate-independent systems can be formulated in a more general way than (1.1) and we refer the reader to the recent monograph by Mielke and Rubicek [20], but also to [18].

We are interested in the limit $\varepsilon \to 0$, where we expect that $u^\varepsilon \overset{2s}{\rightharpoonup} u$ in the two-scale sense, which will be specified below. The limit function u usually lies in a different Banach space $\mathscr{B}$ than the sequence u^ε. Nevertheless,we expect that u is the solution of a new equation on $\mathscr{B}$ of the form

$$0 \in \partial\Psi(\partial_t u) + D\mathcal{E}(u)\,, \tag{1.2}$$

where again $\mathcal{E} : [0,T] \times \mathscr{B} \to \overline{\mathbb{R}}$ is a proper, quadratic functional and $\Psi : \mathscr{B} \to \overline{\mathbb{R}}$ is proper and 1-homogeneous.

In this work, we focus on the 1-homogeneous functional Ψ_ε, as the homogenization of quadratic functionals is well understood (see [32, 33] and references therein). More precisely, we consider the case of a (stationary, ergodic) random measure μ_ω and set $\mu_\omega^\varepsilon(A) := \varepsilon^n \mu_\omega(\varepsilon^{-1}A)$. Let $\boldsymbol{Q} \subset \mathbb{R}^n$ be a bounded domain and let $(\Omega, \mathcal{B}(\omega), \mathcal{P})$ be a probability space with an ergodic dynamical system τ. Taking a family $\mathcal{C}(x,\omega) \subset \mathbb{R}^D$ of closed and convex subsets of $\mathbb{R}^D$, where $(x,\omega) \in \boldsymbol{Q} \times \Omega$, we introduce the convex sets

$$\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega) := \left\{u \in L^p(\boldsymbol{Q}; \mu_\omega^\varepsilon; \mathbb{R}^D) \,:\, u(x) \in \mathcal{C}(x, \tau_{\frac{x}{\varepsilon}}\omega) \text{ for } \mu_\omega^\varepsilon\text{-a.e. } x \in \boldsymbol{Q}\right\}, \qquad \text{for } \omega \in \Omega\,,$$
$$\mathscr{C}_p(\boldsymbol{Q} \times \Omega) := \left\{u \in L^p(\boldsymbol{Q}; L^p(\Omega; \mu_{\mathcal{P}}; \mathbb{R}^D)) \,:\, u(x,\omega) \in \mathcal{C}(x,\omega) \text{ for } \mathcal{L} \times \mu_{\mathcal{P}}\text{-a.e. } (x,\omega)\right\},$$

and the functions

$$\psi_\varepsilon(x,u) = \sup_{\sigma\in\mathcal{C}(x,\tau_{\frac{x}{\varepsilon}}\omega)} \sigma\cdot u\,.$$

We then consider the family of functionals

$$\Psi_{\varepsilon,\omega}(u) := \sup_{\sigma\in\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)} \int_{\boldsymbol{Q}} u(x)\sigma(x)\, d\mu_\omega^\varepsilon(x) \quad \text{and} \quad \Psi(u) := \sup_{\sigma\in\mathscr{C}(\boldsymbol{Q}\times\Omega)} \int_{\boldsymbol{Q}}\int_\Omega u\cdot\sigma\, d\mu_{\mathcal{P}} dx\,.$$

The major issues that will be studied in Section 5 are the following: consider $u^\varepsilon \in \partial\Psi_{\varepsilon,\omega}(0) = \mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)$ for $\varepsilon\to 0$ that weakly two-scale converges to u. By Theorem 5.6 it then follows that $u\in\partial\Psi(0) = \mathscr{C}_p(\boldsymbol{Q}\times\Omega)$. This can be considered as a kind of stability result for the sequence $\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)$. Lemmas 5.8–5.10 show that

$$\liminf_{\varepsilon\to 0} \Psi_{\varepsilon,\omega}(v^\varepsilon) \geq \Psi(v) \qquad \text{whenever } v^\varepsilon \xrightharpoonup{2s} v\,. \tag{1.3}$$

Theorem 5.6 then yields equality in (1.3) for $v^\varepsilon \xrightarrow{2s} v$ strongly in the two-scale sense. In Sections 6–9, we provide three applications of Lemmas 5.8–5.10, namely in case of Prandtl-Reuss plasticity and Coulomb-friction on a macroscopic and on a microscopic level.

The results obtained in this paper are linked to the theory of evolutionary Γ-convergence, which could be applied in the periodic setting. Using evolutionary Γ-convergence, most of the results obtained in this paper could be proved easily in the periodic setting, in particular Theorem 5.6. The theory of evolutionary Γ-convergence has its roots in a work by Sandier and Serfaty [24] and has been applied quite successfull to the homogenization of rate-independent systems within the periodic setting, compare e.g. [8]. A summary on the applications of evolutionary Γ-convergence can be found in a recent work by Mielke [19].

In the periodic setting, one benefits from the existence of the so called unfolding operator: Given a periodic measure $\mu^\varepsilon(A) := \varepsilon^n\mu_0(\varepsilon^{-1}A)$, the unfolding operators $\mathcal{T}_\varepsilon$ are uniformly bounded linear opertors from $L^2(Q;\mu^\varepsilon)$ onto $L^2(Q\times Y;\mathcal{L}\times\mu_0)$. Thus, the sequence of solutions $u^\varepsilon\in L^2(Q;\mu^\varepsilon)$ can be interpreted as a sequence of functions in $\mathcal{T}_\varepsilon u^\varepsilon \in \mathscr{H} := L^2(Q\times Y;\mathcal{L}\times\mu_0)$, a spacethat is independent of ε. One equally might consider $\mathcal{E}_\varepsilon$ and Ψ_ε as functionals on $\mathscr{H}$. Given the assumption that $\mathcal{E}_\varepsilon\to\mathcal{E}$ and $\Psi_\varepsilon\to\Psi$ on $\mathscr{H}$ in the Mosco-sense, one easily obtains that the limit function u is an energetic solution to $(\mathscr{H},\mathcal{E},\Psi)$. Note that Mosco-convergence of $\Psi_\varepsilon\to\Psi$ implies that the limit (1.3) automatically holds.

In the stochastic setting, the unfolding operator can formally be defined as the adjoint of the mapping $f(x,\omega)\mapsto f(x,\tau_{\frac{x}{\varepsilon}}\omega)$, but this operator in general is no more continuous. Since the stochastic setting lacks of a continuous unfolding operator, bounded sequences in $\mathscr{H}_\varepsilon$ are no longer bounded in $\mathscr{H}$. this makes it impossible to consider Mosco-convergence of Ψ_ε. We are thus pushed to develop other methods, where we exploit the characterization of convex sets by linear functionals.

An example from plasticity theory As an example for applications of the theory developed below, we mention here the Prandtl-Reuss equations of plasticity on a bounded domain $\boldsymbol{Q}\subset\mathbb{R}^n$ and on a time interval $[0,T]$:

$$\left.\begin{aligned} -\nabla\cdot\sigma^\varepsilon &= f\,, & \sigma^\varepsilon &= C_\varepsilon^{-1}e^\varepsilon\,,\\ \nabla^s u^\varepsilon + \nabla^s u_{Dir} &= e^\varepsilon + p^\varepsilon\,, & \partial_t p^\varepsilon &\in \partial\psi_\varepsilon^*(\sigma^\varepsilon - B_\varepsilon p^\varepsilon)\,, \end{aligned}\right\} \text{ on } [0,T]\times\boldsymbol{Q}\,. \tag{1.4}$$

Here, C_ε is the elasticity modul from Hook's law, B_ε is the hardening parameter and ψ_ε is the flow rule function. All these parameters strongly depend on the underlying material.

We assume for the moment, that C_ε and B_ε are scalar functions and thus isotropic. Given $C : \mathbb{R}^n \to \mathbb{R}$, $B : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R}^n \times \mathbb{R}^{n\times n}_s \to \mathbb{R}$, we define for $\varepsilon > 0$ the scaled quantities $C_\varepsilon(x) = C(\frac{x}{\varepsilon})$, $B_\varepsilon(x) = B(\frac{x}{\varepsilon})$ and $\psi_\varepsilon(x,u) = \psi(\frac{x}{\varepsilon}, u)$.

The limit $\varepsilon \to 0$ of system (1.4) was recently studied in the periodic setting, starting from a work by Alber [1] and continued by Visintin [28, 29], Alber and Nesenenko [2, 21], Schweizer and Veneroni [25] and others (see [11] for more references). A result on the stochastic homogenization of (1.4) was obtained in [12]. Under the assumption of an *averaging property,* it was shown there, that $u^\varepsilon \to u$ strongly in $L^2(0,T;L^2(\boldsymbol{Q}))$ as $\varepsilon \to 0$, where u solves the limit system

$$-\nabla \cdot \Sigma(\nabla^s u) = f \tag{1.5}$$

for some hysteresis operator $\Sigma : H^1(0,T;\mathbb{R}^{n\times n}_s) \to H^1(0,T;\mathbb{R}^{n\times n}_s)$, which depends on the "unscaled" parameters C, B and ψ. It was shown in [11] that the stochastic setting satisfies this averaging property.

In spite of our expectations, equation (1.5) does not have the form (1.2). On the other hand, the structure of equation (1.5) is not surprising since the concept of hysteresis was introduced to deal with (macroscopic) rate independent memory effects that arise from hidden (microscopic or fast) material variables. In this work, the limit problem of (1.4) will first have the form (1.2) but we will see that it can be rewritten in the form (1.5).

The difference between the result in [11] and in the current work are two-fold. First, the results in [11] apply to arbitrary convex functions ψ, while the present work is focused on 1-homogeneous ψ. Second, the present work allows for a dependence of the parameters B, C and ψ on the macroscopic variable $x \in \boldsymbol{Q}$, which is not the case in [11].

Structure of the article The structure of the article is as follows. In the next section, we introduce basic concepts that are needed throughout the rest of this work, i.e. we introduce some notations for function spaces and concepts like energetic solutions to rate independent systems, ergodicity and random closed sets. In Section 3, we introduce some geometric examples to which we can apply the theory outlined in Sections 4 and 5. We introduce the concept of stochastic two-scale convergence in Section 4 while we introduce the central concept of this work, namely the weak two-scale convergence of convex sets, in Section 5. In Sections 6–9, we apply the theory of Sections 4 and 5 to Prandtl-Reuss plasticity, Coulomb friction on the surface of an elastic body and Coulomb friction on microscopic fissures.

2 Notations and Preliminaries

2.1 General notations

Given a Radon-measure μ on a Borel-measurable set $\boldsymbol{U} \subset \mathbb{R}^n$, we write $L^p(\boldsymbol{U};\mathbb{R}^D;\mu)$, $1 \le p < \infty$ for the set of measurable $\mathbb{R}^D$-valued functions such that $\int_{\boldsymbol{U}} |f|^p d\mu$ exists. If $\mu = \mathcal{L}$ is the

Lebesgue-measure, we omit μ and simply write $L^p(\boldsymbol{U};\mathbb{R}^D)$. If $D = 1$, we write $L^p(\boldsymbol{U};\mu)$ and similarly, we write $L^p(\boldsymbol{U})$ if no confusion occurs.

For any Banach space $\mathscr{B}$ with norm $\|\cdot\|_{\mathscr{B}}$, we denote by $L^p(\boldsymbol{U};\mathscr{B})$, $1 \le p < \infty$, the usual Bochner space of functions $f : \boldsymbol{U} \to \mathscr{B}$ such that $\int_{\boldsymbol{U}} \|f\|_{\mathscr{B}}^p \, d\mathcal{L}$ exists. By $L^\infty(\boldsymbol{U};\mathscr{B})$, we denote the space of functions, that are bounded almost everywhere. We say that $f \in W^{1,p}(0,T;\mathscr{B})$ for $1 \le p \le \infty$, if $f, \partial_t f \in L^p(0,T;\mathscr{B})$. We denote by $W^{1,p}(0,T;\mathscr{B})$ the space of functions $u \in L^p(0,T;\mathscr{B})$ such that also $\partial_t u \in L^p(0,T;\mathscr{B})$.

Given a vector space V, we call $\mathscr{L}(V,V)$ the space of all linear mappings from V to V.

Given a functional $\mathcal{E} : [0,T] \times \mathscr{B} \to \mathbb{R}$ we define

$$\mathcal{D}(\mathcal{E}(t)) := \{u \in \mathscr{B} : |\mathcal{E}(t,u)| < \infty\} .$$

2.2 Rate-independent systems

We collect some results on existence and uniqueness of solutions for rate-independent systems of the form

$$0 \in \partial\Psi(\partial_t u) + D\mathcal{E}(t,u(t)) \tag{2.1}$$

on a Banach space $\mathscr{B}$. In particular, we consider the case of a 1-homogeneous convex functional $\Psi : \mathscr{B} \to \mathbb{R}_+$, i.e. $\Psi(\lambda v) = \lambda\Psi(v)$ for all $\lambda \ge 0$ and of a quadratic energy $\mathcal{E}$. It is well known that under these conditions (2.1) has the following reformulation (see [20] or [18] Sections 2 and 4).

Definition 2.1. Let $\mathscr{B}$ be a Banach space, $\mathcal{E} : [0,T]\times\mathscr{B} \to \mathbb{R}$ be lower semicontinuous and $\Psi : \mathscr{B} \to \mathbb{R}$ be convex, lower semicontinuous and 1-homogeneous. We say that $u \in C^{Lip}(0,T;\mathscr{B})$ is an *energetic solution* to $(\mathscr{B},\mathcal{E},\Psi)$, resp. (2.1), if the following two conditions hold for every $t \in [0,T]$:

$$\mathcal{E}(t,u(t)) \le \mathcal{E}(t,\hat{u}) + \Psi(\hat{u} - u(t)) \qquad \forall \hat{u} \in \mathscr{B}\,, \tag{2.2}$$

$$\mathcal{E}(t,u(t)) + \int_0^t \Psi(\partial_t u) = \mathcal{E}(0,u(0)) + \int_0^t \partial_s\mathcal{E}(s,u(s))\,ds\,. \tag{2.3}$$

Condition (2.2) is called *stability condition* and equation (2.3) is called *global energy-balance*. Condition (2.2) can be reformulated [18, Section 2] into

$$\Psi^*(-D\mathcal{E}(t,u(t))) = 0 \qquad \text{or} \qquad -D\mathcal{E}(t,u(t)) \in \partial\Psi(0)\,. \tag{2.4}$$

The following lemma states that we can weaken (2.3). It is proved for example in [18], Step 5 of the proof of Theorem 2.1.

Lemma 2.2. *Let $u \in C^{Lip}(0,T;\mathscr{B})$ satisfy (2.2) or (2.4). Furthermore, let*

$$\mathcal{E}(t,u(t)) + \int_0^t \Psi(\partial_t u) \le \mathcal{E}(0,u(0)) + \int_0^t \partial_s\mathcal{E}(s,u(s))\,ds \quad \forall t \in [0,T]\,. \tag{2.5}$$

Then, u is an energetic solution to $(\mathscr{B},\mathcal{E},\Psi)$.

The following existence result will be sufficient for our applications.

Theorem 2.3. *Let $\mathscr{H}_1$, $\mathscr{H}_2$ be Hilbert spaces such that $\mathscr{H}_1 \hookrightarrow \mathscr{H}_2$ continuously, i.e. $\|u\|_{\mathscr{H}_2} \le C_{12}\|u\|_{\mathscr{H}_1}$. Let*

$$\mathcal{E} : [0,T] \times \mathscr{H}_2 \to \mathbb{R} \qquad \textit{with} \qquad \mathcal{E}(t,u) = \frac{1}{2}\|u\|^2_{\mathscr{H}_1} + K(u) + \langle l(t), u\rangle_{\mathscr{H}_1} + f(t)\,,$$

where $K : \mathscr{H}_2 \to (-\infty, +\infty]$ is a convex functional, $l \in H^1(0,T;\mathscr{H}_1)$ and $f \in W^{1,1}(0,T)$. Let $\Psi : \mathscr{H}_2 \to \mathbb{R}$ be a proper convex 1-homogeneous functional. Finally, let $u_0 \in \mathscr{H}_1$ such that (2.2) or (2.4) holds for $t = 0$ and $u(t = 0) = u_0$. Then, there exists a unique energetic solution to $(\mathscr{H}_2, \mathcal{E}, \Psi)$ with $u(0) = u_0$, satisfying the apriori estimates

$$\mathcal{E}(t,u(t)) + \int_0^t \Psi(s, \partial_s u(s))\, ds = \mathcal{E}(0,u(0)) + \int_0^t \partial_s \mathcal{E}(s,u(s))\, ds\,, \tag{2.6}$$

$$\|u\|_{C^{Lip}([0,T];\mathscr{H}_1)} + \|\partial_t u\|_{L^\infty([0,T];\mathscr{H}_1)} \le C(C_{12}, \|l\|_{H^1(0,T;\mathscr{H}_1)}) \left(\mathcal{E}(0,u(0)) + 1\right)\,, \tag{2.7}$$

where $C(C_{12}, \|l\|_{H^1(0,T;\mathscr{H}_1)})$ depends only on C_{12}, $\|l\|_{H^1(0,T;\mathscr{H}_1)}$. Furthermore, $u \in C([0,T];\mathscr{H}_2)$ depends Lipschitz-continuously on $l \in L^1(0,T;\mathscr{H}_2)$ and on $u_0 \in \mathscr{H}_1$.

Proof. By our assumptions, the functional $\mathcal{E}(t,\cdot)$ is α-convex for all t in the sence of [18], Section 3.5. From [18] Theorems 3.4 and 5.2 we get existence of an energetic solution $u \in C^{0,1}([0,T];\mathscr{H}_2)$. The estimate (2.7) follows from the proof of Theorem 3.4 in [18] on noting that we obtain for $\Lambda_1 := \|l\|_{H^1(0,T;\mathscr{H}_1)} > 0$

$$\begin{aligned}\frac{1}{2}\|y(t) - y(s)\|^2_{\mathscr{H}_2} \le \frac{C_{12}}{2}\|y(t) - y(s)\|^2_{\mathscr{H}_1} &\le C_{12}\Lambda_1 \int_s^t \|y(t) - y(\tau)\|_{\mathscr{H}_2} d\tau \\ &\le C_{12}^2 \Lambda_1 \int_s^t \|y(t) - y(\tau)\|_{\mathscr{H}_1} d\tau\,. \end{aligned}\tag{2.8}$$

The Lipschitz-continuous dependence of $u \in C([0,T];\mathscr{H}_1)$ on $l \in L^1(0,T;\mathscr{H}_2)$ and on $u_0 \in \mathscr{H}_1$ follows from (2.8). □

2.3 Ergodic dynamical systems

Throughout this paper, we follow the setting of Papanicolaou and Varadhan [23] and make the following assumptions.

Assumption 2.4. *Let $(\Omega, \mathcal{B}_\Omega, \mathcal{P})$ be a probability space with countably generated σ-algebra $\mathcal{B}_\Omega$. Further, we assume we are given a family $(\tau_x)_{x\in\mathbb{R}^n}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$, having the properties of a* dynamical system *on $(\Omega, \mathcal{B}_\Omega, \mathcal{P})$, i.e. they satisfy (i)-(iii):*

(i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ *(Group property)*

(ii) $\mathcal{P}(\tau_{-x}B) = \mathcal{P}(B) \quad \forall x \in \mathbb{R}^n,\ B \in \mathcal{B}_\Omega$ *(Measure preserving)*

(iii) $A : \mathbb{R}^n \times \Omega \to \Omega \qquad (x,\omega) \mapsto \tau_x\omega$ *is measurable (Measurablility of evaluation)*

We finally assume that the system $(\tau_x)_{x\in\mathbb{R}^n}$ is ergodic. This means that for every measurable function $f : \Omega \to \mathbb{R}$ there holds

$$[f(\omega) = f(\tau_x\omega)\ \forall x \in \mathbb{R}^n\,,\ a.e.\ \omega \in \Omega] \Rightarrow [f(\omega) = const \text{ for } \mathcal{P} - a.e.\ \omega \in \Omega]\,. \tag{2.9}$$

Remark. 1. An equivalent characterization of ergodicity is the following: For every $\mathcal{B}_\Omega$-measurable set $B \subset \Omega$ holds

$$[\mathcal{P}\left((\tau_x(B) \cup B)\backslash(\tau_x(B) \cap B)\right) = 0 \ \forall x \in \mathbb{R}^n] \Rightarrow [\mathcal{P}(B) \in \{0,1\}] . \tag{2.10}$$

2. In some application, the notion of ergodic dynamical system is given for $(\tau_x)_{x\in\mathbb{Z}^n}$. This definition is analogous to the above definitions with $\mathbb{R}^n$ replaced by $\mathbb{Z}^n$.

From [5] Theorem 4.13 we get that $L^p(\Omega)$ is separable for every $1 \le p < \infty$. For a set X, a function $f : \Omega \to X$ and $\omega \in \Omega$, the function $f_\omega : \mathbb{R}^d \to X$, $f_\omega(x) := f(\tau_x\omega)$ is called a realization (or the ω-realization) of f. The following *ergodic theorem* states that almost all realizations of integrable functions are integrable. The first part up to (2.11) is standard and (to the author's knowledge) due to Tempel'man [26]. It can also be found e.g. in [7, 33]. The second part is an immediate consequence.

Theorem 2.5 (Ergodic Theorem [33])**.** *Let Assumption 2.4 hold for* $(\Omega, \mathcal{B}_\Omega, \mu, \tau)$*. Let* $f \in L^1(\Omega)$*. Then, for almost all* $\omega \in \Omega$ *it holds:* $f(\tau_{\frac{x}{\varepsilon}}\omega) \in L^1_{loc}(\mathbb{R}^n)$ *for all* $\varepsilon > 0$ *and for all bounded open sets* $\boldsymbol{Q} \subset \mathbb{R}^n$ *it holds*

$$\lim_{\varepsilon\to 0} \int_{\boldsymbol{Q}} f(\tau_{\frac{x}{\varepsilon}}\omega)\, dx = \lim_{\varepsilon\to 0} \int_{\boldsymbol{Q}} f_\omega(\frac{x}{\varepsilon})\, dx = \mathcal{L}(\boldsymbol{Q}) \int_\Omega f(\omega)\, d\mu(\omega) . \tag{2.11}$$

Furthermore, for all $f \in L^p(\Omega)$*,* $1 \le p \le \infty$ *and a.e.* $\omega \in \Omega$ *holds* $f(\tau_x\omega) \in L^p_{loc}(\mathbb{R}^n)$ *and for* $1 \le p < \infty$ *holds* $f(\tau_{\frac{x}{\varepsilon}}\omega) \rightharpoonup \int_\Omega f$ *weakly in* $L^p_{loc}(\mathbb{R}^n)$ *as* $\varepsilon \to 0$*.*

We say that the realization $f_\omega$is ergodic if (2.11) holds.

2.4 Stationary random measures

Let $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$ be a probability space with dynamical system satisfying Assumption 2.4 and let $\mathcal{M}(\mathbb{R}^n)$ be the set of Radon measures on $\mathbb{R}^n$ equipped with the Vague topology.

Definition 2.6. A *random measure* is a mapping $\mu_\bullet : \Omega \to \mathcal{M}(\mathbb{R}^n)$, $\omega \mapsto \mu_\omega$ such that $\omega \mapsto \mu_\omega(A)$ is measurable for all Borel sets $A \subset \mathbb{R}^n$. This is equivalent with the measurability of $\mu_\bullet$ with respect to the Vague topology on $\mathcal{M}(\mathbb{R}^n)$. A random measure is called stationary, if $\mu_{\tau_x\omega}(A) = \mu_\omega(A + x)$ for all Borel sets $A \subset \mathbb{R}^n$. The *intensity* $\lambda(\mu_\omega)$ is defined by:

$$\lambda(\mu_\omega) := \int_\Omega \int_{\mathbb{R}^n} \chi_{[0,1]^n}(\tau_x\omega, x) d\mu_\omega(x)\, d\mu(\omega) = \mu_{\mathcal{P}}(\Omega) . \tag{2.12}$$

Theorem 2.7 (Mecke [17, 7]: Existence of Palm measure)**.** *Let* $\omega \mapsto \mu_\omega$ *be a stationary random measure. Then there exists a unique measure* $\mu_{\mathcal{P}}$ *on* Ω *such that*

$$\int_\Omega \int_{\mathbb{R}^n} f(x, \tau_x\omega)\, d\mu_\omega(x) d\mathcal{P}(\omega) = \int_{\mathbb{R}^n} \int_\Omega f(x, \omega)\, d\mu_{\mathcal{P}}(\omega) dx$$

for all $\mathcal{L} \times \mu_{\mathcal{P}}$*-measurable non negative functions and all* $\mathcal{L} \times \mu_{\mathcal{P}}$*- integrable functions* f*. Furthermore for all* $A \subset \Omega$*,* $u \in L^1(\Omega, \mu_{\mathcal{P}})$ *there holds*

$$\mu_{\mathcal{P}}(A) = \int_\Omega \int_{\mathbb{R}^n} g(s)\chi_A(\tau_s\omega) d\mu_\omega(s) d\mathcal{P} \tag{2.13}$$

$$\int_\Omega u(\omega) d\mu_{\mathcal{P}} = \int_\Omega \int_{\mathbb{R}^n} g(s) u(\tau_s\omega) d\mu_\omega(s) d\mathcal{P} \tag{2.14}$$

for an arbitrary $g \in L^1(\mathbb{R}^n, \mathcal{L})$ *with* $\int_{\mathbb{R}^n} g(x)dx = 1$ *and* $\mu_{\mathcal{P}}$ *is* σ*-finite.*

A few properties of the Palm measure and its underlying σ-algebra seem to be noteworthy:

Remark 2.8. a) Setting $g(s) := \chi_{[0,1]^n}(s)$, the Palm measure can equally be defined through (2.13).

b) For the constant measure $\omega \mapsto \mathcal{L}$, we simply find $\mu_{\mathcal{P}} = \mu$, the original probability measure. This is a direct consequence of (2.13), Fubini's theorem and Assumption 2.4.

c) For a random measure $\omega \mapsto \mu_\omega$, we may assume that $\Omega \subset \mathcal{M}(\mathbb{R}^n)$ and $\mu_{\mathcal{P}}$ is a measure with respect to the Borel-algebra on Ω (see [9], Section 3). Then, Ω is a separable metric space.

d) By comment c), it follows from [4] Theorems 67.2 and 68.1 (see also [9]) for all $1 \le p < \infty$ and all $k \in \mathbb{N}$ that the spaces $L^p(\Omega, \mu_{\mathcal{P}})$ and $C_b(\Omega)$ are separable and that $C_b(\Omega) \hookrightarrow L^p(\Omega, \mu_{\mathcal{P}})$ densely ([5] Theorem 4.13).

2.5 The Ergodic Theorem

Let Assumption 2.4 hold for $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$. Given a stationary random measure μ_ω, we introduce the scaled measure μ_ω^ε through

$$\mu_\omega^\varepsilon(A) := \varepsilon^n \mu_\omega(\varepsilon^{-1} A) . \tag{2.15}$$

We cite the following generalization of Theorem 2.5:

Theorem 2.9 (Ergodic Theorem [7]). *Let Assumption 2.4 hold for $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$. Let μ_ω be a stationary random measure with finite intensity and Palm measure $\mu_{\mathcal{P}}$. Then, for all $g \in L^1(\Omega, \mu_{\mathcal{P}})$ there holds $\mathcal{P}$ almost surely*

$$\lim_{\varepsilon \to 0} \int_A g(\tau_{\frac{x}{\varepsilon}} \omega) d\mu_\omega^\varepsilon(x) = |A| \int_\Omega g(\omega) d\mu_{\mathcal{P}}(\omega) \tag{2.16}$$

for all bounded Borel sets A.

At this point, we note that in [7] this theorem is provided only for A being a convex set containing an open ball around 0. However, the theorem can be generalized to arbitrary Borel sets by first considering simplices A. Such simplices can be extended to convex sets containing an open ball around zero. The statement then follows from the linearity of (2.16) in the characteristic function of A. The ergodic theorem only holds for function on Ω. Nevertheless, it motivates the following generalization of the concept of ergodicity:

Definition 2.10. Let $f \in L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}})$ for some $1 \le p < \infty$. We say that f is a *p-ergodic function* if for a.e. $\omega \in \Omega$ it holds that $f_\omega^\varepsilon(x) := f(x, \tau_{\frac{x}{\varepsilon}} \omega)$ is measurable for all $\varepsilon > 0$ and

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\boldsymbol{Q}} f(x, \tau_{\frac{x}{\varepsilon}} \omega) \, d\mu_\omega^\varepsilon(x) &= \int_{\boldsymbol{Q}} \int_\Omega f(x, \tilde{\omega}) \, d\mu_{\mathcal{P}}(\tilde{\omega}) \, dx \, , \\ \lim_{\varepsilon \to 0} \int_{\boldsymbol{Q}} \left| f(x, \tau_{\frac{x}{\varepsilon}} \omega) \right|^p \, d\mu_\omega^\varepsilon(x) &= \int_{\boldsymbol{Q}} \int_\Omega |f(x, \tilde{\omega})|^p \, d\mu_{\mathcal{P}}(\tilde{\omega}) \, dx \, . \end{aligned} \tag{2.17}$$

We call ω, resp. f_ω, an *ergodic realization* of f, if (2.17) holds.

The rest of this section deals with the identification of a suitably large class of ergodic functions.

Lemma 2.11. *Let Assumption 2.4 hold for* $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$. *Let* $\boldsymbol{Q} \subset \mathbb{R}^n$ *be a bounded domain and let* $A \subset \boldsymbol{Q} \times \Omega$ *be a* $\mathcal{B}_{\boldsymbol{Q}} \times \mathcal{B}_\Omega$*-measurable set. Then, the characteristic function* $\chi_A(x, \omega)$ *satisfies*

$$\lim_{\varepsilon \to 0} \int_{\boldsymbol{Q}} \chi_A(x, \tau_{\frac{x}{\varepsilon}} \omega)\, d\mu_\omega^\varepsilon(x) = \int_{\boldsymbol{Q}} \int_\Omega \chi_A(x, \tilde{\omega})\, d\mu_{\mathcal{P}}(\tilde{\omega})\, dx \quad \textit{for a.e. } \omega \in \Omega\,. \tag{2.18}$$

Proof. Due to Theorem 2.5, the statement is evident for $A = A_Q \times A_\Omega$, where $A_Q \in \mathcal{B}_{\boldsymbol{Q}}$ and $A_\Omega \in \mathcal{B}_\Omega$ are measurable sets. In general, A has the form

$$A = \bigcup_{i \in \mathbb{N}} A_i \text{ with } A_i = A_{i,Q} \times A_{i,\Omega} \text{ for } i \in \mathbb{N}, \text{ where } A_{i,Q} \in \mathcal{B}_{\boldsymbol{Q}} \text{ and } A_{i,\Omega} \in \mathcal{B}_\Omega \tag{2.19}$$

are measurable sets. After countably many operations, we can assume that $A_i \cap A_j = \varnothing$ and $A_{i,Q} \cap A_{j,Q} = \varnothing$ for all $i \neq j$. Note that (2.18) then holds for χ_{A_i} for all $i \in \mathbb{N}$.

Since $\bigcup_{i \in \mathbb{N}} A_{i,Q} \subset \boldsymbol{Q}$ and since this union is disjoint, we find $\lim_{J \to \infty} \sum_{i=J}^\infty \mathcal{L}(A_{i,Q}) = 0$. Thus, for each $n \in \mathbb{N}$, there exists $J_n \in \mathbb{N}$ such that $\mathcal{L}(\tilde{A}_{n,Q}) < \frac{1}{n}$ with $\tilde{A}_{n,Q} = \bigcup_{i=J_n}^\infty A_{i,Q}$. We set $\hat{A}_n := \bigcup_{i=1}^{J_n - 1} A_i$ and obtain for a set $\Omega_n \subset \Omega$ of full $\mathcal{P}$-measure such that for all $\omega \in \Omega_n$

$$\begin{aligned}
\int_{\boldsymbol{Q}} \int_\Omega \chi_{\hat{A}_n}(x, \tilde{\omega})\, d\mu_{\mathcal{P}}(\tilde{\omega})\, dx &= \lim_{\varepsilon \to 0} \int_{\boldsymbol{Q}} \chi_{\hat{A}_n}(x, \tau_{\frac{x}{\varepsilon}} \omega)\, d\mu_\omega^\varepsilon(x) \leq \limsup_{\varepsilon \to 0} \int_{\boldsymbol{Q}} \chi_A(x, \tau_{\frac{x}{\varepsilon}} \omega)\, d\mu_\omega^\varepsilon(x) \\
&\leq \limsup_{\varepsilon \to 0} \left(\int_{\boldsymbol{Q}} \chi_{\hat{A}_n}(x, \tau_{\frac{x}{\varepsilon}} \omega)\, d\mu_\omega^\varepsilon(x) + \int_{\tilde{A}_{n,Q}} d\mu_\omega^\varepsilon \right) \\
&\leq \lim_{\varepsilon \to 0} \int_{\boldsymbol{Q}} \chi_{\hat{A}_n}(x, \tau_{\frac{x}{\varepsilon}} \omega)\, d\mu_\omega^\varepsilon(x) + \mathcal{L}(\tilde{A}_{n,Q}) \mu_{\mathcal{P}}(\Omega) \\
&= \int_{\boldsymbol{Q}} \int_\Omega \chi_{\hat{A}_n}(x, \tilde{\omega})\, d\mu_{\mathcal{P}}(\tilde{\omega})\, dx + \frac{1}{n} \mu_{\mathcal{P}}(\Omega)\,.
\end{aligned}$$

Since $n \in \mathbb{N}$ was arbitrary and $\chi_{\hat{A}_n} \nearrow \chi_A$ pointwise, we obtain

$$\int_Q \int_\Omega \chi_A = \lim_{n \to \infty} \int_Q \int_\Omega \chi_{\hat{A}_n} = \lim_{\varepsilon \to 0} \int_Q \chi_A(x, \tau_{\frac{x}{\varepsilon}} \omega)\, d\mu_\omega^\varepsilon(x)$$

for all $\omega \in \tilde{\Omega}$ with $\tilde{\Omega} := \bigcap_{n \in \mathbb{N}} \Omega_n$. Since $\mathcal{P}(\tilde{\Omega}) = 1$, the statement follows. □

Lemma 2.12. *Let Assumption 2.4 hold for* $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$. *Let* $\boldsymbol{Q} \subset \mathbb{R}^n$ *be a bounded domain and let* $f \in L^p(\boldsymbol{Q} \times \Omega; \mathcal{L} \otimes \mu_{\mathcal{P}}) \cap L^\infty(\boldsymbol{Q} \times \Omega; \mathcal{L} \otimes \mu_{\mathcal{P}})$, $1 \leq p < \infty$. *Then, f has a* $\mathcal{B}_{\boldsymbol{Q}} \times \mathcal{B}_\Omega$*-measurable representative which is an ergodic function.*

Proof. The function f has a Borel-measurable representative. Furthermore, we can assume that this representative is bounded. The statement now follows from the fact that for every $\delta > 0$ we find piecewise constant Borel-measurable functions f_1^δ, f_2^δ such that $f_1^\delta \leq f \leq f_2^\delta$ and $\sup_{\boldsymbol{Q} \times \Omega} |f_1^\delta - f_2^\delta| < \delta$. □

2.6 A particular probability space

We provide a construction of a probability space which will be used below. We therefore consider Ω_0 a separable (or compact) metric space with a probability measure $\mathcal{P}_0$. Then, we consider $\Omega := \Omega_0^{\mathbb{N}}$ and write $\omega = (\omega_i)_{i\in\mathbb{N}}$ for all $\omega \in \Omega$. If d_0 denotes the metric on Ω_0, we define the metric on Ω through

$$d(\omega_1, \omega_2) = \sum_{i=1}^{\infty} 2^{-i} \frac{d_0(\omega_{1,i}, \omega_{2,i})}{1 + d_0(\omega_{1,i}, \omega_{2,i})} .$$

This topology is generated by the open sets $A \times \Omega_0^{\mathbb{N}}$, where for some $n > 0$, $A \subset \Omega_0^n$ is an open set. In case Ω_0 is separable (compact), the space Ω is separable (compact), too (see [15]).

In a next step, note that the sets of the form $A\times\Omega_0^{\mathbb{N}} = A\times\Omega$ together with their complements form an *algebra* in Ω, which we denote $\mathcal{R}$. For any set $A \subset \Omega$ of the form $A = \tilde{A} \times \Omega_0^{\mathbb{N}}$, where $\tilde{A} \subset \Omega_0^n$ is measurable for some $0 < n < \infty$, we define

$$\mathcal{P}(A) = \mathcal{P}_0^n(\tilde{A}) , \text{ where } \mathcal{P}_0^n \text{ denotes the classical product measure on } \Omega_0^n .$$

We make the observation that $\mathcal{P}$ is additive, positive and $\mathcal{P}(\varnothing) = 0$. Next, let $(A_j)_{j\in\mathbb{N}}$ be an increasing sequence of sets in $\mathcal{R}$ such that $A := \bigcup_j A_j \in \mathcal{R}$. Then, there exists $0 < n < \infty$ and $\tilde{A}_1 \subset \Omega_0^n$ such that $\tilde{A}_1 \times \Omega_0^{\mathbb{N}} = A_1 \subset A_2 \subset \cdots \subset A \subset \Omega_0^n \times \Omega_0^{\mathbb{N}} = \Omega$. Furthermore, for every $j > 1$, there exists $\tilde{A}_j$ such that $A_j = \tilde{A}_j \times \Omega_0^{\mathbb{N}}$ and there exists $\tilde{A} \subset \Omega_0^n$ such that $A = \tilde{A} \times \Omega_0^{\mathbb{N}}$. Since A_j is an increasing sequence, also $\tilde{A}_j$ must be increasing and $\tilde{A} = \bigcup_j \tilde{A}_j$. Therefore, $\mathcal{P}(A_j) = \mathcal{P}_0(\tilde{A}_j) \to \mathcal{P}_0(\tilde{A}) = \mathcal{P}(A)$. We have thus shown that $\mathcal{P} : \mathcal{R} \to [0,1]$ can be extended to a measure on the Borel-σ-Algebra on Ω (See [4] Theorem 6-2).

The same considerations hold, if we consider $\Omega := \Omega_0^{\mathbb{Z}^n}$. We write $\omega = (\omega_i)_{i\in\mathbb{Z}^n}$ for $\omega \in \Omega$ and define for $x \in \mathbb{Z}^n$ the mapping

$$\tau_x : \Omega \to \Omega , \qquad \omega \mapsto \tau_x\omega , \quad \text{where } \omega_i \mapsto \omega_{i+x} .$$

Then, $(\tau_x)_{x\in\mathbb{Z}^n}$ form a dynamical system on Ω with respect to $\mathbb{Z}^n$. We set $Y := [0,1[^n$ with $\hat{\Omega} := Y \times \Omega$ and write $(y, \omega) \in \hat{\Omega}$ if $y \in Y$ and $\omega \in \Omega$. As a measure on $\hat{\Omega}$, we consider $\hat{\mathcal{P}} := \mathcal{L} \times \mathcal{P}$. For $x \in \mathbb{R}^n$ we use the unique decomposition $x = \lfloor x \rfloor + x_Y$, where $\lfloor x \rfloor \in \mathbb{Z}^n$ and $x_Y \in Y$. Then, for $x \in \mathbb{R}^n$, we define the mapping

$$\hat{\tau}_x : \hat{\Omega} \to \hat{\Omega} , \qquad \hat{\omega} \mapsto \tau_x\hat{\omega} , \quad \text{where } \omega_i \mapsto \omega_{i+\lfloor y+x \rfloor} \text{ and } y \mapsto y + x - \lfloor y + x \rfloor .$$

The most important result of this subsection is the following.

Lemma 2.13. *The family $\hat{\tau}$ is ergodic.*

Proof. It is known that the family τ is ergodic on Ω. Now, let $A \subset \hat{\Omega}$ be invariant, i.e. $\mathcal{P}\left((A \cup \hat{\tau}_x A) \setminus (A \cap \hat{\tau}_x A)\right) = 0$ for all $x \in \mathbb{R}^n$. This is equivalent with

$$\int_Y \int_\Omega \chi_A(y,\omega) + \chi_{\hat{\tau}_x A}(y,\omega) - 2\chi_A(y,\omega)\chi_{\hat{\tau}_x A}(y,\omega) d\mathcal{P}(\omega) dy = 0 . \tag{2.20}$$

For fixed $y \in Y$ we obtain

$$\int_\Omega \chi_A(y,\omega) + \chi_A(y,\tau_x\omega) - 2\chi_A(y,\omega)\chi_A(y,\tau_x\omega) d\mathcal{P}(\omega) = 0 \qquad \forall x \in \mathbb{Z}^n$$

Since τ is ergodic on Ω, it follows by (2.9) and positivity of the integrand that $\chi_A(y,\cdot)$ is constant in Ω for a.e. $y \in Y$. More precisely, we obtain $\chi_A(y,\tau_x\omega) = \chi_A(y,\omega)$ almost surely for all $x \in \mathbb{Z}^n$. But then, (2.20) yields

$$0 \le \chi_A(y,\omega) + \chi_A(y+x,\omega) - 2\chi_A(y,\omega)\chi_A(y+x,\omega) = 0 \qquad \forall x \in \mathbb{R}^n$$

for a.e. $(y,\omega) \in \hat{\Omega}$. This implies $\mathcal{P}(A) \in \{0,1\}$. □

3 Examples for stochastic geometries and random measures

In this section, we give some concrete examples of stationary ergodic measures in order to demonstrate the large range of geometric settings that are captured by Assumption 2.4. We start with the periodic case, as this is the case most familiar to the homogenization community and also the easiest setting from the point of view of description. We then go on with general stochastic geometries and finally discuss the case of a random checkerboard.

3.1 The periodic case

In [31], Zhikov introduced two-scale convergence for periodic measures. This work was a straight generalization of Allaire's definition in [3] and is (to the author's knowledge) the most general definition of periodic two-scale convergence up to now. The notation and the formulation of the results show a significant similarity with the notation used in the definition of stochastic two-scale convergence in [33]. Of course, one expects that the case of a periodic measure should be a special case of a stochastic measure. In fact, shifting a Y-periodic measure by $y \in Y = [0,1[^n$, we can consider Y as a probability space.

To be more specific, let μ_0 be a $\mathbb{Z}^n$-periodic measure in $\mathbb{R}^n$ which is a measure satisfying $\mu_0(\cdot) = \mu_0(\cdot + x)$ for all $x \in \mathbb{Z}^n$. We consider $Y = [0,1[^n$ equiped with the Euclidean topology on the torus. Consider the family of mappings $\tau_x : y \mapsto [(y+x) \bmod \mathbb{Z}^n]$ for every $x \in \mathbb{R}^n$ and note that $\tau_x : Y \to Y$ satisfies the Assumptions 2.4(i)-(iii). Defining

$$\iota : Y \to \mathcal{M}(\mathbb{R}^n)\,, \quad y \mapsto \mu_y(\cdot) := \mu_0(\cdot + y)\,,$$

we can prove the following lemma.

Lemma 3.1. *[9, Lemma 3.5]The mapping ι is a homeomorphism.*

Thus, setting $\Omega := [0,1[^n$ with the topology of the torus, $\mathcal{B}_\Omega := \mathcal{B}_{[0,1[^n}$ and $\mathcal{P} = \mathcal{L}$ with τ as above, we note that $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$ satisfies Assumption 2.4. Furthermore, $y \mapsto \mu_y$ is a stationary random measure with $\mu_{\mathcal{P}} = \mu_0|_{[0,1[^n}$. Denoting $C^k_{per}(Y)$ the set of k-times differentiable functions on Y which are Y-periodic, we note that $C^k(\Omega) = C^k_b(\Omega) = C^k_{per}(Y)$.

Without giving a proof, we state that the concept of two-scale convergence introduced below in Definition 4.2 is equivalent with the following definition from [31].

Definition 3.2. Let $1 < p < \infty$. Let μ_0 be a periodic Radon measure on $\mathbb{R}^n$ and set $\mu^\varepsilon(A) := \varepsilon^n \mu_0(\varepsilon^{-1}A)$. Let $u^\varepsilon \in L^p(\boldsymbol{Q};\mu^\varepsilon)$ for all $\varepsilon > 0$. We say that (u^ε) converges (weakly) in two scales to $u \in L^p(\boldsymbol{Q}; L^p(Y,\mu_0))$ and write $u^\varepsilon \stackrel{2s}{\rightharpoonup} u$ if for all $\phi \in C_0(\boldsymbol{Q}; C_{per}(Y))$ there holds

$$\lim_{\varepsilon\to 0} \int_{\boldsymbol{Q}} u^\varepsilon(x)\phi(x,\frac{x}{\varepsilon})d\mu^\varepsilon = \int_{\boldsymbol{Q}}\int_\Omega u(x,y)\phi(x,y)\,d\mu_0(y)\,dx\,.$$

Choosing $\mu_0 = \mathcal{L}$, Definition 3.2 is equivalent with the original definition of two-scale convergence by Allaire [3].

3.2 The random checkerboard

We study the checker board construction of i.i.d. random variables, since this is a commonly used example for an ergodic stochastic setting in homogenization.

Defining $Y := [0,1[^n$, we consider $\mathbb{R}^n$ to be partitioned into unit cubes $\mathcal{C}_z := z + Y$ for $z \in \mathbb{Z}^n$. Like in Section 3.1, we equip Y with the topology of the torus. We then consider the sets

$$\tilde{\Omega} := \{u \in L^\infty(\mathbb{R}^n) \,|\, u|_{\mathcal{C}_z} \equiv c_z\,,\ \text{with } c_z \in [0,1] \text{ for every } z \in \mathbb{Z}^n\} \tag{3.1}$$

$$\Omega := \left\{u \in L^\infty(\mathbb{R}^n) \,|\, \exists \xi \in Y \text{ s.t. } u(.-\xi) \in \tilde{\Omega}\right\}. \tag{3.2}$$

For $u \in \Omega$ we denote the (unique) ξ from (3.2) as $\xi(u)$.

Since $L^1(\mathbb{R}^n)$ is separable, we infer from [5], Theorem III.28, that $L^\infty(\mathbb{R}^n)$ with the weak$\ast$-topology is metrizable. Given a countable and dense subset $(\phi_i)_{i\in\mathbb{N}}$ of $L^1(\mathbb{R}^n)$, a metric d on $L^\infty(\mathbb{R}^n)$ is given by

$$d(u,v) := \sum_{i=1}^\infty \frac{1}{2^i} \frac{|\langle u-v,\phi_i\rangle|}{1+|\langle u-v,\phi_i\rangle|}.$$

It is straight forward to verify that Ω with the metric d is isomorph with $[0,1[^n \times [0,1]^{\mathbb{Z}^n}$. Thus, by Section 2.6 $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$, with $\mathcal{P} = \mathcal{L}|_{[0,1]^n} \otimes \left(\mathcal{L}|_{[0,1]}\right)^{\mathbb{Z}^n}$, satisfies Assumption 2.4.

3.3 Stochastic geometries

In this section, we describe how random measures in the sense of Definition 2.6 can be derived from *random sets*. Let $\mathcal{F}(\mathbb{R}^n)$ denote the set of all closed sets in $\mathbb{R}^n$. We write

$$\mathcal{F}_V := \{F \in \mathcal{F}(\mathbb{R}^n) \ | \ F \cap V \neq \emptyset\} \quad \text{if } V \subset \mathbb{R}^n \quad \text{is an open set}, \tag{3.3}$$

$$\mathcal{F}^K := \{F \in \mathcal{F}(\mathbb{R}^n) \ | \ F \cap K = \emptyset\} \quad \text{if } K \subset \mathbb{R}^n \quad \text{is a compact set}. \tag{3.4}$$

The topology on created by the sets $\mathcal{F}_V$, $\mathcal{F}^K$ is the *Fell-topology* $\mathcal{T}_\mathcal{F}$ and $(\mathcal{F}(\mathbb{R}^n), \mathcal{T}_\mathcal{F})$ is compact, Hausdorff and separable[16]. The *Matheron-σ-field* $\sigma_\mathcal{F}$ is the Borel-σ-algebra created by the Fell-topology.

Definition 3.3. a) Let $(\Omega, \sigma, \mathcal{P})$ be a probability space. Then a *Random Closed Set (RACS)* is a measurable mapping

$$A : (\Omega, \sigma, \mathcal{P}) \longrightarrow (\mathcal{F}, \sigma_\mathcal{F})$$

b) A random closed set is called stationary if its characteristic functions $\chi_{A(\omega)}$ are stationary, i.e. they satisfy $\chi_{A(\omega)}(x) = \chi_{A(\tau_x\omega)}(0)$ for almost every $\omega \in \Omega$ for almost all $x \in \mathbb{R}^n$.

c) A random closed set $M : (\Omega, \sigma, P) \longrightarrow (\mathcal{F}, \sigma_\mathcal{F})$ $\ \omega \mapsto M(\omega)$ is called a *Random closed C^k-Manifold* if $M(\omega)$ is a piecewise C^k-manifold for P almost every ω.

For more information, the reader is referred to [16]. The importance of the concept of random geometries stems from the following Lemma by Zähle. It states that every random closed set induces a random measure. Thus, every stationary ergodic RACS induces a stationary ergodic random measure.

Lemma 3.4 ([30] Theorem 2.1.3 resp. Corollary 2.1.5)**.** *Let $\mathcal{F}_m \subset \mathcal{F}$ be the space of closed m-dimensional sub manifolds of $\mathbb{R}^n$ such that the corresponding Hausdorff measure is locally finite. Then, the σ-algebra $\sigma_{\mathcal{F}} \cap \mathcal{F}_m$ is the smallest such that*

$$M_B : \mathcal{F}_m \to \mathbb{R} \quad M \mapsto \mathcal{H}^m(M \cap B)$$

is measurable for every measurable and bounded $B \subset \mathbb{R}^n$.

This means that

$$M_{\mathbb{R}^n} : \mathcal{F}_m \to \mathcal{M}(\mathbb{R}^n) \quad M \mapsto \mathcal{H}^m(M \cap \cdot)$$

is measurable with respect to the σ-algebra created by the Vague topology on $\mathcal{M}(\mathbb{R}^n)$. Hence a random closed set always induces a random measure. Based on Lemma 3.4 and on Palm-theory, the following usefull result was obtained in [9] (See Lemma 2.14 and Section 3.1 therein).

Theorem 3.5. *Let (Ω, σ, P) be a probability space with an ergodic dynamical system τ. Let $A : (\Omega, \sigma, P) \longrightarrow (\mathcal{F}, \sigma_{\mathcal{F}})$ be a stationary random closed m-dimensional C^k-Manifold.*

a) There exists a separable metric space $\tilde{\Omega}$ with an ergodic dynamical system $\tilde{\tau}$ and a mapping $\tilde{A} : (\tilde{\Omega}, \mathcal{B}_{\tilde{\Omega}}, \mathcal{P}) \to (\mathcal{F}, \sigma_{\mathcal{F}})$ such that A and $\tilde{A}$ have the same law and such that $\tilde{A}$ still is stationary. Furthermore, $(x, \omega) \mapsto \tau_x\omega$ is continuous. We identify $\tilde{\Omega} = \Omega$, $\tilde{A} = A$ and $\tilde{\tau} = \tau$.

b) The mapping

$$\mu_\bullet : \Omega \to \mathcal{M}(\mathbb{R}^n), \quad \omega \mapsto \mu_\omega(\cdot) := \mathcal{H}^m(M \cap \cdot)$$

is a stationary random measure on $\mathbb{R}^n$ and there exists a corresponding Palm-measure $\mu_{\mathcal{P}}$ if and only if $\mu_\bullet$ has finite intensity.

c) There exists a measurable set $\hat{A} \subset \Omega$, called the prototype *of A, such that $\chi_{A(\omega)}(x) = \chi_{\hat{A}}(\tau_x\omega)$ for $\mathcal{L} + \mu_\omega$-almost every x and $\mathcal{P}$-almost surely. The Palm-measure $\mu_{\mathcal{P}}$ of μ_ω concentrates on $\hat{A}$, i.e. $\mu_{\mathcal{P}}(\Omega \backslash \hat{A}) = 0$.*

d) If A is a random closed m-dimensional C^k-manifold, then $\mathcal{P}(\hat{A}) = 0$.

4 Two-scale convergence

4.1 Time independent case

Let Assumption 2.4 hold for $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$ and let $\omega \mapsto \mu_\omega$ be a stationary random measure with μ_ω^ε and $\mu_{\mathcal{P}}$ defined through (2.15) and (2.13). The product σ-algebra $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$ is countably generated and therefore, the space $L^p(\boldsymbol{Q} \times \Omega)$ is separable ([5] Theorem 4.13). In particular, for every $1 \le p < \infty$, there exists a countable dense subset of finite step-functions in $L^p(\boldsymbol{Q} \times \Omega)$.

Remark 4.1. For $1 \le p < \infty$ let $\Phi_p := (\phi_i)_{i \in \mathbb{N}}$ be a countable dense subset of $L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}}))$ such that every $\phi \in \Phi_p$ is a finite $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$ -step-function. By Lemma 2.11, every $\phi \in \Phi_p$ is an ergodic function. Since the countable union of $\mathcal{P}$-null-sets is a $\mathcal{P}$-null set, there exists a set $\Omega_{\Phi_p} \subset \Omega$ with $\mathcal{P}(\Omega_{\Phi_p}) = 1$ such that all $\phi \in \Phi_p$ satisfy (2.17) (i.e. they admit ergodic realizations) for all $\omega \in \Omega_{\Phi_p}$.

The choice of the family Φ_p is closely related to Allaire's problem [3] of identifying the class of "admissible" functions in $L^2(\boldsymbol{Q}\times[0,1]^n)$. Note that even in the periodic setting, given $\phi\in L^2(\boldsymbol{Q}\times Y)$, it is by no means clear whether

$$\lim_{\varepsilon\to0}\int_{\boldsymbol{Q}}\phi^2(x,\frac{x}{\varepsilon})\,dx=\int_{\boldsymbol{Q}}\int_Y\phi^2(x,y)\,dy\,dx\,.$$

Indeed, it is not even clear, whether $\phi(x,\frac{x}{\varepsilon})$ is measurable (See [3] the discussion after Definition 1.4 and Proposition 5.8).

Definition 4.2. Let $1<p,q<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. Let Φ_p be the set of Remark 4.1 and let $\omega\in\Omega_{\Phi_p}$. Let $u^\varepsilon\in L^q(\boldsymbol{Q};\mu_\omega^\varepsilon)$ for all $\varepsilon>0$. We say that (u^ε) converges (weakly) in two scales to $u\in L^q(\boldsymbol{Q};L^q(\Omega,\mu_{\mathcal{P}}))$ and write $u^\varepsilon\stackrel{2s}{\rightharpoonup}u$ if for all $\phi\in\Phi_p$ there holds with $\phi_{\omega,\varepsilon}(x):=\phi(x,\tau_{\frac{x}{\varepsilon}}\omega)$

$$\lim_{\varepsilon\to0}\int_{\boldsymbol{Q}}u^\varepsilon\phi_{\omega,\varepsilon}d\mu_\omega^\varepsilon=\int_{\boldsymbol{Q}}\int_\Omega u\phi\,d\mu_{\mathcal{P}}\,d\mathcal{L}\,.$$

Furthermore, we say that u^ε converges strongly in two scales to u, written $u^\varepsilon\stackrel{2s}{\to}u$, if for all weakly two-scale converging sequences $v^\varepsilon\in L^p(\boldsymbol{Q};\mu_\omega^\varepsilon)$ with $v^\varepsilon\stackrel{2s}{\rightharpoonup}v\in L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}}))$ as $\varepsilon\to0$ there holds

$$\lim_{\varepsilon\to0}\int_{\boldsymbol{Q}}u^\varepsilon v^\varepsilon d\mu_\omega^\varepsilon=\int_{\boldsymbol{Q}}\int_\Omega uv\,d\mu_{\mathcal{P}}\,d\mathcal{L}\,.$$

Remark 4.3. a) Note that $\phi_{\omega,\varepsilon}\stackrel{2s}{\to}\phi$ strongly in two scales by definition.

b) If $f\in L^p(\boldsymbol{Q}\times\Omega;\mathcal{L}\otimes\mu_{\mathcal{P}})$ is a p-ergodic function and f_ω is an ergodic realization of f and all $(f\varphi)_\omega$ are ergodic realizations of $f\varphi$, $\varphi\in\Phi_p$, then

$$\lim_{\varepsilon\to0}\int_{\boldsymbol{Q}}u^\varepsilon f_{\omega,\varepsilon}d\mu_\omega^\varepsilon=\int_{\boldsymbol{Q}}\int_\Omega uf\,d\mu_{\mathcal{P}}\,d\mathcal{L}$$

for all $u^\varepsilon\stackrel{2s}{\rightharpoonup}u$. This means we can always extend our class of test-functions by countably many functions, losing only a set of Ω with $\mathcal{P}$-measure 0.

The definition of strong two-scale convergence makes sense in view of classical strong convergence. The proof of part 1. is very similar to [33]. Part 2. is easy to prove.

Lemma 4.4. *1. Let $\omega\in\Omega$, $1<p<\infty$ and $u^\varepsilon\in L^p(\boldsymbol{Q};\mu_\omega^\varepsilon)$ be a sequence of functions such that $\|u^\varepsilon\|_{L^p(\boldsymbol{Q};\mu_\omega^\varepsilon)}\le C$ for some $C>0$ independent of ε. Then there exists a subsequence of $(u^{\varepsilon'})_{\varepsilon'\to0}$ and $u\in L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}}))$ such that $u^{\varepsilon'}\stackrel{2s}{\rightharpoonup}u$ and*

$$\|u\|_{L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}}))}\le\liminf_{\varepsilon'\to0}\left\|u^{\varepsilon'}\right\|_{L^p(\boldsymbol{Q};\mu_\omega^{\varepsilon'})}\,.\qquad(4.1)$$

2. Let $\mu_\omega^\varepsilon=\mathcal{L}$ for all $\varepsilon>0$ and let $u^\varepsilon\in L^p(\boldsymbol{Q})$ such that $u^\varepsilon\to u\in L^p(\boldsymbol{Q})$ strongly. Then $u^\varepsilon\stackrel{2s}{\to}u$.

Proof. 1. Let $(\phi_k)_{k\in\mathbb{N}}$ be an enumeration of Φ_q. For fixed $k\in\mathbb{N}$, we obtain that

$$\begin{aligned}\limsup_{\varepsilon\to0}\left|\int_Q u^\varepsilon(x)\phi_k(x,\tau_{\frac{x}{\varepsilon}}\omega)d\mu_\omega^\varepsilon(x)\right|&\le C\limsup_{\varepsilon\to0}\left(\int_Q\left(\phi_k(x,\tau_{\frac{x}{\varepsilon}}\omega)\right)^q d\mu_\omega^\varepsilon(x)\right)^{\frac{1}{q}}\\&=C\left\|\phi_k\right\|_{L^q(\boldsymbol{Q}\times\Omega;\mathcal{L}\otimes\mu_{\mathcal{P}})}\,.\end{aligned}$$

Therefore, we can use Cantor's diagonalization argument to construct a subsequence $u^{\varepsilon'}$ of u^ε such that

$$\int_Q u^{\varepsilon'}(x)\phi_k(x,\tau_{\frac{x}{\varepsilon}}\omega)d\mu_\omega^{\varepsilon'}(x) \to L_k \quad \text{as } \varepsilon' \to 0$$

and L_k is linear in $\phi_k \in \Phi_q$. Therefore, there exists $u \in L^p(Q; L^p(\Omega;\mu_{\mathcal{P}}))$ such that

$$L_{k,j} = \int_Q \int_\Omega u(x,\tilde{\omega})\phi_k(x,\tilde{\omega})d\mu_{\mathcal{P}}(\tilde{\omega})dx \qquad \forall k \in \mathbb{N}.$$

Since Φ_q is dense in $L^q(Q; L^q(\Omega;\mu_{\mathcal{P}}))$, the function u is unique.

2. This follows from the fact that every weakly converging sequence $v^\varepsilon \in L^2(\boldsymbol{Q})$ with $v^\varepsilon \rightharpoonup v$ converges weakly in two scales to the same function v. □

Lemma 4.5. *Let $u \in L^p(\boldsymbol{Q}; L^p(\Omega;\mu_{\mathcal{P}}))$. Then, for almost every $\omega \in \Omega$, there exists a sequence $u^\varepsilon \in L^p(\boldsymbol{Q};\mu_\omega^\varepsilon)$ such that $u^\varepsilon \overset{2s}{\rightharpoonup} u$ as $\varepsilon \to 0$.*

A similar result by Allaire (see [3]) states that every $u \in L^2(\boldsymbol{Q}; L^2([0,1[^n))$ is obtained as a (periodic) two-scale limit.

Proof. Let $u \in L^p(\boldsymbol{Q}; L^p(\Omega;\mu_{\mathcal{P}}))$. For p and $q = \frac{p-1}{p}$ let Φ_p and Φ_q be the family of functions with Ω_{Φ_p} and Ω_{Φ_q} from Remark 4.1. For $\tilde{\Omega} := \Omega_{\Phi_p} \cap \Omega_{\Phi_q}$ and $\omega \in \tilde{\Omega}$, we create the sequence u^ε by the following algorithm.

1. Chose $u_0 \in \Phi_p$ and $\varepsilon_0 > 0$ arbitrarily.
2. Let $n \in \mathbb{N}$ and assume u_{n-1} is chosen. There exists $u_n \in \Phi_p$ with $\|u - u_n\|_{\mathscr{B}} \le \frac{1}{n}$, and $\varepsilon_n > 0$ with $\varepsilon_n \le \varepsilon_{n-1}$ such that for all $\phi_j \in \Phi_q$ with $1 \le j \le n$ there holds

$$\left|\int_{\boldsymbol{Q}} u_n(\tau_{\frac{x}{\varepsilon}}\omega)\phi_j(\tau_{\frac{x}{\varepsilon}}\omega)d\mu_\omega^\varepsilon(x) - \int_\Omega u_n\phi_j d\mu_{\mathcal{P}}\right| \le \frac{1}{n} \qquad \forall \varepsilon < \varepsilon_n\,.$$

3. Finally, set $u^\varepsilon(x) = u_{n-1}(\tau_{\frac{x}{\varepsilon}}\omega)$ for $\varepsilon_{n-1} \ge \varepsilon > \varepsilon_n$.
4. Continue with 2.

The constructed sequence u^ε has the property that $u^\varepsilon \overset{2s}{\rightharpoonup} u$. □

Lemma 4.6. *Let $N \in \mathbb{N}$ and let $A \in L^\infty(\boldsymbol{Q}; L^\infty(\Omega; \mathscr{L}(\mathbb{R}^N,\mathbb{R}^N)))$ be symmetric and assume A is $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$ -measurable. We furthermore assume the existence of a constant $\alpha > 0$ such that*

$$\alpha|\xi|^2 \le \xi A(x,\omega)\xi \le \frac{1}{\alpha}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n \text{ and for } \mathcal{L}\times\mu_{\mathcal{P}}\text{-a.e. } (x,\omega) \in \boldsymbol{Q}\times\Omega\,. \tag{4.2}$$

Then, for almost all $\omega \in \Omega$ there holds: For all sequences $u^\varepsilon \in L^2(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^N)$ with weak two-scale limit $u \in L^2\boldsymbol{Q} \times L^2(\Omega;\mu_{\mathcal{P}};\mathbb{R}^N))$ there holds with $A_{\varepsilon,\omega}(x) := A(x,\tau_{\frac{s}{x}}\omega)$

$$\liminf_{\varepsilon\to 0} \int_{\boldsymbol{Q}} u^\varepsilon \cdot (A_{\varepsilon,\omega}u^\varepsilon)\, d\mu_\omega^\varepsilon \ge \int_{\boldsymbol{Q}}\int_\Omega u\cdot(Au)\, d\mu_{\mathcal{P}}\,.$$

Proof. Let Φ_2 be the family of functions with Ω_{Φ_2} from Remark 4.1. Since A is symmetric and strictly positive definite, there exists $A^{\frac{1}{2}} \in L^\infty(\boldsymbol{Q}; L^\infty(\Omega; \mathscr{L}(\mathbb{R}^N, \mathbb{R}^N)))$ such that $A^{\frac{1}{2}T} A^{\frac{1}{2}} = A$ and such that A is $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$ -measurable. Then, $A^{\frac{1}{2}}\Phi_2$ is a family of ergodic functions, i.e.

$$\lim_{\varepsilon\to 0} \int_{\boldsymbol{Q}} u^\varepsilon \left(A^{\frac{1}{2}}\phi\right)_{\omega,\varepsilon} d\mu_\omega^\varepsilon = \int_{\boldsymbol{Q}} \int_\Omega u A^{\frac{1}{2}} \phi \, d\mu_{\mathcal{P}} \, d\mathcal{L} \qquad \forall \phi \in \Phi_2$$

by Remark 4.3. Thus, there exists $\tilde{\Omega} \subset \Omega$ with $\mathcal{P}(\tilde{\Omega}) = 1$ such that $u^\varepsilon \overset{2s}{\rightharpoonup} u$ implies $A^{\frac{1}{2}}_{\varepsilon,\omega} u^\varepsilon \overset{2s}{\rightharpoonup} A^{\frac{1}{2}} u$ for all $\omega \in \tilde{\Omega}$. Using (4.1) from Lemma 4.4, this concludes the proof. □

4.2 Weak two-scale convergence for time-dependent functions

We are also interested in the convergence behavior of functions $u^\varepsilon : [0,T] \to L^p(\boldsymbol{Q}, \mu_\omega^\varepsilon)$. More precisely, we make the following definition:

Definition 4.7. Let $1 < r, r', p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r'} + \frac{1}{r} = 1$. Let Φ_q be the set of Remark 4.1 and let $\omega \in \Omega_{\Phi_q}$. Let $u^\varepsilon \in L^r(0,T; L^p(\boldsymbol{Q}; \mu_\omega^\varepsilon))$ for all $\varepsilon > 0$. We say that (u^ε) converges (weakly) in two scales to $u \in L^r(0,T; L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}})))$ and write $u^\varepsilon \overset{2s}{\rightharpoonup} u$ if for all continuous and piecewise affine functions $\phi : [0,T] \to \mathbb{R}\Phi_q$ there holds with $\phi_{\omega,\varepsilon}(t,x) := \phi(t, x, \tau_{\frac{x}{\varepsilon}}\omega)$

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\boldsymbol{Q}} u^\varepsilon \phi_{\omega,\varepsilon} d\mu_\omega^\varepsilon dt = \int_0^T \int_{\boldsymbol{Q}} \int_\Omega u\phi \, d\mu_{\mathcal{P}} \, dx \, .dt$$

Notation 4.8. Throuout this subsection, we frequently write $\mathscr{B}_0 = L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}}))$ with $\mathscr{B}_0^* = L^q(\boldsymbol{Q}; L^q(\Omega, \mu_{\mathcal{P}}))$ and $\mathscr{B}_{\varepsilon,\omega} = L^p(\boldsymbol{Q}; \mu_\omega^\varepsilon)$ with $\mathscr{B}_{\varepsilon,\omega}^* = L^q(\boldsymbol{Q}; \mu_\omega^\varepsilon)$. We denote by $\langle u, v\rangle_{\mathscr{B},\mathscr{B}^*}$and $\langle u^\varepsilon, v^\varepsilon\rangle_{\mathscr{B}_\varepsilon, \mathscr{B}_\varepsilon^*,\omega}$ the corresponding dual pairings and by $\left(\mathfrak{B}_{\varepsilon,\omega}^* \phi\right)(x) := \phi(x, \tau_{\frac{x}{\varepsilon}}\omega)$ the natural mapping from $\mathscr{B}_0$ to $\mathscr{B}_{\varepsilon,\omega}$. If no confusion occurs, we drop the index ω.

Lemma 4.9. *Let $1 < p, r < \infty$ and $T > 0$. Then, every sequence $(u^\varepsilon \in L^r(0,T; L^p(\boldsymbol{Q}; \mu_\omega^\varepsilon)))_{\varepsilon>0}$ satisfying $\|u^\varepsilon\|_{L^r(0,T;L^p(\boldsymbol{Q};\mu_\omega^\varepsilon))} \le C$ for some $C > 0$ independent from ε has a weakly two-scale convergent subsequence with limit function $u \in L^r(0,T; L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}})))$. Furthermore, if $\|\partial_t u^\varepsilon\|_{L^r(0,T;L^p(\boldsymbol{Q};\mu_\omega^\varepsilon))} \le C$ uniformly for $1 < p \le \infty$, then also $\|\partial_t u\|_{L^r(0,T;L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}})))} \le C$ and $\partial_t u^\varepsilon \overset{2s}{\rightharpoonup} \partial_t u$ in the sense of Definition 4.7 a) as well as $u^\varepsilon(t) \overset{2s}{\rightharpoonup} u(t)$ for all $t \in [0,T]$.*

Proof. We make use of the notation 4.8. We may assume that Φ_q is a $\mathbb{Q}$-vectorspace. Given $T > 0$, we fix the timesteps $\tau_{i,k} = \frac{k}{2^i}T$ for $i \in \mathbb{N}$ and $0 \le k \le 2^i$. Then, $L^{r'}(0,T; \mathscr{B}_0^*)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) has a countable dense subset of piecewise constant functions of the form

$$\phi = \sum_{k=0}^{2^i-1} \chi_{[\tau_{i,k}, \tau_{i,k+1})} v_k \quad \text{for some } i \in \mathbb{N} \text{ and some } (v_k)_{k=0,\dots,2^i-1} \subset \mathscr{B}_0^* \,. \tag{4.3}$$

We set $U_{i,k}^\varepsilon = \int_{\tau_{i,k}}^{\tau_{i,k+1}} u^\varepsilon$ and observe that $U_{i,k}^\varepsilon = U_{i+1,2k}^\varepsilon + U_{i+1,2k+1}^\varepsilon$. By induction over $i \in \mathbb{N}$ and the Cantor-argument, we can extract a subsequense of u^ε as $\varepsilon \to 0$ such that $U_{i,k}^\varepsilon \overset{2s}{\rightharpoonup} U_{i,k}$ for all $i \in \mathbb{N}$, $k = 0, \dots, 2^i - 1$. This sequence then satisfies $U_{i,k} = U_{i+1,2k} + U_{i+1,2k+1}$. For every ϕ of the form (4.3) we find

$$\int_0^T \langle u^\varepsilon, \mathfrak{B}_\varepsilon^* \phi\rangle_{\mathscr{B}_\varepsilon, \mathscr{B}_\varepsilon^*} \to l(\phi) := \sum_{k=0}^{2^i-1} \langle U_{i,k}, v_k\rangle_{\mathscr{B},\mathscr{B}^*} \,, \tag{4.4}$$

and l is linear. Furthermore, since $\int_0^T \langle u^\varepsilon, \mathfrak{B}_\varepsilon^* \phi\rangle_{\mathscr{B}_\varepsilon,\mathscr{B}_\varepsilon^*} \le C \|\mathfrak{B}_\varepsilon^*\phi\|_{L^{r'}(0,T;\mathscr{B}_\varepsilon^*)}$, we infer from the ergodic theorem that $|l(\phi)| \le C\|\phi\|_{L^{r'}(0,T;\mathscr{B}^*)}$. Since l is linear, it can be extended to a bounded linear functional on $L^{r'}(0,T;\mathscr{B}^*)$ by the Hahn-Banach extension theorem. Thus, there exists $u \in L^r(0,T;\mathscr{B}_0)$, such that $l(\phi) = \int_0^T \langle u,\phi\rangle_{\mathscr{B},\mathscr{B}^*}$. Since the set of functions ϕ having the form (4.3) is dense in $L^{r'}(0,T;\mathscr{B}_0^*)$, this u is unique and we conclude $u^\varepsilon \overset{2s}{\rightharpoonup} u$. Finally, we can approximate any piecewise affine and continuous function $\phi : [0,T] \to \mathbb{R}\Phi_p$ uniformly by piecewise constant functions. Therefore, we get

$$\int_0^T \langle u^\varepsilon, \mathfrak{B}_\varepsilon^*\phi\rangle_{\mathscr{B}_\varepsilon,\mathscr{B}_\varepsilon^*} \to l(\phi)$$

for all such piecewise affine functions ϕ.

Now, let $\|\partial_t u^\varepsilon\|_{L^r(0,T;\mathscr{B}_\varepsilon)} \le C$ uniformly for $1 < p \le \infty$. Then, also $\|u^\varepsilon\|_{C([0,T];\mathscr{B}_\varepsilon)} \le C$ uniformly. By the above calculations, there exists $u_t \in L^r(0,T;\mathscr{B})$ and a further subsequence of u^ε, such that $\partial_t u^\varepsilon \overset{2s}{\rightharpoonup} u_t$, $u^\varepsilon(0) \overset{2s}{\rightharpoonup} u(0)$ and $u^\varepsilon(T) \overset{2s}{\rightharpoonup} u(T)$ as $\varepsilon \to 0$. Chosing an arbitrary piecewise affine and continuous function $\phi : [0,T] \to \mathbb{R}\Phi_p$, we obtain

$$\begin{aligned}\int_0^T \langle u_t,\phi\rangle_{\mathscr{B},\mathscr{B}^*} \leftarrow \int_0^T \langle \partial_t u^\varepsilon, \mathfrak{B}_\varepsilon^*\phi\rangle_{\mathscr{B}_\varepsilon,\mathscr{B}_\varepsilon^*} &= \langle u^\varepsilon, \mathfrak{B}_\varepsilon^*\phi\rangle_{\mathscr{B}_\varepsilon,\mathscr{B}_\varepsilon^*}\Big|_0^T - \int_0^T \langle u^\varepsilon, \mathfrak{B}_\varepsilon^*\partial_t\phi\rangle_{\mathscr{B}_\varepsilon,\mathscr{B}_\varepsilon^*}\\ &\to \langle u,\phi\rangle_{\mathscr{B},\mathscr{B}^*}\Big|_0^T - \int_0^T \langle u,\partial_t\phi\rangle_{\mathscr{B},\mathscr{B}^*}\,,\end{aligned}$$

where we used (4.4) for $u^\varepsilon(\cdot)$. Thus, we find that $\partial_t u = u_t$.

Finally, we get for every $\tau \in (0,T]$:

$$u^\varepsilon(\tau) = u^\varepsilon(0) + \int_0^\tau \partial_t u^\varepsilon \overset{2s}{\rightharpoonup} u(0) + \int_0^\tau \partial_t u = u(\tau)\,.$$

□

Lemma 4.10. *Let $1 < p < \infty$ and let ϕ_q and Ω_{ϕ_q}, $\frac{1}{p} + \frac{1}{q} = 1$, be given by Remark 4.1. Given $\omega \in \Omega_{\Phi_q}$, let $u^\varepsilon \in C^{Lip}(0,T;L^p(\boldsymbol{Q};\mu_\omega^\varepsilon))$ for all $\varepsilon > 0$ such that $\|u^\varepsilon\|_{C^{Lip}(0,T;L^p(\boldsymbol{Q};\mu_\omega^\varepsilon))} \le C$ for some C independent from $\varepsilon > 0$. Then, there exists $u \in C^{Lip}(0,T;L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}})))$ and a subsequence $u^{\varepsilon'}$ of u^ε such that $u^{\varepsilon'}(t) \overset{2s}{\rightharpoonup} u(t)$ for all $t \in [0,T]$.*

Proof. By the uniform Lipschitz bound, we find that $\|\partial_t u^\varepsilon\|_{L^p(0,T;L^p(\boldsymbol{Q};\mu_\omega^\varepsilon))} \le C$ for all $\varepsilon > 0$. By Lemmas 4.9 and 4.4, we obtain a subsequence $u^{\varepsilon'}$ and $u \in C(0,T;L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}})))$ such that $u^{\varepsilon'}(t) \overset{2s}{\rightharpoonup} u(t)$ for all $t \in [0,T]$. We observe

$$\|u(t_1) - u(t_2)\|_{L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}}))} \le \sup_{\varepsilon>0} \|u^\varepsilon(t_1) - u^\varepsilon(t_2)\|_{L^p(\boldsymbol{Q};\mu_\omega^\varepsilon)} \le C|t_1 - t_2|\,,$$

and therefore $u \in C^{Lip}([0,T];L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}})))$. □

5 Weakly two-scale converging convex sets

5.1 The main result on weakly converging convex sets

Let $\boldsymbol{Q} \subset \mathbb{R}^n$ be a bounded domain with the Borel-σ-algebra $\mathcal{B}_Q$ and let Assumption 2.4 hold for $(\Omega,\mathcal{B}_\Omega,\mathcal{P},\tau)$ and let $\omega \mapsto \mu_\omega$ be a stationary random measure with μ_ω^ε and $\mu_{\mathcal{P}}$

defined through (2.15) and (2.13). Let furthermore $D \in \mathbb{N}$. We provide a class of convex sets $\mathscr{C}_p^\varepsilon \subset L^p(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D)$ with a convex set $\mathscr{C}_p \subset L^p(\boldsymbol{Q};L^p(\Omega;\mu_{\mathcal{P}};\mathbb{R}^D))$ such that $\mathscr{C}_p^\varepsilon \stackrel{2s}{\rightharpoonup} \mathscr{C}_p$ in the following sense.

Definition 5.1. Let $1 < p < \infty$. For each $\varepsilon > 0$ let $\mathscr{C}_p^\varepsilon \subset L^p(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D)$ be a closed and convex set and let $\mathscr{C}_p \subset L^p(\boldsymbol{Q};L^p(\Omega;\mu_{\mathcal{P}};\mathbb{R}^D))$ be closed and convex. We say that $\mathscr{C}_p^\varepsilon$ weakly two-scale converges to $\mathscr{C}_p$, written $\mathscr{C}_p^\varepsilon \stackrel{2s}{\rightharpoonup} \mathscr{C}_p$, if for every weakly two-scale converging sequence $\left(u^\varepsilon \in \mathscr{C}_p^\varepsilon\right)_{\varepsilon>0}$ with $u^\varepsilon \stackrel{2s}{\rightharpoonup} u \in L^p(\boldsymbol{Q};L^p(\Omega;\mu_{\mathcal{P}};\mathbb{R}^D))$ there follows $u \in \mathscr{C}_p$.

As a special case, we introduce weak convergence of convex sets.

Definition 5.2. Let $1 < p < \infty$. For each $\varepsilon > 0$ let $\mathscr{C}_p^\varepsilon \subset L^p(\boldsymbol{Q};\mathbb{R}^D)$ be a closed and convex set and let $\mathscr{C}_p \subset L^p(\boldsymbol{Q};\mathbb{R}^D)$ be closed and convex. We say that $\mathscr{C}_p^\varepsilon$ weakly converges to $\mathscr{C}_p$, written $\mathscr{C}_p^\varepsilon \rightharpoonup \mathscr{C}_p$, if for every weakly converging sequence $\left(u^\varepsilon \in \mathscr{C}_p^\varepsilon\right)_{\varepsilon>0}$ with $u^\varepsilon \rightharpoonup u \in L^p(\boldsymbol{Q};\mathbb{R}^D)$ there follows $u \in \mathscr{C}_p$.

We will prove a weak two-scale convergence result for convex sets within the following setting.

Assumption 5.3. *For $\mathcal{L}\times\mu_{\mathcal{P}}$-almost every $(x,\omega) \in \boldsymbol{Q}\times\Omega$ let $\mathcal{C}(x,\omega) \subset \mathbb{R}^D$ be a closed convex set in $\mathbb{R}^D$ with $0 \in \mathcal{C}(x,\omega)$ for all x,ω and define*

$$\mathcal{X}_{\mathcal{C}} : \boldsymbol{Q}\times\Omega\times\mathbb{R}^D \to \mathbb{R}\,, \quad (x,\omega,u) \mapsto \operatorname{dist}\left(u,\mathcal{C}(x,\omega)\right)\,.$$

For all $u \in \mathbb{R}^D$, the function $\mathcal{X}_{\mathcal{C}}(\cdot,\cdot,u)$ is $\mathcal{B}_Q\otimes\mathcal{B}_\Omega$-measurable on $\boldsymbol{Q}\times\Omega$.

Assumption 5.4. *Let Assumption 5.3 hold. We additionaly assume there are functions $O : \boldsymbol{Q}\times\Omega \to \mathbb{R}^D$ and $r : \boldsymbol{Q}\times\Omega \to \mathbb{R}_+$ such that $B_{r(x,\omega)}(O(x,\omega)) \subset \mathcal{C}(x,\omega)$ for $\mathcal{L}\times\mu_{\mathcal{P}}$-almost every $(x,\omega) \in \boldsymbol{Q}\times\Omega$. Assume further that for all finite step-functions φ, the product $O\varphi$ is a 1-ergodic function. We furthermore make the following regularity assumptions: The function $R : (x,\omega) \mapsto r(x,\omega)^{-1}$ is $\mathcal{B}_Q\otimes\mathcal{B}_\Omega$-measurable and an element of $L^q(\boldsymbol{Q};L^q(\Omega,\mu_{\mathcal{P}}))$ for $\frac{1}{p}+\frac{1}{q}=1$.*

Introducing the sets

$$\begin{aligned}\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega) &:= \left\{u \in L^p(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D) \,:\, u(x) \in \mathcal{C}(x,\tau_{\frac{x}{\varepsilon}}\omega) \text{ for } \mu_\omega^\varepsilon\text{-a.e. } x \in \boldsymbol{Q}\right\}, \qquad \text{for } \omega\in\Omega\,,\\ \mathscr{C}_p(\boldsymbol{Q}\times\Omega) &:= \left\{u \in L^p(\boldsymbol{Q};L^p(\Omega;\mu_{\mathcal{P}};\mathbb{R}^D)) \,:\, u(x,\omega) \in \mathcal{C}(x,\omega) \text{ for } \mathcal{L}\times\mu_{\mathcal{P}}\text{-a.e. } (x,\omega)\right\},\end{aligned} \tag{5.1}$$

we note that these sets are closed in the respective Banach spaces. Given a convex set $\mathcal{C} \subset \mathbb{R}^D$ containing an open ball around $B_r(0)$ for some $r > 0$, the vector

$$a(u,\mathcal{C}) := \operatorname{argmin}\left\{\operatorname{dist}(a,u) \,:\, a \in \mathcal{C}\right\} \tag{5.2}$$

is uniquely defined for all $u \in \mathbb{R}^D$. We first observe the following:

Lemma 5.5. *Let Assumption 5.3 hold and let $a(u,\mathcal{C})$ be defined through (5.2). Let $\varphi \in L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}}))$ be a finite step function and let $a(\varphi)(x,\omega) = a(\varphi(x,\omega),\mathcal{C}(x,\omega))$. Then, for all $1 \le p < \infty$ it holds that $a(\varphi) \in \mathscr{C}_p(\boldsymbol{Q}\times\Omega)$ and admits a $\mathcal{B}_Q\otimes\mathcal{B}_\Omega$-measurable and p-ergodic representative. The set $\Phi_p^{\mathcal{C}} := a(\Phi_p)$ is dense in $\mathscr{C}_p(\boldsymbol{Q}\times\Omega)$ and all $\phi \in \Phi_p^{\mathcal{C}}$ are p-ergodic.*

We shift the proof of Lemma 5.5 to Section 5.4. The main result of this section is the following theorem, which is proved in Section 5.3.

Theorem 5.6. *Let Assumption 5.4 hold and let* $\mathscr{C}_p(\boldsymbol{Q}\times\Omega)$ *and* $\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)$ *be given through (5.1). Then, there exists a set* $\tilde\Omega\subset\Omega$ *with* $\mathcal{P}(\tilde\Omega)=1$ *such that* $\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)\overset{2s}{\rightharpoonup}\mathscr{C}_p(\boldsymbol{Q}\times\Omega)$ *for all* $\omega\in\tilde\Omega$.

Corollary 5.7. *Let Assumption 5.4 hold, let* $\mu_\omega=\mathcal{L}$ *for all* $\omega\in\Omega$ *and let* $\mathscr{C}_p(\boldsymbol{Q}\times\Omega)$ *and* $\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)$ *be given through (5.1). Let*

$$\mathscr{C}_p(\boldsymbol{Q}):=\mathrm{cl}_{L_2(\boldsymbol{Q})}\left\{u\in L^p(\boldsymbol{Q})\,:\,\exists v\in\mathscr{C}_p(\boldsymbol{Q}\times\Omega)\,:\,u=\int_\Omega v\,d\mu\right\}. \tag{5.3}$$

Then there exists a set $\tilde\Omega\subset\Omega$ *with* $\mathcal{P}(\tilde\Omega)=1$ *such that* $\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)\rightharpoonup\mathscr{C}_p(\boldsymbol{Q})$ *in the sense of Definition 5.2 for all* $\omega\in\tilde\Omega$.

Proof. Note that $\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)\overset{2s}{\rightharpoonup}\mathscr{C}_p(\boldsymbol{Q}\times\Omega)$ according to Theorem 5.6. Furthermore, $u^\varepsilon\overset{2s}{\rightharpoonup}\tilde u$ weakly in two scales implies $u^\varepsilon\rightharpoonup u:=\int_\Omega\tilde u\,d\mu$ weakly in $L^p(\boldsymbol{Q};\mathbb{R}^D)$ with $u\in\mathscr{C}_p(\boldsymbol{Q})$. □

5.2 Convergence of 1-homogeneous functionals

Given a reflexive Banach spaces $\mathscr{B}$, the closed convex subsets containing 0 can be identified with 1-homogeneous functionals on the dual space. We make the following two observations:

Lemma 5.8. *Let Assumption 5.3 hold and let* $\mathscr{C}_q(\boldsymbol{Q}\times\Omega)$ *and* $\mathscr{C}_q^\varepsilon(\boldsymbol{Q},\omega)$ *be given through (5.1). Let* $\frac1p+\frac1q=1$ *and* $\Psi_\varepsilon:L^p(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D)\to\mathbb{R}$ *and* $\Psi:L^p(\boldsymbol{Q}\times\Omega;\mathcal{L}\times\mu_{\mathcal{P}};\mathbb{R}^D)\to\mathbb{R}$ *be given through*

$$\Psi_{\varepsilon,\omega}(u):=\sup_{\sigma\in\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)}\int_{\boldsymbol{Q}}u\cdot\sigma d\mu_\omega^\varepsilon\,,\qquad\Psi(u):=\sup_{\sigma\in\mathscr{C}_p(\boldsymbol{Q}\times\Omega)}\int_{\boldsymbol{Q}}\int_\Omega u\cdot\sigma d\mu_{\mathcal{P}}d\mathcal{L}\,. \tag{5.4}$$

Then, for every sequence $u^\varepsilon\in L^p(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D)$ *such that* $u^\varepsilon\overset{2s}{\rightharpoonup}u\in L^p(\boldsymbol{Q};L^p(\Omega;\mu_{\mathcal{P}};\mathbb{R}^D))$ *there holds*

$$\liminf_{\varepsilon\to0}\Psi_{\varepsilon,\omega}(u^\varepsilon)\geq\Psi(u)\,.$$

Furthermore, if Assumption 5.4 holds and if $u^\varepsilon\overset{2s}{\to}u\in L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}};\mathbb{R}^D))$ *strongly in two scales, we find*

$$\lim_{\varepsilon\to0}\Psi_{\varepsilon,\omega}(u^\varepsilon)=\Psi(u)\,.$$

The proof is similar to the proof of the following more general, time dependent result.

Lemma 5.9. *Let Assumption 5.3 hold and let* $\mathscr{C}_q(\boldsymbol{Q}\times\Omega)$ *and* $\mathscr{C}_q^\varepsilon(\boldsymbol{Q},\omega)$ *be given through (5.1). Let* $\frac1p+\frac1q=1$, $1<p,q<\infty$, *and* $\Psi_\varepsilon:L^p(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D)\to\mathbb{R}$ *and* $\Psi:L^p(\boldsymbol{Q}\times\Omega;\mathcal{L}\times\mu_{\mathcal{P}};\mathbb{R}^D)\to\mathbb{R}$ *be given through (5.4). Then, for every sequence* $a^\varepsilon\in L^p(0,T;L^p(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D))$ *such that* $a^\varepsilon\overset{2s}{\rightharpoonup}a\in L^p(0,T;L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}};\mathbb{R}^D)))$ *there holds*

$$\liminf_{\varepsilon\to0}\int_0^T\Psi_\varepsilon(a^\varepsilon)\geq\int_0^T\Psi(a)\,.$$

Furthermore, if Assumption 5.4 holds and $a^\varepsilon \stackrel{2s}{\to} a \in L^p(0,T;L^p(\boldsymbol{Q};L^p(\Omega,\mu_{\mathcal{P}})))$ strongly in two scales, we find

$$\lim_{\varepsilon\to 0}\int_0^T \Psi_{\varepsilon,\omega}(a^\varepsilon) = \int_0^T \Psi(a)\,.$$

Proof. We write $\mathscr{C} := \mathscr{C}_q(\boldsymbol{Q}\times\Omega)$ and $\mathscr{C}_\varepsilon := \mathscr{C}_q^\varepsilon(\boldsymbol{Q},\omega)$. Let Φ_q be the countable dense subset of $L^q(\boldsymbol{Q};L^q(\Omega,\mu_{\mathcal{P}}))$ announced in Remark 4.1. By Lemma 5.5, we may assume that $\Phi_q\cap\mathscr{C}$ is dense in $\mathscr{C}$. We denote by $PL(0,T;\Phi_q,\mathscr{C})$ the piecewise linear functions over $(0,T)$ with values in $(\mathbb{R}\Phi_q)\cap\mathscr{C}$. By density of $PL(0,T;\Phi_q,\mathscr{C})$ in $L^p(0,T;\mathscr{C})$ and

$$\int_0^T \Psi(a) = \sup_{v\in L^q(0,T;\mathscr{C})}\int_0^T\int_{\boldsymbol{Q}}\int_\Omega av\,d\mu_{\mathcal{P}}d\mathcal{L}\,,$$

we choose for every $\delta>0$ a function $v_\delta\in PL(0,T;\Phi_q,\mathscr{C})$ such that $\int_0^T\Psi(a) < \int_0^T\langle a,v_\delta\rangle_{\mathscr{B},\mathscr{B}^*}+\delta$. Since v_δ is piecewise linear with values in $\mathbb{R}\Phi_q$, we find that $v_\delta^\varepsilon(t,x) := v_\delta(t,x,\tau_{\frac{x}{\varepsilon}}\omega)$ satisfies $v_\delta^\varepsilon\in L^q(0,T;\mathscr{C}_\varepsilon)$ and

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_0^T\Psi_\varepsilon(a^\varepsilon) &= \liminf_{\varepsilon\to 0}\left(\sup_{v\in L^q(0,T;\mathscr{C}_\varepsilon)}\int_0^T\int_{\boldsymbol{Q}} a^\varepsilon v\,d\mu_\omega^\varepsilon\right)\\ &\geq \liminf_{\varepsilon\to 0}\int_0^T\int_{\boldsymbol{Q}} a^\varepsilon v_\delta^\varepsilon\,d\mu_\omega^\varepsilon\\ &= \int_0^T\int_{\boldsymbol{Q}}\int_\Omega av_\delta\,d\mu_{\mathcal{P}}d\mathcal{L} > \int_0^T\Psi(a)-\delta\,.\end{aligned}$$

As δ was arbitrary, this concludes the proof of the first part.

Let $a^\varepsilon\stackrel{2s}{\to}a$. For every $\varepsilon>0$ let $v^\varepsilon\in L^q(0,T;\mathscr{C}_\varepsilon)$ be defined through

$$v^\varepsilon = \operatorname{argmax}\left\{\int_0^T\int_{\boldsymbol{Q}} a^\varepsilon v\,d\mu_\omega^\varepsilon\,:\,v\in L^q(0,T;\mathscr{C}_\varepsilon)\right\}.$$

Then, there exists $v\in L^q(0,T;L^q(\boldsymbol{Q};L^q(\Omega,\mu_{\mathcal{P}})))$ such that $v^\varepsilon\stackrel{2s}{\rightharpoonup}v$ weakly in two scales with $v\in L^q(0,T;\mathscr{C})$ due to Theorem 5.6. We obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^T\Psi_{\varepsilon,\omega}(a^\varepsilon) &= \lim_{\varepsilon\to 0}\left(\int_0^T\int_{\boldsymbol{Q}} a^\varepsilon v^\varepsilon\,d\mu_\omega^\varepsilon\right)\\ &= \int_0^T\int_{\boldsymbol{Q}}\int_\Omega av\,d\mu_{\mathcal{P}}d\mathcal{L} \leq \int_0^T\Psi(a)\,.\end{aligned}$$

□

Lemma 5.10. *Let Assumption 5.3 hold and let $\mathscr{C}_p(\boldsymbol{Q})$ and $\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)$ be given through (5.1) and (5.1). Let $\frac{1}{p}+\frac{1}{q}=1$, $1<p,q<\infty$, and $\Psi_\varepsilon : L^q(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D)\to\mathbb{R}$ and $\Psi : L^q(\boldsymbol{Q}\times\Omega,\mathcal{L}\times\mu_{\mathcal{P}};\mathbb{R}^D)\to\mathbb{R}$ be given through*

$$\Psi_{\varepsilon,\omega}(u) := \sup_{\sigma\in\mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)}\int_{\boldsymbol{Q}} u\cdot\sigma d\mathcal{L}\,,\qquad \Psi(u) := \sup_{\sigma\in\mathscr{C}_p(\boldsymbol{Q})}\int_{\boldsymbol{Q}} u\cdot\sigma d\mathcal{L}\,. \tag{5.5}$$

Then, for every $u^\varepsilon\in L^q(\boldsymbol{Q};\mu_\omega^\varepsilon;\mathbb{R}^D)$ such that $u^\varepsilon\to u$ strongly in $L^q(\boldsymbol{Q})$ there holds

$$\lim_{\varepsilon\to 0}\Psi_{\varepsilon,\omega}(u^\varepsilon) = \Psi(u)\,.$$

Proof. The statement follows from Corollary 5.7, the definition of $\mathscr{C}_p(\boldsymbol{Q})$ in (5.3) and Lemma 5.8. □

5.3 Proof of Theorem 5.6

Note the similarities between the statement of Theorem 5.6 and Mazur's Lemma. The proof of Mazur's Lemma ist based on the Hyperplane Separation Theorem and the idea that every convex set $\mathcal{C}$ in a Banach space $\mathscr{B}$ is fully characterized by the set of all hyperplanes that do not intersect with $\mathcal{C}$. In particular, it is possible to characterize a convex set in $\mathscr{B}$ by a family $\Phi \subset \mathscr{B}^*$ of bounded affine functionals. In our case, $\mathscr{B} = L^p(\boldsymbol{Q}; L^p(\Omega; \mu_{\mathcal{P}}; \mathbb{R}^D)$.

For technical reasons we will only provide a countable set $\Phi_p^{\mathcal{C}}$ of hyperplanes in $\mathscr{B}^*$. We will then show that the limit of any weakly two-scale converging sequence $u^\varepsilon \in \mathscr{C}_p^\varepsilon(\boldsymbol{Q}, \omega)$ lies on the "correct side" of each of these hyperplanes. Finally, note that we cannot use the linear functionals used in the proof of the Hyperplane-separation Theorem (which are based on the Minkowsky functional), since we do not know whether these would be ergodic. Instead, we will use a different construction.

Proof of Theorem 5.6. We assume, that $O(x,\omega) \equiv 0$. The statement for general O follows from a consideration of the shifts $\mathcal{C}(x,\omega) \rightsquigarrow \mathcal{C}(x,\omega) - O(x,\omega)$ and $u^\varepsilon(x) \rightsquigarrow u^\varepsilon(x) - O(x, \tau_{\frac{x}{\varepsilon}}\omega)$.

Step 1: Recall the definition of $a(u,\mathcal{C})$ from (5.2). We define the set $\mathcal{C}^c := \{u \in \mathbb{R}^D : u \notin \mathcal{C}\}$ and study the case $u \in \mathcal{C}^c$. Since $\mathcal{C}$ is convex and $0 \in \mathcal{C}$, the angle between $a(u,\mathcal{C})$ and $(u - a(u,\mathcal{C}))$ lies in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and thus $|a(u,\mathcal{C})|_2 < |u|_2$ and $(u - a(u,\mathcal{C}))\cdot a(u,\mathcal{C}) > 0$. Therefore, the function

$$\mathfrak{U}_{\mathcal{C}} : u \mapsto \mathfrak{U}_{\mathcal{C}}(u) := \frac{u - a(u,\mathcal{C})}{a(u,\mathcal{C}) \cdot (u - a(u,\mathcal{C}))}$$

is well defined on $\mathcal{C}^c$. By definition of $\mathfrak{U}_{\mathcal{C}}$, we find

$$\mathfrak{U}_{\mathcal{C}}(u) \cdot u = 1 + \frac{(u - a(u,\mathcal{C}))^2}{(u \cdot a(u,\mathcal{C}) - a(u,\mathcal{C})^2)} \geq 1 + \frac{\operatorname{dist}(u,\mathcal{C})}{|a(u,\mathcal{C})|_2} \geq 1 + \frac{\operatorname{dist}(u,\mathcal{C})}{|u|_2}. \tag{5.6}$$

Furthermore, $\mathfrak{U}_{\mathcal{C}}(u) \cdot a(u,\mathcal{C}) = 1$. Since the hyperplane through $a(u,\mathcal{C})$ with outer normal $\mathfrak{U}_{\mathcal{C}}(u)$ is tangential to the convex set $\mathcal{C}$ we find

$$\mathfrak{U}_{\mathcal{C}}(u) \cdot w \leq \mathfrak{U}_{\mathcal{C}}(u) \cdot a(u,\mathcal{C}) = 1 \text{ for all } w \in \mathcal{C}, \quad \text{implying} \quad |\mathfrak{U}_{\mathcal{C}}(u)| \leq \frac{1}{r}. \tag{5.7}$$

Note that the second inequality follows from the fact that $B_r(0) \subset \mathcal{C}$, i.e. $r\,|\mathfrak{U}_{\mathcal{C}}| \leq |\mathfrak{U}_{\mathcal{C}}|^2 \leq 1$.

Step 2: We set $\mathfrak{U}_{\mathcal{C}}(u) = 0$ and $a(u) = u$ if $u \in \mathcal{C}$. Let $f \in \Phi_p$. By Lemma 5.5, the function $a(f)$ is measurable and p-ergodic. Since f is piecewise constant, Assumption 5.4 guaranties that

$$\mathcal{X}_f(x,\omega) := \mathcal{X}_{\mathcal{C}}(x,\omega,f(x,\omega)), \quad A_f := \mathcal{X}_f^{-1}(\mathbb{R}\backslash\{0\})$$

and

$$\chi_{A_f}(x,\omega) := \begin{cases} 0 & \text{if } \mathcal{X}_f(x,\omega) = 0 \\ 1 & \text{else} \end{cases},$$

are $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$-measurable, where $\chi_{A_f} = 1$ if and only if $(x,\omega,f(x,\omega)) \in \mathcal{C}^c(\boldsymbol{Q}\times\Omega)$. Defining the function

$$\mathfrak{U}(f)(x,\omega) := \begin{cases} \mathfrak{U}_{\mathcal{C}(x,\omega)}(f(x,\omega)) & \text{if } (x,\omega,f(x,\omega)) \in \mathcal{C}^c(\boldsymbol{Q}\times\Omega) \\ 0 & \text{otherwise} \end{cases}, \tag{5.8}$$

we see that $\mathfrak{U}(f)$ is measurable. Furthermore, $|\mathfrak{U}(f)| \le R$ for all f due to (5.7) (with R given in Assumption 5.4) and

$$\|\mathfrak{U}(f)\|_{L^q(\boldsymbol{Q};L^q(\Omega,\mu_{\mathcal{P}}))} \le K := \|R\|_{L^q(\boldsymbol{Q};L^q(\Omega,\mu_{\mathcal{P}}))} \quad \text{for all } f\,. \tag{5.9}$$

Step 4: For any $f \in L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}}; \mathbb{R}^D))$, estimate (5.6) yields the following pointwise inequality:

$$f(x,\omega)\,\mathfrak{U}(f)(x,\omega) \ge \chi_{A_f}(x,\omega) + l_f(x,\omega) \qquad \forall (x,\omega)\,, \tag{5.10}$$

where

$$l_f(x,\omega) := \begin{cases} \frac{\operatorname{dist}(f(x,\omega),\mathcal{C}(x,\omega))}{|f(x,\omega)|_2} & \text{for } f(x,\omega) \ne 0 \\ 0 & f(x,\omega) = 0 \end{cases}.$$

We find $l_f(x,\omega) \le 1$ for all $(x,\omega) \in \boldsymbol{Q} \times \Omega$. Since $\mathcal{C}(x,\omega)$ is convex, we find that $\mathcal{X}_{\mathcal{C}}(x,\omega,\cdot)$ is lower semicontinuous for fixed (x,ω). Since convergence in L^p implies pointwise convergence along a subsequence, we find that also

$$L_\bullet : L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}}; \mathbb{R}^D)) \to \mathbb{R}\,, \qquad f \mapsto L_f := \int_{\boldsymbol{Q}} \int_\Omega l_f d\mu_{\mathcal{P}} d\mathcal{L} \quad \text{is l.s.c.}\,. \tag{5.11}$$

For functions $g \in \mathscr{C}_p(\boldsymbol{Q} \times \Omega)$, i.e. $g(x,\omega) \in \mathcal{C}(x,\omega)$ for $\mathcal{L} \times \mu_{\mathcal{P}}$ almost all (x,ω), we find by (5.7) and (5.8)

$$\int_{\boldsymbol{Q}} \int_\Omega g\, \mathfrak{U}(f) d\mu_{\mathcal{P}} d\mathcal{L} \le \int_{\boldsymbol{Q}} \int_\Omega \chi_{A_f}\,. \tag{5.12}$$

Step 5: We assume $u \notin \mathscr{C}_p(\boldsymbol{Q} \times \Omega)$. Then (by a contradiction argument) we find that $L_u > 0$. Furthermore, for every $\delta > 0$ we find $\phi_\delta \in \Phi_p$ satisfying $\|\phi_\delta - u\|_{L^p} \le \delta$. Using the above results, we conclude:

$$\begin{aligned} \int_{\boldsymbol{Q}} \int_\Omega \mathfrak{U}(\phi_\delta) u\, d\mu_{\mathcal{P}} d\mathcal{L} &= \int_{\boldsymbol{Q}} \int_\Omega \mathfrak{U}(\phi_\delta)\phi_\delta\, d\mu_{\mathcal{P}} d\mathcal{L} + \int_{\boldsymbol{Q}} \int_\Omega \mathfrak{U}(\phi_\delta)\,(u - \phi_\delta)\, d\mu_{\mathcal{P}} d\mathcal{L} \\ &\overset{(5.10)}{\ge} \left(\int_{\boldsymbol{Q}} \int_\Omega \chi_{A_{\phi\delta}}\, d\mu_{\mathcal{P}} d\mathcal{L} + L_{\phi_\delta} \right) - K\delta\,. \end{aligned}$$

with K independent from δ by (5.9). Since $\liminf_{\delta\to 0} L_{\phi_\delta} \ge L_u > 0$ by (5.11), there exists $\phi_\delta \in \Phi_p$ such that

$$\int_{\boldsymbol{Q}} \int_\Omega \mathfrak{U}(\phi_\delta) u\, d\mu_{\mathcal{P}} d\mathcal{L} > \int_{\boldsymbol{Q}} \int_\Omega \chi_{A_{\phi_\delta}}\, d\mu_{\mathcal{P}} d\mathcal{L}\,.$$

Step 6: Let $f \in \Phi_p$. It is our aim to use $\mathfrak{U}(f)$ as a testfunction for two-scale convergence. However, since $\mathfrak{U}(f)$ probably is unbounded, we define

$$f_n(x,\omega) = \begin{cases} f(x,\omega) & \text{if } |\mathfrak{U}(f)(x,\omega)| \in [n-1, n) \\ 0 & \text{else} \end{cases}$$

Since $\mathfrak{U}(f)$ is $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$-measurable, f_n is a finite step function for all $n \in \mathbb{N}$ and $f = \sum_n f_n$ and $\mathfrak{U}(f) = \sum_n \mathfrak{U}(f_n)$. Finally, note that for all $\omega \in \Omega$ such that the realizations $\mathfrak{U}(f_n)_\omega$ and $\chi_{A_{fn,\omega}}$ are measurable for all n, and for all $g^\varepsilon \in \mathscr{C}_p^\varepsilon(\boldsymbol{Q},\omega)$, i.e. $g^\varepsilon(x) \in \mathcal{C}(x, \tau_{\frac{x}{\varepsilon}}\omega)$, we find again by (5.7) and (5.8)

$$\int_{\boldsymbol{Q}} g^\varepsilon(x)\, \mathfrak{U}(f_n)(x, \tau_{\frac{x}{\varepsilon}}\omega) d\mu_\omega^\varepsilon(x) \le \int_{\boldsymbol{Q}} \chi_{A_{fn}}(x, \tau_{\frac{x}{\varepsilon}}\omega) d\mu_\omega^\varepsilon(x)\,. \tag{5.13}$$

Let $u^\varepsilon \in L^p(\boldsymbol{Q}, \mu_\omega^\varepsilon)$ with $u^\varepsilon \in \mathscr{C}_p^\varepsilon(\boldsymbol{Q}, \omega)$ for all $\varepsilon > 0$ be a sequence of functions that weakly two-scale converges to $u \in L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}}))$. We find that for all $\phi \in \Phi_p$

$$\begin{aligned}
\int_{\boldsymbol{Q}} \int_\Omega \mathfrak{U}(\phi_n) u \, d\mu_{\mathcal{P}} d\mathcal{L} &= \lim_{\varepsilon \to 0} \int_{\boldsymbol{Q}} \mathfrak{U}(\phi_n)(x, \tau_{\frac{x}{\varepsilon}} \omega) u^\varepsilon(x) d\mu_\omega^\varepsilon(x) \\
&\overset{(5.9)-(5.11)}{\leq} \lim_{\varepsilon \to 0} \int_{\boldsymbol{Q}} \chi_{A_{\phi n}}(x, \tau_{\frac{x}{\varepsilon}} \omega) d\mu_\omega^\varepsilon(x) \\
&= \int_{\boldsymbol{Q}} \int_\Omega \chi_{A_{\phi n}} \, d\mu_{\mathcal{P}} d\mathcal{L} \qquad (5.14)
\end{aligned}$$

This implies

$$\int_{\boldsymbol{Q}} \int_\Omega \mathfrak{U}(\phi) u \, d\mu_{\mathcal{P}} d\mathcal{L} \leq \int_{\boldsymbol{Q}} \int_\Omega \chi_{A_\phi} \, d\mu_{\mathcal{P}} d\mathcal{L}$$

and by Step 5, we obtain $u \in \mathscr{C}_p(\boldsymbol{Q} \times \Omega)$. □

5.4 Proof of Lemma 5.5

We use the theory of set valued measurable mappings from [27].

Definition 5.11 ([27] Definition III.10)**.** Let $(U, \mathcal{F})$ be a measurable space and let X be a separable metric space. Let Γ be a set-valued map from U onto the closed subsets of X. Then, Γ is called measurable if $u \mapsto \operatorname{dist}(x, \Gamma(u))$ is measurable for all $x \in X$.

Theorem 5.12 ([27] Theorem III.9 and Proposition III.13)**.** *Let $(U, \mathcal{F})$ be a measurable space and let X be a separable metric space. Let Γ be a measurable set-valued map from U onto the closed subsets of X. Then the following holds:*

1. *There exists a measurable function $\sigma : U \to X$ with $\sigma(u) \in \Gamma(u)$ for all $u \in U$.*
2. *The graph $\operatorname{Gr}(\Gamma) := \{(u, x) \in U \times X : x \in \Gamma(u)\}$ is measurable with respect to $\mathcal{F} \otimes \mathcal{B}_X$, where $\mathcal{B}_X$ is the Borel-algebra on X.*

The following result is an inverse statement of Theorem 5.12.

Theorem 5.13 ([27] Theorem III.30)**.** *Let $(U, \mathcal{F}, \mu)$ be a complete measure space (that is $\mathcal{F}$ is complete w.r.t. μ) with σ-finite μ and let X be a separable metric space. Let Γ be a set-valued map from U onto the closed subsets of X. If the graph $\operatorname{Gr}(\Gamma) := \{(u, x) \in U \times X : x \in \Gamma(u)\}$ is measurable with respect to $\mathcal{F} \otimes \mathcal{B}_X$, then Γ is measurable.*

We will prove Lemma 5.5 using Theorems 5.12 and 5.13.

To this aim, let φ be a finite step function. Then, the map $(x, \omega) \to \operatorname{dist}(\varphi(x, \omega), \mathcal{C}(x, \omega))$ is measurable by Assumption 5.4. For every (x, ω), we define the set

$$B(x, \omega) := \left\{ u \in \mathbb{R}^D : |u - \varphi(x, \omega)| \leq \operatorname{dist}(\varphi(x, \omega), \mathcal{C}(x, \omega)) \right\} .$$

The graph of B is measurable since

$$\operatorname{dist}(\tilde{u}, B(x, \omega)) = \max \{ |\tilde{u} - \varphi(x, \omega)| - \operatorname{dist}(\varphi(x, \omega), \mathcal{C}(x, \omega)) \, , 0 \}$$

is measurable for all $\tilde{u} \in \mathbb{R}^D$. Therefore, by Theorem 5.12, the graphs of Γ and B are measurable and thus

$$\mathrm{Gr}(\Gamma) \cap \mathrm{Gr}(B) = \{\, (\, x, \omega, a(\varphi(x,\omega))\,)\,\} \subset (\boldsymbol{Q} \times \Omega) \times \mathbb{R}^D$$

is measurable. Theorem 5.13 now states that function $(x,\omega) \mapsto a(\varphi(x,\omega))$ is measurable with respect to the completion of $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$ in $\mathcal{L} \otimes \mu_{\mathcal{P}}$.

It remains to prove the existence of $\Phi_p^{\mathcal{C}}$. For this aim, given any $\phi \in \Phi_p$, let $a(\phi) \in L^p(\boldsymbol{Q}; L^p(\Omega, \mu_{\mathcal{P}}))$ be given through $a(\phi)(x,\omega) = a(\phi(x,\omega), \mathcal{C}(x,\omega))$. By the above considerations, we find that $a(\phi)$ is ergodic. Let $u \in \mathscr{C}_p$ and $\phi \in \Phi_p$ with $\|u - \phi\|_{L^p(\boldsymbol{Q}; L^p(\Omega,\mu_{\mathcal{P}}))} \le \delta$. Then, as $u(x,\omega) \in \mathcal{C}(x,\omega)$ for almost all (x,ω), we find $|u(x,\omega) - \phi(x,\omega)| \ge |u(x,\omega) - a(\phi)(x,\omega)|$ and thus $\|u - a(\phi)\|_{L^p(\boldsymbol{Q}; L^p(\Omega,\mu_{\mathcal{P}}))} \le \|u - \phi\|_{L^p(\boldsymbol{Q}; L^p(\Omega,\mu_{\mathcal{P}}))} \le \delta$. Thus, we set $\Phi_p^{\mathcal{C}} := a(\Phi_p)$.

6 The Prandtl-Reuss plasticity equations

We study the stochastic homogenization problem of the Prandtl-Reuss plasticity equations:

$$\left.\begin{aligned} -\nabla \cdot \sigma^\varepsilon &= f\,, & \sigma^\varepsilon &= C_\varepsilon^{-1} e^\varepsilon\,, \\ \nabla^s u^\varepsilon + \nabla^s u_{Dir} &= e^\varepsilon + p^\varepsilon\,, & \partial_t p^\varepsilon &\in \partial\psi_\varepsilon^*(\sigma^\varepsilon - B_\varepsilon p^\varepsilon)\,, \end{aligned}\right\} \text{ on } [0,T] \times \boldsymbol{Q}\,. \tag{6.1}$$

Here, we look for u^ε having boundary values $u^\varepsilon|_{\partial \boldsymbol{Q}} = 0$. Therefore, u_{Dir} prescribes the boudary values of $u^\varepsilon + u_{Dir}$. Problem (6.1) consists of a force balance equation for σ^ε, Hooke's law $\sigma^\varepsilon \propto e^\varepsilon$, the decomposition of the strain $\nabla^s(u^\varepsilon + u_{Dir})$ into a plastic part p^ε and an elastic part e^ε as well as the flow rule for $\partial_t p^\varepsilon$. Here, ∇^s denotes the symmetric part of the gradient. Note that (6.1) has only two independent variables: u^ε and p^ε.

We want to study the homogenization of (6.1) with help of the concepts developed in Sections 4 and 5. Let Assumption 2.4 hold for $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$. As a random measure, we consider $\omega \mapsto \mathcal{L}$, i.e. we assume that $\mathcal{P} = \mu_{\mathcal{P}}$ by Remark 2.8 a).

6.1 Function spaces and preliminaries

In what follows, we will study suitable subspaces of $L^2(\Omega)$. Most of these spaces have been introduced in [11]. We denote by $L^2_{loc}(\mathbb{R}^n; \mathbb{R}^{n\times n})$ the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $f|_{\boldsymbol{U}} \in L^2(\boldsymbol{U}; \mathbb{R}^{n\times n})$ for every bounded set $\boldsymbol{U}$ and we define

$$\begin{aligned} L^2_{pot,loc}(\mathbb{R}^n) &:= \left\{ u \in L^2_{loc}(\mathbb{R}^n; \mathbb{R}^{n\times n}) \mid \forall \boldsymbol{U} \text{ bounded domain, } \exists \varphi \in H^1(\boldsymbol{U}; \mathbb{R}^n) : u = \nabla\varphi \right\}, \\ L^2_{sol,loc}(\mathbb{R}^n) &:= \left\{ u \in L^2_{loc}(\mathbb{R}^n; \mathbb{R}^{n\times n}) \mid \int_{\mathbb{R}^n} u \cdot \nabla\varphi = 0 \ \forall \varphi \in C^1_c(\mathbb{R}^n) \right\}. \end{aligned}$$

We can then define similar spaces on Ω through

$$\begin{aligned} L^2_{pot}(\Omega) &:= \left\{ u \in L^2(\Omega; \mathbb{R}^{n\times n}) : u_\omega \in L^2_{pot,loc}(\mathbb{R}^n) \text{ for } \mathcal{P}-\text{a.e. } \omega \in \Omega \right\}, \\ L^2_{sol}(\Omega) &:= \left\{ u \in L^2(\Omega; \mathbb{R}^{n\times n}) : u_\omega \in L^2_{sol,loc}(\mathbb{R}^n) \text{ for } \mathcal{P}-\text{a.e. } \omega \in \Omega \right\}, \\ \mathcal{V}^2_{pot}(\Omega) &:= \left\{ u \in L^2_{pot}(\Omega) : \int_\Omega u\, d\mathcal{P} = 0 \right\}, \end{aligned} \tag{6.2}$$

From [32], we know that the above spaces are closed and that $L^2(\Omega;\mathbb{R}^n) = \mathcal{V}_{pot}(\Omega) \oplus L^2_{sol}(\Omega)$. For $a \in \mathbb{R}^{n\times n}$ we write $a^s := \frac{1}{2}(a + a^T)$. For sufficiently smooth functions u we write $\nabla^s u := \frac{1}{2}(\nabla u + \nabla u^T)$.

We will need the following Korn inequality on $L^2_{pot}(\Omega;\mathbb{R}^n)$ from [11].

Lemma 6.1. *For all $v \in \mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n)$ holds*

$$\|v\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \le 2\,\|v^s\|_{L^2(\Omega;\mathbb{R}^{n\times n})}\,. \tag{6.3}$$

The following Lemma is well known in the periodic case [3] but also in the stochastic setting [33]. Howver, since the proof in [33] heavily uses topological assumptions which we do not have in our setting, we provide a new proof.

Lemma 6.2. *If $u^\varepsilon \in H^1_0(\boldsymbol{Q};\mathbb{R}^n)$ for all ε with $\|\nabla u^\varepsilon\|_{L^2(\boldsymbol{Q})} < C$ for C independent from $\varepsilon > 0$ then there exists a subsequence denoted by u^ε and functions $u \in H^1_0(\boldsymbol{Q};\mathbb{R}^n)$ and $v \in L^2(\boldsymbol{Q}; L^2_{pot}(\Omega;\mathbb{R}^n))$ such that*

$$u^\varepsilon \stackrel{2s}{\rightharpoonup} u \quad \text{and} \quad \nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla u + v \qquad \text{as } \varepsilon \to 0\,.$$

Proof. **Step 1:** From Lemma 4.4 we obtain that $u^\varepsilon \stackrel{2s}{\rightharpoonup} u \in L^2(Q;L^2(\Omega))$ along a subsequence. We first show that u does not depend on the Ω-coordinate using ergodicity (2.9). To this aim, consider a set $A \subset \Omega$ and the characteristic function $\phi(\omega) = \chi_A(\omega)$. For any $\psi \in C^\infty_c(\boldsymbol{Q})$ we find that $\psi u^\varepsilon \stackrel{2s}{\rightharpoonup} u$. Thus, for any $a \in \mathbb{R}^n$ it holds

$$\begin{aligned}
&\int_{\boldsymbol{Q}}\int_\Omega (u(x,\tau_a\omega) - u(x,\omega))\psi(x)\phi(\omega)\,d\mathcal{P}(\omega)\,dx\\
&\qquad = \int_{\boldsymbol{Q}}\int_\Omega u(x,\omega)\psi(x)\left(\phi(\tau_{-a}\omega) - \phi(\omega)\right)\,d\mathcal{P}(\omega)\,dx\\
&\qquad = \lim_{\varepsilon\to 0}\int_{\boldsymbol{Q}} u^\varepsilon(x)\psi(x)\left(\phi(\tau_{\frac{-\varepsilon a + x}{\varepsilon}}\omega) - \phi(\tau_{\frac{x}{\varepsilon}}\omega)\right)\,dx\\
&\qquad = \lim_{\varepsilon\to 0}\int_{\boldsymbol{Q}} \left(u^\varepsilon(x+\varepsilon a)\psi(x+\varepsilon a) - u^\varepsilon(x)\psi(x)\right)\phi(\tau_{\frac{x}{\varepsilon}}\omega)\,dx\,.
\end{aligned}$$

Due to the apriori bounds, the family $u^\varepsilon\psi$ is compact in $L^2(\boldsymbol{Q})$. Therefore, the Riesz-characterization of compact sets in $L^2(\boldsymbol{Q})$ yields that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left|\int_{\boldsymbol{Q}} \left(u^\varepsilon(x+\varepsilon a)\psi(x+\varepsilon a) - u^\varepsilon(x)\psi(x)\right)\phi(\tau_{\frac{x}{\varepsilon}}\omega)\,dx\right|\\
&\qquad \le \sup_{\varepsilon\to 0}\left\|\left(u^\varepsilon\psi\right)(\cdot+\varepsilon a) - \left(u^\varepsilon\psi\right)(\cdot)\right\|_{L^2(Q)}\left\|\phi(\tau_{\frac{x}{\varepsilon}}\omega)\right\|_{L^2(\boldsymbol{Q})} \to 0\,.
\end{aligned}$$

Since a was arbitray, it follows for all $x \in \mathbb{R}^n$ that $u(x,\cdot) = \mathit{const}$.

Step 2: From Lemma 4.4 we obtain that $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} v \in L^2(Q;L^2(\Omega))$ along a subsequence. Let we consider a countable set $\Phi_{sol} \subset L^2_{sol}(\Omega)$ which is dense in $L^2_{sol}(\Omega)$. Then, for all $b \in \Phi_{sol}$ and all $\psi \in C^\infty_c(\boldsymbol{Q})$, we find

$$\int_{\boldsymbol{Q}} \left(\psi(x)\nabla u^\varepsilon(x) + u^\varepsilon(x)\nabla\psi(x)\right)\cdot b(\tau_{\frac{x}{\varepsilon}}\omega)\,dx = \int_{\boldsymbol{Q}} \nabla\left(u^\varepsilon(x)\psi(x)\right)\cdot b(\tau_{\frac{x}{\varepsilon}}\omega)\,dx = 0\,.$$

We take the limit $\varepsilon \to 0$ on the left hand side and obtain

$$\int_{\boldsymbol{Q}} (\psi(x)v(x,\tilde{\omega}) + u\nabla\psi(x)) \cdot b(\tilde{\omega})\, d\mathcal{P}(\tilde{\omega})\, dx = 0\,.$$

After integration by parts, this implies

$$\int_{\boldsymbol{Q}} \psi(x)\,(\nabla u(x) - v(x,\tilde{\omega})) \cdot b(\tilde{\omega})\, d\mathcal{P}(\tilde{\omega})\, dx = 0\,.$$

As $\psi \in C_c^\infty(\boldsymbol{Q})$ and $b \in \Phi_{sol}$ were arbitrary and since $\Phi_{sol} \subset L^2_{sol}(\Omega)$ is dense, the last equation implies that $\nabla u(x) - v(x,\cdot) \in \mathcal{V}_{pot}(\Omega)$ for almost all $x \in \boldsymbol{Q}$. □

Lemma 6.3 (Existence of small potentials, see [11])**.** *Let $v \in \mathcal{V}^2_{pot}(\Omega)$. Then, almost every $\omega \in \Omega$ it holds that for every $\varepsilon > 0$ there exists $\phi_{\omega,\varepsilon,v} \in H^1(Q)$ such that $\nabla\phi_{\omega,\varepsilon,v}(x) = v(\tau_{\frac{x}{\varepsilon}}\omega)$ and $\|\phi_{\omega,\varepsilon,v}\|_{L^2(Q)} \to 0$ as $\varepsilon \to 0$.*

Finally, we will need the following simple result.

Corollary 6.4. *Let $u^\varepsilon \in L^2(\boldsymbol{Q};\mathbb{R}^{n\times n})$ and $u \in L^2(\boldsymbol{Q}\times\Omega)$ such that $u^\varepsilon \overset{2s}{\rightharpoonup} u$ as $\varepsilon \to \infty$. Then also $(u^\varepsilon)^s \overset{2s}{\rightharpoonup} u^s$.*

6.2 The homogenization result

For the formulation of the homogenization result, we make the following assumptions.

1. Let $(x,\omega) \to \mathcal{C}(x,\omega) \subset \mathbb{R}^{n\times n}_s$ for $(x,\omega) \in \boldsymbol{Q}\times\Omega$ be a family of closed convex sets satisfying Assumption 5.4. We define the functionals

$$\psi^*(x,\omega,v) = \begin{cases} 0 & v \in \mathcal{C}(x,\omega) \\ +\infty & v \notin \mathcal{C}(x,\omega) \end{cases}, \qquad \psi(x,\omega,z) := \sup_{v\in\mathcal{C}(x,\omega)} v\cdot z$$
$$\text{and} \quad \psi^*_{\varepsilon,\omega}(x,v) = \psi^*(x,\tau_{\frac{x}{\varepsilon}}\omega,v)\,, \qquad \psi_{\varepsilon,\omega}(x,z) = \psi(x,\tau_{\frac{x}{\varepsilon}}\omega,z)\,.$$

2. We assume that $C, B \in L^\infty(\boldsymbol{Q}; L^\infty(\Omega; \mathscr{L}(\mathbb{R}^{n\times n}_s, \mathbb{R}^{n\times n}_s)))$ are symmetric C, B are is $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$-measurable. We further assume the existence of a constants $\gamma, \beta > 0$ such that

$$\left.\begin{aligned} \gamma\,|\xi|^2 \le \xi C(x,\omega)\xi \le \frac{1}{\gamma}\,|\xi|^2 \\ \beta\,|\xi|^2 \le \xi B(x,\omega)\xi \le \frac{1}{\beta}\,|\xi|^2 \end{aligned}\right\} \quad \forall \xi \in \mathbb{R}^n \text{ and for a.e. } (x,\omega) \in \boldsymbol{Q}\times\Omega\,. \tag{6.4}$$

   Given $\omega \in \Omega$ and $\varepsilon > 0$, we set $C_{\varepsilon,\omega}(x) := C(x,\tau_{\frac{x}{\varepsilon}}\omega)$ and $B_{\varepsilon,\omega}(x) := B(x,\tau_{\frac{x}{\varepsilon}}\omega)$.

3. We denote by $\overset{2s}{\rightharpoonup}$ two-scale convergence with respect to the random measure $\omega \mapsto \mathcal{L}$, $\mu_{\mathcal{P}} = \mathcal{P}$ and $p = 2$. With regard to Theorem 5.6, we consider a set $\tilde{\Omega} \subset \Omega$ of full measure, such that for all $\omega \subset \tilde{\Omega}$

   (a) Remark 4.1 holds for a countable dense set $\Phi_2 \subset L^2(\boldsymbol{Q}; L^2(\Omega))$. Note that Φ_2^N is dense in $L^2(\boldsymbol{Q}; L^2(\Omega;\mathbb{R}^N))$ for all $N \in \mathbb{N}$.

(b) Lemmas 4.10 and 6.2 are applicable and Lemma 4.6 holds for C and B. Furthermore, we claim that the realizations $C_{\varepsilon,\omega}(x) := C(x, \tau_{\frac{x}{\varepsilon}}\omega)$ and $B_{\varepsilon,\omega}(x) := B(x, \tau_{\frac{x}{\varepsilon}}\omega)$ are ergodic.

(c) $\mathscr{C}_2^\varepsilon(\boldsymbol{Q},\omega)$ and $\mathscr{C}_2(\boldsymbol{Q}\times\Omega)$ defined in (5.1) satisfy $\mathscr{C}_2^\varepsilon(\boldsymbol{Q},\omega) \xrightarrow{2s} \mathscr{C}_2(\boldsymbol{Q}\times\Omega)$.

4. Let $p_0^\varepsilon \in L^2(\boldsymbol{Q};\mathbb{R}_s^{n\times n})$ be such that $p_0^\varepsilon \xrightarrow{2s} p_0 \in L^2(\boldsymbol{Q}\times\Omega)$ strongly in two scales and satisfy the following: If for $\varepsilon > 0$, $u_0^\varepsilon \in H_0^1(\boldsymbol{Q})$ is the solution of the elliptic problem

$$-\nabla\cdot(\nabla^s u_0^\varepsilon + \nabla^s u_{Dir}(0) - p_0^\varepsilon) = f(0)$$

then there holds $(\nabla^s u_0^\varepsilon + \nabla^s u_{Dir}(0) - p_0^\varepsilon) - B_\varepsilon p_0^\varepsilon \in \mathscr{C}_2^\varepsilon$ for all $\varepsilon > 0$. Furthermore, there holds $\nabla u_0^\varepsilon \xrightarrow{2s} \nabla u_0 + \upsilon_0$ strongly in two scales for some $u_0 \in H_0^1(\boldsymbol{Q})$ and $\upsilon_0 \in L^2(\boldsymbol{Q}; L^2_{pot}(\Omega))$.

Remark. Condition 4. is satisfied by $p_0^\varepsilon \equiv 0$ for all $\varepsilon > 0$.

Theorem 6.5. *Let 1.-4. hold. There exists a unique solution*

$$(u,v,p) \in H^1(\,0,T;\, H^1(\boldsymbol{Q}) \times \mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{R}_s^{n\times n})\,)$$

to the problem

$$\begin{aligned} -\nabla\cdot\int_\Omega \sigma\, d\mathcal{P} = f \qquad & on\ [0,T]\times\boldsymbol{Q} \\ \nabla^s u + \nabla^s u_{Dir} = C\sigma + p - v^s\,, \quad \partial_t p \in \partial\psi^*(\sigma - Bp) \qquad & on\ [0,T]\times\boldsymbol{Q}\times\Omega \end{aligned} \tag{6.5}$$

with $p(0,\cdot) = p_0(\cdot)$. Furthermore, for every $\omega \in \tilde\Omega$ it holds: For each $\varepsilon > 0$ there exists a unique solution $(u^\varepsilon, p^\varepsilon) \in H^1(0,T;\, H_0^1(\boldsymbol{Q}) \times L^2(\boldsymbol{Q})\,)$ with $p^\varepsilon(0,x) = p_0(x,\tau_{\frac{x}{\varepsilon}}\omega)$ to (6.1) and as $\varepsilon \to 0$ it holds that

$$u^\varepsilon(t) \xrightharpoonup{2s} u(t)\,, \quad \nabla u^\varepsilon(t) \xrightharpoonup{2s} \nabla u(t) + v(t) \quad and \quad p^\varepsilon(t) \xrightharpoonup{2s} p(t) \quad \forall t \in [0,T]\,.$$

We follow [11] and decouple (6.5) into macro- and microscopic processes. As announced in the introduction, the macroscopic behavior is discribed by a hysteresis operator.

Theorem 6.6. *For $x \in \boldsymbol{Q}$ define the operator*

$$\begin{aligned} \Sigma(x,\cdot):\, H^1(0,T;\mathbb{R}_s^{n\times n}) &\to H^1(0,T;\mathbb{R}_s^{n\times n}) \\ \xi &\mapsto \int_\Omega \sigma_\xi\, d\mathcal{P} \end{aligned}$$

where $\sigma_\xi \in H^1(0,T;L^2_{sol}(\Omega;\mathbb{R}^n))$, $p_\xi \in H^1(0,T;L^2(\Omega;\mathbb{R}_s^{n\times n}))$ and $v_\xi \in H^1(0,T;L^2_{pot}(\Omega;\mathbb{R}^n))$ solve

$$\xi = C\sigma_\xi + p_\xi - v_\xi^s\,, \quad \partial_t p_\xi \in \partial\psi^*(\sigma_\xi - Bp_\xi)\,, \quad p_\xi(0,\cdot) = p_0(x,\cdot)\,. \tag{6.6}$$

Then, for all $x \in \boldsymbol{Q}$, $\Sigma(x,\cdot)$ is well defined on

$$\left\{\xi \in H^1(0,T;\mathbb{R}_s^{n\times n}) \,:\, \sigma_\xi(0) - Bp_\xi(0) \in \partial\Psi(0)\right\}$$

and continuous with respect to the weak topology on $H^1(0,T;\mathbb{R}_s^{n\times n})$. Furthermore, for $u \in H_0^1(\boldsymbol{Q})$ the following two statements are equivalent:

1. *There exists* $(v,p) \in H^1(0,T;\, L^2_{pot}(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{R}^{n\times n}_s))$ *such that* (u,v,p) *solves (6.5) with* $p(0,\cdot) = p_0(\cdot)$.

2. u *solves*
$$-\nabla\cdot\Sigma(\nabla^s u + \nabla^s u_{Dir}) = f$$
in the weak sense.

6.3 Proof of Teorem 6.5

We set

$$\begin{aligned}\mathscr{H}_2 &= L^2(\boldsymbol{Q}) \times L^2(\boldsymbol{Q};\mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n)) \times L^2(\boldsymbol{Q};L^2(\Omega;\mathbb{R}^{n\times n}_s))\,,\\ \mathscr{H}_1 &= H^1_0(\boldsymbol{Q}) \times L^2(\boldsymbol{Q};\mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n)) \times L^2(\boldsymbol{Q};L^2(\Omega;\mathbb{R}^{n\times n}_s))\,,\end{aligned}$$

and
$$\mathscr{H}_1^\varepsilon = H^1_0(\boldsymbol{Q}) \times L^2(\boldsymbol{Q};\mathbb{R}^{n\times n}_s) \quad\text{and}\quad \mathscr{H}_2^\varepsilon = L^2(\boldsymbol{Q}) \times L^2(\boldsymbol{Q};\mathbb{R}^{n\times n}_s) \quad \forall\varepsilon > 0\,.$$
We note that for the functionals Ψ and $\Psi_{\varepsilon,\omega}$ defined in (5.4) satisfy

$$\Psi(p) = \int_{\boldsymbol{Q}}\int_\Omega \psi(x,\omega,p(x,\omega))\,, \qquad\qquad \Psi_{\varepsilon,\omega}(p^\varepsilon) = \int_{\boldsymbol{Q}} \psi_{\varepsilon,\omega}(x,p^\varepsilon(x))\,. \tag{6.7}$$

We now define the family of functionals

$$\mathcal{E}_{\varepsilon,\omega} : [0,T]\times\mathscr{H}_2^\varepsilon \to \mathbb{R}\,, \qquad (t,u,p) \mapsto \frac{1}{2}\int_{\boldsymbol{Q}} \left(p : (B_{\varepsilon,\omega}p) + \sigma : (C_{\varepsilon,\omega}\sigma) - 2f(t)u\right)\,, \tag{6.8}$$

where $C_{\varepsilon,\omega}\,\sigma := \nabla u + \nabla u_{Dir}(t) - p$. The expression

$$\|(u,p)\|^2_\varepsilon := \int_{\boldsymbol{Q}} \left(p : (B_{\varepsilon,\omega}p) + (\nabla u - p) : \left(C_{\varepsilon,\omega}\left(\nabla u - p\right)\right)\right)$$

defines a norm on $\mathscr{H}_1^\varepsilon$. This norm is equivalent with $\|\cdot\|_{H^1_0(\boldsymbol{Q})\times L^2(\boldsymbol{Q})}$, since convergence with respect to $\|\cdot\|_\varepsilon$ implies convergence with respect to $\|\cdot\|_{H^1_0(\boldsymbol{Q})\times L^2(\boldsymbol{Q})}$ and vice versa. Since $u_{Dir} \in H^1(0,T;H^1(\boldsymbol{Q}))$ it holds that $\int_{\boldsymbol{Q}} |\nabla u_{Dir}|^2 \in W^{1,1}(0,T)$. Therefore, we can apply Theorem 2.3 to get existence of a unique energetic solution $(u^\varepsilon,p^\varepsilon) \in C^{Lip}(0,T;\mathscr{H}_1^\varepsilon)$ to $(\mathscr{H}_2^\varepsilon,\mathcal{E}_{\varepsilon,\omega},\Psi_{\varepsilon,\omega})$. The derivative $D\mathcal{E}_{\varepsilon,\omega} = (D_p\mathcal{E}_{\varepsilon,\omega}, D_u\mathcal{E}_{\varepsilon,\omega})$ can be easily obtained to be

$$D_u\mathcal{E}(u^\varepsilon,p^\varepsilon) = -f(t) - \nabla\cdot\left[C^{-1}_{\varepsilon,\omega}\left(\nabla u^\varepsilon + \nabla u_{Dir}(t) - p^\varepsilon\right)\right]\,, \qquad D_p\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon,p^\varepsilon) = B_{\varepsilon,\omega}p^\varepsilon - \sigma^\varepsilon\,.$$

With regard to Section 2.2, we see that $(u^\varepsilon_\omega,p^\varepsilon_\omega)$ is a solution to $(6.1)_\omega$ if and only if $(u^\varepsilon_\omega,p^\varepsilon_\omega)$ satisfy
$$0 \in \partial\Psi_{\varepsilon,\omega}(\partial_t p^\varepsilon_\omega(t)) + D\mathcal{E}_{\varepsilon,\omega}(t,u^\varepsilon_\omega(t),p^\varepsilon_\omega(t)) \quad \text{for a.e. } t\in[0,T]\,.$$
From Theorem 2.3, we obtain an estimate

$$\|(u^\varepsilon,p^\varepsilon)\|_{C^{Lip}([0,T];\mathscr{H}_1^\varepsilon)} + \|(\partial_t u^\varepsilon,\partial_t p^\varepsilon)\|_{L^\infty([0,T];\mathscr{H}_1^\varepsilon)} \le C\,,$$

where C only depends on $\|f\|_{L^2(\boldsymbol{Q})}$, $\|u_{Dir}\|_{H^1(0,T;H^1(\boldsymbol{Q}))}$ and $\|p_0^\varepsilon\|_{L^2(\boldsymbol{Q})} \to \|p_0\|_{L^2(\boldsymbol{Q}\times\Omega)}$ and is thus independent from ε. By Theorem 4.10 and Lemma 6.2, there exists $(u,v,p) \in$

$W^{1,\infty}([0,T];\mathscr{H}_1)$ and $\sigma \in W^{1,\infty}([0,T];L^2(\boldsymbol{Q}\times\Omega))$ and a subsequence, still denote $(u^\varepsilon, p^\varepsilon)$, such that for all $t \in [0,T]$

$$
\begin{aligned}
&u^\varepsilon(t) \to u(t) && \text{strongly in } L^2(\boldsymbol{Q}), \quad \partial_t u^\varepsilon(t) \rightharpoonup \partial_t u(t) \quad \text{weakly in } L^2(\boldsymbol{Q}),\\
&p^\varepsilon(t) \overset{2s}{\rightharpoonup} p(t) \qquad \nabla u^\varepsilon(t) \overset{2s}{\rightharpoonup} \nabla u(t) + v(t) && \sigma^\varepsilon(t) \overset{2s}{\rightharpoonup} \sigma(t) \quad \text{weakly in two scales,}\\
&\partial_t p^\varepsilon(t) \overset{2s}{\rightharpoonup} \partial_t p(t) \quad \partial_t \nabla u^\varepsilon(t) \overset{2s}{\rightharpoonup} \partial_t \nabla u(t) + \partial_t v(t) && \partial_t \sigma^\varepsilon(t) \overset{2s}{\rightharpoonup} \partial_t \sigma(t) \quad \text{weakly in two scales,}
\end{aligned}
\tag{6.9}
$$

Next, we define the functional

$$
\mathcal{E} : [0,T]\times\mathscr{B} \to \mathbb{R}, \qquad (t,u,p,v) \mapsto \frac{1}{2}\int_{\boldsymbol{Q}}\int_\Omega \left(p : (Bp) + \sigma : (C\sigma) - 2f(t)u\right), \tag{6.10}
$$

where $C\sigma = \nabla^s(u + u_{Dir}) + v^s - p$. The derivative $D\mathcal{E} = (D_u\mathcal{E}, D_v\mathcal{E}, D_p\mathcal{E})$ can be easily obtained to be

$$
D_u\mathcal{E} = -f(t) - \nabla\cdot\int_\Omega \sigma, \tag{6.11}
$$

$$
D_v\mathcal{E} = \mathbb{P}_{pot}\left(C^{-1}\left(\nabla^s(u + u_{Dir}(t)) + v^s - p\right)\right) \quad \text{and} \quad D_p\mathcal{E} = Bp - \sigma. \tag{6.12}
$$

Here, $\mathbb{P}_{pot} : L^2(\boldsymbol{Q}; L^2(\Omega;\mathbb{R}^{n\times n}_s)) \to L^2(\boldsymbol{Q}; \mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n)^s)$ is the orthogonal projection. From the convergences (6.9) and Lemma 4.6, we obtain

$$
\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u^\varepsilon(t), p^\varepsilon(t)) \geq \mathcal{E}(t, u(t), v(t), p(t)).
$$

From Assumption 4. of the initial conditions, we obtain

$$
\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(0, u^\varepsilon(0), p^\varepsilon(0)) = \mathcal{E}(0, u(0), v(0), p(0)).
$$

Due to linearity, it is easy to verify that

$$
\lim_{\varepsilon\to 0}\int_0^t \partial_t\mathcal{E}_\varepsilon(s, u^\varepsilon(s), p^\varepsilon(s))\,ds = \int_0^t \partial_t\mathcal{E}(s, u(s), v(s), p(s))\,ds.
$$

Furthermore, from Lemma 5.9 and the convergence (6.9) we obtain

$$
\liminf_{\varepsilon\to 0}\int_0^t \Psi_\varepsilon(\partial_t p^\varepsilon) \geq \int_0^t \Psi(\partial_t p) \quad \forall t \in [0,T].
$$

Using the fact that $(u^\varepsilon, p^\varepsilon)$ is an energetic solution of $(\mathscr{H}_1^\varepsilon, \mathcal{E}_\varepsilon, \Psi_\varepsilon)$ and the last four convergence results, we obtain for all $t \in [0,T]$

$$
\mathcal{E}(t, u(t), v(t), p(t)) + \int_0^t \Psi(\partial_t p) \leq \mathcal{E}(0, u(0), v(0), p(0)) - \int_0^t \partial_t\mathcal{E}(s, u(s), v(s), p(s))\,ds.
$$

It only remains to show that (u, v, p) satisfies (2.4) for all $t \in [0,T]$ with $\mathcal{E}$ and Ψ defined above. Then, from Lemma 2.2, we get that (u, v, p) is an energetic solution of $(\mathscr{H}_2, \mathcal{E}, \Psi)$. From (6.11)–(6.12) we then obtain that (u, v, p) solves (6.5).

We chose test functions $\psi \in \mathcal{V}^2_{pot}(\Omega)$ with small potentials $\phi_{\omega,\varepsilon,\psi}$ as in Lemma 6.3 and $\varphi_1, \varphi_2 \in C^1_0(\boldsymbol{Q})$. For all $t \in [0,T]$ we find

$$\int_{\boldsymbol{Q}} f(t) \cdot \varphi_1 = \lim_{\varepsilon\to 0} \int_{\boldsymbol{Q}} f(t) \cdot (\varphi_1 + \varphi_2 \phi_{\omega,\varepsilon,\psi}) = \lim_{\varepsilon\to 0} \int_{\boldsymbol{Q}} \sigma^\varepsilon(t) : (\nabla\varphi_1 + \varphi_2 \psi_\omega(x/\varepsilon) + \phi_{\omega,\varepsilon,\psi} \nabla\varphi_2)$$
$$= \int_{\boldsymbol{Q}} \int_\Omega \sigma(t) : (\nabla\varphi_1 + \varphi_2 \nabla_\omega \psi) \ .$$

We conclude $-\nabla \cdot \int_\Omega \sigma(t) = f(t)$ and $\sigma(t) \in L^2(\boldsymbol{Q}; L^2_{sol}(\Omega))$ for a.e. $t \in [0,T]$ which is equivalent with

$$D_u \mathcal{E}(t, u(t), v(t), p(t)) = 0\,, \qquad D_v \mathcal{E}(t, u(t), v(t), p(t)) = 0\,.$$

Furthermore, $B_\varepsilon p^\varepsilon - \sigma^\varepsilon \overset{2s}{\rightharpoonup} Bp - \sigma$. Since $(p^\varepsilon, u^\varepsilon)$ are energetic solutions, there holds $B_\varepsilon p^\varepsilon - \sigma^\varepsilon \in \mathscr{C}^\varepsilon_2(Q, \omega)$ and from Theorem 5.6 we conclude that

$$D_p \mathcal{E}(t, u(t), v(t), p(t)) \in \mathscr{C}_2(\boldsymbol{Q} \times \Omega) = \partial_p \Psi(0)\,.$$

Therefore, we obtain that $\Psi^*((u,v,p)(t)) = 0$ for all $t \in [0,T]$. Lemma 2.2 yields that (u,v,p) is an energetic solution to (6.5). This concludes the proof.

6.4 Proof of Theorem 6.6

Like in the proof of Theorem 6.5, we can prove the following: for fixed $x \in \boldsymbol{Q}$ and $\xi \in H^1(0,T; \mathbb{R}^{n\times n}_s)$ it holds that $\sigma_\xi \in L^2(0,T; L^2_{sol}(\Omega; \mathbb{R}^n))$, $p_\xi \in L^2(0,T; L^2(\Omega; \mathbb{R}^{n\times n}_s))$ and $v_\xi \in L^2(0,T; \mathcal{V}^2_{pot}(\Omega; \mathbb{R}^n))$ solve (6.6) if and only if (p_ξ, v_ξ) is the unique energetic solution to $\mathscr{H}_2 = \mathscr{H}_1 = L^2(\Omega; \mathbb{R}^{n\times n}_s) \times \mathcal{V}^2_{pot}(\Omega; \mathbb{R}^n)$,

$$\Psi_{x,\xi}(\partial_t p_\xi) = \int_\Omega \psi(x, \cdot, \partial_t p_\xi)\, d\mu_{\mathcal{P}}$$
$$\mathcal{E}_{x,\xi}(t, p_\xi, v_\xi) = \frac{1}{2} \int_{\boldsymbol{Q}} \int_\Omega (p_\xi : (B(x,\cdot) p_\xi) + \sigma_\xi : (C(x,\cdot)\sigma_\xi)) \ ,$$

where $C(x,\cdot)\,\sigma_\xi = \xi + v^s_\xi - p_\xi$. Furthermore, Theorem 2.3 yields the continuity of $\Sigma(x,\cdot)$ with respect to the weak topologies. The equivalence of the formulations 1. and 2. is easy to verify.

7 Coulomb-friction on a rough surface

7.1 Formulation of the problem and homogenization result

In this section, we investigate homogenization of the following problem: We define

$$\boldsymbol{Q} := [-1,1]^{n-1} \times [0,1] \subset \mathbb{R}^n \text{ with } \Gamma_N := [-1,1]^{n-1} \times \{0\} \text{ and } \Gamma_D := \partial\boldsymbol{Q} \backslash \Gamma_N$$

and denote ν the outer normal of Γ_N. We then consider the following elasticity problem with mixed boundary-conditions

$$-\nabla \cdot (A\nabla^s u) = f \quad \text{on } \boldsymbol{Q}, \qquad -\nu \cdot A\nabla^s u \in \partial\psi(\partial_t u) \quad \text{on } \Gamma_N, \qquad u = u_{Dir} \quad \text{on } \Gamma_D\,. \qquad (7.1)$$

We denote by $H^1_{0,D}(\boldsymbol{Q})$ the space of all functions $u \in H^1(\boldsymbol{Q})$ such that $u|_{\Gamma_D} = 0$ and set $\|u\|_{H^1_{0,D}(\boldsymbol{Q})} := \|\nabla u\|_{L^2(\boldsymbol{Q})}$. A function $u \in H^1(0,T;H^1_{0,D}(\boldsymbol{Q}))$ is called a weak solution to (7.1) if and only if

$$\int_{\Gamma_N} \partial\psi(\partial_t u)\varphi + \int_{\boldsymbol{Q}} \nabla^s\varphi : (A\nabla^s(u+u_{Dir})) = \int_{\boldsymbol{Q}} f\cdot\varphi \qquad \forall\varphi \in H^1_{0,D}(\boldsymbol{Q}),$$

where $u_{Dir} \in H^1(0,T;H^1(\boldsymbol{Q}))$ with $u_{Dir}(t)|_{\Gamma_N} \equiv 0$ for all t. Like in the previous section, $A : \boldsymbol{Q} \to \mathbb{R}^{n\times n}_s$ will be a spatially heterogeneous coefficient matrix. Furthermore, ψ will also be spatially heterogeneous.

More precisely, let Assumption 2.4 hold both for $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$ and for $(\Omega_\gamma, \mathcal{B}_{\Omega,\gamma}, \mathcal{P}_\gamma, \tau_\gamma)$, where τ_γ is a $n-1$-dimensional ergodic dynamical system on Ω_γ. We consider the random measures $\omega \mapsto \mathcal{L}^n|_{\boldsymbol{Q}}$ on Ω and $\omega_\gamma \mapsto \mathcal{L}^{n-1}|_\Gamma$ on Ω_γ with Palm measures $\mathcal{P} = \mu_{\mathcal{P}}$ and $\mathcal{P}_\gamma = \mu_{\gamma,\mathcal{P}}$ respectively (see Remark 2.8 a)).

For the formulation of the homogenization result, we make the following assumptions.

1. Let $A \in L^\infty(\boldsymbol{Q}; L^\infty(\Omega; \mathscr{L}(\mathbb{R}^{n\times n}_s, \mathbb{R}^{n\times n}_s)))$ be symmetric a.e. and $\mathcal{B}_Q \otimes \mathcal{B}_\Omega$-measurable. Assume the existence of a constant $\alpha > 0$ such that

$$\alpha\,|\xi|^2 \le \xi A(x,\omega)\xi \le \frac{1}{\alpha}\,|\xi|^2 \qquad \forall\xi \in \mathbb{R}^n \text{ and for a.e. } (x,\omega) \in \boldsymbol{Q}\times\Omega\,. \tag{7.2}$$

   Given $\omega \in \Omega$ and $\varepsilon > 0$, we set $A_{\varepsilon,\omega}(x) := A(x, \tau_{\frac{x}{\varepsilon}}\omega)$.

2. Let $\mathcal{C} : \Gamma\times\Omega_\gamma \to 2^{\mathbb{R}^n}$ be a family of closed convex sets satisfying Assumption 5.4 with $\mathbb{R}^n$ replaced by $\mathbb{R}^{n-1}$, $\boldsymbol{Q}$ replaced by Γ and Ω replaced by Ω_γ. We define

$$\mathcal{I}_{\mathcal{C}}(x) = \left\{ v \in \mathbb{R}^n \,:\, \exists u \in L^1(\Omega_\gamma)\,\text{s.t.}\, v = \int_{\Omega_\gamma} u\, d\mathcal{P}_\gamma \text{ and } u(\omega) \in \mathcal{C}(x,\omega) \text{ a.s.} \right\} \tag{7.3}$$

   and the functions

$$\psi(x,z) := \sup_{v\in\mathcal{I}_{\mathcal{C}}(x)} v\cdot z\,, \qquad \psi_{\varepsilon,\omega_\gamma}(x,z) = \sup_{v\in\mathcal{C}(x,\tau_{\gamma,\frac{x}{\varepsilon}}\omega_\gamma)} v\cdot z\,.$$

   Recalling the definition of Ψ and Ψ_ε in (5.5) with $\boldsymbol{Q}$ replaced by Γ, we then find

$$\Psi(u) := \sup_{\sigma\in\mathscr{C}_2(\Gamma)} \int_\Gamma u\sigma = \int_\Gamma \psi(\cdot,u)\,, \qquad \Psi_\varepsilon(u) := \sup_{\sigma\in\mathscr{C}^\varepsilon_2(\Gamma,\omega_\gamma)} \int_\Gamma u\sigma = \int_\Gamma \psi_{\varepsilon,\omega_\gamma}(\cdot,u)\,.$$

Theorem 7.1. *Let 1.-3. hold. For almost all $\omega \in \tilde\Omega$ and $\omega_\gamma \in \tilde\Omega_\gamma$ the following holds: For every $f_\varepsilon \in H^1(0,T;L^2(\boldsymbol{Q}))$ and every $u^\varepsilon_0 \in H^1(\boldsymbol{Q})$ satisfying*

$$-\nabla\cdot(A_{\varepsilon,\omega}\nabla u^\varepsilon_0) = f_\varepsilon(0)\,, \qquad -\nu\cdot A_{\varepsilon,\omega}\nabla u^\varepsilon_0 \in \mathscr{C}^\varepsilon_2(\Gamma,\omega_\gamma)\,. \tag{7.4}$$

there exists a unique weak solution $u^\varepsilon \in H^1(0,T;H^1(\Omega))$ to the problem

$$-\nabla\cdot(A_{\varepsilon,\omega}\nabla u^\varepsilon) = f_\varepsilon \quad \text{on } \boldsymbol{Q}, \qquad -\nu\cdot A_{\varepsilon,\omega}\nabla u^\varepsilon \in \partial\psi_{\varepsilon,\omega,\gamma}(\partial_t u^\varepsilon) \quad \text{on } \Gamma_N, \qquad u^\varepsilon = u_{Dir} \quad \text{on } \Gamma_D\,, \tag{7.5}$$

satisfying the initial condition $u^\varepsilon(0) = u^\varepsilon_0$.

Furthermore, if $f_\varepsilon \rightharpoonup f$ in $H^1(0,T;L^2(\boldsymbol{Q}))$ and $u_0^\varepsilon \rightharpoonup u_0$ in $H^1(\boldsymbol{Q})$ as $\varepsilon \to 0$, then there exists $u \in H^1(0,T;H^1(\Omega))$ such that $u^\varepsilon \rightharpoonup u$ weakly in $H^1(0,T;H^1(\Omega))$ as $\varepsilon \to 0$ and u is the unique weak solution of

$$-\nabla\cdot(A_{hom}\nabla u) = f \quad \text{on } \boldsymbol{Q}, \qquad -\nu\cdot A_{hom}\nabla u \in \partial\psi(\partial_t u) \quad \text{on } \Gamma_N, \qquad u = u_{Dir} \quad \text{on } \Gamma_D\,. \tag{7.6}$$

Here, A_{hom} is defined through

$$(A_{hom})_{ij} = \min_{v,w\in L^2_{pot}(\Omega)} \int_\Omega (v_i + e_i)A(v_j + e_j)\,, \tag{7.7}$$

where v_i is the unique minimizer of $\int_\Omega (v + e_i)A(v + e_i)$.

7.2 Proof of Theorem 7.1

We consider the following function spaces: Let $L^2(\Gamma_N)$ be the space of square-integrable functions with respect to the Lebesgue measure on Γ_N. Furthermore, let $H^{\frac{1}{2}}(\Gamma_N)$ be the trace space of $H^1_{0,D}(\boldsymbol{Q})$ on Γ_N. Then, the operator

$$H^{\frac{1}{2}}(\Gamma_N) \to H^1_{0,D}(\boldsymbol{Q})\,, \qquad u_\gamma \mapsto \hat{u}_\gamma\,,$$

were $\hat{u}_\gamma$ solves

$$-\Delta\hat{u}_\gamma = 0\,, \quad \text{on } \boldsymbol{Q}\,, \qquad \hat{u}_\gamma|_{\Gamma_N} = u_\gamma\,, \qquad \hat{u}_\gamma|_{\Gamma_D} = 0\,,$$

is well defined.

Lemma 7.2. *For $u \in H^1_{0,D}(\boldsymbol{Q})$ set $u_\gamma := u|_{\Gamma_N}$ and $u_q := u - \hat{u}_\gamma$. Then , the mapping $u \mapsto (u_\gamma, u_q)$ is an isomorphism $H^1_{0,D}(\boldsymbol{Q}) \to H^{1/2}(\Gamma_N) \otimes H^1_0(\boldsymbol{Q})$ and becomes an isometry with respect to the norms*

$$\|u\|_{H^1_{0,D}(\boldsymbol{Q})} := \|\nabla u\|_{L^2(\boldsymbol{Q})} \quad \text{and} \quad \|(u_\gamma, u_q)\|_{H^{1/2}(\Gamma_N)\otimes H^1_0(\boldsymbol{Q})} := \|\nabla(u_q + \hat{u}_\gamma)\|_{L^2(\boldsymbol{Q})}\,.$$

Proof. Let $u \in H^1_{0,D}(\boldsymbol{Q})$. Then $u_\gamma \in H^{1/2}(\Gamma_N)$ and $u_q \in H^1_0(\boldsymbol{Q})$. For $(u_\gamma, u_q) \in H^{1/2}(\Gamma_N) \otimes H^1_0(\boldsymbol{Q})$, we note that $u_q + \hat{u}_\gamma \in H^1_{0,D}(\boldsymbol{Q})$. □

Thus, for $u \in H^1_{0,D}(\boldsymbol{Q})$ we equally write (u_q, u_γ) and identify $\hat{u}_\gamma \simeq u_\gamma$ if this will not cause confusion.

For simplicity of notation, we write:

1. $\tilde{\mathscr{H}} = L^2(\boldsymbol{Q}) \times L^2(\Gamma_N)$, $\mathscr{H}_2 = \tilde{\mathscr{H}} \times L^2(\boldsymbol{Q}; \mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n))$ and $\mathscr{H}_2^\varepsilon = L^2(\boldsymbol{Q}) \times L^2(\Gamma_N)$ for all $\varepsilon > 0$.
2. $\mathscr{H}_1 = H^1_0(\boldsymbol{Q}) \times H^{1/2}(\Gamma_N) \times L^2(\boldsymbol{Q}; \mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n))$ and $\mathscr{H}_1^\varepsilon = H^1_0(\boldsymbol{Q}) \times H^{1/2}(\Gamma_N)$

Existence of solutions for the ε-problem We define the family of functionals

$$\mathcal{E}_{\varepsilon,\omega} : [0,T] \times \mathscr{H}_2^\varepsilon \to \mathbb{R}\,, \tag{7.8}$$

$$(t, u_q, u_\gamma) \mapsto \frac{1}{2}\int_{\boldsymbol{Q}} \nabla^s u : (A_{\varepsilon,\omega}\nabla^s (u + 2u_{Dir}(t))) - \int_{\boldsymbol{Q}} f_\varepsilon (u + u_{Dir}(t))\,, \tag{7.9}$$

From Lemma 7.2, we obtain that

$$\|u\|_{\mathscr{H}_1^\varepsilon} := \int_{\boldsymbol{Q}} \nabla^s (u_q + \tilde{u}_\gamma) : (A_{\varepsilon,\omega}\nabla^s (u_q + \tilde{u}_\gamma)) = \int_{\boldsymbol{Q}} \nabla^s u : (A_{\varepsilon,\omega}\nabla^s u)$$

is an equivalent norm on $\mathscr{H}_1^\varepsilon$. We find

$$D_{u_q}\mathcal{E}_{\varepsilon,\omega} = \nabla\cdot (A_{\varepsilon,\omega}\nabla^s (u_q + u_{Dir}(t) + \tilde{u}_\gamma)) - f_\varepsilon\,, \qquad D_{u_\gamma}\mathcal{E} = -\nu\cdot A_{\varepsilon,\omega}\nabla^s (u_q + u_{Dir}(t) + \tilde{u}_\gamma)\,.$$

Furthermore, due to our assumptions, u_0^ε satisfies (2.4) for every $\varepsilon > 0$. Therefore, Theorem 2.3 yields existence of a unique energetic solution $u^\varepsilon \in C^{Lip}([0,T];\mathscr{H}_1^\varepsilon)$ to

$$0 \in \partial\Psi_{\varepsilon,\omega}(\partial_t u_\gamma^\varepsilon) + D\mathcal{E}_{\varepsilon,\omega}(t, u^\varepsilon)\,.$$

The solution u^ε also solves $(7.5)_\omega$.

Passage to the limit $\varepsilon \to 0$ From Theorem 2.3, we get uniform bounds $\|u^\varepsilon\|_{C^{Lip}([0,T];\mathscr{H}_1^\varepsilon)} \le C$. From Lemma 6.2 and Lemma 4.10, we find a subsequence, still labeled u^ε, and functions $u \in C^{Lip}(0,T;H^1_{0,D}(\boldsymbol{Q}))$, $v \in C^{Lip}(0,T;\mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n))$ such that $u^\varepsilon(t) \rightharpoonup u(t)$ weakly in $H^1(\boldsymbol{Q})$ and $\nabla u^\varepsilon(t) \overset{2s}{\rightharpoonup} \nabla u(t) + v(t)$ for every $t \in [0,T]$. By Rellich's embedding theorem, $u_\gamma^\varepsilon(t) \to u_\gamma(t)$ strongly in $L^2(\Gamma_N)$ for every $t \in [0,T]$.

We define the functional

$$\mathcal{E} : [0,T] \times \mathscr{B} \to \mathbb{R}\,, \tag{7.10}$$

$$(t, u_q, v, u_\gamma) \mapsto \frac{1}{2}\int_{\boldsymbol{Q}}\int_\Omega [\nabla^s u + v^s] : [A\,(\nabla^s (u + 2u_{Dir}(t)) + v^s)] - \int_{\boldsymbol{Q}} f(u + u_{Dir}(t))\,, \tag{7.11}$$

and note that the above convergences of u^ε imply (by Lemma 4.6) that

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u^\varepsilon(t)) \ge \mathcal{E}(t, u(t), v(t))\,,$$

as well as (by Lemmas 5.9 and 5.10)

$$\liminf_{\varepsilon\to 0} \int_0^t \Psi_\varepsilon(\partial_t u^\varepsilon) \ge \int_0^t \Psi(\partial_t u) \quad \forall t \in [0,T]\,.$$

Furthermore, for $t = 0$ we obtain due to (7.4) that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(0, u^\varepsilon(0)) = \mathcal{E}(0, u(0), v(0))\,.$$

The last three convergence results imply

$$\mathcal{E}(t, u(t), v(t)) + \int_0^t \Psi(\partial_t u) \le \mathcal{E}(0, u(0), v(0)) + \int_0^t \partial_t\mathcal{E}(s, u(s), v(s))\,ds\,. \tag{7.12}$$

Stability Now, let $\phi_1 \in H^1_{0,D}(\boldsymbol{Q})$, $\varphi \in C^1_0(\boldsymbol{Q})$, $\psi \in L^2_{pot}(\Omega)$ with a potential $\phi_{\varepsilon,\omega,\psi}$ from Lemma 6.3 and set $\varphi_\varepsilon(x) := \varphi(x)\phi_{\varepsilon,\omega,\psi}$. By the strong convergences $\varphi_\varepsilon \to 0$, $u^\varepsilon \to u$ and Lemma 5.9 we then obtain

$$\begin{aligned}\mathcal{E}(t,u(t),v(t)) &\le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t,u^\varepsilon(t)) \le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t,\phi_1+\varphi_\varepsilon) + \Psi_\varepsilon(\phi_1+\varphi_\varepsilon-u^\varepsilon(t)) \\ &= \mathcal{E}(t,\phi_1,\psi) + \Psi(\phi_1 - u(t)) \,.\end{aligned}$$

Since $\nabla_\omega\varphi$ for $\varphi \in C^1_b(\Omega;\mathbb{R}^n)$ are dense in $L^2_{pot}(\Omega;\mathbb{R}^n)$, we obtain that for all $\phi \in H^1_{0,D}(\boldsymbol{Q})$, $w \in L^2(\boldsymbol{Q};L^2_{pot}(\Omega;\mathbb{R}^n))$ there holds

$$\mathcal{E}(t,u(t),v(t)) \le \mathcal{E}(t,\phi,w) + \Psi(\phi - u(t)) \,. \tag{7.13}$$

Thus, by Lemma 2.2, (u,v) is an energetic solution to $(\mathscr{H}_2,\mathcal{E},\Psi)$.

Macroscopic model The derivative $D\mathcal{E} = (D_{uq}\mathcal{E}, D_v\mathcal{E}, D_{u\gamma}\mathcal{E})$ can be easily obtained to be

$$D_{uq}\mathcal{E} = -f(t) - \nabla\cdot\left[A\left(\nabla^s\left(u_q + u_{Dir}(t) + u_\gamma\right) + v^s\right)\right] , \tag{7.14}$$
$$D_v\mathcal{E} = \mathbb{P}_{pot}\left(A\left(\nabla^s\left(u_q + u_{Dir}(t) + u_\gamma\right) + v^s\right)\right) \quad \text{and} \tag{7.15}$$
$$D_{u\gamma}\mathcal{E} = -\nu\cdot A\left(\nabla^s\left(u_q + u_{Dir}(t) + u_\gamma\right) + v^s\right) \tag{7.16}$$

Here, $\mathbb{P}_{pot} : L^2(\boldsymbol{Q};L^2(\Omega;\mathbb{R}^{n\times n}_s)) \to L^2(\boldsymbol{Q};\mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n)^s)$ is the orthogonal projection.

From $D_v\mathcal{E} = 0$, we obtain that

$$v = \sum_{j=1}^n \partial_j\left(u_q + u_{Dir}(t) + u_\gamma\right)\phi_j \,,$$

where $\phi_j \in L^2(\boldsymbol{Q};\mathcal{V}^2_{pot}(\Omega;\mathbb{R}^n))$ is the unique minimizer of

$$\mathcal{E}_j(\phi) := \frac{1}{2}\int_{\boldsymbol{Q}}\int_\Omega \left[e_j + \phi^s\right] : \left[A\,\left(e_j + \phi^s\right)\right] .$$

Pluging this information into (7.12)–(7.13), we find that $u \in C^{Lip}([0,T];H^1_{0,D}(\boldsymbol{Q}))$ is an energetic solution to $(\tilde{\mathscr{H}},\tilde{\mathcal{E}},\Psi)$, where

$$\tilde{\mathcal{E}}(t,u) :=\mapsto \frac{1}{2}\int_{\boldsymbol{Q}}\int_\Omega \left[\nabla^s\left(u_q + u_{Dir}(t) + u_\gamma\right)\right] : \left[A_{hom}\,\nabla^s\left(u_q + u_{Dir}(t) + u_\gamma\right)\right] - \int_{\boldsymbol{Q}} f u \,,$$

and A_{hom} is defined through (7.7).

8 Random Fissures

In this section, we will provide the theory that is necessary to formulate and to prove the results of Section 9.

8.1 Geometric construction

As a special case of stochastic geometries, we introduce random fissures. Let Assumption 2.4 hold for $(\Omega, \mathcal{B}_\Omega, \mathcal{P}, \tau)$.

Let $0 < r < \frac{1}{2}$, $B_r := B_r(0)$ and $B := B_{\frac{1}{2}}(0)$. Let $\Gamma_* = \{0\} \times D$ for $D = \{x' \in \mathbb{R}^{n-1} : \|x'\| < r\}$. Assume there exists a set of parameters $U \subset \mathbb{R}^N$, $N \in \mathbb{N}$, and a function $f : \overline{B} \times \overline{U} \to \overline{B}$ such that U is a bounded domain and such that

1. f is continuous
2. For every $y \in U$, the mapping $f(\cdot, y)$ lies in $C^2(\overline{B})$ and
$$\sup_{y \in U} \|f(\cdot, y)\|_{C^2(\overline{B})} < \infty .$$
3. For every $y \in U$, the function
$$f(\cdot, y) : \mathbb{R}^n \to \mathbb{R}^n , \qquad x \mapsto \begin{cases} f(x, y) & \text{if } x \in \overline{B} \\ x & \text{else} \end{cases}$$
lies in $C^2(\mathbb{R}^n)$.

We introduce the sets $A := \{0, 1\} \times \overline{U}$ and $\Omega := [0, 1[^n \times A^{\mathbb{Z}^n}$ and write every $\omega \in \Omega$ in the form $\omega = (y, a)$ with $y \in [0, 1[^n$, $a \in A^{\mathbb{Z}^n}$. Given $j \in \mathbb{Z}^n$, we refer to the j-th coordinate of a by $a_j = (a_{j,1}, a_{j,2})$ with $a_{j,1} \in \{0, 1\}$, $a_{j,2} \in \overline{U}$. Given any probability measure $\mathcal{P}_A$ on A and the Lebesgue measure $\mathcal{L}$, we define the measure $\mathcal{P} := \mathcal{L} \otimes \bigotimes_{j \in \mathbb{Z}^n} \mathcal{P}_A$ on Ω. Given $\omega \in \Omega$ we introduce the set
$$\Gamma(\omega) := y + \prod_{j \in \mathbb{Z}^n : a_{j,1} = 1} f(\Gamma_*, a_{j,2}) .$$

8.2 Sobolev spaces

We now focus on Sobolev spaces on random fisures. The construction of such spaces goes back to [14], where it was used in context of random tessellations, and was later used in [9, 10] in a similar setting. Since random fissures also share all properties of random tessallations that where needed in [9, 13] and results proved there also hold in the current setting. Given a random fissure $\Gamma = \Gamma(\omega)$ with $G := \mathbb{R}^n \backslash \Gamma$, we define the following spaces:

$$C^1(G) := \{u : \mathbb{R}^n \to \mathbb{R} \mid u|_G \in C^1(G)\} , \qquad C^1_0(\boldsymbol{Q} \cap G) := \{u \in C^1(G) \mid u|_{\partial \boldsymbol{Q}} = 0\} .$$

Using ν_Γ we define the *trace operators*:

$$\pm : \quad C(G) \to C(\Gamma_{ns}) , \qquad u_\pm(x) := \lim_{t \downarrow 0} u(x \pm t\nu_\Gamma(x)) \text{ for } x \in \Gamma_{ns} ,$$

where $\Gamma_{ns} = \Gamma \backslash \gamma$ denotes the non-singular part of Γ. On noting that the operator $[\![u]\!] := u_+ - u_-$ is well defined for $u \in C^1_b(G)$, we define the norm

$$\|u\|_{H^1(\boldsymbol{Q} \cap G)} := \left(\int_{\boldsymbol{Q}} u^2 \, d\mathcal{L} + \int_{\boldsymbol{Q} \cap G} |\nabla u|^2 \, d\mathcal{L} + \int_\Gamma [\![u]\!]^2 d\mathcal{H}^{n-1} \right)^{\frac{1}{2}}$$

and define $H^1(\boldsymbol{Q}\cap G)$ and $H_0^1(\boldsymbol{Q}\cap G)$ as the closure of $C_b^1(G)$ and $C_0^1(\boldsymbol{Q}\cap G)$ with respect to $\|\cdot\|_{H^1(\boldsymbol{Q}\cap G)}$. Note in this context that $[\![\cdot]\!]$ extends to an operator

$$[\![\cdot]\!]:\, H^1(G)\to H^{\frac{1}{2}}(\Gamma)\,.$$

For every $\phi\in H^1(G)$ and $\psi\in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ there holds:

$$\int_G \phi\nabla\cdot\psi d\mathcal{L} = -\int_G (\nabla\phi)\cdot\psi d\mathcal{L} - \int_\Gamma [\![\phi]\!]\psi\cdot\nu_\Gamma d\mathcal{H}^{n-1} \tag{8.1}$$

Writing $\mu_\Gamma(B) := \mathcal{H}^{n-1}(\Gamma\cap B)$, we then find the following trivial fact:

$$H^1(\boldsymbol{Q}\cap G) = \Bigg\{u\in L^2(\boldsymbol{Q})\;\Big|\;\exists Du\in L^2(\boldsymbol{Q},\mathcal{L}|_G+\mu_\Gamma)^n\,: \\ \int_{\boldsymbol{Q}} u\nabla\cdot\phi d\mathcal{L} = \int_{\boldsymbol{Q}}\phi\cdot Du\, d(\mathcal{L}|_G+\mu_\Gamma)\;\forall\phi\in C_0^\infty(\boldsymbol{Q})^n\Bigg\}. \tag{8.2}$$

This motivates the following definitions:

$$\begin{aligned}
&L^2(\boldsymbol{Q},G,\Gamma)^n := L^2(\boldsymbol{Q},\mathcal{L}|_G+\mu_\Gamma)^n\,,\\
&L^2_{pot}(\boldsymbol{Q},G,\Gamma) := \{\phi\in L^2(\boldsymbol{Q},G,\Gamma)^n\;\big|\;\exists u\in H^1(\boldsymbol{Q}\cap G)\,:\,Du=\phi\}\,,\\
&L^2_{sol}(\boldsymbol{Q},G,\Gamma) := \left\{\phi\in L^2(\boldsymbol{Q},G,\Gamma)^n\;\Big|\forall u\in H_0^1(\boldsymbol{Q}\cap G)\,:\,\int_{\boldsymbol{Q}\cap G}\phi\cdot\nabla u\,d\mathcal{L}+\int_\Gamma\phi\cdot[\![u]\!]\nu_\Gamma d\mu_\Gamma=0\right\}.
\end{aligned}$$

for a bounded and open $\boldsymbol{Q}\subset\mathbb{R}^n$. Moreover, we define

$$\begin{aligned}
L^2_{loc}(G,\Gamma)^n &:= \{\phi\;\big|\;\phi\in L^2(U,\mathcal{L}|_G+\mu_\Gamma)^n\;\forall\text{ open and bounded }U\subset\mathbb{R}^n\}\,,\\
L^2_{loc,pot}(G,\Gamma) &:= \{\phi\in L^2_{loc}(G,\Gamma)^n\;\big|\;\phi\in L^2_{pot}(U,G,\Gamma)\;\forall\text{ open and bounded }U\subset\mathbb{R}^n\}\\
L^2_{loc,sol}(G,\Gamma) &:= \{\phi\in L^2_{loc}(G,\Gamma)^n\;\big|\;\phi\in L^2_{sol}(U,G,\Gamma)\;\forall\text{ open and bounded }U\subset\mathbb{R}^n\}
\end{aligned}$$

Lemma 8.1 (Orthogonal decomposition Lemma [9, Lemma 4.10])**.** *Let $\boldsymbol{Q}\subset\mathbb{R}^n$ be a bounded domain. Then*

$$L^2(\boldsymbol{Q},G,\Gamma) = L^2_{pot}(\boldsymbol{Q},G,\Gamma)\oplus L^2_{sol}(\boldsymbol{Q},G,\Gamma)$$

and for every $\phi\in L^2_{sol}(\boldsymbol{Q},G(\omega),\Gamma(\omega))$ holds $\nabla\cdot\phi=0$ on $G(\omega)$ in the sense of distribution.

Sobolev spaces on Γ_* We denote $\partial\Gamma_* := \{0\}\times\partial D$ and $H_0^{\frac{1}{2}}(\Gamma_*) := \{[\![u]\!]\,:\,u\in H^1(B\backslash\Gamma_*)\}$. As a norm on $H_0^{\frac{1}{2}}(\Gamma_*)$ we chose

$$\|u\|_{H_0^{\frac{1}{2}}(\Gamma_*)} := \inf\left\{\|\tilde u\|_{H^1(B\backslash\Gamma_*)}\,:\,[\![\tilde u]\!]=u\right\}\,. \tag{8.3}$$

We note in this context that $C_0^1(\Gamma_*) := \{g\in C^1(\Gamma_*)\,:\,g|_{\partial\Gamma_*}=0\}$, is dense in $H_0^{\frac{1}{2}}(\Gamma_*)$ since $C^1(B\backslash\Gamma_*)\cap H^1(B\backslash\Gamma_*)$ is dense in $H^1(B\backslash\Gamma_*)$. However, we still have to show that $H_0^{\frac{1}{2}}(\Gamma_*)$ is a Hilbert space.

Lemma 8.2. *The linear operator* $H_0^{\frac{1}{2}}(\Gamma_*) \to H^1(B\backslash\Gamma_*)$, $g \mapsto u_g$ *given through*

$$-\Delta u_g = 0 \quad \text{on } B\backslash\Gamma_*\,, \qquad [\![u_g]\!]|_{\Gamma_*} = g\,, \qquad u_g|_{\partial B} = 0$$

is continuous and the space $H_0^{\frac{1}{2}}(\Gamma_*)$ *with norm (8.3) is a Hilbert space. Furthermore, it holds*

$$\|[\![u]\!]\|_{L^2(\Gamma_*)} + \|\nabla u\|_{L^2(B\backslash\Gamma_*)} \le C\,\|\nabla^s u\|_{L^2(B\backslash\Gamma_*)}\,. \tag{8.4}$$

Proof. Let $\tilde g \in H^1(B\backslash\Gamma_*)$ be a minimizer of $\|g\|_{H_0^{\frac{1}{2}}(\Gamma_*)}$. There exists a continuous operator $\tilde g \mapsto \hat g \in H^1(B)$ such that $\hat g|_{\partial B} = \tilde g|_{\partial B}$ and $\|\hat g\|_{H^1(B)} \le C\,\|\tilde g\|_{H^1(B\backslash B_r)}$. We now solve the problem

$$-\Delta \hat u_g = -\Delta(-\tilde g + \hat g) \quad \text{on } B\backslash\Gamma_*\,, \qquad [\![\hat u_g]\!]|_{\Gamma_*} = 0\,, \qquad \hat u_g|_{\partial B} = 0\,,$$

which has a unique solution. Setting $u_g = \hat u_g + \tilde g - \hat g$ the operator $g \mapsto u_g$ is continuous by construction. The space $H_0^{\frac{1}{2}}(\Gamma_*)$ is complete since for any Cauchy sequence $(g_n)_{n\in\mathbb{N}}$, also $(u_{g_n})_{n\in\mathbb{N}}$ is a Cauchy sequence.

We introduce $Z_D := [-\frac{1}{2}, \frac{1}{2}]\times D$. In order to prove (8.4), assume there exists a sequence $u_n \in H^1(B\backslash\Gamma_*)$ such that $\|[\![u_n]\!]\|_{L^2(\Gamma_*)} + \|\nabla u_n\|_{L^2(B\backslash\Gamma_*)} = 1$ for all $n \in \mathbb{N}$ but with $\|\nabla^s u_n\|_{L^2(B\backslash\Gamma_*)} \to 0$. Without loss of generality, we may assume that $\int_{B\backslash Z_D} u_n = 0$. From classical Korn's inequality, we can deduce that $\nabla u_n \to 0$ in $L^2(B\backslash U)$ for all open sets $\Gamma_* \subset U \subset B$. By Sobolev's inequality, we obtain that $u_n \to u$ in $L^2(B\backslash U)$ and that u is constant. In particular, we obtain $u_n \to 0$ in $H^1(B\backslash Z_D)$. Furthermore, we find

$$\|u_n\|_{L^2(B\cap Z_D)} \le C\left(\|\nabla u_n\|_{L^2(B\cap Z_D)} + \|u_n\|_{H^{1/2}(B\cap\partial Z_D)}\right) \le C\left(\|\nabla^s u_n\|_{L^2(B\cap Z_D)} + \|u_n\|_{H^1(B\backslash Z_D)}\right).$$

Therefore, $u_n \to u$ in $L^2(B)$ and $\nabla u_n \to 0$ in $L^2(B\backslash\Gamma_*)$. This implies $\|[\![u_n]\!]\|_{L^2(\Gamma_*)} \to 0$, which is a contradiction to the initial assumptions. □

Random fissures Let $\Gamma(\omega)$ be a random fissure. Then, according to Lemma 3.4, $\mu_{\Gamma(\omega)} := \mathcal{H}^{n-1}(\Gamma(\omega)\cap\cdot)$ is a random measures with corresponding Palm measure $\mu_{\Gamma,\mathcal{P}}$. By Theorem 3.5, there exists a prototype $\tilde\Gamma \subset \Omega$ of $\Gamma(\omega)$ such that $\mu_{\Gamma,\mathcal{P}}$ concentrates on $\tilde\Gamma$. We set $\tilde G := \Omega\backslash\tilde\Gamma$ and $G(\omega) := \mathbb{R}^n\backslash\Gamma(\omega)$. The measures $\mu_\omega(B) := \mathcal{L}(B\cap G(\omega))$ also define a stationary random measure with $\mu_\omega = \mathcal{L}|_G$ and Palm measure $\mu := \mathcal{P}|_{\tilde G}$ (see Theorem 3.5).

We set $L^2(G,\Gamma) := L^2(\Omega, \mu + \mu_{\Gamma,\mathcal{P}})^n$ and

$$L^2_{pot}(\Omega, G, \Gamma) := \left\{u \in L^2(\Omega, \mu + \mu_{\Gamma,\mathcal{P}})^n \;\middle|\; u(\tau_x\omega) \in L^2_{loc,pot}(G(\omega), \Gamma(\omega)) \text{ for } \mu\text{-a.e. } \omega\right\}, \tag{8.5}$$

$$L^2_{sol}(\Omega, G, \Gamma) := \left\{u \in L^2(\Omega, \mu + \mu_{\Gamma,\mathcal{P}})^n \;\middle|\; u(\tau_x\omega) \in L^2_{loc,sol}(G(\omega), \Gamma(\omega)) \text{ for } \mu\text{-a.e. } \omega\right\}. \tag{8.6}$$

From Section 4 of [9], we know that both $L^2_{pot}(G,\Gamma)$ and $L^2_{sol}(G,\Gamma)$ are nonempty. Unfortunately, there exists no orthogonal decomposition result similar to Lemma 8.1. However, we can get the following result:

Lemma 8.3. *[9, Lemma 4.13]* $L^2_{sol}(G,\Gamma)$ *and* $L^2_{pot}(G,\Gamma)$ *are closed subspaces of* $L^2(G,\Gamma)$ *and*

$$L^2_{sol}(G,\Gamma)^\perp \subset L^2_{pot}(G,\Gamma)\,.$$

8.3 Two-scale convergence

Given any random hyperface $\Gamma(\omega)$ with $G(\omega) := \mathbb{R}^n \backslash \Gamma(\omega)$ and any $\varepsilon > 0$, we set $G^\varepsilon(\omega) := \varepsilon G(\omega)$ and $\Gamma^\varepsilon(\omega) := \varepsilon\Gamma(\omega)$. We then define

$$\begin{aligned} \mu_\omega^\varepsilon(B) &:= \varepsilon^n \mu_\omega(\varepsilon^{-1}B)\,, \\ \mu_{\Gamma(\omega)}^\varepsilon(B) &:= \varepsilon^n \mu_{\Gamma(\omega)}(\varepsilon^{-1}B) = \varepsilon\mathcal{H}^{n-1}(\Gamma^\varepsilon(\omega)\cap B)\,. \end{aligned}$$

We aim for a two-scale-convergence result in the spirit of Lemma 6.2 for functions with jumps. To this aim, we need some Poincaré inequalities and some Rellich-type embedding results for spaces $H^1(\boldsymbol{Q}\cap G^\varepsilon(\omega))$.

Lemma 8.4. *Let $\Gamma(\omega)$ be given by the construction in Section 8.1.*

1. There exists $0 < M < \infty$ such that for almost every $\omega \in \Omega$ there holds

$$\#\{\Gamma(\omega)\cap(t,t+s)\} \le M \text{ for almost all } t\in\mathbb{R}^n \text{ and } s\in S^{n-1}\,, \tag{8.7}$$

where (x,y) is the line segment between $x,y\in\mathbb{R}^n$ and $\#$ is the cardinality of a set.

2. There exists a constant $C > 0$ such that for all $\varepsilon > 0$, all $\omega\in\Omega$ and all $u \in H^1(Q\backslash\Gamma^\varepsilon(\omega))$, there holds

$$\left\|\frac{1}{\varepsilon}[\![u]\!]\right\|_{L^2(\Gamma^\varepsilon(\omega))} + \|\nabla u\|_{L^2(Q\backslash\Gamma^\varepsilon(\omega))} \le C\,\|\nabla^s u\|_{L^2(Q\backslash\Gamma^\varepsilon(\omega))}\,.$$

3. The space $H^1(Q\backslash\Gamma^\varepsilon(\omega))$ is isomorph to $H^1(Q)\times H_0^{\frac12}(\Gamma^\varepsilon(\omega))$. For $u\in H^1(Q\backslash\Gamma^\varepsilon(\omega))$, we equally write $u = (u_q, u_\gamma)$ with $u_q\in H^1(Q)$ and $[\![u_\gamma]\!]\in H_0^{\frac12}(\Gamma^\varepsilon(\omega))$.

Proof. The first statement is evident. Concerning the proof of 2. and 3. note that this follows from Lemma 8.2 in combination with a simple scaling argument. □

The following two results were proved for so called random tessellations, a special case of random hyperfaces. However, the original proof does not require Γ to be a tessellation, but only that Γ is a hyperface that satisfies Condition (8.7). A further generalization, going beyond Condition (8.7), can be found in [10].

Proposition 8.5 (Compactness property [14])**.** *Let $\boldsymbol{Q}$ be a bounded domain in $\mathbb{R}^n$. A fissure Γ satisfying Condition (8.7) has the following compactness property: For any $s\in]0,\frac12[$ exists a constant C_s independent on ε such that for every $\varepsilon>0$ and every $\phi^\varepsilon\in H_0^1(\boldsymbol{Q}\cap G^\varepsilon)$:*

$$\|\phi^\varepsilon\|^2_{H_0^s(\boldsymbol{Q})} \le C_s\left(\int_{\boldsymbol{Q}\cap G^\varepsilon}|\nabla\phi^\varepsilon|^2 d\mathcal{L} + \varepsilon^{-1}\int_{\boldsymbol{Q}\cap\Gamma^\varepsilon}[\![\phi^\varepsilon]\!]^2 d\mathcal{H}^{n-1}\right)\,. \tag{8.8}$$

Furthermore, for every $\phi^\varepsilon\in H^1(\boldsymbol{Q}\cap G^\varepsilon)$ there holds

$$\|\phi^\varepsilon\|^2_{H^s(\boldsymbol{Q})} \le C_s\left(\int_{\boldsymbol{Q}\cap G^\varepsilon}|\nabla\phi^\varepsilon|^2 d\mathcal{L} + \varepsilon^{-1}\int_{\boldsymbol{Q}\cap\Gamma^\varepsilon}[\![\phi^\varepsilon]\!]^2 d\mathcal{H}^{n-1} + \left(\int_{\boldsymbol{Q}\cap G^\varepsilon}\phi^\varepsilon d\mathcal{L}\right)^2\right)\,. \tag{8.9}$$

The last Proposition implies the following important consequence:

Lemma 8.6. *Let $v\in L^2_{pot}(G,\Gamma)$ and let $\boldsymbol{Q}\subset\mathbb{R}^n$ a bounded domain and $\omega\in\Omega$ such that v_ω has the ergodicity property. Then, for every $\varepsilon>0$ there exists $\phi_\varepsilon\in H^1(\boldsymbol{Q}\cap G^\varepsilon(\omega))$ such that $D\phi_\varepsilon(x) = w(\tau_{\frac{x}{\varepsilon}}\omega)$ and $\|\phi_\varepsilon\|_{L^2(\boldsymbol{Q})}\to 0$ as $\varepsilon\to\infty$.*

Proof. By definition of $L^2_{pot}(G,\Gamma)$, we obtain a sequence $\phi_\varepsilon \in H^1(\boldsymbol{Q}\cap G^\varepsilon(\omega))$ such that $\nabla\phi_\varepsilon(x) = v|_G(\tau_{\frac{x}{\varepsilon}}\omega)$ and $[\![\phi_\varepsilon]\!](x) = \varepsilon v|_\Gamma(\tau_{\frac{x}{\varepsilon}}\omega)$ and $\int_{\boldsymbol{Q}}\phi_\varepsilon = 0$. Thus, by (8.8) we find

$$\begin{aligned}\limsup_{\varepsilon\to0}\|\phi^\varepsilon\|^2_{H^s(\boldsymbol{Q})} &\le C\limsup_{\varepsilon\to0}\left(\int_{\boldsymbol{Q}\cap G^\varepsilon}|\nabla\phi^\varepsilon|^2 d\mathcal{L} + \varepsilon^{-1}\int_{\boldsymbol{Q}\cap\Gamma^\varepsilon}[\![\phi^\varepsilon]\!]^2 d\mathcal{H}^{n-1}\right)\\ &\le C\limsup_{\varepsilon\to0}\left(\int_{\boldsymbol{Q}\cap G^\varepsilon}v^2(\tau_{\frac{x}{\varepsilon}}\omega)d\mathcal{L} + \varepsilon\int_{\boldsymbol{Q}\cap\Gamma^\varepsilon}v^2(\tau_{\frac{x}{\varepsilon}}\omega)d\mathcal{H}^{n-1}\right)\\ &= C\,\|v\|^2_{L^2(G,\Gamma)}\,,\end{aligned}$$

such that ϕ_ε is precompact in $L^2(\boldsymbol{Q})$. For any $\psi\in H^1_0(\boldsymbol{Q})^n\cap C_b(\boldsymbol{Q})^n$ we obtain

$$\begin{aligned}\int_{\boldsymbol{Q}}\phi\nabla\cdot\psi &= \lim_{\varepsilon\to0}\int_{\boldsymbol{Q}}\phi_\varepsilon\nabla\cdot\psi = \lim_{\varepsilon\to0}\int_{\boldsymbol{Q}\cap G^\varepsilon(\omega)}\nabla\phi_\varepsilon\cdot\psi - \int_{\boldsymbol{Q}\cap\Gamma^\varepsilon(\omega)}[\![\phi_\varepsilon]\!]\nu_{\Gamma^\varepsilon}\cdot\psi\\ &= \lim_{\varepsilon\to0}\int_{\boldsymbol{Q}\cap G^\varepsilon(\omega)}v(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x) - \int_{\boldsymbol{Q}\cap\Gamma^\varepsilon(\omega)}\varepsilon v(\tau_{\frac{x}{\varepsilon}}\omega)\nu_{\Gamma^\varepsilon}\cdot\psi(x)\\ &= \int_{\boldsymbol{Q}}\left(\int_G v + \int_\Gamma v\right)\psi = 0\,,\end{aligned}$$

where we used that constants are in $L^2_{sol}(G,\Gamma)$. Thus, $\phi_\varepsilon\rightharpoonup0$ in $L^2(\boldsymbol{Q})$ which implies $\phi_\varepsilon\to0$ as $\varepsilon\to0$ due to precompactness of ϕ_ε. □

From Proposition 8.5, we also obtain the following two-scale convergence result.

Proposition 8.7. *[9] For a random tessellation $(G(\omega),\Gamma(\omega))$ that fulfills the compactness property 8.5 in $\mathbb{R}^n$ with $\boldsymbol{Q}\subset\mathbb{R}^n$ bounded and open and fixed $\omega\in\Omega$ let $u^\varepsilon\in H^1_0(\boldsymbol{Q}\cap G^\varepsilon(\omega))$ with*

$$\|\nabla u^\varepsilon\|^2_{L^2(\boldsymbol{Q}\cap G^\varepsilon(\omega))} + \frac{1}{\varepsilon}\|[\![u^\varepsilon]\!]\|^2_{L^2(\boldsymbol{Q}\cap\Gamma^\varepsilon(\omega))} \le C$$

Then there are $u\in H^1_0(\boldsymbol{Q})$ and $u_1\in L^2(\boldsymbol{Q},\mathcal{L};L^2_{pot}(G,\Gamma))$ such that as $\varepsilon\to0$:

$$\begin{aligned}u^\varepsilon &\to u \qquad \text{in } L^2(\boldsymbol{Q}))\,,\\ \nabla u^\varepsilon &\overset{2s}{\rightharpoonup} \nabla u + u_1\big|_G\,,\\ \frac{1}{\varepsilon}[\![u^\varepsilon]\!]\nu_{\Gamma^\varepsilon(\omega)} &\overset{2s}{\rightharpoonup} u_1\big|_\Gamma\,.\end{aligned} \tag{8.10}$$

9 Coulomb-friction on a microstructure

We study the stochastic homogenization of a problem of elasticity with cracks and friction. A more general problem has been studied in the periodic setting, refer to [6, 22].

Let $\Gamma:\Omega\to\mathcal{F}(\mathbb{R}^n)$ be the random fissure constructed in Section 8.1 with normal field $\nu_{\Gamma(\omega)}$. We then consider the following problem:

$$-\nabla\cdot(A_\varepsilon\nabla u^\varepsilon) = f \qquad \text{on } \boldsymbol{Q}\backslash\Gamma^\varepsilon(\omega)\,, \tag{9.1}$$

$$(\nu_{\Gamma^\varepsilon}(A_\varepsilon\nabla u^\varepsilon)) \in \frac{1}{\varepsilon}\partial\psi_\varepsilon([\![\partial_t u^\varepsilon]\!]) \qquad \text{on } \Gamma^\varepsilon(\omega)\,. \tag{9.2}$$

We additionally prescribe the boundary values through $u^\varepsilon|_{\partial\boldsymbol{Q}} = u_{Dir}|_{\partial\boldsymbol{Q}}$ and demand that $[\![u^\varepsilon]\!]_n = [\![u^\varepsilon]\!]\cdot\nu_{\Gamma^\varepsilon} \geq 0$ and $u_{Dir}(0) = 0$. In order to formulate (9.1)–(9.2) in a weak sense, we define $G(\omega) := \mathbb{R}^n\backslash\Gamma(\omega)$ and recall the definition of $H^1(\boldsymbol{Q}\cap G)$ in (8.2). The weak formulation of our problem then reads as follows: Find $u^\varepsilon \in H_0^1(Q\cap G^\varepsilon(\omega))$ such that $[\![u^\varepsilon]\!]_n := [\![u^\varepsilon]\!]\cdot\nu_{\Gamma^\varepsilon} \geq 0$ holds almost everywhere and such that

$$\int_{\boldsymbol{Q}\backslash\Gamma^\varepsilon} \left(A_\varepsilon \nabla\left(u^\varepsilon + u_{Dir}\right)\right) : \nabla\varphi + \int_{\Gamma^\varepsilon} \frac{1}{\varepsilon}\partial\psi_\varepsilon\left([\![\partial_t u^\varepsilon]\!]\right)\cdot[\![\varphi]\!] = \int_{\boldsymbol{Q}} f\cdot\varphi \qquad \forall\varphi\in H_0^1(Q\cap G^\varepsilon(\omega))\,. \quad (9.3)$$

Let $\mu_{\Gamma(\omega)}(B) := \mathcal{H}^{n-1}(B\cap\Gamma(\omega))$ be the Hausdorff-measure on $\Gamma(\omega)$ with the scaled measure $\mu^\varepsilon_{\Gamma(\omega)}(B) := \varepsilon^n\mu_{\Gamma(\omega)}(\varepsilon^{-1}B)$ and $\mu_{\Gamma,\mathcal{P}}$ the Palm measure for $\mu_{\Gamma(\omega)}$ defined through (2.13).

9.1 Formulation of the homogenization result

For the formulation of the homogenization result, we make the following assumptions.

1. Let $A \in L^\infty(\boldsymbol{Q}; L^\infty(\Omega; \mathscr{L}(\mathbb{R}^{n\times n}_s, \mathbb{R}^{n\times n}_s)))$ be symmetric a.e. and $\mathcal{B}_Q\otimes\mathcal{B}_\Omega$-measurable. Assume the existence of a constant $\alpha > 0$ such that

$$\alpha\left|\xi\right|^2 \leq \xi A(x,\omega)\xi \leq \frac{1}{\alpha}\left|\xi\right|^2 \qquad \forall\xi\in\mathbb{R}^n \text{ and for a.e. } (x,\omega)\in\boldsymbol{Q}\times\Omega\,. \quad (9.4)$$

   Given $\omega\in\Omega$ and $\varepsilon > 0$, we set $A_{\varepsilon,\omega}(x) := A(x, \tau_{\frac{x}{\varepsilon}}\omega)$.

2. Let $\mathcal{C} : \boldsymbol{Q}\times\tilde{\Gamma} \to 2^{\mathbb{R}^n}$ be a family of closed convex sets satisfying Assumption 5.4, where $\tilde{\Gamma}\subset\Omega$ is the prototype of $\Gamma(\omega)$. We define the functions

$$\psi(x,\omega,z) := \sup_{v\in\mathcal{C}(x,\omega)} v\cdot z\,, \qquad \psi_{\varepsilon,\omega}(x,z) = \sup_{v\in\mathcal{C}(x,\tau_{\frac{x}{\varepsilon}}\omega)} v\cdot z\,.$$

   Recalling the definition of Ψ, Ψ_ε in (5.5), we then find

$$\Psi(u) := \sup_{\sigma\in\mathscr{C}_2(\boldsymbol{Q}\times\tilde{\Gamma})} \int_{\boldsymbol{Q}}\int_{\tilde{\Gamma}} u\sigma d\mu_{\mathcal{P}} dx\,, \qquad \Psi_\varepsilon(u) := \sup_{\sigma\in\mathscr{C}^\varepsilon_2(Q,\omega)} \int_{\Gamma^\varepsilon(\omega)} u\sigma d\mu^\varepsilon_{\Gamma(\omega)}\,.$$

3. For simplicity, we assume that $u_0^\varepsilon \in H_0^1(Q)$ solves

$$-\nabla\cdot\left(A_\varepsilon\nabla\left(u_0^\varepsilon + u_{Dir}(0)\right)\right) = f \quad \text{on } \boldsymbol{Q}\backslash\Gamma^\varepsilon(\omega)\,.$$

   Thus, we assume there are initially no jumps of u^ε accross $\Gamma^\varepsilon(\omega)$.

Theorem 9.1. *Let 1.–3. hold. There exists a unique solution $u\in C^{Lip}([0,T]; H_0^1(\boldsymbol{Q}))$ and $v\in C^{Lip}([0,T]; L^2(\boldsymbol{Q}; L^2_{pot}(\Omega, G, \Gamma)))$ with $v(0) = 0$ to*

$$-\nabla\cdot\int_\Omega\left(A\left(\nabla u + \nabla u_{Dir} + v\right)\right) = f \qquad \text{on } \boldsymbol{Q}\times[0,T]\,,$$

$$\int_0^T\int_{\boldsymbol{Q}}\int_\Omega\left(\nabla u + \nabla u_{Dir} + v\right)A\mathrm{w} + \int_0^T\int_{\boldsymbol{Q}}\int_\Gamma\partial\psi\left(\partial_t v\right)\mathrm{w} = 0$$

for all w $\in L^2(0,T;L^2(\boldsymbol{Q};L^2_{pot}(\Omega,G,\Gamma)))$ *with* $v|_\Gamma\cdot\nu_\Gamma\geq 0$ *a.e. on* $\boldsymbol{Q}\times\Omega$.

Furthermore, for almost all $\omega\in\Omega$ *it holds: For every* $\varepsilon>0$ *there exists a weak solution* $u^\varepsilon\in C^{Lip}(H^1_0(Q\cap G^\varepsilon(\omega)))$ *to (9.3) such that* $[\![u^\varepsilon]\!]_n=[\![u^\varepsilon]\!]\cdot\nu_{\Gamma^\varepsilon}\geq 0$ *holds almost everywhere and such that* $u^\varepsilon(0)=u_0$. *As* $\varepsilon\to 0$, *it holds that for all* $t\in[0,T]$*:*

$$\begin{aligned} u^\varepsilon(t)&\to u(t)\qquad \text{in } L^2(\boldsymbol{Q}))\,,\\ \nabla u^\varepsilon(t)&\overset{2s}{\rightharpoonup}\nabla u(t)+v(t)\big|_G\,,\\ \frac{1}{\varepsilon}[\![u^\varepsilon(t)]\!]\nu_{\Gamma^\varepsilon(\omega)}&\overset{2s}{\rightharpoonup}v(t)\big|_\Gamma\,. \end{aligned}\tag{9.5}$$

9.2 Proof of homogenization result

The proof is very similar to Sections 6.3 and 7.2, and we only provide a short skech. With the definition of $L^2_{pot}(\Omega,G,\Gamma)$ in (8.5), we consider the following function spaces:

1. $\mathscr{H}_2=L^2(\boldsymbol{Q})\times L^2_{pot}(\Omega,G,\Gamma)$ and $\mathscr{H}_2^\varepsilon=L^2(\boldsymbol{Q})\times L^2(\Gamma^\varepsilon(\omega);\mu^\varepsilon_{\Gamma(\omega)})$ for all $\varepsilon>0$.
2. $\mathscr{H}_1=H^1_0(\boldsymbol{Q})\times L^2(\boldsymbol{Q};L^2_{pot}(\Omega,G,\Gamma))$ and $\mathscr{H}_1^\varepsilon=H^1_0(\boldsymbol{Q})\times H^{1/2}_0(\Gamma^\varepsilon(\omega))$

With the notation from Lemma 8.4, we define the functional

$$\begin{aligned} \mathcal{E}_\varepsilon:[0,T]\times\mathscr{H}_2^\varepsilon&\to\mathbb{R}\\ (t,u_q,u_\gamma)&\mapsto\frac{1}{2}\int_{\boldsymbol{Q}}(\nabla u_q+\nabla u_\gamma+\nabla u_{Dir}(t))A_{\varepsilon,\omega}(\nabla u_q+\nabla u_\gamma+\nabla u_{Dir}(t)-2f(t))\\ &\qquad+\int_{\Gamma^\varepsilon(\Omega)}K([\![u_\gamma]\!]\cdot\nu_{\Gamma^\varepsilon(\omega)})\,, \end{aligned}$$

where $K(u)=0$ if $u\geq 0$ and $K(u)=\infty$ if $u<0$. Then, Theorem 2.3 yields the existence of a unique energetic solution $u^\varepsilon\in C^{0,1}([0,T];\mathscr{H}_1^\varepsilon)$ to $(\mathscr{H}_1^\varepsilon,\mathcal{E}_\varepsilon,\Psi_\varepsilon)$. A calculation similar to Section 7.2 shows that u^ε is a weak solution to (9.3). The apriori estimates from Theorem 2.3 Lemma 8.4, Proposition 8.7 and Lemma 4.10 provide a subsequence of $u^\varepsilon=\left(u^\varepsilon_q,u^\varepsilon_\gamma\right)$ and functions $u\in C^{lip}([0,T];H^1_0(\boldsymbol{Q}))$, $v\in C^{lip}([0,T];L^2_{pot}(\Omega,G,\Gamma))$ such that for all $t\in[0,T]$ the limit (9.5) holds.

It remains to verify that (u,v) is the unique energetic solution to an apropriate limit problem $(\mathscr{H}_1,\mathcal{E},\Psi)$. The natural canditate for the energy functional is

$$\begin{aligned} \mathcal{E}:[0,T]\times\mathscr{H}_2&\to\mathbb{R}\\ (t,u,v)&\mapsto\frac{1}{2}\int_{\boldsymbol{Q}}\int_\Omega(\nabla u+v+\nabla u_{Dir}(t))A(\nabla u+v+\nabla u_{Dir}(t)-2f(t))\\ &\qquad+\int_{\boldsymbol{Q}}\int_\Gamma K(v\cdot\nu_\Gamma)\,. \end{aligned}$$

The passage to the limit in the energy inequality follows along the lines of the proof of Theorem 7.1. Here, we additionally use Lemma 5.8 to obtain

$$0=\lim_{\varepsilon\to 0}\int_{\Gamma^\varepsilon(\Omega)}K([\![u^\varepsilon_\gamma]\!]\cdot\nu_{\Gamma^\varepsilon(\omega)})=\liminf_{\varepsilon\to 0}\int_{\Gamma^\varepsilon(\Omega)}K([\![u^\varepsilon_\gamma]\!]\cdot\nu_{\Gamma^\varepsilon(\omega)})\geq\int_{\boldsymbol{Q}}\int_\Gamma K(v\cdot\nu_\Gamma)\geq 0\,.$$

In order to pass to the limit in the stability condition, we use the form (2.4) and observe that this is equivalent with

$$
\begin{aligned}
-\nabla\cdot\left(A_{\varepsilon,\omega}(\nabla u_q^\varepsilon+\nabla u_\gamma^\varepsilon+\nabla u_{Dir}(t))\right)&=f(t)\,,\\
a^\varepsilon:=\left(A_{\varepsilon,\omega}(\nabla u_q^\varepsilon+\nabla u_\gamma^\varepsilon+\nabla u_{Dir}(t))\right)\cdot\nu_{\Gamma^\varepsilon(\omega)}&\in\partial\Psi_\varepsilon(0)\,.
\end{aligned}
\tag{9.6}
$$

As $\varepsilon\to 0$, we obtain for almost all $t\in[0,T]$ that $a^\varepsilon(t)\xrightarrow{2s}a_t$ for some $a_t\in L^2(\boldsymbol{Q};L^2(\Gamma;\mu_{\mathcal{P}}))$. Given $\varphi_1,\varphi_2\in C_c^\infty(\boldsymbol{Q})$ and $\hat{v}\in L^2_{pot}(\Omega,G,\Gamma)$, we use $\varphi^\varepsilon(x):=\varphi_1(x)+\varphi_2(x)\phi_{\varepsilon,\omega,\hat{v}}(x)$ as a test function in (9.6), where $\phi_{\varepsilon,\omega,\hat{v}}$ is the potential from Lemma 8.6, and obtain

$$
\begin{aligned}
&\int_{\boldsymbol{Q}}(\nabla u_q^\varepsilon+\nabla u_\gamma^\varepsilon+\nabla u_{Dir}(t))A_{\varepsilon,\omega}(\nabla\varphi_1+\phi_{\varepsilon,\omega,\hat{v}}\nabla\varphi_2+\varphi_2(x)\hat{v}(\tau_{\frac{x}{\varepsilon}}\omega))\\
&\qquad\qquad-\int_{\Gamma^\varepsilon(\omega)}a^\varepsilon\varphi_2(x)\hat{v}(\tau_{\frac{x}{\varepsilon}}\omega)d\mu^\varepsilon_{\Gamma(\omega)}=\int_{\boldsymbol{Q}}f\cdot\varphi^\varepsilon(x)\,.
\end{aligned}
$$

In the limit $\varepsilon\to 0$, the last equation separates into

$$
\begin{aligned}
\int_{\boldsymbol{Q}}\int_\Omega(\nabla u+v+\nabla u_{Dir}(t))A\nabla\varphi_1&=\int_{\boldsymbol{Q}}f\cdot\varphi_1(x)\,,\\
\int_{\boldsymbol{Q}}\int_\Omega(\nabla u+v+\nabla u_{Dir}(t))A\,\hat{v}\varphi_2-\int_{\boldsymbol{Q}}\int_\Gamma a_t\varphi_2\hat{v}d\mu_{\mathcal{P}}&=0\,.
\end{aligned}
$$

From the second equation, we infer that for almost every $x\in\boldsymbol{Q}$ it holds $(\nabla u(x)+v(x,\cdot)+\nabla u_{Dir}(t,x))\in L^2_{sol}(\Omega,G,\Gamma)$ with $a_t(x,\cdot)=(\nabla u(x)+v(x,\cdot)+\nabla u_{Dir}(t,x))\cdot\nu_\Gamma$. Furthermore, we find by (9.6) and Theorem 5.6 that $a_t\in\partial\Psi(0)$. We infer $D_v\mathcal{E}(t,u(t),v(t))\in\partial\Psi(0)$ and the limit (u,v) satisfies the stability condition for $(\mathscr{H}_1,\mathcal{E},\Psi)$.

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