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On maximal parabolic regularity for non-autonomous parabolic operators

Karoline Disser[1], A. F. M. ter Elst[2], Joachim Rehberg[1]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: karoline.disser@wias-berlin.de
joachim.rehberg@wias-berlin.de

[2] University of Auckland
Department of Mathematics
Private bag 92019
Auckland 1142
New Zealand
E-Mail: terelst@math.auckland.ac.nz

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider linear inhomogeneous non-autonomous parabolic problems associated to sesquilinear forms, with discontinuous dependence of time. We show that for these problems, the property of maximal parabolic regularity can be extrapolated to time integrability exponents $r \neq 2$. This allows us to prove maximal parabolic L^r-regularity for discontinuous non-autonomous second-order divergence form operators in very general geometric settings and to prove existence results for related quasilinear equations.

Contents

1 Introduction

In this paper we are interested in maximal parabolic regularity for non-autonomous parabolic equations like

$$u'(t) + \mathcal{A}(t)u(t) = f(t),$$

for almost every $t \in (0,T)$, where $u(0) = 0$, $f \in L^r((0,T);X)$ and the operators $\mathcal{A}(t)$ all have the same domain of definition D in a Banach space X. If the operator function $\mathcal{A}(\cdot)$ is constant, these equations may be solved using the concept of maximal parabolic regularity, see for example [DeS], [AB], [Are], [Lam], [DV], [Dor], [Wei], [DHP], [ABHR], [DMR], [PSi]. This theory extends to cases in which the dependence

$$(0,T) = J \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D;X) \tag{1}$$

is continuous, see [Gri1, Chapter VI], [PSc], [Ama3] and the survey in [Sch] and it is a powerful tool for solving corresponding nonlinear equations, see [CL], [Prü], [Ama4], [HiR], [HR1], [HR3], [KPW], [EMR].

If the continuity of (1) is violated, things are much less understood, the only classical exception being the case that the summabilility exponent in time, r, equals 2 and that X and D are Hilbert spaces (see Proposition 4.1 below). For results in the Banach space case, we refer to [AL] and [Grö] and for relevant recent achievements see [ACFP], [ACFP, Proposition 1.3], [ADLO], [GV], [Fac1], [Fac2] and the references therein. These results are mostly to be seen as perturbation results, with respect to an *autonomous* parabolic operator.

The spirit of our paper is perturbative in a different sense: not the operator is changed, but the Banach space. This enables us to extrapolate maximal parabolic regularity for whole classes of non-autonomous operators simultaneously. Remarkably, Gröger proved in [Grö] that maximal parabolic $L^r(J;W^{-1,q}_{\mathfrak{D}})$-regularity for second-order divergence-form operators is preserved in case of non-smooth, time-dependent coefficients, if one deviates from $q = r = 2$ to temporal and spatial integrability exponents $q = r$ in an interval $[2, 2+\varepsilon)$ that depends on the ellipticity constant and the L^∞-norm of the coefficient function. Here, we provide an abstract extrapolation strategy that includes general non-autonomous operators corresponding to sesquilinear forms, and we extend the results of Gröger to indices $q \neq r \in (2-\varepsilon, 2+\varepsilon)$ and more general geometric settings for mixed boundary conditions.

In the first part of the paper, we develop an abstract framework which allows to show maximal parabolic $L^r(J;X)$-regularity for *non-autonomous* operators in a Banach space X and for some $r \in (1,\infty)$, provided that one knows maximal parabolic regularity for suitable *autonomous* operators. Later on we specify X to be $W^{-1,q}_{\mathfrak{D}}(\Omega)$ (see Definition 5.3 below) and $\mathcal{A}(t)$ to be a second-order divergence operator $-\nabla \cdot \mu_t \nabla + I$. We aim at situations in which the map in (1) is substantially discontinuous, which means that it is allowed to be discontinuous in every point t of the time interval J, and does *not* satisfy the already weak condition of relative continuity in [ACFP]. A prototype for this is the following: for each time $t \in J$ there is a moving subdomain $\Omega_t \subset \Omega$ on which the coefficient function μ_t is constant and it takes a different constant value on $\Omega \setminus \Omega_t$, see Section 9.

The Banach spaces X of type $W^{-1,q}_{\mathfrak{D}}(\Omega)$ turn out to be well suited for the treatment of elliptic and parabolic problems if these are combined with inhomogeneous Neumann boundary conditions (cf. [Lio, Section 3.3] for $q = 2$) or if right hand sides of distributional type appear, e.g. surface densities, which may even vary their position in time, cf. [HR1], [HR3]. Note that there is often an intrinsic connection between (spatial) jumps in the coefficient function and the appearance of surface densities as parts of the right hand side (see [Tam, Chapter 1]). Interestingly, these spaces are also adequate for the treatment of control problems, see [KPV], [CCK], [KR], [HMRR]. The advantage of $W^{-1,q}_{\mathfrak{D}}(\Omega)$ over $W^{-1,2}_{\mathfrak{D}}(\Omega)$ is that the domain of elliptic divergence operators continuously embeds into a Hölder space if q is larger than the space dimension (cf. [HMRS] [ER2]), which is helpful when considering quasilinear problems, see Section 8 below and cf. [Prü], [HiR], [HR1] in the continuous setting. Moreover, in contrast to the $L^p(\Omega)$ spaces, the space $W^{-1,q}_{\mathfrak{D}}(\Omega)$ satisfies the property that the domains of divergence operators $-\nabla \cdot (\mu_t \nabla)$ coincide at different points in time even if the discontinuities of the coefficient functions μ_t move in Ω, see the examples in Section 6, taken from [ERS], [DKR].

We next give an outline of the paper. We first recall preliminary results on maximal parabolic regularity and a quantitative version of Sneiberg's extrapolation principle. Then we prove an interpolation result for different spaces of maximal parabolic regularity, i.e. we prove the interpolation identity

$$[L^{r_0}(J;D)\cap W^{1,r_1}_0(J;X), L^{r_1}(J;E)\cap W^{1,r_2}_0(J;Y)]_\theta = L^r(J;[D,E]_\theta)\cap W^{1,r}_0(J;[X,Y]_\theta), \quad (2)$$

in which $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ and (D, X) and (E, Y) form a *pair of common maximal parabolic regularity*, see Definition 3.1 below. Having this at hand, one can extrapolate maximal parabolic regularity from one setting to 'neighbouring ones', see Theorem 3.4.

Then in Section 4 we treat linear, non-autonomous parabolic equations in the classical Hilbert space setting as in [DL, Section XVIII.3]. We show that the exponent of time integrability extrapolates from 2 to $r \neq 2$ without losing maximal parabolic regularity and provide quantitative estimates on the size of r in Theorem 4.2. A detailed motivation for these type of results is given at the end of Section 4. In Section 5 we introduce the precise setting of the elliptic differential operators in divergence form with mixed boundary conditions which we use in the remainder of the paper. In Section 6 we use recent results on elliptic regularity [HJKR] and autonomous parabolic regularity [ABHR], [EHT] to obtain maximal parabolic regularity for autonomous operators in the $W^{-1,q}$-scale, even for some $q < 2$. Next in Section 7 we exploit (2) in order to achieve maximal parabolic regularity for non-autonomous operators in the $W^{-1,q}$-scale. To be precise, in Theorem 7.3 we show that non-autonomous operators $\frac{\partial}{\partial t} - \nabla \cdot (\mu_t \nabla)$ satisfy maximal parabolic $L^r(J; W^{-1,q}_{\mathfrak{D}}(\Omega))$-regularity if $r \in (2 - r_0, 2 + r_0)$ and $q \in (2 - \varepsilon, 2 + \varepsilon)$. The coefficient function $t \mapsto \mu_t$ can be discontinuous in time and space. As in the preceding articles [HJKR] and [ABHR], the geometric setting for the domains and boundary parts is extremely wide: the domains may even fail to be Lipschitz and the 'Dirichlet' boundary part $\mathfrak{D}$ is only required to be Ahlfors–David regular. In Section 8, the main results from previous sections are applied to related quasilinear problems. In Section 9, we give an example of a non-autonomous parabolic operator with discontinuous coefficients in both space and time to which Theorem 7.3 applies.

2 Preliminaries

Throughout this paper let $T > 0$ and set $J = (0, T)$. Let us start by introducing the following (standard) definition.

Definition 2.1. If X is a Banach space and $r \in (1, \infty)$, then we denote by $L^r(J; X)$ the space of X-valued functions f on J which are Bochner-measurable and for which $\int_J \|f(t)\|_X^r \, dt$ is finite. We define $W^{1,r}(J; X) := \{u \in L^r(J; X) : u' \in L^r(J; X)\}$, where u' is to be understood as the time derivative of u in the sense of X-valued distributions (cf. [Ama1, Section III.1]). Moreover, we introduce the subspace

$$W_0^{1,r}(J; X) := \{\psi \in W^{1,r}(J; X) : \psi(0) = 0\}.$$

We equip this subspace always with the norm $v \mapsto \|v'\|_{L^r(J;X)}$.

In this paper we consider the following notion of maximal parabolic regularity in the non-autonomous case.

Definition 2.2. Let X, D be Banach spaces with D densely and continuously embedded in X. Let $J \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D; X)$ be a bounded and measurable map and suppose that the operator $\mathcal{A}(t)$ is closed in X for all $t \in J$. Let $r \in (1, \infty)$. Then we say that the family $\{\mathcal{A}(t)\}_{t \in J}$ satisfies **(non-autonomous) maximal parabolic $L^r(J; X)$-regularity**, if for any $f \in L^r(J; X)$ there is a unique function $u \in L^r(J; D) \cap W_0^{1,r}(J; X)$ which satisfies

$$u'(t) + \mathcal{A}(t)u(t) = f(t) \tag{3}$$

for almost all $t \in J$. We write

$$\mathrm{MR}_0^r(J; D, X) := L^r(J; D) \cap W_0^{1,r}(J; X)$$

for the space of maximal parabolic regularity. The norm of $u \in \mathrm{MR}_0^r(J; D, X)$ is

$$\|u\|_{\mathrm{MR}_0^r(J;D,X)} = \|u\|_{L^r(J;D)} + \|u'\|_{L^r(J;X)}.$$

Then $\mathrm{MR}_0^r(J; D, X)$ is a Banach space.

If all operators $\mathcal{A}(t)$ are equal to one (fixed) operator $\mathcal{A}_0$, and there exists an $r \in (1, \infty)$ such that $\{\mathcal{A}(t)\}_{t \in J}$ satisfies maximal parabolic $L^r(J; X)$-regularity, then $\{\mathcal{A}(t)\}_{t \in J}$ satisfies maximal parabolic $L^s(J; X)$-regularity for all $s \in (1, \infty)$ and we say that $\mathcal{A}_0$ satisfies **maximal parabolic regularity on** X. In all what follows, we denote the mapping which assigns to the right hand side $f \in L^r(J; X)$ in (3) the solution $u \in \mathrm{MR}_0^r(J; D, X)$ by $\big(\frac{\partial}{\partial t} + \mathcal{A}(\cdot)\big)^{-1}$ and $\big(\frac{\partial}{\partial t} + \mathcal{A}_0\big)^{-1}$, respectively.

The L^p-spaces satisfy optimal interpolation properties with respect to the complex interpolation method.

Proposition 2.3. *Let X, Y be two Banach spaces which form an interpolation couple. Further, let $r_0, r_1 \in [1, \infty)$, $\theta \in (0,1)$ and set $r = \left(\frac{1-\theta}{r_0} + \frac{\theta}{r_1}\right)^{-1}$. Then*

$$[L^{r_0}(J;X), L^{r_1}(J;Y)]_\theta = L^r(J;[X,Y]_\theta)$$

with **equality** *of norms.*

Proof. See [BL, Theorem 5.1.2]. □

We continue by quoting Sneiberg's extrapolation principle.

Theorem 2.4. *Let F_1, F_2, Z_1, Z_2 be Banach spaces such that (F_1, F_2) and (Z_1, Z_2) are interpolation couples. Assume $\mathcal{R} \in \mathcal{L}(F_1; Z_1) \cap \mathcal{L}(F_2; Z_2)$ and put*

$$\gamma := \max\big(\|\mathcal{R}\|_{F_1 \to Z_1}, \|\mathcal{R}\|_{F_2 \to Z_2}\big).$$

Suppose that for one $\theta \in (0,1)$ the operator $\mathcal{R} \colon [F_1, F_2]_\theta \to [Z_1, Z_2]_\theta$ is a topological isomorphism and let $\beta \geq \|\mathcal{R}^{-1}\|_{[Z_1,Z_2]_\theta \to [F_1,F_2]_\theta}$. Then one has the following.

(a) *If $\tilde\theta \in (0,1)$ and*

$$|\theta - \tilde\theta| < \frac{\min(\theta, 1-\theta)}{1 + \beta\gamma},$$

then $\mathcal{R} \colon [F_1, F_2]_{\tilde\theta} \to [Z_1, Z_2]_{\tilde\theta}$ remains surjective.

(b) *If $\tilde\theta \in (0,1)$ and*

$$|\theta - \tilde\theta| \leq \frac{1}{6}\,\frac{\min(\theta, 1-\theta)}{1 + 2\beta\gamma},$$

then $\mathcal{R} \colon [F_1, F_2]_{\tilde\theta} \to [Z_1, Z_2]_{\tilde\theta}$ remains an isomorphism and

$$\|\mathcal{R}^{-1}\|_{[Z_1,Z_2]_{\tilde\theta} \to [F_1,F_2]_{\tilde\theta}} \leq 8\beta.$$

Proof. Essentially, the theorem was discovered by Sneiberg [Sne] and elaborated in more detail in [VV]. The explicit quantitative estimates as quoted here were worked out only recently in [Ege, Propositions 1.3.27 and 1.3.25]. □

3 Pairs of common maximal parabolic regularity

The aim of this section is to provide an abstract setting in which the property of non-autonomous maximal parabolic regularity can be extrapolated by using Sneiberg's theorem.

Definition 3.1. Let X, Y, D, E be Banach spaces, with continuous embeddings $X \hookrightarrow Y$, $D \hookrightarrow X$ and $E \hookrightarrow Y$. Suppose that D is dense in X and E is dense in Y. Then we say that the tuples (D, X) and (E, Y) form a **pair of common maximal parabolic regularity** if there is an operator $\mathcal{B} \in \mathcal{L}(E; Y)$ such that $D = \{x \in E \cap X : \mathcal{B}x \in X\}$, the operator $\mathcal{B}$ satisfies maximal parabolic regularity on Y, and the restriction $\mathcal{B}|_D$ satisfies $\mathcal{B}|_D \in \mathcal{L}(D; X)$ and maximal parabolic regularity on X.

For convenience, as is common for interpolation results, we sometimes extend the notation $\mathcal{B}$ to $\mathcal{B}|_D$ or other restrictions of $\mathcal{B}$ in this section.

If (D, X) and (E, Y) form a pair of common maximal parabolic regularity, then the spaces of maximal parabolic regularity interpolate as follows.

Lemma 3.2. *Assume that (D, X) and (E, Y) form a pair of common maximal parabolic regularity. Let $r_0, r_1 \in [1, \infty)$. Then*

$$[\mathrm{MR}_0^{r_0}(J; D, X)), \mathrm{MR}_0^{r_1}(J; E, Y)]_\theta = \mathrm{MR}_0^r(J; [D, E]_\theta, [X, Y]_\theta) \tag{4}$$

for all $\theta \in (0, 1)$, where $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$.

Proof. The pair $\big(\mathrm{MR}_0^{r_0}(J; D, X)), \mathrm{MR}_0^{r_1}(J; E, Y)\big)$ is an interpolation couple since both spaces continuously inject into $\mathrm{MR}_0^{\min(r_0, r_1)}(J; E, Y)$. It follows from [HR1, Lemma 5.3] that $\mathcal{B}$ satisfies maximal parabolic regularity on $[X, Y]_\theta$, so

$$\frac{\partial}{\partial t} + \mathcal{B} \colon \mathrm{MR}_0^r(J; [D, E]_\theta, [X, Y]_\theta) \to L^r(J; [X, Y]_\theta) \tag{5}$$

is an isomorphism. Here we used that (D, X) and (E, Y) form a pair of common maximal parabolic regularity.

On the other hand, $\frac{\partial}{\partial t} + \mathcal{B}$ is an isomorphism from $\mathrm{MR}_0^{r_0}(J; D, X)$ onto $L^{r_0}(J; X)$ and from $\mathrm{MR}_0^{r_1}(J; E, Y)$ onto $L^{r_1}(J; Y)$. Hence by interpolation, the operator

$$\frac{\partial}{\partial t} + \mathcal{B} \colon [\mathrm{MR}_0^{r_0}(J; D, X), \mathrm{MR}_0^{r_1}(J; E, Y)]_\theta \to [L^{r_0}(J; X), L^{r_1}(J; Y)]_\theta, \tag{6}$$

is an isomorphism. The statement follows from combining (5), (6) and Proposition 2.3. □

This interpolation result together with Theorem 2.4 enables us to extrapolate maximal parabolic regularity to non-autonomous parabolic operators. We need a simple lemma.

Lemma 3.3. *Let X, Y, D, E be Banach spaces, with continuous embeddings $X \hookrightarrow Y$, $D \hookrightarrow X$ and $E \hookrightarrow Y$. Suppose D is dense in X and E is dense in Y. Let $\{\mathcal{B}(t)\}_{t \in J}$ be a subset of $\mathcal{L}(E; Y)$ satisfying $\sup_{t \in J} \|\mathcal{B}(t)\|_{E \to Y} < \infty$. Assume $\mathcal{B}(t)|_D \in \mathcal{L}(D; X)$ for all $t \in J$. Moreover, suppose that $\sup_{t \in J} \|\mathcal{B}(t)\|_{D \to X} < \infty$. Let $\theta \in (0, 1)$. Then one has the following.*

(a) *If $t \in J$, then $\mathcal{B}(t)|_{[D,E]_\theta} \in \mathcal{L}([D, E]_\theta; [X, Y]_\theta)$. Moreover*

$$\sup_{t \in J} \|\mathcal{B}(t)\|_{[D;E]_\theta \to [X;Y]_\theta} < \infty. \tag{7}$$

(b) *Suppose $J \ni t \mapsto \mathcal{B}(t)|_D$ is measurable. Then the map $J \ni t \mapsto \mathcal{B}(t)\psi \in [X, Y]_\theta$ is measurable for all $\psi \in [D, E]_\theta$.*

(c) *Let $r \in (1, \infty)$. Then the map*

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot) \colon \mathrm{MR}_0^r(J; [D, E]_\theta, [X, Y]_\theta) \to L^r(J; [X, Y]_\theta) \tag{8}$$

is well defined.

Proof. '(a)'. This is well known from complex interpolation.

'(b)'. By assumption, for every $\psi \in D$ the map $J \ni t \mapsto \mathcal{B}(t)\psi \in X$ is measurable. Since the inclusion $X \hookrightarrow [X, Y]_\theta$ is continuous, also the map $J \ni t \mapsto \mathcal{B}(t)\psi \in [X, Y]_\theta$ is measurable for all $\psi \in D$. Let now $\psi \in [D, E]_\theta$. Because D is dense in $[D, E]_\theta$ by [Tri, Theorem 1.9.3], there is a sequence $\{\psi_n\}_n$ in D which converges to ψ in $[D, E]_\theta$. But then (7) implies that the function $J \ni t \mapsto \mathcal{B}(t)\psi \in [X, Y]_\theta$ is the pointwise limit of the functions $J \ni t \mapsto \mathcal{B}(t)\psi_n \in [X, Y]_\theta$. Hence it is measurable.

'(c)'. One can easily deduce from Statement (b) that for every $\eta \in (0, 1)$ and $v \in L^r(J; [D, E]_\eta)$ the map

$$J \ni t \mapsto \mathcal{B}(t)v(t) \in [X, Y]_\eta$$

is measurable. Since $\sup_{t\in J} \|\mathcal{B}(t)\|_{[D;E]_\theta\to[X;Y]_\theta} < \infty$, it follows that (8) is well defined. □

The main theorem of this section is the following.

Theorem 3.4. *Suppose the tuples (D, X) and (E, Y) form a pair of common maximal parabolic regularity. For all $t \in J$ let $\mathcal{B}(t) \in \mathcal{L}(E; Y)$ and suppose that $\mathcal{B}(t)|_D \in \mathcal{L}(D; X)$. Assume that the maps*

$$J \ni t \mapsto \mathcal{B}(t)|_D \in \mathcal{L}(D; X) \quad \text{and} \quad J \ni t \mapsto \mathcal{B}(t) \in \mathcal{L}(E; Y)$$

are measurable and $\sup_{t\in J}(\|\mathcal{B}(t)\|_{D\to X} + \|\mathcal{B}(t)\|_{E\to Y}) < \infty$. Let $r_0, r_1 \in (1, \infty)$ and $\theta \in (0, 1)$. Set $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Suppose that

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\colon \mathrm{MR}_0^r(J; [D, E]_\theta, [X, Y]_\theta) \to L^r(J; [X, Y]_\theta) \tag{9}$$

is a topological isomorphism. Then there exists an $\varepsilon \in (0, \min(\theta, 1 - \theta))$ such that

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\colon \mathrm{MR}_0^s(J; [D, E]_{\tilde\theta}, [X, Y]_{\tilde\theta}) \to L^s(J; [X, Y]_{\tilde\theta})$$

is a topological isomorphism for all $\tilde\theta \in (\theta - \varepsilon, \theta + \varepsilon)$, where $s := \left(\frac{1-\tilde\theta}{r_0} + \frac{\tilde\theta}{r_1}\right)^{-1}$

Proof. The operators

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\colon \mathrm{MR}_0^{r_0}(J; D, X) \to L^{r_0}(J; X)$$

and

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\colon \mathrm{MR}_0^{r_1}(J; E, Y) \to L^{r_1}(J; Y)$$

are continuous and their norms are bounded by $\max(1, \gamma)$, where $\gamma = \sup_{t\in J} \|\mathcal{B}(t)\|_{D\to X} + \sup_{t\in J} \|\mathcal{B}(t)\|_{E\to Y}$. Moreover, since the tuples (D, X) and (E, Y) form a pair of common maximal parabolic regularity by assumption, we may apply Lemma 3.2 to obtain the interpolation identity (4). Using also Proposition 2.3, one can rewrite (9) as a topological isomorphism

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\colon [\mathrm{MR}_0^{r_0}(J; D, X), \mathrm{MR}_0^{r_1}(J; E, Y)]_\theta \to [L^{r_0}(J; X), L^{r_1}(J; Y)]_\theta. \tag{10}$$

By Theorem 2.4 the isomorphism in (10) remains a topological isomorphism, if θ is replaced by $\tilde{\theta} \in (0,1)$, r is replaced by $s = \left(\frac{1-\tilde{\theta}}{r_0} + \frac{\tilde{\theta}}{r_1}\right)^{-1}$ and $\tilde{\theta}$ is sufficiently close to θ. Then the theorem follows by applying again Lemma 3.2 and Proposition 2.3. □

As a simple consequence to Theorem 3.4, we obtain the following.

Corollary 3.5. *Let X, D be Banach spaces with D densely and continuously embedded in X. Let $J \ni t \mapsto \mathcal{B}(t) \in \mathcal{L}(D;X)$ be a bounded and measurable map and suppose that the operator $\mathcal{B}(t)$ is closed in X for all $t \in J$. Let $\mathcal{K} \in \mathcal{L}(D;X)$ and suppose that $\mathcal{K}$ satisfies maximal parabolic regularity on X. Let $r \in (1,\infty)$. Suppose that $\{\mathcal{B}(t)\}_{t\in J}$ satisfies maximal parabolic $L^r(J;X)$-regularity. Then there exists an open interval $I \subset (1,\infty)$ with $r \in I$ such that $\{\mathcal{B}(t)\}_{t\in J}$ satisfies maximal parabolic $L^s(J;X)$-regularity for all $s \in I$.*

Proof. Choose $D = E$ and $X = Y$ in Lemma 3.2 and Theorem 3.4. □

Theorem 3.4 and Corollary 3.5 show the abstract principle we use in the sequel for the extrapolation of maximal $L^r(J;X)$-regularity. In Corollary 3.5 the space D and X are connected via *some* autonomous reference operator $\mathcal{K}$ that has maximal regularity. Then Corollary 3.5 gives that for every non-autonomous operator family $J \ni t \mapsto \mathcal{B}(t) \in \mathcal{L}(D;X)$ with maximal parabolic $L^r(J;X)$-regularity the regularity extrapolates around r. We expect that the interpolation formula (4) is of independent interest and may serve for other purpose also in different contexts. In Section 4 we apply this principle to non-autonomous families of operators in Hilbert spaces generated by families of sesquilinear forms, and in Sections 5–7 to non-autonomous elliptic differential operators in Sobolev spaces. In Theorem 3.4 and Corollary 3.5, however, the quantitative estimates of Theorem 2.4 are lost, as (4) holds only with equivalence of norms and the exact constants seem to be hard to control (cf. Remark 4.4). In order to get some quantitative results in the specific settings of Sections 4 and 7, we will use direct proofs based on Theorem 2.4. The following definition and interpolation result will be useful tools.

Definition 3.6. Let X, D be Banach spaces with D densely and continuously embedded in X. Let $\mathcal{K} \in \mathcal{L}(D;X)$ and suppose that $\mathcal{K}$ satisfies maximal parabolic regularity on X. For all $r \in (1,\infty)$, denote by $\mathrm{MR}_0^r(J;D,X)^\sim$ the space $\mathrm{MR}_0^r(J;D,X)$ equipped with the norm

$$u \mapsto \|u\|_{\mathrm{MR}_0^r(J;D,X)^\sim} = \|(\frac{\partial}{\partial t} + \mathcal{K})u\|_{L^r(J;X)}.$$

It will be clear from the context which operator $\mathcal{K}$ is involved. Obviously

$$\frac{\partial}{\partial t} + \mathcal{K} \colon \mathrm{MR}_0^r(J;D,X)^\sim \to L^r(J;X) \tag{11}$$

is an *isometric* isomorphism. For all $r \in (1,\infty)$ define

$$C_{\mathcal{K}}^r := \|(\frac{\partial}{\partial t} + \mathcal{K})^{-1}\|_{L^r(J;X)\to\mathrm{MR}_0^r(J;D,X)}. \tag{12}$$

Then

$$\|u\|_{\mathrm{MR}_0^r(J;D,X)} \le C_{\mathcal{K}}^r \, \|(\frac{\partial}{\partial t} + \mathcal{K})u\|_{L^r(J;X)} = C_{\mathcal{K}}^r \, \|u\|_{\mathrm{MR}_0^r(J;D,X)^\sim} \tag{13}$$

and

$$\begin{aligned}\|u\|_{\mathrm{MR}_0^r(J;D,X)^\sim} &= \|(\frac{\partial}{\partial t} + \mathcal{K})u\|_{L^r(J;X)} \\ &\le \|u'\|_{L^r(J;X)} + \|\mathcal{K}\|_{D\to X}\, \|u\|_{\mathrm{MR}_0^r(J;D,X)} \\ &\le (1 \vee \|\mathcal{K}\|_{D\to X})\, \|u\|_{\mathrm{MR}_0^r(J;D,X)} \end{aligned} \tag{14}$$

for all $u \in \mathrm{MR}_0^r(J;D,X)$. So the two norms are equivalent.

Lemma 3.7. *Adopt the notation as in Definition 3.6. Let $\theta \in (0,1)$ and $r_0, r_1 \in (1,\infty)$. Then*

$$[\mathrm{MR}_0^{r_0}(J;D,X)^\sim, \mathrm{MR}_0^{r_1}(J;D,X)^\sim]_\theta = \mathrm{MR}_0^r(J;D,X)^\sim \tag{15}$$

with **equality** *of norms, where $r \in (1,\infty)$ is such that $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$.*

Proof. By Proposition 2.3 one has the interpolation identity

$$[L^{r_0}(J;X), L^{r_1}(J;X)]_\theta = L^r(J;X), \tag{16}$$

with equality of norms. Then the *isometric* isomorphism (11) carries (16) over to (15), with equality of norms. $\square$

4 Maximal parabolic regularity and form methods

In this section, we will investigate maximal parabolic regularity in a Hilbert space setting. For convenience, recall the following classical existence result, which will serve as the starting point for the remainder of this paper.

Proposition 4.1. *Let V and H be Hilbert spaces with V densely and continuously embedded into H. For every $t \in J$ let $\mathfrak{s}_t$ be a sesquilinear form on V. Let $c_\bullet, c^\bullet > 0$. Suppose that the map $t \mapsto \mathfrak{s}_t[\varphi,\psi]$ from J into $\mathbb{C}$ is measurable for all $\psi,\varphi \in V$. Suppose that $\operatorname{Re}\mathfrak{s}_t[\psi,\psi] \ge c_\bullet \|\psi\|_V^2$ and $|\mathfrak{s}_t[\psi,\varphi]| \le c^\bullet\, \|\psi\|_V\, \|\varphi\|_V$ for all $\varphi,\psi \in V$ and $t \in J$. For all $t \in J$ let $\mathcal{B}(t)\colon V \to V^*$ be the linear operator which is induced by the sesquilinear form $\mathfrak{s}_t$. Then for all $f \in L^2(J;V^*)$ there exists a unique $u \in \mathrm{MR}_0^2(J;V,V^*)$ such that*

$$u'(t) + \mathcal{B}(t)u(t) = f(t) \tag{17}$$

for a.e. $t \in J$. Moreover,

$$\|u\|_{L^2(J;V)} \le \frac{1}{c_\bullet}\|f\|_{L^2(J;V^*)} \quad \text{and} \quad \|u'\|_{L^2(J;V^*)} \le \big(1 + \frac{c^\bullet}{c_\bullet}\big)\|f\|_{L^2(J;V^*)}. \tag{18}$$

In particular,

$$\|u\|_{\mathrm{MR}_0^2(J;V,V^*)} \le \frac{1 + c_\bullet + c^\bullet}{c_\bullet}\, \|f\|_{L^2(J;V^*)} \tag{19}$$

Proof. The existence and uniqueness in (17) follows from [DL, Section XVIII.3 Remark 9]. We next prove the estimates (18). If $\tau \in \overline{J}$, then the energy equality

$$\frac{1}{2}\|u(\tau)\|_H^2 + \operatorname{Re}\int_0^\tau \mathfrak{s}_t[u(t),u(t)]\,dt = \operatorname{Re}\int_0^\tau \langle f(t),u(t)\rangle_{V^*\times V}\,dt$$

follows from (17), cf. [DL, Section XVIII.3 Equation 3.86]. Using the uniform coercivity, this gives

$$\begin{aligned} c_\bullet \int_0^T \|u(t)\|_V^2\,dt &\le \operatorname{Re}\int_0^T \mathfrak{s}_t[u(t),u(t)]\,dt \\ &\le \operatorname{Re}\int_0^T \langle f(t),u(t)\rangle_{V^*\times V}\,dt \\ &\le \int_0^T \|f(t)\|_{V^*}\,\|u(t)\|_V\,dt \le \|f\|_{L^2(J;V^*)}\,\|u\|_{L^2(J;V)}, \end{aligned}$$

which proves the first inequality in (18). Note that $\|\mathcal{B}(t)\|_{V\to V^*} \le c^\bullet$ for all $t \in J$. Therefore

$$\begin{aligned} \|u'\|_{L^2(J;V^*)} &\le \|f\|_{L^2(J;V^*)} + \|\mathcal{B}(\cdot)u(\cdot)\|_{L^2(J;V^*)} \\ &\le \|f\|_{L^2(J;V^*)} + c^\bullet\|u\|_{L^2(J;V)} \le \big(1+\frac{c^\bullet}{c_\bullet}\big)\|f\|_{L^2(J;V^*)} \end{aligned}$$

and the second inequality in (18) follows. □

Adopt the notation and assumptions of Proposition 4.1. We are interested in the problem for which $r \in (1,\infty)\setminus\{2\}$ the map

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\colon \mathrm{MR}_0^r(J;V,V^*) \to L^r(J;V^*) \tag{20}$$

is still an isomorphism.

Let $\mathcal{J}\colon V \to V^*$ be the duality map obtained by the Riesz representation theorem. The defining relation is

$$\langle \mathcal{J}\psi,\varphi\rangle_{V^*\times V} = (\psi,\varphi)_V \tag{21}$$

for all $\psi,\varphi \in V$. It follows from (21) that the operator $\mathcal{J}\colon V \to V^*$ is the operator associated with the sesquilinear form which is the scalar product in V. Note that the sesquilinear form is bounded, with constant 1, and has coercivity constant which is also equal to 1. One deduces from Proposition 4.1 that for all $f \in L^2(J;V^*)$ the equation

$$u' + \mathcal{J}u = f$$

admits exactly one solution $u \in \mathrm{MR}_0^2(J;V,V^*)$. Consequently, the operator $\mathcal{J}$ satisfies maximal parabolic regularity on V^*. Therefore we can apply Corollary 3.5 with $\mathcal{K} = \mathcal{J}$. It follows for the operator family $\{\mathcal{B}(t)\}_{t\in J}$ in Proposition 4.1 that there is an open interval $I \ni 2$ such that the map in (20) is still an isomorphism for all $r \in I$. Using Lemma 3.7 and the Sneiberg theorem we also prove a quantitative result on I. The main theorem of this section is the following.

Theorem 4.2. *Adopt the notation and assumptions of Propositions 4.1. Let $r_0 \in (2,\infty)$, $r_1 \in (1,2)$ and define $\theta \in (0,1)$ such that $\frac{1}{2} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Set*

$$C_{\mathcal{J}} = \max_{j\in\{0,1\}} \|\big(\frac{\partial}{\partial t} + \mathcal{J}\big)^{-1}\|_{L^{r_j}(J;V^*)\to \mathrm{MR}_0^{r_j}(J;V,V^*)}.$$

Let $\tilde\theta \in (0,1)$. Let $r \in (1,\infty)$ be such that $\frac{1}{r} = \frac{1-\tilde\theta}{r_0} + \frac{\tilde\theta}{r_1}$. Then one has the following.

(a) *If*

$$|\theta - \tilde\theta| < \frac{\min(\theta, 1-\theta)}{1 + (1 + \frac{1+c^\bullet}{c_\bullet}) \max(1, c^\bullet) C_{\mathcal{J}}},$$

then $\{\mathcal{B}(t)\}_{t\in J}$ satisfies maximal $L^r(J;V^*)$-regularity.

(b) *If*

$$|\theta - \tilde\theta| \le \frac{1}{6}\,\frac{\min(\theta, 1-\theta)}{1 + 2(1 + \frac{1+c^\bullet}{c_\bullet}) \max(1, c^\bullet) C_{\mathcal{J}}},$$

then

$$\|\big(\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\big)^{-1}\|_{L^r(J;V^*)\to \mathrm{MR}_0^r(J;V,V^*)^\sim} \le 8\,\frac{1 + c_\bullet + c^\bullet}{c_\bullet},$$

where the norm on $\mathrm{MR}_0^r(J;V,V^*)^\sim$ is defined using the operator $\mathcal{J}$.

For the proof, we first show that injectivity is preserved.

Lemma 4.3. *Adopt the notation and assumptions of Propositions 4.1. Then the map*

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\colon \mathrm{MR}_0^r(J;V,V^*) \to L^r(J;V^*)$$

is injective for all $r \in (1,\infty)$.

Proof. For all $t \in J$ the operator $\mathcal{B}(t)$ is accretive on V^*. Then the claim follows from [ACFP, Proposition 3.2]. □

Proof of Theorem 4.2. '(a)'. We apply Theorem 2.4 of Sneiberg with the spaces $F_1 = \mathrm{MR}_0^{r_0}(J;V,V^*)^\sim$, $F_2 = \mathrm{MR}_0^{r_1}(J;V,V^*)^\sim$, $Z_1 = L^{r_0}(J;V^*)$ and $Z_2 = L^{r_1}(J;V^*)$. Then $[F_1,F_2]_\theta = \mathrm{MR}_0^2(J;V,V^*)^\sim$ and $[Z_1,Z_2]_\theta = L^2(J;V^*)$, with equality of norms by Lemma 3.7 and Proposition 2.3. Moreover, $\frac{\partial}{\partial t} + \mathcal{B}(\cdot)$ is an isomorphism from $\mathrm{MR}_0^2(J;V,V^*)$ onto $L^2(J;V^*)$ by Proposition 4.1.

We first estimate $\beta = \|\big(\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\big)^{-1}\|_{[Z_1,Z_2]_\theta\to[F_1,F_2]_\theta}$. Let $u \in [F_1,F_2]_\theta$. Then

$$\|u\|_{[F_1,F_2]_\theta} = \|u\|_{\mathrm{MR}_0^2(J;V,V^*)^\sim} \le \|u\|_{\mathrm{MR}_0^2(J;V,V^*)} \le \Big(\frac{1 + c_\bullet + c^\bullet}{c_\bullet}\Big)\|\Big(\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\Big)u\|_{L^2(J;V^*)},$$

where we used (14) and (19) in the two inequalities. So $\beta \le \frac{1+c_\bullet+c^\bullet}{c_\bullet}$.

Next we estimate

$$\gamma := \max_{j\in\{1,2\}} \|\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\|_{\mathrm{MR}_0^{r_j}(J;V,V^*)^\sim\to L^{r_j}(J;V^*)},$$

which is the second input for the calculation of the admissible interpolation parameters in Theorem 2.4. Let $j \in \{1,2\}$ and $u \in \mathrm{MR}_0^{r_j}(J;V,V^*)$. Then

$$\|\Big(\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\Big)u\|_{L^{r_j}(J;V^*)} \le (1 \vee c^\bullet)\,\|u\|_{\mathrm{MR}_0^{r_j}(J;V,V^*)} \le C_{\mathcal{J}}^{r_j}\,\|u\|_{\mathrm{MR}_0^{r_j}(J;V,V^*)^\sim},$$

where we used (13) in the second step. Hence

$$\gamma \le \max(1, c^\bullet)\, C_{\mathcal{J}}.$$

The operator

$$\frac{\partial}{\partial t} + \mathcal{B}(\cdot)\colon \mathrm{MR}_0^r(J;V,V^*) \to L^r(J;V^*)$$

is obviously continuous, and, by Lemma 4.3, it is injective. Moreover, it is surjective by Theorem 2.4(a) for the claimed interpolation parameters. Then Theorem 4.2 follows from the open mapping theorem.

'(b)'. This is proved analogously by using Theorem 2.4(b). □

Remark 4.4. It would be very interesting to get upper and lower bounds on the exponents r, dependent of the constants $c_\bullet$ and $c^\bullet$. The difficulty is to get explicit estimates of $C_{\mathcal{J}}^r$ dependent on r. For results related to this problem, we refer to [CV].

We continue with some motivation for Theorem 4.2. In Proposition 4.1 one has $f \in L^2(J;V^*)$ and therefore $u \in \mathrm{MR}^2(J;V,V^*) \subset C(J;H)$. So u is continuous from J into H. In many settings, it turns out that $f \in L^r(J;V^*)$ for some $r > 2$, because in many applications like spatial discretization, one is confronted with step functions. Then Theorem 4.2 gives that $u \in \mathrm{MR}_0^r(J;V,V^*)$. We show in the next proposition that then the function $u\colon J \to H$ is Hölder continuous. Moreover, the orbits are relatively compact in H, if the embedding $V \hookrightarrow V^*$ is compact. This additional regularity can be essential for studying related semi- and quasilinear problems (cf. Section 8), large-time asymptotic behaviour (cf. [Sch]), or for optimal control problems with tracking type objective functional (cf. [HMRR]).

Proposition 4.5. *Let V and H be Hilbert spaces with V densely and continuously embedded into H. Then one has the following.*

(a) *If $r \in (1,\infty)$ and $\theta \in (0, 1-\frac{1}{r})$, then $\mathrm{MR}_0^r(J;V,V^*) \hookrightarrow C(\overline{J};(V^*,V)_{1-\frac{1}{r},r})$.*

(b) *If $r \in (1,\infty)$ and $\theta \in (0, 1-\frac{1}{r})$, then $\mathrm{MR}_0^r(J;V,V^*) \hookrightarrow C^\alpha(J;(V^*,V)_{\theta,1})$, where $\alpha = 1 - \frac{1}{r} - \theta$.*

(c) *If $r \in (2,\infty)$, then $\mathrm{MR}_0^r(J;V,V^*) \hookrightarrow C^\alpha(J;H)$ for all $\alpha \in (0, \frac{1}{2} - \frac{1}{r})$.*

(d) *If $r \in (2,\infty)$ and V embeds compactly into V^*, then $\mathrm{MR}_0^r(J;V,V^*) \hookrightarrow C(\overline{J};H)$ and the embedding is compact.*

Proof. '(a)'. See [Ama1, Theorem III.4.10.2].

'(b)'. See [Ama2, Theorem 3], or [DER, Lemma 2.11(b)] for an elementary proof.

‘(c)’. Set $\theta = 1 - \frac{1}{r} - \alpha$. Then $\theta \in (\frac{1}{2}, 1)$. It follows from Statement (b) that $\mathrm{MR}_0^r(J; V, V^*) \hookrightarrow C^\alpha(J; (V^*, V)_{\theta,1})$. Moreover, $(V^*, V)_{\theta,1} \hookrightarrow (V^*, V)_{\frac{1}{2},2} = [V^*, V]_{\frac{1}{2}} = H$. Hence $\mathrm{MR}_0^r(J; V, V^*) \hookrightarrow C^\alpha(J; H)$.

‘(d)’. Choose $\alpha \in (0, \frac{1}{2} - \frac{1}{r})$. Set $\theta = 1 - \frac{1}{r} - \alpha$ as in the proof of Statement (c). Since V embeds compactly into V^*, the embedding $(V^*, V)_{\theta,1} \hookrightarrow (V^*, V)_{\frac{1}{2},2}$ is compact by [BL, Section 3.8]. Then the statement follows by the Ascoli theorem for vector-valued functions, see [Lan, Section III.3]. □

5 Elliptic differential operators

In the sequel we will apply the above results for the derivation of regularity for non-autonomous parabolic differential operators on Sobolev spaces. We first introduce the underlying elliptic setting.

Throughout the rest of this paper we fix a bounded open set $\Omega \subset \mathbb{R}^d$, where $d \geq 2$. Let $\mathfrak{D}$ be a closed subset of the boundary $\partial\Omega$ (to be understood as the Dirichlet boundary part). Regarding our geometric setting, we suppose the following general conditions.

Assumption 5.1.

(i) For every $x \in \overline{\partial\Omega \setminus \mathfrak{D}}$ there exists an open neighbourhood $\mathfrak{U}_x$ of x and a bi-Lipschitz map ϕ_x from $\mathfrak{U}_x$ onto the cube $K := (-1, 1)^d$, such that the following three conditions are satisfied:

$$\begin{aligned} \phi_x(x) &= 0, \\ \phi_x(\mathfrak{U}_x \cap \Omega) &= \{x \in K : x_d < 0\}, \\ \phi_x(\mathfrak{U}_x \cap \partial\Omega) &= \{x \in K : x_d = 0\}. \end{aligned}$$

(ii) The set $\mathfrak{D}$ satisfies the *Ahlfors–David condition*, that is there are $c_0, c_1 > 0$ and $r_{AD} > 0$, such that

$$c_0 r^{d-1} \leq \mathcal{H}_{d-1}(\mathfrak{D} \cap B(x, r)) \leq c_1 r^{d-1} \tag{22}$$

for all $x \in \mathfrak{D}$ and $r \in (0, r_{AD}]$, where $\mathcal{H}_{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure and $B(x, r)$ denotes the ball with centre x and radius r.

(iii) The set Ω is a d-set in the sense of Jonsson–Wallin [JW, Chapter II]. This condition is, however, **not** needed if the coefficient function μ or $(\mu_t)_{t \in J}$ is hermitian, which we introduce below. That is, if $\mu(x)$ is symmetric or $\mu_t(x)$ is symmetric for all $x \in \Omega$ and $t \in J$, then we do not need that Ω is a d-set.

Remark 5.2.

(a) We emphasize that the cases $\mathfrak{D} = \partial\Omega$ or $\mathfrak{D} = \emptyset$ are allowed.

(b) Condition (22) means that $\mathfrak{D}$ is a $(d-1)$-set in the sense of Jonsson–Wallin [JW, Chapter II].

(c) On the set $\partial\Omega \cap \big(\bigcup_{x\in\overline{\partial\Omega\setminus\mathfrak{D}}} \mathfrak{U}_x\big)$ the measure $\mathcal{H}_{d-1}$ is equal to the surface measure σ, which can be constructed via the bi-Lipschitz charts ϕ_x around these boundary points, see [EG, Section 3.3.4 C] or [HR2, Section 3]. In particular, (22) assures that $\sigma\big(\mathfrak{D} \cap \big(\bigcup_{x\in\overline{\partial\Omega\setminus\mathfrak{D}}} \mathfrak{U}_x\big)\big) > 0$ if $\mathfrak{D} \neq \emptyset$ and $\mathfrak{D} \neq \partial\Omega$.

(d) Condition (iii) excludes *outward* cusps, but *inward* cusps are allowed. This condition is only used in Proposition 6.4.

Definition 5.3. For all $q \in [1,\infty)$ we define the space $W^{1,q}_{\mathfrak{D}}(\Omega)$ as the completion of

$$C^\infty_{\mathfrak{D}}(\Omega) := \{\psi|_\Omega : \psi \in C^\infty_c(\mathbb{R}^d) \text{ and } \operatorname{supp}(\psi) \cap \mathfrak{D} = \emptyset\}$$

with respect to the (standard) norm $\psi \mapsto \big(\int_\Omega |\nabla\psi|^q + |\psi|^q\big)^{1/q}$. If $q \in (1,\infty)$ then the (anti-) dual of this space will be denoted by $W^{-1,q'}_{\mathfrak{D}}(\Omega)$, where $1/q + 1/q' = 1$. Here, the dual is to be understood with respect to the extended L^2 scalar product, or, in other words, $W^{-1,q'}_{\mathfrak{D}}(\Omega)$ is the space of continuous antilinear functionals on $W^{1,q}_{\mathfrak{D}}(\Omega)$.

Since the domain Ω is kept fixed, we omit the symbol 'Ω' in the sequel if no confusion is possible. For example, we write $W^{1,q}_{\mathfrak{D}}$ instead of $W^{1,q}_{\mathfrak{D}}(\Omega)$.

The spaces $W^{1,q}_{\mathfrak{D}}$ and the spaces $W^{-1,q}_{\mathfrak{D}}$ interpolate with respect to the complex interpolation functor.

Lemma 5.4. *Adopt Assumption 5.1. Let $q_1, q_2 \in (1,\infty)$ and $\theta \in (0,1)$. Then*

$$[W^{1,q_1}_{\mathfrak{D}}, W^{1,q_2}_{\mathfrak{D}}]_\theta = W^{1,q}_{\mathfrak{D}} \quad \textit{and}$$
$$[W^{-1,q_1}_{\mathfrak{D}}, W^{-1,q_2}_{\mathfrak{D}}]_\theta = W^{-1,q}_{\mathfrak{D}},$$

where $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

Proof. See [HJKR] Theorem 3.3 and Corollary 3.4. □

Remark 5.5. By [ABHR, Lemma 3.2] there exists an extension operator $\mathcal{E} \colon W^{1,q}_{\mathfrak{D}}(\Omega) \to W^{1,q}_{\mathfrak{D}}(\mathbb{R}^d)$, which is universal in $q \in [1,\infty)$. Therefore one has the usual Sobolev embeddings available, including compactness.

We now turn to the definition of the elliptic divergence form operators that will be investigated.

For all $c_\bullet, c^\bullet > 0$ we denote by $\mathcal{E}(c_\bullet, c^\bullet)$ the set of all measurable $\mu \colon \Omega \to \mathbb{R}^{d\times d}$ such that

$$\operatorname{Re}(\mu(x)\xi,\xi)_{\mathbb{C}^d} \geq c_\bullet|\xi|^2 \quad \text{and} \quad \|\mu(x)\|_{\mathcal{L}(\mathbb{C}^d)} \leq c^\bullet$$

for all $\xi \in \mathbb{C}^d$ and almost all $x \in \Omega$. Moreover, define

$$\mathcal{E} = \bigcup_{c_\bullet, c^\bullet > 0} \mathcal{E}(c_\bullet, c^\bullet),$$

the set of all elliptic coefficient functions.

Definition 5.6. For all $q \in (1,\infty)$ and $\mu \in \mathcal{E}$ define the operator $\mathcal{A}_q = \mathcal{A}_q(\mu) \colon W^{1,q}_{\mathfrak{D}} \to W^{-1,q}_{\mathfrak{D}}$ by

$$\langle \mathcal{A}_q(\mu)\psi, \varphi \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,q'}_{\mathfrak{D}}} := \int_\Omega \mu \nabla\psi \cdot \overline{\nabla\varphi},$$

where $\psi \in W^{1,q}_{\mathfrak{D}}$ and $\varphi \in W^{1,q'}_{\mathfrak{D}}$. Then

$$\|\mathcal{A}_q\|_{W^{1,q}_{\mathfrak{D}} \to W^{-1,q}_{\mathfrak{D}}} \le \|\mu\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{C}^d))}$$

for all $q \in (1,\infty)$ by Hölder's inequality.

Moreover, for all $q \in [2,\infty)$ define the operator $\widetilde{\mathcal{A}}_q = \widetilde{\mathcal{A}}_q(\mu)$ in $W^{-1,q}_{\mathfrak{D}}$ by

$$\mathrm{Dom}(\widetilde{\mathcal{A}}_q) = \{\psi \in W^{-1,q}_{\mathfrak{D}} \cap W^{1,2}_{\mathfrak{D}} : \mathcal{A}_2\psi \in W^{-1,q}_{\mathfrak{D}}\}$$

and $\widetilde{\mathcal{A}}_q = \mathcal{A}_2|_{\mathrm{Dom}(\widetilde{\mathcal{A}}_q)}$.

Define the sesquilinear form $\mathfrak{s} = \mathfrak{s}_\mu \colon W^{1,2}_{\mathfrak{D}} \times W^{1,2}_{\mathfrak{D}} \to \mathbb{C}$ by

$$\mathfrak{s}(\psi,\varphi) = \int_\Omega \mu\nabla\psi \cdot \overline{\nabla\varphi}.$$

Then $\mathfrak{s}$ is coercive and continuous. Let $A = A(\mu)$ be the m-sectorial operator in L^2 associated to $\mathfrak{s}$. So $\mathrm{Dom}\, A = \{\psi \in W^{1,2}_{\mathfrak{D}} : \mathcal{A}_2\psi \in L^2\}$ and $A = \mathcal{A}_2|_{\mathrm{Dom}\, A}$. Let S be the semigroup generated by $-A$. Then S extends consistently to a contraction semigroup $S^{(q)}$ on L^q for all $q \in [1,\infty)$. Moreover, $S^{(q)}$ is a holomorphic semigroup on L^q for all $q \in [1,\infty)$ by [ER1] Theorem 3.1. For all $q \in [1,\infty)$ we denote by $-A_q = -A_q(\mu)$ the generator of $S^{(q)}$.

Clearly if $p, q \in (1,\infty)$ and $p \le q$, then $\mathcal{A}_q = \mathcal{A}_p|_{W^{1,q}_{\mathfrak{D}}}$. Thus the graph of $\mathcal{A}_p$ is an extension of the graph of $\mathcal{A}_q$. As a consequence, $W^{1,q}_{\mathfrak{D}} \subset \mathrm{Dom}\, \widetilde{\mathcal{A}}_q$ for all $q \in [2,\infty)$.

Similarly, the graph of A_p is an extension of the graph of A_q if $p, q \in [1,\infty)$ and $p \le q$.

6 Maximal parabolic regularity for differential operators

In this section we prove that there exists a $q_0 \in (1,2)$ such that the operator $\mathcal{A}_q + I$ or $\widetilde{\mathcal{A}}_q + I$ satisfies maximal parabolic regularity on the space $W^{1,q}_{\mathfrak{D}}$ for all $q \in (q_0,\infty)$. A key tool is a recent solution of the Kato square root problem. In order to determine q_0 we need a definition.

Definition 6.1. Let $\mu \in \mathcal{E}$. We call a number $q \in (1,\infty)$ an **isomorphism index** for the coefficient function μ if

$$\mathcal{A}_q + I \colon W^{1,q}_{\mathfrak{D}} \to W^{-1,q}_{\mathfrak{D}}$$

is a topological isomorphism. We denote by $\mathfrak{I}_\mu$ the set of isomorphism indices for μ. Although the set $\mathfrak{I}_\mu$ also depends on the set $\mathfrak{D}$, we suppress the dependence of $\mathfrak{D}$ in the notation.

If $q \in (1,\infty)$ and $\mu \in \mathcal{E}$, then by duality one obviously has $q \in \mathfrak{I}_\mu$ if and only if $q' \in \mathfrak{I}_{\mu^T}$. This allows to concentrate to all $q \in [2,\infty)$.

Lemma 6.2. *Let $\mu \in \mathcal{E}$ and $q \in (2,\infty)$. Then one has the following.*

(a) *$q \in \mathfrak{I}_\mu$ if and only if the operator $\mathcal{A}_q + I$ is surjective.*

(b) *If $q \in \mathfrak{I}_\mu$, then $\operatorname{Dom} \widetilde{\mathcal{A}}_q = W^{1,q}_{\mathfrak{D}}$.*

Proof. '(a)'. Let $\psi \in \ker(\mathcal{A}_q + I)$. Then $\psi \in W^{1,q}_{\mathfrak{D}} \subset W^{1,2}_{\mathfrak{D}}$ and $\int_\Omega \mu \nabla\psi \cdot \overline{\nabla\varphi} + \int_\Omega \psi\,\overline{\varphi} = 0$ for all $\varphi \in C^\infty_{\mathfrak{D}}$. By continuity and density the latter is then also valid for all $\varphi \in W^{1,2}_{\mathfrak{D}}$, in particular for $\varphi = \psi$. Since μ is elliptic one deduces that $\psi = 0$. So the operator $\mathcal{A}_q + I$ is injective for all $q \in [2,\infty)$.

'(b)'. Let $\psi \in \operatorname{Dom} \widetilde{\mathcal{A}}_q$. Then $\psi \in W^{-1,q}_{\mathfrak{D}} \cap W^{1,2}_{\mathfrak{D}}$ and $\mathcal{A}_2\psi \in W^{-1,q}_{\mathfrak{D}}$. Since $\mathcal{A}_q + I$ is surjective, there exists a $\varphi \in W^{1,q}_{\mathfrak{D}}$ such that $(\mathcal{A}_q + I)\varphi = (\mathcal{A}_2 + I)\psi$. Then $\varphi \in W^{1,2}_{\mathfrak{D}}$ and $\mathcal{A}_2\varphi = \mathcal{A}_q\varphi$. So $(\mathcal{A}_2 + I)\varphi = (\mathcal{A}_2 + I)\psi$. Since $(\mathcal{A}_2 + I)$ is injective, one deduces that $\psi = \varphi \in W^{1,q}_{\mathfrak{D}}$. So $\operatorname{Dom} \widetilde{\mathcal{A}}_q \subset W^{1,q}_{\mathfrak{D}}$. The reverse inclusion is trivial. □

The next proposition states that the set $\mathfrak{I}_\mu$ is always non-empty and open.

Proposition 6.3. *Adopt Assumption 5.1. Then for all $\mu \in \mathcal{E}$ the set $\mathfrak{I}_\mu$ is an open interval which contains 2. Moreover, for all $c_\bullet, c^\bullet > 0$ there are $\varepsilon, \delta > 0$ such that $(2-\delta, 2+\varepsilon) \subset \mathfrak{I}_\mu$ for all $\mu \in \mathcal{E}(c_\bullet, c^\bullet)$, and, in addition,*

$$\sup_{\mu \in \mathcal{E}(c_\bullet, c^\bullet)} \|(\mathcal{A}_q(\mu) + I)^{-1}\|_{W^{-1,q}_{\mathfrak{D}} \to W^{1,q}_{\mathfrak{D}}} < \infty.$$

Proof. If follows from Lemma 5.4 that $\mathfrak{I}_\mu$ is connected. Moreover, $2 \in \mathfrak{I}_\mu$ by the Lax–Milgram theorem. For the other assertions, see [HJKR, Theorem 5.6 and Remark 5.7]. □

Assumption 5.1 implies that the Kato problem for the operator A_q with real measurable coefficients and boundary conditions is solved on L^q for all $q \in (1,2]$.

Proposition 6.4. *Adopt Assumption 5.1. Let $\mu \in \mathcal{E}$ and $q \in (1,2]$. Then $\operatorname{Dom} A_q^{1/2} = W^{1,q}_{\mathfrak{D}}$. Moreover, the operator $(A_q + I)^{1/2}$ is a topological isomorphism from $W^{1,q}_{\mathfrak{D}}$ onto L^q. Hence its adjoint map $((A_q + I)^{1/2})'$ is a topological isomorphism from L^p onto $W^{-1,p}_{\mathfrak{D}}$ for all $p \in [2,\infty)$.*

Proof. The case $q = 2$ is proved in [EHT, Main Theorem 4.1]. The general case is in [ABHR, Theorem 5.1]. □

Let $\mu \in \mathcal{E}$ and $q \in (1,\infty)$. If $\operatorname{Dom} A_q^{1/2} = W^{1,q}_{\mathfrak{D}}$, then it follows from the closed graph theorem that the operator $(A_q + I)^{1/2}$ is a topological isomorphism from $W^{1,q}_{\mathfrak{D}}$ onto L^q. As a consequence we can split the problem whether $\mathcal{A}_q + I$ is an isomorphism from $W^{1,q}_{\mathfrak{D}}$ onto $W^{-1,q}_{\mathfrak{D}}$ into two parts: from $W^{1,q}_{\mathfrak{D}}$ into L^q and from L^q into $W^{-1,q}_{\mathfrak{D}}$.

Theorem 6.5. *Adopt Assumption 5.1.*

(a) *Let $q \in [2,\infty)$ and $\mu \in \mathcal{E}$. Then $q \in \mathfrak{I}_\mu$ if and only if $\operatorname{Dom} A_q^{1/2} = W^{1,q}_{\mathfrak{D}}$.*

(b) *Let* $\mu \in \mathcal{E}$ *and* $q \in \mathfrak{I}_\mu$. *Then*

$$\operatorname{Dom} A_q = \{\psi \in W^{1,q}_{\mathfrak{D}} : \mathcal{A}_q \psi \in L^q\}$$

and $A_q \psi = \mathcal{A}_q \psi$ *for all* $\psi \in \operatorname{Dom} A_q$.

(c) *For all* $c_\bullet, c^\bullet > 0$ *there exists an* $\varepsilon > 0$ *such that* $\operatorname{Dom} A_q^{1/2} = W^{1,q}_{\mathfrak{D}}$ *for all* $q \in (1, 2+\varepsilon)$ *and* $\mu \in \mathcal{E}(c_\bullet, c^\bullet)$.

Proof. '(a)'. Suppose that $\operatorname{Dom} A_q^{1/2} = W^{1,q}_{\mathfrak{D}}$. Then $(A_q + I)^{1/2}$ is an isomorphism from $W^{1,q}_{\mathfrak{D}}$ onto L^q. Write $p = q'$. Then it follows from Proposition 6.4 that $\operatorname{Dom} A_p(\mu^T)^{1/2} = W^{1,p}_{\mathfrak{D}}$ and the operator $(A_p(\mu^T) + I)^{1/2}$ is a topological isomorphism from $W^{1,p}_{\mathfrak{D}}$ onto L^p. Let $\psi \in W^{1,q}_{\mathfrak{D}}$. We first show that

$$\langle (A_q + I)^{1/2}\psi, (A_p(\mu^T) + I)^{1/2} u\rangle_{L^q \times L^p} = \langle (\mathcal{A}_q + I)\psi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} \tag{23}$$

for all $u \in W^{1,p}_{\mathfrak{D}}$. Let $u \in \operatorname{Dom}(A_2(\mu^T))$. Then $u \in \operatorname{Dom}(A_p(\mu^T))$ and

$$\begin{aligned}
\langle (A_q + I)^{1/2}\psi, (A_p(\mu^T) + I)^{1/2} u\rangle_{L^q \times L^p} &= \langle (A_2 + I)^{1/2}\psi, (A_2(\mu^T) + I)^{1/2} u\rangle_{L^2 \times L^2} \\
&= (\psi, (A_2(\mu^T) + I)u)_{L^2} \\
&= \sum_{k,l=1}^{d} \int_\Omega \mu_{kl}\, (\partial_k \psi)\, \overline{\partial_l u} + (\psi, u)_{L^2} \\
&= \langle (\mathcal{A}_q + I)\psi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}}.
\end{aligned}$$

So (23) is valid for all $u \in \operatorname{Dom}(A_2(\mu^T))$. Clearly $u \mapsto \langle (A_q + I)^{1/2}\psi, (A_p(\mu^T) + I)^{1/2} u\rangle_{L^q \times L^p}$ is continuous from $\operatorname{Dom}(A_p(\mu^T) + I)^{1/2}$ into $\mathbb{C}$. Also $u \mapsto \langle (\mathcal{A}_q + I)\psi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}}$ is continuous from $W^{1,p}_{\mathfrak{D}}$ into $\mathbb{C}$ and hence from $\operatorname{Dom}(A_p(\mu^T) + I)^{1/2}$ into $\mathbb{C}$. Since $\operatorname{Dom} A_2(\mu^T)$ is dense in $\operatorname{Dom} A_p(\mu^T)$, and hence in $\operatorname{Dom}(A_p(\mu^T) + I)^{1/2}$, it follows by continuity that (23) is valid for all $u \in \operatorname{Dom}(A_p(\mu^T) + I)^{1/2} = W^{1,p}_{\mathfrak{D}}$.

Let $\varphi \in W^{-1,q}_{\mathfrak{D}}$. By the last part of Proposition 6.4 there exists a $\tau \in L^q$ such that $\langle \tau, (A_p(\mu^T) + I)^{1/2} u\rangle_{L^q \times L^p} = \langle \varphi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}}$ for all $u \in W^{1,p}_{\mathfrak{D}}$. By assumption there exists a $\psi \in W^{1,q}_{\mathfrak{D}}$ such that $(A_q + I)^{1/2}\psi = \tau$. Then

$$\begin{aligned}
\langle \varphi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} &= \langle \tau, (A_p(\mu^T) + I)^{1/2} u\rangle_{L^q \times L^p} \\
&= \langle (A_q + I)^{1/2}\psi, (A_p(\mu^T) + I)^{1/2} u\rangle_{L^q \times L^p} = \langle (\mathcal{A}_q + I)\psi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}}
\end{aligned}$$

for all $u \in W^{1,p}_{\mathfrak{D}}$, where we used (23) in the last step. So $\varphi = (\mathcal{A}_q + I)\psi$ and $(\mathcal{A}_q + I)$ is surjective. Therefore $q \in \mathfrak{I}_\mu$ by Lemma 6.2(a).

Next let $q \in \mathfrak{I}_\mu$. We shall show that $W^{1,q}_{\mathfrak{D}} = \operatorname{Dom} A_q^{1/2}$. Let $\psi \in W^{1,q}_{\mathfrak{D}}$. Then $\psi \in W^{1,2}_{\mathfrak{D}}$ and $(\mathcal{A}_q + I)\psi \in W^{-1,q}_{\mathfrak{D}}$. By the last part of Proposition 6.4 there exists a $\tau \in L^q$ such that $\langle \tau, (A_p(\mu^T) + I)^{1/2} u\rangle_{L^q \times L^p} = \langle (\mathcal{A}_q + I)\psi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}}$ for all $u \in W^{1,p}_{\mathfrak{D}}$, where $p = q'$. Let

$u \in \operatorname{Dom} A_2(\mu^T)$. Then

$$\begin{aligned}\langle \tau, (A_p(\mu^T) + I)^{1/2} u \rangle_{L^q \times L^p} &= \langle (\mathcal{A}_q + I)\psi, u \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} \\ &= \langle (\mathcal{A}_2 + I)\psi, u \rangle_{W^{-1,2}_{\mathfrak{D}} \times W^{1,2}_{\mathfrak{D}}} \\ &= (\psi, (A_2(\mu^T) + I)u)_{L^2} \\ &= \langle \psi, (A_p(\mu^T) + I)u \rangle_{L^q \times L^p}.\end{aligned}$$

Since $\operatorname{Dom} A_2$ is a core for A_p one deduces that

$$\langle \tau, (A_p(\mu^T) + I)^{1/2} u \rangle_{L^q \times L^p} = \langle \psi, (A_p(\mu^T) + I)u \rangle_{L^q \times L^p}$$

for all $u \in \operatorname{Dom} A_p(\mu^T)$. Hence $\langle \tau, v \rangle_{L^q \times L^p} = \langle \psi, (A_p(\mu^T) + I)^{1/2} v \rangle_{L^q \times L^p}$ for all $v \in \operatorname{Dom}(A_p(\mu^T))^{1/2}$. This implies that $\psi \in \operatorname{Dom}(((A_p(\mu^T) + I)^{1/2})^*) = \operatorname{Dom} A_q^{1/2}$.

Conversely, let $\psi \in \operatorname{Dom} A_q^{1/2}$. Then $(A_q + I)^{1/2}\psi \in L^q$. By the last part of Proposition 6.4 there exists a $\varphi \in W^{-1,q}_{\mathfrak{D}}$ such that $\langle (A_q + I)^{1/2}\psi, (A_p(\mu^T) + I)^{1/2} u \rangle_{L^q \times L^p} = \langle \varphi, u \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}}$ for all $u \in W^{1,p}_{\mathfrak{D}}$. Since $q \in \mathfrak{I}_\mu$, the operator $\mathcal{A}_q + I$ is surjective. Hence there exists a $\tau \in W^{1,q}_{\mathfrak{D}}$ such that $(\mathcal{A}_q + I)\tau = \varphi$. Now let $u \in \operatorname{Dom} A_2(\mu^T)$. Then

$$\begin{aligned}(\psi, (A_2(\mu^T) + I)u)_{L^2} &= \langle \psi, (A_p(\mu^T) + I)u \rangle_{L^q \times L^p} \\ &= \langle (A_q + I)^{1/2}\psi, (A_p(\mu^T) + I)^{1/2} u \rangle_{L^q \times L^p} \\ &= \langle \varphi, u \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} \\ &= \langle (\mathcal{A}_q + I)\tau, u \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} \\ &= \langle (\mathcal{A}_2 + I)\tau, u \rangle_{W^{-1,2}_{\mathfrak{D}} \times W^{1,2}_{\mathfrak{D}}} \\ &= (\tau, (A_2(\mu^T) + I)u)_{L^2}\end{aligned}$$

Since $(A_2(\mu^T) + I)$ is surjective, it follows that $\psi = \tau \in W^{1,q}_{\mathfrak{D}}$.

'(b)'. Again write $p = q'$. First suppose that $q \geq 2$. Let $\psi \in \operatorname{Dom} A_q$. Then $\psi \in \operatorname{Dom} A_q^{1/2} = W^{1,q}_{\mathfrak{D}}$ by Statement (a). Moreover, $\psi \in \operatorname{Dom} A_2$. If $u \in \operatorname{Dom}(A_2(\mu^T))^{1/2}$ then $u \in \operatorname{Dom}(A_p(\mu^T))^{1/2}$ and (23) gives

$$\begin{aligned}\langle (\mathcal{A}_q + I)\psi, u \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} &= ((A_2 + I)^{1/2}\psi, (A_2(\mu^T) + I)^{1/2} u)_{L^2} \\ &= ((A_2 + I)\psi, u)_{L^2} = \langle (A_q + I)\psi, u \rangle_{L^q \times L^p}.\end{aligned}$$

Hence $\mathcal{A}_q \psi = A_q \psi \in L^q$.

Conversely, let $\psi \in W^{1,q}_{\mathfrak{D}}$ and suppose that $\mathcal{A}_q \psi \in L^q$. Write $\tau = (\mathcal{A}_q + I)\psi$. Then

$$\begin{aligned}\langle \tau, u \rangle_{L^q \times L^p} &= \langle (\mathcal{A}_q + I)\psi, u \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} \\ &= \langle (A_q + I)^{1/2}\psi, (A_p(\mu^T) + I)^{1/2} u \rangle_{L^q \times L^p}\end{aligned}$$

for all $u \in W^{1,p}_{\mathfrak{D}} = \operatorname{Dom}(A_p(\mu^T) + I)^{1/2}$, where we used (23). It follows that $(A_q + I)^{1/2}\psi \in \operatorname{Dom}((A_p(\mu^T)^{1/2} + I)^*) = \operatorname{Dom}(A_q + I)^{1/2}$. Hence $\psi \in \operatorname{Dom}((A_q + I)^{1/2}\, (A_q + I)^{1/2}) = \operatorname{Dom} A_q$.

Now suppose that $q \leq 2$. Let $\psi \in \operatorname{Dom} A_q$. Then $\psi \in \operatorname{Dom} A_q^{1/2} = W^{1,q}_{\mathfrak{D}}$ by Proposition 6.4. If $u \in \operatorname{Dom} A_p(\mu^T)$ then by the above $u \in W^{1,p}_{\mathfrak{D}}$ and $A_p(\mu^T)u = \mathcal{A}_p(\mu^T)u$. So

$$\begin{aligned}\langle A_q\psi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} &= \langle A_q\psi, u\rangle_{L^q \times L^p} = \langle \psi, A_p(\mu^T)u\rangle_{L^q \times L^p} = \langle \psi, \mathcal{A}_p(\mu^T)u\rangle_{L^q \times L^p} \\ &= \langle \psi, \mathcal{A}_p(\mu^T)u\rangle_{W^{1,q}_{\mathfrak{D}} \times W^{-1,p}_{\mathfrak{D}}} = \langle \mathcal{A}_q\psi, u\rangle_{L^q \times L^p}\end{aligned}$$

and

$$\langle A_q\psi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} = \langle \mathcal{A}_q\psi, u\rangle_{L^q \times L^p}. \tag{24}$$

Since $\operatorname{Dom} A_p(\mu^T)$ is dense in $\operatorname{Dom} A_p(\mu^T)^{1/2} = W^{1,p}_{\mathfrak{D}}$, one deduces that (24) is valid for all $u \in W^{1,p}_{\mathfrak{D}}$. So $\mathcal{A}_q\psi = A_q\psi \in L^q$.

Conversely, suppose that $\psi \in W^{1,q}_{\mathfrak{D}}$ and $\mathcal{A}_q\psi \in L^q$. Let $u \in \operatorname{Dom} A_p(\mu^T)$. Then again by the above

$$\begin{aligned}\langle \psi, A_p(\mu^T)u\rangle_{L^q \times L^p} &= \langle \psi, \mathcal{A}_p(\mu^T)u\rangle_{L^q \times L^p} = \langle \psi, \mathcal{A}_p(\mu^T)u\rangle_{W^{1,q}_{\mathfrak{D}} \times W^{-1,p}_{\mathfrak{D}}} \\ &= \langle \mathcal{A}_q\psi, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} = \langle \mathcal{A}_q\psi, u\rangle_{L^q \times L^p}.\end{aligned}$$

So $\psi \in \operatorname{Dom}((A_p(\mu^T))^*) = \operatorname{Dom} A_q$ and the proof of Statement (b) is complete.

'(c)'. This follows from Statement (a) and Proposition 6.3. □

We next present a few illustrative examples with explicit subsets of the set $\mathfrak{I}_\mu$. Note that the requirements on the geometry of Ω and the Dirichlet boundary part $\mathfrak{D}$, as well as on the coefficient function μ is much higher in the examples than in our general assumptions.

Example 6.6. Assume that Ω is a C^1-domain and that $\mathfrak{D} = \partial\Omega$ or $\mathfrak{D} = \emptyset$ (pure Dirichlet or pure Neumann condition). If $\mu \in \mathcal{E}$ is uniformly continuous on Ω, then $\mathfrak{I}_\mu = (1, \infty)$ by [ADN, Section 15] or [Mor, pages 156–157].

The conclusion remains true, if there is a C^1-subdomain Λ with positive distance to the boundary, such that $\mu|_\Lambda$ and $\mu|_{\Omega \setminus \overline{\Lambda}}$ are uniformly continuous, see [ERS, Theorem 1.1 and Remark 3.15].

Example 6.7. Assume that Ω is a Lipschitz graph-domain (see [Gri2, Definition 1.2.1.1]). There are equivalent terminologies for this notion: strong Lipschitz domain in [Maz, Section 1.1.8] and Ω possesses the uniform cone property in [Gri2, Section 1.2.2]. Suppose that $\mu \in \mathcal{E}$ takes symmetric matrices as values. Then, under the same continuity properties for μ as in Example 6.6 (both cases), the (open) set $\mathfrak{I}_\mu$ contains the interval $[2, 3]$ both in the pure Dirichlet case (that is $\mathfrak{D} = \partial\Omega$), and in the pure Neumann case (that is $\mathfrak{D} = \emptyset$), see [ERS]. Moreover, one cannot replace 3 by a larger number, independent of Ω and μ. For the pure Dirichlet Laplacian this result was already proved in [JK, Theorem 1.1(c) and Theorem 1.2(a)], and for the pure Neumann Laplacian in [Zan].

Example 6.8. In [DKR] there are given a huge variety of domains $\Omega \subset \mathbb{R}^3$, Dirichlet boundary parts $\mathfrak{D}$ and (possibly discontinuous – even up to the boundary) elliptic coefficient functions μ, such that $\mathfrak{I}_\mu$ contains the interval $[2, 3]$. In particular, it is allowed that $\mathfrak{D} \cap \overline{\partial\Omega \setminus \mathfrak{D}}$ is not empty, i.e. the Dirichlet boundary part meets the Neumann part.

For all $q \in (1,\infty)$ we consider the operator $\mathcal{A}_q$ as a densely defined operator in $W^{-1,q}_{\mathfrak{D}}$ with domain $W^{1,p}_{\mathfrak{D}}$.

Let $\mu \in \mathcal{E}$ and $q \in \mathfrak{I}_\mu \cup [2,\infty)$. Write $p = q'$. Then it follows from Proposition 6.4 and Theorem 6.5(a) that $(A_p(\mu^T) + I)^{1/2} \colon W^{1,p}_{\mathfrak{D}} \to L^p$ is a topological isomorphism. Let $((A_p(\mu^T) + I)^{1/2})' \colon L^q \to W^{-1,q}_{\mathfrak{D}}$ be the adjoint of the operator. Then $((A_p(\mu^T) + I)^{1/2})'$ is an isomorphism too. We use the isomorphism $((A_p(\mu^T) + I)^{1/2})'$ to transfer the C_0-semigroup $S^{(q)}$ on L^q to a C_0-semigroup $T^{(q)}$ on $W^{-1,q}_{\mathfrak{D}}$. Explicitly, for all $t \in (0,\infty)$ define $T^{(q)}_t \colon W^{-1,q}_{\mathfrak{D}} \to W^{-1,q}_{\mathfrak{D}}$ by

$$T^{(q)}_t = ((A_p(\mu^T) + I)^{1/2})' \, S^{(q)}_t \Big(((A_p(\mu^T) + I)^{1/2})' \Big)^{-1}. \tag{25}$$

Then $T^{(q)}$ is a C_0-semigroup on $W^{-1,q}_{\mathfrak{D}}$. Clearly $((A_p(\mu^T) + I)^{1/2})'$ is an extension of the operator $(A_q + I)^{1/2}$ and hence $\Big(((A_p(\mu^T) + I)^{1/2})' \Big)^{-1}$ is an extension of $(A_q + I)^{-1/2}$. Since $(A_q + I)^{-1/2}$ and $S^{(q)}_t$ commute for all $t > 0$, it follows that $T^{(q)}_t$ is an extension of $S^{(q)}_t$ for all $t > 0$.

We denote the generator of $T^{(q)}$ by $-B_q = -B_q(\mu)$. Obviously $T^{(q_1)}$ is consistent with $T^{(q_2)}$ for all $q_1, q_2 \in \mathfrak{I}_\mu \cup [2,\infty)$. Hence the graph of B_{q_1} is an extension of the graph of B_{q_2} and

$$\{(\psi, B_{q_2}\psi) : \psi \in \operatorname{Dom} B_{q_2}\} = \{(\psi, B_{q_1}\psi) : \psi \in \operatorname{Dom} B_{q_2}\} \cap \Big(W^{-1,q_2}_{\mathfrak{D}} \times W^{-1,q_2}_{\mathfrak{D}} \Big) \tag{26}$$

if $q_1 \le q_2$.

Lemma 6.9. *Adopt Assumption 5.1. Let μ be in $\mathcal{E}$.*

(a) *If $q \in \mathfrak{I}_\mu \cup [2,\infty)$, then* $\operatorname{Dom} B_q = \operatorname{Dom} A_q^{1/2}$.

(b) *If $q \in \mathfrak{I}_\mu$, then $B_q = \mathcal{A}_q$.*

(c) *If $q \in [2,\infty)$, then $B_q = \widetilde{\mathcal{A}}_q$.*

Proof. '(a)'. By definition of the semigroup $T^{(q)}$ it follows that

$$\operatorname{Dom} B_q = \{((A_p(\mu^T) + I)^{1/2})' v : v \in \operatorname{Dom} A_q\}.$$

Write $p = q'$. If $v \in \operatorname{Dom} A_q^{1/2}$ and $u \in W^{1,p}_{\mathfrak{D}}$, then

$$\begin{aligned} \langle ((A_p(\mu^T) + I)^{1/2})' v, u \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} &= (v, (A_p(\mu^T) + I)^{1/2} u)_{L^q \times L^p} \\ &= ((A_q + I)^{1/2} v, u)_{L^q \times L^p} = \langle (A_q + I)^{1/2} v, u \rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}}. \end{aligned}$$

So $((A_p(\mu^T) + I)^{1/2})' v = (A_q + I)^{1/2} v$ for all $v \in \operatorname{Dom} A_q^{1/2}$. Hence

$$\operatorname{Dom} B_q = \{((A_p(\mu^T) + I)^{1/2})' v : v \in \operatorname{Dom} A_q\} = \operatorname{Dom} A_q^{1/2}.$$

This proves Statement (a).

'(b)'. Suppose $q \in \mathfrak{I}_\mu$. Then $\operatorname{Dom} B_q = \operatorname{Dom} A_q^{1/2} = W^{1,q}_{\mathfrak{D}} = \operatorname{Dom} \mathcal{A}_q$ by Statement (a), Theorem 6.5(a) and Proposition 6.4. If $v \in \operatorname{Dom} A_q^{1/2}$ and $u \in W^{1,p}_{\mathfrak{D}}$, then

$$\begin{aligned}\langle ((A_p(\mu^T) + I)^{1/2})'(A_q + I)^{1/2} v, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}} &= ((A_q + I)^{1/2} v, (A_p(\mu^T) + I)^{1/2} u)_{L^q \times L^p} \\ &= \langle (\mathcal{A}_q + I)v, u\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,p}_{\mathfrak{D}}},\end{aligned}$$

where the last equality is (23). So $((A_p(\mu^T) + I)^{1/2})'(A_q + I)^{1/2} v = (\mathcal{A}_q + I)v$ for all $v \in \operatorname{Dom} A_q^{1/2}$. Hence $((A_p(\mu^T)+I)^{1/2})'(A_q+I)v = (\mathcal{A}_q+I)(A_q+I)^{1/2} v$ for all $v \in \operatorname{Dom} A_q$. Using again that $((A_p(\mu^T) + I)^{1/2})'v = (A_q + I)^{1/2} v$ one deduces that

$$\begin{aligned}&(\mathcal{A}_q + I)\Big(((A_p(\mu^T) + I)^{1/2})'v\Big) \\ &\quad = (\mathcal{A}_q + I)(A_q + I)^{1/2} v \\ &\quad = ((A_p(\mu^T) + I)^{1/2})'(A_q + I)v \\ &\quad = ((A_p(\mu^T) + I)^{1/2})'(A_q + I)\Big(((A_p(\mu^T) + I)^{1/2})'\Big)^{-1}\Big(((A_p(\mu^T) + I)^{1/2})'v\Big) \\ &\quad = (B_q + I)\Big(((A_p(\mu^T) + I)^{1/2})'v\Big)\end{aligned}$$

for all $v \in \operatorname{Dom} A_q$. Hence $B_q = \mathcal{A}_q$.

'(c)'. Let $q \in [2, \infty)$. It follows from (26) and Statement (b) that

$$\operatorname{Dom} B_q = \{\psi \in W^{-1,q}_{\mathfrak{D}} : B_2\psi \in W^{-1,q}_{\mathfrak{D}}\} = \{\psi \in W^{-1,q}_{\mathfrak{D}} : \mathcal{A}_2\psi \in W^{-1,q}_{\mathfrak{D}}\} = \widetilde{\mathcal{A}}_q.$$

So $B_q = \widetilde{\mathcal{A}}_q$. □

In (25) the topological isomorphism $((A_p(\mu^T) + I)^{1/2})'$ was used to define the C_0-semigroup $T^{(q)}$ from the C_0-semigroup $S^{(q)}$. It then transfers properties of the generator of $S^{(q)}$ to properties of the generator of $T^{(q)}$.

Theorem 6.10. *Adopt Assumption 5.1. Let $\mu \in \mathcal{E}$ and $q \in \mathfrak{I}_\mu \cup [2, \infty)$. Then the operator $B_q + I$ satisfies maximal parabolic regularity on the space $W^{-1,q}_{\mathfrak{D}}$.*

Proof. The semigroup $S^{(q)}$ is a contraction semigroup, hence the operator $A_q + I$ has maximal parabolic regularity in the space L^q by Lamberton [Lam]. Since $((A_p(\mu^T)+I)^{1/2})'$ is a topological isomorphism from L^q onto $W^{-1,q}_{\mathfrak{D}}$, where $p = q'$, it follows from (25) that the operator $B_q + I$ satisfies maximal parabolic regularity on the space $W^{-1,q}_{\mathfrak{D}}$. □

Note that $B_q = \mathcal{A}_q$ for all $q \in \mathfrak{I}_\mu$ in the next theorem. The case $q \in [2, \infty)$ in the next corollary has been proved before in [ABHR, Theorem 11.5].

Corollary 6.11. *Adopt Assumption 5.1. Then for all $c_\bullet, c^\bullet > 0$ there exists a $\delta \in (0, 1)$ such that the operator $B_q + I$ satisfies maximal parabolic regularity on the space $W^{-1,q}_{\mathfrak{D}}$ for all $q \in (2 - \delta, \infty)$ and $\mu \in \mathcal{E}(c_\bullet, c^\bullet)$.*

Proof. This follows immediately from Proposition 6.3 and Theorem 6.10. □

Remark 6.12. It is clear that there is an asymmetry in the cases $q \in [2,\infty)$ and $q \in (1,2]$. In the first case, maximal parabolic regularity holds for the operators $\widetilde{\mathcal{A}}_q + I$ on $W^{-1,q}_{\mathfrak{D}}$ by Theorem 6.10 and Lemma 6.9(c), even if the domain of this operator is unknown. On the contrary, in the case $q < 2$, we can only prove maximal parabolic regularity for the operator $\mathcal{A}_q + I$ in $W^{-1,q}_{\mathfrak{D}}$ if $q \in \mathfrak{I}_\mu$. This is a severe restriction on q, see Examples 6.6–6.8. It is an open problem whether in Corollary 6.11 maximal parabolic regularity for $\mathcal{A}_q + I$ is valid on $W^{-1,q}_{\mathfrak{D}}$ for all $q \in (1,2)$.

7 Time dependent coefficients

We next consider coefficient functions which also may depend on time. Let $\mu\colon J \to \mathcal{E}$ be a function. We frequently write $\mu_t = \mu(t)$ for all $t \in J$. Note that $\mu_t \in L^\infty(\Omega;\mathbb{R}^{d\times d}) \subset L^1(\Omega;\mathbb{R}^{d\times d})$ for all $t \in J$. We say that μ **is L^1-measurable** if the map $t \mapsto \mu_t$ is measurable as a map from J into $L^1(\Omega;\mathbb{R}^{d\times d})$.

In the main result of this section we require measurability of the coefficient function only in the space $L^1(\Omega;\mathbb{R}^{d\times d})$. This allows that μ_t is discontinuous in the space variable for each $t \in J$. Note that the set of point in Ω where μ_t is discontinuous may depend on t. In general the map $t \mapsto \mu_t$ from J into $L^\infty(\Omega;\mathbb{R}^{d\times d})$ is discontinuous at *every* time point t and therefore it cannot be measurable. An example is mentioned in the introduction and it will be considered in more detail in Section 9.

Lemma 7.1. *Adopt Assumption 5.1. Let $c^\bullet > 0$ and $\mu\colon J \to \bigcup_{c_\bullet > 0} \mathcal{E}(c_\bullet, c^\bullet)$ be an L^1-measurable map. Let $q, r \in (1,\infty)$. Then one has the following.*

(a) *The map $t \mapsto \mathcal{A}_q(\mu_t)\psi$ is (strongly) measurable from J into $W^{-1,q}_{\mathfrak{D}}$ for all $\psi \in W^{1,q}_{\mathfrak{D}}$.*

(b) *The map $\frac{\partial}{\partial t} + \mathcal{A}_q(\mu(\cdot)) + I$ is a bounded linear map from $\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$ into $L^r(J; W^{-1,q}_{\mathfrak{D}})$ with norm at most $1 + c^\bullet$.*

Proof. Let $\psi, \varphi \in C^\infty_{\mathfrak{D}}(\Omega)$. Then the map

$$\rho \mapsto \int_\Omega \rho\, \nabla\psi \cdot \overline{\nabla\varphi}$$

is continuous from $L^1(\Omega,\mathbb{R}^{d\times d})$ into $\mathbb{C}$. Since μ is L^1-measurable, also the map

$$t \mapsto \langle \mathcal{A}_q(\mu_t)\psi, \varphi\rangle_{W^{-1,q}_{\mathfrak{D}} \times W^{1,q}_{\mathfrak{D}}} \tag{27}$$

from J into $\mathbb{C}$ is well defined, bounded and measurable. Since $C^\infty_{\mathfrak{D}}(\Omega)$ is dense in $W^{1,q}_{\mathfrak{D}}$ and $W^{1,q'}_{\mathfrak{D}}$ and μ is bounded, the map (27) is measurable for all $\psi \in W^{1,q}_{\mathfrak{D}}$ and $\varphi \in W^{1,q'}_{\mathfrak{D}}$. Therefore one obtains the weak measurability of the map $J \ni t \mapsto \mathcal{A}_q(\mu_t)\psi \in W^{-1,q}_{\mathfrak{D}}$ for all $\psi \in W^{1,q}_{\mathfrak{D}}$, which implies also the strong measurability, since the space $W^{-1,q}_{\mathfrak{D}}$ is separable. This proves the first statement. The second one is easy. □

The first main result of this section is as follows. In order to get good estimates, we use again the normed spaces $\mathrm{MR}^r_0(J; W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}})^\sim$ introduced in Definition 3.6.

Theorem 7.2. *Let $c_\bullet, c^\bullet \in (0,\infty)$ with $c_\bullet \leq 1 \leq c^\bullet$. Let $s \in (2,\infty)$ and put*

$$C_{\mathcal{J},s} = \max_{r\in\{s,s'\}} \|\big(\frac{\partial}{\partial t} + \mathcal{J}\big)^{-1}\|_{L^r(J;W^{-1,2}_{\mathfrak{D}})\to \mathrm{MR}^r_0(J;W^{1,2}_{\mathfrak{D}},W^{-1,2}_{\mathfrak{D}})},$$

where $\mathcal{J}\colon W^{1,2}_{\mathfrak{D}} \to W^{-1,2}_{\mathfrak{D}}$ is the duality map. Define

$$\kappa_s := \frac{1}{12}\,\frac{1}{1+2(1+\frac{1+c_\bullet+c^\bullet}{c_\bullet})\max(1,c^\bullet)C_{\mathcal{J},s}} \quad \text{and} \quad r_0 := \big(\frac{1}{2} - \kappa_s(1-\frac{2}{s})\big)^{-1}.$$

Then for every L^1-measurable $\mu\colon J \to \mathcal{E}(c_\bullet,c^\bullet)$ and $r \in (r_0', r_0)$, the family $\{\mathcal{A}_2(\mu_t)+I\}_{t\in J}$ has maximal $L^r(J;W^{-1,2}_{\mathfrak{D}})$-regularity and

$$\|\Big(\frac{\partial}{\partial t} + \mathcal{A}_2(\mu(\cdot)) + I\Big)^{-1}\|_{L^r(J;W^{-1,2}_{\mathfrak{D}})\to \mathrm{MR}^r_0(J;W^{1,2}_{\mathfrak{D}},W^{-1,2}_{\mathfrak{D}})^\sim} \leq 8\frac{1+c_\bullet+c^\bullet}{c_\bullet},$$

where the norm on $\mathrm{MR}^r_0(J;W^{1,2}_{\mathfrak{D}},W^{-1,2}_{\mathfrak{D}})^\sim$ is defined using the operator $\mathcal{J}$.

Proof. We wish to apply Theorem 4.2. Let $V = W^{1,2}_{\mathfrak{D}}$. Then $V^* = W^{-1,2}_{\mathfrak{D}}$. Note that $\frac{1}{2} = \frac{1-\theta}{s} + \frac{\theta}{s'}$ if $\theta = \frac{1}{2}$. Let $\mu\colon J \to \mathcal{E}(c_\bullet,c^\bullet)$ be an L^1-measurable map. For all $t \in J$ define $\mathfrak{s}_t\colon V\times V \to \mathbb{C}$ by

$$\mathfrak{s}_t[\psi,\varphi] = \mathfrak{s}_{\mu(t)}[\psi,\varphi] + (\psi,\varphi)_{L^2}.$$

Then $t \mapsto \mathfrak{s}_t[\psi,\varphi]$ is measurable from J into $\mathbb{C}$ for all $\psi,\varphi \in V$. Moreover, $\operatorname{Re}\mathfrak{s}_t[\psi,\psi] \geq c_\bullet\|\psi\|_V^2$ and $|\mathfrak{s}_t[\psi,\varphi]| \leq c^\bullet\,\|\psi\|_V\,\|\varphi\|_V$ for all $\varphi,\psi \in V$ and $t \in J$. If $t \in J$, then $\mathcal{A}_2(\mu(t))+I$ is the operator induced by the sesquilinear form $\mathfrak{s}_t$.

All the assumptions of Theorem 4.2 are satisfied. If $\tilde\theta \in (\frac{1}{2}-\kappa_s, \frac{1}{2}+\kappa_s)$, then it follows from Theorem 4.2(b) that the isomorphism property is preserved. Then the assertion follows by using the identity $\frac{1}{r} = \frac{1-\tilde\theta}{s} + \frac{\tilde\theta}{s'}$. □

The second main result of this section is that non-autonomous maximal $L^r(J;W^{-1,q}_{\mathfrak{D}})$-regularity extrapolates in both temporal and spatial integrability scales, given by r and q. Again, quantitative estimates as in (31) below, are based on the constants $C^r_{\mathcal{K}}$ in (12), corresponding to a suitable autonomous reference operator $\mathcal{K}$.

Theorem 7.3. *Suppose Assumption 5.1 is satisfied. Let $c_\bullet, c^\bullet > 0$. Then there are open intervals $\mathcal{I}_1,\mathcal{I}_2 \subset (1,\infty)$ with $2 \in \mathcal{I}_1$ and $2 \in \mathcal{I}_2$ such that for all $r \in \mathcal{I}_1$, $q \in \mathcal{I}_2$ and L^1-measurable $\mu\colon J \to \mathcal{E}(c_\bullet,c^\bullet)$ the family $\{\mathcal{A}_q(\mu_t)+I\}_{t\in J}$ has maximal parabolic $L^r(J;W^{-1,q}_{\mathfrak{D}})$-regularity. So*

$$\frac{\partial}{\partial t} + \mathcal{A}_q(\mu(\cdot)) + I\colon \mathrm{MR}^r_0(J;W^{1,q}_{\mathfrak{D}},W^{-1,q}_{\mathfrak{D}}) \to L^r(J;W^{-1,q}_{\mathfrak{D}})$$

is a topological isomorphism.

Proof. Define $\delta\colon\Omega \to \mathbb{R}^{d\times d}$ by $\delta(x) = I$, the identity matrix, for all $x \in \Omega$. Then $\delta \in \mathcal{E}$ and $A(\delta) = -\Delta$, the minus Laplacian. Let $\mathcal{J}\colon W^{1,2}_{\mathfrak{D}} \to W^{-1,2}_{\mathfrak{D}}$ be the duality mapping. Then $\langle\mathcal{J}\psi,\varphi\rangle_{W^{-1,2}_{\mathfrak{D}}\times W^{1,2}_{\mathfrak{D}}} = (\psi,\varphi)_{W^{1,2}_{\mathfrak{D}}} = (\psi,\varphi)_{L^2(\Omega)} + \sum_{k=1}^d(\partial_k\psi,\partial_k\varphi)_{L^2(\Omega)} = \langle(\mathcal{A}_2(\delta)+I)\psi,\varphi\rangle_{W^{-1,2}_{\mathfrak{D}}\times W^{1,2}_{\mathfrak{D}}}$ for all $\psi,\varphi \in W^{1,2}_{\mathfrak{D}}$. So $\mathcal{J} = \mathcal{A}_2(\delta)+I$.

If follows from Proposition 6.3, Lemma 6.9(b) and Theorem 6.10 that there exists a $q_0 \in (2,\infty)$ such that $\mathcal{A}_q(\delta) + I \colon W^{1,q}_{\mathfrak{D}} \to W^{-1,q}_{\mathfrak{D}}$ is an isomorphism and the operator $\mathcal{A}_q(\delta) + I$ satisfies maximal parabolic regularity on the space $W^{-1,q}_{\mathfrak{D}}$ for all $q \in (q_0', q_0)$. By Proposition 6.3, there is a $q_1 \in (2, q_0]$ such that $[q_1', q_1] \subset \mathfrak{I}_{\tilde\mu}$ for all $\tilde\mu \in \mathcal{E}(c_\bullet, c^\bullet)$.

For all $q \in [q_1', q_1]$, we choose $\mathcal{K}_q = \mathcal{A}_q(\delta) + I$ as the autonomous reference operator for the spaces $\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim$ in Definition 3.6. Note that in case $q = 2$ the reference operator is $\mathcal{K}_2 = \mathcal{A}_2(\delta) + I = \mathcal{J}$, which was used in Theorem 7.2. For all $s \in (2,\infty)$ and $\alpha \in (0,1)$ with $\frac{1}{r} = \frac{1-\alpha}{s} + \frac{\alpha}{s'}$ we obtain by Lemma 3.7 that

$$[\mathrm{MR}^{s'}_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim, \mathrm{MR}^{s}_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim]_\alpha = \mathrm{MR}^{r}_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim \tag{28}$$

with equality of norms. Moreover, for each $q \in [q_1', q_1]$ and $r \in (1,\infty)$ the norms on $\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$ and $\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim$ are equivalent. Hence it suffices to prove the theorem with $\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$ replaced by $\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim$.

By Theorem 7.2, there is an $r_0 \in (2,\infty)$ such that for all L^1-measurable $\mu \colon J \to \mathcal{E}(c_\bullet, c^\bullet)$ and $r \in [r_0', r_0]$ the map

$$\frac{\partial}{\partial t} + \mathcal{A}_2(\mu(\cdot)) + I \colon \mathrm{MR}^r_0(J; W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}})^\sim \to L^r(J; W^{-1,2}_{\mathfrak{D}})$$

is a topological isomorphism with

$$\|\Big(\frac{\partial}{\partial t} + \mathcal{A}_2(\mu(\cdot)) + I\Big)^{-1}\|_{L^r(J; W^{-1,2}_{\mathfrak{D}}) \to \mathrm{MR}^r_0(J; W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}})^\sim} \le 8\frac{1 + c_\bullet + c^\bullet}{c_\bullet}. \tag{29}$$

These will be the important inverse bounds to apply Theorem 2.4.

Next, let $q \in \{q_1', q_1\}$. We need a suitable bound on the operator norms

$$\gamma_{q,r} := \|\frac{\partial}{\partial t} + \mathcal{A}_q(\mu(\cdot)) + I\|_{\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim \to L^r(J; W^{-1,q}_{\mathfrak{D}})},$$

uniformly in $r \in [r_0', r_0]$. Since both the spaces $\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim$ and $L^r(J; W^{-1,q}_{\mathfrak{D}})$ form exact complex interpolation scales in r by (28) and Proposition 2.3, it follows by interpolation that

$$\gamma_{q,r} \le \max_{r \in \{r_0', r_0\}} \gamma_{q,r}.$$

Now let $r \in \{r_0', r_0\}$. Then it follows from Lemma 7.1(b) and (13) that

$$\begin{aligned}\|(\frac{\partial}{\partial t} + \mathcal{A}_q(\mu(\cdot)) + I)u\|_{L^r(J; W^{-1,q}_{\mathfrak{D}})} &\le (1 + c^\bullet)\, \|u\|_{\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})} \\ &\le (1 + c^\bullet)\, C^r_{\mathcal{K}_q}\, \|u\|_{\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim}\end{aligned}$$

for all $u \in \mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$. So $\gamma_{q,r} \le (1 + c^\bullet)\, C^r_{\mathcal{K}_q}$. Set

$$\gamma_0 := \max_{q \in \{q_1', q_1\}} \max_{r \in \{r_0', r_0\}} (1 + c^\bullet)\, C^r_{\mathcal{K}_q}.$$

Then we proved that

$$\|\frac{\partial}{\partial t} + \mathcal{A}_q(\mu(\cdot)) + I\|_{\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})^\sim \to L^r(J; W^{-1,q}_{\mathfrak{D}})} \le \gamma_0 \tag{30}$$

for all $r \in [r_0', r_0]$, $q \in \{q_1', q_1\}$ and L^1-measurable $\mu\colon J \to \mathcal{E}(c_\bullet, c^\bullet)$.

Let
$$\kappa = \frac{1}{12} \frac{1}{1 + 2(1 + 8\frac{1+c_\bullet+c^\bullet}{c_\bullet})\gamma_0} \quad \text{and} \quad q_2 = \Big(\frac{1}{2} - \kappa(1 - \frac{2}{q_1})\Big)^{-1}. \tag{31}$$

Finally, let $\mu\colon J \to \mathcal{E}(c_\bullet, c^\bullet)$ be L^1-measurable. Let $r \in [r_0', r_0]$ and $q \in [q_1', q_1]$. Then there exists a $\tilde\theta \in [\frac{1}{2} - \kappa, \frac{1}{2} + \kappa]$ such that $\frac{1}{q} = \frac{1-\tilde\theta}{q_1} + \frac{\tilde\theta}{q_1'}$. Note that $\frac{1}{2} = \frac{1-\theta}{q_1} + \frac{\theta}{q_1'}$ with $\theta = \frac{1}{2}$. We apply Theorem 2.4 with $F_1 = \mathrm{MR}_0^r(J; W_{\mathfrak{D}}^{1,q_1'}, W_{\mathfrak{D}}^{-1,q_1'})^\sim$, $F_2 = \mathrm{MR}_0^r(J; W_{\mathfrak{D}}^{1,q_1}, W_{\mathfrak{D}}^{-1,q_1})^\sim$, $Z_1 = L^r(J; W_{\mathfrak{D}}^{-1,q_1'})$, $Z_2 = L^r(J; W_{\mathfrak{D}}^{-1,q_1})$ and $\theta = \frac{1}{2}$. Note that we have the estimates (30) and (29). Since $|\tilde\theta - \frac{1}{2}| \le \kappa$ one deduces from Theorem 2.4 that

$$\frac{\partial}{\partial t} + \mathcal{A}_q(\mu(\cdot)) + I\colon \mathrm{MR}_0^r(J; W_{\mathfrak{D}}^{1,q}, W_{\mathfrak{D}}^{-1,q})^\sim \to L^r(J; W_{\mathfrak{D}}^{-1,q})$$

is a topological isomorphism. This completes the proof of Theorem 7.3. □

8 Quasilinear equations

In this section we are interested in quasilinear, non-autonomous equations of the form

$$u'(t) - \nabla \cdot \big(\sigma(u(t))\mu_t \nabla u(t)\big) + u(t) = f(t); \quad u(0) = 0.$$

The main result is the following.

Theorem 8.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $\mathfrak{D} \subset \partial\Omega$ be closed. Suppose Assumption 5.1 is satisfied. Let $c_\bullet, c^\bullet > 0$ and $\mu\colon J \to \mathcal{E}(c_\bullet, c^\bullet)$ an L^1-measurable map. Let $\sigma_\bullet, \sigma^\bullet \in (0, \infty)$ with $\sigma_\bullet \le \sigma^\bullet$. Let $\sigma\colon \mathbb{R} \mapsto [\sigma_\bullet, \sigma^\bullet]$ be a continuous function. Then there exists an $r_0 \in (2, \infty)$ such that for all $r \in (2, r_0)$ and $f \in L^r(J; W_{\mathfrak{D}}^{-1,2})$ there exists a $u \in L^r(J; W_{\mathfrak{D}}^{1,2}) \cap W_0(J; W_{\mathfrak{D}}^{-1,2})$ such that*

$$u'(t) + \mathcal{A}_2(\sigma(u(t))\, \mu_t)u(t) + u(t) = f(t) \tag{32}$$

in $W_{\mathfrak{D}}^{-1,2}$ for almost every $t \in J$.

Proof. By Theorem 7.2 there exist $r_0 \in (2, \infty)$ and $\beta' > 0$ such that for every L^1-measurable $\mu\colon J \to \mathcal{E}(\sigma_\bullet\, c_\bullet, \sigma^\bullet\, c^\bullet)$ the family $\{\mathcal{A}_2(\mu(t)) + I\}_{t \in J}$ has maximal $L^r(J; W_{\mathfrak{D}}^{-1,2})$-regularity and

$$\Big\|\Big(\frac{\partial}{\partial t} + \mathcal{A}_2(\mu(\cdot)) + I\Big)^{-1}\Big\|_{L^r(J; W_{\mathfrak{D}}^{-1,2}) \to \mathrm{MR}_0^r(J; W_{\mathfrak{D}}^{1,2}, W_{\mathfrak{D}}^{-1,2})} \le \beta'$$

for all $r \in [2, r_0]$.

Now let $r \in (2, r_0]$, $f \in L^r(J; W_{\mathfrak{D}}^{-1,2})$ and $\mu\colon J \to \mathcal{E}(c_\bullet, c^\bullet)$ be an L^1-measurable map. We wish to define a map $\Psi\colon C(\overline{J}; L^2) \to C(\overline{J}; L^2)$. Let $v \in C(\overline{J}; L^2)$. Then $\sigma(v(t, \cdot))\, \mu_t(\cdot) \in \mathcal{E}(\sigma_\bullet\, c_\bullet, \sigma^\bullet\, c^\bullet)$ for almost every $t \in J$ and $t \mapsto \sigma(v(t))\, \mu_t$ is L^1-measurable. Hence there exists a unique $u \in \mathrm{MR}_0^r(J; W_{\mathfrak{D}}^{1,2}, W_{\mathfrak{D}}^{-1,2})$ such that

$$u'(t) + \mathcal{A}_2(\sigma(v(t))\, \mu_t)u(t) + u(t) = f(t)$$

for almost every $t \in J$. Then $u \in C(\overline{J}; L^2)$ by Proposition 4.5(d). Define $\Psi(v) = u$. Then $\Psi(C(\overline{J}; L^2)) \subset \mathrm{MR}_0^r(J; W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}})$ is relatively compact in $C(\overline{J}; L^2)$ by Proposition 4.5(d). We next show that Ψ is continuous. Then the theorem follows from Schauder's fixed point theorem.

Let $v, v_1, v_2, \ldots \in C(\overline{J}; L^2)$ and suppose that $\lim_{n\to\infty} v_n = v$ in $C(\overline{J}; L^2)$. For all $n \in \mathbb{N}$ let $u_n = \Psi(v_n)$ and $u = \Psi(v)$. Then $u_n, u \in \mathrm{MR}_0^r(J; W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}})$,

$$u_n'(t) + \mathcal{A}_2(\sigma(v_n(t))\,\mu_t)u_n(t) + u_n(t) = f(t)$$

and

$$u'(t) + \mathcal{A}_2(\sigma(v(t)\,\mu_t)u(t) + u(t) = f(t)$$

for almost every $t \in J$ and all $n \in \mathbb{N}$. Subtracting gives

$$\begin{aligned}(u - u_n)'(t) + \mathcal{A}_2(\sigma(v(t))\,\mu_t)((u - u_n)(t)) + (u - u_n)(t) \\ = \mathcal{A}_2(\sigma(v_n(t))\,\mu_t)u_n(t) - \mathcal{A}_2(\sigma(v(t))\,\mu_t)u_n(t)\end{aligned} \tag{33}$$

for almost every $t \in J$ and all $n \in \mathbb{N}$. Since $\lim v_n = v$ in $C(\overline{J}; L^2)$, also $\lim v_n = v$ in $L^2(J; L^2) = L^2(J \times \Omega; \mathbb{C})$. Hence passing to a subsequence, if necessary, we may assume that $\lim_{n\to\infty} v_n(t, x) = v(t, x)$ for almost every $(t, x) \in J \times \Omega$. For all $n \in \mathbb{N}$ define $\tilde{u}_n \in L^r(J; W^{-1,2}_{\mathfrak{D}})$ by

$$\tilde{u}_n(t) = \mathcal{A}_2(\sigma(v_n(t))\,\mu_t)u_n(t) - \mathcal{A}_2(\sigma(v(t))\,\mu_t)u_n(t).$$

We shall show that $\lim \tilde{u}_n = 0$ weakly in $L^r(J; W^{-1,2}_{\mathfrak{D}})$. Let $w \in L^{r'}(J; W^{1,2}_{\mathfrak{D}})$. Then

$$\begin{aligned}&|\langle \tilde{u}_n, w\rangle_{L^r(J;W^{-1,2}_{\mathfrak{D}})\times L^{r'}(J;W^{1,2}_{\mathfrak{D}})}| \\
&= \Big| \int_0^T \int_\Omega \Big(\sigma(v_n(t,x)) - \sigma(v(t,x))\Big)\Big(\mu_t(x)\,\nabla u_n(t,x)\Big) \cdot \overline{\nabla w(t,x)}\,dx\,dt \\
&\le c^\bullet \int_0^T \Big(\int_\Omega \Big|\big(\sigma(v_n(t,x)) - \sigma(v(t,x))\big)\,\nabla w(t,x)\Big|^2 dx\Big)^{1/2} \Big(\int_\Omega |\nabla u_n(t)|^2\Big)^{1/2} dt \\
&\le c^\bullet \Big(\int_0^T \Big(\int_\Omega \Big|\big(\sigma(v_n(t,x)) - \sigma(v(t,x))\big)\,\nabla w(t,x)\Big|^2 dx\Big)^{r'/2}\Big)^{1/r'} \|u_n\|_{L^r(J;W^{1,2}_{\mathfrak{D}})}.\end{aligned}$$

Obviously $\|u_n\|_{L^r(J;W^{1,2}_{\mathfrak{D}})} \le \|u_n\|_{\mathrm{MR}_0^r(J;W^{1,2}_{\mathfrak{D}},W^{-1,2}_{\mathfrak{D}})} \le \beta' \, \|f\|_{L^r(J;W^{-1,2}_{\mathfrak{D}})}$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} |\langle \tilde{u}_n, w\rangle_{L^r(J;W^{-1,2}_{\mathfrak{D}})\times L^{r'}(J;W^{1,2}_{\mathfrak{D}})}| = 0$ by the Lebesgue dominated convergence theorem. So $\lim \tilde{u}_n = 0$ weakly in $L^r(J; W^{-1,2}_{\mathfrak{D}})$. But

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}_2(\sigma(v(t,\cdot))\mu_t) + I\Big)(u - u_n)(t) = \tilde{u}_n(t)$$

for almost every $t \in J$ and all $n \in \mathbb{N}$ by (33). Also $u - u_n \in \mathrm{MR}_0^r(J; W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}})$ for all $n \in \mathbb{N}$. Hence by maximal parabolic regularity $\lim_{n\to\infty} u - u_n = 0$ weakly in $\mathrm{MR}_0^r(J; W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}})$. In addition the embedding of $\mathrm{MR}_0^r(J; W^{1,2}_{\mathfrak{D}}, W^{-1,2}_{\mathfrak{D}})$ into $C(\overline{J}; L^2)$ is compact by Proposition 4.5(d). So $\lim_{n\to\infty} u - u_n = 0$ in $C(\overline{J}; L^2)$ and the continuity of Ψ follows. □

Corollary 8.2. *If the right hand side f in* (32) *belongs to a space $L^r(J; W^{-1,q}_{\mathfrak{D}})$ and $r, q > 2$ are sufficiently close to* 2*, then every solution u provided by the theorem belongs to the space* $\mathrm{MR}^r_0(J; W^{1,q}_{\mathfrak{D}}, W^{-1,q}_{\mathfrak{D}})$.

Proof. The coefficient function $t \mapsto \sigma(u(t))\mu_t$ satisfies the assumptions of Theorem 7.3. □

It is unclear whether (32) has a unique solution.

9 Example with non-smooth coefficients in space and time

Let Ω and $\mathfrak{D}$ satisfy Assumption 5.1. Let $\Omega_0 \subset \Omega$ be an open non-empty set such that $\overline{\Omega_0} \subset \Omega$. Define a reference coefficient function $\mu_0 \colon J \to \mathbb{R}^{d\times d}$ by

$$\mu_0(x) = \begin{cases} 1\, I_d & \text{if } x \in \overline{\Omega_0}, \\ 2\, I_d & \text{if } x \in \Omega \setminus \Omega_0, \end{cases}$$

where I_d denotes the identity matrix in $\mathbb{R}^d$. For all $t \in J$ let $\Psi_t \colon \Omega \to \Omega$ be a map and set $\mu_t = \mu_0 \circ \Psi_t$. Suppose that $t \mapsto \mu_t$ is L^1-measurable. Then by Theorem 7.3 the operator family $\{\mathcal{A}(\mu_t) + I\}_{t\in J}$ satisfies maximal parabolic $L^s(J; W^{-1,q}_{\mathfrak{D}})$-regularity for all $s, q \in (1, \infty)$ sufficiently close to 2.

More specifically, consider the case in which $\Omega_t = \Psi_t(\Omega_0)$ is an open subset of Ω for all $t \in J$ such that $\Omega_{t_1} \neq \Omega_{t_2}$ for all $t_1, t_2 \in J$ with $t_1 \neq t_2$. Then the map $t \mapsto \mu_t$ from J into $L^\infty(\Omega; \mathbb{R}^{d\times d})$ is discontinuous at every point $t \in J$ and it is straightforward to show that also the map $t \mapsto \mathcal{A}_2(\mu_t) + I$ from J into $\mathcal{L}(W^{1,2}_{\mathfrak{D}}; W^{-1,2}_{\mathfrak{D}})$ is discontinuous. We remark that the analysis for this problem is known to be complicated already in case of elliptic equations, see the discussion in [EKRS] for relatively simple geometries of interfaces. Moreover, it represents a challenge also in numerics, see e.g. [AL]. In many interesting cases, the movement of the subdomain Ω_t is not determined by an 'outer' law, but may depend on the underlying physical/chemical process itself. Mathematically, this leads to a free boundary problem where for example Ψ_t depends on the solution u. Particular (simple) cases may then be covered by Theorem 8.1 to obtain existence and regularity of a solution.

10 Concluding remarks

Remark 10.1. It is possible to carry our results over to real spaces: in case of the real space $W^{1,q}_{\mathfrak{D},\mathbb{R}}$ one identifies its dual with the elements of $W^{-1,q'}_{\mathfrak{D}}$ which take real values for real functions from $W^{1,q}_{\mathfrak{D}}$. Then one applies the 'complex' result. This is enabled by the fact that, in case of real coefficients, the corresponding operators map the real subspace onto the 'real' subspace of the image.

Remark 10.2. We expect that our abstract results in Section 3 have further applications in the field of maximal parabolic regularity for non-autonomous parabolic equations. For example, one could investigate non-autonomous problems in the $X = L^p(\Omega)$-setting, cf. [ADLO, Section 5], [Fac1] and [Fac2]. Maximal parabolic regularity for autonomous elliptic, second-order divergence-form operators A on $L^p(\Omega)$, with $p \in (1, \infty)$, can be shown under Assumption 5.1(i). In this case, however, it is very difficult to determine the exact domain $D(t)$ of an operator $A(t)$, if the coefficient function is spatially discontinuous. The condition $D(t) = D(0)$ in our results then generically excludes settings like the one in Section 9. At the same time, recent optimal results in [Fac1] on varying domains with $D(t)$ require some continuity in time and regularity in space which also do not cover this setting. In particular, we highlight that in [Fac2] it is shown that the extrapolation for Lions' result, Proposition 4.1, is impossible in an L^p-setting.

Remark 10.3. The abstract results in Section 3 and applications to non-autonomous forms in Section 4 naturally include systems of equations. For the more specific setting in Sections 5–7, one has the required elliptic $W^{1,q}$-regularity for systems (see [HJKR, Section 6] or [BMMM, Section 7]), but presently the corresponding maximal parabolic regularity results are an open problem.

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