Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Thermistor systems of $p(x)$-Laplace-type with discontinuous exponents via entropy solutions

Miroslav Bulíček[1], Annegret Glitzky[2], Matthias Liero[2]

submitted: April 20, 2016

[1] Mathematical Institute of Charles University
Sokolovská 83
18675 Praha 8
Czech Republic
E-mail: mbul8060@karlin.mff.cuni.cz

[2] Weierstrass Institute
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: annegret.glitzky@wias-berlin.de
matthias.liero@wias-berlin.de

No. 2247
Berlin 2016

2010 *Mathematics Subject Classification.* 35J92, 35Q79, 35J57, 80A20.

Key words and phrases. Sobolev spaces with variable exponent, existence of weak solution, entropy solution, thermistor system, $p(x)$-Laplacian, heat transfer.

Acknowledgments. M. Bulíček thanks the project MORE (ERC-CZ project no. LL1202) financed by the Ministry of Education, Youth and Sports, Czech Republic. A. Glitzky was supported by ERC under project AnaMultiScale (AdG 267802) and M. Liero gratefully acknowledges the funding from the Einstein Foundation Berlin within the Research Center MATHEON under the ECMath Project SE2 "Electrothermal modeling of large-area OLEDs".

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We show the existence of solutions to a system of elliptic PDEs, that was recently introduced to describe the electrothermal behavior of organic semiconductor devices. Here, two difficulties appear: (i) the elliptic term in the current-flow equation is of $p(x)$-Laplacian-type with discontinuous exponent p, which limits the use of standard methods, and (ii) in the heat equation, we have to deal with an a priori L^1 term on the right hand side describing the Joule heating in the device. We prove the existence of a weak solution under very weak assumptions on the data. Our existence proof is based on Schauder's fixed point theorem and the concept of entropy solutions for the heat equation. Here, the crucial point is the continuous dependence of the entropy solutions on the data of the problem.

1 Introduction

In this paper we study a system of partial differential equations, that describes the electrothermal behavior of organic semiconductor devices. It consists of the current-flow equation for the electrostatic potential φ coupled to the heat equation with Joule heat term for the temperature θ, namely

$$-\nabla \cdot A(x, \theta, \nabla\varphi) = 0, \tag{1a}$$

$$-\nabla \cdot (\lambda(x)\nabla\theta) = H(x, \theta, \nabla\varphi) \tag{1b}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$. The current flux $A : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is given via

$$A(x, \theta, z) := \sigma_0(x)F(x, \theta)|z|^{p(x)-2}z, \quad \text{where} \quad F(x, \theta) = \exp\left[-\beta(x)\Big(\frac{1}{\theta} - \frac{1}{\theta_{\mathrm{a}}}\Big)\right] \tag{2}$$

describes an Arrhenius-like temperature law. The Joule heat term in the right-hand side of (1b) $H : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to [0, \infty)$ is defined by

$$H(x, \theta, z) := (1-\eta(x, \theta, z))A(x, \theta, z) \cdot z, \tag{3}$$

where $\eta(x, \theta, z) \in [0, 1]$ is the light-outcoupling factor. In particular, the equation (1a) is of $p(x)$-Laplace-type, where $x \mapsto p(x)$ is measurable and satisfies $1 < \operatorname{ess\,inf}_{x\in\Omega} p(x) \leq \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$.

The system is complemented by Dirichlet and homogeneous Neumann boundary conditions for φ and Robin boundary conditions for θ, viz.

$$\varphi = \varphi^D \text{ on } \Gamma_D \quad \text{and} \quad A(x, \theta, \nabla\varphi) \cdot \nu = 0 \text{ on } \Gamma_N, \tag{4a}$$

$$-\lambda(x)\nabla\theta \cdot \nu = \kappa(x)(\theta - \theta_{\mathrm{a}}) \text{ on } \Gamma := \partial\Omega. \tag{4b}$$

Systems of the form (1) model materials conducting both heat and electrical current and for which the electrical conductivity in the definition of A strongly depends on the temperature. Devices of this type are called thermistors, see [3, 4]. Recently, systems of the form (1) with the specific constitutive law in (2) were introduced in [11] to describe electrothermal effects, such as self-heating and inhomogeneous current distributions, in organic, i.e. carbon-based, semiconductor devices, see also [6]. For example, Organic Light-Emitting Diodes (OLEDs) are thin-film heterostructures based on organic molecules or polymers, where each functional layer has, in general, its own current-voltage characteristics and material parameters. In particular, the exponent $p(x)$, which describes the non-Ohmic behavior of each layer, changes abruptly from one material to another. In electrodes, the typically used parameter is $p(x) = 2$, while organic layers feature significantly larger values, e.g. $p(x) \approx 9$ (see [6]). The material function β, which is related to the so-called activation energy in an Arrhenius law, is positive in organic layers. This, however, leads to a positive feedback as the electrical conductivity increases with rising temperature and in turn the power dissipation increases with the electrical current. This mechanism was identified in [6] as the cause of the appearance of different operation modes and accompanying unpleasant brightness inhomogeneities in large-area OLEDs.

The analytical difficulties of the problem in (1)–(4) arise from two issues: First, the exponent function $x \mapsto p(x)$ is discontinous and in general only measurable. Second, the source term H in the right hand side of (1b) is only in the space $L^1(\Omega)$ for functions φ in the energy space associated with the differential operator in the left hand side of (1a).

In [8] these issues were overcome in the two-dimensional case and for a piecewise constant $x \mapsto p(x)$ satisfying $p(x) \geq 2$ by showing improved integrability of the gradient of the electrostatic potential, i.e. $\nabla\varphi \in L^{sp(\cdot)}(\Omega)^d$ for some $s > 1$. The latter is proved using Caccioppoli estimates and a Gehring-type lemma. This significantly helps to deal with the right hand side H in the heat equation in the existence proof, since then we have a priori control of $H(\cdot, \theta, \nabla\varphi)$ in $L^s(\Omega)$. In particular, one does not need to face the problem of concentration effects and correspondingly the presence of a singular measure. However, this approach heavily relies on the use of the Poincaré inequality, which does in general not hold for discontinuous exponents p, see [5, Sec. 8.2]. Moreover, the extension to higher dimension and ranges of p that are realistic for organic devices is unclear.

To tackle higher spatial dimensions an approach based on regularization and Galerkin approximation was discussed in [2]. Therein, a regularized version of the system in (1) was introduced, where the crucial term H is approximated so that it remains bounded. The existence of solutions to the regularized problem is proven by Galerkin approximations. By using suitable test functions in the weak formulation of the regularized version of (1) uniform estimates for φ and θ independent of the regularization parameter $\varepsilon > 0$ were derived, which allows to pass to the limit $\varepsilon \to 0$ and to obtain weak solutions of (1).

In this paper, we present a different existence proof using the concept of entropy solutions with Robin boundary conditions (4b) and Schauder's fixed-point theorem. More precisely, for a given temperature distribution $\widetilde{\theta}$ we obtain a unique solution $\varphi = \varphi(\widetilde{\theta})$ of the current flow equation (1a). Using φ and $\widetilde{\theta}$ in H, we then solve the heat equation to obtain an entropy solution θ. The map $\widetilde{\theta} \mapsto \theta$ is continuous and hence has a fixed point in a suitable compact set $\mathcal{M} \subset L^1(\Omega)$. Here, the crucial point is the continuous dependence of the entropy solutions on given right hand sides, see Lemma 3.4.

Outline of the paper. We start in Section 2 with fixing the notation, introducing the underlying assumptions and function spaces and formulating our main result concerning the existence of weak solutions to the coupled $p(x)$-thermistor model (1). Section 3 covers the solvability of the two subproblems, i.e. the current-flow and heat equation. Finally, in Section 4 we verify the solvability of the coupled system via Schauder's fixed-point theorem. In the Appendix we collect and proof needed results for entropy solutions to linear elliptic problems with boundary conditions of the form (4b).

2 Preliminaries and main result

2.1 Assumptions on the data

Here, we collect the essential assumptions for the analytical investigations:

(A1) The domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitzian domain and Γ_D and Γ_N are disjoint open subsets of $\Gamma := \partial\Omega$ satisfying $\mathrm{mes}(\Gamma_D) > 0$ and $\overline{\Gamma_D \cup \Gamma_N} = \Gamma$.

(A2) The function $x \mapsto p(x)$ is measurable (we write $p \in \mathcal{P}(\Omega)$) and $p : \Omega \to (1, \infty)$ fulfills $1 < p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x), \quad p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$.

(A3) The Dirichlet datum satisfies $\varphi^D \in L^\infty(\Omega)$ and $\int_\Omega |\nabla\varphi^D|^{p(x)} \mathrm{d}x < \infty$.

(A4) The electrical flux function $A : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is of the form $A(x, \theta, z) = \sigma_0(x)F(x, \theta)|z|^{p(x)-2}z$, where $\sigma_0 \in L^\infty(\Omega)$ satisfies $0 < \underline{\sigma_0} \leq \sigma_0 \leq \overline{\sigma_0}$ a.e. on Ω. The Arrhenius factor is of the form $F(x, \theta) = \exp\big[-\beta(x)\big(\frac{1}{\theta} - \frac{1}{\theta_\mathrm{a}}\big)\big]$ with $\beta \in L^\infty_+(\Omega)$ and the constant $\theta_\mathrm{a} > 0$ is the ambient temperature.

(A5) The heat conductivity λ satisfies $\lambda \in L^\infty(\Omega)$ and $\lambda \geq \lambda_0 > 0$ a.e. on Ω. The heat transfer coefficient κ is such that $\kappa \in L^\infty_+(\Gamma)$ and $\|\kappa\|_{L^1(\Gamma)} > 0$.

(A6) The light-outcoupling factor $\eta = \eta(x, \theta, z)$ is such that $\eta : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function and $\eta(x, \theta, z) \in [0, 1]$ holds f.a.a. $x \in \Omega$ and $\forall(\theta, z) \in \mathbb{R}_+ \times \mathbb{R}^d$.

The assumptions (A4) ensure important structural properties of the function A. They imply that A is a Carathéodory function with the growth

$$|A(x, \theta, z)| \leq c_g |z|^{p(x)-1} \quad \forall(\theta, z) \in \mathbb{R}_+ \times \mathbb{R}^d, \text{ f.a.a. } x \in \Omega. \tag{5}$$

Due to the inequality (see [12, Chapter 10]) for z_1, $z_2 \in \mathbb{R}^d$,

$$\big(|z_1|^{p-2}z_1 - |z_2|^{p-2}z_2\big) \cdot (z_1 - z_2) \geq \begin{cases} 2^{2-p}|z_1 - z_2|^p & \text{if } p \geq 2, \\ (p-1)(1+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_1 - z_2|^2 & \text{if } p \in [1, 2], \end{cases}$$

the function $A(x, \theta, \cdot)$ ist strictly monotone in the third argument, meaning that

$$\big(A(x, \theta, z_1) - A(x, \theta, z_2)\big) \cdot (z_1 - z_2) > 0 \quad \forall\theta \in \mathbb{R}_+, \forall z_1 \neq z_2 \in \mathbb{R}^d, \text{ f.a.a. } x \in \Omega. \tag{6}$$

Additionally, there exists a constant $c_\mathrm{a} > 0$ such that

$$A(x, \theta, z) \cdot z \geq c_\mathrm{a}|z|^{p(x)} \quad \text{f.a.a. } x \in \Omega, \forall\theta \geq \theta_\mathrm{a} > 0, \quad \forall z \in \mathbb{R}^d \tag{7}$$

is fulfilled. (The lower estimate $\theta \geq \theta_\mathrm{a}$ for solutions θ of the heat equation can be obtained in the setting of entropy solutions for the heat flow equation by Lemma 3.5.)

2.2 Function spaces

For constant $p \in [1, \infty]$, we use the classical Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$. Following [5] we work with the generalized function spaces $L^{p(\cdot)}(\Omega)$, where we assume that the bounded variable exponents $p \in \mathcal{P}(\Omega)$ satisfy

$$1 < p_- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) \leq p_+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x) < \infty.$$

The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u for which the modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} \,\mathrm{d}x$$

is finite. With the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}} := \inf \Big\{ \tau > 0 : \ \rho_{p(\cdot)}\Big(\frac{u}{\tau}\Big) \leq 1 \Big\}$$

$L^{p(\cdot)}(\Omega)$ is a Banach space and [5, Lemma 3.2.5] ensures for all $u \in L^{p(\cdot)}(\Omega)$ the estimates

$$\min\Big\{\rho_{p(\cdot)}(u)^{\frac{1}{p_-}}, \rho_{p(\cdot)}(u)^{\frac{1}{p_+}}\Big\} \leq \|u\|_{L^{p(\cdot)}} \leq \max\Big\{\rho_{p(\cdot)}(u)^{\frac{1}{p_-}}, \rho_{p(\cdot)}(u)^{\frac{1}{p_+}}\Big\}. \tag{8}$$

We work with a definition of generalized Sobolev spaces that will be appropriate for our problem. We emphasize that the spaces introduced here are not necessarily equivalent to the standard Sobolev spaces with the variable exponent in [5]. This is necessary since in our case the exponent p is discontinuous and varies over a large range, see also [2]. For a given $p \in \mathcal{P}(\Omega)$ we define the generalized Sobolev space $W^{1,p(\cdot)}(\Omega)$ and equip it with the following norm

$$\begin{aligned} W^{1,p(\cdot)}(\Omega) &:= \Big\{ u \in W^{1,p_-}(\Omega) : \rho_{p(\cdot)}(|\nabla u|) < \infty \Big\}, \\ \|u\|_{W^{1,p(\cdot)}} &:= \|u\|_{W^{1,p_-}} + \|\nabla u\|_{L^{p(\cdot)}}. \end{aligned}$$

In the case $1 < p_- \leq p_+ < \infty$ the space $W^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space, since $L^{p(\cdot)}$ has the same properties. We introduce the subspace

$$W_D^{1,p(\cdot)}(\Omega) := \{ u \in W^{1,p(\cdot)}(\Omega) : \ u = 0 \text{ on } \Gamma_D \}.$$

Since we assume that Γ_D is of positive $(d-1)$-dimensional measure, we have the equivalent norms

$$C_1 \|u\|_{W^{1,p(\cdot)}} \leq \|\nabla u\|_{L^{p(\cdot)}} \leq C_2 \|u\|_{W^{1,p(\cdot)}}, \quad u \in W_D^{1,p(\cdot)}(\Omega). \tag{9}$$

Indeed, we can use the facts that the classical Sobolev space $W_D^{1,p_-}(\Omega)$ satisfies the Poincaré inequality and that the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is continuously embedded into $L^{p_-}(\Omega)$ to obtain for arbitrary $u \in W_D^{1,p(\cdot)}(\Omega)$

$$\begin{aligned} \|u\|_{W^{1,p(\cdot)}} &= \|u\|_{W^{1,p_-}} + \|\nabla u\|_{L^{p(\cdot)}} \\ &\leq c(\|\nabla u\|_{L^{p_-}} + \|\nabla u\|_{L^{p(\cdot)}}) \leq c\|\nabla u\|_{L^{p(\cdot)}} \leq c\|u\|_{W^{1,p(\cdot)}}. \end{aligned}$$

Furthermore, we denote by $H^1(\Omega)$ the usual Hilbert space. By means of the assumption (A5) the estimates

$$\underline{\alpha}\|\theta\|_{H^1}^2 \leq \int_\Omega \lambda|\nabla\theta|^2 \mathrm{d}x + \int_\Gamma \kappa\theta^2 \mathrm{d}\Gamma \leq \overline{\alpha}\|\theta\|_{H^1}^2, \quad \theta \in H^1(\Omega) \tag{10}$$

with constants $\underline{\alpha}$, $\overline{\alpha} > 0$ are satisfied.

Moreover, the dual space of a Banach space X is denoted by X^*. In our estimates, positive constants, which may depend at most on the data of our problem, are denoted by c. In particular, we allow them to change from line to line.

2.3 Main result

Theorem 2.1 *We assume (A1)–(A6). Then the $p(x)$-thermistor problem (1)–(4) has a (weak) solution (φ,θ) with $\varphi \in \varphi^D + W_D^{1,p(\cdot)}(\Omega)$ and $\theta \in W^{1,q}(\Omega)$ for all $q \in [1, \frac{d}{d-1})$. Especially, θ is an entropy solution (see Definition 3.2) to the heat equation (1b).*

3 Solvability of subproblems

3.1 Existence of solutions to the current flow equation

In the first step, we turn our attention to the current-flow equation (1a) for the potential φ. In particular, we consider an arbitrary but fixed θ, which is assumed to lie in the set of relevant temperature distributions given by

$$\Theta := \{\theta \in L^1(\Omega) \,:\, \theta \geq \theta_\mathrm{a} \text{ a.e. on } \Omega\}. \tag{11}$$

For fixed $\theta \in \Theta$, we introduce the operator $\mathcal{A}_\theta : \varphi^D + W_D^{1,p(\cdot)}(\Omega) \to W_D^{1,p(\cdot)}(\Omega)^*$ and consider the following problem: Find $\varphi \in \varphi^D + W_D^{1,p(\cdot)}(\Omega)$ such that

$$\langle \mathcal{A}_\theta(\varphi), v\rangle_{W_D^{1,p(\cdot)}} := \int_\Omega A(x,\theta,\nabla\varphi)\cdot\nabla v \,\mathrm{d}x = 0 \quad \forall\, v \in W_D^{1,p(\cdot)}(\Omega), \tag{12}$$

which corresponds to finding a weak solution $\varphi \in \varphi^D + W_D^{1,p(\cdot)}(\Omega)$ of the current-flow equation (1a) with boundary conditions (4a) and fixed temperature distribution $\theta \in \Theta$.

Theorem 3.1 *We assume (A1)–(A4). Let $\theta \in \Theta$ be a fixed given function. Then (12) has exactly one solution φ, and for almost all $x \in \Omega$ this solution satisfies*

$$\operatorname*{ess\,inf}_{x\in\Omega} \varphi^D \leq \varphi(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} \varphi^D. \tag{13}$$

Moreover, there are constants $c_\varphi > 0$, $c_\mathrm{int} > 0$, and $c_A > 0$ depending only on the data (Ω, φ^D, $\underline{\sigma_0}$, $\overline{\sigma_0}$, θ_a, and β) but not on θ, such that (with $p'(x) = \frac{p(x)}{p(x)-1}$)

$$\int_\Omega |\nabla\varphi|^{p(x)}\,\mathrm{d}x \leq c_\mathrm{int}, \quad \|\varphi\|_{W^{1,p(\cdot)}} \leq c_\varphi, \quad \|A(\cdot,\theta,\nabla\varphi)\|_{L^{p'(\cdot)}} \leq c_A. \tag{14}$$

Proof. 1. Bounds. The uniform bounds in (13) and (14) (except that for A) are exactly obtained as in Step 1 of the proof of [8, Lemma 3.1]. The bound for A results then directly from the growth condition (5),

$$\rho_{p'(\cdot)}(|A(\cdot,\theta,\nabla\varphi)|) \le \int_\Omega c_g^{p'(x)}|\nabla\varphi|^{p(x)}\,\mathrm{d}x \le c\rho_{p(\cdot)}(|\nabla\varphi|), \tag{15}$$

from the bound for $\nabla\varphi$ and (8). Especially, the map $\varphi^{\mathrm{D}} + W_D^{1,p(\cdot)}(\Omega) \ni \varphi \mapsto A(\cdot,\theta,\nabla\varphi) \in L^{p'(\cdot)}(\Omega)^d$ maps bounded sets in bounded sets.

2. Existence. Due to (6), $\mathcal{A}_\theta$ is strictly monotone. To prove the demi-continuity of $\mathcal{A}_\theta$, we have to show that $\varphi_n - \varphi \to 0$ in $W_D^{1,p(\cdot)}(\Omega)$ implies $\mathcal{A}_\theta\varphi_n - \mathcal{A}_\theta\varphi \rightharpoonup 0$ in $W_D^{1,p(\cdot)}(\Omega)^*$.

Let $\varphi_n - \varphi \to 0$ in $W_D^{1,p(\cdot)}(\Omega)$. Then, by Step 1, the set $\{A(\cdot,\theta,\nabla\varphi_n)\}$ is bounded and weakly compact in $L^{p'(\cdot)}(\Omega)^d$. To verify the weak convergence $A(\cdot,\theta,\nabla\varphi_n) \rightharpoonup A(\cdot,\theta,\nabla\varphi)$ in $L^{p'(\cdot)}(\Omega)^d$, by [7, Lemma 5.4, Chapter 1], it suffices to show for each convergent subsequence $\{A(\cdot,\theta,\nabla\varphi_{n_k})\}$ of $\{A(\cdot,\theta,\nabla\varphi_n)\}$ that $A(\cdot,\theta,\nabla\varphi_{n_k}) \rightharpoonup A(\cdot,\theta,\nabla\varphi)$ in $L^{p'(\cdot)}(\Omega)^d$. Let $w \in L^{p'(\cdot)}(\Omega)^d$ be the weak limit of such a subsequence $\{A(\cdot,\theta,\nabla\varphi_{n_k})\}$. Since $\varphi_n - \varphi \to 0$ in $W_D^{1,p(\cdot)}(\Omega)$ there exists a subsequence $\{\varphi_{n_{k_l}}\}$ of $\{\varphi_{n_k}\}$ such that $\nabla\varphi_{n_{k_l}}$ converges a.e. in Ω to $\nabla\varphi$. As A is a Caratheodory function it follows that $A(\cdot,\theta,\nabla\varphi_{n_{k_l}}) \to A(\cdot,\theta,\nabla\varphi)$ a.e. in Ω. As a subsequence of $\{A(\cdot,\theta,\nabla\varphi_{n_k})\}$ the sequence $\{A(\cdot,\theta,\nabla\varphi_{n_{k_l}})\}$ has the weak limit w in $L^{p'(\cdot)}(\Omega)^d$, and therefore in $L^{p_+/(p_+-1)}(\Omega)^d$, too. By [7, Lemma 1.19, Chap. 2] we obtain $A(\cdot,\theta,\nabla\varphi) = w$, and thus for the whole sequence $A(\cdot,\theta,\nabla\varphi_n) \rightharpoonup A(\cdot,\theta,\nabla\varphi)$ in $L^{p'(\cdot)}(\Omega)^d$. Since for all $v \in W_D^{1,p(\cdot)}(\Omega)$ we have $\nabla v \in L^{p(\cdot)}(\Omega)^d$ it follows

$$\langle \mathcal{A}_\theta\varphi_n - \mathcal{A}_\theta\varphi, v\rangle_{W_D^{1,p(\cdot)}} = \int_\Omega (A(x,\theta,\nabla\varphi_n) - A(x,\theta,\nabla\varphi))\cdot\nabla v\,\mathrm{d}x \to 0$$

for all $v \in W_D^{1,p(\cdot)}(\Omega)$ and thus $\mathcal{A}_\theta\varphi_n - \mathcal{A}_\theta\varphi \rightharpoonup 0$ in $W_D^{1,p(\cdot)}(\Omega)^*$ as $\varphi_n - \varphi \to 0$ in $W_D^{1,p(\cdot)}(\Omega)$. Additionally, the demi-continuity implies the radial continuity of $\mathcal{A}_\theta$.

Next, we prove the coercivity of $\mathcal{A}_\theta$. We apply (7) and (5), convexity of $z \mapsto |z|^p$ as well as (pointwise) Young's inequality to estimate

$$\begin{aligned}
&\langle \mathcal{A}_\theta\varphi, \varphi - \varphi^D\rangle_{W_D^{1,p(\cdot)}} \\
&\quad\ge \int_\Omega \left(c_a|\nabla\varphi|^{p(x)} - c_g|\nabla\varphi|^{p(x)-1}|\nabla\varphi^D|\right)\mathrm{d}x \\
&\quad\ge \int_\Omega \left(\overline{c}|\nabla(\varphi-\varphi^D)|^{p(x)} - c|\nabla\varphi^D|^{p(x)} - c|\nabla(\varphi-\varphi^D)|^{p(x)-1}|\nabla\varphi^D|\right)\mathrm{d}x \\
&\quad\ge \frac{\overline{c}}{2}\rho_{p(\cdot)}(|\nabla(\varphi-\varphi^D)|) - c\rho_{p(\cdot)}(|\nabla\varphi^D|).
\end{aligned} \tag{16}$$

By assumption (A3), the term $\rho_{p(\cdot)}(|\nabla\varphi^D|)$ is bounded. By assumption (A1) $\mathrm{mes}(\Gamma_D) > 0$, thus the seminorm $\|\nabla(\cdot)\|_{L^{p(\cdot)}}$ is an equivalent norm on $W_D^{1,p(\cdot)}(\Omega)$, compare (9). According to (8) we can estimate $\rho_{p(\cdot)}(|\nabla(\varphi-\varphi^D)|)$ from below, either by $\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}}^{p_+}$ or $\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}}^{p_-}$. Note that both exponents are strictly greater than 1. Dividing the previous estimate (16) by $\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}}$ the right hand side goes to $+\infty$ if $\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}} \to \infty$ which guarantees that the operator $\mathcal{A}_\theta$ is coercive.

In summary, the main theorem of monotone operators (see [7]) ensures the existence of a solution to (12). Since $\mathcal{A}_\theta$ is strictly monotone, this solution is unique. □

3.2 Entropy solutions of the heat equation

For the solvability of the second equation with right hand side in $L^1(\Omega)$ we use the concept of entropy solutions. In the case of Dirichlet boundary conditions this theory is well presented in the survey [13], for nonlinear problems see [1, 10].

We consider the stationary heat flow equation with Robin boundary conditions and right hand side $f \in L^1(\Omega)$ as well as boundary data $g \in L^1(\Gamma)$, namely

$$\begin{aligned} -\nabla \cdot (\lambda(x)\nabla\theta) &= f(x) && \text{in } \Omega, \\ -\lambda(x)\nabla\theta \cdot \nu &= \kappa(x)\theta - g(x) && \text{on } \Gamma. \end{aligned} \tag{17}$$

For $k > 0$, we define the truncation $C_k : \mathbb{R} \to [-k, k]$ by

$$C_k(s) := \max\{-k, \min\{s, k\}\}$$

and introduce $\mathcal{V}^{1,2}(\Omega) := \{\theta : \Omega \to \mathbb{R} \text{ measurable}, C_k(\theta) \in H^1(\Omega)\ \forall k > 0\}$.

Definition 3.2 *Let $f \in L^1(\Omega)$, $g \in L^1(\Gamma)$. A function $\theta \in \mathcal{V}^{1,2}(\Omega)$ is called an entropy solution to (17) if*

$$\int_\Omega \lambda\nabla\theta \cdot \nabla C_k(\theta - \omega)\,dx + \int_\Gamma \Big(\kappa\theta - g\Big)C_k(\theta - \omega)\,d\Gamma \le \int_\Omega f C_k(\theta - \omega)\,dx \tag{18}$$

for all $k > 0$ and all $\omega \in H^1(\Omega) \cap L^\infty(\Omega)$.

Note that for $\omega \in H^1(\Omega)$ with $\|\omega\|_{L^\infty(\Omega)} \le c$ we have $\|\omega\|_{L^\infty(\Gamma)} \le c$ with the same constant.

Theorem 3.3 *We assume (A1) and (A5). Let $f \in L^1(\Omega)$, $g \in L^1(\Gamma)$. Then there exists a unique entropy solution θ to (17). This entropy solution belongs to $W^{1,q}(\Omega)$, for all $1 \le q < \frac{d}{d-1}$. Especially, there are constants $c_{Eq} > 0$ not depending on f and g such that*

$$\|\theta\|_{W^{1,q}} \le c_{Eq}(\|f\|_{L^1} + \|g\|_{L^1(\Gamma)}), \quad 1 \le q < \tfrac{d}{d-1}.$$

We give the proof of Theorem 3.3 in Appendix A and finalize this subsection by two lemmas with special properties of entropy solutions to (17).

Lemma 3.4 *We assume (A1) and (A5). Let $f^l \to f$ in $L^1(\Omega)$, $g^l \to g$ in $L^1(\Gamma)$. Then the corresponding entropy solutions θ^l to (17) converge weakly in $W^{1,q}(\Omega)$ to the entropy solution θ for data f and g, $1 \le q < \frac{d}{d-1}$.*

Proof. 1. Since the θ^l are entropy solutions to (17) for f^l and g^l by (18) with $\omega = 0$ we find

$$\underline{\alpha}\|C_k(\theta^l)\|_{H^1}^2 \le k(\|f^l\|_{L^1} + \|g^l\|_{L^1(\Gamma)}) \le kc \quad \forall l, \tag{19}$$

where the unified constant $c > 0$ results from the fact that $f^l \to f$ in $L^1(\Omega)$, $g^l \to g$ in $L^1(\Gamma)$. As in Step 2 of the proof of Theorem 3.3 (see Appendix A) we then verify for all exponents $q < \frac{d}{d-1}$ that

$$\int_\Omega (|\nabla\theta^l|^q + |\theta^l|^q)\,\mathrm{d}x \le c_q(\|f^l\|_{L^1} + \|g^l\|_{L^1(\Gamma)})^q \le C_q \quad \forall l.$$

In summary we find a $\theta \in W^{1,q}(\Omega)$ such that for a (non-relabeled) subsequence we have

$$\begin{aligned}
&\theta^l \rightharpoonup \theta \text{ in } W^{1,q}(\Omega), \quad \theta^l \to \theta \text{ in } L^1(\Omega), \quad \theta^l \to \theta \text{ a.e. in } \Omega,\\
&C_k(\theta^l) \rightharpoonup C_k(\theta) \text{ in } H^1(\Omega), \quad C_k(\theta^l) \to C_k(\theta) \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega,\\
&C_k(\theta^l) \to C_k(\theta) \text{ in } L^2(\Gamma) \text{ and a.e. on } \Gamma \text{ for all } k > 0.
\end{aligned} \tag{20}$$

By the weak lower semicontinuity of the norm and $C_k(\theta^l) \rightharpoonup C_k(\theta)$ in $H^1(\Omega)$ from (19) it results

$$\underline{\alpha}\|C_k(\theta)\|_{H^1}^2 \le kc, \quad \forall k > 0.$$

2. To verify that θ is the entropy solution to (17) with data f and g we prove now as in [10] the strong convergence $C_k(\theta^l) \to C_k(\theta)$ in $H^1(\Omega)$. Let $2h > k$ and take $\omega^l := C_h(\theta^l) - C_{k/2}(\theta^l) + C_{k/2}(\theta)$ as test function for the entropy solution θ^l in (18) with data f^l and g^l. Then for $m := h + 2k$ we have $\nabla C_k(\theta^l - \omega^l) = 0$ on the set $\{x \in \Omega : |\theta^l(x)| > m\}$. Therefore it follows from the entropy formulation

$$\int_\Omega \lambda\nabla C_m(\theta^l)\cdot\nabla C_k(\theta^l - \omega^l)\,\mathrm{d}x + \int_\Gamma (\kappa\theta^l - g)C_k(\theta^l - \omega^l)\,\mathrm{d}\Gamma \le \int_\Omega f^l C_k(\theta^l - \omega^l)\,\mathrm{d}x.$$

Splitting the volume integral on the left hand side in integrals on $\Omega_k^l := \{x \in \Omega : 2|\theta^l(x)| > k\}$ and $\Omega \setminus \Omega_k^l$ (where $\nabla C_k(\theta^l - \omega^l) = \nabla\big(C_{k/2}(\theta^l) - C_{k/2}(\theta)\big)$ holds true) it results

$$\begin{aligned}
\int_\Omega \lambda\nabla C_m(\theta^l)\cdot\nabla C_k(\theta^l - \omega^l)\,\mathrm{d}x \ge& \int_\Omega \lambda\nabla C_{k/2}(\theta^l)\cdot\nabla\big(C_{k/2}(\theta^l) - C_{k/2}(\theta)\big)\,\mathrm{d}x\\
&- \int_{\Omega_k^l} \lambda|\nabla C_m(\theta^l)||\nabla C_{k/2}(\theta)|\,\mathrm{d}x.
\end{aligned}$$

Using the entropy formulation we obtain

$$\begin{aligned}
&\int_\Omega \lambda|\nabla\big(C_{k/2}(\theta^l) - (C_{k/2}(\theta)\big)|^2\,\mathrm{d}x\\
&\le \int_{\Omega_k^l} \lambda|\nabla C_m(\theta^l)||\nabla C_{k/2}(\theta)|\,\mathrm{d}x + \int_\Omega f^l C_k(\theta^l - \omega^l)\,\mathrm{d}x + \int_\Gamma g^l C_k(\theta^l - \omega^l)\,\mathrm{d}\Gamma\\
&\quad - \int_\Gamma \kappa\theta^l C_k(\theta^l - \omega^l)\,\mathrm{d}\Gamma - \int_\Omega \lambda\nabla C_{k/2}(\theta)\cdot\nabla\big(C_{k/2}(\theta^l) - C_{k/2}(\theta)\big)\,\mathrm{d}x,
\end{aligned} \tag{21}$$

where, due to (20), the last term obviously converges to zero for $l \to \infty$. Additionally, for every fixed h, the convergences (20) ensure the weak convergences for $l \to \infty$

$$C_k\big(\theta^l - C_h(\theta^l)\big) \rightharpoonup C_k\big(\theta - C_h(\theta)\big) \text{ in } H^1(\Omega),$$

$$C_k\big(\theta^l - \omega^l\big) = C_k\big(\theta^l - C_h(\theta^l) + C_{k/2}(\theta^l) - C_{k/2}(\theta)\big) \rightharpoonup C_k\big(\theta - C_h(\theta)\big) \text{ in } H^1(\Omega)$$

as well as corresponding convergences a.e. in Ω and Γ for a further non-relabeled subsequence. Because of the estimates

$$\begin{aligned}
\int_\Omega f^l C_k(\theta^l - \omega^l)\,\mathrm{d}x &\leq k\|f^l - f\|_{L^1} + \int_\Omega f C_k(\theta^l - \omega^l)\,\mathrm{d}x,\\
\int_\Gamma g^l C_k(\theta^l - \omega^l)\,\mathrm{d}\Gamma &\leq k\|g^l - g\|_{L^1(\Gamma)} + \int_\Gamma g C_k(\theta^l - \omega^l)\,\mathrm{d}\Gamma,\\
\Big|\int_\Gamma \kappa\theta^l C_k(\theta^l - \omega^l)\,\mathrm{d}\Gamma\Big| &\leq \|\kappa\|_{L^\infty(\Gamma)}\|\theta^l\|_{L^{\frac{2d}{2d-1}}(\Gamma)}\|C_k(\theta^l - \omega^l)\|_{L^{2d}(\Gamma)}
\end{aligned}$$

we therefore conclude with $f^l \to f$ in $L^1(\Omega)$, $g^l \to g$, in $L^1(\Gamma)$, (20), and Lebesgue's dominated convergence theorem, in the limit $l \to \infty$

$$\begin{aligned}
\lim_{l\to\infty} \int_\Omega f^l C_k(\theta^l - \omega^l)\,\mathrm{d}x &= \int_\Omega f C_k(\theta - C_h(\theta))\,\mathrm{d}x,\\
\lim_{l\to\infty} \int_\Gamma g^l C_k(\theta^l - \omega^l)\,\mathrm{d}\Gamma &= \int_\Gamma g C_k(\theta - C_h(\theta))\,\mathrm{d}\Gamma,\\
\lim_{l\to\infty} \Big|\int_\Gamma \kappa\theta^l C_k(\theta^l - \omega^l)\,\mathrm{d}\Gamma\Big| &\leq \widehat{c}\|C_k(\theta - C_h(\theta))\|_{L^{2d}(\Gamma)},
\end{aligned} \tag{22}$$

where $\widehat{c} := \|\kappa\|_{L^\infty(\Gamma)}\|\theta\|_{L^{\frac{2d}{2d-1}}(\Gamma)}$. According to Lebesque's theorem, we have that the terms on the right hand sides in (22) converge to zero for $h \to \infty$. Hence, we can fix a sufficiently large $h_\varepsilon > 0$ such that

$$\int_\Omega f C_k\big(\theta - C_{h_\varepsilon}(\theta)\big)\,\mathrm{d}x + \int_\Gamma g C_k\big(\theta - C_{h_\varepsilon}(\theta)\big)\,\mathrm{d}\Gamma + \widehat{c}\|C_k(\theta - C_{h_\varepsilon}(\theta))\|_{L^{2d}(\Gamma)} \leq \varepsilon. \tag{23}$$

With this h_ε and $m = m_\varepsilon = 2k + h_\varepsilon$ we estimate the remaining term in the right hand side of (21). Since $|\nabla C_m(\theta^l)|$ is bounded in $L^2(\Omega)$ and $\chi_{\{|\theta^l|>k/2\}}|\nabla C_{k/2}(\theta)| \to 0$ in $L^2(\Omega)$ we verify

$$\lim_{l\to\infty} \int_{\{|\theta^l|>k/2\}} \lambda|\nabla C_m(\theta^l)||\nabla C_{k/2}(\theta)|\,\mathrm{d}x = 0.$$

Therefore, passing to the limit $l \to \infty$ in (21) and using (23) it results

$$\begin{aligned}
&\lim_{l\to\infty} \int_\Omega \lambda\big|\nabla\big(C_{k/2}(\theta^l) - C_{k/2}(\theta)\big)\big|^2\,\mathrm{d}x\\
&\leq \int_\Omega f C_k\big(\theta - C_{h_\varepsilon}(\theta)\big)\,\mathrm{d}x + \int_\Gamma g C_k\big(\theta - C_{h_\varepsilon}(\theta)\big)\,\mathrm{d}\Gamma + \widehat{c}\|C_k(\theta - C_{h_\varepsilon}(\theta))\|_{L^{2d}(\Gamma)} \leq \varepsilon,
\end{aligned}$$

where $\varepsilon > 0$ can be made arbitrarily small by increasing h_ε. This yields

$$\lim_{l\to\infty} \int_\Omega \lambda|\nabla\big(C_{k/2}(\theta^l) - C_{k/2}(\theta)\big)|^2 \,\mathrm{d}x = 0,$$

and together with $\lambda \geq \lambda_0 > 0$ and $C_{k/2}(\theta^l) \to C_{k/2}(\theta)$ in $L^2(\Omega)$ we obtain $C_{k/2}(\theta^l) \to C_{k/2}(\theta)$ in $H^1(\Omega)$ which then implies again for a non-relabeled subsequence $\nabla C_{k/2}(\theta^l) \to \nabla C_{k/2}(\theta)$ a.e. in Ω. These arguments hold true for all $k > 0$ and enable us to verify similarly to Step 5 in the proof of Theorem 3.3 that θ is an entropy solution for the data f and g.

3. Since the entropy solution to (17) with data f and g is unique, not only a subsequence but the whole sequence converges (weakly) to θ in $W^{1,q}(\Omega)$. □

Lemma 3.5 *We assume (A1) and (A5). Let $f \in L^1_+(\Omega)$ and $g = \kappa\theta_\mathrm{a}$ with $\theta_\mathrm{a} = const > 0$. Then the entropy solution θ to (17) fulfills $\theta \geq \theta_\mathrm{a}$ a.e. on Ω.*

Proof. Let $f_n := C_n(f) \in L^\infty(\Omega)$, $g_n := g = \kappa\theta_\mathrm{a}$ and let $\theta_n \in H^1(\Omega)$ be the unique weak solution to (17) with data f_n and g_n. We test (17) by $-(\theta_n - \theta_\mathrm{a})^- = \min\{\theta - \theta_\mathrm{a}, 0\}$ and find

$$\int_\Omega \lambda|\nabla(\theta_n - \theta_\mathrm{a})^-|^2 \,\mathrm{d}x + \int_\Gamma \kappa((\theta_n - \theta_\mathrm{a})^-)^2 \,\mathrm{d}\Gamma \leq 0$$

implying that $\theta_n \geq \theta_\mathrm{a}$ a.e. on Ω.

We fix $k > \theta_\mathrm{a}$. Since $C_k(\theta_n) \to C_k(\theta)$ in $L^1(\Omega)$ as $n \to \infty$ (note that $\theta_n \to \theta$ in $L^1(\Omega)$ due to Lemma 3.4) we find a subsequence $\{n_l\}$ such that $C_k(\theta_{n_l}) \to C_k(\theta)$ a.e. in Ω which implies together with $\theta_{n_l} \geq \theta_\mathrm{a}$ a.e. on Ω that $C_k(\theta) \geq \theta_\mathrm{a}$ a.e. in Ω. Especially this ensures $\theta \geq 0$ a.e. in Ω because of $\theta_\mathrm{a} > 0$. Additionally, by our choice $k > \theta_\mathrm{a}$ we have $\theta \geq C_k(\theta) \geq \theta_\mathrm{a}$ a.e. in Ω. □

4 Solution of the coupled system via Schauder's fixed-point theorem

In this section we proof our main result, Theorem 2.1.

1. Definition of the fixed-point map. Let

$$\mathcal{M} := \{\theta \in L^1(\Omega) : \|\theta\|_{W^{1,1}} \leq c_Q,\, \theta \geq \theta_\mathrm{a}\},$$

where $c_Q > 0$ will be fixed in (26). We consider the fixed-point map $Q : \mathcal{M} \to \mathcal{M}$ with $\theta = Q(\widetilde{\theta})$ defined by θ being the unique entropy solution of

$$\begin{aligned} -\nabla\cdot(\lambda\nabla\theta) &= H(x,\widetilde{\theta},\nabla\varphi) && \text{in } \Omega,\\ -\lambda\nabla\theta\cdot\nu &= \kappa(\theta-\theta_\mathrm{a}) && \text{on } \Gamma, \end{aligned} \tag{24}$$

where H is defined in (3) and the function $\varphi = \varphi(\widetilde{\theta})$ is the unique weak solution to

$$\begin{aligned} -\nabla\cdot A(x,\widetilde{\theta},\nabla\varphi) &= 0 \quad \text{in } \Omega,\\ A(x,\widetilde{\theta},\nabla\varphi)\cdot\nu &= 0 \quad \text{on } \Gamma_N, \quad \varphi = \varphi^D \quad \text{on } \Gamma_D. \end{aligned} \tag{25}$$

Since $\widetilde{\theta} \in \mathcal{M}$ we have $\widetilde{\theta} \in \Theta$ and the existence and uniqueness of a weak solution $\varphi \in \varphi^D + W_D^{1,p(\cdot)}(\Omega)$ of the current flow equation (25) follows from Theorem 3.1. From (14) in this theorem we find $\|H(\cdot, \widetilde{\theta}, \nabla\varphi)\|_{L^1} \le c_H$. With $f := H(\cdot, \widetilde{\theta}, \nabla\varphi) \in L^1(\Omega)$ and $g := \kappa\theta_{\mathrm{a}} \in L^1(\Gamma)$ Theorem 3.3 and Sobolev's embedding result give a unique entropy solution θ of (24) with

$$\|\theta\|_{W^{1,1}(\Omega)} \le c_{E1}(c_H + \|\kappa\theta_a\|_{L^1(\Gamma)}) =: c_Q \tag{26}$$

for all $\varphi = \varphi(\widetilde{\theta})$ with $\widetilde{\theta} \in \mathcal{M}$. Here, $c_{E1} > 0$ comes from Theorem 3.3. Finally, Lemma 3.5 ensures that $\theta \ge \theta_{\mathrm{a}}$. Thus, we obtain that $\theta = Q(\widetilde{\theta}) \in \mathcal{M}$.

2. Existence of a solution. The continuity of the mapping $Q : \mathcal{M} \to \mathcal{M}$ will be proven in Lemma 4.1. Since for all $\theta \in \mathcal{M}$ the norm $\|\theta\|_{W^{1,1}}$ is uniformly bounded, the compact embedding of $W^{1,1}(\Omega)$ in $L^1(\Omega)$ gives the desired compactness of the convex and nonempty set $\mathcal{M} \subset L^1(\Omega)$. Therefore Schauder's fixed-point theorem ensures the solvability of the coupled problem (1)–(4). This finishes the proof of Theorem 2.1. □

Lemma 4.1 *We assume (A1) – (A6). The fixed-point map $Q : \mathcal{M} \to \mathcal{M}$ defined in Step 1 of the proof of Theorem 2.1 is continuous.*

Proof. Let $\widetilde{\theta}$, $\widetilde{\theta}_n \in \mathcal{M}$ with $\widetilde{\theta}_n \to \widetilde{\theta}$ in $L^1(\Omega)$. We denote by φ_n the unique solution to (25) with $\widetilde{\theta}_n$ instead of $\widetilde{\theta}$ as fixed argument in A. We have to show that $\theta_n = Q(\widetilde{\theta}_n) \to \theta = Q(\widetilde{\theta})$ in $L^1(\Omega)$.

1. Convergences of subsequences. Theorem 3.1, the growth properties of A and η and Theorem 3.3 ensure for all $\varphi_n = \varphi(\widetilde{\theta}_n)$ and $\theta_n = Q(\widetilde{\theta}_n)$ the uniform estimates

$$\begin{gathered} \|\varphi_n\|_{W^{1,p(\cdot)}} \le c_\varphi, \quad \|A(\cdot, \widetilde{\theta}_n, \nabla\varphi_n)\|_{L^{p'(\cdot)}} \le c_A, \quad \tfrac{1}{p(x)} + \tfrac{1}{p'(x)} = 1, \\ \|\theta_n\|_{W^{1,q_0}} \le \widetilde{c}_{Eq_0}, \quad q_0 := \tfrac{2d}{2d-1}. \end{gathered} \tag{27}$$

The estimates in (27) guarantee for some $\overline{\varphi} \in W^{1,p(\cdot)}(\Omega)$, $\overline{A} \in L^{p'(\cdot)}(\Omega)^d$, and $\overline{\theta} \in W^{1,q_0}(\Omega)$ and for a (not-relabeled) subsequence the weak convergences

$$\begin{gathered} \varphi_n \rightharpoonup \overline{\varphi} \text{ in } W^{1,p(\cdot)}(\Omega), \quad A(\cdot, \widetilde{\theta}_n, \nabla\varphi_n) \rightharpoonup \overline{A} \text{ in } L^{p'(\cdot)}(\Omega)^d, \\ \theta_n \rightharpoonup \overline{\theta} \text{ in } W^{1,q_0}(\Omega). \end{gathered} \tag{28}$$

The growth condition (5) gives $|A(x, \widetilde{\theta}_n, \nabla\overline{\varphi}) - A(x, \widetilde{\theta}, \nabla\overline{\varphi})| \le c|\nabla\overline{\varphi}|^{p(x)-1}$. Hence, we have an integrable majorant for the integrand $|A(x, \widetilde{\theta}_n, \nabla\overline{\varphi}) - A(x, \widetilde{\theta}, \nabla\overline{\varphi})|^{p'(x)}$. Since $\widetilde{\theta}_n \to \widetilde{\theta}$ in $L^1(\Omega)$ and A is a Caratheodory function, this integrand converges to 0 a.e. on Ω for an again non-relabeled subsequence. Thus, Lebesgue's theorem on dominated convergence gives

$$\int_\Omega |A(x, \widetilde{\theta}_n, \nabla\overline{\varphi}) - A(x, \widetilde{\theta}, \nabla\overline{\varphi})|^{p'(x)} \,\mathrm{d}x \to 0.$$

Exploiting the monotonicity of A in the last argument and using that φ_n is the weak solution to (25) with $\widetilde{\theta}_n$ as argument in A, we derive

$$\begin{aligned} 0 &\le \int_\Omega \big(A(x, \widetilde{\theta}_n, \nabla\varphi_n) - A(x, \widetilde{\theta}_n, \nabla\overline{\varphi})\big) \cdot \nabla(\varphi_n - \overline{\varphi}) \,\mathrm{d}x \\ &= 0 - \int_\Omega A(x, \widetilde{\theta}_n, \nabla\overline{\varphi}) \cdot \nabla(\varphi_n - \overline{\varphi}) \,\mathrm{d}x \to 0 \end{aligned}$$

since $\nabla\varphi_n - \nabla\overline{\varphi} \rightharpoonup 0$ in $L^{p(\cdot)}(\Omega)^d$ and $A(\cdot,\widetilde{\theta}_n,\nabla\overline{\varphi}) \to A(\cdot,\widetilde{\theta},\nabla\overline{\varphi})$ in $L^{p'(\cdot)}(\Omega)^d$. This guarantees (for the subsequence)

$$\int_\Omega \big(A(x,\widetilde{\theta}_n,\nabla\varphi_n) - A(x,\widetilde{\theta}_n,\nabla\overline{\varphi})\big)\cdot\nabla(\varphi_n-\overline{\varphi})\,\mathrm{d}x \to 0. \tag{29}$$

Due to (6) the integrand in (29) is nonnegative which implies (together with $\widetilde{\theta}_n \geq \theta_a$)

$$\begin{aligned} &\big(A(\cdot,\widetilde{\theta}_n,\nabla\varphi_n) - A(\cdot,\widetilde{\theta}_n,\nabla\overline{\varphi})\big)\cdot\nabla(\varphi_n-\overline{\varphi}) \to 0 \quad \text{in } L^1(\Omega),\\ &\big(|\nabla\varphi_n|^{p(\cdot)-2}\nabla\varphi_n - |\nabla\overline{\varphi}|^{p(\cdot)-2}\nabla\overline{\varphi}\big)\cdot\nabla(\varphi_n-\overline{\varphi}) \to 0 \quad \text{in } L^1(\Omega). \end{aligned} \tag{30}$$

Using the strict monotonicity and arguing similar to [14, p. 50f] we obtain the convergence $\nabla\varphi_{n_l} \to \nabla\overline{\varphi}$ a.e. on Ω, which together with $\widetilde{\theta}_{n_l} \to \widetilde{\theta}$ a.e. on Ω for a subsequence and the Caratheodory property of A gives $\overline{A} = A(\cdot,\widetilde{\theta},\nabla\overline{\varphi})$. We obtain

$$\int_\Omega A(x,\widetilde{\theta},\nabla\overline{\varphi})\cdot\nabla v\,\mathrm{d}x = \lim_{n_l\to\infty}\int_\Omega A(x,\widetilde{\theta}_{n_l},\nabla\varphi_{n_l})\cdot\nabla v\,\mathrm{d}x = 0 \quad \forall v\in W^{1,p(\cdot)}_D(\Omega).$$

By Theorem 3.1, the weak solution to (25) with $\widetilde{\theta}$ as second argument in A is unique such that we find $\overline{\varphi} = \varphi = \varphi(\widetilde{\theta})$.

2. Weak convergence of the whole sequence $\varphi_n \rightharpoonup \varphi$ in $W^{1,p(\cdot)}(\Omega)$. To verify the weak convergence $\varphi_n \rightharpoonup \varphi$ in the reflexive Banach space $W^{1,p(\cdot)}(\Omega)$ of the whole sequence and not only of the subsequence given in (28) we apply [7, Lemma 5.4, Chap. 1]. We have to show that for every weakly convergent subsequence $\varphi_{n_k} \rightharpoonup \widehat{\varphi}$ it holds true that $\widehat{\varphi} = \varphi$: If there is a subsequence $\varphi_{n_k} \rightharpoonup \widehat{\varphi}$ in $W^{1,p(\cdot)}(\Omega)$ then the arguments of Step 1 ensure again non-relabeled subsequences such that $\nabla\varphi_{n_k} \to \nabla\widehat{\varphi}$ a.e. on Ω and $A(\cdot,\widetilde{\theta}_{n_k},\nabla\varphi_{n_k}) \rightharpoonup A(\cdot,\widetilde{\theta},\nabla\widehat{\varphi})$ in $L^{p'(\cdot)}(\Omega)^d$. And $\widehat{\varphi}$ would solve (25), with $\widetilde{\theta}$ as second argument in A. Since the weak solution to (25) is unique – we have $\widehat{\varphi} = \varphi$. Thus, the convergence $\varphi_n \rightharpoonup \varphi$ in $W^{1,p(\cdot)}(\Omega)$ is valid for the whole sequence and not only a subsequence.

3. Further convergences for subsequences. Let $w \in L^\infty(\Omega)$ be arbitrarily given. Then by (5), $|(A(x,\widetilde{\theta}_n,\nabla\varphi) - A(x,\widetilde{\theta},\nabla\varphi))w|^{p'(x)} \leq c|\nabla\varphi|^{p(x)}\|w\|_{L^\infty}^{p'(x)}$ gives an integrable majorant and we have for a non-relabeled subsequence that $A(x,\widetilde{\theta}_n,\nabla\varphi) \to A(x,\widetilde{\theta},\nabla\varphi)$ for a.a. $x\in\Omega$ (remember $\widetilde{\theta}_n \to \widetilde{\theta}$ a.e. in Ω, A Caratheodory function). Therefore it results for this subsequence by Lebesgue's dominated convergence theorem $A(\cdot,\widetilde{\theta}_n,\nabla\varphi)w \to A(\cdot,\widetilde{\theta},\nabla\varphi)w$ in $L^{p'(\cdot)}(\Omega)^d$. Together with $\nabla\varphi_n \rightharpoonup \nabla\varphi$ in $L^{p(\cdot)}(\Omega)^d$ we obtain

$$\int_\Omega A(x,\widetilde{\theta}_n,\nabla\varphi)\cdot\nabla(\varphi_n-\varphi)w\,\mathrm{d}x \to 0.$$

Since this argument holds true for all $w\in L^\infty(\Omega)$, we find

$$A(\cdot,\widetilde{\theta}_n,\nabla\varphi)\cdot\nabla(\varphi_n-\varphi) \rightharpoonup 0 \quad \text{in } L^1(\Omega).$$

This together with (30) ensures the weak convergence

$$A(\cdot,\widetilde{\theta}_n,\nabla\varphi_n)\cdot\nabla(\varphi_n-\varphi) \rightharpoonup 0 \quad \text{in } L^1(\Omega). \tag{31}$$

The weak convergence of $A(\cdot,\widetilde{\theta}_n,\nabla\varphi_n)$ in $L^{p'(\cdot)}(\Omega)^d$ and $w\nabla\varphi\in L^{p(\cdot)}(\Omega)^d$ with $w\in L^\infty(\Omega)$ guarantee

$$A(\cdot,\widetilde{\theta}_n,\nabla\varphi_n)\cdot\nabla\varphi \rightharpoonup A(\cdot,\widetilde{\theta},\nabla\varphi)\cdot\nabla\varphi \quad \text{in } L^1(\Omega)$$

which with (31) for the subsequence leads to the weak convergence

$$h_n := A(\cdot,\widetilde{\theta}_n,\nabla\varphi_n)\cdot\nabla\varphi_n \rightharpoonup A(\cdot,\widetilde{\theta},\nabla\varphi)\cdot\nabla\varphi =: h \quad \text{in } L^1(\Omega).$$

Since a sequence of functions $\{h_n\}\subset L^1(\Omega)$, which converges weakly to $h\in L^1(\Omega)$ is uniformly equi integrable, the sequence $\{A(\cdot,\widetilde{\theta}_n,\nabla\varphi_n)\cdot\nabla\varphi_n\}$ is uniformly equi integrable. Due to the pointwise a.e. convergence of $\widetilde{\theta}_n(x)\to\widetilde{\theta}(x)$, $\nabla\varphi_n(x)\to\nabla\varphi(x)$ a.e. in Ω and the Caratheodory property of A we find $h_n\to h$ a.e. in Ω for the subsequence.

Next, we apply Vitali's theorem which tells us that for a sequence of functions $\{h_n\}\subset L^1(\Omega)$, which converges pointwise a.e. in Ω to h the following two assertions are equivalent: (i) The sequence $\{h_n\}$ is uniformly equi integrable and (ii) $h_n\to h$ in $L^1(\Omega)$.

Setting $f_n := (1-\eta(\cdot,\widetilde{\theta}_n,\nabla\varphi_n))h_n$ and $f := (1-\eta(\cdot,\widetilde{\theta},\nabla\varphi))h$ we estimate for a non-relabeled subsequence

$$\|f_n-f\|_{L^1} \le \int_\Omega \big|\eta(\cdot,\widetilde{\theta}_n,\nabla\varphi_n)-\eta(\cdot,\widetilde{\theta},\nabla\varphi)\big||h|\,\mathrm{d}x + \int_\Omega \big|1-\eta(\cdot,\widetilde{\theta}_n,\nabla\varphi_n)\big||h_n-h|\,\mathrm{d}x.$$

Since the second term on the right hand side can be estimated by $c\|h_n-h\|_{L^1}\to 0$ and the first term tends to zero by Lebesgue's dominated convergence theorem (η is bounded, Caratheodory, $\widetilde{\theta}_n\to\widetilde{\theta}$, $\nabla\varphi_n\to\nabla\varphi$ a.e. on Ω for the subsequence) we conclude that $f_n\to f$ in $L^1(\Omega)$. To summarize, for the right hand sides of the equation (24) we have for the last subsequence the strong convergence

$$f_n = H(\cdot,\widetilde{\theta}_n,\nabla\varphi_n) \to H(\cdot,\widetilde{\theta},\nabla\varphi) = f \quad \text{in } L^1(\Omega).$$

According to Lemma 3.4 we find for the entropy solutions θ_n and θ of (24) with right hand sides f_n and f, respectively, that $\theta_n\rightharpoonup\theta$ in $W^{1,q_0}(\Omega)$. By Theorem 3.3 the solution to (24) with right hand side f is unique, thus, with (28) it follows that $\theta_n = Q(\widetilde{\theta}_n)\rightharpoonup\overline{\theta}=\theta=Q(\widetilde{\theta})$ in $W^{1,q_0}(\Omega)$, for the subsequence in (28).

4. Convergences of the whole sequence $\theta_n\rightharpoonup\theta$ in $W^{1,q_0}(\Omega)$ and $\theta_n\to\theta$ in $L^1(\Omega)$. Similar to Step 2, we have to show that for each weakly convergent subsequence $\theta_{n_k}\rightharpoonup\widehat{\theta}$ in the reflexive Banach space $W^{1,q_0}(\Omega)$ the identity $\widehat{\theta}=\theta$ is fulfilled. We can argue as in Step 3 to find for not-relabeled subsequences $f_{n_k}=H(\cdot,\widetilde{\theta}_{n_k},\nabla\varphi_{n_k})\to H(\cdot,\widetilde{\theta},\nabla\varphi)=f$ in $L^1(\Omega)$ such that the uniqueness result in Theorem 3.3 and Lemma 3.4 ensure $\theta_{n_k}\rightharpoonup\theta=\widehat{\theta}$ in $W^{1,q_0}(\Omega)$. This leads to the convergence of the whole sequence $\theta_n\rightharpoonup\theta$ in $W^{1,q_0}(\Omega)$ and by the compact embedding of $W^{1,q_0}(\Omega)$ into $L^1(\Omega)$ we obtain the strong convergence of the whole sequence in $L^1(\Omega)$ which proves the continuity of the operator Q. □

A Proof of Theorem 3.3

In the case of Dirichlet boundary conditions the theory of entropy solutions is presented in the survey [13], for nonlinear problems see [1, 10]. To provide a complete theory for our type of boundary conditions in (17) we give a proof by adapting the techniques in [10, 13].

For the uniqueness proof of Theorem 3.3 we need that weak solutions to (17) for more regular data f and g are admissible test functions ω in the sense of (18), which we state first.

Lemma A.1 *We assume (A1) and (A5). Let $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Gamma)$. Then the weak solution to (17) belongs to $L^\infty(\Omega)$ and $\|\theta\|_{L^\infty} \le C(f,g)$.*

Proof. We test (17) by θ^{r-1}, $r = 2^k$, $k \in \mathbb{N}$, and use the notation $v := \theta^{\frac{r}{2}}$,

$$\begin{aligned}\underline{\alpha}\|v\|_{H^1}^2 &\le \|f\|_{L^\infty} \int_\Omega \theta^{r-1}\,\mathrm{d}x + c\|g\|_{L^\infty(\Gamma)} \int_\Gamma \theta^{r-1}\,\mathrm{d}\Gamma \\ &\le c\|f\|_{L^\infty} \int_\Omega (1+v^2)\,\mathrm{d}x + c\|g\|_{L^\infty(\Gamma)} \int_\Gamma (1+v^2)\,\mathrm{d}\Gamma.\end{aligned} \tag{32}$$

By the Gagliardo-Nirenberg inequality, the trace inequality, and Young's inequality we find for all $\varepsilon > 0$ a $c_\varepsilon > 0$ such that

$$\|v\|_{L^2}^2 \le c\|v\|_{L^1}^{\frac{4}{d+2}} \|v\|_{H^1}^{\frac{2d}{d+2}} \le \varepsilon\|v\|_{H^1}^2 + c_\varepsilon\|v\|_{L^1}^2,$$

$$\|v\|_{L^2(\Gamma)}^2 \le c\|v\|_{L^2}\|v\|_{H^1} \le c\|v\|_{L^1}^{\frac{2}{d+2}} \|v\|_{H^1}^{\frac{2d+2}{d+2}} \le \varepsilon\|v\|_{H^1}^2 + c_\varepsilon\|v\|_{L^1}^2.$$

From (32) it results with the same constant $c_0(f,g) \ge 2$ for $r = 2^k$, $k \in \mathbb{N}$, the estimate

$$\|v\|_{L^2}^2 \le \|v\|_{H^1}^2 \le \frac{1}{2} c_0(f,g)(1 + \|v\|_{L^1}^2).$$

Setting $a_k := 1 + \|\theta\|_{L^{2^k}}^{2^k}$ the previous estimates ensure the recursion

$$a_k \le c_0(f,g) a_{k-1}^2 \le c_0(f,g)^{1+2} a_{k-2}^4 \le c_0(f,g)^{1+2+\dots+2^{k-2}} a_1^{2^{k-1}} \le c_0(f,g)^{2^k} a_1^{2^k}.$$

The starting estimate for a_1 is obtained by testing (17) by θ. Applying embedding and trace inequality as well as Young's inequality gives

$$\underline{\alpha}\|\theta\|_{H^1}^2 \le \int_\Omega f\theta\,\mathrm{d}x + \int_\Gamma g\theta\,\mathrm{d}\Gamma \le \frac{\underline{\alpha}}{2}\|\theta\|_{H^1}^2 + c_{\underline{\alpha}}(\|f\|_{L^2}^2 + \|g\|_{L^2(\Gamma)}^2).$$

This ensures that $a_1 = 1 + \|\theta\|_{L^2}^2 \le c_1(f,g)$ which finishes the proof. □

Proof of Theorem 3.3: 1. We use a proof by approximation. Let $f_n := C_n(f) \in L^\infty(\Omega)$, $g_n := C_n(g) \in L^\infty(\Gamma)$ and let $\theta_n \in H^1(\Omega)$ be the unique solution to

$$\begin{aligned} -\nabla\cdot(\lambda\nabla\theta_n) &= f_n && \text{in } \Omega, \\ -\lambda\nabla\theta_n\cdot\nu &= \kappa\theta_n - g_n && \text{on } \Gamma. \end{aligned} \tag{33}$$

For $k > 0$ we use for (33) the test function $C_k(\theta_n) \in H^1(\Omega)$ and find

$$\begin{aligned} \underline{\alpha}\|C_k(\theta_n)\|^2_{H^1} &\le \int_\Omega \lambda |\nabla C_k(\theta_n)|^2 \,\mathrm{d}x + \int_\Gamma \kappa C_k(\theta_n)^2 \,\mathrm{d}\Gamma \\ &\le \int_\Omega \lambda \nabla\theta_n \cdot \nabla C_k(\theta_n) \,\mathrm{d}x + \int_\Gamma \kappa \theta_n C_k(\theta_n) \,\mathrm{d}\Gamma \\ &= \int_\Omega f_n C_k(\theta_n) \,\mathrm{d}x + \int_\Gamma g_n C_k(\theta_n) \,\mathrm{d}\Gamma \le k(\|f\|_{L^1} + \|g\|_{L^1(\Gamma)}). \end{aligned} \tag{34}$$

Therefore, for all fixed $k > 0$ the sequence $\{C_k(\theta_n)\}$ is bounded in $H^1(\Omega)$.

2. According to Sobolev's embedding result, for $d > 2$ we have additionally

$$\frac{\underline{\alpha}}{c_S}\Big(\int_\Omega |C_k(\theta_n)|^{\frac{2d}{d-2}} \mathrm{d}x\Big)^{\frac{d-2}{d}} \le k(\|f\|_{L^1} + \|g\|_{L^1(\Gamma)}).$$

Since $|C_k(\theta_n)| = k$ on $\Omega_{k,n} := \{x \in \Omega : |\theta_n(x)| \ge k\}$ it results

$$\mathrm{mes}(\Omega_{k,n}) \le c\Big(\frac{\|f\|_{L^1} + \|g\|_{L^1(\Gamma)}}{k}\Big)^{\frac{d}{d-2}},$$

where $c > 0$ does not depend on n and k. [9, Chap. 2, p. 105] ensures that

$$\int_\Omega |\theta_n|^{\widetilde{q}} \,\mathrm{d}x \le c_{\widetilde{q}}(\|f\|_{L^1} + \|g\|_{L^1(\Gamma)})^{\widetilde{q}} \quad \text{for } \widetilde{q} < \tfrac{d}{d-2}. \tag{35}$$

For $d = 2$ this estimate is valid for all $\widetilde{q} < \infty$. Similar to [13, p. 34], but now exploiting (34) in the case of Robin boundary conditions we verify that

$$\int_\Omega |\nabla\theta_n|^q \,\mathrm{d}x \le c_q(\|f\|_{L^1} + \|g\|_{L^1(\Gamma)})^q \quad \text{for } q < \tfrac{d}{d-1}. \tag{36}$$

Thus, $\|\theta_n\|_{W^{1,q}}$ is uniformly bounded and we find a subsequence again denoted by $\{\theta_n\}$ and $\theta \in W^{1,q}(\Omega)$ with $\theta_n \rightharpoonup \theta$ in $W^{1,q}(\Omega)$, $\theta_n \to \theta$ in $L^q(\Omega)$ and a.e. in Ω.

3. Due to our construction, $f_n \to f$ in $L^1(\Omega)$ and f_n is a Cauchy sequence in $L^1(\Omega)$ and $g_n \to g$ in $L^1(\Gamma)$ and g_n is a Cauchy sequence in $L^1(\Gamma)$. Repeating Step 2 of the proof for $\theta_n - \theta_m$ with $f_n - f_m$, $g_n - g_m$ (weak solutions of a linear problem) we find

$$\int_\Omega |\nabla(\theta_n - \theta_m)|^q \,\mathrm{d}x \le c_q(\|f_n - f_m\|_{L^1} + \|g_n - g_m\|_{L^1(\Gamma)})^q$$

and similar estimates of the form (35) for the corresponding differences. Therefore $\{\theta_n\}$ is a Cauchy sequence in $W^{1,q}(\Omega)$ for all $q < \frac{d}{d-1}$ and $\{\theta_n\}$ converges strongly to θ in $W^{1,q}(\Omega)$. For a subsequence, $\nabla\theta_n \to \nabla\theta$ a.e. in Ω and $\theta_n \to \theta$ a.e. on Γ.

4. Using (35) and (36) for $1 \le \widetilde{q} = q < \frac{d}{d-1}$ and the convergence $\theta_n \to \theta$ in $W^{1,q}(\Omega)$, the estimate for $\|\theta\|_{W^{1,q}}$ of Theorem 3.3 is verified.

5. Next, we show that θ is an entropy solution of (17) for data f and g. From the previous convergences we additionally obtain for a subsequence that $\nabla C_k(\theta_n) \to \nabla C_k(\theta)$

a.e. in Ω, $C_k(\theta_n) \to C_k(\theta)$ in $L^2(\Omega)$ and $C_k(\theta_n) \to C_k(\theta)$ in $L^2(\Gamma)$ for all $k > 0$. Fatou's Lemma implies

$$\int_\Omega \lambda_0 |\nabla C_k(\theta)|^2 \,\mathrm{d}x + \int_\Gamma \kappa C_k(\theta)^2 \,\mathrm{d}\Gamma \\ \leq \liminf_{n\to\infty} \int_\Omega \lambda_0 |\nabla C_k(\theta_n)|^2 \,\mathrm{d}x + \lim_{n\to\infty} \int_\Gamma \kappa C_k(\theta_n)^2 \,\mathrm{d}\Gamma \leq k(\|f\|_{L^1} + \|g\|_{L^1(\Gamma)})$$

which ensures that $\theta \in \mathcal{V}^{1,2}(\Omega)$. Let now $k > 0$ and $\omega \in H^1(\Omega) \cap L^\infty(\Omega)$ be fixed. We use the test function $C_k(\theta - \omega) \in H^1(\Omega)$ for the weak formulation of (33) and obtain

$$\int_\Omega \lambda \nabla\theta_n \cdot \nabla C_k(\theta_n - \omega) \,\mathrm{d}x + \int_\Gamma \kappa\theta_n C_k(\theta_n - \omega) \,\mathrm{d}\Gamma \\ = \int_\Omega f_n C_k(\theta_n - \omega) \,\mathrm{d}x + \int_\Gamma g_n C_k(\theta_n - \omega) \,\mathrm{d}\Gamma.$$

We write the terms on the left hand side in the form

$$\int_\Omega \lambda \nabla\theta_n \cdot \nabla C_k(\theta_n - \omega) \,\mathrm{d}x = \int_\Omega \lambda |\nabla C_k(\theta_n - \omega)|^2 \,\mathrm{d}x + \int_\Omega \lambda \nabla\omega \cdot \nabla C_k(\theta_n - \omega) \,\mathrm{d}x,$$

$$\int_\Gamma \kappa\theta_n C_k(\theta_n - \omega) \,\mathrm{d}\Gamma = \int_\Gamma \kappa(\theta_n - \omega) C_k(\theta_n - \omega) \,\mathrm{d}\Gamma + \int_\Gamma \kappa\omega C_k(\theta_n - \omega) \,\mathrm{d}\Gamma.$$

On the one hand, by Fatou's lemma we verify

$$\int_\Omega \lambda |\nabla C_k(\theta - \omega)|^2 \,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega \lambda |\nabla C_k(\theta_n - \omega)|^2 \,\mathrm{d}x,$$

$$\int_\Gamma \kappa(\theta - \omega) C_k(\theta - \omega) \,\mathrm{d}\Gamma \leq \liminf_{n\to\infty} \int_\Gamma \kappa(\theta_n - \omega) C_k(\theta_n - \omega) \,\mathrm{d}\Gamma.$$

On the other hand, Lebesgue's theorem on dominated convergence implies

$$\lim_{n\to\infty} \int_\Omega f_n C_k(\theta_n - \omega) \,\mathrm{d}x = \int_\Omega f C_k(\theta - \omega) \,\mathrm{d}x, \\ \lim_{n\to\infty} \int_\Gamma g_n C_k(\theta_n - \omega) \,\mathrm{d}\Gamma = \int_\Gamma g C_k(\theta - \omega) \,\mathrm{d}\Gamma.$$

Next, the weak convergence of $C_k(\theta_n)$ to $C_k(\theta)$ in $H^1(\Omega)$ ensures

$$\lim_{n\to\infty} \int_\Omega \lambda \nabla\omega \cdot \nabla C_k(\theta_n - \omega) \,\mathrm{d}x = \int_\Omega \lambda \nabla\omega \cdot \nabla C_k(\theta - \omega) \,\mathrm{d}x, \\ \lim_{n\to\infty} \int_\Gamma \kappa\omega C_k(\theta_n - \omega) \,\mathrm{d}\Gamma = \int_\Gamma \kappa\omega C_k(\theta - \omega) \,\mathrm{d}\Gamma.$$

Collecting and balancing all the limit terms we end up with

$$\int_\Omega \lambda \nabla\theta \cdot \nabla C_k(\theta - \omega) \,\mathrm{d}x + \int_\Gamma \kappa\theta C_k(\theta - \omega) \,\mathrm{d}\Gamma \leq \int_\Omega f C_k(\theta - \omega) \,\mathrm{d}x + \int_\Gamma g C_k(\theta - \omega) \,\mathrm{d}\Gamma$$

meaning that θ is an entropy solution of (17).

6. Finally, if there would be another entropy solution $\widetilde{\theta}$ besides θ we argue with a proof by approximation. Let for the subsequence $\{n\}$ from Step 3 of the above proof $f_n := C_n(f) \in L^\infty(\Omega)$, $g_n := C_n(g) \in L^\infty(\Gamma)$ and let θ_n be the corresponding unique weak solution to (17). According to Lemma A.1, θ_n belongs to $H^1(\Omega) \cap L^\infty(\Omega)$ and is an admissible choice for ω in the definition of the entropy solution $\widetilde{\theta}$ in (18). Hence it results

$$\int_\Omega \lambda\nabla\widetilde{\theta} \cdot \nabla C_k(\widetilde{\theta} - \theta_n)\,\mathrm{d}x + \int_\Gamma (\kappa\widetilde{\theta} - g)C_k(\widetilde{\theta} - \theta_n)\,\mathrm{d}\Gamma \le \int_\Omega f C_k(\widetilde{\theta} - \theta_n)\,\mathrm{d}x.$$

$C_k(\widetilde{\theta} - \theta_n) \in H^1(\Omega)$ yields as test function in the weak formulation of (33) the relation

$$\int_\Omega \lambda\nabla\theta_n \cdot \nabla C_k(\widetilde{\theta} - \theta_n)\,\mathrm{d}x + \int_\Gamma (\kappa\theta_n - g_n)C_k(\widetilde{\theta} - \theta_n)\,\mathrm{d}\Gamma = \int_\Omega f_n C_k(\widetilde{\theta} - \theta_n)\,\mathrm{d}x.$$

Subtracting the above estimates and using the equivalent norm in $H^1(\Omega)$ we derive

$$\underline{\alpha}\|C_k(\widetilde{\theta} - \theta_n)\|^2_{H^1} \le k(\|f - f_n\|_{L^1} + \|g - g_n\|_{L^1(\Gamma)}).$$

Similar to Step 2 (now for $\widetilde{\theta} - \theta_n$, $f - f_n$, $g - g_n$), we obtain for $1 < \widetilde{q} < \frac{d}{d-2}$ ($\widetilde{q} < \infty$ if $d = 2$) and $1 \le q < \frac{d}{d-1}$

$$\int_\Omega |\widetilde{\theta}-\theta_n|^{\widetilde{q}}\,\mathrm{d}x \le c_{\widetilde{q}}(\|f-f_n\|_{L^1} + \|g-g_n\|_{L^1(\Gamma)})^{\widetilde{q}} \to 0,$$
$$\int_\Omega |\nabla(\widetilde{\theta}-\theta_n)|^q\,\mathrm{d}x \le c_q(\|f-f_n\|_{L^1} + \|g - g_n\|_{L^1(\Gamma)})^q \to 0$$

since f_n approximates f in $L^1(\Omega)$ and g_n approximates g in $L^1(\Gamma)$. By Step 3, $\theta_n \to \theta$ in $W^{1,q}(\Omega)$ and thus we get $\theta = \widetilde{\theta}$ and the entropy solution θ is unique. □

Acknowledgment

M. Bulíček thanks the project MORE (ERC-CZ project no. LL1202) financed by the Ministry of Education, Youth and Sports, Czech Republic. A. Glitzky was supported by ERC under project AnaMultiScale (AdG 267802) and M. Liero gratefully acknowledges the funding from the Einstein Foundation Berlin within the Research Center MATHEON under the ECMath Project SE2 "Electrothermal modeling of large-area OLEDs". The authors thank Jens Frehse for stimulating discussions and helpful comments.

References

[1] L. Boccardo, T. Gallouët and L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **13** (1996), 539–551.

[2] M. Bulíček, A. Glitzky and M. Liero, *Systems describing electrothermal effects with $p(x)$-Laplace like structure for discontinuous variable exponents*, Preprint 2206, Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin, 2016.

[3] G. Cimatti, Remark on the existence and uniqueness for the thermistor problem under mixed boundary conditions, *Quart. Appl. Math.*, **47** (1989), 117–121.

[4] G. Cimatti, Remark on the thermistor problem with rapidly growing conductivity, *Applicable Analysis*, **80** (2001), 133–140.

[5] L. Diening, P. Harjulehto, P. Hästö and M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Berlin, 2011.

[6] A. Fischer, T. Koprucki, K. Gärtner, J. Brückner, B. Lüssem, K. Leo, A. Glitzky and R. Scholz, Feel the heat: Nonlinear electrothermal feedback in organic LEDs, *Adv. Funct. Mater.*, **24** (2014), 3367–3374.

[7] H. Gajewski, K. Gröger and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974.

[8] A. Glitzky and M. Liero, *Analysis of p(x)-Laplace thermistor models describing the electrothermal behavior of organic semiconductor devices*, Preprint 2143, Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin, 2015.

[9] A. Kufner, O. John and S. Fučik, *Function spaces*, Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1977.

[10] C. Leone and A. Porretta, Entropy solutions for nonlinear elliptic equations in L^1, *Nonlinear Analysis, Theory, Methods & Applications*, **32** (1998), 325–334.

[11] M. Liero, T. Koprucki, A. Fischer, R. Scholz and A. Glitzky, p-Laplace thermistor modeling of electrothermal feedback in organic semiconductor devices, *Zeitschrift für Angewandte Mathematik und Physik*, **66** (2015), 2957–2977.

[12] P. Lindqvist, *Notes on the p-Laplace equation*, Report 102, University of Jyväskyla, Department of Mathematics and Statistics, ISBN: 951-39-2586-2, 2006.

[13] L. Orsina, Elliptic equations with measure data, 2013, URL http://www1.mat.uniroma1.it/people/orsina/AS1213/AS1213.pdf.

[14] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, International Series of Numerical Mathematics 153, Springer, Basel, 2013.