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# Effect of chromatic dispersion on multimode laser dynamics: Delay differential model

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**Abstract**

A set of differential equations with distributed delay is derived for modeling of multimode ring lasers with intracavity chromatic dispersion. Analytical stability analysis of continuous wave regimes is performed and it is demonstrated that sufficiently strong anomalous dispersion can destabilize these regimes.

# 1 Introduction

Mode-locking is a powerful technique to generate ultrashort optical pulses, which are used in numerous applications [4]. In particular, monolithic quantum well and quantum dot semiconductor lasers are compact, low cost, and efficient sources of short optical pulses in the picosecond and subpicosecond range [1, 7, 13]. The most common way to model these lasers is based on the use of the so-called traveling wave equations [2,3,6,12], governing the space-time evolution of the amplitudes of two counter-propagating waves in the laser cavity coupled to the carrier density in the semiconductor medium. An alternative and more simple approach to the analysis of mode-locking in semiconductor lasers is based on a system of delay differential equations (DDEs) proposed in [14–16]. This approach assuming unidirectional lasing in a ring cavity and Lorentzian profile of the spectral filtering allows to perform a detailed stability and bifurcation analysis of different mode-locking regimes [14, 15]. Furthermore, under certain simplifying approximations analytical stability analysis of the fundamental and harmonic mode-locking regimes is possible [8, 15]. Later a modification of DDE model was applied to describe the characteristic features of Fourier domain mode-locked (FDML) [10] and sliding frequency mode-locked [11] regimes in frequency swept lasers used in optical coherence tomography. However, despite of a remarkable success of the DDE model in describing the dynamics of FDML lasers, this model does not take into account such important characteristics of these lasers as chromatic dispersion of the long fiber delay line. On the other hand it was recently shown experimentally that in the anomalous dispersion domain the dispersion of the fiber delay line can destabilize a continuous wave (CW) regime of the laser leading to a chaotic behavior [9]. In order to fill this gap, in this paper we develop a new model of an FDML laser that takes into account chromatic dispersion of the fiber delay line. This is a system of delay differential equations that in addition to a fixed delay contains a distributed delay term and can be reduced to an infinite chain of delay differential equations with a single fixed delay. Using the developed DDE model we analyze the stability of CW operation regimes in an FDML laser and find a modulational instability caused by the presence of anomalous chromatic dispersion in the fiber delay line.

# 2 Model equations

We consider a frequency swept ring-cavity laser consisting of a short semiconductor optical amplifier (SOA) gain section, linear frequency selective spectral filter, and a long dispersive fiber delay line [10]. A schematic representation of this laser studied experimentally in [10] is given in Fig. 1.

Let us start the derivation of our model by considering the propagation of light in the fiber delay line. We assume that the fiber operates in a linear dispersive regime when the Kerr nonlinearity can be neglected and consider chromatic dispersion caused by a Lorentzian absorption line with the central frequency detuned with respect to the reference frequency associated with the central wavelength of the laser amplification line. Using this approach we can describe both the case of normal dispersion (blue-shifted absorption line, positive detuning $\Omega > 0$) and anomalous dispersion case (red-shifted absorption line, negative detuning $\Omega < 0$). Propagation

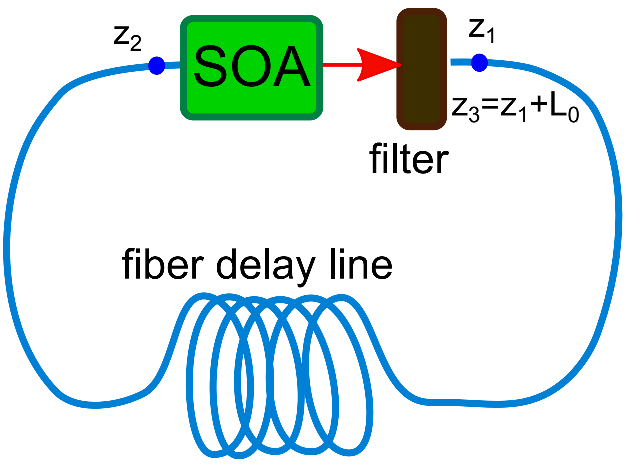


Figure 1: Schematic representation of a ring-cavity FDML laser. SOA – semiconductor optical amplifier acting as gain medium, $z$ is the coordinate along the cavity axis and $L_0$ is the total cavity length.

of light in a dispersive fiber with detuned Lorentzian absorption line can be described by a set of two partial differential equations for the slowly varying electric field $E(\tau, z)$ and polarization $P(\tau, z)$ envelopes:

$$\frac{\partial E}{\partial z} = P, \quad \frac{\partial P}{\partial \tau} = -(\Gamma + i\Omega)P - a\Gamma E, \tag{1}$$

where $z$ is the coordinate along the fiber, $\tau = t - z/v_{gr}$ is the retarded time, and $v_{gr}$ is the group velocity of light in the fiber. The system (1) can be rewritten as a single equation for the electric field envelope:

$$\frac{\partial^2 E}{\partial z \partial \tau} = -(\Gamma + i\Omega)\frac{\partial E}{\partial z} - a\Gamma E.$$

Applying Fourier transform to this equation we get ordinary differential equation for Fourier-transformed field $\hat{E}(\omega, z)$:

$$[\Gamma + i(\Omega - \omega)] \frac{d\hat{E}}{dz} = -a\Gamma \hat{E}.$$

Integrating this equation along the coordinate $z$ we obtain

$$\hat{E}(\omega, z_2) = \exp\left[\frac{-aL\Gamma}{\Gamma + i(\Omega - \omega)}\right] \hat{E}_1(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-aL\Gamma}{\Gamma + i(\Omega - \omega)}\right)^k E(\xi, z_1),$$

where the longitudinal coordinate $z = z_1$ ($z = z_2$) corresponds to the beginning (the end) of the fiber delay line of the length $L = z_2 - z_1$. Finally, by applying inverse Fourier transform to the above expression we get

$$E(\tau, z_2) = E(\tau, z_1) + P_1(\tau), \tag{2}$$

with

$$P_1(\tau) = \sum_{k=1}^{\infty} \frac{(-aL\Gamma)^k}{k!(k-1)!} \int_{-\infty}^{\tau} e^{-(\Gamma+i\Omega)(\tau-s)} (\tau - s)^{k-1} E_1(s, z_1) ds =$$

$$-aL\Gamma \int_{-\infty}^{\tau} e^{-(\Gamma+i\Omega)(\tau-s)} \frac{J_1\left[\sqrt{4aL\Gamma(\tau - s)}\right]}{\sqrt{aL\Gamma(\tau - s)}} E(s, z_1) ds, \tag{3}$$

where $J_1$ is the Bessel function of the first order.

The next step of the derivation is to describe the propagation of the electromagnetic field through the SOA gain section and then through a tunable filter with the Lorentzian profile having halfwidth $\gamma$ and detuning $w$. To this end we express the envelope $E(\tau, z_3)$ of the electric field output from the filter in terms of the electric field envelope at the entrance of the SOA section [15]:

$$\frac{dE(\tau, z_3)}{d\tau} + (\gamma - iw)E(\tau, z_3) = \gamma\sqrt{\kappa}e^{(1-i\alpha)G(t)/2}E(\tau, z_2), \tag{4}$$

where $\alpha$ is the linewidth enhancement (Henry) factor in the SOA section, the saturable gain $G = \int_{z_2}^{z_3} n(\tau, z)dz$ is given by the integral of the carrier density $n$ along this section, and $\kappa$ accounts for non-resonant cavity losses per cavity round trip due to linear absorption in SOA and radiation output. The saturable gain obeys the ordinary differential equation:

$$\frac{dG}{d\tau} = \gamma_g \left[g_0 - G - (e^G - 1)|E(\tau, z_2)|^2\right]. \tag{5}$$

Finally substituting Eq. (2) into (4) and (5) and using the boundary condition $E(\tau, z_1) = E(\tau - T, z_3)$, where $T$ is the cold cavity round trip time, we obtain the following set of two differential equations

$$\frac{dA}{d\tau} + (\gamma - iw)A = \gamma\sqrt{\kappa}e^{(1-i\alpha)G/2}\left[A(\tau - T) + P_1(\tau - T)\right], \tag{6}$$

$$\frac{dG}{d\tau} = \gamma_g \left[g_0 - G - (e^G - 1)\left|A(\tau - T) + P_1(\tau - T)\right|^2\right], \tag{7}$$

with

$$P_1(\tau) = -aL\Gamma \int_{-\infty}^{\tau} e^{-(\Gamma + i\Omega)(\tau - s)} \frac{J_1\left[\sqrt{4aL\Gamma(\tau - s)}\right]}{\sqrt{aL\Gamma(\tau - s)}} A(s)ds. \tag{8}$$

where $A(\tau) = E(\tau, z_3)$.

Alternatively, using the expansion in (3) we can represent the term $P_1(t)$, containing distributed delay, using the following infinite hierarchy of ordinary differential equations:

$$\begin{aligned}
\frac{dP_1}{d\tau} &= -(\Gamma + i\Omega)P_1 - aL\Gamma(E_1(\tau - T) + P_2(\tau - T)),\\
\frac{dP_2}{d\tau} &= -(\Gamma + i\Omega)P_2 - \frac{aL\Gamma}{2}(E_1(\tau - T) + P_3(\tau - T)),\\
&\cdots,\\
\frac{dP_k}{d\tau} &= -(\Gamma + i\Omega)P_k - \frac{aL\Gamma}{k}(E_1(\tau - T) + P_{k+1}(\tau - T)),\\
&\cdots.
\end{aligned}$$

In numerical simulations one can truncate this infinite chain by setting $P_N = 0$ for certain $N > 1$ and integrate the resulting equations together with (6) and (7).

## 3 CW regimes and their stability

Let us consider a CW solution in the form $A(\tau) = A_0 e^{i\nu\tau}$ and $G(\tau) = G_0$. Then, using expansion in (3) we get

$$P_1(t) = \left(e^{-aL\Gamma/[\Gamma + i(\Omega + \nu)]} - 1\right) A_0 e^{i\nu\tau} = P_0 e^{i\nu\tau}.$$

Substituting this relation into (6) and (7) we obtain the following expression for the CW intensity

$$|A_0|^2 = \frac{g_0 - G_0}{e^{G_0} - 1} e^{\frac{2aL\Gamma^2}{\Gamma^2 + (\Omega+\nu)^2}}, \tag{9}$$

where frequencies $\nu$ of the CW solutions are obtained by solving the transcendental equation:

$$\tan\left[\frac{\alpha G_0}{2} + \nu T - \frac{a\Gamma L(\nu+\Omega)}{\Gamma^2 + (\Omega+\nu)^2}\right] = \frac{w-\nu}{\gamma}. \tag{10}$$

In Eqs. (9) and (10) the stationary value of the saturable gain is given by

$$G_0 = \ln\left[\frac{\gamma^2 + (w-\nu)^2}{\gamma^2 \kappa} e^{\frac{2aL\Gamma^2}{\Gamma^2 + (\Omega+\nu)^2}}\right]. \tag{11}$$

We note the expressions for $|A_0|^2$ and $G_0$ in Eqs. (9) and (11) do not contain explicitly the cavity round trip time

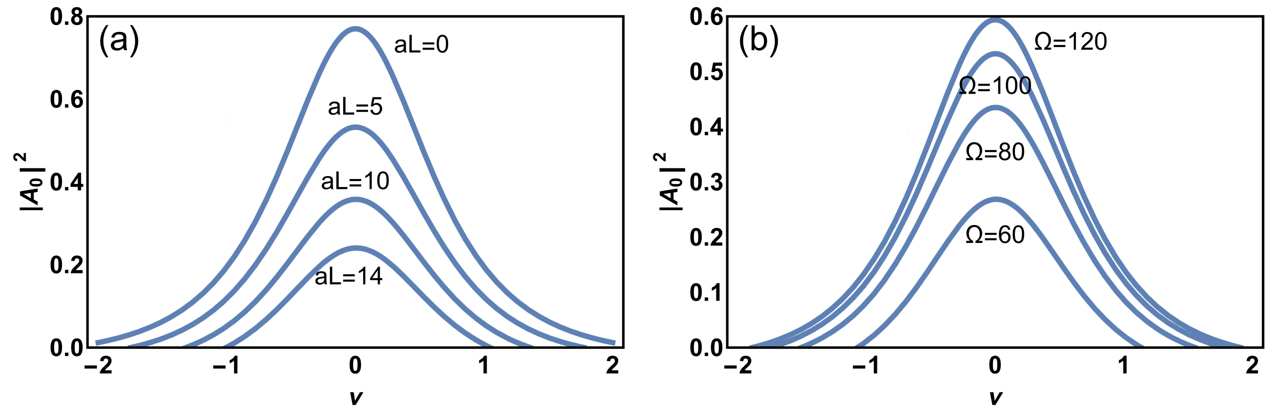


Figure 2: Dependence of the CW solution intensity $|A_0|^2$ on the frequency shift $\nu$ for different fiber length $L$ and $\Omega = 100.0$ (a) and frequency detunings $\Omega$ of the absorption line and $aL = 5.0$ (b). Other parameter values: $\gamma = 1$, $g_0 = 3.0$, $\kappa = 0.3$, $w = 0$, and $\Gamma = 20.0$. Decrease (increase) of the CW intensity with the fiber delay line length $L$ (detuning $\Omega$) is related to the increase (decrease) of the losses introduced by the absorption line in the fiber.

$T$ and the linewidth enhancement factor $\alpha$. However, for any finite delay time $T < \infty$ only a discrete set of the frequencies $\nu$ satisfies Eq. (10) and, hence, corresponds to CW solutions (or longitudinal modes) of the laser. The frequency separation between these modes is of order $1/T$ at large $T$. Therefore, in the limit $T \to \infty$ the frequency separations between the modes become negligibly small and each of the curves $|A_0(\nu)|^2$ shown in Fig. 2 becomes densely filled with discrete points corresponding to CW solutions of the system for a fixed parameter set.

Next, let us perform linear stability analysis of the CW solution (9)-(11) in the limit $T \to \infty$. We linearize the system near the steady state $A = \left(A_0 + \delta A e^{\lambda\tau}\right) e^{i\nu t}$, $G = G_0 + \delta G e^{\lambda\tau}$, and $P_1 = \left(P_0 + \delta P e^{\lambda\tau}\right) e^{i\nu t}$. Substituting into (3) we get

$$\delta P = \delta A \left(e^{-aL\Gamma/[\Gamma+\lambda+i(\Omega+\nu)]} - 1\right),$$

Taking this relation into account we can obtain the following characteristic equation for the eigenvalues $\lambda$ describing the stability of CW solution:

$$a(\lambda)Y(\lambda)^2 + b(\lambda)Y(\lambda) + c(\lambda) = 0,$$

where $Y(\lambda) = e^{-\lambda T}$, and

$$a(\lambda) = -\gamma^2 \kappa e^{G_0 - \frac{2a\Gamma L(\Gamma+\lambda)}{(\Gamma+\lambda)^2+(\nu+\Omega)^2}} \left[\lambda + \gamma_g \left(1 + |A_0|^2 e^{-\frac{2a\Gamma^2 L}{\Gamma^2+(\nu+\Omega)^2}}\right)\right],$$

$$c(\lambda) = -\left[(\gamma+\lambda)^2 + (w-\nu)^2\right] \left[\lambda + \gamma_g \left(1 + |A_0|^2 e^{G_0 - \frac{2a\Gamma^2 L}{\Gamma^2+(\nu+\Omega)^2}}\right)\right],$$

$$b(\lambda) = e^{-\frac{a\Gamma L(\Gamma+\lambda)}{(\Gamma+\lambda)^2+(\Omega+\nu)^2}} \left\{p(\lambda)\cos\left[\Psi + \Theta(\lambda)\right] + q(\lambda)\sin\left[\Psi + \Theta(\lambda)\right]\right\},$$

where

$$p(\lambda) = \gamma e^{G_0/2}\sqrt{\kappa} \left\{2(\gamma_g+\lambda)(\gamma+\lambda) + \gamma_g |A_0|^2 e^{-\frac{2a\Gamma^2 L}{\Gamma^2+(\nu+\Omega)^2}} \left[\left(e^{G_0}+1\right)(\lambda+\gamma) + \alpha\left(e^{G_0}-1\right)(\nu-w)\right]\right\},$$

$$q(\lambda) = \gamma e^{G_0/2}\sqrt{\kappa} \left\{2(\gamma_g+\lambda)(w-\nu) + \gamma_g |A_0|^2 e^{-\frac{2a\Gamma^2 L}{\Gamma^2+(\nu+\Omega)^2}} \left[\left(e^{G_0}+1\right)(w-\nu) + \alpha\left(e^{G_0}-1\right)(\gamma+\lambda)\right]\right\},$$

and

$$\Psi = \frac{\alpha G_0}{2} + \nu T, \quad \Theta(\lambda) = \frac{a\Gamma L(\nu+\Omega)}{(\Gamma+\lambda)^2+(\nu+\Omega)^2}.$$

Finally, in the limit of large delay time $T \to \infty$ we can represent the eigenvalues belonging to the pseudo-continuous spectrum in the form $\lambda = i\mu + \frac{\Lambda}{T} + \mathcal{O}(1/T^2)$ with real $\mu$ [17]. Then, keeping only the single leading term $i\mu$ in $a(\lambda)$,$b(\lambda)$, $c(\lambda)$ and two leading terms $i\mu + \frac{\Lambda}{T}$ in $Y(\lambda)$, we obtain two branches of pseudo-continuous spectrum given by

$$\Lambda_\pm(\mu) + i\mu T = -\ln\left[\frac{-b(i\mu) \pm \sqrt{b(i\mu)^2 - 4a(i\mu)c(i\mu)}}{2a(i\mu)}\right]. \quad (12)$$

Real parts of these eigenvalue branches are shown in Fig. 3, where the modulational instability corresponds to the anomalous dispersion regime.

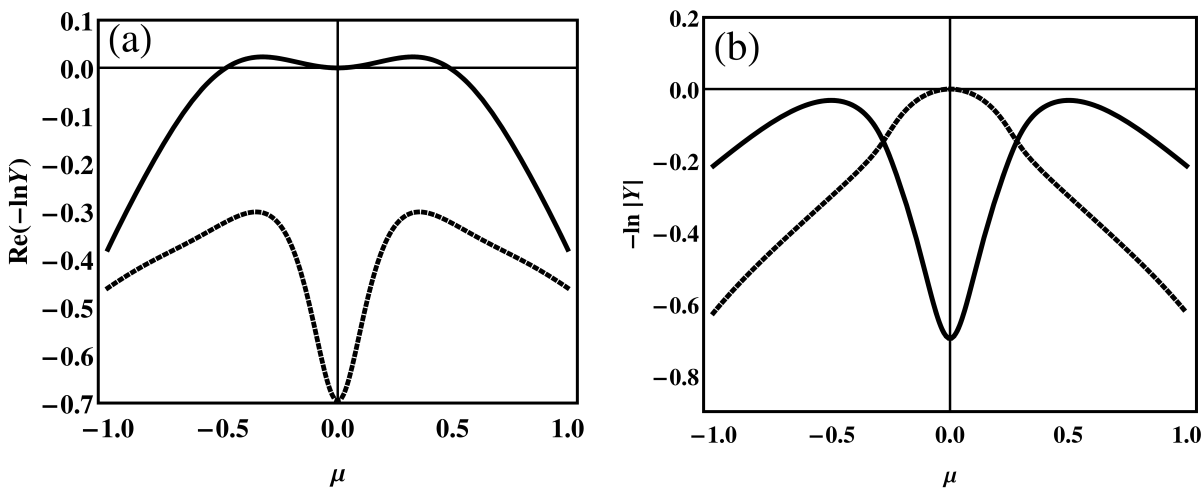


Figure 3: Real parts of the two branches of the pseudo-continuous spectrum $\mathrm{Re}\Lambda_\pm(\mu)$ calculated using Eq. (12) for the case of anomalous, $\Omega = -2$, (a) and normal dispersion, $\Omega = 2$, (b). Other parameter values: $\gamma = 1$, $g_0 = 3.0$, $\alpha = 2$, $\kappa = 0.2$, $\gamma_g = 0.1$, $aL = 80.0$, $w = 0$, $\nu = 0$, and $\Gamma = 0.05$. It is seen that changing the sign of the frequency detuning $\Omega$ leads to a modulational instability of the CW regime.

As it can be seen from Fig. 3, appearance of the modulational instability is associated with the change of the sign of the curvature of one of the two eigenvalue branches at the origin $\mu = 0$, i.e. $y''(0) > 0$, where

$y(\mu) = -Re \ln Y$. In the case $\nu = w = 0$ by assuming $\mu \ll 1$ we can obtain from (12) the expansion of the function $Y(\mu) = 1 + ik_1\mu + k_2\mu^2 + O(\mu^3)$, where $k_1 = \frac{1}{\gamma} + \frac{aL\Gamma(-\Gamma^2-2\alpha\Gamma\Omega+\Omega^2)}{(\Gamma^2+\Omega^2)^2}$ and $k_2$ are real. Therefore, we get the expansion $y(\mu) = k_{thr}\mu^2 + O(\mu^3)$ with $k_{thr} = -(\frac{k_1^2}{2} + k_2)$ satisfying

$$k_{thr} = -\frac{1}{2\gamma^2} - \frac{\alpha D_2}{2} + \frac{aL\Gamma^2(-r_s + 2aL(1+\alpha)^2\Gamma^2\Omega^2 r_u)}{(\Gamma^2+\Omega^2)^4}, \tag{13}$$

where $D_2 = \frac{2aL\Gamma\Omega(-3\Gamma^2+\Omega^2)}{(\Gamma^2+\Omega^2)^3}$ characterizes the second order dispersion at $w = 0$, and

$$r_s = 3\Omega^4 + 2\Gamma^2\Omega^2 - \Gamma^4, \qquad r_u = 1 + \frac{2\kappa}{-\kappa + \exp\frac{aL\Gamma^2}{\Gamma^2+\Omega^2}} + \frac{1}{g_0 + \ln\kappa - 2aL\Gamma^2/(\Gamma^2+\Omega^2)} > 0$$

above the lasing threshold. In the case when $\Omega > \Gamma\sqrt{3}$ we have $\mathrm{sign}\, D_2 = \mathrm{sign}\,\Omega$ and $r_s > 0$. Therefore, the instability condition $y''(\mu) > 0$ implies $k_{thr} > 0$. One can see that the term $r_u$ in (13) destabilizes the CW regime for any type of dispersion (normal or anomalous) in a small vicinity of the lasing threshold where $0 < g_0 + \ln\kappa - 2aL\Gamma^2/(\Gamma^2+\Omega^2) \ll 1$ and $k_{thr} > 0$. For larger $g_0$ the term $r_u$ becomes sufficiently small and the CW regime gains stability ($k_{thr} < 0$) till another modulational instability threshold is reached ($k_{thr} > 0$) and the CW regime is destabilized once again. Since $r_s > 0$, the necessary condition for the modulational instability at $\nu = 0$ becomes

$$\alpha D_2 < -\frac{1}{\gamma^2}, \tag{14}$$

which resembles the modulational instability condition in the complex Ginzburg-Landau equation.

## 4 Conclusion

To conclude, we have developed a DDE model of an FDML laser taking into account the chromatic dispersion of the fiber delay line and studied analytically the stability of CW regimes in this model. We have demonstrated that in the anomalous dispersion regime a CW operation can be destabilized, which is in agreement with the recent experimental results [9]. A standard approach to describe second-order dispersion in fiber and some other types of lasers is based on incorporation of the second order derivative into the Haus-type laser master equation. This approach is not applicable to the case of DDE laser model because of the destabilizing effect of the second order derivative at high frequencies. Instead, we use basic physical intuition to account for the second and higher order dispersion by considering a detuned absorption line, which (depending on the sign of the detuning) gives either anomalous, or normal dispersion. We derive a modification of the DDE FDML laser model [10] satisfying automatically the causality principle and containing a distributed delay term similar to the one introduced in [5]. However, unlike the model discussed in [5], our model provides an explicit analytical description of the linear response function in (3) and, therefore, allows to perform analytical stability analysis of the CW solutions in the limit of large delay time. Apart from FDML lasers the approach discussed in this paper can be applied to study the effect of chromatic dispersion on the characteristics of mode-locked photonic crystal [5] and other types of multimode lasers.

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