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Existence of weak solutions for the Cahn-Hilliard reaction model including elastic effects and damage

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Abstract. In this paper, we introduce and study analytically a vectorial Cahn-Hilliard reaction model coupled with rate-dependent damage processes. The recently proposed Cahn-Hilliard reaction model can e.g. be used to describe the behavior of electrodes of lithium-ion batteries as it includes both the intercalation reactions at the surfaces and the separation into different phases. The coupling with the damage process allows considering simultaneously the evolution of a damage field, a second important physical effect occurring during the charging or discharging of lithium-ion batteries.

Mathematically, this is realized by a Cahn-Larché system with a non-linear Newton boundary condition for the chemical potential and a doubly non-linear differential inclusion for the damage evolution. We show that this system possesses an underlying generalized gradient structure which incorporates the non-linear Newton boundary condition. Using this gradient structure and techniques from the field of convex analysis we are able to prove constructively the existence of weak solutions of the coupled PDE system.

1. Introduction

Lithium-ion batteries belong to the most promising technologies to store energy. They are used as well for small electronic devices as for electric cars or the storage of renewable energies. Due to the increasing demand of such batteries, it is important to develop and study mathematical models in order to understand the charging and discharging process. During the last years it was observed that the classical battery models (like e.g. a shrinking core model) do not predict the right behavior for lithium-ion batteries [1].

As an alternative to the classical models, Singh et al. [29] and Zeng et al. [31] proposed to use an extended phase-field model of Cahn-Hilliard or Cahn-Larché type. Their idea bases on the fact that LiFePO4 has the strong tendency to separate in a lithium rich and a lithium poor phase. The model takes inherently care of the intercalation reactions at the phase boundaries. Singh et al. and Zeng et al. introduced a non-linear Newton boundary condition for the chemical potential reflecting the chemical reactions on the surface using generalized Butler-Volmer kinetics. That model is sometimes called Cahn-Hilliard reaction (CHR) model. The fundamental difference to the classical Cahn-Larché model is the new chemically active boundary condition instead of the classical no-flux condition.

In recent years, the classical Cahn-Larché model describing phase-separation in elastic materials was studied intensively. In particular, the existence, the uniqueness, the regularity and the long-time behavior of solutions were investigated (see [4, 11] and the references therein). Often the main idea for the analysis was to write the equations as an H^{-1}-gradient flow. This bases essentially on the mass conservation of the solution which allows proving a priori estimates. However, in the CHR model the mass will not be conserved in general due to the chemical reactions at the boundary. For this reason, our approach to handle the CHR model analytically is to use a non-quadratic dissipation potential and to introduce a corresponding generalized gradient structure instead of the H^{-1}-gradient structure (see [20, 21] for details on generalized gradient structures). This ansatz naturally takes into account the non-linear Newton boundary condition and circumvents the analytical problem with the non-conserved mass. To the best of our knowledge, this additional structure of the CHR model was not known before.

In [13] and [14], the Cahn-Larché model was expanded to describe also damage of an alloy using a scalar damage variable z. The damage is driven by a rate-dependent dissipation potential. Assuming that the damage process is unidirectional and that the damage variable lies in a fixed interval, i.e. $[0, 1]$, we are confronted with the mathematical challenging task to deal with inequality constraints guaranteeing $\partial_t z \leq 0$ and $z \in [0, 1]$. In the context of lithium-ion batteries, the charging behavior is expected to depend strongly on the damage of the material. For this reason we include such a damage variable in the CHR model [29, 31]. The main objective of this work is to prove the existence of a weak solution of this coupled model. As in [14], we are also able to deal with the physical meaningful gradient term $|\nabla z|^2$ in the damage energy density (see [9]). Hence, we do not need to restrict to $|\nabla z|^p$ ($p >$ space dimension).

During the last years, many damage models based on the model of Frémond and Nedjar [10] describing damage in concrete (see also [9]) were investigated. The models divide basically in two types, the rate-independent [18, 17, 30] and the rate-dependent models [2, 13, 14, 15]. Both types can be coupled with different other equations to describe e.g. different phases [13, 14], inertia [16] or thermal effects [19, 24, 25, 5].

To prove existence of a weak solution of the CHR model we will discretize the equations in time using the implicit Euler scheme. For analytical reason, we add a viscosity term $\nu\partial_t c$ to the chemical potential in the first instance. This will result in additional regularity of the solution which is important to pass to the limit of the Euler discretization. The technique we use to construct a solution can also be used to compute a solution numerically. As far as we know, there is yet only one numerical scheme developed for the CHR model [6] which is based on another discretization. It does not take into account the natural gradient structure of the system.

This paper is divided into five sections. In section 2, we present the model including its viscous approximation. Section 3 deals with the functional replacing the H^{-1}-norm of the usual Cahn-Larché framework. In section 4, we prove the main existence result for the viscous case using an implicit time-discretization before we pass to the vanishing viscosity limit in section 5.

2. Model

In the following, we will present the model in some more details. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$. The time-dependent state of the system is described by a triple (u, c, z) where $u\colon \Omega \to \mathbb{R}^n$ describes the deformation of the elastic material, $c\colon \Omega \to \mathbb{R}^N$ the chemical concentration field and $z\colon \Omega \to [0,1]$ the damage variable. Here, $z = 1$ refers to undamaged material whereas $z = 0$ refers to completely damaged material. To describe the state of the system we use a generalized Ginzburg-Landau free energy functional of the form

$$\tilde{\mathcal{E}}\colon H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega;\mathbb{R}^N) \times H^1(\Omega) \to \mathbb{R}$$

$$\tilde{\mathcal{E}}(u,c,z) := \int_\Omega \frac{1}{2}\mathbf{\Gamma}\nabla c : \nabla c + \frac{1}{2}\mathbf{K}\nabla z \cdot \nabla z + W^{\mathrm{ch}}(c) + W^{\mathrm{el}}(e(u),c,z)\mathrm{d}x.$$

Here, the term W^{ch} accounts for the chemical energy density and W^{el} for the elastically stored energy density. We consider the linearized strain tensor $e(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ since we assume the deformations to be sufficiently small. The first two terms in the functional penalize rapid spatial changes of the concentration and the damage profile.

A typical form of the chemical free energy density is a double well-potential like $W^{\mathrm{ch}}(c) = (1 - |c|^2)^2$. The elastically stored energy density is often modeled as $W^{\mathrm{el}}(e,c,z) = (\Phi(z)+\eta)\,\hat{W}^{\mathrm{el}}(e,c)$ with a continuously differentiable and monotonically increasing function $\Phi : [0,1] \to [0,\infty)$ with $\Phi(0) = 0$ and an elastically stored energy of the undamaged material of the form

$$\hat{W}^{\mathrm{el}}(e,c) = \frac{1}{2}\left(e - e^\star(c)\right) : \mathbb{C}(c)\left(e - e^\star(c)\right).$$

Here, $e^\star(c)$ is the eigenstrain which is usually linear and $\mathbb{C}(c) \in \mathcal{L}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a concentration dependent, symmetric and positive definite fourth order stiffness tensor. The parameter $\eta > 0$ is introduced to ensure that the system is not degenerating (compare assumption (A3) at the end of this section).

As usual, the chemical potential μ is defined as the variational derivative of the energy functional in direction of c, i.e.

$$\mu = -\operatorname{div}(\mathbf{\Gamma}\nabla c) + W^{\mathrm{ch}}_c(c) + W^{\mathrm{el}}_c(e(u),c,z) + \nu\partial_t c.$$

Note that we have added the viscosity term $\nu\partial_t c$ to capture viscous effects. This term can physically be interpreted as a microforce (see [8, 22]). It is also useful for analytical reasons (see proof of Theorem 4). Then, the mass balance in the bulk material leads to the viscous Cahn-Hilliard equation

$$\partial_t c = \Delta\mu. \tag{1}$$

At the surface, chemical reactions imply a flux into the domain. This is described using an Arrhenius type law [29] by a nonlinear Newton boundary condition of the form

$$\nabla\mu\nu_b = R(c,\mu) \tag{2}$$

where ν_b denotes the outer normal on the boundary $\partial\Omega$ and $R(c,\mu)$ the reaction rate. This function may be of the form

$$R(c,\mu) = g\Bigg(c, R_{\text{ins}}\Bigg(1-\exp\left(\frac{\sum_{i=1}^N(\mu_i-\mu_{e,i})}{kT}\right)\Bigg)\Bigg) \text{ with } \mu=(\mu_1,\dots,\mu_N), \mu_e=(\mu_{e,1},\dots,\mu_{e,N}),$$

where $g:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ and $R_{\text{ins}}:\mathbb{R}\to\mathbb{R}^N$. Here, R_{ins} denotes the rate for the insertion rate, μ_e the external potential, k the Boltzmann constant and T the temperature. In our work, we assume a polynomial growth condition for R, see (A8) to (A12). To allow for an exponential growth, we have to modify the function R_{ins} outside some compact interval $[-c_1, c_2]$.

Furthermore, we assume the system to be in the quasi-static equilibrium, i.e.

$$\operatorname{div}(\sigma) = 0$$

where $\sigma = W_e^{\text{el}}(e(u),c,z)$ denotes the stress tensor. The damage process is described by the dissipation potential (compare [13, 14] for more details on the damage process)

$$\tilde{\mathcal{R}}\colon L^2(\Omega)\to\mathbb{R}$$
$$\tilde{\mathcal{R}}(\dot z) := \int_\Omega \rho(\dot z)\mathrm{d}x$$

with $\rho(\dot z) := -\alpha\dot z + \frac{\beta}{2}\dot z^2$ and $\alpha,\beta>0$. We assume the damage process to be irreversible, i.e. $\partial_t z\le 0$. To take into account the irreversibility and the natural constraint $z\ge 0$ we add the corresponding indicator functions to the energy functional and the dissipation potential

$$\mathcal{E}(u,c,z) := \tilde{\mathcal{E}}(u,c,z) + \int_\Omega I_{[0,\infty)}(z)\mathrm{d}x, \quad \mathcal{R}(\dot z) := \tilde{\mathcal{R}}(\dot z) + \int_\Omega I_{(-\infty,0]}(\dot z)\mathrm{d}x$$

where I_A denotes the indicator function with $I_A(x)=0$ if $x\in A$ and $I_A(x)=\infty$ otherwise. Then, the evolution of the damage variable z can be described as a doubly nonlinear differential inclusion

$$0 \in \partial_z^{\text{Cl}}\mathcal{E}(u,c,z) + \partial_{\dot z}\mathcal{R}(\partial_t z).$$

Due to the constraints on z and $\partial_t z$ the functionals are non-smooth and we need to use the (Clarke) subdifferential.

Remark that the CHR model without damage process is a special case of the presented model for $W^{\text{el}}(e,c,z) = W^{\text{el}}(e,c)$ and $z(0)=0$. Thus, all presented results also hold for the usual CHR model.

Let us summarize the complete model with initial and boundary conditions:

$$\begin{aligned}
\partial_t c &= \Delta\mu && \text{in } \Omega_T,\\
\mu &= -\operatorname{div}(\mathbf{\Gamma}\nabla c) + W_c^{\mathrm{ch}}(c) + W_c^{\mathrm{el}}(e(u),c,z) + \nu\partial_t c && \text{in } \Omega_T,\\
\operatorname{div}(W_e^{\mathrm{el}}(e(u),c,z)) &= 0 && \text{in } \Omega_T\\
0 &\in \partial_z^{\mathrm{Cl}}\mathcal{E}(u(t),c(t),z(t)) + \partial_{\dot z}\mathcal{R}(\dot z(t)) && \text{in } \Omega_T,\\
\nabla c\,\nu_b &= 0 && \text{on } \Gamma,\\
\nabla\mu\,\nu_b &= R(c,\mu) && \text{on } \Gamma,\\
\sigma\,\nu_b &= 0 && \text{on } \Gamma,\\
\nabla z\cdot\nu_b &= 0 && \text{on } \Gamma,\\
c(0) &= c_0 && \text{in } \Omega,\\
z(0) &= z_0 && \text{in } \Omega.
\end{aligned}$$

To prove existence of weak solutions, we first add a regularization term in the energy. Therefore, we introduce the regularized energy for $\varepsilon > 0$

$$\mathcal{E}_\varepsilon\colon W^{1,p}(\Omega;\mathbb{R}^n)\times H^1(\Omega;\mathbb{R}^N)\times W^{1,p}(\Omega)\to\mathbb{R}\cup\{\infty\}$$

$$\mathcal{E}_\varepsilon(u,c,z) = \mathcal{E}(u,c,z) + \frac{\varepsilon}{p}\int_\Omega |e(u)|^p + |\nabla z|^p \mathrm{d}x$$

for some $p > n$. Later, we will also use the regularized energy without the constraint on z, i.e.

$$\tilde{\mathcal{E}}_\varepsilon(u,c,z) \coloneqq \tilde{\mathcal{E}}(u,c,z) + \frac{\varepsilon}{p}\int_\Omega |e(u)|^p + |\nabla z|^p \mathrm{d}x.$$

The corresponding regularized equations read

$$\begin{aligned}
\partial_t c &= \Delta\mu && \text{in } \Omega_T,\\
\mu &= -\operatorname{div}(\mathbf{\Gamma}\nabla c) + W_c^{\mathrm{ch}}(c) + W_c^{\mathrm{el}}(e(u),c,z) + \nu\partial_t c && \text{in } \Omega_T,\\
\operatorname{div}(\sigma) + \varepsilon\operatorname{div}(|e(u)|^{p-2}e(u)) &= 0 && \text{in } \Omega_T,\\
0 &\in \partial_z^{\mathrm{Cl}}\mathcal{E}_\varepsilon(u(t),c(t),z(t)) + \partial_{\dot z}\mathcal{R}(\dot z(t)) && \text{in } \Omega_T,\\
\nabla c\,\nu_b &= 0 && \text{on } \Gamma,\\
\nabla\mu\,\nu_b &= R(c,\mu) && \text{on } \Gamma,\\
(\sigma + \varepsilon|e(u)|^{p-2}e(u))\,\nu_b &= 0 && \text{on } \Gamma,\\
\nabla z\cdot\nu_b &= 0 && \text{on } \Gamma,\\
c(0) &= c_0 && \text{in } \Omega,\\
z(0) &= z_0 && \text{in } \Omega.
\end{aligned}$$

To simplify the notation we will assume $\mathbf{\Gamma} = \mathbf{K} = \mathrm{Id}$. The presented results can easily be adapted to any positive definite diagonal matrices.

Note that all presented results can be proven similarly prescribing a Dirichlet boundary condition $u = b$ on $\Gamma_D\times[0,T]$ where $\Gamma_D\subset\Gamma$ is a part of the boundary with $\mathcal{H}^{n-1}(\Gamma_D) > 0$. Then, the Neumann boundary condition $(\sigma + \varepsilon|e(u)|^{p-2}e(u))\,\nu_b = 0$ is only necessary on the remaining part $(\Gamma\setminus\Gamma_D)\times[0,T]$ of the boundary.

To conclude this section, we list all assumptions on the involved functions which we need in order to prove our results. Let $2^\star$ denote the critical Sobolev exponent and

$$2^\# = \begin{cases} \frac{2n-2}{n-2} & \text{for } n > 2\\ \text{arbitrary} & \text{for } n = 2\\ \infty & \text{for } n = 1\end{cases}$$

the critical exponent for the trace operator to be continuous as an operator from $H^1(\Omega;\mathbb{R}^N)$ to $L^{2^\#}(\Gamma;\mathbb{R}^N)$ (see e.g. [26]). The Euclidean norm is always denoted by $|\cdot|$ whereas the 1-norm in $\mathbb{R}^N$ is written as $|\cdot|_1$. The unit vectors of the standard basis of $\mathbb{R}^n$ are written as e^k. During the whole work, $C > 0$ and $C_1 > 0$ denote constants which may vary from line to line.

We assume that there exist constants $\eta, \delta, C, C_1 > 0$ with $\delta < \min\left(\frac{2^\star}{2}, 2^\# - 1\right)$ such that the following (in-)equalities hold for all $e, e_1, e_2 \in \mathbb{R}^{n\times n}$, $c, \mu, \mu_1, \mu_2 \in \mathbb{R}^N$, and $z \in [0,1]$.

(i) Elastically stored energy density:

$$
\begin{aligned}
& W^{\mathrm{el}} \in C^1\left(\mathbb{R}^{n\times n} \times \mathbb{R}^N \times \mathbb{R}; [0,\infty)\right) \text{ with} \\
& W^{\mathrm{el}}(e,c,z) = W^{\mathrm{el}}(e^T,c,z), && \text{(A1)}\\
& W^{\mathrm{el}}(e,c,z) \le C(|e|^2 + |c|^2 + 1), && \text{(A2)}\\
& \eta|e_1 - e_2|^2 \le \left(W_e^{\mathrm{el}}(e_1,c,z) - W^{\mathrm{el}}(e_2,c,z)\right) : (e_1 - e_2), && \text{(A3)}\\
& |W_e^{\mathrm{el}}(e_1 + e_2, c, z)| \le C(W_e^{\mathrm{el}}(e_1,c,z) + |e_2| + 1), && \text{(A4)}\\
& |W_c^{\mathrm{el}}(e,c,z)| \le C(|e|^2 + |c|^2 + 1), && \text{(A5)}\\
& |W_c^{\mathrm{el}}(e,c,z)| \le C(|e| + |c|^2 + 1), && \text{(A5')}\\
& |W_z^{\mathrm{el}}(e,c,z)| \le C(|e|^2 + |c|^2 + 1). && \text{(A6)}
\end{aligned}
$$

(ii) Chemical energy density:

$$
\begin{aligned}
& W^{\mathrm{ch}} \in C^1(\mathbb{R};\mathbb{R}^N) \text{ with } W^{\mathrm{ch}}(c) > -C, \\
& |W_c^{\mathrm{ch}}(c)| \le C(|c|^{\frac{2^\star}{2}-\delta} + 1) && \text{(A7)}
\end{aligned}
$$

(iii) Reaction rate:
Let $G \in C^1(\mathbb{R}^N \times \mathbb{R}^N)$ with the following properties:

- The reaction rate $R(c,\mu) := G_\mu(c,\mu)$ is strictly decreasing, uniformly in c, i.e.

$$\Big(R(c,\mu_1) - R(c,\mu_2)\Big) \cdot (\mu_1 - \mu_2) \le -C_1|\mu_1 - \mu_2|^2 \tag{A8}$$

  and

$$|R(c, \pm e^k)| \le C \tag{A9}$$

  for $k = 1, \dots, N$.
- The growth condition

$$|R(c,\mu)| \le C\left(1 + |c|^{2^\# - \delta - 1} + |\mu|^{2^\# - \delta - 1}\right), \tag{A10}$$

  is satisfied.

Note that the monotonicity (A8) and the boundedness (A9) implies $|R(c,0)| \le C$ and thus, by Young's inequality we can extract from (A8) the estimate

$$-\mu \cdot R(c,\mu) \ge C\left(|\mu|^2 - C_1\right). \tag{A11}$$

In addition, due the growth condition (A10) the following estimate is fulfilled:

$$|G(c,\mu)| \le C\left(1 + |c|^{2^\# - \delta} + |\mu|^{2^\# - \delta}\right). \tag{A12}$$

3. Legendre-Fenchel transform and subdifferentials

In this section, we will collect some basic facts about the Legendre-Fenchel transform and the subdifferential. We will apply these results to a special functional which is important to generate the generalized gradient flow structure of the equations.

We define the proper, partly convex, and lower semi-continuous functional

$$\mathcal{A}\colon L^{2^{\#}-\delta}(\Gamma;\mathbb{R}^N)\times L^2(\Omega;\mathbb{R}^N)\to\mathbb{R}\cup\{\infty\}$$
$$\mathcal{A}(c,v)=\begin{cases}\frac{1}{2}(\nabla v,\nabla v)_{L^2}-\int_\Gamma G(c,v)\mathrm{d}\omega & \text{if } v\in H^1(\Omega;\mathbb{R}^N)\\ \infty & \text{otherwise.}\end{cases} \tag{3}$$

We observe that the boundary integral $\int_\Gamma G(c,v)\mathrm{d}\omega$ is finite for $v\in H^1(\Omega;\mathbb{R}^N)$ and $c\in L^{2^{\#}-\delta}(\Gamma;\mathbb{R}^N)$ due to the assumption (A12) on G.

Furthermore, we introduce the continuous operator

$$\begin{aligned}\mathcal{B}\colon L^{2^{\#}-\delta}(\Gamma;\mathbb{R}^N)\times H^1(\Omega;\mathbb{R}^N)&\to H^1(\Omega;\mathbb{R}^N)'\\ \langle\mathcal{B}(c,u),v\rangle=\int_\Omega\nabla u:\nabla v\mathrm{d}x&-\int_\Gamma R(c,u)\cdot v\,d\omega.\end{aligned} \tag{4}$$

For fixed $c\in L^{2^{\#}-\delta}(\Gamma;\mathbb{R}^N)$, the operator $\mathcal{B}_c\coloneqq\mathcal{B}(c,\cdot)$ is strictly monotone, bounded, and coercive due to the assumptions on R. Here, the coercivity can be proven using assumption (A11) to obtain

$$\begin{aligned}\langle\mathcal{B}(c,u),u\rangle&=\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^N)}-\int_\Gamma R(c,u)\cdot u\,d\omega\\ &\geq\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^N)}+C\left(\|u\|^2_{L^2(\Gamma;\mathbb{R}^N)}-C_1\mathcal{H}^{n-1}(\Gamma)\right)\end{aligned} \tag{5}$$

for any $u\in H^1(\Omega;\mathbb{R}^N)$. Then, a special variant of Poincaré's inequality, namely

$$\|u\|_{H^1(\Omega;\mathbb{R}^N)}\leq C\left(\|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)}+\|u\|_{L^q(\Gamma;\mathbb{R}^N)}\right) \tag{6}$$

for any $1\leq q\leq 2^{\#}$ (see e.g. Theorem 1.32 of [26]) implies the coercivity. Altogether, there exists a strictly monotone, bounded, and demicontinuous inverse operator $\mathcal{B}_c^{-1}\colon H^1(\Omega;\mathbb{R}^N)'\to H^1(\Omega;\mathbb{R}^N)$ (see e.g. Theorem 2.14 of [26]).

Lemma 1. *The operator $\bar{\mathcal{B}}\colon L^{2^{\#}-\delta}(\Gamma;\mathbb{R}^N)\times H^1(\Omega;\mathbb{R}^N)'\to H^1(\Omega;\mathbb{R}^N)$,*

$$\bar{\mathcal{B}}(c,v^\star)=\mathcal{B}_c^{-1}(v^\star)$$

is bounded and continuous.

Proof. The boundedness can directly be seen from estimate (5) and Poincaré's inequality (6) since we have for $u=\bar{\mathcal{B}}(c,v^\star)$

$$\begin{aligned}\|\bar{\mathcal{B}}(c,v^\star)\|^2_{H^1(\Omega;\mathbb{R}^N)}&\leq C\left(\|\bar{\mathcal{B}}(c,v^\star)\|^2_{L^2(\Gamma;\mathbb{R}^N)}+\|\nabla\bar{\mathcal{B}}(c,v^\star)\|^2_{L^2(\Omega;\mathbb{R}^N)}\right)\\ &\leq C\left(\langle\mathcal{B}(c,\bar{\mathcal{B}}(c,v^\star)),\bar{\mathcal{B}}(c,v^\star)\rangle+1\right)\\ &\leq C\left(\|v^\star\|_{H^1(\Omega;\mathbb{R}^N)'}\|\bar{\mathcal{B}}(c,v^\star)\|_{H^1(\Omega;\mathbb{R}^N)}+1\right)\end{aligned}$$

and thus

$$\|\bar{\mathcal{B}}(c,v^\star)\|_{H^1(\Omega;\mathbb{R}^N)}\leq C(\|v^\star\|_{H^1(\Omega;\mathbb{R}^N)'}+1). \tag{7}$$

In order to show the continuity, let $\{(c_n,v_n^\star)\}\subset L^{2^{\#}-\delta}(\Gamma;\mathbb{R}^N)\times H^1(\Omega;\mathbb{R}^N)'$ be a norm converging sequence with limit $(c,v^\star)$. Then, the sequence $\bar{\mathcal{B}}(c_n,v_n^\star)$ is bounded in $H^1(\Omega;\mathbb{R}^N)$ and there exists a weakly convergent subsequence $v_{n_k}\coloneqq\bar{\mathcal{B}}(c_{n_k},v_{n_k}^\star)\rightharpoonup v$ in $H^1(\Omega;\mathbb{R}^N)$. Due to the monotonicity of $\mathcal{B}_c$ and the continuity of $\mathcal{B}$, it follows for any $w\in H^1(\Omega;\mathbb{R}^N)$

$$\begin{aligned}0&\leq\langle\mathcal{B}(c_{n_k},v_{n_k})-\mathcal{B}(c_{n_k},w),v_{n_k}-w\rangle\\ &=\langle v_{n_k}^\star-\mathcal{B}(c,w),v_{n_k}-w\rangle+\langle\underbrace{\mathcal{B}(c,w)-\mathcal{B}(c_{n_k},w)}_{\to 0\text{ in }H^1(\Omega;\mathbb{R}^N)'},v_{n_k}-w\rangle\\ &\to\langle v^\star-\mathcal{B}(c,w),v-w\rangle.\end{aligned}$$

Then, Minty's trick (see e.g. Lemma 2.13 of [26]) yields $v^\star=\mathcal{B}(c,v)$ which is equivalent to $v=\bar{\mathcal{B}}(c,v^\star)$. By contradiction we see that in fact the whole sequence $\{v_n\}$ is weakly converging to v and it only remains to show the strong convergence of v_n in $H^1(\Omega;\mathbb{R}^N)$. The compactness of the trace operator from $H^1(\Omega;\mathbb{R}^N)$ to $L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)$ implies the strong convergence of v_n in $L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)$. The growth condition (A12) yields the strong convergence of $R(c_n,v_n)$ in $L^{(2^\#-\delta)'}(\Gamma;\mathbb{R}^N)$. Using the definition of $\bar{\mathcal{B}}$ and (4), we observe

$$\langle \bar{v}^\star, \bar{\mathcal{B}}(c,\bar{v}^\star)\rangle = \langle \mathcal{B}(c,\bar{\mathcal{B}}(c,\bar{v}^\star)), \bar{\mathcal{B}}(c,\bar{v}^\star)\rangle = (\nabla\bar{\mathcal{B}}(c,\bar{v}^\star), \nabla\bar{\mathcal{B}}(c,\bar{v}^\star))_{L^2} - \int_\Gamma R(c,\bar{\mathcal{B}}(c,\bar{v}^\star))\cdot\bar{\mathcal{B}}(c,\bar{v}^\star)\,d\omega \tag{8}$$

for any $\bar{v}^\star \in H^1(\Omega;\mathbb{R}^N)'$. Thus, we conclude

$$\|\nabla\bar{\mathcal{B}}(c_n,v_n^\star)\|_{L^2}^2 = \langle v_n^\star, v_n\rangle + \int_\Gamma R(c_n,v_n)\cdot v_n\,d\omega \to \langle v^\star, v\rangle + \int_\Gamma R(c,v)\cdot v\,d\omega = \|\nabla\bar{\mathcal{B}}(c,v^\star)\|_{L^2}^2$$

which shows the strong convergence of $\bar{\mathcal{B}}(c_n,v_n^\star)$ in $H^1(\Omega;\mathbb{R}^N)$. Hence, the continuity of $\bar{\mathcal{B}}$ is proven. □

Note that from (7) and continuity of the trace operator we obtain

$$\|\bar{\mathcal{B}}(c,0)\|_{L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)} \le C\|\bar{\mathcal{B}}(c,0)\|_{H^1(\Omega;\mathbb{R}^N)} \le C. \tag{9}$$

Next, we will study the relation between $\mathcal{A}$ and $\mathcal{B}$. For the proper, convex, and lower semicontinuous functional $\mathcal{A}_c := \mathcal{A}(c,\cdot)$, the following relationship is well-known for $v \in \mathrm{dom}(\mathcal{A}_c)$ (see e.g. [23])

$$v^\star \in \partial\mathcal{A}_c(v) \Leftrightarrow \langle v^\star, w\rangle_{L^2(\Omega;\mathbb{R}^N)} \le \mathrm{d}^+\mathcal{A}_c(v;w) \text{ for all } w \in L^2(\Omega;\mathbb{R}^N) \tag{10}$$

where d^+ denotes the right-hand directional derivative. For $v,w \in H^1(\Omega;\mathbb{R}^N)$ the directional derivative of operator $\mathcal{A}_c$ at v in the direction w exists and it holds $\mathrm{d}\mathcal{A}_c(v;w) = \langle \mathcal{B}_c(v), w\rangle$. Thus, the relationship (10) turns into

$$v^\star \in \partial\mathcal{A}_c(v) \Leftrightarrow \langle v^\star, w\rangle_{L^2(\Omega;\mathbb{R}^N)} = \mathrm{d}\mathcal{A}_c(v;w) = \langle \mathcal{B}(c,v), w\rangle_{H^1(\Omega;\mathbb{R}^N)'} \text{ for all } w \in H^1(\Omega;\mathbb{R}^N). \tag{11}$$

Hence, we observe that the weak formulation of (1) and (2) is equivalent to $-\partial_t c(t) \in \partial\mathcal{A}_{c(t)}(\mu(t))$ for almost all $t \in (0,T)$.

Since we want to use variational methods in the following, the (partial) Legendre-Fenchel transform will occur naturally. The transform is defined as

$$\begin{aligned} &\mathcal{A}^\star\colon L^{2^\#-\delta}(\Gamma;\mathbb{R}^N) \times L^2(\Omega;\mathbb{R}^N) \to \mathbb{R}\cup\{\infty\} \\ &\qquad \mathcal{A}^\star(c,v^\star) = \mathcal{A}_c^\star(v^\star) = \sup_{v\in L^2(\Omega;\mathbb{R}^N)} \left\{(v^\star,v)_{L^2(\Omega;\mathbb{R}^N)} - \mathcal{A}_c(v)\right\} \\ &\qquad\qquad = \sup_{v\in H^1(\Omega;\mathbb{R}^N)} \left\{(v^\star,v)_{L^2(\Omega;\mathbb{R}^N)} - \mathcal{A}_c(v)\right\}. \end{aligned}$$

Since $\mathcal{A}_c$ is proper, convex, and lower semi-continuous and since $L^2(\Omega;\mathbb{R}^N)$ is reflexive, we obtain the equivalence (see e.g. Proposition 4.4.4 of [27])

$$\mathcal{A}_c(v) < \infty,\ v^\star \in \partial\mathcal{A}_c(v) \Leftrightarrow \mathcal{A}_c^\star(v^\star) < \infty,\ v \in \partial\mathcal{A}_c^\star(v^\star). \tag{12}$$

As a next step, we will examine the map $\mathcal{A}_c^\star$ in more detail.

Let $(c,v^\star) \in L^{2^\#-\delta}(\Gamma;\mathbb{R}^N) \times L^2(\Omega;\mathbb{R}^N)$ be fixed. The necessary condition for $v \in H^1(\Omega;\mathbb{R}^N)$ to be a maximizer of the strict concave functional $(v^\star,v)_{L^2(\Omega;\mathbb{R}^N)} - \mathcal{A}_c(v)$ is

$$\langle \mathcal{B}_c(v), \xi\rangle_{H^1(\Omega;\mathbb{R}^N)'} = (v^\star,\xi)_{L^2} \quad \text{for all } \xi \in H^1(\Omega;\mathbb{R}^N).$$

Since $\mathcal{B}_c$ is invertible, the unique solution of this equation is given by $v = \bar{\mathcal{B}}(c,v^\star)$ and since the functional is strict concave, v must be the unique global maximizer. Thus, it is

$$\begin{aligned} \mathcal{A}^\star(c,v^\star) &= (v^\star, \bar{\mathcal{B}}(c,v^\star))_{L^2} - \mathcal{A}(c,\bar{\mathcal{B}}(c,v^\star)) \\ &= (v^\star, \bar{\mathcal{B}}(c,v^\star))_{L^2} - \frac{1}{2}(\nabla\bar{\mathcal{B}}(c,v^\star), \nabla\bar{\mathcal{B}}(c,v^\star))_{L^2} + \int_\Gamma G(c,\bar{\mathcal{B}}(c,v^\star))\mathrm{d}\omega. \end{aligned}$$

Using the equation (8), this can be reformulated as

$$\mathcal{A}^\star(c,v^\star) = \frac{1}{2}(\bar{\mathcal{B}}(c,v^\star),v^\star)_{L^2} + \int_\Gamma \left(G(c,\bar{\mathcal{B}}(c,v^\star)) - \frac{1}{2}R(c,\bar{\mathcal{B}}(c,v^\star))\cdot\bar{\mathcal{B}}(c,v^\star)\right) \mathrm{d}\omega$$

and

$$\mathcal{A}^\star(c,v^\star) = \frac{1}{2}(\nabla\bar{\mathcal{B}}(c,v^\star),\nabla\bar{\mathcal{B}}(c,v^\star))_{L^2} + \int_\Gamma \big(G(c,\bar{\mathcal{B}}(c,v^\star)) - R(c,\bar{\mathcal{B}}(c,v^\star))\cdot\bar{\mathcal{B}}(c,v^\star)\big) \mathrm{d}\omega. \quad (13)$$

In particular, it is $\mathcal{A}_c^\star(v^\star) < \infty$ for every $v^\star \in L^2(\Omega;\mathbb{R}^N)$. Hence, due to the equivalences in (12) and in (11), $v \in \partial\mathcal{A}_c^\star(v^\star)$ implies $v \in H^1(\Omega;\mathbb{R}^N)$ and $v = \bar{\mathcal{B}}(c,v^\star)$. In particular, $\partial\mathcal{A}_c^\star(v^\star)$ is a singleton.

Furthermore, we see that $\mathcal{A}^\star$ is continuous as $\bar{\mathcal{B}}$ in $H^1(\Omega;\mathbb{R}^N)$ and $\mathcal{A}|_{L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)\times H^1(\Omega;\mathbb{R}^N)}$ are continuous.

Since $\mathcal{A}_c^\star$ is convex on $L^2(\Omega;\mathbb{R}^N)$ and since $\partial\mathcal{A}_c^\star(v^\star)$ is a singleton, $\mathcal{A}_c^\star$ is Gateaux differentiable (see e.g. Corollary 4.2.5 of [3]) in the sense that the right-hand directional derivative is linear and continuous. Obviously, then the restriction $\mathcal{A}_c^\star|_{H^1(\Omega;\mathbb{R}^N)}$ has also a linear and continuous right-hand directional derivative and thus, for each $v^\star \in H^1(\Omega;\mathbb{R}^N)$ the subdifferential $\partial\mathcal{A}_c^\star|_{H^1(\Omega;\mathbb{R}^N)}(v^\star) \subset H^1(\Omega;\mathbb{R}^N)'$ is also a singleton. To be precise, it is

$$\partial\mathcal{A}_c^\star|_{H^1(\Omega;\mathbb{R}^N)}(v^\star) = \left\{\bar{\mathcal{B}}(c,v^\star)\right\}. \quad (14)$$

To conclude the analysis of the functional $\mathcal{A}^\star$, we prove some estimates of the Legendre-Fenchel transform.

Lemma 2. *The functional $\mathcal{A}_c^\star(v^\star) + \mathcal{A}_c(\bar{\mathcal{B}}(c,0))$ is uniformly bounded from below, i.e.*

$$0 \le \frac{1}{n}\Big|\int_\Omega v^\star \mathrm{d}x\Big|_1 \le \mathcal{A}_c^\star(v^\star) + \mathcal{A}_c(\bar{\mathcal{B}}(c,0)) + C \quad (15)$$

for any $c \in L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)$ and any $v^\star \in L^2(\Omega;\mathbb{R}^N)$.

Proof. From (A8) we infer

$$R(c,t(\mu\mp e^k)\pm e^k)\cdot t(\mu\mp e^k) \le R(c,\pm e^k)\cdot t(\mu\mp e^k) - C_1 t^2|\mu\mp e^k|^2$$

for any standard basis vector $e^k \in \mathbb{R}^N$. In consequence, it follows with (A9)

$$\int_0^1 R(c,t(\mu\mp e^k)\pm e^k)\cdot(\mu\mp e^k)\,\mathrm{d}t \le \int_0^1 R(c,\pm e^k)\cdot(\mu\mp e^k)\,\mathrm{d}t \le C|\mu\mp e^k|.$$

For fixed $c \in \mathbb{R}^N$ and any $\mu \in \mathbb{R}^N$, we estimate

$$\begin{aligned} G(c,\pm e^k) - G(c,\mu) &= -\int_0^1 R(c,t(\mu\mp e^k)\pm e^k)\cdot(\mu\mp e^k)\,\mathrm{d}t \\ &\ge -C|\mu\mp e^k|. \end{aligned}$$

Thus, for any $c \in L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)$ and $v^\star \in L^2(\Omega;\mathbb{R}^N)$, we obtain due to (9)

$$\begin{aligned} \mathcal{A}_c^\star(v^\star) &\ge (v^\star,\pm e^k)_{L^2} - \mathcal{A}_c(\pm e^k) = \pm\int_\Omega v^\star\cdot e^k \mathrm{d}x + \int_\Gamma G(c,\pm e^k)\mathrm{d}\omega \\ &\ge \pm\int_\Omega v^\star\cdot e^k\mathrm{d}x + \int_\Gamma \big(G(c,\bar{\mathcal{B}}(c,0)) - C|\bar{\mathcal{B}}(c,0)\mp e^k|\big)\,\mathrm{d}\omega \\ &\ge \pm\int_\Omega v^\star\cdot e^k\mathrm{d}x - A_c(\bar{\mathcal{B}}(c,0)) - C. \end{aligned}$$

Hence, it follows

$$0 \le \frac{1}{n}\Big|\int_\Omega v^\star\mathrm{d}x\Big|_1 \le \mathcal{A}_c^\star(v^\star) + \mathcal{A}_c(\bar{\mathcal{B}}(c,0)) + C.$$

In particular, $\mathcal{A}_c^\star(v^\star) + \mathcal{A}_c(\bar{\mathcal{B}}(c,0))$ is uniformly bounded from below. ◻

4. Viscous Cahn-Hilliard reaction equations

As in [11], we will consider boundary conditions of the second type for u and thus u can only be determined uniquely up to translations and infinitesimal rotations. Therefore, we introduce the space of infinitesimal rigid displacements

$$\begin{aligned} X_{\mathrm{ird},p} \coloneqq \Big\{ & u \in W^{1,p}(\Omega;\mathbb{R}^n) | \text{there exists } b \in \mathbb{R}^n \text{ and a skew symmetric matrix} \\ & \mathbf{A} \in \mathbb{R}^{n\times n} \text{ with } u(x) = \mathbf{A}x + b \Big\} \end{aligned}$$

which is known to be the null space of the symmetrized gradient $e(v)$, i.e. of the operator

$$\begin{aligned} e\colon W^{1,p}(\Omega;\mathbb{R}^n) &\to L^p(\Omega;\mathbb{R}^{n\times n}) \\ v &\mapsto \frac{1}{2}\left(\nabla v + \nabla v^T\right). \end{aligned}$$

We define the quotient space $\dot W^{1,p}(\Omega;\mathbb{R}^n) \coloneqq W^{1,p}(\Omega;\mathbb{R}^n)/X_{\mathrm{ird},p}$ and set $e([v]) \coloneqq e(v)$ for $[v] \in \dot W^{1,p}(\Omega;\mathbb{R}^n)$ and any $v \in [v]$. This is possible since $e(v)$ does not depend on the choice of the representative of $[v]$. For $p = 2$ we denote the quotient space as usual by $\dot H^1(\Omega;\mathbb{R}^n) \coloneqq \dot W^{1,2}(\Omega;\mathbb{R}^n)$. Since $H^1(\Omega;\mathbb{R}^n)$ is a Hilbert space, the quotient space is isomorphic to the orthogonal complement

$$\dot H^1(\Omega;\mathbb{R}^n) \cong X_{\mathrm{ird},2}^{\perp} \coloneqq \left\{ u \in H^1(\Omega;\mathbb{R}^n) | (u,v)_{H^1} = 0 \text{ for all } v \in X_{\mathrm{ird},2} \right\}.$$

In order to simplify the notation, let us introduce some more spaces

$$\begin{aligned} \mathcal{C} &= L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^N)) \\ \mathcal{U}_\varepsilon &= L^\infty(0,T;\dot W^{1,p}(\Omega;\mathbb{R}^n)) \\ \mathcal{U}_0 &= L^\infty(0,T;\dot H^1(\Omega;\mathbb{R}^n)) \\ \mathcal{Z}_\varepsilon &= L^\infty(0,T;W^{1,p}_+(\Omega)) \cap H^1(0,T;L^2(\Omega)) \\ \mathcal{Z}_0 &= L^\infty(0,T;H^1_+(\Omega)) \cap H^1(0,T;L^2(\Omega)) \\ \mathcal{M} &= L^2(0,T;H^1(\Omega;\mathbb{R}^N)) \end{aligned}$$

for $\varepsilon > 0$. Furthermore, let us write $p_\varepsilon = \begin{cases} p & \text{for } \varepsilon > 0 \\ 2 & \text{for } \varepsilon = 0. \end{cases}$ These notations allow us to define a solution for $\varepsilon > 0$ and $\varepsilon = 0$ in a unified way.

Definition 3. Let be $\varepsilon \geq 0$ and $\nu > 0$. A tuple $q = (u, c, \mu, z) \in \mathcal{U}_\varepsilon \times \mathcal{C} \times \mathcal{M} \times \mathcal{Z}_\varepsilon$ is called a weak solution of the viscous Cahn-Hilliard reaction equations with elasticity and damage if the following properties are satisfied:

(i) The initial conditions are fulfilled, i.e. $c(0) = c_0$ and $z(0) = z_0$ and the damage process is irreversible, i.e. $\partial_t z \leq 0$.

(ii) It holds
$$-(\partial_t c(t), \xi)_{L^2} = \int_\Omega \nabla\mu(t) : \nabla\xi \mathrm{d}x - \int_\Gamma R(c(t),\mu(t)) \cdot \xi \mathrm{d}\omega \tag{16}$$
for all $\xi \in H^1(\Omega;\mathbb{R}^N)$ and almost all $t \in (0,T)$.

(iii) It holds
$$(\mu(t),\xi)_{L^2} = \int_\Omega \nabla c(t) : \nabla\xi + \left(W^{\mathrm{ch}}_c(c(t)) + W^{\mathrm{el}}_c(e(u(t)), c(t), z(t)) + \nu\partial_t c(t)\right) \cdot \xi \mathrm{d}x \tag{17}$$
for all $\xi \in H^1(\Omega;\mathbb{R}^N)$ and for almost all $t \in (0,T)$.

(iv) It holds

$$\int_\Omega W_e^{\mathrm{el}}(e(u(t)),c(t),z(t)):e(\xi)+\varepsilon|e(u(t))|^{p-2}e(u(t)):e(\xi)\mathrm{d}x=0 \tag{18}$$

for all $\xi\in W^{1,p_\varepsilon}(\Omega;\mathbb{R}^n)$ and for almost all $t\in[0,T]$.

(v) There exists a function $r\in L^1\left(0,T;\left(W^{1,p_\varepsilon}(\Omega)\cap L^\infty(\Omega)\right)'\right)$ such that it holds

$$\begin{aligned}&\int_\Omega\left(\varepsilon|\nabla z(t)|^{p-2}+1\right)\nabla z(t)\cdot\nabla\xi+\left(W_z^{\mathrm{el}}(e(u(t)),c(t),z(t))-\alpha+\beta\partial_t z(t)\right)\xi\mathrm{d}x\\&\geq-\langle r(t),\xi\rangle\end{aligned} \tag{19}$$

for all $\xi\in W_-^{1,p_\varepsilon}(\Omega)\cap L^\infty(\Omega)$ and almost all $t\in(0,T)$ and

$$\langle r(t),\xi-z(t)\rangle\leq 0$$

for all $\xi\in W_+^{1,p_\varepsilon}(\Omega)\cap L^\infty(\Omega)$ and almost all $t\in(0,T)$.

(vi) The energy inequality

$$\begin{aligned}&\mathcal{E}_\varepsilon(u(t),c(t),z(t))\\&\quad+\int_0^t-\alpha\partial_t z(s)+\beta|\partial_t z(s)|^2+\mathcal{A}(c(s),\mu(s))+\mathcal{A}^\star(c(s),-\partial_t c(s))+\nu\|\partial_t c(s)\|^2_{L^2(\Omega)}\mathrm{d}s\\&\leq\mathcal{E}_\varepsilon(u_0,c_0,z_0)\end{aligned} \tag{20}$$

is fulfilled for almost all $t\in[0,T]$ and all $u_0\in\dot{W}^{1,p_\varepsilon}(\Omega;\mathbb{R}^n)$.

Let us give some comments on the solution concept for the damage equation using the energy inequality. It was introduced in [13] since (until now) it seems to be impossible to show $\partial_t z(t)\in W^{1,p_\varepsilon}(\Omega)$ which would be necessary in order to understand the differential inclusion in a classical way (compare the comments on this concept in [18] and the original derivation in [13]). If the solution has this additional regularity, the above defined notation of a solution is equivalent to the usual concept of the differential inclusion.

The rest of this paper is concerned with the construction of such a solution either in the case $\varepsilon>0$ or $\varepsilon=0$.

Theorem 4 (Existence for $\varepsilon>0$). *Let the assumptions* (A1)-(A10) *be true. For each $\varepsilon>0$, $\nu>0$ and each $c_0\in H^1(\Omega;\mathbb{R}^N)$ and $z_0\in W^{1,p}(\Omega)$ with $0\leq z_0\leq 1$, there exists a weak solution q as defined in Definition 3. Additionally, it is possible to choose $r\in L^\infty(0,T;L^s(\Omega))$ for $s=\frac{1}{2}\min(2^\star,p)$.*

Using this result, we will be able to consider the limit $\varepsilon\to 0$ to show that there also exists a solution for the case $\varepsilon=0$. Remark that we still require ν to be positive. Later, we will see that this yields the required regularity in order to perform the limit process. Altogether, this will result in the following theorem.

Theorem 5 (Existence for $\varepsilon=0$). *Let the assumptions* (A1)-(A4), (A5') *and* (A6)-(A10) *be true. For each $\nu>0$ and $\varepsilon=0$ and each $c_0\in H^1(\Omega;\mathbb{R}^N)$ and $z_0\in H^1(\Omega)$ with $0\leq z_0\leq 1$, there exists a weak solution q as defined in Definition 3. It is possible to choose $r\in L^s(0,T;L^1(\Omega))$ for any $1\leq s<\infty$.*

The proof of the existence of a solution of the viscous problem is divided into several steps. For some details we refer to [13] and [14]. In this section, we will always assume $\varepsilon>0$.

4.1. **Construction of time-discrete solutions.** Let be $M\in\mathbb{N}$ and set $\tau=\frac{T}{M}$. Furthermore, let u_0 be a minimizer in $\dot{W}^{1,p}(\Omega;\mathbb{R}^n)$ of the functional $u\mapsto\mathcal{E}(u,c_0,z_0)$. Then, we set $(u_M^0,c_M^0,z_M^0)=(u_0,c_0,z_0)$, define the convex closed subsets

$$Q=:\left\{(u,c,z)\in\dot{W}^{1,p}(\Omega;\mathbb{R}^n)\times H^1(\Omega;\mathbb{R}^N)\times W^{1,p}(\Omega)|0\leq z\leq 1\right\}$$

and

$$Q_M^m:=\left\{(u,c,z)\in Q|z\leq z_M^{m-1}\right\}$$

and the discrete energy functional

$$
\begin{aligned}
&\mathbb{E}^m_M \colon Q^m_M \to \mathbb{R}\\
\mathbb{E}^m_M(u,c,z) = \tilde{\mathcal{E}}_\varepsilon(u,c,z) + \tau\tilde{\mathcal{R}}\left(\frac{z-z^{m-1}_M}{\tau}\right) &+ \tau\mathcal{A}^\star_{c^{m-1}_M}\left(-\frac{c-c^{m-1}_M}{\tau}\right) + \tau\mathcal{A}_{c^{m-1}_M}\left(\bar{\mathcal{B}}(c^{m-1}_M,0)\right)\\
&+ \frac{\nu}{2\tau}\|c-c^{m-1}_M\|^2_{L^2}.
\end{aligned}
$$

Remark 6. The use of $\mathcal{A}^\star$ is the natural generalization of the H^{-1}-norm which is well-known for the study of the Cahn-Hilliard equation. This functional incorporates the nonlinear gradient flow originating from the Newton boundary condition for the potential.

Lemma 7 (Existence of minimizers). *For each $(u^{m-1}_M, c^{m-1}_M, z^{m-1}_M) \in Q$, there exists a minimizer $(u^m_M, c^m_M, z^m_M) \in Q^m_M$ of the functional $\mathbb{E}^m_M$, i.e.*

$$\mathbb{E}^m_M(u^m_M, c^m_M, z^m_M) = \inf_{(u,c,z)\in Q^m_M} \mathbb{E}^m_M(u,c,z).$$

Proof. We will show that the functional is coercive and sequentially weakly lower semi-continuous. For the coercivity let $(u_k, c_k, z_k)_k \subset Q^m_M$ be a sequence and $C > 0$ with $\mathbb{E}^m_M(u_k,c_k,z_k) \le C$. Then, it is due to the non-negativity of W^{el}, the inequality $W^{\mathrm{ch}}(c) \ge -C$ and the inequality (15)

$$
\begin{aligned}
&\frac{1}{2}\|\nabla c_k\|^2_{L^2} + \frac{\varepsilon}{p}\left(\|e(u_k)\|^p_{L^p} + \|\nabla z_k\|^p_{L^p}\right) + \frac{1}{n}\left|\int_\Omega c_k \mathrm{d}x\right|_1\\
&\le \frac{1}{2}\|\nabla c_k\|^2_{L^2} + \frac{\varepsilon}{p}\left(\|e(u_k)\|^p_{L^p} + \|\nabla z_k\|^p_{L^p}\right) + \frac{\tau}{n}\left|\int_\Omega \frac{c_k - c^{m-1}_M}{\tau}\mathrm{d}x\right|_1 + \frac{1}{n}\left|\int_\Omega c^{m-1}_M \mathrm{d}x\right|_1\\
&\le \mathbb{E}^m_M(u_k,c_k,z_k) + \frac{1}{n}\left|\int_\Omega c^{m-1}_M\mathrm{d}x\right|_1 + C \le C.
\end{aligned}
$$

Thus, we can apply Poincaré's inequality to obtain the boundedness of the sequence c_k in $H^1(\Omega;\mathbb{R}^N)$. Together with the constraint $0 \le z_k \le 1$, the sequence z_k is bounded in $W^{1,p}(\Omega)$. The sequence u_k is bounded in $\dot{W}^{1,p}(\Omega;\mathbb{R}^n)$ by Korn's inequality for L^p (see e.g. [7]).

The lower semi-continuity follows as in [13] since the Legendre-Fenchel conjugate is always lower semi-continuous. ☐

Now, we set $(u^m_M, c^m_M, z^m_M) = \arg\min_{(u,c,z)\in Q^m_M} \mathbb{E}^m_M(u,c,z)$ and

$$\mu^m_M = \bar{\mathcal{B}}\left(c^{m-1}_M, -\frac{c^m_M - c^{m-1}_M}{\tau}\right). \tag{21}$$

In order to shorten the notation, we denote the vector of unknowns by $q^m_M = (u^m_M, c^m_M, \mu^m_M, z^m_M)$ and introduce

$$t_M(t) = \min_{1\le m\le M}\{m\tau | t \le m\tau\}$$

and

$$t^-_M(t) = t_M(t) - \tau.$$

Furthermore, we introduce the left-continuous and right-continuous piecewise constant interpolants defined by

$$
\begin{aligned}
q_M(t) &= q^m_M && \text{if } (m-1)\tau < t \le m\tau,\\
q^-_M(t) &= q^{m-1}_M && \text{if } (m-1)\tau \le t < m\tau,
\end{aligned}
$$

for $m = 1,\dots,M$ with $q_M(0) = q^0_M$ and $q^-_M(T) = q^M_M$. Then,

$$\hat{q}_M(t) = q_M(t)\frac{t - t^-_M(t)}{\tau} - q^-_M(t)\frac{t - t_M(t)}{\tau}$$

is the piecewise linear interpolant.

As a next step, we investigate the Euler-Lagrange equations which are satisfied by the approximations:

Lemma 8 (Euler-Lagrange equations). *The approximations q_M, q_M^- and $\hat{q}_M$ satisfy the following properties:*

(i) *It holds*
$$-(\partial_t\hat{c}_M(t),\xi)_{L^2} = \int_\Omega \nabla\mu_M(t) : \nabla\xi \mathrm{d}x - \int_\Gamma R(c_M^-(t),\mu_M(t))\cdot\xi\mathrm{d}\omega$$
for all $\xi \in H^1(\Omega;\mathbb{R}^N)$ and for almost all $t \in [0,T]$.

(ii) *It holds*
$$(\mu_M(t),\xi)_{L^2} = \int_\Omega \nabla c_M(t) : \nabla\xi + \big(W_c^{\mathrm{ch}}(c_M(t)) + W_c^{\mathrm{el}}(e(u_M(t)),c_M(t),z_M(t)) + \nu\partial_t\hat{c}_M(t)\big)\cdot\xi\mathrm{d}x \quad (22)$$
for all $\xi \in H^1(\Omega;\mathbb{R}^N)$ and for almost all $t \in [0,T]$.

(iii) *It holds*
$$\int_\Omega W_e^{\mathrm{el}}(e(u_M(t)),c_M(t),z_M(t)) : e(\xi) + \varepsilon|e(u_M(t))|^{p-2}e(u_M(t)) : e(\xi)\mathrm{d}x = 0 \quad (23)$$
for all $\xi \in W^{1,p}(\Omega;\mathbb{R}^n)$ and for almost all $t \in [0,T]$.

(iv) *It holds*
$$0 \le \int_\Omega \big(\varepsilon|\nabla z_M(t)|^{p-2}+1\big)\nabla z_M(t)\cdot\nabla\xi + \big(W_z^{\mathrm{el}}(e(u_M(t)),c_M(t),z_M(t)) + \beta\partial_t\hat{z}_M(t) - \alpha\big)\xi\mathrm{d}x \quad (24)$$
for almost all $t\in[0,T]$ and for all $\xi \in W^{1,p}(\Omega)$ such that there exists a constant $\nu > 0$ with $0 \le \nu\xi + z_M(t) \le z_M^-(t)$ a.e. in Ω.

(v) *The discrete energy inequality*
$$\begin{aligned}&\mathcal{E}_\varepsilon(u_M(t),c_M(t),z_M(t)) + \int_0^{t_M(t)} \mathcal{R}(\partial_t\hat{z}_M) + \mathcal{A}^\star_{c_M^-}(-\partial_t\hat{c}_M) + \mathcal{A}_{c_M^-}\big(\bar{\mathcal{B}}(c_M^-,0)\big) + \frac{\nu}{2}\|\partial_t\hat{c}_M\|^2_{L^2}\mathrm{d}s \\ &\le \mathcal{E}_\varepsilon(\hat{u}_0,c_0,z_0)\end{aligned} \quad (25)$$
is fulfilled for all $t\in[0,T]$ and all $\hat{u}_0 \in \dot{W}^{1,p}(\Omega;\mathbb{R}^n)$.

Proof. Using the assumptions (A1) to (A7) on the constitutive energy densities, we compute the Gateaux derivatives of $\tilde{\mathcal{E}}_\varepsilon$ as
$$\begin{aligned}\langle \mathrm{d}_u\tilde{\mathcal{E}}_\varepsilon(u,c,z),[\xi]\rangle &= \int_\Omega W_e^{\mathrm{el}}(e(u),c,z) : e(\xi) + \varepsilon|e(u)|^{p-2}e(u) : e(\xi)\mathrm{d}x &&\text{for } [\xi] \in \dot{W}^{1,p}(\Omega;\mathbb{R}^n)\\ \langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(u,c,z),\xi\rangle &= \int_\Omega \nabla c : \nabla\xi + W_c^{\mathrm{ch}}(c)\cdot\xi + W_c^{\mathrm{el}}(e(u),c,z)\cdot\xi\mathrm{d}x &&\text{for } \xi \in H^1(\Omega;\mathbb{R}^N) \quad (26)\\ \langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(u,c,z),\xi\rangle &= \int_\Omega \big(\varepsilon|\nabla z|^{p-2}+1\big)\nabla z\cdot\nabla\xi + W_z^{\mathrm{el}}(e(u),c,z)\xi\mathrm{d}x &&\text{for } \xi \in W^{1,p}(\Omega).\end{aligned}$$

(i) This is fulfilled by construction (see (21)).

(ii) Since (u_M^m, c_M^m, z_M^m) is a minimizer of the discrete energy functional $\mathbb{E}_M^m$ we have
$$0 \in \partial_c^{\mathrm{Cl}}\left(\tilde{\mathcal{E}}_\varepsilon(u_M,c_M,z_M) + \tau\mathcal{A}^\star_{c_M^-}\Big|_{H^1(\Omega;\mathbb{R}^N)}(-\partial_t\hat{c}_M) + \tau\frac{\nu}{2}\|\partial_t\hat{c}_M\|^2_{L^2(\Omega)}\right).$$
Thus, using the chain rule it is
$$\mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(u_M,c_M,z_M) + \nu(\partial_t\hat{c}_M,\cdot)_{L^2(\Omega;\mathbb{R}^N)} \in \partial\mathcal{A}^\star_{c_M^-}\Big|_{H^1(\Omega;\mathbb{R}^N)}(-\partial_t\hat{c}_M).$$
The observation (14) implies
$$(\bar{\mathcal{B}}(c_M^-,-\partial_t\hat{c}_M),v)_{L^2} = \langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(u_M,c_M,z_M),v\rangle + \nu(\partial_t\hat{c}_M,v)_{L^2}$$

for all $v \in H^1(\Omega;\mathbb{R}^N)$. Thus, the definition of μ_M and the Gateaux derivative (26) result in equation (22).

(iii) Let be $\xi \in W^{1,p}(\Omega;\mathbb{R}^n)$ and $[\xi]$ the corresponding equivalence class in $\dot{W}^{1,p}(\Omega;\mathbb{R}^n)$ with $\xi \in [\xi]$. The necessary condition $\langle \mathrm{d}_u \tilde{\mathcal{E}}_\varepsilon(u_M, c_M, z_M), [\xi]\rangle = 0$ immediately yields equation (23) for all $\xi \in W^{1,p}(\Omega;\mathbb{R}^n)$.

(iv) The necessary condition for z_M^m to be a minimizer in the convex set $0 \le z \le z_M^{m-1}$ is

$$\left\langle \mathrm{d}_z \tilde{\mathcal{E}}_\varepsilon(u_M^m, c_M^m, z_M^m), \xi - z_M^m \right\rangle + \left\langle \mathrm{d}_{\dot{z}} \tilde{\mathcal{R}}\left(\frac{z_M^m - z_M^{m-1}}{\tau}\right), \xi - z_M^m \right\rangle \ge 0$$

for all $\xi \in W^{1,p}(\Omega)$ with $0 \le \xi \le z_M^{m-1}$ which is equivalent to condition (24).

(v) To prove the discrete energy inequality we test $\mathbb{E}_M^m$ with $(u_M^{m-1}, c_M^{m-1}, z_M^{m-1})$. Using $\mathcal{A}_c^\star(0) = -\mathcal{A}_c(\bar{\mathcal{B}}(c,0))$, this yields

$$\begin{aligned}
&\mathcal{E}_\varepsilon(u_M^m, c_M^m, z_M^m) + \tau \mathcal{R}\left(\frac{z_M^m - z_M^{m-1}}{\tau}\right) + \tau \mathcal{A}^\star_{c_M^{m-1}}\left(-\frac{c_M^m - c_M^{m-1}}{\tau}\right) \\
&\quad + \tau \mathcal{A}_{c_M^{m-1}}(\bar{\mathcal{B}}(c_M^{m-1}, 0)) + \frac{\nu}{2\tau}\|c_M^m - c_M^{m-1}\|^2_{L^2(\Omega;\mathbb{R}^N)} \\
&= \mathbb{E}_M^m(u_M^m, c_M^m, z_M^m) \\
&\le \mathbb{E}_M^m(u_M^{m-1}, c_M^{m-1}, z_M^{m-1}) \\
&= \mathcal{E}_\varepsilon(u_M^{m-1}, c_M^{m-1}, z_M^{m-1}).
\end{aligned}$$

Summing up these inequalities from 1 to M leads to the desired discrete energy inequality. Remark that we can choose an arbitrary value $u_M^0 \in \dot{W}^{1,p}(\Omega;\mathbb{R}^n)$.

□

4.2. **A priori estimates and convergence.** The energy estimate leads to the following a priori estimates.

Corollary 9 (A priori estimates). *There exists a constant $C > 0$ - depending on ε and ν - such that*

(i) $\|u_M\|_{L^\infty(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))} \le C$,

(ii) $\|c_M\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))} \le C$,

(iii) $\|z_M\|_{L^\infty(0,T;W^{1,p}(\Omega))} \le C$,

(iv) $\|\partial_t \hat{c}_M\|_{L^2(\Omega_T;\mathbb{R}^N)} \le C$,

(v) $\|\partial_t \hat{z}_M\|_{L^2(\Omega_T)} \le C$ *and*

(vi) $-C \le \int_0^T \mathcal{A}^\star_{c_M^-}(-\partial_t \hat{c}_M) + \mathcal{A}_{c_M^-}(\bar{\mathcal{B}}(c_M^-, 0))\mathrm{d}t \le C$

holds for all $M \in \mathbb{N}$.

Proof. Using the energy estimate (25) and (15), we immediately obtain the properties (iv),(v) and (vi) and the boundedness of $\nabla c_M(t)$ in $L^2(\Omega;\mathbb{R}^{N\times n})$. With the inequality (15), we find

$$\begin{aligned}
\left|\int_\Omega c_M(t)\mathrm{d}x\right|_1 &\le \tau \sum_{i=1}^{\frac{t_M(t)}{\tau}} \left|\int_\Omega \frac{c_M^i - c_M^{i-1}}{\tau}\mathrm{d}x\right|_1 + \int_\Omega |c_0|_1 \mathrm{d}x \\
&\le n\tau \sum_{i=1}^{\frac{t_M(t)}{\tau}} \left(\mathcal{A}^\star_{c_M^{i-1}}\left(-\frac{c_M^i - c_M^{i-1}}{\tau}\right) + \mathcal{A}_{c_M^{i-1}}(\bar{\mathcal{B}}(c_M^{i-1}, 0))\right) + \tau M C + \|c_0\|_{L^1(\Omega;\mathbb{R}^N)} \\
&= n\int_0^{t_M(t)} \mathcal{A}^\star_{c_M^-}(-\partial_t \hat{c}_M) + \mathcal{A}_{c_M^-}(\bar{\mathcal{B}}(c_M^-, 0))\mathrm{d}t + TC + \|c_0\|_{L^1(\Omega;\mathbb{R}^N)}.
\end{aligned}$$

Thus, the mean value of $c_M(t)$ is bounded in $L^\infty(0,T;\mathbb{R}^N)$ and Poincaré's inequality yields the property (ii). The property (iii) follows from the constraint $0 \le z_M(t) \le 1$ and the boundedness of $\nabla z_M(t)$ in $L^p(\Omega;\mathbb{R}^n)$. Property (i) is a consequence of Korn's inequality. □

Using compactness arguments we can extract weakly (weakly$\star$, resp.) convergent subsequences due to the a priori estimates. The details of the proof of the following lemma can be found in [13].
Lemma 10 (Weak convergence). *There exists a subsequence $\{M_k\}$ and elements $u \in \mathcal{U}_\varepsilon$, $c \in \mathcal{C}$ and $z \in \mathcal{Z}_\varepsilon$ such that it holds*

(i) $u_{M_k} \overset{\star}{\rightharpoonup} u$ *in* $L^\infty(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))$

(ii) $c_{M_k}, c^-_{M_k} \overset{\star}{\rightharpoonup} c$ *in* $L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))$
$c_{M_k}, c^-_{M_k} \to c$ *in* $L^q(0,T;L^{2^\star-\delta}(\Omega;\mathbb{R}^N))$
$c_{M_k}(t), c^-_{M_k}(t) \rightharpoonup c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$ *a.e.*
$c_{M_k}, c^-_{M_k} \to c$ *a.e. in* Ω_T
$\hat{c}_{M_k} \rightharpoonup c$ *in* $H^1(0,T;L^2(\Omega;\mathbb{R}^N))$

(iii) $z_{M_k}, z^-_{M_k} \overset{\star}{\rightharpoonup} z$ *in* $L^\infty(0,T;W^{1,p}(\Omega))$
$z_{M_k}, z^-_{M_k} \to z$ *in* $L^q(0,T;C(\Omega))$
$z_{M_k}(t), z^-_{M_k}(t) \rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$ *a.e.*
$z_{M_k}, z^-_{M_k} \to z$ *a.e. in* Ω_T
$\hat{z}_{M_k} \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$

for any $1 \le q < \infty$ *and* $\delta > 0$.

Proof. The proof bases merely on the bounds provided in Corollary 9 and follows the lines of [13]. The only differences are the second convergence statements of (ii) and (iii). But these are just direct consequences of the Aubin-Lions-Simon theorem (see e.g. [28]). □

For the convergence results obtained in the previous lemma only the boundedness of the energy functional was needed. As a next step, we will sharpen these convergence results using the discrete Euler-Lagrange equations. Basically, we will perform the same steps as in [13] but we have to change the order of the steps carefully as we don't have the weak convergence of μ_{M_k} yet. To simplify the notation, we will index the subsequences just by M instead of M_k.
Lemma 11 (Strong convergence of u_M). *There exists a further subsequence such that it holds*

$$u_M, u^-_M \to u \qquad \text{in } L^p(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))$$
$$e(u_M), e(u^-_M) \to e(u) \qquad \text{a.e. in } \Omega_T$$

and

$$u_M(t) \to u(t) \qquad \text{in } \dot{W}^{1,p}(\Omega;\mathbb{R}^n) \text{ for almost all } t \in [0,T].$$

Proof. In each inner product space, the elementary inequality

$$C_{uc}\|x-y\|^q \le \left\langle \left(\|x\|^{q-2}x - \|y\|^{q-2}y\right), x-y \right\rangle \tag{27}$$

is valid for $q \ge 2$ and some positive constant $C_{uc} > 0$ (see e.g. [13]). We will apply this inequality for $q = p$ in the space of $n \times n$ matrices. Testing equation (23) by any representative ξ of $u_M(t) - u(t)$, this leads together with (A3) to

$$\begin{aligned}
&\eta\|e(u_M)-e(u)\|^2_{L^2(\Omega_T;\mathbb{R}^{n\times n})} + \varepsilon C_{uc}\|e(u_M)-e(u)\|^p_{L^p(\Omega_T;\mathbb{R}^{n\times n})} \\
&\le \int_{\Omega_T} \left(W^{\mathrm{el}}_e(e(u_M),c_M,z_M) - W^{\mathrm{el}}_e(e(u),c_M,z_M)\right) : (e(u_M)-e(u))\mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon \int_{\Omega_T} \left(|e(u_M)|^{p-2}e(u_M) - |e(u)|^{p-2}e(u)\right) : (e(u_M)-e(u))\mathrm{d}x\mathrm{d}t \\
&= \underbrace{\int_{\Omega_T} W^{\mathrm{el}}_e(e(u_M),c_M,z_M) : e(\xi) + \varepsilon|e(u_M)|^{p-2}e(u_M) : e(\xi)\mathrm{d}x\mathrm{d}t}_{=0 \text{ by } (23)} \\
&\quad - \int_{\Omega_T} W^{\mathrm{el}}_e(e(u),c_M,z_M) : e(u_M-u)\mathrm{d}x\mathrm{d}t - \varepsilon\int_{\Omega_T} |e(u)|^{p-2}e(u) : (e(u_M)-e(u))\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{28}$$

Due to Lebesque's generalized convergence theorem and the assumptions (A2) and (A4), the sequence $W^{\mathrm{el}}_e(e(u),c_M,z_M)$ is strongly converging in $L^2(\Omega_T;\mathbb{R}^{n\times n})$ and thus the weak convergence $e(u_M) \rightharpoonup e(u)$ in $L^p(\Omega_T;\mathbb{R}^{n\times n})$ shows that the right hand side of (28) converges to zero. Therefore, we get the strong convergence $e(u_M) \to e(u)$ in $L^p(\Omega_T;\mathbb{R}^{n\times n})$. Since it is $u_M(t), u(t) \in \dot{W}^{1,p}(\Omega;\mathbb{R}^n)$, Korn's inequality yields the convergence in $L^p(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))$.

Due to the continuity of the translation operator, it is clear that u_M^- also converges strongly to u in $L^p(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))$. To see this, let $\bar{\varepsilon} > 0$ be given. Then, we choose N such that $\|u_M - u\|^p_{L^p(0,T;\dot{W}^{1,p}(\Omega))} \leq \frac{\bar{\varepsilon}}{3^p}$ and $\|u_M - u_{\bar{M}}\|^p_{L^p(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))} \leq \frac{\bar{\varepsilon}}{3^p}$ for all $M, \bar{M} \geq N$. Denote by $C > 0$ the bound in $L^p(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))$ of the sequence u_M. Then, it holds with Jensen's inequality

$$\begin{aligned}
&\int_0^T \|u(t) - u_M^-(t)\|^p_{\dot{W}^{1,p}(\Omega;\mathbb{R}^n)}\mathrm{d}t \\
&\leq 3^{p-1}\int_0^T \|u(t) - u_N(t)\|^p + \left\|u_N(t) - u_N\left(t - \frac{T}{M}\right)\right\|^p + \left\|u_N\left(t - \frac{T}{M}\right) - u_M\left(t - \frac{T}{M}\right)\right\|^p \mathrm{d}t \\
&\leq \frac{\bar{\varepsilon}}{3} + 3^{p-1}\sum_{k=1}^N 2^p \|u_N^k\|^p \frac{T}{M} + \frac{\bar{\varepsilon}}{3} \\
&\leq \frac{2}{3}\bar{\varepsilon} + \frac{1}{3}6^p \frac{N}{M}\|u_N\|^p_{L^p(0,T;\dot{W}^{1,p}(\Omega))} \\
&\leq \bar{\varepsilon}
\end{aligned} \tag{29}$$

for all $M \geq \max\left(N, \frac{6^p}{\bar{\varepsilon}}NC^p\right)$. Here, we used the convention $u_M(t) = 0$ for $t < 0$.

Clearly, we can now extract a subsequence such that $e(u_M)$ and $e(u_M^-)$ converge to $e(u)$ almost everywhere in Ω_T and $u_M(t) \to u(t)$ in $\dot{W}^{1,p}(\Omega;\mathbb{R}^n)$ for almost all $t \in [0,T]$. □

Remark, that the last part of the proof did not use any fact from the specific differential equation. It bases just on the convergence of u_M in $L^p(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))$. It can be modified without any changes to an arbitrary Banach space X instead of $\dot{W}^{1,p}(\Omega;\mathbb{R}^n)$.

Using the discrete energy inequality (24) of Lemma 8, we can also conclude the strong convergence of z_M in $L^q(0,T;W^{1,p}(\Omega))$.

Lemma 12 (Strong convergence of z_M). *It holds $z_M, z_M^- \to z$ in $L^q(0,T;W^{1,p}(\Omega))$ for any $1 \leq q < \infty$ and $z_M(t) \to z(t)$ in $W^{1,p}(\Omega)$ for almost all $t \in [0,T]$ for the same subsequence as before.*

Proof. The proof is the same as in [13] since it just uses the differential inclusion for the damage variable. □

Now, we are able to prove an energy estimate which is stronger than inequality (25). In comparison to [13], we have weaker requirements at the moment since we have only the strong convergence of c_M in $L^q(0,T;L^{2^\star-\delta}(\Omega;\mathbb{R}^N))$ and not in $L^q(0,T;H^1(\Omega;\mathbb{R}^N))$ and since we have no weak convergence of μ_M. But this is sufficient for the proof. Remark at this stage, that we need the exponent $\frac{2^\star}{2} - \delta$ in the assumption (A7) with positive δ.

For this second energy estimate, not only the Legendre-Fenchel transform $\mathcal{A}^\star$ but also the operator $\mathcal{A}$ itself comes into play naturally.

Lemma 13 (Precise energy inequality). *For every $M \in \mathbb{N}$ and every $t \in [0,T]$, there exists $\kappa_M(t) \in \mathbb{R}$ such that the approximations fulfill*

$$\begin{aligned}
&\mathcal{E}_\varepsilon(u_M(t), c_M(t), z_M(t)) + \int_0^{t_M(t)} \mathcal{A}_{c_M^-}(\mu_M) + \mathcal{A}^\star_{c_M^-}(-\partial_t \hat{c}_M)\mathrm{d}t \\
&\quad + \int_0^{t_M(t)}\int_\Omega -\alpha\partial_t \hat{z}_M + \beta|\partial_t \hat{z}_M|^2 + \nu|\partial_t \hat{c}_M|^2 \mathrm{d}x\mathrm{d}t \leq \mathcal{E}_\varepsilon(u_0, c_0, z_0) + \kappa_M(t)
\end{aligned} \tag{30}$$

for any $u_0 \in \dot{W}^{1,p}(\Omega;\mathbb{R}^n)$ with $\kappa_M(t) \to 0$ for $M \to \infty$. Here, κ_M is independent of the choice of u_0.

Proof. Since it is $\mathbb{E}_M^m(u_M^m, c_M^m, z_M^m) \le \mathbb{E}_M^m(u_M^{m-1}, c_M^m, z_M^m)$, we obtain with the chain rule

$$\begin{aligned}\mathcal{E}_\varepsilon(u_M^m, c_M^m, z_M^m) &\le \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, c_M^m, z_M^m)\\
&= \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, c_M^{m-1}, z_M^{m-1})\\
&\quad + \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, c_M^m, z_M^{m-1}) - \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, c_M^{m-1}, z_M^{m-1})\\
&\quad + \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, c_M^m, z_M^m) - \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, c_M^m, z_M^{m-1})\\
&= \tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, c_M^{m-1}, z_M^{m-1})\\
&\quad + \int_{(m-1)\tau}^{\tau} \Big\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, \hat{c}_M(s), z_M^{m-1}), \partial_t\hat{c}_M(s)\Big\rangle \mathrm{d}s\\
&\quad + \int_{(m-1)\tau}^{\tau} \Big\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(u_M^{m-1}, c_M^m, \hat{z}_M(s)), \partial_t\hat{z}_M(s)\Big\rangle \mathrm{d}s.\end{aligned}$$

Taking the discrete Euler-Lagrange equations into account, a summation from $m = 1$ to $\frac{t_M(t)}{\tau}$ leads to

$$\begin{aligned}&\mathcal{E}_\varepsilon(u_M(t), c_M(t), z_M(t)) - \mathcal{E}_\varepsilon(u_0, c_0, z_0)\\
&\quad\le \underbrace{\int_0^{t_M(t)} \Big\langle \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(u_M^-, \hat{c}_M, z_M^-) - \mathrm{d}_c\tilde{\mathcal{E}}_\varepsilon(u_M, c_M, z_M), \partial_t\hat{c}_M\Big\rangle \mathrm{d}t}_{=:\kappa_M^1(t)}\\
&\qquad + \int_0^{t_M(t)} \Big\langle \mathrm{d}_c\tilde{\mathcal{E}}(u_M, c_M, z_M), \partial_t\hat{c}_M\Big\rangle \mathrm{d}t\\
&\qquad + \underbrace{\int_0^{t_M(t)} \Big\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(u_M^-, c_M, \hat{z}_M) - \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(u_M, c_M, z_M), \partial_t\hat{z}_M\Big\rangle \mathrm{d}t}_{=:\kappa_M^2(t)}\\
&\qquad + \int_0^{t_M(t)} \Big\langle \mathrm{d}_z\tilde{\mathcal{E}}_\varepsilon(u_M, c_M, z_M), \partial_t\hat{z}_M\Big\rangle \mathrm{d}t\\
&\quad\le -\int_0^{t_M(t)} (\mu_M, -\partial_t\hat{c}_M)_{L^2} + \nu\|\partial_t\hat{c}_M\|_{L^2}^2 + \langle \mathrm{d}_{\dot z}\tilde{\mathcal{R}}(\partial_t\hat{z}_M), \partial_t\hat{z}_M\rangle \mathrm{d}t + \kappa_M^1(t) + \kappa_M^2(t)\end{aligned}$$

for any $u_0 \in \dot{W}^{1,p}(\Omega;\mathbb{R}^n)$. Since it is $(\mu_M(t), -\partial_t\hat{c}_M(t))_{L^2} = \mathcal{A}_{c_M^-}(\mu_M(t)) + \mathcal{A}^\star_{c_M^-}(-\partial_t\hat{c}_M(t))$ due to the definition of μ_M and $\mathcal{A}^\star_{c_M^-}$, it remains only to find $\kappa_M(t)$ with $\kappa_M^1(t) + \kappa_M^2(t) \le \kappa_M(t) \to 0$.

Due to the convexity of $x \mapsto |x|^p$ and $x \mapsto |x|^2$, we have the estimates

$$\big(|\nabla\hat{z}_M)|^{p-2}\nabla\hat{z}_M - |\nabla z_M|^{p-2}\nabla z_M\big) \cdot \nabla\partial_t\hat{z}_M \le 0$$

and

$$(\nabla\hat{z}_M - \nabla z_M) \cdot \nabla\partial_t\hat{z}_M \le 0.$$

Thus, we conclude

$$\kappa_M^2(t) \le \int_0^{t_M(t)} \int_\Omega \big(W_z^{\mathrm{el}}(u_M^-, c_M, \hat{z}_M) - W_z^{\mathrm{el}}(u_M, c_M, z_M)\big)\,\partial_t\hat{z}_M \mathrm{d}x\mathrm{d}t \to 0$$

using Lebeque's generalized convergence theorem. Similarly, we find

$$\begin{aligned}\kappa_M^1(t) \le \int_0^{t_M(t)} \int_\Omega &\big(W_c^{\mathrm{el}}(u_M^-, \hat{c}_M, z_M^-) - W_c^{\mathrm{el}}(u_M, c_M, z_M)\big) \cdot \partial_t\hat{c}_M\\
&+ \big(W_c^{\mathrm{ch}}(\hat{c}_M) - W_c^{\mathrm{ch}}(c_M)\big) \cdot \partial_t\hat{c}_M \mathrm{d}x\mathrm{d}t \to 0.\end{aligned}$$

□

Corollary 14 (Weak convergence of μ_M). *There exists a constant $C > 0$ - only depending on ε and ν - with*

$$\|\mu_M\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^N))} \leq C,$$

a function $\mu \in L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ and a subsequence with $\mu_M \rightharpoonup \mu$ in $L^2(0,T;H^1(\Omega;\mathbb{R}^N))$.

Proof. Since the sequence $\kappa_M(T)$ of the previous lemma is bounded, we find a constant $C_2 > 0$ with

$$\int_0^T \mathcal{A}_{c_M^-}(\mu_M) + \mathcal{A}^\star_{c_M^-}(-\partial_t \hat{c}_M)\mathrm{d}t \leq C_2.$$

Using the definition (21) of μ_M, the definition (3) of $\mathcal{A}$, the reformulation (13) for $\mathcal{A}^\star$ and the assumption (A11) on R, we obtain

$$\begin{aligned}
C_2 &\geq \int_0^T \mathcal{A}_{c_M^-}(\mu_M) + \mathcal{A}^\star_{c_M^-}(-\partial_t \hat{c}_M)\mathrm{d}t \\
&= \int_0^T (\nabla\mu_M, \nabla\mu_M)_{L^2(\Omega;\mathbb{R}^N)}\mathrm{d}t - \int_0^T \int_\Gamma R(c_M^-, \mu_M)\cdot\mu_M \mathrm{d}\omega\mathrm{d}t \\
&\geq \int_0^T (\nabla\mu_M, \nabla\mu_M)_{L^2(\Omega;\mathbb{R}^N)} + C(\mu_M, \mu_M)_{L^2(\Gamma;\mathbb{R}^N)}\mathrm{d}t - CC_1T\mathcal{H}^{n-1}(\Gamma).
\end{aligned}$$

The variant (6) of Poincaré's inequality yields the desired boundedness. The existence of μ and the weak convergent subsequence is then clear. □

Now, we are able to strengthen the convergence result for c_M.

Lemma 15 (Strong convergence of c_M). *The subsequences c_M and c_M^- strongly converge to c in $L^q(0,T;H^1(\Omega;\mathbb{R}^N))$ and in $L^q(0,T;L^{2^\#}(\Gamma;\mathbb{R}^N))$ for any $1 \leq q < \infty$.*

Proof. The previously obtained convergence results allow us to pass to the limit in (22). Thus, we see that μ and c fulfill

$$\int_0^T (\mu,\xi)_{L^2(\Omega;\mathbb{R}^N)}\mathrm{d}t = \int_0^T \int_\Omega \nabla c : \nabla\xi + \left(W_c^{\mathrm{ch}}(c) + W_c^{\mathrm{el}}(e(u),c,z) + \nu\partial_t c\right)\cdot\xi\mathrm{d}x \tag{31}$$

for any $\xi \in L^2(0,T;H^1(\Omega;\mathbb{R}^N))$. Now, we use c_M as test functions in (22) and pass to the limit. Then, we find

$$\begin{aligned}
&\int_0^T \|\nabla c_M\|^2_{L^2}\mathrm{d}t \\
&\quad= \int_0^T \int_\Omega \mu_M\cdot c_M - \left(W_c^{\mathrm{ch}}(c_M) + W_c^{\mathrm{el}}(e(u_M),c_M,z_M) + \nu\partial_t\hat{c}_M\right)\cdot c_M\mathrm{d}x\mathrm{d}t \\
&\quad\to \int_0^T \int_\Omega \mu\cdot c - \left(W_c^{\mathrm{ch}}(c) + W_c^{\mathrm{el}}(e(u),c,z) + \nu\partial_t c\right)\cdot c\mathrm{d}x\mathrm{d}t \\
&\quad= \int_0^T \|\nabla c\|^2_{L^2}\mathrm{d}t.
\end{aligned}$$

Thus, due to the uniform convexity of $L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ the convergence of c_M is in fact strong in $L^2(0,T;H^1(\Omega;\mathbb{R}^N))$. Together with the boundedness in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))$, this yields the convergence in $L^q(0,T;H^1(\Omega;\mathbb{R}^N))$ for any $1 \leq q < \infty$ and consequently also in $L^q(0,T;L^{2^\#}(\Gamma;\mathbb{R}^N))$. The convergence of c_M^- follows again by the continuity of the translation operator (compare (29) and the comment after the proof of Lemma 11). □

The lower semi-continuity and the partial convexity of $\mathcal{A}$ and $\mathcal{A}^\star$ allows us now to prove the (norm $\times$ weak) lower semi-continuity of the integral operators $(c,\mu) \mapsto \mathcal{I}_\mathcal{A}(c,\mu) := \int_0^T \mathcal{A}(c(t),\mu(t))\mathrm{d}t$ and $(\partial_t c,\mu) \mapsto \mathcal{I}_{\mathcal{A}^\star} := \int_0^T \mathcal{A}^\star(c(t),-\partial_t c(t))\mathrm{d}t$.

Lemma 16 ((norm $\times$ weak) lower semi-continuity of the integral operators). *It holds*

$$\int_0^T \mathcal{A}(c,\mu)\mathrm{d}t \leq \liminf_{M\to\infty} \int_0^T \mathcal{A}(c_M^-,\mu_M)\mathrm{d}t$$

and

$$\int_0^T \mathcal{A}^\star(c,-\partial_t c)\mathrm{d}t \leq \liminf_{M\to\infty} \int_0^T \mathcal{A}^\star(c_M^-,-\partial_t \hat{c}_M)\mathrm{d}t.$$

Proof. We define the mappings

$$\begin{aligned}\bar{\mathcal{A}}\colon L^1(\Gamma;\mathbb{R}^N) \times H^1(\Omega;\mathbb{R}^N) &\to \mathbb{R}\\ \bar{\mathcal{A}}(w,v) = \mathcal{A}\left(\left(|w_1|^{\frac{1}{2^\#-\delta}}\operatorname{sgn}(w_1),\ldots,|w_N|^{\frac{1}{2^\#-\delta}}\operatorname{sgn}(w_N)\right),v\right)\end{aligned}$$

and

$$\begin{aligned}\bar{\mathcal{A}}^\star\colon L^1(\Gamma;\mathbb{R}^N) \times L^2(\Omega;\mathbb{R}^N) &\to \mathbb{R}\\ \bar{\mathcal{A}}^\star(w,v^\star) = \mathcal{A}^\star\left(\left(|w_1|^{\frac{1}{2^\#-\delta}}\operatorname{sgn}(w_1),\ldots,|w_N|^{\frac{1}{2^\#-\delta}}\operatorname{sgn}(w_N)\right),v^\star\right).\end{aligned}$$

These mappings are continuous and thus Carathéodory functions. Furthermore, for fixed $w \in L^1(\Gamma;\mathbb{R}^N)$ the mappings $v \mapsto \bar{\mathcal{A}}(w,v)$ and $v^\star \mapsto \bar{\mathcal{A}}^\star(w,v^\star)$ are convex. To simplify the notation, we set $(c_1,\ldots,c_N) = (|w_1|^{\frac{1}{2^\#-\delta}}\operatorname{sgn}(w_1),\ldots,|w_N|^{\frac{1}{2^\#-\delta}}\operatorname{sgn}(w_N))$ in the following. Due to the definition of $\mathcal{A}^\star$ and $\bar{\mathcal{B}}$, it is for any $v \in H^1(\Omega;\mathbb{R}^N)$

$$-\mathcal{A}(c,\bar{\mathcal{B}}(c,0)) = \mathcal{A}^\star(c,0) \geq -\mathcal{A}(c,v).$$

Thus, with estimate (A12) on G and (9), we find the lower bound

$$\begin{aligned}\bar{\mathcal{A}}(w,v) = \mathcal{A}(c,v) \geq \mathcal{A}(c,\bar{\mathcal{B}}(c,0)) &= \frac{1}{2}\|\nabla\bar{\mathcal{B}}(c,0)\|_{L^2}^2 - \int_\Gamma G(c,\bar{\mathcal{B}}(c,0))\mathrm{d}\omega\\ &\geq -C\left(1+\|c\|_{L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)}^{2^\#-\delta} + \|\bar{\mathcal{B}}(c,0)\|_{L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)}^{2^\#-\delta}\right)\\ &\geq -C\left(1+\|w\|_{L^1(\Gamma;\mathbb{R}^N)}\right).\end{aligned}$$

Using additionally the estimates (15) on $\mathcal{A}^\star$ and (9) on $\|\nabla\bar{\mathcal{B}}(c,0)\|_{L^2(\Omega;\mathbb{R}^N)}$, we can also bound $\bar{\mathcal{A}}^\star$ from below

$$\begin{aligned}\bar{\mathcal{A}}^\star(w,v^\star) = \mathcal{A}^\star(c,v^\star) &\geq -\mathcal{A}(c,\bar{\mathcal{B}}(c,0)) - C\\ &= -\frac{1}{2}\|\nabla\bar{\mathcal{B}}(c,0)\|_{L^2(\Omega)} + \int_\Gamma G(c,\bar{\mathcal{B}}(c,0))\mathrm{d}\omega - C\\ &\geq -C\left(1+\|c\|_{L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)}^{2^\#-\delta}\right)\\ &= -C\left(1+\|w\|_{L^1(\Gamma;\mathbb{R}^N)}\right).\end{aligned}$$

Altogether, we can apply Theorem 3.5.50 of [12] to obtain the (norm $\times$ weak) lower semi-continuity of the integral operators

$$\begin{aligned}\mathcal{I}_{\bar{\mathcal{A}}}\colon L^1(0,T;L^1(\Gamma;\mathbb{R}^N)) \times L^1(0,T;H^1(\Omega;\mathbb{R}^N)) &\to \mathbb{R}\\ (w,v) &\mapsto \int_0^T \bar{\mathcal{A}}(w(t),v(t))\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{\bar{\mathcal{A}}^\star}\colon L^1(0,T;L^1(\Gamma;\mathbb{R}^N)) \times L^1(0,T;L^2(\Omega;\mathbb{R}^N)) &\to \mathbb{R}\\ (w,v^\star) &\mapsto \int_0^T \bar{\mathcal{A}}^\star(w(t),v^\star(t))\mathrm{d}t\end{aligned}$$

which finishes the proof. $\square$

Before we present the proof of the main result of this section, we cite two lemmas from [13] which will help us to pass to the limit in the variational inequality.

Lemma 17 ([13], Lemma 5.2). *Let be $p > n$, $q > 1$ and $f, \zeta \in L^q(0,T;W^{1,p}_+(\Omega))$ with $\{\zeta = 0\} \supseteq \{f = 0\}$. Furthermore, let be $\{f_M\}_{M\in\mathbb{N}} \subseteq L^q(0,T;W^{1,p}_+(\Omega))$ be a sequence with $f_M(t) \rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ as $M \to \infty$ for a.e. $t \in [0,T]$. Then, there exists a sequence $\{\zeta_M\}_{M\in\mathbb{N}} \subseteq L^q(0,T;W^{1,p}_+(\Omega))$ and constants $\nu_{M,t} > 0$ such that*

(i) $\zeta_M \to \zeta$ *in* $L^q(0,T;W^{1,p}(\Omega))$ *as* $M \to \infty$,
(ii) $\zeta_M \le \zeta$ *a.e. in* Ω_T *for all* $M \in \mathbb{N}$,
(iii) $\nu_{M,t}\zeta_M(t) \le f_M(t)$ *a.e. in* Ω *for a.e.* $t \in [0,T]$ *and for all* $M \in \mathbb{N}$.

If, in addition, $\zeta \le f$ a.e. in Ω_T then condition(iii) *can be refined to*
(iii)' $\zeta_M \le f_M$ a.e. in Ω_T for all $M \in \mathbb{N}$.

Lemma 18 ([13], Lemma 5.3). *Let be $f \in L^{p'}(\Omega;\mathbb{R}^n)$, $g \in L^1(\Omega)$ and $z \in W^{1,p}_+(\Omega)$ with $f \cdot \nabla z \ge 0$ a.e. in Ω and $\{f = 0\} \supseteq \{z = 0\}$ in an a.e. sense. Furthermore, we assume that*

$$\int_\Omega f \cdot \nabla\zeta + g\zeta \mathrm{d}x \ge 0 \quad \textit{for all } \zeta \in W^{1,p}_-(\Omega) \textit{ with } \{\zeta = 0\} \supseteq \{z = 0\}.$$

Then

$$\int_\Omega f \cdot \nabla\zeta + g\zeta \mathrm{d}x \ge \int_{\{z=0\}} \max\{g, 0\}\zeta \mathrm{d}x \quad \textit{for all } \zeta \in W^{1,p}_-(\Omega).$$

Finally, we have now all ingredients to prove the main existence result for $\varepsilon > 0$ since the (weak $\times$ norm) lower semi-continuity of $\mathcal{A}$ and $\mathcal{A}^\star$ allows us to conclude that the pair (c, μ) solves the continuity equation for the concentration with the nonlinear Newton boundary condition for the chemical potential.

Proof of Theorem 4. Using the Fenchel-Young inequality, the (norm $\times$ weak) lower semi-continuity provided by Lemma 16, the weak semi-continuity of the norm and testing equation (22) with μ_M and (31) with μ, we estimate

$$\begin{aligned}
&\nu \int_0^T (-\partial_t c, \mu)_{L^2(\Omega;\mathbb{R}^N)} \mathrm{d}t \\
&\le \nu \int_0^T \mathcal{A}_c(\mu) + \mathcal{A}^\star_c(-\partial_t c) \mathrm{d}t \\
&\le \nu \liminf_{M\to\infty} \int_0^T \mathcal{A}(c^-_M, \mu_M)\mathrm{d}t + \nu \liminf_{M\to\infty} \int_0^T \mathcal{A}^\star(c^-_M, -\partial_t \hat{c}_M)\mathrm{d}t \\
&\le \nu \liminf_{M\to\infty} \left(\int_0^T \mathcal{A}(c^-_M, \mu_M)\mathrm{d}t + \int_0^T \mathcal{A}^\star(c^-_M, -\partial_t \hat{c}_M)\mathrm{d}t \right) \\
&= \liminf_{M\to\infty} \int_0^T \nu(-\partial_t \hat{c}_M, \mu_M)_{L^2(\Omega;\mathbb{R}^N)} \mathrm{d}t \\
&= \liminf_{M\to\infty} \Bigg(\int_0^T \int_\Omega \nabla c_M : \nabla \mu_M \qquad (32) \\
&\qquad + \big(W^{\mathrm{ch}}_c(c_M) + W^{\mathrm{el}}_c(e(u_M), c_M, z_M)\big) \cdot \mu_M \mathrm{d}x\mathrm{d}t - \|\mu_M\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^N))} \Bigg) \\
&= \int_0^T \int_\Omega \nabla c : \nabla \mu + \big(W^{\mathrm{ch}}_c(c) + W^{\mathrm{el}}_c(e(u), c, z)\big) \cdot \mu \mathrm{d}x\mathrm{d}t + \liminf_{M\to\infty} \Big(-\|\mu_M\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^N))} \Big) \\
&= \int_0^T \int_\Omega \nabla c : \nabla \mu + \big(W^{\mathrm{ch}}_c(c) + W^{\mathrm{el}}_c(e(u), c, z)\big) \cdot \mu \mathrm{d}x\mathrm{d}t - \limsup_{M\to\infty} \Big(\|\mu_M\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^N))} \Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T \int_\Omega \nabla c : \nabla \mu + \left(W_c^{\mathrm{ch}}(c) + W_c^{\mathrm{el}}(e(u), c, z)\right) \cdot \mu \mathrm{d}x\mathrm{d}t - \liminf_{M\to\infty} \left(\|\mu_M\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^N))}\right) \\
&\leq \int_0^T \int_\Omega \nabla c : \nabla \mu + \left(W_c^{\mathrm{ch}}(c) + W_c^{\mathrm{el}}(e(u), c, z)\right) \cdot \mu \mathrm{d}x\mathrm{d}t - \|\mu\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^N))} \\
&= \nu \int_0^T (-\partial_t c, \mu)_{L^2(\Omega;\mathbb{R}^N)} \mathrm{d}t.
\end{aligned}
$$

Thus, in fact all inequality signs are equality signs. In particular, it is $\int_0^T (-\partial_t c, \mu)_{L^2(\Omega;\mathbb{R}^N)}\mathrm{d}t = \int_0^T \mathcal{A}_c(\mu) + \mathcal{A}_c^\star(-\partial_t c)\mathrm{d}t$. Hence, the Fenchel-Young inequality implies

$$
(-\partial_t c(t), \mu(t))_{L^2(\Omega;\mathbb{R}^N)} = \mathcal{A}(c(t), \mu(t)) + \mathcal{A}^\star(c(t), -\partial_t c(t))
$$

for almost all $t \in [0,T]$ which is equivalent to the weak formulation (16) of the first equation.

The validity of equation (17) can be obtained from (31) using a density argument. Similarly, we see that we can pass to the limit in equation (23) to obtain (18).

As a third step, we want to prove the energy inequality. The basis for the proof is the precise discrete energy inequality of Lemma 13. Due to the convergence results and the assumptions (A2) and (A7), Lebesque's generalized convergence theorem yields $\mathcal{E}_\varepsilon(q_M(t)) \to \mathcal{E}_\varepsilon(q(t))$ for almost all $t \in [0,T]$. The weak convergence of $\partial_t \hat{c}_M$, $\partial_t \hat{z}_M$ and μ_M lead with the lower semi-continuity of the L^2-norm and of the integral functionals $\mathcal{I}_\mathcal{A}$ and $\mathcal{I}_{\mathcal{A}^\star}$ to

$$
\begin{aligned}
&\int_0^t \int_\Omega -\alpha \partial_t z + \beta |\partial_t z|^2 \mathrm{d}x + \mathcal{A}(c,\mu) + \mathcal{A}^\star(c, -\partial_t c)\mathrm{d}t \\
&\quad = \int_0^t \int_\Omega -\alpha \partial_t z \mathrm{d}x\mathrm{d}t + \beta \|\partial_t z\|^2_{L^2(0,t;L^2(\Omega))} + \int_0^t \mathcal{A}(c,\mu)\mathrm{d}t + \int_0^t \mathcal{A}^\star(c, -\partial_t c)\mathrm{d}t \\
&\quad \leq \liminf_{M\to\infty} \int_0^t \int_\Omega -\alpha \partial_t \hat{z}_M + \beta |\partial_t \hat{z}_M|^2 \mathrm{d}x + \mathcal{A}(c_M^-, \mu_M) + \mathcal{A}^\star(c_M^-, -\partial_t \hat{c}_M)\mathrm{d}t.
\end{aligned}
$$

Thus, we can perform the limit process $M \to \infty$ in the discrete energy inequality (30) and obtain the desired energy inequality (20) for the continuous case.

It remains to prove the existence of $r \in L^\infty(0,T;L^q(\Omega))$ with $q = \frac{1}{2}\min(2^\star, p)$ satisfying (v) of Definition (3). Therefore, let be $\xi \in L^p(0,T;W_-^{1,p}(\Omega))$ with $\{\xi = 0\} \supseteq \{z = 0\}$. Applying Lemma 17 provides us with a sequence $\xi_M \in L^p(0,T;W_-^{1,p}(\Omega))$ and constants $\nu_{M,t} > 0$ such that it holds

$$
\xi_M \to \xi \quad \text{in } L^p(0,T;W^{1,p}(\Omega))
$$

and

$$
0 \geq \nu_{M,t}\xi_M(t) \geq -z_M(t) \quad \text{a.e. in } \Omega \text{ for almost all } t \in [0,T].
$$

Thus, we can use $\xi_M(t)$ as a test function in (24). An integration from 0 to T leads to

$$
\int_0^T \int_\Omega \left(\varepsilon |\nabla z_M|^{p-2} + 1\right) \nabla z_M \cdot \nabla \xi_M + \left(W_z^{\mathrm{el}}(e(u_M), c_M, z_M) + \beta \partial_t \hat{z}_M - \alpha\right) \xi_M \mathrm{d}x\mathrm{d}t \geq 0.
$$

In this inequality, we can pass to the limit and obtain

$$
\int_0^T \int_\Omega \left(\varepsilon |\nabla z|^{p-2} + 1\right) \nabla z \cdot \nabla \xi + \left(W_z^{\mathrm{el}}(e(u), c, z) + \beta \partial_t z - \alpha\right) \xi \mathrm{d}x\mathrm{d}t \geq 0.
$$

Clearly, this leads for almost all $t \in [0,T]$ to

$$
\int_\Omega \left(\varepsilon |\nabla z|^{p-2} + 1\right) \nabla z \cdot \nabla \xi + \left(W_z^{\mathrm{el}}(e(u), c, z) + \beta \partial_t z - \alpha\right) \xi \mathrm{d}x \geq 0
$$

for all $\xi \in W^{1,p}_-(\Omega)$ with $\{\xi = 0\} \supseteq \{z(t) = 0\}$. Thus, the requirements of Lemma 18 are exactly fulfilled and we conclude

$$\begin{aligned}
&\int_\Omega \left(\varepsilon|\nabla z|^{p-2} + 1\right) \nabla z \cdot \nabla \xi + \left(W^{\mathrm{el}}_z(e(u), c, z) + \beta\partial_t z - \alpha\right) \xi \mathrm{d}x \\
&\quad\geq \int_{\{z(t)=0\}} \left(W^{\mathrm{el}}_z(e(u), c, z) + \beta\partial_t z - \alpha\right)^+ \xi \mathrm{d}x \\
&\quad\geq \int_{\{z(t)=0\}} \left(W^{\mathrm{el}}_z(e(u), c, z)\right)^+ \xi \mathrm{d}x
\end{aligned} \tag{33}$$

for any $\xi \in W^{1,p}_-(\Omega)$ and almost all $t \in [0, T]$. Now, we define $r(t) = -\chi_{z(t)=0} \left(W^{\mathrm{el}}_z(e(u), c, z)\right)^+$ which fulfills inequality (19) by the above construction. Furthermore, it holds $r \in L^\infty(0, T; L^q(\Omega))$ and for any $\xi \in L^2_+(\Omega)$ and almost all $t \in [0, T]$, it follows

$$\int_\Omega r(t) \left(\xi - z(t)\right) \mathrm{d}x = -\int_{\{z(t)=0\}} \left(W^{\mathrm{el}}_z(e(u), c, 0)\right)^+ \xi \mathrm{d}x \leq 0. \tag{34}$$

□

We conclude this section with two remarks. First, we want to emphasize that the argument in (32) essentially uses the regularization with $\nu > 0$ of the Cahn-Hilliard reaction equation. Without this trick we are not able to show, that the pair $(\partial_t c, \mu)$ solves the equation $\partial_t c = \Delta\mu$ with the nonlinear Newton boundary condition $\nabla\mu\nu = R(c, \mu)$ since we have neither the weak convergence of $\partial_t \hat{c}_M(t)$ and $\mu_M(t)$ for almost all t nor the strong convergence of $R(c^-_M, \mu_M)$ on $\Gamma \times [0, T]$.

The second comment concerns the special choice of r. In particular, it is $r = 0$ if we assume $W^{\mathrm{el}}_z(e, c, 0) \leq 0$. This corresponds to the situation in which one can use a simpler approach to deal with the constraint $z \geq 0$ (compare [18] and [17]). It is possible to neglect the constraint in the construction of the sequence z_M and show afterwards using a comparison argument that the constraint $z \geq 0$ is fulfilled naturally. Nevertheless, in many physically meaningful models it is $W^{\mathrm{el}}_z(e, c, 0) > 0$.

5. The limit $\varepsilon \to 0$

With the results obtained in the previous section we like to study the limit $\varepsilon \to 0$. As mentioned before, this is not the complete vanishing viscosity limit since we keep the viscosity ν positive. The energy inequality (20) constitutes the starting point for the analysis. Therefore, let $z_0 \in H^1(\Omega)$ be the initial damage profile and let $z^0_\varepsilon \in W^{1,p}(\Omega)$ be an approximating sequence with $z^0_\varepsilon \to z_0$ in $H^1(\Omega)$ and $\varepsilon\|\nabla z^0_\varepsilon\|^p_{L^p(\Omega;\mathbb{R}^n)} \to 0$. In the following, we will denote by a subscript ε a solution as constructed in the previous section belonging to $\varepsilon > 0$. For $\varepsilon \in (0, 1]$ we have for any $u_0 \in \dot{W}^{1,p}(\Omega; \mathbb{R}^n)$

$$\begin{aligned}
\mathcal{E}_\varepsilon(u_\varepsilon(t), c_\varepsilon(t), z_\varepsilon(t)) &+ \int_0^t \int_\Omega -\alpha\partial_t z_\varepsilon + \beta|\partial_t z_\varepsilon|^2 \mathrm{d}x + \mathcal{A}_{c_\varepsilon}(\mu_\varepsilon) + \mathcal{A}^\star_{c_\varepsilon}(-\partial_t c_\varepsilon) + \nu\|\partial_t c_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^N)} \mathrm{d}s \\
&\leq \mathcal{E}_\varepsilon(u_0, c_0, z^0_\varepsilon) \\
&\leq \mathcal{E}(u_0, c_0, z^0_\varepsilon) + \frac{1}{p}\|e(u_0)\|^p_{L^p(\Omega;\mathbb{R}^{n\times n})} + \frac{\varepsilon}{p}\|\nabla z^0_\varepsilon\|^p_{L^p(\Omega;\mathbb{R}^n)} \\
&\leq C.
\end{aligned}$$

This leads to the following a priori estimates.

Lemma 19. *There exists a constant $C > 0$ - only depending on ν - such that*

(i) $\|u_\varepsilon\|_{L^\infty(0,T;\dot{H}^1(\Omega))} \leq C$,
(ii) $\varepsilon^{\frac{1}{p}}\|u_\varepsilon\|_{L^\infty(0,T;\dot{W}^{1,p}(\Omega))} \leq C$,
(iii) $\|c_\varepsilon\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))} \leq C$,
(iv) $\|z_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq C$,
(v) $\varepsilon^{\frac{1}{p}}\|z_\varepsilon\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq C$,
(vi) $\|\partial_t c_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^N)} \leq C$,
(vii) $\|\partial_t z_\varepsilon\|_{L^2(\Omega_T)} \leq C$ *and*
(viii) $\|\mu_\varepsilon\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^N))} \leq C$

holds for all $\varepsilon \in (0,1]$.

Proof. This follows from the above mentioned a priori estimate (compare the proof of Corollaries 9 and 14). ☐

Using standard compactness arguments we can extract weakly convergent subsequences which we again index by ε.

Lemma 20. *There exists a subsequence and a tuple* $q = (u, c, z, \mu) \in \mathcal{U}_0 \times \mathcal{C} \times \mathcal{Z}_0 \times \mathcal{M}$ *such that it holds*

(i) $u_\varepsilon \overset{\star}{\rightharpoonup} u$ *in* $L^\infty(0,T;\dot{H}^1(\Omega;\mathbb{R}^n))$
$\varepsilon^{\frac{1}{p-1}} e(u_\varepsilon) \to 0$ *in* $L^\infty(0,T;L^p(\Omega;\mathbb{R}^{n\times n}))$

(ii) $c_\varepsilon \overset{\star}{\rightharpoonup} c$ *in* $L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))$
$c_\varepsilon \to c$ *in* $L^q(0,T;L^{2^\star-\delta}(\Omega;\mathbb{R}^N))$
$c_\varepsilon \to c$ *in* $L^{2^\#-\delta}(0,T;L^{2^\#-\delta}(\Gamma;\mathbb{R}^N))$
$c_\varepsilon(t) \rightharpoonup c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$ *a.e.*
$c_\varepsilon \to c$ *a.e. in* Ω_T
$c_\varepsilon \rightharpoonup c$ *in* $H^1(0,T;L^2(\Omega;\mathbb{R}^N))$

(iii) $z_\varepsilon \overset{\star}{\rightharpoonup} z$ *in* $L^\infty(0,T;H^1(\Omega))$
$z_\varepsilon \to z$ *in* $L^q(0,T;L^{2^\star-\delta}(\Omega))$
$\varepsilon^{\frac{1}{p-1}} \nabla z_\varepsilon \to 0$ *in* $L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))$
$z_\varepsilon(t) \rightharpoonup z(t)$ *in* $H^1(\Omega)$ *a.e.*
$z_\varepsilon \to z$ *a.e. in* Ω_T
$z_\varepsilon \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$

(iv) $\mu_\varepsilon \rightharpoonup \mu$ *in* $L^2(0,T;H^1(\Omega;\mathbb{R}^N))$

for any $1 \le q < \infty$ *and* $\delta > 0$.

Proof. (i) and (iv) follows directly from the a priori estimates (i), (ii) and (viii) of Lemma 19. The convergence results for c and z except for the third statement of (ii) and (iii) follows exactly the lines of the proof of the convergence of c_M in Lemma 10 since we have the same a priori estimates. The third statement of (iii) is a direct consequence of the a priori estimate (v) of Lemma 19. Last but not least, to prove the strong convergence of c_ε in $L^{2^\#-\delta}(0,T;L^{2^\#-\delta}(\Gamma;\mathbb{R}^N))$ we use the compactness of the trace operator. The weak convergence $c_\varepsilon(t) \to c(t)$ in $H^1(\Omega;\mathbb{R}^N)$ a.e. implies the strong convergence $c_\varepsilon(t) \to c(t)$ in $L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)$ a.e. Thus, using the boundedness $\|c_\varepsilon(t)\|_{L^{2^\#-\delta}(\Gamma;\mathbb{R}^N)} \le C\|c_\varepsilon(t)\|_{H^1(\Omega;\mathbb{R}^N)} \le C$ we can apply Lebesgue's dominated convergence theorem to obtain the desired convergence. ☐

For u_ε we can strengthen the convergence result using a similar argument as in the proof of Lemma 11.

Lemma 21. *There exists a subsequence with*

$$\begin{aligned} u_\varepsilon &\to u && \textit{in } L^q(0,T;\dot{H}^1(\Omega;\mathbb{R}^n)) \\ u_\varepsilon(t) &\to u(t) && \textit{in } \dot{H}^1(\Omega;\mathbb{R}^n) \textit{ a.e.} \\ e(u_\varepsilon) &\to e(u) && \textit{a.e. in } \Omega_T \end{aligned}$$

for any $1 \le q < \infty$.

Proof. We would like to use a representative of the equivalence class $u_\varepsilon - u$ as a test function in (23) but since the representatives of $u(t)$ are not necessarily in $W^{1,p}(\Omega)$ we need to approximate u. Therefore, let $\tilde{u}_k \in L^p(0,T;\dot{W}^{1,p}(\Omega;\mathbb{R}^n))$ be an approximating sequence of u, i.e.

$$\tilde{u}_k \to u \text{ in } L^2(0,T;\dot{H}^1(\Omega;\mathbb{R}^n)) \text{ for } k \to \infty. \tag{35}$$

Since k is independent of ε, we can choose k_ε with

$$\varepsilon^{\frac{1}{p}} \|e(\tilde{u}_{k_\varepsilon})\|_{L^p(\Omega_T;\mathbb{R}^{n\times n})} \to 0 \text{ and } k_\varepsilon \to \infty \text{ for } \epsilon \to 0. \tag{36}$$

We use any representative ξ of $u_\varepsilon - \tilde{u}_{k_\varepsilon}$ as a test function in equation (23) and integrate from 0 to T. Then, the assumption (A3) and the inequality (27) yield

$$
\begin{aligned}
&\frac{\eta}{2}\|e(u)-e(u_\varepsilon)\|^2_{L^2(\Omega_T;\mathbb{R}^{n\times n})}\\
&\leq \eta\|e(u)-e(\tilde{u}_{k_\varepsilon})\|^2_{L^2(\Omega_T;\mathbb{R}^{n\times n})} + \eta\|e(u_\varepsilon)-e(\tilde{u}_{k_\varepsilon})\|^2_{L^2(\Omega_T;\mathbb{R}^{n\times n})} + \varepsilon C_{uc}\|e(u_\varepsilon)-e(\tilde{u}_{k_\varepsilon})\|^p_{L^p(\Omega_T;\mathbb{R}^{n\times n})}\\
&\leq \eta\|e(u)-e(\tilde{u}_{k_\varepsilon})\|^2_{L^2(\Omega_T;\mathbb{R}^{n\times n})}\\
&\quad + \int_{\Omega_T}\big(W^{\mathrm{el}}_e(e(u_\varepsilon),c_\varepsilon,z_\varepsilon) - W^{\mathrm{el}}_e(e(\tilde{u}_{k_\varepsilon}),c_\varepsilon,z_\varepsilon)\big) : (e(u_\varepsilon)-e(\tilde{u}_{k_\varepsilon})\mathrm{d}x\mathrm{d}t\\
&\quad + \varepsilon\int_{\Omega_T}\big(|e(u_\varepsilon|^{p-2})e(u_\varepsilon) - |e(\tilde{u}_{k_\varepsilon})|^{p-2}e(\tilde{u}_{k_\varepsilon})\big) : (e(u_\varepsilon)-e(\tilde{u}_{k_\varepsilon}))\,\mathrm{d}x\mathrm{d}t\\
&= \eta\|e(u)-e(\tilde{u}_{k_\varepsilon})\|^2_{L^2(\Omega_T;\mathbb{R}^{n\times n})} \qquad (37)\\
&\quad \underbrace{+ \int_{\Omega_T} W^{\mathrm{el}}_e(e(u_\varepsilon),c_\varepsilon,z_\varepsilon) : (e(u_\varepsilon-\tilde{u}_{k_\varepsilon}) + \varepsilon|e(u_\varepsilon)|^{p-2}e(u_\varepsilon) : (e(u_\varepsilon-\tilde{u}_{k_\varepsilon}))\mathrm{d}x\mathrm{d}t}_{=0 \text{ by } (22)}\\
&\quad - \int_{\Omega_T} W^{\mathrm{el}}_e(e(\tilde{u}_{k_\varepsilon}),c_\varepsilon,z_\varepsilon) : (e(u_\varepsilon)-e(\tilde{u}_{k_\varepsilon}))\mathrm{d}x\mathrm{d}t\\
&\quad \underbrace{- \varepsilon\int_{\Omega_T} |e(\tilde{u}_{k_\varepsilon})|^{p-2}e(\tilde{u}_{k_\varepsilon}) : (e(u_\varepsilon)-e(\tilde{u}_{k_\varepsilon}))\mathrm{d}x\mathrm{d}t}_{=:I_1}\,.
\end{aligned}
$$

Due to the a priori estimate 19 (ii) and the convergence in (36), we conclude

$$
\begin{aligned}
|I_1| &\leq \varepsilon\|e(\tilde{u}_{k_\varepsilon})\|^{p-1}_{L^p(\Omega_T;\mathbb{R}^{n\times n})}\|e(u_\varepsilon)-e(\tilde{u}_{k_\varepsilon})\|_{L^p(\Omega_T;\mathbb{R}^{n\times n})}\\
&\leq \Bigg(\underbrace{\varepsilon^{\frac{1}{p}}\|e(\tilde{u}_{k_\varepsilon})\|_{L^p(\Omega_T;\mathbb{R}^{n\times n})}}_{\to 0 \text{ by } (36)}\Bigg)^{p-1}\Bigg(\underbrace{\varepsilon^{\frac{1}{p}}\|e(u_\varepsilon)\|_{L^p(\Omega_T;\mathbb{R}^{n\times n})}}_{\leq C \text{ by } 19 \text{ (ii)}} + \underbrace{\varepsilon^{\frac{1}{p}}\|e(\tilde{u}_{k_\varepsilon})\|_{L^p(\Omega_T;\mathbb{R}^{n\times n})}}_{\to 0 \text{ by } (36)}\Bigg)\,.
\end{aligned}
$$

Furthermore, Lebesgue's generalized convergence theorem leads with Lemma 20 to the strong convergence

$$W^{\mathrm{el}}_e(e(\tilde{u}_{k_\varepsilon}),c_\varepsilon,z_\varepsilon) \to W^{\mathrm{el}}_e(e(u),c,z)$$

in $L^2(\Omega_T;\mathbb{R}^{n\times n})$. The weak-star convergence of u_ε in $L^\infty(0,T;\dot{H}^1(\Omega;\mathbb{R}^n))$ yields together with the convergence in (35) the weak convergence

$$e(u_\varepsilon)-e(\tilde{u}_{k_\varepsilon}) \rightharpoonup 0$$

in $L^2(\Omega_T;\mathbb{R}^{n\times n})$. Thus, all terms on the right hand side of inequality (37) converge to zero and it follows the strong convergence $e(u_\varepsilon)\to e(u)$ in $L^2(\Omega_T;\mathbb{R}^{n\times n})$. As it is $u(t)\in\dot{H}^1(\Omega;\mathbb{R}^n)$ and $u_\varepsilon(t)\in\dot{W}^{1,p}(\Omega;\mathbb{R}^n)\subset\dot{H}^1(\Omega;\mathbb{R}^n)$ for a.e. $t\in[0,T]$, Korn's inequality yields the convergence $u_\varepsilon\to u$ in $L^2(0,T;\dot{H}^1(\Omega;\mathbb{R}^n))$ and the boundedness statement in Lemma 19 (i) yields the convergence in $L^q(0,T;\dot{H}^1(\Omega;\mathbb{R}^n))$ for any $q\in[1,\infty)$. The convergences $u_\varepsilon(t)\to u(t)$ in $\dot{H}^1(\Omega;\mathbb{R}^n)$ for a.e. t and $e(u_\varepsilon)\to e(u)$ a.e. in Ω_T are then immediate consequences. □

Now, we are in the position to prove the existence result also for $\varepsilon=0$.

Proof of Theorem 5. With the previously obtained convergence results and Lebesgue's generalized convergence theorem, we can perform the limit $\varepsilon\to 0$ in the integrated version of equation (17) without problems for any $\xi\in L^2(0,T;H^1(\Omega;\mathbb{R}^N))$. For $\xi\in L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^n))$ we can also pass to the limit in the integrated version of (18). Due to the a priori estimate 19 (ii) this leads to

$$\int_0^T\int_\Omega W^{\mathrm{el}}_e(e(u),c,z) : e(\xi)\mathrm{d}x\mathrm{d}t = 0$$

and a density argument shows also the validity of (18) for all $\xi \in H^1(\Omega;\mathbb{R}^n)$. Due to the stronger assumption (A5'), we can use c_ε as a test function in (17) and perform the limit. This yields

$$\begin{aligned}\int_0^T \|\nabla c_\varepsilon\|_{L^2}^2 \mathrm{d}t &= \int_0^T \int_\Omega \big(\mu_\varepsilon - W_c^{\mathrm{ch}}(c_\varepsilon) - W_c^{\mathrm{el}}(e(u_\varepsilon), c_\varepsilon, z_\varepsilon) - \nu\partial_t c_\varepsilon\big) \cdot c_\varepsilon \mathrm{d}x\mathrm{d}t \\ &\to \int_0^T \int_\Omega \big(\mu - W_c^{\mathrm{ch}}(c) - W_c^{\mathrm{el}}(e(u), c, z) - \nu\partial_t c\big) \cdot c\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \|\nabla c\|_{L^2}^2 \mathrm{d}t.\end{aligned}$$

Thus, we have the strong convergence $c_\varepsilon \to c$ in $L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ and the equation (16) can be shown exactly as in the proof of Theorem 4.

As a next step, we will show the existence of a function r satisfying the conditions of Definition 3 (v). We recall the choice $r_\varepsilon = -\chi_{z_\varepsilon(t)=0}\left(W^{\mathrm{el}}(e(u_\varepsilon), c_\varepsilon, 0)\right)^+$ from the proof of Theorem 4. Since $\chi_{z_\varepsilon(t)=0}$ is bounded in $L^\infty(\Omega_T)$, we can extract a weak-$\star$ convergent subsequence with limit $\chi \in L^\infty(\Omega_T)$, i.e. $\chi_{z_\varepsilon(t)=0} \stackrel{\star}{\rightharpoonup} \chi$ in $L^\infty(\Omega_T)$. Now, let $\xi \in L^p(0,T;W_-^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ be a test function. From Lemma 20 we know

$$\left|\int_{\Omega_T} \varepsilon|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla\xi \mathrm{d}x\mathrm{d}t\right| \le \varepsilon\|\nabla z_\varepsilon\|_{L^p(\Omega_T)}^{p-1}\|\nabla\xi\|_{L^p(\Omega_T)} \to 0.$$

Thus, taking the limit $\varepsilon \to 0$ in the integrated version of inequality (33), we conclude

$$\int_{\Omega_T} \nabla z \cdot \nabla\xi + \big(W_z^{\mathrm{el}}(e(u), c, z) + \beta\partial_t z - \alpha\big)\,\xi\mathrm{d}x\mathrm{d}t \ge \int_{\Omega_T} \chi\left(W_z^{\mathrm{el}}(e(u), c, 0)\right)^+ \xi\mathrm{d}x\mathrm{d}t.$$

We set $r = \chi\left(W_z^{\mathrm{el}}(e(u), c, 0)\right)^+$. Then, it holds $r \in L^q(0,T;L^1(\Omega))$ for any $1 \le q < \infty$ and

$$\int_\Omega \nabla z \cdot \nabla\xi + \big(W_z^{\mathrm{el}}(e(u), c, z) + \beta\partial_t z - \alpha\big)\,\xi\mathrm{d}x \ge \int_\Omega r\xi\mathrm{d}x$$

for any $\xi \in H_-^1(\Omega) \cap L^\infty(\Omega)$ and almost all $t \in [0,T]$. Now, let be $\xi \in H_+^1(\Omega) \cap L^\infty(\Omega)$ and $\zeta \in L^\infty(0,T)$ with $\zeta \ge 0$ a.e. on $[0,T]$. Then, inequality (34) yields

$$\begin{aligned}0 &\ge \int_0^T \left(\int_\Omega r_\varepsilon(t)(\xi - z_\varepsilon(t))\mathrm{d}x\right)\zeta(t)\mathrm{d}t = \int_{\Omega_T} r_\varepsilon(\xi - z_\varepsilon)\zeta\mathrm{d}x\mathrm{d}t \\ &\to \int_{\Omega_T} r(\xi - z)\zeta\mathrm{d}x\mathrm{d}t = \int_0^T \left(\int_\Omega r(t)(\xi - z(t))\mathrm{d}x\right)\zeta(t)\mathrm{d}t.\end{aligned}$$

Since ζ was an arbitrary non-negative function, it also holds $0 \ge \int_\Omega r(t)(\xi - z(t))\mathrm{d}x$ for almost all $t \in [0,T]$.

Thus, it only remains to prove the energy inequality. This can again be done similarly as in the proof of Theorem 4. Due to the weak convergence of $\partial_t c_\varepsilon$, $\partial_t z_\varepsilon$ and μ_ε in $L^2(0,T;L^2))$ and of $\nabla c_\varepsilon(t)$ and $\nabla z_\varepsilon(t)$ in L^2 for a.e. $t \in [0,T]$, the semi-continuity of the L^2-norm and of $\mathcal{A}$ and $\mathcal{A}^\star$ yield

(compare proof of Theorem 4)

$$
\begin{aligned}
\mathcal{E}_0(u(t),c(t),z(t)) &+ \int_0^t \int_\Omega -\alpha\partial_t z + \beta|\partial_t z|^2 \mathrm{d}x + \mathcal{A}(c,\mu) + \mathcal{A}^\star(c,-\partial_t c) + \nu\|\partial_t c\|^2_{L^2(\Omega;\mathbb{R}^N)} \mathrm{d}s \\
&\leq \liminf_{\varepsilon\searrow 0}\Bigg(\mathcal{E}_0(u_\varepsilon(t),c_\varepsilon(t),z_\varepsilon(t)) \\
&\quad + \int_0^t \int_\Omega -\alpha\partial_t z_\varepsilon + \beta|\partial_t z_\varepsilon|^2 \mathrm{d}x + \mathcal{A}(c_\varepsilon,\mu_\varepsilon) + \mathcal{A}^\star(c_\varepsilon,-\partial_t c_\varepsilon) + \nu\|\partial_t c_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^N)} \mathrm{d}s \Bigg) \\
&\leq \liminf_{\varepsilon\searrow 0}\Bigg(\mathcal{E}_\varepsilon(u_\varepsilon(t),c_\varepsilon(t),z_\varepsilon(t)) \\
&\quad + \int_0^t \int_\Omega -\alpha\partial_t z_\varepsilon + \beta|\partial_t z_\varepsilon|^2 \mathrm{d}x + \mathcal{A}(c_\varepsilon,\mu_\varepsilon) + \mathcal{A}^\star(c_\varepsilon,-\partial_t c_\varepsilon) + \nu\|\partial_t c_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^N)} \mathrm{d}s \Bigg) \\
&\leq \liminf_{\varepsilon\searrow 0} \mathcal{E}_\varepsilon(u_0,c_0,z_\varepsilon^0) \\
&= \mathcal{E}(u_0,c_0,z_0)
\end{aligned}
$$

for any $u_0 \in \dot{W}^{1,p}(\Omega;\mathbb{R}^n)$. The density of $\dot{W}^{1,p}(\Omega;\mathbb{R}^n)$ in $\dot{H}^1(\Omega;\mathbb{R}^n)$ shows also the validity of the inequality for $u_0 \in \dot{H}^1(\Omega;\mathbb{R}^n)$. □

6. Conclusion

In this paper, we have shown how to use a generalized gradient structure to deal with a non-linear Newton boundary condition for the potential in the Cahn-Larché framework. In particular, we have presented how this gradient structure can be used in order to construct a weak solution by a time-discretization. For the passage of the limit from the discretization to the continuous solution we use an additional viscosity term to regularize the solution in the first instance.

Furthermore, we have proven that the whole procedure can be performed simultaneously with the discretization of an additional doubly nonlinear differential inclusion. In this way, we were able to construct a solution of a coupled nonlinear system of evolution equations.

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