Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

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Study of the bifurcation of a multiple limit cycle of the second kind by means of a Dulac-Cherkas function: A case study

Klaus R. Schneider[1], Alexander Grin[2]

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[1] Weierstraß-Institut
Mohrenstraße 39
10117 Berlin, Germany
E-Mail: Klaus.Schneider@wias-berlin.de

[2] Yanka Kupala State University of Grodno
Ozheshko Street 22
230023 Grodno, Belarus
E-Mail: grin@grsu.by

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. We consider a generalized pendulum equation depending on the scalar parameter μ having for $\mu = 0$ a limit cycle Γ of the second kind and of multiplicity three. We study the bifurcation behavior of Γ for $-1 \leq \mu \leq (\sqrt{5}+3)/2$ by means of a Dulac-Cherkas function.

1. INTRODUCTION

We consider planar autonomous differential systems

$$\frac{dx}{dt} = P(x, y, \mu), \qquad \frac{dy}{dt} = Q(x, y, \mu) \tag{1.1}$$

depending on the scalar parameter $\mu \in \mathcal{I}$, where $P, Q : \mathbb{R} \times \mathbb{R} \times \mathcal{I} \to \mathbb{R}$ are periodic in the first variable with period 2π. Under this assumption we can identify the phase space of (1.1) with the cylinder $\mathcal{Z} := \mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ is the unit circle. Interpreting x as arclength φ on $\mathbb{S}^1$ we will use in the following the notation

$$\frac{d\varphi}{dt} = P(\varphi, y, \mu), \qquad \frac{dy}{dt} = Q(\varphi, y, \mu). \tag{1.2}$$

It is well-known that to given μ the qualitative behavior of the trajectories of system (1.2) on $\mathcal{Z}$ is determined by the singular orbits, namely equilibria, limit cycles and orbits connecting equilibria (homoclinic and heteroclinic orbits). We note that in the case of a cylindric phase space we have to distinguish two types of limit cycles. A limit cycle $\Gamma(\mu)$ of system (1.2) on $\mathcal{Z}$ is called a limit cycle of the first kind, if $\Gamma(\mu)$ is contractible to a point on $\mathcal{Z}$, $\Gamma(\mu)$ is called a limit cycle of the second kind, if $\Gamma(\mu)$ surrounds the cylinder $\mathcal{Z}$, that means, it is not contractible to a point on $\mathcal{Z}$ [1, 2]. Analogously, there are two types of homoclinic orbits: homoclinic orbits of the first kind (contractible to a point) and homoclinic orbits of the second kind which surround the cylinder $\mathcal{Z}$. A crucial problem in the qualitative theory is to establish and to locate global singular trajectories and to estimate their number. A traditional approach to prove the existence of at least one limit cycle for planar system consists in constructing a Poincaré-Bendixson region whose boundaries are crossed transversally by the trajectories. In a recent paper by Giacomini and Grau [6] an algorithm has been provided to construct such boundaries in the form of conics. Another method to estimate the number of limit cycles consists in constructing a Dulac function. This method has been extended by Cherkas [3]. The advantage of his approach consists in the possibility to derive results on the existence of a unique limit cycle in some annular region and to determine its stability. In our paper [7] we called the corresponding functions as Dulac-Cherkas functions. A method to construct such functions numerically is described in [4]. The method of Dulac-Cherkas function can also be applied to study limit cycles of the second kind on a cylinder [5].

If we are interested in the dependence of the phase portrait of (1.2) on the parameter $\mu \in \mathcal{I}$, where $\mathcal{I}$ is some interval, then we have to determine those values of μ in $\mathcal{I}$ which are related to a possible change of the phase portrait and we have to study the changes of the phase portrait when μ passes such critical parameter value. In this paper we are interested in the change of the phase portrait when the critical parameter value is related to the existence of a limit cycle of the second kind of multiplicity three. Our goal is to show by means of a special example that a Dulac-Cherkas function can be used to study the bifurcation locally and globally.

For this purpose we consider the scalar second order equation

$$\frac{d^2\varphi}{dt^2} - \sum_{i=0}^{3} g_i(\varphi,\mu)\Big(\frac{d\varphi}{dt}\Big)^i = 0, \tag{1.3}$$

where μ is a scalar real parameter belonging to some interval $\mathcal{I}$, the functions $g_i : \mathbb{R}\times\mathcal{I} \to \mathbb{R}$, $i = 0, 1, 2, 3$, are defined as follows

$$g_0(\varphi,\mu) := \frac{1}{4}g(\varphi,\mu)\big[g^2(\varphi,\mu) + \mu(\sin\varphi - 6)\big],$$

$$g_1(\varphi,\mu) := \frac{3}{2}\big[g^2(\varphi,\mu) + \mu(\sin\varphi - 2)\big],$$
$$g_2(\varphi,\mu) := 3g(\varphi,\mu),$$
$$g_3(\varphi,\mu) := 2,$$

where

$$g(\varphi,\mu) := 2(1 + \mu\cos\varphi).$$

Equation (1.3) can be viewed as a generalized pendulum equation. It can be rewritten as the first order system

$$\begin{aligned} \frac{d\varphi}{dt} =& y, \\ \frac{dy}{dt} =& \frac{1}{4}g(\varphi,\mu)\big[g^2(\varphi,\mu) + \mu(\sin\varphi - 6)\big] \\ &+ \frac{3}{2}\big[g^2(\varphi,\mu) + \mu(\sin\varphi - 2)\big]y + 3g(\varphi,\mu)y^2 + 2y^3. \end{aligned} \tag{1.4}$$

For $\mu = 0$, system (1.4) takes the form

$$\frac{d\varphi}{dt} = y, \quad \frac{dy}{dt} = 2(y+1)^3,$$

that means that the second equation can be solved separately. The second equation has the unique equilibrium $y = -1$ which is an unstable equilibrium of multiplicity three. Hence, we can conclude that system (1.4) has for $\mu = 0$ a unique singular trajectory which is the unstable limit cycle

$$\Gamma(0) := \{(\varphi, y) \in \mathcal{Z} : y = -1\}$$

of the second kind of multiplicity three. To study the bifurcation behavior of $\Gamma(0)$ when μ passes the critical value 0 we will use the method of Dulac-Cherkas functions.

The paper is structured as follows: In the next section we describe the method of Dulac-Cherkas functions. In order to be able to apply it to system (1.4) we determine in section 3 an interval $\mathcal{I}_1 \subset \mathcal{I}$ containing the origin such that system (1.4) has no equilibrium for $\mu \in \mathcal{I}_1$. In section 4 we prove some general properties of limit cycles of system (1.4). In the last section we construct a Dulac-Cherkas function for system (1.4) and apply it to study the bifurcation behavior of the limit cycle $\Gamma(0)$ of the second kind of multiplicity three when μ crosses 0.

2. Method of Dulac-Cherkas functions

We consider on the cylinder $\mathcal{Z}$ the system

$$\frac{d\varphi}{dt} = P(\varphi, y, \mu), \quad \frac{dy}{dt} = Q(\varphi, y, \mu) \tag{2.1}$$

under the assumption

(A_1). $P, Q : \mathcal{Z} \times \mathcal{I} \to \mathbb{R}$ are continuous, continuously differentiable in the first two variables and 2π-periodic in the first variable.

Let f be the vector field defined by (2.1), let $\mathcal{D}$ be a subregion of $\mathcal{Z}$.

Definition 2.1. *Suppose hypothesis (A_1) to be valid. A function $B : \mathcal{D} \times \mathcal{I} \to \mathbb{R}$ having the same smoothness as P, Q and with the properties*

(i). $B(\varphi, y, \mu) = B(\varphi + 2\pi, y, \mu) \quad \forall(\varphi, y, \mu) \in \mathcal{D} \times \mathcal{I}$,
(ii). $div(Bf) \equiv (gradB, f) + Bdivf \geq 0 \ \ (\leq 0) \ \ in \ \ \mathcal{D}$ *for* $\mu \in \mathcal{I}$, *where* $div(Bf)$ *vanishes only on a subset of* $\mathcal{D}$ *of measure zero*

is called a Dulac function of system (2.1) in $\mathcal{D}$ for $\mu \in \mathcal{I}$.

The following generalization of a Dulac function is basically due to L. Cherkas [3], hence we call it Dulac-Cherkas function.

Definition 2.2. *Suppose hypothesis (A_1) to be valid. A function $\Psi : \mathcal{D} \times \mathcal{I} \to \mathbb{R}$ having the same smoothness as P, Q and with the properties*

(i). $\Psi(\varphi, y, \mu) = \Psi(\varphi + 2\pi, y, \mu) \quad \forall(\varphi, y, \mu) \in \mathcal{D} \times \mathcal{I}$.
(ii). *For $\mu \in \mathcal{I}$ the set*

$$\mathcal{W}(\mu) := \{(\varphi, y) \in \mathcal{D} : \Psi(\varphi, y, \mu) = 0\}$$

has measure zero.
(iii). *There is a real number $\kappa \neq 0$ such that for $\mu \in \mathcal{I}$*

$$\Phi(\varphi, y, \mu, \kappa) := (grad\Psi, f) + \kappa\Psi divf \geq 0 \ \ (\leq 0) \quad \textit{in } \mathcal{D}, \tag{2.2}$$

where the set

$$\mathcal{V}_\kappa(\mu) := \{(\varphi, y) \in \mathcal{D} : \Phi(\varphi, y, \mu, \kappa) = 0\}$$

has the properties
(a). *$\mathcal{V}_\kappa(\mu)$ has measure zero.*
(b). *If $\Gamma(\mu)$ is a limit cycle of (2.1), then it holds $\Gamma(\mu) \cap V_\kappa(\mu) \neq \Gamma(\mu)$.*

(iv).

$$(grad\Psi, f)_{|\mathcal{W}(\mu)} \neq 0 \tag{2.3}$$

is called a Dulac-Cherkas function of (2.1) in $\mathcal{D}$ for $\mu \in \mathcal{I}$.

Remark 2.3. *If Φ does not depend on y, that is*

$$\Phi(\varphi, y, \mu, \kappa) = \Phi_0(\varphi, \mu, \kappa)$$

and if Φ_0 vanishes only in finitely many points $\varphi_i(\mu, \kappa)$, then the conditions on the set $\mathcal{V}_\kappa(\mu)$ are fulfilled.

Remark 2.4. *From (2.3) it follows that the curves belonging to the set $\mathcal{W}(\mu)$ are crossed transversally by the trajectories of system (2.1), and that no equilibrium is located on $\mathcal{W}(\mu)$.*

Since we are focusing on limit cycles of the second kind we assume for the following

(A_2). The boundary of the region $\mathcal{D}$ consists of two closed curves Δ_u (upper closed curve) and Δ_l (lower closed curve) surrounding the cylinder $\mathcal{Z}$ and which have no common point.

The following results which have been proved in [5] show how the topological structure of $\mathcal{W}(\mu)$ influences the topological structure of the set of limit cycles of the second kind.

Theorem 2.5. *Suppose the hypotheses (A_1) and (A_2) to be valid. Let Ψ be a Dulac-Cherkas function of (2.1) in $\mathcal{D}$ for $\mu \in \mathcal{I}$. If the set $\mathcal{W}(\mu)$ is empty, then system (2.1) has at most one limit cycle of the second kind in $\mathcal{D}$.*

Now we assume

(A_3). The set $\mathcal{W}(\mu)$ consists in $\mathcal{D}$ of s isolated closed curves $\mathcal{W}_1(\mu), \mathcal{W}_2(\mu), ..., \mathcal{W}_s(\mu)$ surrounding the cylinder $\mathcal{Z}$ and which do not touch the boundaries Δ_u and Δ_l.

Without loss of generality we assume the following ordering of these curves: $\mathcal{W}_i(\mu)$ is located above $\mathcal{W}_{i+1}(\mu)$ on $\mathcal{Z}$. With this ordering we associate the following notation: The region on $\mathcal{Z}$ between $\mathcal{W}_i(\mu)$ and $\mathcal{W}_{i+1}(\mu)$ is denoted by $\mathbb{A}_i(\mu)$, the region between Δ_u and $\mathcal{W}_1(\mu)$ is denoted by $\mathbb{A}_0(\mu)$, the region between $\mathcal{W}_s(\mu)$ and Δ_l is denoted by $\mathbb{A}_s(\mu)$. Fig. 1 illustrates the case $s = 2$.

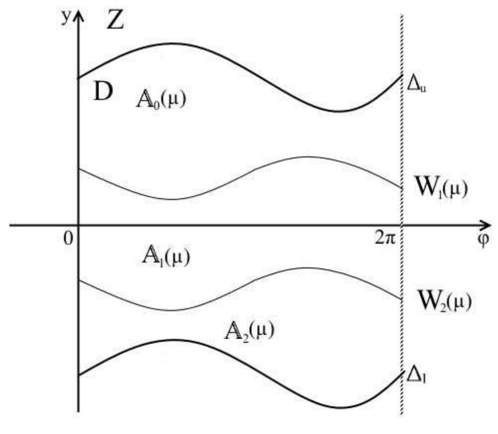

FIGURE 1. Regions $\mathbb{A}_i$ in the case $s = 2$.

Theorem 2.6. *Assume the hypotheses $(A_1) - (A_3)$ to be valid, and that $\mathcal{D}$ contains no equilibrium of (2.1). Then system (2.1) has at least s but at most $s + 2$ limit cycles of the second kind in $\mathcal{D}$, more precisely, the region $\mathbb{A}_i(\mu), i = 1, ..., s - 1,$ contains a unique limit cycle $\Gamma_i(\mu)$ of the second kind, each of the regions $\mathbb{A}_0(\mu)$ and $\mathbb{A}_s(\mu)$ may contain a unique limit cycle of the* *second kind. Furthermore, the limit cycle $\Gamma_i(\mu)$ in $\mathbb{A}_i(\mu)$ is hyperbolic and asymptotically stable (unstable) if*

$$\kappa\Phi(\varphi, y, \mu, \kappa)\Psi(\varphi, y, \mu) \;<\; 0\;(>0) \quad \textit{in} \quad \mathbb{A}_i(\mu).$$

Remark 2.7. *The condition that $\mathcal{D}$ contains no equilibrium is essential. If we suppose that for $\mu_0 < \mu < \mu_1$ there exists a Dulac-Cherkas function $\Psi(\varphi, y, \mu)$ of (2.1) in $\mathcal{D}$ to which there belong two closed curves $\mathcal{W}_1(\mu)$ and $\mathcal{W}_2(\mu)$ bounding the region $\mathbb{A}_1(\mu)$ which contains no equilibrium of (2.1) but a family $\Gamma(\mu)$ of limit cycles of the second kind. If for $\mu = \mu_1$ there arises a multiple equilibrium in $\mathbb{A}_1(\mu)$ with a homoclinic orbit, then it may happen that the family $\Gamma(\mu)$ of limit cycles can not be continued for $\mu > \mu_1$.*

3. Equilibria of system (1.4)

In order to be able to apply Theorem 2.6 to system (1.4) we have to exclude the existence of an equilibrium. Therefore, our goal in this section is to determine an interval $\mathcal{I}_1$ containing the origin such that system (1.4) has no equilibrium for $\mu \in \mathcal{I}_1$.
It is obvious that all equilibria of system (1.4) lie on the φ-axis and that their location is determined by the equations

$$g(\varphi, \mu) := 2(1 + \mu\cos\varphi) = 0, \tag{3.1}$$

$$g^2(\varphi, \mu) + \mu(\sin\varphi - 6) = 0. \tag{3.2}$$

First we consider equation (3.1). By inspection we get immediately the result

Lemma 3.1. *Equation (3.1) has no root for $|\mu| < 1$, for $\mu = -1$ it has the double root $\varphi = 0$, for $\mu = 1$ it has the double root $\varphi = \pi$, for $|\mu| > 1$ it has two simple roots.*

Now we study equation (3.2). From (3.2) we obtain

Lemma 3.2. *Equation (3.2) has no root for $\mu \leq 0$.*

In what follow we want to determine a positive number μ_1 with the properties: equation (3.2) has no root for $\mu < \mu_1$, equation (3.2) has for $\mu = \mu_1$ a real root.
From the relations

$$g^2(0, \mu) - 6\mu = g^2(2\pi, \mu) - 6\mu = 4\mu^2 + 2\mu + 4 > 0 \quad \text{for} \quad \mu > 0,$$
$$g^2(\pi, \mu) - 6\mu = 2(2\mu^2 - 7\mu + 2), g^2(\pi, 0.5) - 3 = -2, g^2(\pi, 4) - 24 = 12$$

we can conclude that there are at least two values of the parameter μ, we denote them by μ_1 and μ_3 satisfying

$$0 < \mu_1 < 0.5, \qquad 0.5 < \mu_3 < 4$$

such that (3.2) has at least two multiple roots of even multiplicity $\varphi_2^{(1)} = \varphi(\mu_1)$ and $\varphi_2^{(3)} = \varphi(\mu_3)$.

A multiple root of equation (3.2) is determined by the equation (3.2) and by the equation

$$-8\sin\varphi(1 + \mu\cos\varphi) + \cos\varphi = 0. \tag{3.3}$$

From (3.3) we get that the multiple root satisfies

$$\cos\varphi \neq 0, \qquad \sin\varphi \neq 0.$$

Thus, we can solve (3.3) for μ,

$$\mu = \frac{\cos\varphi - 8\sin\varphi}{8\sin\varphi\cos\varphi}, \tag{3.4}$$

and substitute this relation into (3.2). By this way we obtain the equation

$$\sin\varphi[\cos\varphi(16\cos + \sin\varphi - 12) + 16(6\sin\varphi - 1)] + \cos\varphi = 0. \tag{3.5}$$

It can be proved that this equation has exactly two roots in the interval $(-\pi, \pi)$. The numerical solution of (3.5) provides in the interval $(-\pi, \pi)$ the real roots

$$\varphi_2^{(1)} \approx -2.9632, \qquad \varphi_2^{(3)} \approx 3.08413.$$

Relation (3.4) yields the corresponding μ-values

$$\mu_1 \approx 0.311448, \qquad \mu_3 \approx 3.17818.$$

It can be shown that these values do not satisfy the equation

$$-8\cos\varphi - \sin\varphi - 8\mu(\cos^2\varphi - \sin^2\varphi) = 0,$$

which is necessary to be a root of order higher than two. Thus, $\varphi_2^{(1)}$ and $\varphi_2^{(3)}$ are double roots of the equation (3.2).
Taking into account Lemma 3.2 we have the result

Lemma 3.3. *The equation (3.2) has no root for $\mu < \mu_1$, for $\mu = \mu_1$ it has a double root, for $\mu_1 < \mu < \mu_3$ it has two simple roots, for $\mu = \mu_3$ it has two simple roots and a double root, for $\mu > \mu_3$ it has four simple roots.*

From Lemma 3.1 and Lemma 3.3 we get

Lemma 3.4. *System (1.4) has no equilibrium for $-1 < \mu < \mu_1$.*

4. General properties of the limit cycles of the second kind

Before we introduce a Dulac-Cherkas function we prove some general results about a limit cycle of the second kind of system (1.4).

First we prove the following basic result

Theorem 4.1. *Let $\Gamma(\mu)$ be a limit cycle of the second kind of system (1.4). Then $\Gamma(\mu)$ cannot meet the φ-axis.*

Proof. We assume that the limit cycle $\Gamma(\mu)$ has the representation

$$\varphi = \bar\varphi(t, t_0, 0, y_0, \mu), \quad y = \bar y(t, t_0, 0, y_0, \mu) \quad \forall t \in \mathbb{R}$$

with

$$\bar\varphi(t_0, t_0, 0, y_0, \mu) = 0, \;\; \bar y(t_0, t_0, 0, y_0, \mu) = y_0,$$
$$\bar\varphi(t, t_0, 0, y_0, \mu) = \bar\varphi(t + T(\mu), t_0, 0, y_0, \mu) \quad \forall t \in \mathbb{R},$$
$$\bar y(t, t_0, 0, y_0, \mu) = \bar y(t + T(\mu), t_0, 0, y_0, \mu) \quad \forall t \in \mathbb{R},$$

where the period $T(\mu)$ is a finite number. For the sequel we suppose $y_0 > 0$, the case $y_0 < 0$ can be treated analogously. We assume that there is a $t_1 > t_0$ with $\bar{y}(t_1, t_0, 0, y_0, \mu) = 0$ and $\bar{y}(t, t_0, 0, y_0, \mu) \neq 0$ for $t_0 < t < t_1$. From $(\bar{\varphi}(t_1, t_0, 0, y_0, \mu), 0) \in \Gamma(\mu)$ we can conclude that $(\bar{\varphi}(t_1, t_0, 0, y_0, \mu), 0)$ is not an equilibrium of system (1.4). Thus we have

$$\frac{d\varphi}{dt}_{|(\bar{\varphi}(t_1,t_0,0,y_0,\mu),0)} = 0, \ \frac{dy}{dt}_{|(\bar{\varphi}(t_1,t_0,0,y_0,\mu),0)} \neq 0,$$

that is, Γ meets the φ-axis vertically. Since $\Gamma(\mu)$ is a closed trajectory there is a further time moment $t_2 > t_1$ with $\bar{y}(t_2, t_0, 0, y_0, \mu) = 0$. From (1.4) we conclude $\bar{\varphi}(t_2, t_0, 0, y_0, \mu) < \bar{\varphi}(t_1, t_0, 0, y_0, \mu)$. In case $0 \leq \bar{\varphi}(t_2, t_0, 0, y_0, \mu) < \bar{\varphi}(t_1, t_0, 0, y_0, \mu)$ we can immediately conclude that $\Gamma(\mu)$ is not a closed orbit. In case $\bar{\varphi}(t_2, t_0, 0, y_0, \mu) < 0$ we consider $\bar{\varphi}(t, t_0, 0, y_0, \mu)$ for $t < t_0$. Since $\Gamma(\mu)$ is a closed trajectory there is a time moment $t_{-1} < t_0$ with $\bar{y}(t_{-1}, t_0, 0, y_0, \mu) = 0$ and $\bar{y}(t, t_0, 0, y_0, \mu) \neq 0$ for $t_{-1} < t < t_0$. In case

$$\bar{\varphi}(t_2, t_0, 0, y_0, \mu)) = \bar{\varphi}(t_{-1}, t_0, 0, y_0, \mu)$$

$\Gamma(\mu)$ is a limit cycle of the first kind. In case

$$\bar{\varphi}(t_2, t_0, 0, y_0, \mu)) \neq \bar{\varphi}(t_{-1}, t_0, 0, y_0, \mu)$$

$\Gamma(\mu)$ is not a closed orbit. Both cases contradict our assumption that $\Gamma(\mu)$ is a limit cycle of the second kind. This proves our theorem. ☐

Analogously we can prove

Theorem 4.2. *Let $\Gamma(\mu_0)$ be a homoclinic orbit of the second kind to the equilibrium $(\varphi(\mu_0), 0)$ of system (1.4). Then $\Gamma(\mu)$ meets the φ-axis only in the point $(\varphi(\mu_0), 0)$.*

Theorem 4.1 and Theorem 4.2 imply the following corollary

Corollary 4.3. *$\frac{d\varphi}{dt}$ does not change its sign on any limit cycle of the second kind and on any homoclinic orbit of the second kind of system (1.4), that is, any limit cycle of the second kind of system (1.4) is entirely located either in the region $y > 0$ or in the region $y < 0$, and any homoclinic orbit of the second kind is located either in the region $y \geq 0$ or in the region $y \leq 0$.*

The following result addresses the existence of limit cycles of the second kind when system (1.4) has no equilibrium.

Theorem 4.4. *For $-1 < \mu < \mu_1$, system (1.4) has at least one limit cycle.*

Proof. From (1.4) we get immediately that to any given μ there is a positive number $C_0(\mu)$ such that all closed curves $y = C$ and $y = -C$ on $\mathcal{Z}$ with $C \geq C_0(\mu)$ are crossed by the trajectories of (1.4) transversally. More precisely, any trajectory of (1.4) which crosses any of these closed curves crosses it for increasing t in the direction of increasing $|y|$. By Lemma 3.4 there is no equilibrium of system (1.4) in the region $\mathbb{B}(\mu)$ bounded by $y = C_0(\mu)$ and $y = -C_0(\mu)$ for $-1 < \mu < \mu_1$. Thus, we may apply the Poincaré-Bendixson Theorem which completes the proof of Theorem 4.4. ☐

Corollary 4.5. *If system (1.4) has a unique limit cycle of the second kind in $\mathbb{B}(\mu)$ then it is orbitally unstable.*

In what follows we use the null-isocline to derive a sufficient condition for system (1.4) to have no limit cycle of the second kind in the region $y > 0$.

We denote by $\mathbb{N}(\mu)$ the curve defined by

$$
\begin{aligned}
\mathbb{N}(\mu) :=& \{(\varphi, y) \in \mathcal{Z} : \frac{dy}{dt} = \frac{1}{4} g(\varphi,\mu)\big[g^2(\varphi,\mu) + \mu(\sin\varphi - 6)\big] \\
&+ \frac{3}{2}\big[g^2(\varphi,\mu) + \mu(\sin\varphi - 2)\big] y + 3g(\varphi,\mu) y^2 + 2y^3 = 0\}.
\end{aligned} \tag{4.1}
$$

It is obvious that any limit cycle $\Gamma(\mu)$ of system (1.4) must either cut the curve $\mathbb{N}(\mu)$ or be a subset of that curve.

From the relation

$$g(\varphi,\mu) := 2(1 - \mu\cos\varphi) > 0 \quad \text{for} \quad \varphi \in [0, 2\pi], |\mu| < 1$$

we can conclude that all coefficients of the polynomial in y for dy/dt in (4.1) are positive for $-1 < \mu < 0$ and $\varphi \in [0, 2\pi]$ such that $\mathbb{N}(\mu)$ is located in the region $y < 0$ for $-1 < \mu < 0$. Thus is holds

Lemma 4.6. *System (1.4) has for $-1 < \mu \leq 0$ no limit cycle in the region $y \geq 0$.*

Since the curve

$$\mathbb{N}(0) = \{(\varphi, y) \in \mathcal{Z} : y = -1\}$$

is completely located in the region $y < -0.9$ we can conclude

Lemma 4.7. *There is a positive number c_1 such that system (1.4) for $|\mu| < c_1$ has no limit cycle in the region $y \geq 0$.*

If we ask for the smallest positive value of μ for which $\mathbb{N}(\mu)$ meets the first time the line $y = 0$ we get from (4.1) that this value coincides with $\mu_1 \approx 0.311448$, where the expression $g^2(\varphi,\mu) + \mu(\sin\varphi - 6)$ has a double root. Hence, it holds

Lemma 4.8. *System (1.4) has for $-1 < \mu \leq \mu_1$ no limit cycle in the region $y \geq 0$.*

Thus we can improve Theorem 4.4.

Theorem 4.9. *For $-1 < \mu < \mu_1$, system (1.4) has at least one limit cycle in the region $y < 0$.*

5. Applying a Dulac-Cherkas function to study the bifurcation behavior of the limit cycle $\Gamma(0)$ of multiplicity three

Now we introduce a Dulac-Cherkas function to study the bifurcation behavior of the limit cycle of the second kind $y = -1$ of multiplicity three. In what follows we will show that the function

$$\Psi(\varphi, y, \mu) := \Big(y + \frac{1}{2} g(\varphi,\mu)\Big)^2 - \mu \equiv \Big(y + 1 + \mu\cos\varphi\Big)^2 - \mu \tag{5.1}$$

fulfills all conditions of Definition 2.2 for $\mu \neq 0$, that is, Ψ is a Dulac-Cherkas function of system (1.4) in $\mathcal{Z}$ for $\mu \neq 0$.

It is obvious that the function Ψ has the required smoothness and periodicity properties described in Definition 2.2. The set

$$\mathcal{W}(\mu) := \{(\varphi, y) \in \mathcal{Z} : \Psi(\varphi, y, \mu) = 0\}$$

consists for $\mu > 0$ of the two curves

$$\mathcal{W}_1(\mu) := \{(\varphi, y) \in \mathcal{Z} : y = -1 - \mu\cos\varphi + \sqrt{\mu} = 0\},$$

$$\mathcal{W}_2(\mu) := \{(\varphi, y) \in \mathcal{Z} : y = -1 - \mu\cos\varphi - \sqrt{\mu} = 0\}$$

surrounding the cylinder, $\mathcal{W}(0)$ represents the closed curve

$$\mathcal{W}(0) := \{(\varphi, y) \in \mathcal{Z} : (y+1)^2 = 0\},$$

and $\mathcal{W}(\mu)$ is empty for $\mu < 0$. Hence, $\mathcal{W}(\mu)$ satisfies the condition (ii) in Definition 2.2. Furthermore, we observe that the topological structure of the set $\mathcal{W}(\mu)$ changes when μ crosses the value 0. Later we will show that this bifurcation behavior of the set $\mathcal{W}(\mu)$ can be exploited to study the bifurcation behavior of the multiple limit cycle $y = -1$ of system (1.4).
To derive an explicit expression for the function

$$\Phi(\varphi, y, \mu, \kappa) := (grad\Psi, f) + \kappa\Psi div f \tag{5.2}$$

introduced in (2.2) we have to calculate the corresponding expressions for $(grad\Psi, f)$ and $\Psi div f$. From the relation

$$f := \Big(y, \frac{1}{4}g\big[g^2 + \mu(\sin\varphi - 6)\big] + \frac{3}{2}\big[g^2 + \mu(\sin\varphi - 2)\big]y + 3gy^2 + 2y^3\Big)$$

and (5.1) we obtain

$$\begin{aligned}(grad\Psi, f) &= 4y^4 + 8gy^3 + \big(6g^2 + \mu(\sin\varphi - 6)\big)y^2 \\ &+ g\Big(2g^2 + \mu(\sin\varphi - 6)\Big)y + \frac{1}{4}g^2(g^2 + \mu(\sin\varphi - 6)),\end{aligned} \tag{5.3}$$

$$\Psi div f = 6y^4 + 12gy^3 + 3\Big(3g^2 + \frac{1}{2}(\sin\varphi - 6)\Big)y^2$$

$$+\frac{3}{2}g\Big(2g^2 + \mu(\sin\varphi - 6)\Big)y + \frac{3}{2}\Big(g^2 + \mu(\sin\varphi - 2)\Big)\Big(\frac{g^2}{4} - \mu\Big).$$

Thus, Φ is a polynomial in y of degree four. Setting $\kappa = -\frac{2}{3}$ in (5.2) we obtain for $\mu \neq 0$

$$\Phi(\varphi, y, \mu, -\frac{2}{3}) \equiv \mu^2(\sin\varphi - 2) < 0.$$

Thus, the condition (iii) in Definition 2.2 is fulfilled.

To verify the condition (iv) we exploit the relation

$$y^2 + gy + \frac{1}{4}g^2 - \mu = 0 \tag{5.4}$$

which characterizes the set $\mathcal{W}(\mu)$. Using (5.4) we obtain from (5.3) for $\mu \neq 0$

$$
\begin{aligned}
&(grad\Psi, f)_{|\mathcal{W}_\mu} \\
&= 4y^2(y^2 + gy + \frac{1}{4}g^2 - \mu) + 4gy^3 + y^2(\mu \sin\varphi + 5g^2 - 2\mu) \\
&+ g(\mu \sin\varphi + 2g^2 - 6\mu)y + \frac{1}{4}g^4 + \frac{1}{4}\mu g^2 \sin\varphi - \frac{3}{2}\mu g^2 \\
&= 4gy(y^2 + gy + \frac{1}{4}g^2 - \mu) + y^2(\mu \sin\varphi + g^2 - 2\mu) \\
&+ g(\mu \sin\varphi + g^2 - 2\mu)y + \frac{1}{4}g^4 + \frac{1}{4}\mu g^2 \sin\varphi - \frac{3}{2}\mu g^2 \qquad (5.5)\\
&= \mu \sin\varphi(y^2 + gy + \frac{1}{4}g^2) + y^2(g^2 - 2\mu) \\
&+ y(g^3 - 2\mu g) + \frac{1}{4}g^4 + \frac{1}{4}\mu g^2 \sin\varphi - \frac{3}{2}\mu g^2 \\
&= g^2(y^2 + gy + \frac{1}{4}g^2 - \mu) - 2\mu y^2 - 2\mu g y - \frac{1}{2}g^2\mu + \mu^2 \sin\varphi \\
&= -2\mu(y^2 + gy + \frac{1}{4}g^2 - \mu) - 2\mu^2 + \mu^2 \sin\varphi = \mu^2(-2 + \sin\varphi) < 0.
\end{aligned}
$$

Therefore, condition (iv) is valid and we can conclude that for $\mu \neq 0$ the function $\Psi(\varphi, y, \mu)$ is a Dulac-Cherkas function for system (1.4).

Now we apply the Dulac-Cherkas function $\Psi(\varphi, y, \mu)$ to study the limit cycles of the second kind of system (1.4) and their bifurcation.
First we study the existence of limit cycles of the second kind of system (1.4) for $\mu < 0$. Since the set $\mathcal{W}(\mu)$ is empty for $\mu < 0$ we can conclude by Theorem 2.5 and Corollary 4.3

Lemma 5.1. *System (1.4) has for $\mu < 0$ at most one limit cycle of the second kind. If this limit cycle exists it is located either in the region $y \geq 0$ or in the region $y \leq 0$.*

If we restrict us to the interval $-1 < \mu < 0$ we can easily check that dy/dt is positive on the line $y = 0$ and negative on the line $y = -7$. Since by Lemma 3.4, system (1.4) has no equilibrium for that μ - interval, it holds

Lemma 5.2. *System (1.4) has a unique limit cycle $\Gamma_u(\mu)$ for $-1 < \mu < 0$. This limit cycle is located in the region bounded by $y = 0$ and $y = -7$, and it is unstable.*

For $\mu > 0$, we denote the region bounded by $\mathcal{W}_1(\mu)$ and $\mathcal{W}_2(\mu)$ by $\mathbb{A}_1(\mu)$, by $\tilde{\mathbb{A}}_0(\mu)$ the region bounded by the φ-axis and the curve $\mathcal{W}_1(\mu)$, and by $\tilde{\mathbb{A}}_2(\mu)$ the region bounded by the curves $\mathcal{W}_2(\mu)$ and $y = -C_0(\mu)$. From (5.5) we get that the vector field f defined by system (1.4) is directed into the region $\mathbb{A}_1(\mu)$ for $\mu > 0$. Taking into account Lemma 3.4 we get by applying Theorem 2.6 the result

Theorem 5.3. *For $0 < \mu < \mu_1 \approx 0.311448$, system (1.4) has in the region $\mathbb{A}_1(\mu)$ a unique hyperbolic asymptotically stable limit cycle $\Gamma_1^s(\mu)$.*

For $0 < \mu < \mu_1$, the vector field on the φ-axis is directed into the region $y > 0$ and on the curve $y = -C_0(\mu)$ the trajectories of system (1.4) leave the region $\tilde{\mathbb{A}}_2(\mu)$ for increasing t. Consequently, using Theorem 2.6 and the Poincaré-Bendixson Theorem we obtain the result

Theorem 5.4. *For $0 < \mu < \mu_1$, system (1.4) has in each of the regions $\tilde{\mathbb{A}}_0(\mu)$ and $\tilde{\mathbb{A}}_2(\mu)$ a unique hyperbolic unstable limit cycle which we denote by $\Gamma_0^u(\mu)$ and $\Gamma_2^u(\mu)$, respectively.*

Fig. 2 illustrates the results of Theorem 5.3 and Theorem 5.4.

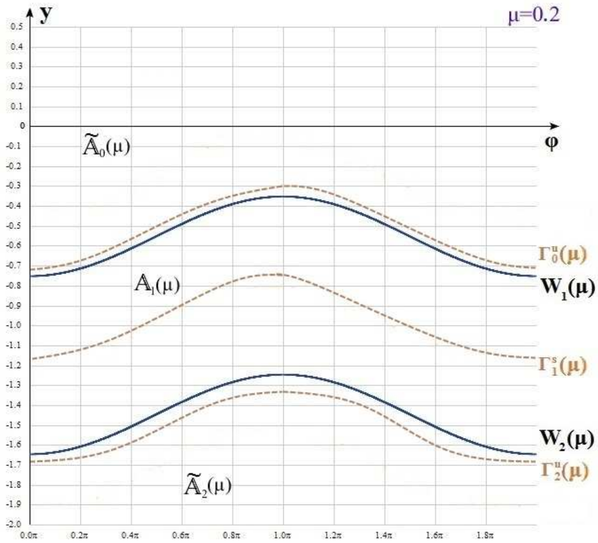

FIGURE 2. Location of the limit cycles $\Gamma_0^u(\mu), \Gamma_1^s(\mu)$ and $\Gamma_2^u(\mu)$ of system (1.4), of the curves $\mathcal{W}_1(\mu)$ and $\mathcal{W}_2(\mu)$ and of the regions $\tilde{\mathbb{A}}_0(\mu)$, $\mathbb{A}_1(\mu)$ and $\tilde{\mathbb{A}}_2(\mu)$ in the case $\mu = 0.2$.

Since for increasing μ the amplitude of the curves $\mathcal{W}_1(\mu)$ and $\mathcal{W}_2(\mu)$ is increasing we can determine the values μ_1^w and μ_2^w of μ, when $\mathcal{W}_1(\mu)$ and $\mathcal{W}_2(\mu)$ meet the φ-axis. We get

$$\mu_1^w = \frac{3-\sqrt{5}}{2} \approx 0.381966, \qquad \mu_2^w = \frac{3+\sqrt{5}}{2} \approx 3.736.$$

Since by Theorem 4.1 no limit cycle of the second kind can meet the φ-axis, we can conclude that μ_1^w and μ_2^w yield an upper bounds for the μ-intervals for which the limit cycles $\Gamma_0^u(\mu)$ and $\Gamma_1^s(\mu)$ can exists.

Theorem 5.5. *The limit cycles $\Gamma_0^u(\mu)$ and $\Gamma_1^s(\mu)$ exist for μ at most in the intervals $(0, \mu_1^w)$ and $(0, \mu_2^w)$, respectively.*

Remark 5.6. *Numerical investigations give a strong evidence that the limit cycles $\Gamma_0^u(\mu)$ bifurcates from the homoclinic orbit of the second kind to the double equilibrium of system (1.4) for $\mu = \mu_1 \approx 0.311448$ and that $\Gamma_1^s(\mu)$ bifurcates from the homoclinic orbit of the second kind to the double equilibrium for $\mu = \mu_3 \approx 3.17818$ for decreasing μ.*

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