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A logistic equation with nonlocal interactions

Luis Caffarelli[1], Serena Dipierro[2], Enrico Valdinoci[3,4,5]

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[1] The University of Texas at Austin
Department of Mathematics and
Institute for Computational Engineering and Sciences
2515 Speedway
Austin
TX 78751
USA
E-Mail: caffarel@math.utexas.edu

[2] Fakultät für Mathematik
Institut für Analysis und Numerik
Otto-von-Guericke-Universität Magdeburg
Universitätsplatz 2
39106 Magdeburg
Germany
E-Mail: serena.dipierro@ed.ac.uk

[3] Weierstrass Institute
Mohrenstrasse 39
10117 Berlin
Germany
E-Mail: Enrico.Valdinoci@wias-berlin.de

[4] Dipartimento di Matematica
Università degli studi di Milano
Via Saldini 50
20133 Milan
Italy

[5] Istituto di Matematica Applicata e Tecnologie Informatiche
Consiglio Nazionale delle Ricerche
Via Ferrata 1
27100 Pavia
Italy
E-Mail: enrico@mat.uniroma3.it

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract. We consider here a logistic equation, modeling processes of nonlocal character both in the diffusion and proliferation terms.

More precisely, for populations that propagate according to a Lévy process and can reach resources in a neighborhood of their position, we compare (and find explicit threshold for survival) the local and nonlocal case.

As ambient space, we can consider:

- bounded domains,
- periodic environments,
- transition problems, where the environment consists of a block of infinitesimal diffusion and an adjacent nonlocal one.

In each of these cases, we analyze the existence/nonexistence of solutions in terms of the spectral properties of the domain. In particular, we give a detailed description of the fact that nonlocal populations may better adapt to sparse resources and small environments.

1. Introduction

In this paper we study stationary solutions for a logistic equation. The solution u can be interpreted, from the point of view of mathematical biology, as the density of a population living in some environment $\Omega \subseteq \mathbb{R}^n$.

In the classical logistic equation (see e.g. [Ver45, MP12, PR20]), the population is supposed to increase proportionally to the resource of the environment (the growing effect being modeled by a nonnegative function σ) and to die when the resources get extinguished (the dying effect being described by a nonnegative function μ). The population is also assumed to diffuse randomly (the random diffusion being modeled by the Laplace operator). These considerations lead to a detailed study of the evolution equation

$$\partial_t u = \Delta u + (\sigma - \mu u)\, u$$

and to the stationary case of equilibrium solution described by the elliptic equation

$$\Delta u + (\sigma - \mu u)\, u = 0.$$

In this paper we will consider two variants of the latter equation, motivated by the nonlocal features of the population.

First of all, the diffusion operator of the population is considered to be nonlocal, that is, we replace the Gaussian diffusion by the one induced by Lévy flights. These types of nonlocal dispersal strategy have been observed in nature and may be related to optimal hunting strategies and adaptation to the environment stimulated by the natural selection, see e.g. [VAB+96, HQD+10] for experimental results and [Alu14] for divulgative explanations of these phenomena in popular magazines. From the mathematical point of view, taking into account this kind of nonlocal diffusion translates in our setting into the analysis of logistic equations driven by fractional Laplace operators.

Moreover, we take into account the possibility that also the increasing rate of the species has a nonlocal character. This feature is motivated in concrete cases by the fact that a population takes advantage not only of the resources that are exactly in the area in which they permanent settle, but also of the ones that are "at their reach" (say, a "giraffe's neck" effect). This nonlocal feature will be modeled for us by the convolution with an integrable kernel (from the mathematical point of view, we remark that the two types of nonlocal operators considered are very different, since the fractional Laplacian causes a loss of differentiability on the function, while the convolution produces a regularizing effect).

The precise mathematical formulation that we consider is the following. Given $s \in (0, 1)$, we consider the fractional Laplacian

$$(-\Delta)^s u(x) := 2s\,(1-s)\,PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy. \tag{1}$$

The notation "PV" denotes, as customary, the singular integral taken in the "principal value" sense, that is

$$PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy := \lim_{\delta\to 0} \int_{\mathbb{R}^n\setminus B_\delta(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy.$$

The constant $s\,(1-s)$ in (1) is just a normalizing factor, to allow ourselves to consider the case $s = 1$ as a limit. Indeed, with this choice,

$$\lim_{s\to 1}(-\Delta)^s u(x) = c_\star \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) =: -\Delta u(x),$$

for a suitable normalizing constant $c_\star > 0$, only depending on n, for any $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

The stationary logistic equation that we study is then

$$-(-\Delta)^s u + (\sigma - \mu u)\,u + \tau(J * u) = 0,$$

where σ, μ and J are nonnegative functions, $\tau \geq 0$ is a constant and $s \in (0, 1]$. As usual, $J * u$ denotes the convolution between two functions, that is, for any $x \in \mathbb{R}^n$,

$$(J * u)(x) := \int_{\mathbb{R}^n} J(x-y)u(y)\,dy.$$

We also assume that the convolution kernel is even and normalized with total mass 1, that is

$$\int_{\mathbb{R}^n} J(x)\,dx = 1 \tag{2}$$

and

$$J(-x) = J(x) \quad \text{for any } x \in \mathbb{R}^n. \tag{3}$$

We consider two types of setting for our equation: the bounded domain with Dirichlet datum (corresponding to a confined environment with hostile surrounding areas) and the periodic case. These two cases will be discussed in detail in the forthcoming subsections.

For recent investigations of different nonlocal equations arising in biological contexts, see e.g. [ABVV10, MPV13, NRRP13, HR14, MV15] and the references therein.

1.1. **Bounded domains with Dirichlet data.** The environment with hostile borders is modeled in our case by the following equation:

$$\begin{cases} (-\Delta)^s u = (\sigma - \mu u)\, u + \tau (J * u) & \text{in } \Omega, \\ u = 0 & \text{outside } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{4}$$

We will present an existence theory for nontrivial solutions and we will compare local and nonlocal behaviors of the population, analyzing their effectiveness in terms of the resource and of the domain.

In further detail, we consider the (possibly fractional) critical Sobolev exponent $2^*_s := 2n/(n-2s)$ and we state a general existence result as follows:

Theorem 1.1. *Let Ω be a bounded Lipschitz domain. Assume that $\sigma \in L^m(\Omega)$, for some $m \in (2^*_s/(2^*_s - 2), +\infty]$, and that $(\sigma + \tau)^3 \mu^{-2} \in L^1(\Omega)$. Then, there exists a solution of (4).*

To study the solutions obtained by Theorem 1.1 it is useful to compare them to the domain using a spectral analysis. For this, we denote by $\lambda_s(\Omega)$ the first Dirichlet eigenvalue for $(-\Delta)^s$ in Ω, i.e.

$$\lambda_s(\Omega) := \inf s\,(1-s) \iint_{Q_\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy,$$

where

$$Q_\Omega := (\Omega \times \mathbb{R}^n) \cup ((\mathbb{R}^n \setminus \Omega) \times \Omega) \tag{5}$$

and the infimum is taken under the conditions that $\|u\|_{L^2(\mathbb{R}^n)} = 1$ and $u = 0$ outside Ω, if $s \in (0,1)$, and, as classical,

$$\lambda_1(\Omega) := c_\star \inf_{\substack{\|u\|_{L^2(\mathbb{R}^n)}=1 \\ u \in H^1_0(\Omega)}} \int_\Omega |\nabla u|^2\, dx.$$

For a detailed study of these eigenvalues (also in the nonlocal case) see for instance Appendix A in [SV13].

The existence of nontrivial solutions to (4) can be characterized in terms of these first eigenvalues: roughly speaking, when the resource σ is too small, the only solution of (4) is the one identically zero, i.e. all the population dies; viceversa, if the resource σ is large enough, there exists a positive solution.

More precisely, we have the following:

Theorem 1.2. *Let Ω be a bounded Lipschitz domain. Assume that $\sigma \in L^m(\Omega)$, for some $m \in (2^*_s/(2^*_s - 2), +\infty]$, and that $(\sigma + \tau)^3 \mu^{-2} \in L^1(\Omega)$. Then:*

- *if $\sup_\Omega \sigma + \tau \leq \lambda_s(\Omega)$ then the only solution of (4) is the one identically zero;*
- *if $\inf_\Omega \sigma \geq \lambda_s(\Omega)$ with strict inequality on a set of positive measure and $\mu \in L^1(\Omega)$, then (4) possesses a solution u such that $u > 0$ in Ω.*

A consequence of Theorem 1.2 is that nonlocal species can better adapt to sparse resources. For instance, there exist examples of disjoint domains Ω_1 and Ω_2 such that the resource in each single Ω_i is not sufficient for the species to survive, but the combined resources in the union of the domains can be used by a nonlocal population efficiently enough. A formal statement goes as follows:

Theorem 1.3. *Let $s \in (0,1)$. Let Ω_1 be a domain in $\mathbb{R}^n$, and Ω_2 be a domain congruent to Ω_1, with $\overline{\Omega_1} \cap \overline{\Omega_2} = \varnothing$. Then, there exists $\sigma \in (0,+\infty)$ such that the only solution of*

$$\begin{cases} (-\Delta)^s u = (\sigma - \mu u)\, u & \text{in } \Omega_i, \\ u = 0 & \text{outside } \Omega_i, \\ u \geq 0 & \text{in } \mathbb{R}^n \end{cases}$$

is the trivial one, for any $i \in \{1,2\}$, but the equation

$$\begin{cases} (-\Delta)^s u = (\sigma - \mu u)\, u & \text{in } \Omega_1 \cup \Omega_2, \\ u = 0 & \text{outside } \Omega_1 \cup \Omega_2, \\ u \geq 0 & \text{in } \mathbb{R}^n \end{cases}$$

admits a positive solution in $\Omega_1 \cup \Omega_2$.

Also, in light of Theorem 1.2 it is interesting to determine for which s positive solutions of (4) may occur. Roughly speaking, when Ω is "small", the strongly diffusive species corresponding to small values of s may be favored. Viceversa, when Ω is "large", the species corresponding to small s may be annihilated. As a prototype example we present the following two results:

Proposition 1.4. *Let Ω be a bounded Lipschitz domain and set*

$$\Omega_r := \{rx, \ x \in \Omega\}.$$

Then the equation

$$\begin{cases} (-\Delta)^s u = (1 - u)\, u + \tau\, (J * u) & \text{in } \Omega_r, \\ u = 0 & \text{outside } \Omega_r, \\ u \geq 0 & \text{in } \mathbb{R}^n \end{cases}$$

admits a nontrivial solution if and only if

$$r > (\lambda_s(\Omega))^{\frac{1}{2s}}.$$

Theorem 1.5. *Fix s, $S \in (0,1]$, with $s < S$. Let Ω be a bounded Lipschitz domain and set*

$$\Omega_r := \{rx, \ x \in \Omega\}.$$

Let also J be a nonnegative function satisfying (2) and (3).

Then there exist $\overline{r} > \underline{r} > 0$ such that

- *if $r \in (0, \underline{r})$, then there exist σ_r, $\tau_r \in (0,+\infty)$ such that the equation*

$$\begin{cases} (-\Delta)^s u = (\sigma_r - u)\, u + \tau_r\, (J * u) & \text{in } \Omega_r, \\ u = 0 & \text{outside } \Omega_r, \\ u \geq 0 & \text{in } \mathbb{R}^n \end{cases} \tag{6}$$

  admits a nontrivial solution while the equation

$$\begin{cases} (-\Delta)^S u = (\sigma_r - u)\, u + \tau_r\, (J * u) & \text{in } \Omega_r, \\ u = 0 & \text{outside } \Omega_r, \\ u \geq 0 & \text{in } \mathbb{R}^n \end{cases} \tag{7}$$

  admits only the trivial solution;
- *viceversa, if $r \in (\overline{r}, +\infty)$ then there exist σ_r, $\tau_r \in (0,+\infty)$ such that equation (6) only admits the trivial solution, while equation (7) admits a nontrivial solution.*

The biological interpretation of Theorem 1.5 is that "large" environments are "more favorable" to "local" populations (namely, the population with faster diffusion related to $(-\Delta)^s$ is extinguished, while the population with slower diffusion related to $(-\Delta)^S$ is still alive); viceversa, "small" environments are "more favorable" to "nonlocal" populations (namely, in this case it is the population with slower diffusion $(-\Delta)^S$ that is extinguished, while the population with faster diffusion $(-\Delta)^s$ persists).

Another relevant question in this framework is whether or not the population fits the resources. An easy observation is that, if $\tau = 0$, the population never overcomes the maximal available resource. This follows from the more general result:

Lemma 1.6. *If $\sigma \in L^\infty(\Omega)$ and u is a solution of*

$$\begin{cases} (-\Delta)^s u = (\sigma - u)\, u + \tau(J * u) & \text{in } \Omega, \\ u = 0 & \text{outside } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^n, \end{cases}$$

then $u \leq \|\sigma\|_{L^\infty(\Omega)} + \tau$.

It is conceivable to think that large resources in a given region favor, at least locally, large density populations. We show indeed that there is a linear dependence on the largeness of the resource and the population density (independently on how large the resource is), according to the following result:

Theorem 1.7. *Let $R > r > 0$. Let Ω be a bounded Lipschitz domain, with $\overline{B_R} \subset \Omega$. Then, there exist $c_o \in (0,1)$ only depending on n, s, R and r, and $M_o > 0$ only depending on n, s and R, such that if $M \geq M_o$ and $\sigma \geq M$ in B_R, then there exists a solution u of*

$$\begin{cases} (-\Delta)^s u = (\sigma - u)\, u + \tau\,(J * u) & \text{in } \Omega, \\ u = 0 & \text{outside } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^n, \end{cases}$$

such that $u \geq c_o M$ in B_r.

Next result stresses the fact that nonlocal populations can efficiently plan their distribution in order to consume and possibly beat the given resources in a given "strategic region" (up to a small error). That is, fixing a region of interest, say the ball B_1, one can find a solution of a (slightly perturbed by an error ε) logistic equation in B_1 which exhausts the resources in B_1 and which vanishes outside B_{R_ε}, for some (possibly large) $R_\varepsilon > 1$. The "strategic plan" in this framework consists in the fact that, in order for the population to consume all the given resource in B_1, the distribution in $B_{R_\varepsilon} \setminus B_1$ must be appropriately adjusted (in particular, the logistic equation is not satisfied in $B_{R_\varepsilon} \setminus B_1$, where the population needs to be "artificially" settled from outside). The detailed statement of such result goes as follows:

Theorem 1.8. *Let $s \in (0,1)$ and $k \in \mathbb{N}$, with $k \geq 2$. Assume that*

$$\inf_{\overline{B_2}} \mu > 0, \qquad \inf_{\overline{B_2}} \sigma > 0,$$

and that σ*,* $\mu \in C^k(\overline{B_2})$*. Fix* $\varepsilon \in (0,1)$*. Then, there exist a nonnegative function* u_ε*,* $R_\varepsilon > 2$ *and* $\sigma_\varepsilon \in C^k(\overline{B_1})$ *such that*

$$(-\Delta)^s u_\varepsilon = (\sigma_\varepsilon - \mu\, u_\varepsilon)\, u_\varepsilon + \tau(J * u_\varepsilon) \qquad \text{in } B_1, \tag{8}$$

$$u_\varepsilon = 0 \qquad \text{in } \mathbb{R}^n \setminus B_{R_\varepsilon}, \tag{9}$$

$$\|\sigma_\varepsilon - \sigma\|_{C^k(\overline{B_1})} \le \varepsilon \tag{10}$$

$$\text{and} \qquad u_\varepsilon \ge \mu^{-1}\sigma_\varepsilon \qquad \text{in } B_1. \tag{11}$$

In light of Lemma 1.6 and Theorems 1.7 and 1.8, a relevant question is also whether or not the population can beat the resource, i.e. whether or not the set $\{u > \sigma\}$ is void. Notice indeed that Lemma 1.6 says that, if $\tau = 0$, this does not occur for constant resources σ. Nevertheless, when the resource is oscillatory, then this phenomenon occurs, thanks to the diffusive terms which allow the species to somewhat attains resources "from somewhere else". Namely we have the following result:

Theorem 1.9. *Let* $R > r > 0$ *and* Ω *be a bounded Lipschitz domain satisfying the exterior ball condition and such that* $\overline{B_R} \subset \Omega$*. Let* M_o *be as in Theorem 1.7.*

Let $\sigma_0 \in C(\overline{\Omega})$ *be such that* $\sigma_0 \ge M_o$ *in* B_R*. Assume also that there exists* $x_0 \in \Omega$ *such that* $\sigma_0(x_0) = 0$*, and, for any* $m \in [0,1]$*, set* $\sigma_m := \sigma + m$*. Then there exists* $m_0 > 0$ *such that for any* $m \in (0, m_0)$ *there exists a solution of*

$$\begin{cases} (-\Delta)^s u = (\sigma_m - u)\, u & \text{in } \Omega, \\ u = 0 & \text{outside } \Omega, \\ u \ge 0 & \text{in } \mathbb{R}^n \end{cases} \tag{12}$$

for which $\{u > \sigma_m\}$ *is nonvoid.*

1.2. **Periodic environments.** We now turn our attention to a periodic environment, i.e. we suppose that σ and μ are periodic with respect to translations in $\mathbb{Z}^n$ and we look for periodic solutions. In this framework, the equation that we take into account is

$$\begin{cases} (-\Delta)^s u = (\sigma - \mu u)\, u + \tau(J * u) & \text{in } \mathbb{R}^n, \\ u(x+k) = u(x) & \text{for any } k \in \mathbb{Z}^n, \\ u > 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{13}$$

We suppose here that σ and μ are bounded and periodic functions (with respect to the lattice $\mathbb{Z}^n$), that μ is positive and bounded away from zero and that J is compactly supported.

In this setting, we obtain the following existence result for periodic solutions:

Theorem 1.10. *Assume that*

$$\text{either } \sigma \text{ is not identically zero or } \tau > 0. \tag{14}$$

Then, there exists a solution of (13).

We remark that the solutions obtained in Theorem 1.10 are in general not constant (for instance, when μ is constant and σ is not). But when both σ and μ are constant then the periodic solutions need also to be constant, according to the following result:

Theorem 1.11. *Let* u *be a positive solution of* $(-\Delta)^s u = (\sigma - \mu u)u + \tau(J * u)$ *in* $\mathbb{R}^n$*. Assume that* u *is periodic with respect to* $\mathbb{Z}^n$ *and that* $\sigma \in (0, +\infty)$*,* $\mu \in (0, +\infty)$ *and* $\tau \in [0, +\infty)$ *are all constant.*

Then, u *is also constant, and constantly equal to* $(\sigma + \tau)/\mu$*.*

1.3. **A transmission problem.** Now, inspired by the recent work in [Kri15], we consider a transmission model in which the population is made of two species (or of one population that adapts to two different environments), one with a local behavior in a domain Ω_1, and one with a nonlocal behavior in a domain Ω_2, with $\Omega_1 \cap \Omega_2 = \varnothing$. The transmission problem occurs between Ω_i and its complement, for $i \in \{1, 2\}$, and it is modeled by positive parameters ν_i.

More precisely, we take two disjoint, bounded and Lipschitz domain Ω_1 and $\Omega_2 \subset \mathbb{R}^n$. We define $\Omega := \Omega_1 \cup \Omega_2$ and

$$\begin{aligned} \mathcal{T}(u) \;:=\; & \frac{1}{2}\int_{\Omega_1} |\nabla u|^2\,dx + \frac{s\,(1-s)}{2}\iint_{\Omega_2\times\Omega_2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & + \sum_{i=1}^{2} \frac{\nu_i\, s_i\,(1-s_i)}{2}\iint_{\Omega_i\times(\mathbb{R}^n\setminus\Omega_i)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_i}}\,dx\,dy + \int_\Omega \frac{\mu\,|u|^3}{3} - \frac{\sigma\,u^2}{2}\,dx. \end{aligned} \tag{15}$$

Here, s, s_1, $s_2 \in (0,1)$, σ, $\mu \in L^\infty(\Omega, [0,+\infty))$ with $\mu \geq \mu_o$, for some $\mu_o > 0$.

In this setting, we have the following existence result:

Theorem 1.12. *The functional $\mathcal{T}$ attains its minimum among the functions $u \in L^2(\Omega)$ for which*

$$\begin{aligned} & \frac{1}{2}\int_{\Omega_1} |\nabla u|^2\,dx + \frac{s\,(1-s)}{2}\iint_{\Omega_2\times\Omega_2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & \qquad + \sum_{i=1}^{2} \frac{\nu_i\, s_i\,(1-s_i)}{2}\iint_{\Omega_i\times(\mathbb{R}^n\setminus\Omega_i)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_i}}\,dx\,dy \;<\; +\infty, \end{aligned}$$

and such that $u = 0$ a.e. outside Ω.

Also, such minimizer is nonnegative.

It is worth to point out that minimizers of $\mathcal{T}$ satisfy the equations

$$\begin{aligned} & -\Delta u + \frac{\nu_1\, s_1\,(1-s_1)}{2}\int_{\mathbb{R}^n\setminus\Omega_1} \frac{u(x)-u(y)}{|x-y|^{n+2s_1}}\,dy \\ & \qquad + \frac{\nu_2\, s_1\,(1-s_2)}{2}\int_{\Omega_2} \frac{u(x)-u(y)}{|x-y|^{n+2s_2}}\,dy = (\sigma-\mu u)\,u \quad \text{in } \Omega_1 \\ \text{and} \quad & 2s\,(1-s)\,PV\int_{\Omega_2} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy + \frac{\nu_1\, s_1\,(1-s_1)}{2}\int_{\Omega_1} \frac{u(x)-u(y)}{|x-y|^{n+2s_1}}\,dy \\ & \qquad + \frac{\nu_2\, s_1\,(1-s_2)}{2}\int_{\mathbb{R}^n\setminus\Omega_2} \frac{u(x)-u(y)}{|x-y|^{n+2s_2}}\,dy = (\sigma-\mu u)\,u \quad \text{in } \Omega_2, \end{aligned} \tag{16}$$

in the weak sense (and also pointwise, by Theorem 5.5(3) in [Kri15] and Theorem 1 in [SV14]).

The biological interpretation of equation (16) is that the population has local behavior in Ω_1, with nonlocal interactions outside Ω_1, and a nonlocal transmission between the domains Ω_1 and Ω_2 takes place. See also [Kri15] for additional comments and motivations.

The existence/nonexistence of nontrivial solutions in dependence of the spectral analysis of the domain will be addressed in the following result. To this end, we define $\lambda_\star(\Omega)$ the first Dirichlet eigenvalue for the operator in (15). Namely, we set

$$\lambda_\star(\Omega) \;:=\; \inf \mathcal{T}_o(u), \tag{17}$$

where

$$\mathcal{T}_o(u) := \int_{\Omega_1} |\nabla u|^2\,dx + s\,(1-s) \iint_{\Omega_2\times\Omega_2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \sum_{i=1}^{2} \nu_i\, s_i\,(1-s_i) \iint_{\Omega_i\times(\mathbb{R}^n\setminus\Omega_i)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_i}}\,dx\,dy,$$

and the infimum in (17) is taken under the conditions that $\|u\|_{L^2(\mathbb{R}^n)} = 1$ and $u = 0$ a.e. outside Ω. In this setting, we obtain a result similar to Theorem 1.2 for the transmission problem in (15):

Theorem 1.13. *In the setting above,*

- *if* $\sup_\Omega \sigma \le \lambda_\star(\Omega)$ *then the only solution of* (16) *is the one identically zero;*
- *if* $\inf_\Omega \sigma \ge \lambda_\star(\Omega)$ *with strict inequality on a set of positive measure then* (16) *possesses a solution* u *such that* $u > 0$ *in* $\Omega_1 \cup \Omega_2$.

1.4. **Organization of the paper.** The rest of the paper is organized as follows: in Section 2 we discuss the existence of a solution by energy minimization and we prove Theorem 1.1.

Then, in Section 3, we discuss the qualitative properties of the solution and we present a proof of Theorem 1.2.

In Sections 4, 5 and 6 we discuss how the population adapts to the resources and we give the proof of Theorem 1.3, Proposition 1.4, Theorem 1.5, Lemma 1.6 and Theorem 1.7.

The strongly nonlocal diffusive strategy is considered in Section 7, where we prove Theorem 1.8.

The case in which the population actually beats the resource is discussed in Section 8, where Theorem 1.9 is proved.

The existence/nonexistence of nontrivial periodic solutions in a periodic environment is taken into account in Section 9 with the proofs of Theorems 1.10 and 1.11.

Then, in Section 10, we consider the transmission problem and we prove Theorems 1.12 and 1.13.

2. Existence theory and proof of Theorem 1.1

The proof of Theorem 1.1 is based on a minimization argument. More precisely, in order to deal with problem (4), if $s \in (0,1)$, given $u \in L^1_{\rm loc}(\mathbb{R}^n)$ with $u = 0$ a.e. outside Ω, we consider the energy functional

$$\mathcal{E}(u) := \frac{s\,(1-s)}{2} \iint_{Q_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega \frac{\mu\,|u|^3}{3} - \frac{\sigma\,u^2}{2} - \frac{\tau\,u\,(J*u)}{2}\,dx,$$

where Q_Ω is defined in (5).

When $s = 1$, instead we consider the standard energy functional

$$\mathcal{E}(u) := \frac{c_\star}{2} \int_\Omega \frac{|\nabla u|^2}{2} + \frac{\mu\,|u|^3}{3} - \frac{\sigma\,u^2}{2} - \frac{\tau\,u\,(J*u)}{2}\,dx,$$

with condition $u \in H^1_0(\Omega)$.

It is worth to point out that solutions of (4) are strictly positive, unless they vanish identically:

Lemma 2.1. *Let* u *be a nonnegative solution of* $(-\Delta)^s u = (\sigma - \mu u)\,u + \tau(J*u)$ *in* Ω*. Then either* $u > 0$ *in* Ω *or it vanishes identically.*

Proof. Suppose that $u(z) = 0$ for some $z \in \Omega$ and, by contradiction, that $u > 0$ in a set of positive measure. Then $u(z+x) - u(z) = u(z+x) \geq 0$ for any $x \in \mathbb{R}^n$, and in fact strictly positive in a set of positive measure. Accordingly, $(-\Delta)^s u(z) < 0$. Nevertheless, from (4), we have that

$$(-\Delta)^s u(z) = (\sigma(z) - \mu(z)u(z))\, u(z) + \tau(J * u)(z) = \tau(J * u)(z) \geq 0,$$

which is a contradiction. □

Equation (4) has a variational structure, according to the following observation:

Lemma 2.2. *The Euler-Lagrange equation associated to the energy functional $\mathcal{E}$ at a non-negative function u is* (4).

Proof. We denote by

$$\mathcal{J}(u) := \int_\Omega \frac{\tau\, u\, (J * u)}{2}\, dx.$$

If $\phi \in C_0^\infty(\Omega)$ and $\epsilon \in (-1, 1)$, we have that

$$\begin{aligned} &\mathcal{J}(u + \epsilon\phi) \\ =\;& \frac{\tau}{2} \int_\Omega (u + \epsilon\phi)(x)\big(J * (u + \epsilon\phi)\big)(x)\, dx \\ =\;& \frac{\tau}{2} \int_\Omega u(x)(J * u)(x) + \epsilon\Big[(u(x)(J * \phi)(x) + \phi(x)(J * u)(x)\Big] + \epsilon^2\phi(x)(J * \phi)(x)\, dx. \end{aligned}$$

As a consequence,

$$\frac{d\mathcal{J}}{d\epsilon}(u + \epsilon\phi)\Big|_{\epsilon=0} = \frac{\tau}{2} \int_\Omega \left(u(x)(J * \phi)(x) + \phi(x)(J * u)(x)\right)\, dx. \tag{18}$$

Now we recall that u and ϕ vanish outside Ω and we use (3) to see that

$$\begin{aligned} \int_\Omega u(x)(J * \phi)(x)\, dx &= \int_{\mathbb{R}^n} u(x) \left(\int_{\mathbb{R}^n} J(x-y)\phi(y)\, dy\right) dx \\ &= \int_{\mathbb{R}^n} \phi(y) \left(\int_{\mathbb{R}^n} J(x-y)u(x)\, dx\right) dy = \int_{\mathbb{R}^n} \phi(y) \left(\int_{\mathbb{R}^n} J(y-x)u(x)\, dx\right) dy \\ &= \int_{\mathbb{R}^n} \phi(y)(J * u)(y)\, dy = \int_\Omega \phi(y)(J * u)(y)\, dy. \end{aligned}$$

Using this into (18) we obtain that

$$\frac{d\mathcal{J}}{d\epsilon}(u + \epsilon\phi)\Big|_{\epsilon=0} = \tau \int_\Omega \phi(x)(J * u)(x)\, dx.$$

With this, the case $s = 1$ is standard, so we consider the case $s \in (0, 1)$. If $\phi \in C_0^\infty(\Omega)$, we have

$$\iint_{Q_\Omega} \frac{\big(u(x) - u(y)\big)\big(\phi(x) - \phi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy = \iint_{\mathbb{R}^{2n}} \frac{\big(u(x) - u(y)\big)\big(\phi(x) - \phi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy,$$

which gives the desired result. □

In the light of Lemma 2.2, to prove existence of solutions, it is useful to look at the minimizing problem for $\mathcal{E}$. We first show the following useful inequality:

Lemma 2.3. *Let v, $w \in L^2(\Omega)$ with $v = 0 = w$ a.e. outside Ω. Then*

$$\int_\Omega v(x)(J * w)(x)\,dx \le \|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}. \tag{19}$$

Proof. By the Hölder Inequality with exponents equal to 2 and the Young Inequality for convolutions with exponents 1 and 2, we have that

$$\begin{aligned}\int_\Omega v(x)(J * w)(x)\,dx &\le \|v\|_{L^2(\Omega)}\|J * w\|_{L^2(\mathbb{R}^n)}\\ &\le \|v\|_{L^2(\Omega)}\|J\|_{L^1(\mathbb{R}^n)}\|w\|_{L^2(\Omega)} = \|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)},\end{aligned}$$

where (2) was also used. This shows (19). □

Then the following existence result holds:

Proposition 2.4. *Let Ω be a bounded Lipschitz domain.*

Assume that $\sigma \in L^m(\Omega)$, for some $m \in (2^*_s/(2^*_s - 2), +\infty]$, and that $(\sigma + \tau)^3\mu^{-2} \in L^1(\Omega)$. Let also

$$p := \frac{2}{1 - \frac{1}{m}}.$$

Then $\mathcal{E}$ attains its minimum among the functions $u \in L^p(\Omega)$ for which

$$\iint_{Q_\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\,dx\,dy < +\infty$$

and such that $u = 0$ a.e. outside Ω.

Moreover, there exists a nonnegative minimizer. Finally, if u is such minimizer, it is a solution of (4).

Proof. We deal with the case $s \in (0, 1)$, since the case $s = 1$ is similar, and simpler. The proof is by direct methods. First, we notice that $p \in [2, 2^*_s)$ and

$$\frac{2}{p} + \frac{1}{m} = 1. \tag{20}$$

By (19) (used here with $v := u$ and $w := u$) we have that

$$\int_\Omega \frac{\tau\,u\,(J * u)}{2}\,dx \le \frac{\tau}{2}\int_\Omega |u|^2\,dx. \tag{21}$$

Furthermore, we use the Young Inequality, with exponents $3/2$ and 3, to see that

$$\frac{(\sigma + \tau)\,u^2}{2} = \frac{\mu^{2/3}|u|^2}{2^{2/3}} \cdot \frac{\sigma + \tau}{2^{1/3}\mu^{2/3}} \le \frac{\mu\,|u|^3}{3} + \frac{(\sigma + \tau)^3}{6\mu^2}. \tag{22}$$

As a consequence of this and (21),

$$\int_\Omega \frac{\mu\,|u|^3}{3} - \frac{\sigma\,u^2}{2} - \frac{\tau\,u\,(J * u)}{2}\,dx \ge -\int_\Omega \frac{(\sigma + \tau)^3}{6\mu^2}\,dx.$$

This implies that

$$\mathcal{E}(u) \ge \frac{s\,(1 - s)}{2}\iint_{Q_\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\,dx\,dy - \kappa,$$

for $\kappa := \|(\sigma+\tau)^3\mu^{-2}\|_{L^1(\Omega)}/6$. So we can take a minimizing sequence u_j. We may suppose that

$$0=\mathcal{E}(0)\geq\mathcal{E}(u_j)\geq\frac{s\,(1-s)}{2}\iint_{Q_\Omega}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\kappa.$$

So we set

$$\|u_j\|:=\sqrt{s\,(1-s)\iint_{Q_\Omega}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}.$$

We obtain that

$$\sqrt{s\,(1-s)\iint_{\mathbb{R}^{2n}}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}=\|u_j\|\leq\sqrt{2\kappa}.$$

Hence, by compactness, up to a subsequence u_j converges to some u in $L^p(\Omega)$ and a.e. in $\mathbb{R}^n$. So we recall (20) and we find that

$$\begin{aligned}\limsup_{j\to+\infty}\int_\Omega\sigma(u_j^2-u^2)\,dx&=\limsup_{j\to+\infty}\int_\Omega\sigma(u_j+u)(u_j-u)\,dx\\&\leq\limsup_{j\to+\infty}\|\sigma\|_{L^m(\Omega)}\|u_j+u\|_{L^p(\Omega)}\|u_j-u\|_{L^p(\Omega)}=0.\end{aligned}$$

Furthermore,

$$\int_\Omega\big(u_j\,(J*u_j)-u\,(J*u)\big)\,dx=\int_\Omega(u_j-u)\,(J*u_j)\,dx+\int_\Omega(J*u_j-J*u)\,u\,dx. \tag{23}$$

Now, by (19) with $v:=u_j-u$ and $w:=u_j$ we obtain

$$\limsup_{j\to+\infty}\int_\Omega(u_j-u)\,(J*u_j)\,dx\leq\limsup_{j\to+\infty}\|u_j-u\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)}=0. \tag{24}$$

Moreover, making again use of (19) with $v:=u$ and $w:=u_j-u$, we have that

$$\begin{aligned}\limsup_{j\to+\infty}\int_\Omega(J*u_j-J*u)\,u\,dx&=\limsup_{j\to+\infty}\int_\Omega\big(J*(u_j-u)\big)\,u\,dx\\&\leq\limsup_{j\to+\infty}\|u_j-u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}=0.\end{aligned} \tag{25}$$

So, from (23), (24) and (25), we conclude that

$$\begin{aligned}&\limsup_{j\to+\infty}\int_\Omega\big(u_j\,(J*u_j)-u\,(J*u)\big)\,dx\\&\quad\leq\limsup_{j\to+\infty}\int_\Omega(u_j-u)\,(J*u_j)\,dx+\limsup_{j\to+\infty}\int_\Omega(J*u_j-J*u)\,u\,dx=0.\end{aligned}$$

Also,

$$\liminf_{j\to+\infty}\iint_{Q_\Omega}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\geq\iint_{Q_\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

$$\text{and}\qquad\liminf_{j\to+\infty}\int_\Omega\frac{\mu\,|u_j|^3}{3}\,dx\geq\int_\Omega\frac{\mu\,|u|^3}{3}\,dx,$$

thanks to the Fatou Lemma. These inequalities imply that

$$\liminf_{j\to+\infty}\mathcal{E}(u_j)\geq\mathcal{E}(u),$$

hence u is the desired minimum.

Also, $\mathcal{E}(|u|) \le \mathcal{E}(u)$, so we can suppose in addition that u is nonnegative. Furthermore, u is a solution of (4) thanks to Lemma 2.2. □

The claim in Theorem 1.1 now follows directly from the one in Proposition 2.4.

3. Qualitative properties and proof of Theorem 1.2

The proof of Theorem 1.2 is based on energy arguments, by using the functional introduced in Section 2. The details are the following:

Proof of Theorem 1.2. Assume that $\sup_\Omega \sigma + \tau \le \lambda_s(\Omega)$. Suppose, by contradiction, that there exists a nontrivial solution to (4). Then, by Lemma 2.1, we have that $u > 0$ in Ω.

We observe that

$$\mu \text{ cannot vanish identically:} \tag{26}$$

otherwise, since $(\sigma+\tau)^3\mu^{-2} \in L^1(\Omega)$, we would have that both σ and τ vanish identically as well, thus $(-\Delta)^s u$ would vanish identically in Ω, which would imply that u vanishes identically.

Therefore, using Lemma 2.3 (with $v := u$ and $w := u$) and recalling (26), we see that

$$\begin{aligned} &\int_\Omega (\sigma - \mu u)\, u^2\, dx + \int_\Omega \tau (J * u) u\, dx \le \int_\Omega (\sigma + \tau - \mu u)\, u^2\, dx \\ &\le \lambda_s(\Omega) \int_\Omega u^2\, dx - \int_\Omega \mu u^3\, dx < \lambda_s(\Omega) \int_\Omega u^2\, dx. \end{aligned} \tag{27}$$

Now, we test (4) against u itself and we use (27) to see that

$$\begin{aligned} \lambda_s(\Omega)\|u\|^2_{L^2(\Omega)} &\le s\,(1-s) \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &= \int_\Omega (\sigma - \mu u)\, u^2\, dx + \int_\Omega \tau (J * u) u\, dx < \lambda_s(\Omega)\|u\|^2_{L^2(\Omega)}. \end{aligned}$$

This is a contradiction and it establishes the first claim in Theorem 1.2.

Now we show the second claim. For this, we suppose $\inf_\Omega \sigma \ge \lambda_s(\Omega)$ with strict inequality on a set of positive measure and we remark that it is enough to show that 0 is not a minimizer. To this goal, we take e to be the first eigenfunction of $(-\Delta)^s$ with Dirichlet datum and $\epsilon > 0$. We recall that $e > 0$ in Ω and it is bounded. Then

$$\begin{aligned} &\mathcal{E}(\epsilon e) \\ =\;& \frac{\epsilon^2}{2}\left[s\,(1-s) \iint_{\mathbb{R}^{2n}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\, dx\, dy - \int_\Omega \sigma e^2\, dx - \int_\Omega \tau (J * e) e\, dx \right] + \frac{\epsilon^3}{3} \int_\Omega \mu\, |e|^3\, dx \\ \le\;& \frac{\epsilon^2}{2} \int_\Omega (\lambda_s(\Omega) - \sigma) e^2\, dx + \frac{\epsilon^3}{3} \int_\Omega \mu\, |e|^3\, dx \\ \le\;& -c_1\epsilon^2 + c_2\epsilon^3, \end{aligned}$$

where

$$c_1 := \frac{1}{2} \int_\Omega (\sigma - \lambda_s(\Omega)) e^2\, dx \quad \text{and} \quad c_2 := \frac{1}{3}\, \|\mu\|_{L^1(\Omega)} \|e_o\|^3_{L^\infty(\Omega)}.$$

Notice that $c_1 \in (0, +\infty)$. So, if ϵ is small, $\mathcal{E}(\epsilon e) < 0 = \mathcal{E}(0)$, showing that 0 is not a minimizer, hence the minimizer of Proposition 2.4 is positive in Ω and it provides a positive solution. □

4. Adaptation to sparse resources and proof of Theorem 1.3

The proof of Theorem 1.3 is based on a spectral analysis and on the use of Theorem 1.2. The details are the following.

Proof of Theorem 1.3. Since the domains are congruent, we have that $\lambda_s(\Omega_1) = \lambda_s(\Omega_2)$. We claim that

$$\lambda_s(\Omega_1 \cup \Omega_2) < \lambda_s(\Omega_1) = \lambda_s(\Omega_2). \tag{28}$$

To prove this, we take e_i to be the first eigenfunction of Ω_i, for $i \in \{1,2\}$, normalized in such a way that $\|e_i\|_{L^2(\mathbb{R}^n)} = \|e_i\|_{L^2(\Omega_i)} = 1$. Let $e := e_1 + e_2$. Then

$$\|e\|^2_{L^2(\Omega_1\cup\Omega_2)} = \|e\|^2_{L^2(\mathbb{R}^n)} = \|e_1\|^2_{L^2(\mathbb{R}^n)} + \|e_2\|^2_{L^2(\mathbb{R}^n)} + 2\int_{\mathbb{R}^n} e_1(x)\, e_2(x)\, dx = 2, \tag{29}$$

since the supports of e_1 and e_2 are disjoint. On the other hand, we know that $e_i > 0$ in Ω_i (see e.g. Corollary 8 in [SV14]), therefore

$$\int_{\Omega_1} \left(\int_{\Omega_2} \frac{e_1(x)\, e_2(y)}{|x-y|^{n+2s}}\, dy \right) dx > 0.$$

Also, since e vanishes outside $\Omega_1 \cup \Omega_2$, we have that

$$\begin{aligned}
&\iint_{Q_{\Omega_1\cup\Omega_2}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \iint_{\mathbb{R}^{2n}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\
&\quad= \iint_{\mathbb{R}^{2n}} \frac{|e_1(x)-e_1(y)|^2 + |e_2(x)-e_2(y)|^2 + 2(e_1(x)-e_1(y))(e_2(x)-e_2(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&\quad= \iint_{Q_{\Omega_1}} \frac{|e_1(x)-e_1(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \iint_{Q_{\Omega_2}} \frac{|e_2(x)-e_2(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\
&\qquad +2 \iint_{\mathbb{R}^{2n}} \frac{(e_1(x)-e_1(y))(e_2(x)-e_2(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&\quad= \frac{\lambda_s(\Omega_1)+\lambda_s(\Omega_2)}{s\,(1-s)} + 2 \iint_{(\Omega_1\times\Omega_2)\cup(\Omega_2\times\Omega_1)} \frac{(e_1(x)-e_1(y))(e_2(x)-e_2(y))}{|x-y|^{n+2s}}\, dx\, dy.
\end{aligned}$$

Now we observe that

$$\iint_{\Omega_1\times\Omega_2} \frac{(e_1(x)-e_1(y))(e_2(x)-e_2(y))}{|x-y|^{n+2s}}\, dx\, dy = -\iint_{\Omega_1\times\Omega_2} \frac{e_1(x)e_2(y)}{|x-y|^{n+2s}}\, dx\, dy < 0.$$

Similarly,

$$\iint_{\Omega_2\times\Omega_1} \frac{(e_1(x)-e_1(y))(e_2(x)-e_2(y))}{|x-y|^{n+2s}}\, dx\, dy = -\iint_{\Omega_2\times\Omega_1} \frac{e_1(y)e_2(x)}{|x-y|^{n+2s}}\, dx\, dy < 0.$$

So we obtain that

$$\iint_{Q_{\Omega_1\cup\Omega_2}} \frac{|e(x)-e(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < \frac{\lambda_s(\Omega_1)+\lambda_s(\Omega_2)}{s\,(1-s)} = \frac{2\lambda_s(\Omega_1)}{s\,(1-s)}.$$

This and (29) imply (28), as desired.

From (28), we can take

$$\sigma \in \big(\lambda_s(\Omega_1\cup\Omega_2),\ \lambda_s(\Omega_1)\big) = \big(\lambda_s(\Omega_1\cup\Omega_2),\ \lambda_s(\Omega_2)\big).$$

Then the claim in Theorem 1.3 follows from Theorem 1.2. ∎

It is worth to notice that Theorem 1.3 relies on a purely nonlocal feature: indeed (28) fails in the local case, since

$$\lambda_1(\Omega_1 \cup \Omega_2) = \lambda_1(\Omega_1) = \lambda_1(\Omega_2). \tag{30}$$

Indeed, to prove (30), one may notice that e_1 is an admissible competitor for $\lambda_1(\Omega_1 \cup \Omega_2)$, hence $\lambda_1(\Omega_1 \cup \Omega_2) \leq \lambda_1(\Omega_1)$. On the other hand if $\phi \in H^1_0(\Omega_1 \cup \Omega_2)$, then $\phi_i := \phi\chi_{\Omega_i} \in H^1_0(\Omega_i)$ for any $i \in \{1, 2\}$ and thus

$$\frac{\int_{\Omega_1\cup\Omega_2} |\nabla\phi(x)|^2\,dx}{\int_{\Omega_1\cup\Omega_2} \phi^2(x)\,dx} = \frac{\int_{\Omega_1} |\nabla\phi_1(x)|^2\,dx + \int_{\Omega_2} |\nabla\phi_2(x)|^2\,dx}{\int_{\Omega_1} \phi_1^2(x)\,dx + \int_{\Omega_2} \phi_2^2(x)\,dx}.$$

Now we observe that if a_1, a_2, b_1 and b_2 are positive and such that $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$, then

$$\frac{a_1 + a_2}{b_1 + b_2} = \frac{b_1(a_1 + a_2)}{b_1(b_1 + b_2)} \geq \frac{a_1b_1 + a_1b_2}{b_1(b_1 + b_2)} = \frac{a_1}{b_1} = \min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\right\}.$$

As a consequence

$$\frac{\int_{\Omega_1\cup\Omega_2} |\nabla\phi(x)|^2\,dx}{\int_{\Omega_1\cup\Omega_2} \phi^2(x)\,dx} \geq \min\left\{\frac{\int_{\Omega_1} |\nabla\phi_1(x)|^2\,dx}{\int_{\Omega_1} \phi_1^2(x)\,dx}, \frac{\int_{\Omega_2} |\nabla\phi_2(x)|^2\,dx}{\int_{\Omega_2} \phi_2^2(x)\,dx}\right\} \geq \lambda_1(\Omega),$$

which shows that $\lambda_1(\Omega_1 \cup \Omega_2) \geq \lambda_1(\Omega)$ and completes the proof of (30).

5. Scaling arguments and proof of Proposition 1.4 and Theorem 1.5

The proof of Proposition 1.4 follows by a simple scaling argument, which we present here for the sake of completeness:

Proof of Proposition 1.4. By scaling, we have that

$$\lambda_s(\Omega_r) = r^{-2s}\lambda_s(\Omega). \tag{31}$$

Also, by Theorem 1.2, a nontrivial solution exists if and only if $1 > \lambda_s(\Omega_r)$. These considerations imply the desired claim. □

The proof of Theorem 1.5 combines scaling arguments and spectral analysis and it is presented here below.

Proof of Theorem 1.5. Up to a translation, we may suppose that $0 \in \Omega$. More precisely, we suppose that $B_{a_1} \subset \Omega \subset B_{a_2}$, for some $a_2 > a_1 > 0$. Then $\lambda_s(B_{a_2}) \leq \lambda_s(\Omega) \leq \lambda_s(B_{a_1})$, that is,

$$c_1\lambda_s(B_1) \leq \lambda_s(\Omega) \leq c_2\lambda_s(B_1),$$

for some $c_2 > c_1 > 0$. Furthermore,

$$\inf_{s\in(0,1]} \lambda_s(B_1) \geq c_3$$

for some $c_3 > 0$. This follows, for instance, from[1] formulas (9) and (10) in [Dyd12]. Furthermore

$$\sup_{s\in(0,1]} \lambda_s(B_1) \leq c_4.$$

[1] Regarding formula (9) of [Dyd12] we remark that the map $(0, 1) \ni s \mapsto \gamma(s) := (12n + 2s(16 - 2n))$ is monotone, therefore

$$\gamma(s) \geq \min\{\gamma(0), \gamma(1)\} = \min\{12n, 8n + 32\} > 0.$$

This and the continuity of the Γ-function in $(0, +\infty)$ imply that the quantity in (9) of [Dyd12] is bounded from below uniformly in $s \in (0, 1]$.

This may be checked by fixing $g \in C_0^\infty(B_{a_1})$ with $\|g\|_{L^2(\mathbb{R}^n)} = 1$, and using that

$$\lambda_s(\Omega) \le s\,(1-s) \iint_{Q_\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le c_5 \|g\|_{C^2(\mathbb{R}^n)},$$

for some $c_5 > 0$.

The above consideration and the scaling property (31) give that

$$c_1\,c_3\,r^{-2s} \le \lambda_s(\Omega_r) \le c_2\,c_4\,r^{-2s},$$

for any $s \in (0,1]$.

Now we fix $s < S \in (0,1]$ and we set

$$\underline{r} := \left(\frac{c_1\,c_3}{c_2\,c_4}\right)^{\frac{1}{2(S-s)}} \quad \text{and} \quad \overline{r} := \left(\frac{c_2\,c_4}{c_1\,c_3}\right)^{\frac{1}{2(S-s)}}.$$

Then, if $r \in (0, \underline{r})$ we have that

$$\lambda_S(\Omega_r) - \lambda_s(\Omega_r) \ge c_1\,c_3\,r^{-2S} - c_2\,c_4\,r^{-2s} > 0,$$

thus we can find σ_r in the interval $\big(\lambda_s(\Omega_r),\, \lambda_S(\Omega_r)\big)$. Moreover, we can also find τ_r such that

$$\lambda_s(\Omega_r) < \sigma_r \le \sigma_r + \tau_r < \lambda_S(\Omega_r).$$

From Theorem 1.2, we have that in this case equation (6) has a nontrivial solution, while (7) only has the trivial solution.

Viceversa, if $r \in (\overline{r}, +\infty)$ then $\lambda_S(\Omega_r) - \lambda_s(\Omega_r) < 0$, hence we can find σ_r in the interval $\big(\lambda_S(\Omega_r),\, \lambda_s(\Omega_r)\big)$ and τ_r such that

$$\lambda_S(\Omega_r) < \sigma_r \le \sigma_r + \tau_r < \lambda_s(\Omega_r).$$

In this case, Theorem 1.2 gives that (7) has a nontrivial solution, while (6) has only the trivial solution. □

6. Fitting the resources and proof of Lemma 1.6 and Theorem 1.7

The proof of Lemma 1.6 is a simple maximum principle, whose details are presented here below for completeness:

Proof of Lemma 1.6. Suppose by contradiction that there exists $x_o \in \Omega$ such that $0 < \max_{\mathbb{R}^n} u - \|\sigma\|_{L^\infty(\Omega)} - \tau = u(x_o) - \|\sigma\|_{L^\infty(\Omega)} - \tau$. Notice that, using (2),

$$(J * u)(x_o) = \int_{\mathbb{R}^n} J(x_o - y) u(y)\,dy \le u(x_o) \int_{\mathbb{R}^n} J(z)\,dz = u(x_o).$$

Then

$$0 \le (-\Delta)^s u(x_o) = (\sigma(x_o) - u(x_o))\,u(x_o) + \tau\,(J * u)(x_o) \le (\sigma(x_o) + \tau - u(x_o))\,u(x_o) < 0,$$

which is a contradiction. □

Now we show that u always fits the "abundant" resources (up to a multiplicative constant):

Proposition 6.1. *Let $R > r > 0$. Let Ω be a bounded Lipschitz domain, with $\overline{B_R} \subset \Omega$. Let u be the minimal solution of*

$$\begin{cases} (-\Delta)^s u = (\sigma - u)\,u + \tau\,(J * u) & \text{in } \Omega,\\ u = 0 & \text{outside } \Omega,\\ u \ge 0 & \text{in } \mathbb{R}^n, \end{cases}$$

according to Proposition 2.4.

Then, there exist $c_o \in (0,1)$ only depending on n, s, R and r, and $M_o > 0$ only depending on n, s and R, such that if $M \geq M_o$ and $\sigma \geq M$ in B_R, then $\mathcal{E}(u) < 0$ and $u \geq c_o M$ in B_r.

Proof. We take e_o to be the first Dirichlet eigenfunction of B_R. Then we have

$$\begin{aligned}
\mathcal{E}(e_o) &= \frac{s\,(1-s)}{2} \iint_{\mathbb{R}^{2n}} \frac{|e_o(x)-e_o(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega \frac{|e_o|^3}{3} - \frac{\sigma\, e_o^2}{2} - \frac{\tau\, e_o\,(J * e_o)}{2}\,dx \\
&= \frac{\lambda_s(B_R)}{2} \int_{B_R} e_o^2\,dx + \int_{B_R} \frac{|e_o|^3}{3} - \frac{\sigma\, e_o^2}{2} - \frac{\tau\, e_o\,(J * e_o)}{2}\,dx \\
&\leq \frac{\lambda_s(B_R) - M}{2} \int_{B_R} e_o^2\,dx + \frac{|B_R|\,\|e_o\|^3_{L^\infty(\mathbb{R}^n)}}{3} - \int_{B_R} \frac{\tau\, e_o\,(J * e_o)}{2}\,dx
\end{aligned}$$

The latter quantity is negative if $M \geq M_o$, for large values of M_o, therefore the energy of the minimizer u is negative and u is not the trivial function.

Consequently, from Proposition 2.4 and Lemma 2.1, we can define

$$\iota := \inf_{B_R} u > 0.$$

In particular, if $\eta \in (0,\, \iota\,\|e_o\|^{-1}_{L^\infty(\mathbb{R}^n)})$ we have that $\eta e_o \leq u$. So we take the first η for which a contact point in Ω occurs (of course, if $\eta e_o \leq u$ for all $\eta > 0$, we obtain the desired result by taking η as large as we wish, hence we can assume that such contact point exists). That is, we have that $\eta e_o \leq u$ and there exists $\bar{x} \in \Omega$ such that $\eta e_o(\bar{x}) = u(\bar{x})$. Since e_o vanishes outside B_R, we have that $\bar{x} \in B_R$. Therefore

$$\begin{aligned}
0 &\geq (-\Delta)^s (u - \eta e_o)(\bar{x}) = (\sigma(\bar{x}) - u(\bar{x}))u(\bar{x}) + \tau\,(J * u)(\bar{x}) - \eta \lambda_s(B_R)\, e_o(\bar{x}) \\
&\geq (\sigma(\bar{x}) - u(\bar{x}))u(\bar{x}) - \lambda_s(B_R)\, u(\bar{x}).
\end{aligned}$$

Accordingly,

$$0 \geq M u(\bar{x}) - u^2(\bar{x}) - \lambda_s(B_R)\, u(\bar{x}) \geq \frac{M}{2} u(\bar{x}) - u^2(\bar{x}),$$

as long as $M \geq M_o$ and M_o is large enough. This says that

$$\frac{M}{2} \leq u(\bar{x}) = \eta e_o(\bar{x}) \leq \eta\,\|e_o\|_{L^\infty(\mathbb{R}^n)}.$$

In particular $\eta \geq M/(2\,\|e_o\|_{L^\infty(\mathbb{R}^n)})$ and therefore, for any $x \in B_r$,

$$u(x) \geq \eta e_o(x) \geq \frac{\inf_{B_r} e_o}{2\,\|e_o\|_{L^\infty(\mathbb{R}^n)}}\, M. \qquad \square$$

Now, Theorem 1.7 follows plainly from Proposition 6.1.

7. Fitting the resources in a nonlocal setting and proof of Theorem 1.8

Now we prove Theorem 1.8, by exploiting a result in [DSV15], joined to a minimization argument.

More precisely, we make use of Theorem 1.1 in [DSV15], which we state here for the convenience of the reader:

Theorem 7.1. *Fix $k \in \mathbb{N}$. Then, given any function $f \in C^k(B_2)$ and any $\varepsilon > 0$, we can find $R_\varepsilon > 2$ and a function $u_\varepsilon \in C_0^s(B_{R_\varepsilon})$ such that*

$$\begin{aligned} &(-\Delta)^s u_\varepsilon = 0 \ \textit{in}\ B_2 \\ \textit{and}\qquad &\|u_\varepsilon - f\|_{C^k(\overline{B_2})} \le \varepsilon. \end{aligned}$$

The details of the proof of Theorem 1.8 now go as follows:

Proof of Theorem 1.8. First of all, we use Theorem 7.1 to find a function w_ε and a radius $R_\varepsilon > 2$ such that

$$\begin{aligned} &(-\Delta)^s w_\varepsilon = 0 \qquad \text{in } B_2, \\ &w_\varepsilon = 0 \qquad \text{in } \mathbb{R}^n \setminus B_{R_\varepsilon}, \\ \text{and}\ \ &\|w_\varepsilon - \mu^{-1}\sigma\|_{C^k(\overline{B_2})} \le \varepsilon. \end{aligned} \tag{32}$$

Let

$$W_\varepsilon := |w_\varepsilon| \ \text{ and } \ \sigma_\varepsilon := \mu w_\varepsilon. \tag{33}$$

Notice that

$$\|\sigma_\varepsilon - \sigma\|_{C^k(\overline{B_1})} = \|\mu(w_\varepsilon - \mu^{-1}\sigma)\|_{C^k(\overline{B_1})} \le C_k\, \|w_\varepsilon - \mu^{-1}\sigma\|_{C^k(\overline{B_1})} \le C_k\varepsilon,$$

for some $C_k > 0$, possibly depending on $\|\mu\|_{C^k(\overline{B_1})}$, and this proves (10) (up to renaming ε).

Moreover, if $x \in \overline{B_2}$,

$$w_\varepsilon \ge \mu^{-1}\sigma - \|w_\varepsilon - \mu^{-1}\sigma\|_{L^\infty(\overline{B_2})} \ge \inf_{B_2} \mu^{-1}\sigma - \varepsilon \ge 0,$$

if we take $\varepsilon > 0$ small enough, therefore

$$W_\varepsilon = w_\varepsilon \text{ in } \overline{B_2}. \tag{34}$$

Accordingly, for any $x \in B_1$,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{W_\varepsilon(x+y) + W_\varepsilon(x-y) - 2W_\varepsilon(x)}{|y|^{n+2s}}\, dy \ &\ge \int_{\mathbb{R}^n} \frac{w_\varepsilon(x+y) + w_\varepsilon(x-y) - 2W_\varepsilon(x)}{|y|^{n+2s}}\, dy \\ &= \int_{\mathbb{R}^n} \frac{w_\varepsilon(x+y) + w_\varepsilon(x-y) - 2w_\varepsilon(x)}{|y|^{n+2s}}\, dy \end{aligned}$$

and thus

$$-(-\Delta)^s W_\varepsilon(x) \ge 0$$

for any $x \in B_1$. As a consequence,

$$f_\varepsilon(x) := \tau(J * W_\varepsilon)(x) - (-\Delta)^s W_\varepsilon(x) \ge 0 \tag{35}$$

for any $x \in B_1$.

By (34), we get that $(-\Delta)^s W_\varepsilon \in L^\infty(B_1)$, and consequently

$$f_\varepsilon \in L^\infty(B_1). \tag{36}$$

Now we introduce the energy functional

$$\mathcal{G}(v) := \frac{s\,(1-s)}{2} \iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \int_{B_1} \frac{\mu|v|^3}{3} + \frac{\sigma_\varepsilon v^2}{2} - f_\varepsilon\, v - \frac{\tau v\,(J * v)}{2}\, dx$$

and we aim to minimize $\mathcal{G}$ among all the functions that vanish outside B_1. For this, we observe that $\mathcal{G}(0) = 0$ and we take a minimizing sequence v_j, namely

$$\lim_{j\to+\infty} \mathcal{G}(v_j) = \inf \mathcal{G}(v), \tag{37}$$

where the infimum is taken among the functions v such that $v = 0$ in $\mathbb{R}^n \setminus B_1$. We observe that, by (10), we know that

$$\inf_{\overline{B_1}} \sigma_\varepsilon > 0.$$

Also, by Lemma 2.3,

$$\int_{B_1} v_j(x)(J * v_j)(x)\, dx \le \|v_j\|^2_{L^2(B_1)}$$

and, by (22) (used here with $\sigma = 1$),

$$\frac{(1+\tau)\, v_j^2}{2} \le \frac{\mu\,|v_j|^3}{3} + \frac{(1+\tau)^3}{6\mu^2}.$$

Using these considerations, we find that

$$\begin{aligned}\int_{B_1} f_\varepsilon\, v_j + \frac{\tau v_j\,(J * v_j)}{2} - \frac{\mu|v_j|^3}{3}\, dx &\le \int_{B_1} \frac{f_\varepsilon^2 + v_j^2}{2} + \frac{\tau v_j^2}{2} - \frac{\mu|v_j|^3}{3}\, dx \\ &\le \int_{B_1} \frac{f_\varepsilon^2}{2} + \frac{(1+\tau)^3}{6\mu^2}\, dx \le C_\varepsilon,\end{aligned}$$

for some $C_\varepsilon > 0$ that does not depend on j. As a consequence,

$$\mathcal{G}(v_j) \ge \frac{s\,(1-s)}{2} \iint_{\mathbb{R}^{2n}} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \int_{B_1} \frac{\sigma_\varepsilon v_j^2}{2}\, dx - C_\varepsilon.$$

This gives that v_j is precompact in $L^2(B_1)$ (see e.g. Theorem 7.1 in [DNPV12]) and so we may suppose, up to a subsequence, that it converges to some $v_\star$ in $L^2(B_1)$ and a.e. in $\mathbb{R}^n$, with $v_\star = 0$ outside B_1.

Therefore, by Fatou Lemma,

$$\begin{aligned}&\liminf_{j\to+\infty} \frac{s\,(1-s)}{2} \iint_{\mathbb{R}^{2n}} \frac{|v_j(x) - v_j(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \int_{B_1} \frac{\mu|v_j|^3}{3} + \frac{\sigma_\varepsilon v_j^2}{2} \\ &\qquad \ge \frac{s\,(1-s)}{2} \iint_{\mathbb{R}^{2n}} \frac{|v_\star(x) - v_\star(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \int_{B_1} \frac{\mu|v_\star|^3}{3} + \frac{\sigma_\varepsilon v_\star^2}{2}.\end{aligned} \tag{38}$$

Also, by weak convergence in $L^2(B_1)$,

$$\lim_{j\to+\infty} \int_{B_1} f_\varepsilon\, v_j\, dx = \int_{B_1} f_\varepsilon\, v_\star\, dx. \tag{39}$$

In addition, by Lemma 2.3,

$$\begin{aligned}&\left| \int_{B_1} v_\star(J * v_\star) - v_j(J * v_j)\, dx \right| \\ \le\ & \int_{B_1} |v_\star - v_j|\, |J * v_\star|\, dx + \int_{B_1} |v_j|\, |J * (v_\star - v_j)|\, dx \\ \le\ & \|v_\star - v_j\|_{L^2(B_1)}\, \|v_\star\|_{L^2(B_1)} + \|v_\star - v_j\|_{L^2(B_1)}\, \|v_j\|_{L^2(B_1)}\end{aligned}$$

that are infinitesimal as $j \to +\infty$. Using this, (37), (38) and (39), we obtain that $v_\star$ is a minimizer for $\mathcal{G}$.

Since $\mathcal{G}(|v|) \le \mathcal{G}(v)$, due to (35), we may also suppose that

$$v_\star \text{ is nonnegative.} \tag{40}$$

The minimization property of $v_\star$ gives that

$$(-\Delta)^s v_\star + \mu v_\star^2 + \sigma_\varepsilon v_\star - f_\varepsilon - \tau(J * v_\star) = 0$$

in B_1. Hence, we define

$$u_\varepsilon := W_\varepsilon + v_\star$$

and, recalling (35), we find that

$$\begin{aligned} & -(-\Delta)^s u_\varepsilon + (\sigma_\varepsilon - \mu\, u_\varepsilon)\, u_\varepsilon + \tau(J * u_\varepsilon) \\ = & -(-\Delta)^s W_\varepsilon + \mu v_\star^2 + \sigma_\varepsilon v_\star - f_\varepsilon - \tau(J * v_\star) + \sigma_\varepsilon W_\varepsilon + \sigma_\varepsilon v_\star \\ & \quad - \mu W_\varepsilon^2 - \mu v_\star^2 - 2\mu W_\varepsilon v_\star + \tau(J * W_\varepsilon) + \tau(J * v_\star) \\ = & \; 2\sigma_\varepsilon v_\star + \sigma_\varepsilon W_\varepsilon - \mu W_\varepsilon^2 - 2\mu W_\varepsilon v_\star. \end{aligned} \tag{41}$$

in B_1. Now we recall (33) and (34) and we find that, in B_1,

$$W_\varepsilon = w_\varepsilon = \mu^{-1}\sigma_\varepsilon.$$

Hence we insert this identity into (41) and we conclude that

$$-(-\Delta)^s u_\varepsilon + (\sigma_\varepsilon - \mu\, u_\varepsilon)\, u_\varepsilon + \tau(J * u_\varepsilon) = 2\sigma_\varepsilon v_\star + \mu^{-1}\sigma_\varepsilon^2 - \mu\mu^{-2}\sigma_\varepsilon^2 - 2\sigma_\varepsilon v_\star = 0$$

in B_1, which establishes (8).

Also, by (32), we have that both W_ε and $v_\star$ vanish outside B_{R_ε}, and this establishes (9). Finally, by (33) and (40),

$$u_\varepsilon \ge W_\varepsilon \ge w_\varepsilon = \mu^{-1}\sigma_\varepsilon,$$

which proves (11). □

8. Beating the resources and proof of Theorem 1.9

The proof of Theorem 1.9 is based on a contradiction and limit argument.

Proof of Theorem 1.9. Let u_m be the solution of (12) provided by Proposition 2.4. If the desired claim were false, we would have that $u_m \le \sigma_m$. Then

$$\big|(-\Delta)^s u_m\big| = (\sigma_m - u_m)\, u_m \le \sigma_m u_m \le \|\sigma_m\|_{L^\infty(\Omega)} \|u_m\|_{L^\infty(\Omega)}.$$

Hence, using Lemma 1.6 with $\tau = 0$,

$$\big|(-\Delta)^s u_m\big| \le \|\sigma_m\|^2_{L^\infty(\Omega)} \le \big(\|\sigma_0\|_{L^\infty(\Omega)} + 1\big)^2.$$

Notice that the latter quantity does not depend on m. Thus, by fractional elliptic regularity (see e.g. Proposition 1.1 in [ROS14] and Lemma 4.3 in [CS11]) we have that u_m converges uniformly in Ω to some u_0 as $m \to 0$, and u_0 solves

$$(-\Delta)^s u_0 = (\sigma_0 - u_0)\, u_0$$

in Ω. By Theorem 1.7, we know that $u_0 > 0$ in B_r. In particular u_0 is not the trivial solution, and so $u_0 > 0$, thanks to Lemma 2.1. Then we have

$$0 < u_0(x_0) = \lim_{m \to 0} u_m(x_0) \le \lim_{m \to 0} \sigma_m(x_0) = \sigma_0(x_0) = 0,$$

which is a contradiction. □

9. Periodic solutions and proof of Theorems 1.10 and 1.11

To prove Theorem 1.10, we consider an auxiliary minimization problem. The functional is tailored in order to be compatible with integer translations and produce solutions of (13) via an Euler-Lagrange equation, tested against periodic test functions.

Here, we assume that J is supported in some ball B_ρ and we let

$$Q := \left(-\frac{1}{2}, \frac{1}{2}\right]^n. \tag{42}$$

We define the energy functional

$$\begin{aligned}\mathcal{F}(v) \;:=\; & \frac{s\,(1-s)}{2}\iint_{\mathbb{R}^n\times Q} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_Q \frac{\mu|v|^3}{3} - \frac{\sigma v^2}{2}\,dx \\ & -\frac{\tau}{2}\iint_{\mathbb{R}^n\times Q} J(x-y)\,v(x)\,v(y)\,dx\,dy.\end{aligned}$$

Then we consider the space X of functions $v \in L^2(Q)$, with $v(x+k) = v(x)$ for any $k \in \mathbb{Z}^n$ and a.e. $x \in \mathbb{R}^n$. We have that $\mathcal{F}$ attains a minimum in X, according to the following result:

Lemma 9.1. *There exists $v_* \in X$ such that $\mathcal{F}(v_*) \le \mathcal{F}(v)$ for every $v \in X$.*

Proof. First of all, we notice that $\mathcal{F}(0) = 0$, so we take a minimizing sequence $v_j \in X$ such that

$$\lim_{j\to+\infty} \mathcal{F}(v_j) = \inf_X \mathcal{F} \tag{43}$$

and we may suppose that

$$\mathcal{F}(v_j) \le 0. \tag{44}$$

Our goal is to obtain estimates that are uniform in j.

Letting $w_j := |v_j|\chi_{B_{\rho+\sqrt{n}}}$ and recalling Lemma 2.3, we see that

$$\begin{aligned}\left|\iint_{\mathbb{R}^n\times Q} J(x-y)\,v_j(x)\,v_j(y)\,dx\,dy\right| &\le \iint_{B_{\rho+\sqrt{n}}\times Q} J(x-y)\,|v_j(x)|\,|v_j(y)|\,dx\,dy \\ &\le \iint_{\mathbb{R}^{2n}} J(x-y)\,w_j(x)\,w_j(y)\,dx\,dy \le \|w_j\|^2_{L^2(B_{\rho+\sqrt{n}})} \le C\,\|v_j\|^2_{L^2(Q)},\end{aligned}$$

for some $C > 0$, possibly depending on ρ and n. Hence,

$$\begin{aligned}&\int_Q \frac{\mu|v_j|^3}{3} - \frac{\sigma v_j^2}{2}\,dx - \frac{\tau}{2}\iint_{\mathbb{R}^n\times Q} J(x-y)\,v_j(x)\,v_j(y)\,dx\,dy \\ &\qquad\ge \int_Q \frac{\mu|v_j|^3}{3} - \frac{\sigma v_j^2}{2}\,dx - \frac{C\tau}{2}\int_Q v_j^2\,dx.\end{aligned} \tag{45}$$

Using this and (22) (with $C\tau$ in the place of τ), we get

$$\begin{aligned}&\int_Q \frac{\mu|v_j|^3}{3} - \frac{\sigma v_j^2}{2}\,dx - \frac{\tau}{2}\iint_{\mathbb{R}^n\times Q} J(x-y)\,v_j(x)\,v_j(y)\,dx\,dy \\ &\qquad\ge -\int_Q \frac{(\sigma + C\tau)^3}{6\mu^2}\,dx =: -\kappa,\end{aligned}$$

where $\kappa > 0$ depends on σ, τ, μ, ρ and n. As a consequence of this and (44), we obtain

$$\frac{s\,(1-s)}{2}\iint_{\mathbb{R}^n\times Q}\frac{|v_j(x)-v_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le \kappa. \tag{46}$$

In addition, utilizing (44) and (45), we have that

$$\int_Q \frac{\mu|v_j|^3}{3}-\frac{\sigma v_j^2}{2}\,dx-\frac{C\tau}{2}\int_Q v_j^2\,dx \le 0,$$

and so, by Hölder Inequality,

$$\int_Q \frac{\mu|v_j|^3}{3}\le \frac{\|\sigma\|_{L^\infty(Q)}+C\tau}{2}\left(\int_Q v_j^3\,dx\right)^{2/3}.$$

Accordingly, $\|v_j\|_{L^3(Q)}$ is bounded uniformly in j and therefore $\|v_j\|_{L^2(Q)}$ is also bounded uniformly in j.

From this and (46), it follows that v_j is precompact in $L^2(Q)$ (see e.g. Theorem 7.1 in [DNPV12]). Thus, up to a subsequence, we may assume that $v_j \to v_*$ in $L^2(Q)$ and a.e. in Q (and thus, by periodicity, a.e. in $\mathbb{R}^n$), as $j\to+\infty$. Notice also that v_* is periodic, since so is v_j. This gives that $v_*\in X$. Furthermore, using the convergence of v_j and Fatou Lemma,

$$\liminf_{j\to+\infty}\frac{s\,(1-s)}{2}\iint_{\mathbb{R}^n\times Q}\frac{|v_j(x)-v_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \ge \frac{s\,(1-s)}{2}\iint_{\mathbb{R}^n\times Q}\frac{|v_*(x)-v_*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

$$\liminf_{j\to+\infty}\int_Q\frac{\mu|v_j|^3}{3}\,dx \ge \int_Q\frac{\mu|v_*|^3}{3}\,dx$$

$$\text{and}\qquad \lim_{j\to+\infty}\int_Q\frac{\sigma v_j^2}{2}\,dx=\int_Q\frac{\sigma v_*^2}{2}\,dx.$$

Moreover,

$$\begin{aligned}&\left|\iint_{\mathbb{R}^n\times Q}J(x-y)\,v_j(x)\,v_j(y)\,dx\,dy-\iint_{\mathbb{R}^n\times Q}J(x-y)\,v_*(x)\,v_*(y)\,dx\,dy\right|\\ \le\;& \iint_{B_{\rho+\sqrt{n}}\times Q}J(x-y)\,\big|v_j(x)\big|\,\big|v_j(y)-v_*(y)\big|\,dx\,dy\\ &\quad+\iint_{B_{\rho+\sqrt{n}}\times Q}J(x-y)\,\big|v_j(x)-v_*(x)\big|\,\big|v_*(y)\big|\,dx\,dy\\ \le\;& C\Big(\|v_j\|_{L^2(Q)}\,\|v_j-v_*\|_{L^2(Q)}+\|v_j-v_*\|_{L^2(Q)}\,\|v_*\|_{L^2(Q)}\Big),\end{aligned}$$

thanks to Lemma 2.3, and the latter quantity is infinitesimal as $j\to+\infty$. These considerations and (43) give that

$$\mathcal{F}(v_*)=\inf_X\mathcal{F},$$

so the desired result follows. □

Now we can complete the proof of Theorem 1.10 by considering the minimizer produced by Lemma 9.1 and by checking that periodic perturbations indeed give (13) as Euler-Lagrange equation.

Proof of Theorem 1.10. Let v_* be as in Lemma 9.1 and $u := |v_*|$. Then

$$\mathcal{F}(u) \le \mathcal{F}(v_*) \le \mathcal{F}(v) \qquad \text{for every } v \in X. \tag{47}$$

Now we take $\psi \in C_0^\infty(Q)$ and we consider its periodic extension in $\mathbb{R}^n$, that is

$$\phi(x) := \sum_{k\in\mathbb{Z}^n} \psi(x+k).$$

Using $v := u + \epsilon\phi$ as test function in (47), we obtain that

$$\begin{aligned} s\,(1-s) \iint_{\mathbb{R}^n\times Q} \frac{\big(u(x)-u(y)\big)\big(\phi(x)-\phi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy + \int_Q \mu u^2\phi - \sigma u\phi\,dx \\ - \frac{\tau}{2} \iint_{\mathbb{R}^n\times Q} J(x-y)\,u(x)\,\phi(y)\,dx\,dy - \frac{\tau}{2} \iint_{\mathbb{R}^n\times Q} J(x-y)\,\phi(x)\,u(y)\,dx\,dy = 0. \end{aligned} \tag{48}$$

Now we write

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}^n} (Q+k)$$

and thus, using the substitutions $\tilde{x} := x-k$ and $\tilde{y} := y-k$,

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad= \sum_{k\in\mathbb{Z}^n} \iint_{\mathbb{R}^n\times(Q+k)} \frac{\big(u(x)-u(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad= \sum_{k\in\mathbb{Z}^n} \iint_{\mathbb{R}^n\times Q} \frac{\big(u(\tilde{x}+k)-u(\tilde{y}+k)\big)\big(\psi(\tilde{x}+k)-\psi(\tilde{y}+k)\big)}{|\tilde{x}-\tilde{y}|^{n+2s}}\,d\tilde{x}\,d\tilde{y} \\ &\quad= \sum_{k\in\mathbb{Z}^n} \iint_{\mathbb{R}^n\times Q} \frac{\big(u(\tilde{x})-u(\tilde{y})\big)\big(\psi(\tilde{x}+k)-\psi(\tilde{y}+k)\big)}{|\tilde{x}-\tilde{y}|^{n+2s}}\,d\tilde{x}\,d\tilde{y} \\ &\quad= \iint_{\mathbb{R}^n\times Q} \frac{\big(u(\tilde{x})-u(\tilde{y})\big)\,\sum_{k\in\mathbb{Z}^n}\big(\psi(\tilde{x}+k)-\psi(\tilde{y}+k)\big)}{|\tilde{x}-\tilde{y}|^{n+2s}}\,d\tilde{x}\,d\tilde{y} \\ &\quad= \iint_{\mathbb{R}^n\times Q} \frac{\big(u(\tilde{x})-u(\tilde{y})\big)\big(\phi(\tilde{x})-\phi(\tilde{y})\big)}{|\tilde{x}-\tilde{y}|^{n+2s}}\,d\tilde{x}\,d\tilde{y}. \end{aligned} \tag{49}$$

Similarly,

$$\begin{aligned} \iint_{\mathbb{R}^{2n}} J(x-y)\,\psi(x)\,u(y)\,dx\,dy = \sum_{k\in\mathbb{Z}^n} \iint_{\mathbb{R}^n\times(Q+k)} J(x-y)\,\psi(x)\,u(y)\,dx\,dy \\ = \sum_{k\in\mathbb{Z}^n} \iint_{\mathbb{R}^n\times Q} J(\tilde{x}-\tilde{y})\,\psi(\tilde{x}+k)\,u(\tilde{y})\,dx\,dy = \iint_{\mathbb{R}^n\times Q} J(\tilde{x}-\tilde{y})\,\phi(\tilde{x})\,u(\tilde{y})\,d\tilde{x}\,d\tilde{y}. \end{aligned} \tag{50}$$

So, we insert (49) and (50) into (48) and we obtain that

$$\begin{aligned} s\,(1-s) \iint_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} \mu u^2\psi - \sigma u\psi\,dx \\ -\tau \iint_{\mathbb{R}^{2n}} J(x-y)\,u(x)\,\psi(y)\,dx\,dy = 0. \end{aligned}$$

This gives that u is a solution of the desired equation in Q (and thus in the whole of $\mathbb{R}^n$, by periodicity).

We also claim that

$$u > 0 \text{ in } \mathbb{R}^n. \tag{51}$$

The proof is by contradiction: if there exists x_o for which $u(x_o) = 0$, then, by Lemma 2.1, we see that u vanishes identically. In particular, by (47),

$$0 = \mathcal{F}(0) = \mathcal{F}(u) \le \mathcal{F}(\epsilon), \tag{52}$$

where $\epsilon > 0$ is a fixed constant. On the other hand,

$$\mathcal{F}(\epsilon) = \frac{c_1\epsilon^3}{3} - \frac{c_2\epsilon^2}{2} - \frac{\tau\epsilon^2}{2},$$

where

$$c_1 := \int_Q \mu\, dx \qquad \text{and } c_2 := \int_Q \sigma\, dx.$$

Notice that $c_3 := \frac{c_2}{2} + \frac{\tau}{2} > 0$, thanks to (14), and thus $\mathcal{F}(\epsilon) = \frac{c_1\epsilon^3}{3} - c_3\epsilon^2 < 0$ for small ϵ. This is in contradiction with (52) and so it proves (51). This completes the proof of Theorem 1.10. □

Now we establish Theorem 1.11 via some algebraic and analytical identities.

Proof of Theorem 1.11. Let Q be as in (42). We define

$$m := \int_Q u(x)\, dx \;\text{ and }\; v(x) := u(x) - m. \tag{53}$$

Notice that

$$m > 0, \tag{54}$$

due to the sign of u, and

$$\int_Q v(x)\, dx = 0. \tag{55}$$

Also, since u is periodic, there exists a minimal point x_o, that is

$$u(x_o) = \min_Q u = \min_{\mathbb{R}^n} u. \tag{56}$$

Thus, since u and v differ by a constant, it follows that

$$v(x_o) = \min_Q v = \min_{\mathbb{R}^n} v.$$

This and (55) give that

$$0 = \int_Q v(x)\, dx \ge v(x_o). \tag{57}$$

Now we point out that, for any $y \in \mathbb{R}^n$,

$$\int_Q u(x+y)\, dx = m, \tag{58}$$

due to (53) and the periodicity of u. Therefore, if we fix $\delta > 0$, we see that

$$\int_Q \left[\int_{\mathbb{R}^n \setminus B_\delta} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy \right] dx = \int_{\mathbb{R}^n \setminus B_\delta} \frac{m + m - 2m}{|y|^{n+2s}}\, dy = 0$$

and so, by taking $\delta \to 0$,

$$\int_Q (-\Delta)^s u(x)\,dx = 0. \tag{59}$$

Moreover, using again (58), we find that

$$\int_Q (J*u)(x)\,dx = \int_Q \left[\int_{\mathbb{R}^n} J(y)\,u(x-y)\,dy\right]\,dx = m\int_{\mathbb{R}^n} J(y)\,dy = m.$$

Using this, (55), (59) and the equation for u, we conclude that

$$\begin{aligned}0 &= \int_Q (-\Delta)^s u(x)\,dx = \int_Q \Big((\sigma-\mu u)u + \tau(J*u)\Big)\,dx = \sigma m - \mu\int_Q u^2\,dx + \tau m\\ &= \sigma m - \mu\int_Q \big(v^2+m^2+2mv\big)\,dx + \tau m = \sigma m - \mu\int_Q v^2\,dx - \mu m^2 + \tau m.\end{aligned}$$

This says that

$$\mu\int_Q v^2\,dx = m\,(\sigma+\tau-\mu m). \tag{60}$$

Now, we observe that,

$$(-\Delta)^s u(x_o) \le 0,$$

thanks to (56).

In addition, from (56) we also deduce that

$$(J*u)(x) = \int_{\mathbb{R}^n} J(y)\,u(x-y)\,dy \ge \int_{\mathbb{R}^n} J(y)\,u(x_o)\,dy = u(x_o),$$

for every $x \in \mathbb{R}^n$. Hence, we compute the equation at x_o and we find that

$$\begin{aligned}0 &\ge (-\Delta)^s u(x_o) = (\sigma-\mu u(x_o))u(x_o) + \tau(J*u)(x_o)\\ &\ge (\sigma-\mu u(x_o))u(x_o) + \tau u(x_o) = u(x_o)\big(\sigma+\tau-\mu u(x_o)\big).\end{aligned}$$

Therefore, since $u(x_o) > 0$, we conclude that

$$\sigma+\tau-\mu u(x_o) \le 0$$

and then

$$\sigma+\tau-\mu m \le \mu\big(u(x_o)-m\big) = \mu\,v(x_o).$$

We insert this into (60) and we recall (54), in order to obtain that

$$\mu\int_Q v^2\,dx = m\,(\sigma+\tau-\mu m) \le m\mu\,v(x_o).$$

Thus, by (57),

$$\mu\int_Q v^2\,dx \le 0,$$

which implies that v vanishes identically. Accordingly, by (53), we obtain that u is constant and constantly equal to m. We insert this information into the equation and we obtain that

$$0 = (\sigma-\mu m)m + \tau(J*m) = (\sigma-\mu m)m + \tau m = (\sigma+\tau-\mu m)m.$$

Recalling (54), we then obtain that $\sigma+\tau-\mu m = 0$ and so $m = (\sigma+\tau)/\mu$, as desired. □

10. A transmission problem and proof of Theorems 1.12 and 1.13

Now we consider the transmission problem introduced in (15) and we prove the existence of minimizers.

Proof of Theorem 1.12. We let u_j be a minimizing sequence. Using the Young Inequality

$$ab \le \frac{a^p}{p} + \frac{(p-1)\, b^{\frac{p}{p-1}}}{p}$$

with exponents $p = 3/2$, $a = 4^{-\frac{2}{3}}\mu^{\frac{2}{3}}u^2$ and $b = 2^{\frac{1}{3}}\mu^{-\frac{2}{3}}\sigma$, we see that

$$\frac{\sigma\, u^2}{2} \le \frac{\mu\, |u|^3}{6} + \frac{2\sigma^3}{3\mu^2}.$$

As a consequence,

$$\begin{aligned} 0 = \mathcal{T}(0) &\ge \mathcal{T}(u_j) \\ &\ge \frac{1}{2}\int_{\Omega_1} |\nabla u_j|^2\, dx + \frac{s\,(1-s)}{2} \iint_{\Omega_2\times\Omega_2} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad + \sum_{i=1}^{2} \frac{\nu_i\, s_i\,(1-s_i)}{2} \iint_{\Omega_i\times(\mathbb{R}^n\setminus\Omega_i)} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s_i}}\, dx\, dy + \int_{\Omega} \frac{\mu\,|u_j|^3}{6} - c_o, \end{aligned}$$

where

$$c_o := \int_{\Omega} \frac{2\sigma^3}{3\mu^2}\, dx.$$

In particular,

$$\frac{\mu_o}{6}\int_{\Omega} |u_j|^3\, dx \le c_o,$$

which gives a uniform bound in j of $\|u_j\|_{L^2(\Omega)}$. Also,

$$\frac{1}{2}\int_{\Omega_1} |\nabla u_j|^2\, dx + \frac{s\,(1-s)}{2} \iint_{\Omega_2\times\Omega_2} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \le c_o,$$

therefore, by compactness (see e.g. Theorem 7.1 in [DNPV12]), we find that, up to a subsequence, $u_j \to u$ in $L^2(\Omega)$ and a.e. in $\Omega_1 \cup \Omega_2$, with ∇u_j converging to ∇u weakly in $L^2(\Omega_1)$, for some function u vanishing outside Ω. From this, the desired result follows. □

The following is a maximum principle related to the transmission problem (15):

Lemma 10.1. *Let u be a nonnegative solution of* (16)*. Then either $u > 0$ in $\Omega_1 \cup \Omega_2$ or it vanishes identically.*

Proof. Assume that u vanishes somewhere in $\Omega_1 \cup \Omega_2$. We claim that

$$\text{if } u \text{ vanishes somewhere in } \Omega_2, \text{ then it vanishes identically in } \Omega_1 \cup (\mathbb{R}^n \setminus \Omega_2). \tag{61}$$

To prove this, we suppose that $u(\bar{x}) = 0$, for some $\bar{x} \in \Omega_2$. Then $\bar{x}$ minimizes u and so

$$PV \int_{\Omega_2} \frac{u(\bar{x})-u(y)}{|\bar{x}-y|^{n+2s}}\, dy \le 0,$$

$$\int_{\Omega_1} \frac{u(\bar{x})-u(y)}{|\bar{x}-y|^{n+2s_1}}\, dy \le 0 \qquad \text{and} \qquad \int_{\mathbb{R}^n\setminus\Omega_2} \frac{u(\bar{x})-u(y)}{|\bar{x}-y|^{n+2s_2}}\, dy \le 0.$$

These inequalities and (16) imply that indeed

$$\int_{\Omega_1} \frac{u(\bar{x}) - u(y)}{|\bar{x} - y|^{n+2s_1}}\, dy = 0 \qquad \text{and} \qquad \int_{\mathbb{R}^n \setminus \Omega_2} \frac{u(\bar{x}) - u(y)}{|\bar{x} - y|^{n+2s_2}}\, dy = 0,$$

and this says that $u(y) = u(\bar{x}) = 0$ in the whole of $\Omega_1 \cup (\mathbb{R}^n \setminus \Omega_2)$, thus proving (61).

Now we show that

$$\text{if } u \text{ vanishes somewhere in } \Omega_1, \text{ then it vanishes identically in } \Omega_2 \cup (\mathbb{R}^n \setminus \Omega_1). \tag{62}$$

To this end, let $x_o \in \Omega_1$ such that $u(x_o) = 0$. In particular, x_o minimizes u, therefore $\Delta u(x_o) \geq 0$,

$$\int_{\mathbb{R}^n \setminus \Omega_1} \frac{u(x_o) - u(y)}{|x_o - y|^{n+2s_1}}\, dy \leq 0 \qquad \text{and} \qquad \int_{\Omega_2} \frac{u(x_o) - u(y)}{|x_o - y|^{n+2s_2}}\, dy \leq 0.$$

These inequalities and (16) imply that

$$\int_{\mathbb{R}^n \setminus \Omega_1} \frac{u(x_o) - u(y)}{|x_o - y|^{n+2s_1}}\, dy = 0 \qquad \text{and} \qquad \int_{\Omega_2} \frac{u(x_o) - u(y)}{|x_o - y|^{n+2s_2}}\, dy = 0.$$

In consequence of these equalities, we conclude that $u(y) = u(x_o) = 0$ for any $y \in (\mathbb{R}^n \setminus \Omega_1) \cup \Omega_2$, and this establishes (62).

Now suppose that u vanishes somewhere in Ω_1 (resp. Ω_2). Then, by (62) (resp., (61)), we know that u vanishes identically in $\Omega_2 \cup (\mathbb{R}^n \setminus \Omega_1)$ (resp., in $\Omega_1 \cup (\mathbb{R}^n \setminus \Omega_2)$). Accordingly, by (61) (resp., (62)), we obtain that u vanishes identically in $\Omega_1 \cup (\mathbb{R}^n \setminus \Omega_2)$ (resp., in $\Omega_2 \cup (\mathbb{R}^n \setminus \Omega_1)$). All in all, we find that u vanishes identically in $\Omega_2 \cup (\mathbb{R}^n \setminus \Omega_1) \cup \Omega_1 \cup (\mathbb{R}^n \setminus \Omega_2) = \mathbb{R}^n$, as desired. □

Now we establish the results related to the spectral analysis of the transmission problem (15):

Proof of Theorem 1.13. We let $e_\star$ be the first eigenfunction of the problem, i.e. the minimizer which attains the infimum in (17). That such minimum is attained follows by a compactness argument, as the one in the proof of Theorem 1.12. By construction,

$$\begin{aligned} &\int_{\Omega_1} \nabla e_\star \cdot \nabla \phi\, dx + s\,(1-s) \iint_{\Omega_2 \times \Omega_2} \frac{(e_\star(x) - e_\star(y))(\phi(x) - \phi(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad + \sum_{i=1}^{2} \nu_i\, s_i\,(1-s_i) \iint_{\Omega_i \times (\mathbb{R}^n \setminus \Omega_i)} \frac{(e_\star(x) - e_\star(y))(\phi(x) - \phi(y))}{|x-y|^{n+2s_i}}\, dx\, dy = \lambda_\star(\Omega) \int_\Omega e_\star\, \phi\, dx \end{aligned}$$

for any test function ϕ, and so

$$\begin{aligned} &\int_{\Omega_1} |\nabla e_\star|^2\, dx + s\,(1-s) \iint_{\Omega_2 \times \Omega_2} \frac{|e_\star(x) - e_\star(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad + \sum_{i=1}^{2} \nu_i\, s_i\,(1-s_i) \iint_{\Omega_i \times (\mathbb{R}^n \setminus \Omega_i)} \frac{|e_\star(x) - e_\star(y)|^2}{|x-y|^{n+2s_i}}\, dx\, dy = \lambda_\star(\Omega) \int_\Omega |e_\star|^2\, dx. \end{aligned} \tag{63}$$

Also, we may assume that $e_\star \geq 0$, since taking the absolute value of a candidate may only decrease the energy, and in fact

$$e_\star > 0 \text{ in } \Omega_1 \cup \Omega_2, \tag{64}$$

thanks to the maximum principle in Lemma 10.1.

Given $M>0$, we set

$$e_M(x) := \begin{cases} e_\star(x) & \text{if } e_\star(x) < M, \\ M & \text{if } e_\star(x) \geq M. \end{cases}$$

By the Fatou Lemma,

$$\liminf_{M\to+\infty} \int_\Omega \sigma\, e_M^2\, dx \geq \int_\Omega \sigma\, e_\star^2\, dx,$$

and therefore

$$\liminf_{M\to+\infty} \int_\Omega \sigma\, e_M^2 - \lambda_\star(\Omega)\, e_\star^2\, dx \geq \int_\Omega (\sigma - \lambda_\star(\Omega))\, e_\star^2\, dx =: c_\star. \tag{65}$$

After these considerations, we proceed with the proof of Theorem 1.13.

First, we suppose that $\sup_\Omega \sigma \leq \lambda_\star(\Omega)$. We aim to show that all solutions of (16) are trivial. Assume, by contradiction, that there exists a nontrivial solution u. Then, by Lemma 10.1, we know that $u>0$ in $\Omega_1 \cup \Omega_2$.

Now, we write the weak formulation of (16) as

$$\begin{aligned} &\int_{\Omega_1} \nabla u \cdot \nabla \phi\, dx + s\,(1-s) \iint_{\Omega_2\times\Omega_2} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad + \sum_{i=1}^{2} \nu_i\, s_i\,(1-s_i) \iint_{\Omega_i\times(\mathbb{R}^n\setminus\Omega_i)} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s_i}}\, dx\, dy \\ &\quad + \int_\Omega \mu\, u^2\, \phi - \sigma\, u^2\, dx \;=\; 0, \end{aligned}$$

for any test function ϕ, and we choose $\phi := u$. Hence, we find that

$$\begin{aligned} &\int_{\Omega_1} |\nabla u|^2\, dx + s\,(1-s) \iint_{\Omega_2\times\Omega_2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad + \sum_{i=1}^{2} \nu_i\, s_i\,(1-s_i) \iint_{\Omega_i\times(\mathbb{R}^n\setminus\Omega_i)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_i}}\, dx\, dy \\ &\quad + \int_\Omega \mu\, |u|^3 - \sigma\, u^2\, dx \;=\; 0. \end{aligned}$$

As a consequence,

$$\begin{aligned} \lambda_\star(\Omega) &\leq \|u\|_{L^2(\Omega)}^{-2}\, \mathcal{T}_o(u) \\ &= \|u\|_{L^2(\Omega)}^{-2} \int_\Omega \sigma\, u^2 - \mu\, |u|^3\, dx \\ &< \|u\|_{L^2(\Omega)}^{-2} \int_\Omega \sigma\, u^2\, dx \\ &\leq \|u\|_{L^2(\Omega)}^{-2} \int_\Omega \lambda_\star(\Omega)\, u^2\, dx \\ &= \lambda_\star(\Omega), \end{aligned}$$

which is a contradiction. This establishes the first claim in Theorem 1.13, so we can now focus on the second claim. To this goal, we now assume that $\inf_\Omega \sigma \geq \lambda_\star(\Omega)$ with strict inequality on a set of positive measure. Therefore, recalling (64), we have that, in this case,

$$c_\star > 0$$

and so, in light of (65), we can fix $M_\star$ sufficiently large such that, for any $M \geq M_\star$,

$$\int_\Omega \sigma\, e_M^2 - \lambda_\star(\Omega)\, e_\star^2\, dx \geq \frac{c_\star}{2} > 0.$$

So, from now on, we can fix $M = M_\star$, and the inequality above holds true. In consequence of these observations and recalling (63), we have that

$$\begin{aligned} \mathcal{T}_o(e_M) &\leq \int_{\Omega_1} |\nabla e_\star|^2\, dx + s\,(1-s) \iint_{\Omega_2\times\Omega_2} \frac{|e_\star(x)-e_\star(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &\qquad + \sum_{i=1}^{2} \nu_i\, s_i\,(1-s_i) \iint_{\Omega_i\times(\mathbb{R}^n\setminus\Omega_i)} \frac{|e_\star(x)-e_\star(y)|^2}{|x-y|^{n+2s_i}}\, dx\, dy \\ &= \lambda_\star(\Omega) \int_\Omega |e_\star|^2\, dx \\ &\leq -\frac{c_\star}{2} + \int_\Omega \sigma\, e_M^2\, dx. \end{aligned}$$

Accordingly, for any $\epsilon > 0$,

$$\begin{aligned} \mathcal{T}(\epsilon e_M) &= \epsilon^2\, \mathcal{T}_o(e_M) + \int_\Omega \epsilon^3\, \frac{\mu\, |e_M|^3}{3} - \epsilon^2\, \frac{\sigma\, e_M^2}{2}\, dx \\ &\leq -\frac{c_\star\, \epsilon^2}{2} + \epsilon^3 \int_\Omega \frac{\mu\, |e_\star|^3}{3}, \end{aligned}$$

which is negative if ϵ is suitably small. As a consequence, $\mathcal{T}(\epsilon e_M) < 0 = \mathcal{T}(0)$, which implies that the trivial function is not a minimizer.

This says that the minimizer does not vanish identically, and so it is positive in $\Omega_1 \cup \Omega_2$, in light of Lemma 10.1. This completes the proof of Theorem 1.13. □

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