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Shape optimisation for a class of semilinear variational inequalities with applications to damage models

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Abstract

The present contribution investigates shape optimisation problems for a class of semilinear elliptic variational inequalities with Neumann boundary conditions. Sensitivity estimates and material derivatives are firstly derived in an abstract operator setting where the operators are defined on polyhedral subsets of reflexive Banach spaces. The results are then refined for variational inequalities arising from minimisation problems for certain convex energy functionals considered over upper obstacle sets in H^1. One particularity is that we allow for dynamic obstacle functions which may arise from another optimisation problems. We prove a strong convergence property for the material derivative and establish state-shape derivatives under regularity assumptions. Finally, as a concrete application from continuum mechanics, we show how the dynamic obstacle case can be used to treat shape optimisation problems for time-discretised brittle damage models for elastic solids. We derive a necessary optimality system for optimal shapes whose state variables approximate desired damage patterns and/or displacement fields.

Contents

1 Introduction

Finding optimal shapes such that a physical system exhibits an intended behaviour is of great interest for plenty of engineering applications. For example design questions arise in the construction of air- and spacecrafts, wind and combustion turbines, wave guides and inductor coils. More examples can be found in [5] and references therein. The physical system is usually modelled by a pde or a coupled pde system supplemented with suitable boundary conditions. In certain cases the state is given as a minimiser of an energy, e.g., an equilibrium state of an elastic membrane, which has to be in some set of admissible states. The solution is then characterised by a variational inequality holding for test-functions on the sets of admissible states.

The treatment of optimal shape and control problems for variational inequalities is substantially more difficult as without constraints, where the sets of admissible states is a linear space. For optimal control problems there exist a rapidly growing literature exploring different types of stationarity conditions and their approximations (see, for instance, [14, 18]). However shape optimisation problems for systems described by variational inequalities are less explored and reveal additional difficulties due to the intricated structure of the set of admissible domains. Some results following the paradigm *first optimise—then discretise* can be found in [22, 21, 15, 19] and for the *first discretise–then optimise* approach we refer to [3, 1, 10].

The main aim of this paper is to establish sensitivity estimates and material derivatives for certain nonlinear elliptic variational inequalities with respect to the domain. Our approach is based on the paradigm first optimise then discretise, thus the sensitivity is derived in the infinite dimensional setting. In order to highlight the main arguments needed in the proof of these main results and to increase their applicability, we investigate the optimisation problems firstly on an abstract operator level formulated over a polyhedric subset K of some reflexive Banach space V. The domain-to-state map is there replaced by a parametrised family of operators $(\mathcal{A}_t)$ and sensitivity estimates are shown in Theorem 3.2 and Theorem 3.3 under general assumption (see Assumption (E) and Assumption (O1)). By strengthen the assumptions (see Assumption (O2)) differentiability with respect to the parameter t has been shown in Theorem 3.5. One crucial requirement is the polyhedricity of the closed convex set K on which the operators are defined. The results are applicable for optimal shape as well as for optimal control problems.

Equipped with the proven abstract results we resort to shape optimisation problems where the state system is a variational inequality of semilinear elliptic type given by

$$u \in K_{\psi_\Omega} \text{ and } \forall \varphi \in K_{\psi_\Omega} : \quad dE(\Omega, u; \varphi - u) \geq 0$$

with the energy

$$E(\Omega, u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{2}|u|^2 + W_\Omega(x, u)\,\mathrm{d}x \quad (\lambda > 0)$$

and the upper obstacle set

$$K_{\psi_\Omega} = \{v \in H^1(\Omega) : v \le \psi_\Omega \text{ a.e. in } \Omega\}.$$

In the classical theory of VI-constrained shape optimisation problems established in [23], linear variational inequalities with constant obstacle and $W_\Omega(x,u) = f(x)$ for some given fixed function $f : D \to \mathbb{R}$ defined on a "larger set" $D \supset \Omega$ have been investigated by means of conical derivatives of projection operators in Hilbert spaces dating back to [18]. For results on topological sensitivity analysis for variational inequalities and numerical implementations we refer to [13] as well as [2].

In our paper we allow for semilinear terms in the variational inequality by including convex contributions to W_Ω with respect to u and also consider a dependence of W_Ω and ψ_Ω on Ω in a quit general sense. As presented in the last section of this work ψ_Ω may itself be a solution of a variational inequality. Such general Ω-dependence of the obstacle will be referred to as "dynamic obstacle" in constrast to the case of a "static obstacle" where $\psi_\Omega(x) = g(x)$ for some fixed function $g : D \to \mathbb{R}$.

To apply the abstract sensitivity results we perform the transformation $u \mapsto y := u - \psi_\Omega$ such that the transformed problem is formulated over the cone $H^1_-(\Omega)$, i.e., the non-positive half space of $H^1(\Omega)$. Existence of the material derivative $\dot{y}$ which turns out to be the unique solution of a variational inequality considered over the cone $T_y(H^1_-(\Omega)) \cap kern(dE(u;\cdot))$ and strong convergence of the corresponding difference quotients are established in Theorem 4.8 and Corollary 4.9. The variational inequality characterising the material derivative $\dot{u}$ is then established in Corollary 4.11. Moreover in the case of a static obstacle and $H^2(\Omega)$-regularity for u we derive relations for the state-shape derivative u' in Theorem 4.15 and Corollary 4.16.

The theorems for the abstract semilinear case are then applied to a specific model problem from continuum damage mechanics. There one considers an elastic solid which undergoes deformation and damage processes in a small strain setting. The state of damage is modelled by a phase field variable χ which influences the material stiffness and which is described by an evolution inclusion forcing the variable χ to be monotonically decreasing in time. We consider a time-discretised version of the evolution system (but we stay continuous in the spatial components) where the damage variable fulfills for all time steps the constraints

$$\chi^N \le \chi^{N-1} \le \ldots \le \chi^0 \le 1 \text{ a.e. in } \Omega.$$

Such constraints lead to N-coupled variational inequalities with dynamic obstacle sets of the type

$$K^{k-1}(\Omega) = \{v \in H^1(\Omega) : v \le \chi^{k-1} \text{ a.e. in } \Omega\}, \quad k = 1, \ldots, N.$$

Our objective is to find an optimal shape Ω such that the associated displacement fields $(\mathbf{u}^k)_{k=1}^N$ and damage phase fields $(\chi^k)_{k=1}^N$ minimise a given tracking type cost functional. We derive relations for the material derivative and establish necessary optimality conditions for optimal shapes which are summarised in Proposition 5.3.

Structure of the paper

In Section 2 we recall some basics notions from convex analysis and shape optimisation theory. We derive tangential and normal cones of K_{ψ_Ω} rigorously and prove polyhedricity of K_{ψ_Ω} in Theorem 2.6. The proofs require careful modifications of arguments from [18, 4, 12] since the underlying space in our case is $H^1(\Omega)$ and not $\mathring{H}^1(\Omega)$.

In Section 3 we establish sensitivity and material derivative results in an abstract operator setting (see Theorem 3.2, Theorem 3.3 and Theorem 3.5). Some results are even applicable to quasi-linear problems such as to p-Laplace equations. The advantage of this approach is that the theorems can be applied to a large class of optimisation problems including shape optimisation and optimal control problems.

This flexibility is demonstrated in Section 4 where semilinear VI-constrained shape optimisation problems with an energy and obstacle of type $E(\Omega, u)$ and K_{ψ_Ω} from above are treated. By applying the abstract results from Section 3 we derive sensitivity estimates for the shape-perturbed problem in Proposition 4.5, material derivatives in Theorem 4.8 and state-shape derivatives in Theorem 4.15.

Finally, in Section 5, we apply the still abstract results from Section 4 to a particular problem in continuum damage mechanics where dynamic obstacles occur.

2 Preliminaries

2.1 Notation and basic relations

For the treatment of variational inequalities we recall certain well-known cones from convex analysis (the definitions can, for instance, be found in [4, Chapter 2.2.4] and [23, Chapter 4.1]). Let $K \subseteq V$ be a subset of a real Banach space V and denote by V^* its topological dual space.

The *radial cone* at $y \in K$ of the set K is defined by

$$C_y(K) := \{w \in V : \exists t > 0, y + tw \in K\}, \tag{1}$$

the *tangent cone* at y as

$$T_y(K) := \overline{C_y(K)}^V \tag{2}$$

and the *normal cone* at y as

$$N_y(K) := \{w^* \in V^* : \forall v \in K, \langle w^*, v - y\rangle_V \leq 0\}. \tag{3}$$

Furthermore we introduce the polar cone of a set K as

$$[K]^\circ := \{w^* \in V^* : \forall v \in K, \langle w^*, v\rangle_V \leq 0\}, \tag{4}$$

and the orthogonal complements of elements $y \in V$ and $y^* \in V^*$

$$\begin{aligned} [y]^\perp &:= \{w^* \in V^* : \langle w^*, y\rangle_V = 0\}, \\ kern(y^*) := [y^*]^\perp &:= \{w \in V : \langle y^*, w\rangle_V = 0\}. \end{aligned}$$

The normal cone may also be written as

$$N_y(K) = [T_y(K)]^\circ = [C_y(K)]^\circ \,. \tag{5}$$

In combination with the bipolar theorem (see [4, Prop. 2.40]) we obtain

$$T_y(K) = [[T_y(K)]^\circ]^\circ = [N_y(K)]^\circ. \tag{6}$$

We recall that a closed convex set $K \subseteq V$ is *polyhedric* if (cf. [14])

$$\forall y \in K,\ \forall w \in N_y(K), \quad \overline{C_y(K) \cap [w]^\perp}^V = T_y(K) \cap [w]^\perp. \tag{7}$$

Note that the inclusion "⊆" is always satisfied above. Due to Mazur's lemma and the convexity of the involved sets, the closure in V can also be taken in the weak topology.

The following lemma shows a useful implication of (7) involving variational inequalities arising from (possibly non-)linear operators.

Lemma 2.1. *Let $K \subseteq V$ be a polyhedric subset.*

(i) Let $\mathcal{A} : K \to V^*$ be an operator and let y be a solution of the following variational inequality

$$y \in K \quad \text{and} \quad \forall \varphi \in K : \ \langle \mathcal{A}(y), \varphi - y \rangle_V \geq 0. \tag{8}$$

Then it holds

$$\overline{C_y(K) \cap \mathit{kern}(\mathcal{A}(y))} = T_y(K) \cap \mathit{kern}(\mathcal{A}(y)). \tag{9}$$

(ii) For all $v \in V$ it holds

$$\overline{C_y(K) \cap [v-y]^\perp} = T_y(K) \cap [v-y]^\perp,$$

where y denotes the projection of v on K.

Proof. To (i): We infer from (8) that $-\mathcal{A}(y) \in N_y(K)$. Thus definition (7) implies

$$\overline{C_y(K) \cap \mathit{kern}(-\mathcal{A}(y))} = T_y(K) \cap \mathit{kern}(-\mathcal{A}(y)).$$

The identity $\mathit{kern}(-\mathcal{A}(y)) = \mathit{kern}(\mathcal{A}(y))$ completes the proof.
To (ii): This follows from $v - y \in N_y(K)$. □

2.2 Polyhedricity of upper obstacle sets in $H^1(\Omega)$

Let us consider an important class of polyhedral subsets which will be utilized in Section 4 where semilinear obstacle problems are treated. Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain and $V = H^1(\Omega)$. Moreover let $\psi \in V$ be a given function. We define the upper obstacle set as

$$K_\psi := \{w \in H^1(\Omega) : \ w \leq \psi \text{ a.e. in } \Omega\}. \tag{10}$$

In the remaining part of this subsection we will sketch the proofs for the characterisation of the tangential and normal cones as well as of the polyhedricity of K_ψ for reader's convenience since such obstacles sets are usually considered in the space $\mathring{H}^1(\Omega)$ in the literature. The adaption to $H^1(\Omega)$ requires some careful modifications in the proofs.

Furthermore we denote with $M_+(\overline{\Omega})$ the Radon measures on $\overline{\Omega}$. The Riesz representation theorem for local compact Hausdorff spaces (see [6, Theorem VIII.2.5]) states that for each non-negative functional $I : C(\overline{\Omega}) \to \mathbb{R}$ there exists a unique Radon measure $\mu \in M_+(\overline{\Omega})$ such that for all $f \in C(\overline{\Omega})$

$$I(f) = \int_{\overline{\Omega}} f \, \mathrm{d}\mu. \tag{11}$$

In the sequel we will use the following notation for the half space

$$H^1_+(\Omega) := \big\{v \in H^1(\Omega) : \ v \geq 0 \text{ a.e. in } \Omega\big\}.$$

With the help of the Riesz representation theorem we are now in the position to give a characterisation of (cf. [4, Chapter 6.4.3] for $\mathring{H}^1(\Omega)$ instead of $H^1(\Omega)$)

$$H^1(\Omega)^*_+ := \left\{ I \in H^1(\Omega)^* : \langle I, v\rangle_{H^1(\Omega)} \geq 0 \text{ for all } v \in H^1_+(\Omega) \right\}.$$

Lemma 2.2. *We have*

$$H^1(\Omega)^*_+ = \left\{ I \in H^1(\Omega)^* : \exists!\, \mu_I \in M_+(\overline{\Omega}),\, \forall v \in H^1(\Omega) \cap C(\overline{\Omega}),\, \langle I, v\rangle_{H^1(\Omega)} = \int_{\overline{\Omega}} v \,\mathrm{d}\mu_I \right\}. \tag{12}$$

Proof. Let $I : H^1(\Omega) \to \mathbb{R}$ be a non-negative functional. Then the restriction $I|_{H^1(\Omega)\cap C(\overline{\Omega})}$ is a non-negative functional on the space $H^1(\Omega) \cap C(\overline{\Omega}) =: Y$.

Now let $y \in Y$ be arbitrary. Then $y^+ := \max\{0, y\}$ and $y^- := \min\{0, y\}$ (defined in a pointwise sense) are also in Y and we find by non-negativity of L:

$$\begin{aligned}
|Ly| = |L(y^+ + y^-)| = |\underbrace{L(y^+)}_{\geq 0} + \underbrace{L(y^-)}_{\leq 0}| &\leq |\underbrace{L(y^+)}_{\geq 0} \underbrace{-L(y^-)}_{\geq 0}| \\
&\leq |L(y^+ - y^-)| = L(|y|) \\
&= \underbrace{L(|y| - \mathbb{1}\|y\|_\infty)}_{\leq 0} + \|y\|_\infty L(\mathbb{1}) \\
&\leq \|y\|_\infty L(\mathbb{1}).
\end{aligned}$$

Thus $I|_Y$ is continuous in the $C(\overline{\Omega})$-topology. Since Y is also dense in $C(\overline{\Omega})$ the functional $I|_Y$ has a unique continuous and non-negative extension $\tilde{I} : C(\overline{\Omega}) \to \mathbb{R}$ over $C(\overline{\Omega})$. By the Riesz representation theorem (see (11)) we find a $\mu \in M_+(\overline{\Omega})$ such that $I(v) = \int_{\overline{\Omega}} v \,\mathrm{d}\mu$ for all $v \in C(\overline{\Omega})$.

Conversely, let I be in the set on the right-hand side of (12). Then we know $\langle I, v\rangle_{H^1(\Omega)} = \int_{\overline{\Omega}} v \,\mathrm{d}\mu_I \geq 0$ for all $v \in Y_+ := \{v \in Y : v \geq 0 \text{ pointwise in } \overline{\Omega}\}$. So by density of Y_+ in $H^1_+(\Omega)$ we obtain $I \in H^1(\Omega)^*_+$. □

Remark 2.3. *Note that, by an abuse of notation, the right-hand side of* (12) *is sometimes written as* $H^1(\Omega)^* \cap M_+(\overline{\Omega})$ *(see, e.g., [4, Chapter 6]).*

For the notion of *capacity of a set*, *quasi-everywhere (q.e.)* and *quasi-continuous representant* where refer to [12, Chapter 3.3]. The following result is an extension of (11) valid for elements from $H^1(\Omega)^*_+$.

Lemma 2.4. *For all* $I \in H^1(\Omega)^*_+$ *and all* $f \in H^1(\Omega)$ *we have* $\tilde{f} \in L_1(\overline{\Omega}, \mu_I)$ *and*

$$\langle I, f\rangle_{H^1(\Omega)} = \int_{\overline{\Omega}} \tilde{f} \mathrm{d}\mu_I, \tag{13}$$

where $\tilde{f}$ *(defined on* $\overline{\Omega}$*) denotes a quasi-continuous representative of* f *and* μ_I *the measure from* (12) *of Lemma 2.2.*

Proof. The proof of this lemma requires some substantial modifications of [4, Lemma 6.56] and references therein which were designed to the situation $V = \mathring{H}^1(\Omega)$. In our case we will need the following auxiliary results:

(a) For an arbitrary $D \subseteq \mathbb{R}^d$ the capacity of D calculates as

$$\operatorname{cap}(D) = \inf \{ \|v\|_{H^1(\mathbb{R}^d)}^2 : v \in H^1(\mathbb{R}^d) \text{ and } v \geq 1 \text{ a.e. in a neighborhood of } D \}.$$

See [12, Proposition 3.3.5] for a proof.

(b) Any function $f \in H^1(\Omega)$ can be approximated by a sequence $\{f_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ in the sense that $f_n \to f$ in $H^1(\mathbb{R}^d)$ as $n \to \infty$ by extending f to $\mathbb{R}^d$ with compact support and then uses an approximation argument via Friedrichs mollifiers.

The proof carried out in the following steps on the basis of [4, Lemma 6.56] and the references therein (see also [12, Théorème 3.3.29] for the case $V = H^1(\mathbb{R}^d)$):

Claim 1: *There exists a sequence $\{f_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ s.t. $f_n|_\Omega \to \tilde{f}$ in $H^1(\Omega)$ and q.e. in $\overline{\Omega}$*

Let $\{f_n\}$ be given by (b). By resorting to a subsequence (we omit the subscript) we may find $\|f_n - f\|_{H^1(\mathbb{R}^d)} \leq 2^{-n} n^{-1}$ and therefore

$$\sum_{n=1}^{\infty} 4^{n+1} \|f_{n+1} - f_n\|_{H^1(\mathbb{R}^d)}^2 \leq \sum_{n=1}^{\infty} 4^{n+1} \big(\|f_{n+1} - f\|_{H^1(\mathbb{R}^d)} + \|f_n - f\|_{H^1(\mathbb{R}^d)} \big)^2 < +\infty. \tag{14}$$

We define

$$B_n := \{ x \in \mathbb{R}^d : |f_{n+1}(x) - f_n(x)| \geq 2^{-n} \}.$$

Since $|f_{n+1} - f_n|$ is a continuous with compact support in $\mathbb{R}^d$, the set B_n is compact and

$$2^{n+1} |f_{n+1} - f_n| \geq 1 \text{ holds in a neighborhood of } B_n.$$

Thus by (a)

$$\operatorname{cap}(B_n) \leq 4^{n+1} \|f_{n+1} - f_n\|_{H^1(\mathbb{R}^d)}^2.$$

Using this estimate, the sub-additivity of the capacity (see [12, Remarque 3.3.10]) and (14), we obtain:

$$\operatorname{cap}\Big(\bigcup_{k=n}^{\infty} B_k \Big) \leq \sum_{k=n}^{\infty} \operatorname{cap}(B_k) \leq \sum_{k=n}^{\infty} 4^{n+1} \|f_{n+1} - f_n\|_{H^1(\mathbb{R}^d)}^2 \to 0 \quad \text{as } n \to \infty. \tag{15}$$

Now let $n \in \mathbb{N}$ and $x \in \overline{\Omega} \setminus \bigcup_{k=n}^{\infty} B_k$ be arbitrary. Then $\{f_k(x)\}_{k \geq n}$ is a Cauchy sequence since for all $m \geq n$:

$$|f_m(x) - f_n(x)| \leq \sum_{k=n}^{m-1} |f_{k+1}(x) - f_k(x)| \leq \sum_{k=n}^{m-1} 2^{-k}.$$

We denote the limit with $\tilde{f}(x)$ and gain for all $N, K \geq n$:

$$|\tilde{f}(x) - f_N(x)| \leq \underbrace{|\tilde{f}(x) - f_{K+1}(x)|}_{\to 0 \text{ as } K \to \infty} + \sum_{k=N}^{K} \underbrace{|f_{k+1}(x) - f_k(x)|}_{\leq 2^{-k} \text{ since } x \in \overline{\Omega} \setminus \bigcup_{k=n}^{\infty} B_k}$$

A limit passage $K \to \infty$ then shows

$$|\tilde{f}(x) - f_N(x)| \leq \sum_{k=N}^{\infty} 2^{-k}.$$

This estimate implies that $\{f_N\}_{N\geq n}$ converges uniformly to $\tilde f$ on the set $\overline{\Omega}\setminus\bigcup_{k=n}^\infty B_k$. Due to (15) we obtain Claim 1.

<u>*Claim 2:*</u> *If cap$(A)=0$ for a Borel set $A\subseteq\overline{\Omega}$ than $\mu_I(A)=0$.*

Let $\varepsilon>0$ be arbitrary. By (a) we find a function $u\in H^1(\mathbb{R}^d)$ such that $\|u\|_{H^1(\Omega)}<\varepsilon$ and $u\geq 1$ a.e. on A_ε where A_ε is a neighborhood of A. Thus there exists a Lipschitz function $f_\varepsilon:\mathbb{R}^d\to[0,1]$ such that

$$f_\varepsilon(x)=\begin{cases}0 & \text{if } x\in\mathbb{R}^d\setminus A_\varepsilon,\\ \in(0,1) & \text{if } x\in A_\varepsilon\setminus A,\\ 1 & \text{if } x\in A.\end{cases}$$

Then $f_\varepsilon-u\leq 0$ a.e. in Ω and by Lemma 2.2

$$\begin{aligned}\mu_I(A)=\int_A \mathbb{1}\mathrm{d}\mu_I\leq\int_{\overline{\Omega}} f_\varepsilon\,\mathrm{d}\mu_I=\langle I,f_\varepsilon\rangle_{H^1(\Omega)}&=\langle I,u\rangle_{H^1(\Omega)}+\underbrace{\langle I,f_\varepsilon-u\rangle_{H^1(\Omega)}}_{\leq 0\text{ since }f_e\leq u\text{ a.e. in }\Omega}\\ &\leq\langle I,u\rangle_{H^1(\Omega)}\\ &\leq\varepsilon\|I\|_{H^1(\Omega)^*}.\end{aligned}$$

The limit passage $\varepsilon\searrow 0$ yields to claim.

<u>*Claim 3:*</u> $f_n\to\tilde f$ in $L^1(\overline{\Omega},\mu_I)$
Lemma 2.2 implies for every $n,m\in\mathbb{N}$

$$\int_{\overline{\Omega}}|f_n-f_m|\mathrm{d}\mu_I=\langle I,|f_n-f_m|\rangle_{H^1(\Omega)}\leq\|I\|_{H^1(\Omega)^*}\|f_n-f_m\|_{H^1(\Omega)},\tag{16}$$

where f_n is the approximation sequence from Claim 1. Since $f_n\to f$ in $H^1(\Omega)$ we obtain from (16) that $\{f_n\}$ is a Cauchy sequence in $L^1(\overline{\Omega},\mu_I)$. Thus there exists a limit element $\tilde g\in L^1(\overline{\Omega},\mu_I)$ and a subsequence (we omit the subscript) such that $f_n\to\tilde g$ in $L^1(\overline{\Omega},\mu_I)$ and pointwise μ_I-a.e. on $\overline{\Omega}$. However, by Claim 1, we already know that f_n converges q.e. to $\tilde f$ on $\overline{\Omega}$ and, by Claim 2, we find that this covergence is also μ_I-a.e. Thus $\tilde f=\tilde g$ μ_I-a.e.

<u>*Conclusion:*</u>
Finally, Lemma 2.2 shows for every $n\in\mathbb{N}$

$$\langle I,f_n\rangle_{H^1(\Omega)}=\int_{\overline{\Omega}} f_n\mathrm{d}\mu_I.$$

With the properties proven above we can pass to the limit $n\to\infty$ and obtain (13). □

We are now in a position to characterise the tangential and normal cones in K_ψ. The proofs of the following results are based on arguments from [18, Lemme 3.1-3.2, Théorème 3.2].

Lemma 2.5. *Let $y\in K_\psi$ and K_ψ be as in* (10)*. Then it holds*

$$T_y(K_\psi)=\big\{u\in H^1(\Omega):\tilde u\leq 0\ \textit{q.e. on}\ \{\tilde y=\tilde\psi\}\big\},\tag{17a}$$
$$N_y(K_\psi)=\big\{I\in H^1(\Omega)^*:I\in H^1(\Omega)^*_+\ \textit{and}\ \mu_I(\{\tilde y<\tilde\psi\})=0\big\},\tag{17b}$$

where $\tilde{y}$ denotes a quasi-continuous representant of y (the same for $\tilde{u}$ and $\tilde{\psi}$) and $\mu_I \in M_+(\overline{\Omega})$ the measure associated to I by Lemma 2.2.

Please notice that the sets

$$\{\tilde{y} = \tilde{\psi}\} := \{x \in \overline{\Omega} : \tilde{y}(x) = \tilde{\psi}(x)\},$$
$$\{\tilde{y} < \tilde{\psi}\} := \{x \in \overline{\Omega} : \tilde{y}(x) < \tilde{\psi}(x)\}$$

are calculated for arguments in $\overline{\Omega}$ (not only in Ω).

Proof. From the definitions (1)-(3) we see that

$$T_y(K_\psi) = T_{y-\psi}(K), \quad N_y(K_\psi) = N_{y-\psi}(K)$$

with $K := \{w \in H^1(\Omega) : w \leq 0 \text{ a.e. in } \Omega\}$. Thus it suffices to prove the assertion for $K_\psi = K$.

We firstly prove (17b).

"$\subseteq$": Let $I \in N_y(K)$. Then by using definition (3) and choosing $v = y + w$ for an arbitrary $w \in H^1(\Omega)$ with $w \leq 0$ a.e. we obtain $\langle I, w\rangle_{H^1(\Omega)} \leq 0$. Thus $I \in H^1(\Omega)^*_+$ and by Lemma 2.2 we find the associated measure μ_I from (12). On the other hand by choosing $v = \psi$ and $v = 2y$ in (3) yields $\langle I, y\rangle_{H^1(\Omega)} = 0$. From Lemma 2.4 we obtain

$$\int_{\overline{\Omega}} \tilde{y}\,\mathrm{d}\mu_I = 0 \quad \text{with a quasi-continuous representant } \tilde{y} \text{ of } y. \tag{18}$$

Since $y \leq 0$ a.e. in Ω we find $\tilde{y} \leq 0$ q.e. in $\overline{\Omega}$ (see [12, Remarque 3.3.6]). This implies in combination with (18) that $\int_{\overline{\Omega}} |\tilde{y}|\,\mathrm{d}\mu_I = 0$. Thus $\int_{\{\tilde{y}<0\}} |\tilde{y}|\,\mathrm{d}\mu_I = 0$ and therefore $\mu_I(\{\tilde{y} < 0\}) = 0$.

"$\supseteq$": Let $I \in H^1(\Omega)^*_+$ with $\mu_I(\{\tilde{y} < 0\}) = 0$. Now let $v \in K$ be arbitrary. The splitting $v = \max\{v, y\} + \min\{0, v - y\}$ implies

$$\begin{aligned}
\langle I, v - y\rangle_{H^1(\Omega)} &= \langle I, \max\{v, y\} - y\rangle_{H^1(\Omega)} + \underbrace{\langle I, \min\{0, v - y\}\rangle_{H^1(\Omega)}}_{\leq 0} \\
&\leq \int_{\{\tilde{y}=0\}} \max\{\tilde{v}, \tilde{y}\} - \tilde{y}\,\mathrm{d}\mu_I + \underbrace{\int_{\{\tilde{y}<0\}} \max\{\tilde{v}, \tilde{y}\} - \tilde{y}\,\mathrm{d}\mu_I}_{=0 \text{ since } \mu_I(\{\tilde{y}<0\})=0} \\
&\leq \int_{\{\tilde{y}=0\}} \underbrace{\max\{\tilde{v}, 0\}}_{=0 \text{ since } v\in K}\,\mathrm{d}\mu_I = 0.
\end{aligned}$$

Hence $I \in N_y(K)$.

Now we prove (17a). By applying the bipolar theorem as in (6) as well as Lemma 2.4, we find

$$\begin{aligned}
T_y(K) &= \Big\{u \in H^1(\Omega) : \int_{\overline{\Omega}} \tilde{u}\,\mathrm{d}\mu_I \leq 0 \text{ for all } I \in H^1(\Omega)^*_+ \text{ with } \mu_I(\{\tilde{y} < 0\}) = 0\Big\} \\
&= \Big\{u \in H^1(\Omega) : \int_{\{\tilde{y}=0\}} \tilde{u}\,\mathrm{d}\mu_I \leq 0 \text{ for all } I \in H^1(\Omega)^*_+ \text{ with } \mu_I(\{\tilde{y} < 0\}) = 0\Big\}.
\end{aligned}$$

From this representation we see that the "$\supseteq$"-inclusion in (17a) is fulfilled. Conversely, let $u \in T_y(K)$. By definition of $T_y(K)$ given in (2) we find a sequence $v_n \in K$ and $t_n > 0$ such that $t_n(v_n - y) \to u$

in $H^1(\Omega)$ as $n \to \infty$. This implies for a subsequence (we omit the subindex) $t_n(\tilde{v}_n - \tilde{y}) \to \tilde{u}$ q.e. in $\overline{\Omega}$. Since $v_n \in K$ we see that

$$t_n(\tilde{v}_n - \tilde{y}) = t_n\tilde{v}_n \le 0 \text{ q.e. on } \{\tilde{y} = 0\}.$$

Thus $\tilde{u} \le 0$ q.e. on $\{\tilde{y} = 0\}$. □

Theorem 2.6. *The set K_ψ is polyhedric.*

Proof. Let y and w as in (7) and let $v \in T_y(K_\psi) \cap [w]^\perp$. Then there exists a sequence $v_n \to v$ strongly in $H^1(\Omega)$ such that $v_n \in C_y(K_\psi)$. Define

$$v_n' := \max\{v_n, v\}.$$

By resorting to quasi-continuous representants we find by Lemma 2.5

$$v \le 0 \text{ q.e. in } \{y = 0\} \quad \text{and} \quad v_n \le 0 \text{ q.e. in } \{y = 0\}$$

and thus

$$v_n' \le 0 \text{ q.e. in } \{y = 0\}.$$

Moreover by definition of v_n'

$$v - v_n' \le 0 \text{ q.e. in } \Omega.$$

Invoking Lemma 2.5 again yield $v_n' \in T_y(K_\psi)$ and $v - v_n' \in T_y(K_\psi)$. Since $w \in N_y(K_\psi)$ we see by (5) that

$$\langle w, v_n' \rangle \le 0 \quad \text{and} \quad \langle w, v - v_n' \rangle \le 0.$$

Taking also $\langle w, v \rangle = 0$ into account we obtain from above that $\langle w, v_n' \rangle = 0$. Thus $v_n' \in C_y(K_\psi) \cap [w]^\perp$. □

2.3 Eulerian semi and shape derivative

We recall some preliminaries from shape optimisation theory. For more details we refer to [5].

Let $X : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field satisfying a global Lipschitz condition: there is a constant $L > 0$ such that

$$|X(x) - X(y)| \le L|x - y| \quad \text{for all } x, y \in \mathbb{R}^d.$$

Then we associate with X the flow Φ_t by solving for all $x \in \mathbb{R}^d$

$$\frac{d}{dt}\Phi_t(x) = X(\Phi_t(x)) \text{ on } [-\tau, \tau], \quad \Phi_0(x) = x. \tag{19}$$

The global existence of the flow $\Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is ensured by the theorem of Picard-Lindelöf.

Subsequently, we restrict ourselves to a special class of vector fields, namely C^k-vector fields with compact support in some fixed set. To be more precise for a fixed open set $D \subseteq \mathbb{R}^d$, we consider vector fields belonging to $C_c^k(D, \mathbb{R}^d)$. We equip the space $C_c^k(D, \mathbb{R}^d)$ respectively $C_c^\infty(D, \mathbb{R}^d)$ with the topology induced by the following family of semi-norms: for each compact $K \subseteq D$ and muli-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k$ we define $\|f\|_{K,\alpha} := \sup_{x\in K} |\partial^\alpha f(x)|$. With this familiy of semi-norms the space $C_c^k(D, \mathbb{R}^d)$ becomes a locally convex vector space.

Next, we recall the definition of the Eulerian semi-derivative.

Definition 2.7. *Let $D \subseteq \mathbb{R}^d$ be an open set. Let $J : \Xi \to \mathbb{R}$ be a shape function defined on a set Ξ of subsets of D and fix $k \geq 1$. Let $\Omega \in \Xi$ and $X \in C_c^k(D, \mathbb{R}^d)$ be such that $\Phi_t(\Omega) \in \Xi$ for all $t > 0$ sufficiently small. Then the Eulerian semi-derivative of J at Ω in direction X is defined by*

$$dJ(\Omega)(X) := \lim_{t \searrow 0} \frac{J(\Phi_t(\Omega)) - J(\Omega)}{t}. \tag{20}$$

(i) The function J is said to be shape differentiable at Ω if $dJ(\Omega)(X)$ exists for all $X \in C_c^\infty(D, \mathbb{R}^d)$ and $X \mapsto dJ(\Omega)(X)$ is linear and continuous on $C_c^\infty(D, \mathbb{R}^d)$.

(ii) The smallest integer $k \geq 0$ for which $X \mapsto dJ(\Omega)(X)$ is continuous with respect to the $C_c^k(D, \mathbb{R}^d)$-topology is called the order of $dJ(\Omega)$.

The set D in the previous definition is usually called hold-all domain or hold-all set or universe.

In the case that the state system is given as a solution of a variational inequality we cannot expect $dJ(\Omega)(X)$ to be linear in X. However we have the following general result:

Lemma 2.8. *Suppose that the Eulerian semi-derivative $dJ(\Omega)(X)$ exists for all $X \in C_c^k(D, \mathbb{R}^d)$. Then $dJ(\Omega)(\cdot)$ is positively 1-homogeneous.*

Proof. Let $\lambda > 0$ be arbitrary. We write $\Phi_t^{\lambda X}$ for the flow induced by λX. By definition (19), we see that $\Phi_t^{\lambda X}$ and $\Phi_{\lambda t}^X$ solve

$$\frac{d}{dt}\Phi_t^{\lambda X}(x) = \lambda X(\Phi_t^{\lambda X}(x)), \quad \frac{d}{dt}\Phi_{\lambda t}^X(x) = \lambda X(\Phi_{\lambda t}^X(x))$$

as well as $\Phi_0^{\lambda X}(x) = x$ and $\Phi_0^X(x) = x$. Uniqueness of the flow implies $\Phi_t^{\lambda X} = \Phi_{\lambda t}^X$. Finally,

$$dJ(\Omega)(\lambda X) = \lim_{t \searrow 0} \frac{J(\Phi_t^{\lambda X}(\Omega)) - J(\Omega)}{t} = \lim_{t \searrow 0} \frac{J(\Phi_{\lambda t}^X(\Omega)) - J(\Omega)}{t} = \lambda\, dJ(\Omega)(X).$$

□

The following result can be found for instance in [5]:

Lemma 2.9. *Let $D \subseteq \mathbb{R}^d$ be open and bounded and suppose $X \in C_c^1(D, \mathbb{R}^d)$.*

(i) We have

$$\begin{aligned} \frac{\partial \Phi_t - I}{t} &\to \partial X && \text{strongly in } C(\overline{D}, \mathbb{R}^{d,d}) \\ \frac{\partial \Phi_t^{-1} - I}{t} &\to -\partial X && \text{strongly in } C(\overline{D}, \mathbb{R}^{d,d}) \\ \frac{\det(\partial \Phi_t) - 1}{t} &\to div(X) && \text{strongly in } C(\overline{D}). \end{aligned}$$

(ii) For all open sets $\Omega \subseteq D$ and all $\varphi \in W_\mu^1(\Omega)$, $\mu \geq 1$, we have

$$\frac{\varphi \circ \Phi_t - \varphi}{t} \to \nabla \varphi \cdot X \qquad \text{strongly in } L_\mu(\Omega). \tag{21}$$

3 Abstract sensitivity analysis

In this section we will derive sensitivity estimates and relations for material derivatives under general conditions. We start in Section 3.1 with minimisers of certain p-coercive energy functionals and deduce a Hölder-type estimate with exponent $1/p$. We present an example which includes the quasi-linear p-Laplacian $-\Delta_p(\cdot) = \operatorname{div}(|\nabla \cdot|^{p-2}\nabla\cdot)$. Then we proceed in Section 3.2 with solutions of monotone operators where we are able to improve the estimates from Subsection 3.1. For the case $p = 2$ we even establish a Lipschitz type sensitivity estimate. Finally in Subsection 3.3 we strengthen the assumptions in order to establish the weak material derivative. A crucial requirement will be the polyhedricity of the underlying set.

In this whole section V will denote a Banach space, $K \subseteq V$ a closed convex subset and $\tau > 0$ a fixed constant.

3.1 Sensitivity result for minimisers of energy functionals

Our starting point is a family of energy functionals

$$E : [0, \tau] \times V \to \mathbb{R},$$

where we denote the set of attained infima at $t \in [0, \tau]$ by

$$X(t) := \big\{u^t \in V : \inf_{\varphi \in K} E(t, \varphi) = E(t, u^t)\big\}. \tag{22}$$

Our aim is to establish a general result showing the convergence of minimisers of $E(t, \cdot)$ to minimisers of $E(0, \cdot)$ as $t \searrow 0$. Before we state our abstract sensitivity result, we recall [20, Theorem 1] which will be used in a subsequent proof:

Theorem 3.1 ([20, Theorem 1]). *Let $[\cdot]$ be a seminorm on V. Let $E : V \to \mathbb{R}$ be an energy functional such that for all $v, w \in K$ the mapping $s \mapsto \gamma(s) := E(sw + (1 - s)v))$ is C^1 on $[0, 1]$. Let us denote by $\mathcal{A} : K \to V^*$ the Gateaux-differential of E which is supposed to be p-coercive on K:*

$$\exists \alpha > 0, \forall u, v \in K, \quad \langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_V \geq \alpha[v - w]^p.$$

Then every minimum u of E on K satisfies:

$$\forall v \in K, \quad \frac{\alpha}{p}[u - v]^p \leq E(u) - E(v).$$

In what follows let E satisfy the following assumption:

Assumption (E) *Suppose that the energy functionals $E(t, \cdot)$ satisfies for a given $p \geq 1$:*

(i) $\exists c_1 > 0, \exists c_2 > 0$, $\forall \varphi \in K$, $E(\cdot, \varphi)$ *is differentiable and*

$$\forall t \in [0, \tau], \quad \partial_t E(t, \varphi) \leq c_1 \|\varphi\|_V^p + c_2;$$

(ii) $\exists c > 0$, $\exists \Lambda > 0$, $\forall \varphi \in K$, $\forall t \in [0, \tau]$,

$$E(t, \varphi) \geq c\|\varphi\|_V^p - \Lambda;$$

(iii) $\forall t \in [0,\tau]$, $E(t,\cdot)$ *is Gateaux-differentiable and*

$$\exists \alpha > 0, \forall u, v \in K, \quad \langle \mathcal{A}_t(u) - \mathcal{A}_t(v), u - v\rangle_V \geq \alpha[v-w]^p,$$

where $\langle \mathcal{A}_t(v), w\rangle_V := dE(t, v; w)$ *and* $[\cdot]$ *is a semi-norm on* V*;*

(iv) $\forall v, w \in K$, $\forall t \in [0,\tau]$,

$$s \mapsto \gamma(s) := E(t, sv + (1-s)w) \text{ is } C^1([0,1])$$

Now we are in the position to state and prove our sensitivity result:

Theorem 3.2. *Let* $E : [0,\tau] \times V \to \mathbb{R}$ *be a family of energy functionals satisfying Assumption (E) and let* $X(t)$ *be non-empty for every* $t \in [0,\tau]$*. Then* $X(t) = \{u^t\}$ *is a singleton and there exists a constant* $c > 0$ *such that for all* $t \in [0,\tau]$*:*

$$[u^t - u^0] \leq ct^{1/p}.$$

Proof. Let $t \in [0,\tau]$ and $u^t \in X(t)$. Let us first show that u^t is bounded in V uniformly in t. According to Assumption (E) (i)-(ii), the definition of u^t and the mean value theorem, we obtain $\eta_t \in (0,t)$ such that

$$\begin{aligned} c\|u^t\|_V^p - \Lambda &\leq E(t, u^t) \\ &\leq E(t, u^0) \\ &= E(0, u^0) + t\partial_t E(\eta_t, u^0) \\ &\leq E(0, u^0) + t\big(c_1\|u^0\|_V^p + c_2\big). \end{aligned} \tag{23}$$

This shows that $\|u^t\|_V \leq C$ for all $t \in [0,\tau]$ for some constant $C > 0$. Furthermore applying Theorem 3.1 by using Assumption (E) (iii)-(iv) shows

$$c[u^t - u^0]^p \leq E(t, u^t) - E(t, u^0), \tag{24}$$

$$c[u^t - u^0]^p \leq E(0, u^0) - E(0, u^t). \tag{25}$$

Adding both inequalities, applying the mean value theorem twice with some $\eta_t, \zeta_t \in (0,t)$ and using Assumption (E) (i) and the estimate (23) yields

$$\begin{aligned} 2c[u^t - u^0]^p &\leq E(t, u^t) - E(t, u^0) + E(0, u^0) - E(0, u^t) \\ &\leq t\big(\partial_t E(\eta_t, u^t) - \partial_t E(\zeta_t, u^0)\big) \\ &\leq tC(\|u^t\|_V^p + \|u^0\|_V^p) \\ &\overset{(23)}{\leq} tC(1 + \|u^0\|_V^p). \end{aligned} \tag{26}$$

This finishes the proof. □

Example (p-Laplace equation)
As an application of Theorem 3.2 let us consider the p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } K = V = \mathring{W}_p^1(\Omega)$$

on a bounded Lipschitz domain Ω and the associated energy given by

$$E(0,\varphi) = \frac{1}{p}\int_\Omega |\nabla\varphi|^p \,\mathrm{d}x - \int_\Omega f\varphi \,\mathrm{d}x, \quad \varphi \in \mathring{W}_p^1(\Omega).$$

By applying integration by substitution, the energy of the perturbed equation transported to Ω via integration by substitution is of the form

$$E(t,\varphi) = \frac{1}{p}\int_\Omega \xi(t)|B(t)\nabla\varphi|^p \,\mathrm{d}x - \int_\Omega f(t)\varphi \,\mathrm{d}x,$$

More generally we assume that $\xi : [0,\tau] \to \mathbb{R}$ and $B : [0,\tau] \to \mathbb{R}^{d\times d}$ are C^1-functions which satisfy $\xi(0) = 1$ and $B(0) = I$. Moreover let $f(0) = f$ and $f(\cdot, x)$ be differentiable and $f'(t) \in L_{p'}(\Omega)$ be uniformly bounded where $p' = p/(p-1)$ denotes the conjugate of p. We check that the assumptions in (E) are satisfied:

Indeed, we have

$$\partial_t E(t,\varphi) = \int_\Omega \xi'(t)\frac{1}{p}|B(t)\nabla\varphi|^p + \xi(t)|B(t)\nabla\varphi|^{p-2}B(t)\nabla\varphi\cdot B'(t)\nabla\varphi \,\mathrm{d}x - \int_\Omega f'(t)\varphi \,\mathrm{d}x.$$

Thus applying Hölder and Young's inequalities we verify Assumption (E) (i):

$$\begin{aligned}
\partial_t E(t,\varphi) &\le \int_\Omega \xi'(t)\frac{1}{p}|B(t)\nabla\varphi|^p + \xi(t)|B(t)\nabla\varphi|^{p-2}B(t)\nabla\varphi\cdot B'(t)\nabla\varphi \,\mathrm{d}x - \int_\Omega f'(t)\varphi \,\mathrm{d}x \\
&\le c\int_\Omega |\nabla\varphi|^p + |f'(t)||\varphi| \,\mathrm{d}x \\
&\le c\|\nabla\varphi\|_{L_p}^p + 1/p'\|f'(t)\|_{L_{p'}}^{p'} + \frac{1}{p}\|\varphi\|_{L_p}^p.
\end{aligned}$$

On the other hand using Young's and Poincaré's inequality with small $\varepsilon > 0$

$$\begin{aligned}
E(t,\varphi) &\ge c\|\nabla\varphi\|_{L_p}^p - 1/p'(p\varepsilon)^{-\frac{1}{p-1}}\|f(t)\|_{L_{p/(p-1)}}^{p'} - \varepsilon\|\varphi\|_{L_p}^p \\
&\ge c_1\|\varphi\|_{W_p^1}^p - c_2 - \varepsilon\|\varphi\|_{L_p}^p.
\end{aligned}$$

Thus we have verified Assumption (E) (ii). Assumption (E) (iii) follows from uniform p-monotonicity of $-\Delta_p(\cdot)$ and Assumption (E) (iv) by direct calculations.

Finally we may use Theorem 3.2 and obtain $\|u^t - u\|_{W_p^1(\Omega)} \le ct^{1/p}$ for some constant $c > 0$ and all sufficiently small $t > 0$. In the case of the usual Laplace equation, that is for $p = 2$, we get $\|u^t - u\|_{H^1(\Omega)} \le ct^{1/2}$.

3.2 Sensitivity result for uniformly monotone operators

In this section we develop sensitivity results for variational inequalites involving uniformly monotone operators. Let V be a normed space and $K \subseteq V$ be a closed convex subset.

Assumption (O1) *Suppose that $(\mathcal{A}_t) : K \to V^*$, $t \in [0,\tau]$ is a family of operators such that for a given $p \ge 1$:*

(i) $\exists\alpha > 0$, $\forall t \in [0,\tau]$, $\forall u, v \in K$:

$$\alpha\|u - v\|_V^p \le \langle \mathcal{A}_t(u) - \mathcal{A}_t(v), u - v\rangle_V;$$

(ii) $\forall u \in K$, $\exists c > 0$, $\forall t \in [0,\tau]$, $\forall v \in K$,

$$|\langle \mathcal{A}_t(u) - \mathcal{A}_0(u), u - v\rangle_V| \le ct\|u - v\|_V.$$

Theorem 3.3. *Suppose that $(\mathcal{A}_t) : K \to V^*$ is a family of operators satisfying Assumption (O1). For every $t > 0$ we denote by $u^t \in K$ a solution of the variational inequality*

$$u^t \in K \text{ and } \forall v \in K,\ \langle \mathcal{A}_t(u^t), v - u^t\rangle_V \ge 0. \tag{27}$$

Then there exists a $c > 0$ such that

$$\forall t \in [0,\tau] : \quad \|u^t - u^0\|_V \le ct^{\frac{1}{p-1}}.$$

Proof. Taking into account Assumption (O1) and (27):

$$\begin{aligned}
\alpha\|u^t - u^0\|_V^p &\le \langle \mathcal{A}_t(u^t) - \mathcal{A}_t(u^0), u^t - u^0\rangle_V \\
&\le -\langle \mathcal{A}_t(u^0), u^t - u^0\rangle_V \\
&= \langle \mathcal{A}_0(u^0), u^t - u^0\rangle_V + \langle \mathcal{A}_0(u^0) - \mathcal{A}_t(u^0), u^t - u^0\rangle_V \\
&\le |\langle \mathcal{A}_0(u^0) - \mathcal{A}_t(u^0), u^t - u^0\rangle_V| \\
&\le ct\|u^t - u^0\|_V.
\end{aligned}$$

□

Remark 3.4. *In the important case $p = 2$ Theorem 3.3 yields a Lipschitz type estimates.*

Example (p-Laplace equation)
It can be checked that the p-Laplace example from Subsection 3.1 where $\mathcal{A}_t$ is given by

$$\langle \mathcal{A}_t(u), \varphi\rangle_{\overset{\circ}{W}{}_p^1} = \int_\Omega \xi(t)|B(t)\nabla u|^{p-2}B(t)\nabla u \cdot B(t)\nabla\varphi - f(t)\varphi \,\mathrm{d}x$$

also fulfills Assumption (O1). Thus in this case Theorem 3.3 gives a sharper estimate than Theorem 3.2.

3.3 Variational inequality for the material derivative

In the previous section we have shown that under certain conditions on $(\mathcal{A}_t)$ satisfied for $p = 2$ the quotient $(u^t - u^0)/t$ stays bounded. In this subsection we additionally assume that V is reflexive and that $K \subseteq V$ is a polyhedric subset. Then there will be a weakly converging subsequence of $(u^t - u^0)/t$ converging to some $z \in V$. If this z is unique the whole sequence converges and additionally satisfies some limiting equation which is the subject of this subsection.

Let $(\mathcal{A}_t)$ be as in Subsection 3.2 and define in accordance with (22) for all $t \in [0,\tau]$ the solution set of the associated variational inequality as

$$X(t) := \{u^t \in K : \ \forall\varphi \in K,\ \langle \mathcal{A}_t(u^t), \varphi - u^t\rangle \ge 0\}. \tag{28}$$

We will write $u := u^0$ and $\mathcal{A} := \mathcal{A}_0$. The variational inequality for the material derivative will be deduced from the following assumptions:

Assumption (O2) *Suppose that the family $(\mathcal{A}_t)$ satisfies* *(i) for all $v, w \in V$ and all $u \in K$,*

$$\langle \partial\mathcal{A}(u)w, v\rangle_V := \lim_{t\searrow 0} \left\langle \frac{\mathcal{A}(u+tw)-\mathcal{A}(u)}{t}, v \right\rangle_V$$

and

$$\langle \mathcal{A}'(u), v\rangle_V := \lim_{t\searrow 0} \left\langle \frac{\mathcal{A}_t(u)-\mathcal{A}(u)}{t}, v \right\rangle_V$$

exist;

(ii) for all null-sequences (t_n), for all sequences (v_n) in V converging weakly to some $v \in V$, for all $u^{t_n} \in X(t_n)$ converging strongly to some $u \in K$, we have

$$\langle \mathcal{A}'(u), v\rangle_V = \lim_{n\to 0} \left\langle \frac{\mathcal{A}_{t_n}(u^{t_n})-\mathcal{A}(u^{t_n})}{t_n}, v_n \right\rangle_V ;$$

(iii) for all null-sequences (t_n), there exists a subsequence (still indexed the same) such that $u^{t_n} \in X(t_n)$ converges storngly to $u \in K$ and $(u_n - u)/t_n$ converges weakly to some $z \in V$ and

$$\langle \partial\mathcal{A}(u)z, z\rangle_V \leq \liminf_{n\to 0} \left\langle \frac{\mathcal{A}(u^{t_n})-\mathcal{A}(u)}{t_n}, \frac{u^{t_n}-u}{t_n} \right\rangle_V$$

and for all (v_n) in V converging strongly to $v \in V$:

$$\langle \partial\mathcal{A}(u)z, v\rangle_V = \lim_{n\to 0} \left\langle \frac{\mathcal{A}(u^{t_n})-\mathcal{A}(u)}{t_n}, v_n \right\rangle_V .$$

Theorem 3.5. *Let V be a reflexive Banach space and $K \subseteq V$ a polyhedric subset. Suppose that $\mathcal{A}_t : K \to V^*$, $t \in [0, \tau]$ is a family of operators satisfying Assumption (O1) for $p = 2$ and (O2). Suppose that $u^t \in X(t)$, i.e., u^t solves*

$$u^t \in K, \quad \langle \mathcal{A}_t(u^t), \varphi - u^t\rangle_V \geq 0 \quad \forall \varphi \in K. \tag{29}$$

Then the material derivative $\dot{u} := \mathrm{weak} - \lim_{t\searrow 0}(u^t - u)/t$ exists and solves

$$\dot{u} \in T_u(K) \cap kern(\mathcal{A}(u)) \text{ and} \tag{30a}$$

$$\forall \varphi \in T_u(K) \cap kern(\mathcal{A}(u)) : \ \langle \partial\mathcal{A}(u)\dot{u}, \varphi - \dot{u}\rangle_V \geq -\langle \mathcal{A}'(u), \varphi - \dot{u}\rangle_V. \tag{30b}$$

Proof. Let us firstly show (30a). We get by (29)

$$\forall \varphi \in K(\Omega) : \quad \langle \mathcal{A}_t(u^t), \varphi - u^t\rangle \geq 0, \tag{31}$$

$$\forall \varphi \in K(\Omega) : \quad \langle \mathcal{A}(u), \varphi - u\rangle \geq 0. \tag{32}$$

Thus testing (31) with u and (32) with u^t and dividing by $t > 0$, we obtain by setting $z^t := (u^t - u)/t$

$$\langle \mathcal{A}_t(u^t), z^t\rangle \leq 0, \qquad \langle \mathcal{A}(u), z^t\rangle \geq 0. \tag{33}$$

By invoking Theorem 3.3 with $p = 2$ we know that $u^t \to u$ strongly in V and that z^t is bounded in V which allows us to choose a weakly convergence subsequence with limit $\dot{u} \in V$. We find (by omitting the subscript)

$$\langle \mathcal{A}_t(u^t), z^t\rangle - \langle \mathcal{A}(u), \dot{u}\rangle$$

$$= \underbrace{\langle \mathcal{A}_t(u^t) - \mathcal{A}(u^t), z^t\rangle}_{\to 0 \text{ by Assumption (O2) (ii)}} + \underbrace{\left\langle \frac{\mathcal{A}(u^t) - \mathcal{A}(u)}{t}, u^t - u\right\rangle}_{\to 0 \text{ by Assumption (O2) (iii)}} + \underbrace{\langle \mathcal{A}(u), z^t - \dot{u}\rangle}_{\to 0}$$

Therefore passing to the limit in (33) gives $0 \le \langle \mathcal{A}(u), \dot{u}\rangle \le 0$ and thus $\dot{u} \in \mathrm{kern}(\mathcal{A}(u))$. Furthermore we know by the definition of the radial cone that $z^t \in C_u(K)$. Taking the weak convergence $z^t \rightharpoonup \dot{u}$ in V and Mazur's Lemma into account we find $\dot{u} \in T_u(K)$. Thus (30a) is proven.

Now we will show (30b) by using (29) and obtain for every $\varphi \in V$:

$$\begin{aligned} \langle \mathcal{A}(u^t) - \mathcal{A}(u), \varphi - u^t\rangle &= \langle \mathcal{A}(u^t) - \mathcal{A}_t(u^t), \varphi - u^t\rangle + \langle \mathcal{A}_t(u^t) - \mathcal{A}(u), \varphi - u^t\rangle \\ &\ge \langle \mathcal{A}(u^t) - \mathcal{A}_t(u^t), \varphi - u^t\rangle - \langle \mathcal{A}(u), \varphi - u^t\rangle. \end{aligned} \tag{34}$$

By definition of the radial cone $C_u(K)$ (see (1)) we find for every $\varphi \in C_u(K)$ a $t^* > 0$ such that for all $t \in [0, t^*]$: $u + t\varphi \in K$. Plugging this test-function into (34) we obtain for all $\varphi \in C_u(K)$

$$\langle \mathcal{A}(u^t) - \mathcal{A}(u), t\varphi - (u^t - u)\rangle \ge \langle \mathcal{A}(u^t) - \mathcal{A}_t(u^t), t\varphi - (u^t - u)\rangle - \langle \mathcal{A}(u), t\varphi - (u^t - u)\rangle. \tag{35}$$

Dividing the previous equation by t^2 and setting $z^t := (u^t - u)/t$, we obtain

$$\left\langle \frac{\mathcal{A}(u^t) - \mathcal{A}(u)}{t}, \varphi - z^t\right\rangle \ge -\left\langle \frac{\mathcal{A}_t(u^t) - \mathcal{A}(u^t)}{t}, \varphi - z^t\right\rangle - \frac{1}{t}\langle \mathcal{A}(u), \varphi - z^t\rangle. \tag{36}$$

Now let $\varphi \in C_u(K) \cap kern(\mathcal{A}(u))$. Then because of $\langle \mathcal{A}(u), \varphi\rangle = 0$ and the definition of $u \in X(0)$ (testing the relation in (28) with u^t), we find

$$-\langle \mathcal{A}(u), \varphi - z^t\rangle \ge 0.$$

Thus (36) reads

$$\left\langle \frac{\mathcal{A}(u^t) - \mathcal{A}(u)}{t}, \varphi - z^t\right\rangle \ge -\left\langle \frac{\mathcal{A}_t(u^t) - \mathcal{A}(u^t)}{t}, \varphi - z^t\right\rangle. \tag{37}$$

Using Assumption (O2) we may take the $\limsup$ on both sides to obtain (note that $-\limsup(...) = \liminf -(...)$)

$$\langle \partial \mathcal{A}(u) z, \varphi - z\rangle \ge -\langle \mathcal{A}'(u), \varphi - z\rangle \quad \forall \varphi \in C_u(K) \cap kern(\mathcal{A}(u)).$$

Via density arguments we obtain the inequality for all $\varphi \in \overline{C_u(K) \cap kern(\mathcal{A}(u))}$. Finally using polyhedricity of K and Lemma 2.1 (i) finish the proof. □

4 A semilinear dynamic obstacle problem

In this section we are going to apply the theorems from Section 3 to generalised obstacle problems with convex energies. present a generalised obstacle problem. It also covers previous results from [23] where the zero obstacle case has been treated as a special case. A non-trivial example from continuum damage mechanics is presented afterward in Section 5.

4.1 Setting and state system

Let $D \subseteq \mathbb{R}^d$ be an open and bounded subset. We consider a convex energy of the following type

$$E(\Omega, \varphi) := \int_\Omega \frac{1}{2}|\nabla\varphi|^2 + \frac{\lambda}{2}|\varphi|^2 + W_\Omega(x, \varphi)\,\mathrm{d}x, \quad \varphi \in H^1(\Omega), \tag{38}$$

where $\Omega \subseteq D$ is a bounded Lipschitz domain and $\lambda > 0$. The energy is minimised over the convex set

$$K_{\psi_\Omega}(\Omega) := \big\{\varphi \in H^1(\Omega) : \varphi \leq \psi_\Omega \text{ a.e. in } \Omega\big\}.$$

A particularity of this setting is that, besides the density function W_Ω, also the obstacle function ψ_Ω is allowed to depend on the shape variable Ω (the precise assumptions are stated below in Assumption (A1)):

dynamic density function:	$\Omega \mapsto W_\Omega$
dynamic obstacle:	$\Omega \mapsto \psi_\Omega \in H^1(\Omega)$

In the special case $\psi_\Omega \equiv 0$ we write $K(\Omega) := K_0(\Omega)$.

Remark 4.1. *(i) An important class which is covered by our setting are static obstacle problems where $\psi_\Omega := \Psi|_\Omega$ with a given function $\Psi \in H^2(D)$.*

(ii) The energy $E(\Omega, \cdot)$ is motivated by time-discretised parabolic problems, where an additional λ-convex non-linearity may be included in E. By choosing a small time step size, the incremental minimisation problem may take the form (38).

In context with time-discretised damage models in Section 5 we are faced with iterative obstacle problems. In this case the obstacle ψ_Ω itself is a solution of a variational inequality describing the damage profile from the previous time step. As we will see it suffices to have $H^1(\Omega)$-regularity of the damage profile provided that the material derivative of the obstacle exists in $H^1(\Omega)$ and the initial value is in $H^2(\Omega)$. We will present this application in the last section.

For later use we recall that the Sobolev exponent 2^* depending on the spatial dimension d to the space $H^1(\Omega)$ is defined as

$$2^* := \begin{cases} \frac{2d}{d-2} & \text{if } d > 2, \\ \textit{arbitrary in } [1, +\infty) & \text{if } d = 2, \\ +\infty & \text{if } d = 1. \end{cases} \tag{39}$$

Its conjugate $(2^*)'$ is given by $\frac{2^*}{2^*-1}$ with the convention that $(2^*)' := 1$ for $2^* = +\infty$. For well-posedness of the state system we require the following assumptions (note that we restrict ourselves to the convex case which will be exploited in the next sections):

Assumption (A1) *For all Lipschitz domains $\Omega \subseteq D$ it holds:*

(i) $W_\Omega(x, \cdot)$ is convex and in $C^1(\mathbb{R})$ for all $x \in \Omega$;

(ii) the following map $H^1(\Omega) \to \mathbb{R}$ is assumed to be continuous (in particular the integral exists)

$$y \mapsto \int_\Omega W_\Omega(x, y(x))\,\mathrm{d}x$$

and bounded from below by

$$\int_\Omega W_\Omega(x, y(x))\,\mathrm{d}x \geq -c(\|y\|_{H^1} + 1);$$

(iii) for all $y, \varphi \in H^1(\Omega)$*:*

$$\int_\Omega \frac{W_\Omega(x, y + t\varphi) - W_\Omega(x, y)}{t}\,\mathrm{d}x \to \int_\Omega \partial_y W_\Omega(x, y)\varphi\,\mathrm{d}x \qquad \text{as } t \searrow 0$$

(in particular the integral on the right-hand side exists);

(iv) $\psi_\Omega \in H^1(\Omega)$.

Remark 4.2. *Assumption (A1) (iii) and the continuity property from (A1) (ii) are satisfied if, e.g., the following growth condition holds: There exist constants* $\epsilon, C > 0$ *and functions* $s \in L_1(\Omega)$ *and* $r \in L_{(2^*)'}(\Omega)$ *such that for all* $x \in \Omega$ *and* $y \in \mathbb{R}$*:*

$$|W_\Omega(x, y)| \leq C|y|^{2^* - \epsilon} + s(x),$$
$$|\partial_y W_\Omega(x, y)| \leq C|y|^{2^* - 1} + r(x).$$

The assumptions in (A1) in combination with the direct method in the calculus of variations imply unique solvability of the variational inequality fulfilled by the minimisers of $E(\Omega, \cdot)$.

Lemma 4.3. *Under Assumption (A1) the energy* (38) *admits for each Lipschitz domain* $\Omega \subseteq D$ *a unique minimum* u *(depending on* Ω*) on* $K_\psi(\Omega)$ *which is given as the unique solution of*

$$\begin{cases} u \in K_{\psi_\Omega}(\Omega) \text{ and } \forall \varphi \in K_{\psi_\Omega}(\Omega) : \\ \int_\Omega \nabla u \cdot \nabla(\varphi - u) + \lambda u(\varphi - u) + w_\Omega(x, u)(\varphi - u)\,\mathrm{d}x \geq 0, \end{cases} \tag{40}$$

where

$$w_\Omega(x, y) := \partial_y W_\Omega(x, y).$$

In the sequel we will treat the variational inequality (40) by making use of the transformation for the state variable and its test-function:

$$y := u - \psi_\Omega \text{ and } \tilde{\varphi} := \varphi - \psi_\Omega.$$

The variation inequality becomes a problem involving the standard obstacle set

$$K(\Omega) := \{\varphi \in H^1(\Omega) : \varphi \leq 0 \text{ a.e. on } \Omega\}.$$

Substituting above tranformation into (40) we obtain the following variational inequality:

$$\begin{cases} y \in K(\Omega) \text{ and } \forall \varphi \in K(\Omega) : \\ \int_\Omega \nabla y \cdot \nabla(\varphi - y) + \lambda y(\varphi - y) + w_\Omega(x, y + \psi_\Omega)(\varphi - y)\,\mathrm{d}x \\ \qquad \geq -\int_\Omega \nabla \psi_\Omega \cdot \nabla(\varphi - y) + \lambda \psi_\Omega(\varphi - y)\,\mathrm{d}x \end{cases} \tag{41}$$

Hence it will suffice to investigate the solution y to deduce properties of the function u.

4.2 Perturbed problem and sensitivity estimates

In this subsection we prove a shape sensitivity result for the variational inequality (41). In what follows let us denote by Φ_t the flow generated by a vector field $X \in C_c^1(D, \mathbb{R}^d)$. For $\Omega \subseteq D$ denote by $\Omega_t := \Phi_t(\Omega)$, $t \geq 0$, the perturbed domains (see Subsection 2.3 for more details).

Perturbed problem
The solution $y_t \in H^1(\Omega_t)$ to the perturbed variational inequality to (41) satisfies

$$\begin{cases} y_t \in K(\Omega_t) \text{ and } \forall \varphi \in K(\Omega_t): \\ \displaystyle\int_{\Omega_t} \nabla y_t \cdot \nabla(\varphi - y_t) + \lambda y_t(\varphi - y_t) + w_{\Omega_t}(x, y_t + \psi_{\Omega_t})(\varphi - y_t)\,\mathrm{d}x \\ \qquad \geq -\displaystyle\int_{\Omega_t} \nabla \psi_{\Omega_t} \cdot \nabla(\varphi - y_t) + \lambda \psi_{\Omega_t}(\varphi - y_t)\,\mathrm{d}x. \end{cases} \tag{42}$$

We will sometimes write $y_t(X) = y_t$ to emphasise the dependence on X. Please note that in general $y_0(X) = y_t(X)$ for all $t \geq 0$ and for all vector fields $X \in C_c^1(D, \mathbb{R}^2)$ with the property $X \cdot n = 0$ on $\partial\Omega$. This implication will be used in the forthcoming Lemma 4.14. Throughout this work we will adopt the following abbreviations:

$$\begin{aligned} w_X^t(x, \varphi) &:= w_{\Omega_t}(\Phi_t(x), \varphi), & W_X^t(x, \varphi) &:= W_{\Omega_t}(\Phi_t(x), \varphi), & \psi_X^t &:= \psi_{\Omega_t} \circ \Phi_t, \\ A(t) &:= \xi(t)(\partial\Phi_t)^{-1}(\partial\Phi_t)^{-T}, & \xi(t) &:= \det \partial\Phi_t, & y^t &:= y_t \circ \Phi_t \end{aligned} \tag{43}$$

and (for $t = 0$)

$$\psi(x) := \psi_\Omega(x), \quad w(x, \varphi) := w_X^0(x, \varphi).$$

From Lemma 2.9 we can directly infer the following convergences and estimates

Lemma 4.4. *Let $X \in C_c^1(D, \mathbb{R}^d)$ be given. Then it holds:*

(i) the convergences as $t \searrow 0$:

$$\frac{A(t) - I}{t} \to A'(0) = \operatorname{div}(X)I - \partial X - (\partial X)^T \qquad \textit{strongly in } C(\overline{D}, \mathbb{R}^{d,d}), \tag{44a}$$

$$\frac{\xi(t) - 1}{t} \to \xi'(0) = \operatorname{div}(X) \qquad \textit{strongly in } C(\overline{D}); \tag{44b}$$

(ii) there is a constant $t^* > 0$ such that

$$\begin{aligned} &\forall t \in [0, t^*], \forall x \in \overline{D}, \forall \zeta \in \mathbb{R}^d, \quad A(t, x)\zeta \cdot \zeta \geq 1/2|\zeta|^2, \\ &\forall t \in [0, t^*], \forall x \in \overline{D}, \quad \xi(t, x) \geq 1/2. \end{aligned}$$

Performing a change of variables and using $(\nabla y) \circ \Phi_t = (\partial\Phi_t)^{-T}\nabla(y \circ \Phi_t)$ it is easy to check that the transported function y^t (which is defined on Ω) satisfies the relation

$$\begin{cases} y^t \in K(\Omega) \text{ and } \forall \varphi \in K(\Omega): \\ \displaystyle\int_\Omega A(t)\nabla y^t \cdot \nabla(\varphi - y^t) + \xi(t)\lambda y^t(\varphi - y^t) + \xi(t) w_X^t(x, y^t + \psi_X^t)(\varphi - y^t)\,\mathrm{d}x \\ \qquad \geq \displaystyle\int_\Omega -A(t)\nabla\psi_X^t \cdot \nabla(\varphi - y^t) - \xi(t)\lambda\psi_X^t(\varphi - y^t)\,\mathrm{d}x. \end{cases} \tag{45}$$

For later usage let us introduce the bilinear form

$$\mathfrak{a}^t(v,w) := \int_\Omega A(t)\nabla v\cdot\nabla w + \xi(t)\lambda vw \,\mathrm{d}x,$$

the operator $\mathcal{A}_t : K_{\psi_\Omega}(\Omega) \to H^1(\Omega)^*$ by

$$\langle \mathcal{A}_t(v), w\rangle_{H^1(\Omega)} := \mathfrak{a}^t(v,w) + \int_\Omega \xi(t) w_X^t(x,v) w \,\mathrm{d}x \tag{46}$$

and the "shifted" operator $\tilde{\mathcal{A}}_t : K(\Omega) \to H^1(\Omega)^*$ by

$$\tilde{\mathcal{A}}_t(v) := \mathcal{A}_t(v+\psi_X^t). \tag{47}$$

By making use of this notation the variational inequality (45) can be recasted as

$$y^t \in K(\Omega) \quad \text{and} \quad \langle \tilde{\mathcal{A}}_t(y^t), \varphi - y^t\rangle_{H^1} \geq 0 \quad \text{for all } \varphi \in K(\Omega). \tag{48}$$

In the following we also write $\mathcal{A} := \mathcal{A}_0$ and $\tilde{\mathcal{A}} := \tilde{\mathcal{A}}_0$.

Sensitivity estimate

Our goal is to apply Theorem 3.3 designed for abstract operators. For this reason we make the following assumption in addition to (A1):

Assumption (A2)

(i) $\forall X \in C_c^1(D,\mathbb{R}^d), \exists c>0, \forall t\in[0,\tau], \forall \chi \in H^1(\Omega)$,

$$\|w_X^t(\cdot,\chi) - w(\cdot,\chi)\|_{L_{(2^*)'}(\Omega)} \leq ct;$$

(ii) $\forall X \in C_c^1(D,\mathbb{R}^d), \exists c>0, \forall t\in[0,\tau], \forall \chi_1,\chi_2 \in H^1(\Omega)$,

$$\|w_X^t(\cdot,\chi_1) - w_X^t(\cdot,\chi_2)\|_{L_{(2^*)'}(\Omega)} \leq c\|\chi_1-\chi_2\|_{H^1(\Omega)};$$

(iii) $\forall X \in C_c^1(D,\mathbb{R}^d), \exists c>0, \forall t\in[0,\tau]$,

$$\|\psi_X^t - \psi\|_{H^1(\Omega)} \leq ct.$$

We are now in the position to prove the following sensitivity result:

Proposition 4.5. *Let the Assumptions (A1)-(A2) be satisfied. Then the family of operators* $(\tilde{\mathcal{A}}_t)$ *defined by* (47) *fulfills*

(i) $\exists \alpha>0$, $\exists t^*>0$, $\forall t\in[0,t^*]$, $\forall v,w\in K(\Omega)$,

$$\alpha\|v-w\|_{H^1(\Omega)}^2 \leq \langle \tilde{\mathcal{A}}_t(v) - \tilde{\mathcal{A}}_t(w), v-w\rangle; \tag{49}$$

(ii) $\forall v \in K(\Omega)$, $\exists c>0$, $\exists t^*>0$, $\forall t\in[0,t^*]$, $\forall w\in K(\Omega)$,

$$|\langle \tilde{\mathcal{A}}_t(v) - \tilde{\mathcal{A}}(v), v-w\rangle| \leq ct\|v-w\|_{H^1(\Omega)}. \tag{50}$$

Proof. To (i): We first show the monotonicity estimate (49). With the help of Lemma 4.4 (ii) and monotonicity of w_X^t in the second variable (see Assumption (A1) (i)) we obtain for all $v, w \in H^1(\Omega)$ and all small $t \geq 0$

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla(v-w)|^2 + \lambda|v-w|^2 \,\mathrm{d}x \\
&\quad \leq \mathfrak{a}^t(v-w, v-w) \qquad (51)\\
&\qquad + \int_\Omega \xi(t)\big(w_X^t(x, v+\psi_X^t) - w_X^t(x, w+\psi_X^t)\big)\big((v+\psi_X^t) - (w+\psi_X^t)\big)\,\mathrm{d}x
\end{aligned}$$

Thus (49) is shown.

To (ii): Let us fix $v \in H^1(\Omega)$. Then by applying Hölder inequality, Sobolev embeddings and the assumptions in (A2) we find for all $w \in H^1(\Omega)$

$$\begin{aligned}
&\langle \tilde{\mathcal{A}}_t(v) - \tilde{\mathcal{A}}(v), v-w\rangle \\
&\leq \underbrace{\int_\Omega (A(t)-I)\nabla v\cdot\nabla(v-w)\,\mathrm{d}x}_{\leq \|A(t)-I\|_{L_\infty}\|\nabla v\|_{L_2}\|\nabla(v-w)\|_{L_2}} + \underbrace{\int_\Omega \lambda(\xi(t)-1)v(v-w)\,\mathrm{d}x}_{\leq \lambda\|\xi(t)-1\|_{L_\infty}\|v\|_{L_2}\|v-w\|_{L_2}} \\
&\quad + \underbrace{\int_\Omega (A(t)\nabla\psi_X^t - \nabla\psi)\cdot\nabla(v-w) + \lambda(\xi(t)\psi_X^t - \psi)(v-w)\,\mathrm{d}x}_{\leq \left(\|A(t)-I\|_{L_\infty}\|\nabla\psi_X^t\|_{L_2} + \|\nabla\psi_X^t - \nabla\psi\|_{L_2} + \lambda\|\xi(t)-1\|_{L^\infty}\|\psi_X^t\|_{L^2} + \lambda\|\psi_X^t-\psi\|_{L_2}\right)\|v-w\|_{H^1}} \\
&\quad + \underbrace{\int_\Omega (\xi(t)-1)w_X^t(x, v+\psi_X^t)(v-w)\,dx}_{\leq \|\xi(t)-1\|_{L^\infty}\|w_X^t(x,v+\psi_X^t)\|_{L_{(2^*)'}}\|v-w\|_{H^1}} \\
&\quad + \underbrace{\int_\Omega (w_X^t(x, v+\psi_X^t) - w_X^t(x, v+\psi))(v-w)\,\mathrm{d}x.}_{\leq \|w_X^t(x,v+\psi_X^t) - w^t(x,v+\psi)\|_{L_{(2^*)'}}\|v-w\|_{H^1} \leq \|\psi_X^t - \psi\|_{H^1}\|v-w\|_{H^1}} \\
&\quad + \underbrace{\int_\Omega (w_X^t(x, v+\psi) - w(x, v+\psi))(v-w)\,\mathrm{d}x.}_{\leq \|w_X^t(x,v+\psi) - w(x,v+\psi)\|_{L_{(2^*)'}}\|v-w\|_{H^1} \leq ct\|v-w\|_{H^1}}
\end{aligned}$$

Taking Lemma 4.4 into account and using Young's inequality, we obtain the assertion. ☐

The desired Lipschitz estimate immediately follows from Theorem 3.3 since Proposition 4.5 proves that Assumption (O1) are satisfied for $p = 2$.

Corollary 4.6. *Under the assumption of Proposition 4.5 there exist $t^* > 0$ and $c > 0$ such that*

$$\|y^t - y\|_{H^1(\Omega)} \leq ct \quad \textit{for all } t \in [0, t^*].$$

4.3 Limiting system for the transformed material derivative

In Corollary 4.6 we have established a Lipschitz estimate for the mapping $t \mapsto y^t$. In this section we are going to prove that there is a unique element $\dot{y}$ in $H^1(\Omega)$ – called the material derivative – such that $(y^t - y)/t$ converges strongly to $\dot{y}$ in $H^1(\Omega)$ which is uniquely determined by a variational inequality.

In order to derive the differentiability of y^t we impose the additional assumptions to (A1) and (A2):

Assumption (A3)

(i) $w(x,\cdot)$ *is of class* $C^1(\mathbb{R})$ *for all* $x \in \Omega$*;*

(ii) for all $X \in C_c^1(D,\mathbb{R}^d)$*, there exists a function* $\dot{w}_X : \Omega \times \mathbb{R} \to \mathbb{R}$ *such that for all* $\varphi_n \to \varphi$ *strongly in* $H^1(\Omega)$ *we have* $\dot{w}_X(\cdot,\varphi) \in L_{(2^*)'}(\Omega)$ *and for all* $t_n \searrow 0$

$$\frac{w_X^{t_n}(\cdot,\varphi_n) - w(\cdot,\varphi_n)}{t_n} \to \dot{w}_X(\cdot,\varphi) \qquad \textit{strongly in } L_{(2^*)'}(\Omega) \textit{ as } n \to \infty;$$

(iii) for any given sequences $\varphi_n \to \varphi$ *in* $H^1(\Omega)$ *and* $t_n \searrow 0$ *with* $(\varphi_n - \varphi)/t_n \rightharpoonup z$ *weakly in* $H^1(\Omega)$*:*

$$\frac{w(\cdot,\varphi_n) - w(\cdot,\varphi)}{t_n} \to \partial_y w(\cdot,\varphi) z \qquad \textit{strongly in } L_{(2^*)'}(\Omega) \textit{ as } n \to \infty;$$

(iv) for all $X \in C_c^1(D,\mathbb{R}^d)$ *there exists a function* $\dot{\psi}_X \in H^1(\Omega)$ *such that*

$$\frac{\psi_X^t - \psi}{t} \to \dot{\psi}_X \qquad \textit{strongly in } H^1(\Omega) \textit{ as } t \searrow 0.$$

Remark 4.7. *(i) Property (iii) from Assumption (A3) is satisfied if, e.g., there exist a constant* $C > 0$ *and a function* $s \in L_{\frac{2^*-1}{2^*-2}}(\Omega)$ *such that for all* $x \in \Omega$ *and* $y \in \mathbb{R}$*:*

$$|\partial_y w(x,y)| \le C|y|^\alpha + s(x)$$

with the exponent $\alpha := \frac{2^*(2^*-1)}{2^*-2}$*. The constant* α *is chosen such that the function* $x \mapsto \partial_y w(x,\varphi(x))z(x)$ *and* $x \mapsto f'(\varphi(x))z(x)$ *are in* $L_{(2^*)'}(\Omega)$ *for given* $\varphi, z \in H^1(\Omega)$*.*

(ii) A useful consequence of properties (ii) and (iii) is the following continuity

$$w_X^{t_n}(\cdot,\varphi_n) \to w(\cdot,\varphi) \qquad \textit{strongly in } L_{(2^*)'}(\Omega) \textit{ as } n \to \infty.$$

for all $\varphi_n \to \varphi$ *strongly in* $H^1(\Omega)$ *and* $t_n \searrow 0$*.*

(iii) Let $X \in C_c^1(D,\mathbb{R}^d)$ *be given. Then we have by using property (iv) from Assumption (A3)*

$$\begin{aligned} \frac{-A(t)\nabla\psi_X^t + \nabla\psi}{t} &\to -A'(0)\nabla\psi - \nabla\dot{\psi}_X \qquad &&\textit{strongly in } L_2(\Omega,\mathbb{R}^d), \\ \frac{-\xi(t)\psi_X^t + \psi}{t} &\to -\xi'(0)\psi - \dot{\psi}_X \qquad &&\textit{strongly in } L_2(\Omega,\mathbb{R}^d). \end{aligned}$$

We are now well-prepared for the derivation of the material derivative.

Theorem 4.8. *Let (A1)-(A3) be satisfied. The weak material derivative* $\dot{y}$ *of* $t \mapsto y^t$ *exists in all directions* $X \in C_c^1(D,\mathbb{R}^d)$ *and is characterised as the unique solution of the following variational inequality*

$$\begin{cases} \dot{y} \in \tilde{S}_y(K) \text{ and } \forall \varphi \in \tilde{S}_y(K) : \\ \langle \partial\tilde{\mathcal{A}}(y)\dot{y}, \varphi - \dot{y}\rangle_{H^1} \ge -\langle \tilde{\mathcal{A}}'(y), \varphi - \dot{y}\rangle_{H^1}, \end{cases} \tag{52}$$

where $\tilde{S}_y(K)$ denotes the closed and convex cone

$$\tilde{S}_y(K) = T_y(K) \cap kern(\tilde{\mathcal{A}}(y)). \tag{53}$$

The functional derivatives $\partial\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ are given by

$$\langle \partial\tilde{\mathcal{A}}(y)\dot{y}, \varphi\rangle = \mathfrak{a}(\dot{y} + \dot{\psi}_X, \varphi) + \int_\Omega \partial_y w(x, y + \psi)\dot{y}\varphi \,\mathrm{d}x, \tag{54}$$

$$\begin{aligned} \langle \tilde{\mathcal{A}}'(y), \varphi\rangle = &\int_\Omega A'(0)\nabla y \cdot \nabla\varphi + \xi'(0)\big(\lambda y + w(x, y + \psi)\big)\varphi \,\mathrm{d}x \\ &+ \int_\Omega \dot{w}_X(x, y + \psi)\varphi + \partial_y w(x, y + \psi)\dot{\psi}_X\varphi \,\mathrm{d}x \\ &+ \int_\Omega A'(0)\nabla\psi \cdot \nabla\varphi + \xi'(0)\lambda\psi\varphi \,\mathrm{d}x. \end{aligned} \tag{55}$$

Proof.
Existence of $\dot{y}$: We want to apply Theorem 3.5. For this we need to check Assumption (O2). To this end we notice that by Corollary 4.6 $y_{t_n} \to u$ strongly and $(y_{t_n} - y)/t_n \rightharpoonup z$ weakly in $H^1(\Omega)$ for a suitable subsequence $t_n \searrow 0$.

- We check (O2) (ii): Let $v_n \rightharpoonup v$ be a given weakly convergence sequence in $H^1(\Omega)$. Then

$$\begin{aligned}
&\left\langle \frac{\tilde{\mathcal{A}}_{t_n}(y^{t_n}) - \tilde{\mathcal{A}}(y^{t_n})}{t_n}, v_n \right\rangle \\
&= \underbrace{\int_\Omega \frac{A(t_n) - I}{t_n}\nabla y^{t_n} \cdot \nabla v_n \,\mathrm{d}x}_{\to \int_\Omega A'(0)\nabla y\cdot\nabla v \,\mathrm{d}x} + \underbrace{\int_\Omega \frac{\xi(t_n) - 1}{t_n}\Big(\lambda y^{t_n} + w_X^{t_n}(x, y^{t_n} + \psi_X^{t_n})\Big)v_n \,\mathrm{d}x}_{\to \int_\Omega \xi'(0)(\lambda y + w(x,y+\psi))v \,\mathrm{d}x \quad \text{by Remark 4.7 (ii)}} \\
&\quad + \underbrace{\int_\Omega \frac{w_X^{t_n}(x, y^{t_n} + \psi_X^{t_n}) - w(x, y^{t_n} + \psi_X^{t_n})}{t_n} v_n \,\mathrm{d}x}_{\to \int_\Omega \dot{w}_X(x,y+\psi)v \,\mathrm{d}x \quad \text{by Assumption (A3) (ii) and (iv)}} \\
&\quad + \underbrace{\int_\Omega \frac{w(x, y^{t_n} + \psi_X^{t_n}) - w(x, y + \psi)}{t_n} v_n \,\mathrm{d}x - \int_\Omega \frac{w(x, y^{t_n} + \psi) - w(x, y + \psi)}{t_n} v_n \,\mathrm{d}x}_{\to \int_\Omega \partial_y w(x,y+\psi)(z+\dot{\psi}_X)v \,\mathrm{d}x - \int_\Omega \partial_y w(x,y+\psi)zv \,\mathrm{d}x = \int_\Omega \partial_y w(x,y+\psi)\dot{\psi}_X v \,\mathrm{d}x \text{ by (A3) (iii)-(iv)}} \\
&\quad + \underbrace{\int_\Omega \frac{A(t_n) - I}{t_n}\nabla\psi_X^{t_n} \cdot \nabla v_n + \frac{\xi(t_n) - 1}{t_n}\psi_X^{t_n} v_n \,\mathrm{d}x}_{\to - \int_\Omega A'(0)\nabla\psi\cdot\nabla v + \xi'(0)\psi v \,\mathrm{d}x}.
\end{aligned}$$

- We check (O2) (iii):

$$\begin{aligned}
&\left\langle \frac{\tilde{\mathcal{A}}(y^{t_n}) - \tilde{\mathcal{A}}(y)}{t_n}, \frac{y^{t_n} - y}{t_n} \right\rangle \\
&= \underbrace{\int_\Omega \left|\nabla\frac{y^{t_n} - y}{t_n}\right|^2 + \lambda\left|\frac{y^{t_n} - y}{t_n}\right|^2 \mathrm{d}x}_{\liminf \geq \int_\Omega |\nabla z|^2 + \lambda|z|^2 \,\mathrm{d}x} + \underbrace{\int_\Omega \frac{w(x, y^{t_n} + \psi) - w(x, y + \psi)}{t_n}\frac{y^{t_n} - y}{t_n} \,\mathrm{d}x}_{\to \int_\Omega \partial_y w(x,y+\psi)|z|^2 \quad \text{by Assumption (A3) (iii)}}
\end{aligned}$$

and for all $\varphi_n \to \varphi$ strongly in $H^1(\Omega)$:

$$\left\langle \frac{\tilde{\mathcal{A}}(y^{t_n}) - \tilde{\mathcal{A}}(y)}{t_n}, \varphi_n \right\rangle$$
$$= \underbrace{\int_\Omega \nabla \frac{y^{t_n} - y}{t_n} \cdot \nabla \varphi_n + \lambda \frac{y^{t_n} - y}{t_n} \varphi_n \, \mathrm{d}x}_{\to \int_\Omega \nabla z \cdot \nabla \varphi + \lambda z \varphi \, \mathrm{d}x} + \underbrace{\int_\Omega \frac{w(x, y^{t_n} + \psi) - w(x, y + \psi)}{t_n} \varphi_n \, \mathrm{d}x}_{\to \int_\Omega \partial_y w(x, y+\psi) z \varphi \quad \text{by Assumption (A3) (iii)}} .$$

■ Property (O2) (i) follows from the above calculations.

Uniqueness of $\dot{y}$: Assume two solutions $\dot{y}$ and $\dot{z}$ for (52). Testing their variational inequalities with $\dot{z}$ and $\dot{y}$, respectively, and adding the result yields

$$\langle \partial \tilde{\mathcal{A}}(y)\dot{y} - \partial \tilde{\mathcal{A}}(y)\dot{z}, \dot{y} - \dot{z} \rangle \leq 0.$$

The left-hand side calculates as

$$\langle \partial \tilde{\mathcal{A}}(y)\dot{y} - \partial \tilde{\mathcal{A}}(y)\dot{z}, \dot{y} - \dot{z} \rangle$$
$$= \mathfrak{a}(\dot{y} - \dot{z}, \dot{y} - \dot{z}) + \int_\Omega \partial_y w(x, y + \psi) |\dot{y} - \dot{z}|^2 \, \mathrm{d}x.$$

Due to the convexity assumption in (A1) (i) we find $\partial_y w \geq 0$ and see that

$$\mathfrak{a}(\dot{y} - \dot{z}, \dot{y} - \dot{z}) \leq 0.$$

We obtain $\dot{y} - \dot{z} = 0$. □

By exploiting the specific structure of $\tilde{\mathcal{A}}_t$ and Assumption (A3) we can even show that the strong material derivative exists.

Corollary 4.9. *We have for all $X \in C^1_c(D, \mathbb{R}^d)$*

$$\frac{y^t_X - y}{t} \to \dot{y}_X \qquad \textit{strongly in } H^1(\Omega). \tag{56}$$

Proof. We test the variational inequality (48) with $\varphi = y^t$ and for $t = 0$ with $\varphi = y$. Adding both inequalities yields

$$\langle \tilde{\mathcal{A}}_t(y^t) - \tilde{\mathcal{A}}(y), y^t - y \rangle \leq 0.$$

Dividing by t^2 and rearranging the terms we obtain by setting $z^t := (y^t - y)/t$

$$\begin{aligned}
&\mathfrak{a}(z^t, z^t) \\
&\leq -\int_\Omega \frac{A(t) - I}{t} \nabla y^t \cdot \nabla z^t \, \mathrm{d}x - \int_\Omega \lambda \frac{\xi(t) - 1}{t} y^t z^t \, \mathrm{d}x \\
&\quad - \int_\Omega \left(\frac{\xi(t) - 1}{t} w^t_X(x, y^t + \psi^t_X) + \frac{w^t_X(x, y^t + \psi^t_X) - w(x, y^t + \psi^t_X)}{t} \right) z^t \, \mathrm{d}x \\
&\quad - \int_\Omega \frac{w(x, y^t + \psi^t_X) - w(x, y + \psi)}{t} z^t \, \mathrm{d}x \\
&\quad - \int_\Omega \frac{A(t)\nabla \psi^t_X - \nabla \psi}{t} \cdot \nabla z^t \, \mathrm{d}x - \int_\Omega \lambda \frac{\xi(t)\psi^t_X - \psi}{t} z^t \, \mathrm{d}x \\
&\quad =: \mathfrak{B}(t).
\end{aligned} \tag{57}$$

The known convergence properties shows as $t \searrow 0$ for a subsequence

$$\mathfrak{B}(t) \to \underbrace{-\langle \tilde{\mathcal{A}}'(y), \dot{y}\rangle - \int_\Omega \partial_y w(x, y+\psi)|\dot{y}|^2 \,\mathrm{d}x - \int_\Omega \nabla\dot{\psi}_X \cdot \nabla\dot{y}\,\mathrm{d}x - \int_\Omega \lambda\dot{\psi}_X\dot{y}\,\mathrm{d}x}_{=:\mathfrak{B}(0)}\,.$$

However testing (52) with $\varphi = 2\dot{y} \in \tilde{S}_y(K)$ we also obtain $\langle \partial\tilde{\mathcal{A}}(y)\dot{y}, \dot{y}\rangle_{H^1} \geq -\langle \tilde{\mathcal{A}}'(y), \dot{y}\rangle_{H^1}$ which is precisely

$$\mathfrak{a}(\dot{y}, \dot{y}) \geq \mathfrak{B}(0).$$

All in all we get

$$\limsup_{t\searrow 0} \mathfrak{a}(z^t, z^t) \leq \limsup_{t\searrow 0} \mathfrak{B}(t) = \mathfrak{B}(0) \leq \mathfrak{a}(\dot{y}, \dot{y}). \tag{58}$$

The weak convergence $z^t \rightharpoonup \dot{y}$ in $H^1(\Omega)$ implies $\liminf_{t\searrow 0} \mathfrak{a}(z^t, z^t) \geq \mathfrak{a}(\dot{y}, \dot{y})$. Together with (58) this gives $\mathfrak{a}(z^t, z^t) \to \mathfrak{a}(\dot{y}, \dot{y})$ as $t \searrow 0$. This finishes the proof. □

Remark 4.10. *If we assume that*

$$\dot{w}_X(x, y) := \mathbf{T}_0(x, y) \cdot X(x) + \mathbf{T}_1(x, y) : \partial X(x) \tag{59}$$

for functions $\mathbf{T}_0(\cdot,\cdot) : \Omega \times \mathbb{R} \to \mathbb{R}^d$ *and* $\mathbf{T}_1(\cdot,\cdot) : \Omega \times \mathbb{R} \to \mathbb{R}^{d\times d}$ *we may rewrite the variational inequality in* (52) *by using Lemma 4.4 as*

$$\begin{aligned} &\mathfrak{a}(\dot{y}, \varphi - \dot{y}) + \int_\Omega \partial_y w(x, y+\psi)\dot{y}(\varphi - \dot{y})\,\mathrm{d}x \\ &\quad\geq \int_\Omega \mathfrak{L}_1(x, y+\psi; \varphi - \dot{y}) : \partial X + \mathfrak{L}_0(x, y+\psi; \varphi - \dot{y}) \cdot X\,\mathrm{d}x \\ &\qquad - \mathfrak{a}(\dot{\psi}, \varphi - \dot{y}) + \int_\Omega \partial_y w(x, y+\psi)\dot{\psi}(\varphi - \dot{y})\,\mathrm{d}x, \end{aligned}$$

where we use the abbreviations

$$\begin{aligned} \mathfrak{L}_1(x, y+\psi; \varphi) :=& -\Big(\nabla(y+\psi)\cdot\nabla\varphi + \Big(\lambda(y+\psi) + w(x, y+\psi)\Big)\varphi\Big)\, I \\ &+ \nabla\varphi \otimes \nabla(y+\psi) + \nabla(\psi + y) \otimes \nabla\varphi - \mathbf{T}_1(x, y+\psi)\varphi, \\ \mathfrak{L}_0(x, y+\psi; \varphi) :=& -\mathbf{T}_0(x, y+\psi)\varphi. \end{aligned}$$

4.4 Limiting system for the material derivative

So far we have derived an equation for $\dot{y}$. Since we are interested in the original problem (40), we may now use Theorem 4.8 and the transformation $y = u - \psi$ to obtain the material derivative equation for (40). It is clear that $\dot{y} = \dot{u} - \dot{\psi}_X$ and we conclude with the following result:

Corollary 4.11. *Under the assumptions (A1)-(A3) the material deriative* $\dot{u} = \dot{u}(X)$ *of solutions of the perturbed problem to* (40) *in direction* $X \in C_c^1(D, \mathbb{R}^d)$ *exists and is given as the solution of the* *following variational inequality:*

$$\begin{cases} \dot{u} \in S_u^X(K_\psi) \ \textit{and} \ \forall \varphi \in S_u^X(K_\psi): \\ \mathfrak{a}(\dot{u}, \varphi - \dot{u}) + \int_\Omega \partial_y w(x,u)\dot{u}(\varphi - \dot{u}) \,\mathrm{d}x \\ \quad \geq - \int_\Omega A'(0)\nabla u \cdot \nabla(\varphi - \dot{u}) + \xi'(0)\big(\lambda u + w(x,u)\big)(\varphi - \dot{u}) \,\mathrm{d}x \\ \qquad - \int_\Omega \dot{w}_X(x,u)(\varphi - \dot{u}) \,\mathrm{d}x, \end{cases} \tag{60}$$

where

$$S_u^X(K_\psi) := T_u(K_\psi) \cap kern(\mathcal{A}(u)) + \dot{\psi}_X.$$

In particular under the additional assumption in Remark 4.10

$$\mathfrak{a}(\dot{u}, \varphi - \dot{u}) + \int_\Omega \partial_y w(x,u)\dot{u}(\varphi - \dot{u}) \,\mathrm{d}x$$
$$\geq \int_\Omega \mathfrak{L}_1(x,u;\varphi - \dot{u}) : \partial X + \mathfrak{L}_0(x,u;\varphi - \dot{u}) \cdot X \,\mathrm{d}x.$$

Proof. We obtain from Theorem 4.8 that $\dot{u} \in \tilde{S}_y(K) + \dot{\psi}_X$ and for all $\varphi \in \tilde{S}_y(K) + \dot{\psi}_X$:

$$\langle \partial\tilde{\mathcal{A}}(u-\psi)(\dot{u} - \dot{\psi}_X), \varphi - \dot{u}\rangle_{H^1} \geq -\langle \tilde{\mathcal{A}}'(u-\psi), \varphi - \dot{u}\rangle_{H^1},$$

which is precisely the inequality in (60).

It remains to show $S_u^X(K_\psi) = \tilde{S}_y(K) + \dot{\psi}_X$ which is equivalent to $T_u(K_\psi) \cap kern(\mathcal{A}(u)) = T_y(K) \cap kern(\tilde{\mathcal{A}}(y))$. Indeed, by definition (47) we find

$$kern(\mathcal{A}(u)) = kern(\tilde{\mathcal{A}}(y))$$

as well as by (1)-(3)

$$T_u(K_\psi) = T_{u-\psi}(K) = T_y(K)$$

□

Note that we get the following characterisation of S_u^X by using Lemma 2.5 and the definition in (53):

$$\begin{aligned} \varphi \in S_u^X(K_\psi) \quad &\Leftrightarrow \quad \varphi - \dot{\psi}_X \in T_u(K_\psi) \cap kern(\mathcal{A}(u)) \\ &\Leftrightarrow \quad \begin{cases} \varphi \in H^1(\Omega) \text{ with } \varphi \leq \dot{\psi}_X \text{ q.e. on } \{u = \psi_\Omega\}, \\ \langle \mathcal{A}(u), \varphi - \dot{\psi}_X\rangle = 0. \end{cases} \end{aligned}$$

Moreover under an additional assumptions we obtain the subsequent translation property:

Lemma 4.12. *Suppose that $u, \psi \in H^2(\Omega)$ and let $\zeta \in H^1(\Omega)$ be with*

$$\tilde{\zeta} = 0 \text{ q.e. on the coincidence set } \{x \in \overline{\Omega} : \tilde{u}(x) = \tilde{\psi}(x)\},$$

where $\tilde{\zeta}$, $\tilde{u}$ and $\tilde{\psi}$ denote quasi-continuous representatives for ζ, u and ψ. Then we have

$$\pm\zeta \in T_u(K_\psi) \cap kern(\mathcal{A}(u)).$$

In particular

$$\zeta + S_u^X(K_\psi) = \mathcal{S}_u^X(K_\psi). \tag{61}$$

Proof. It is clear from the assumption that $\pm\tilde{\zeta} = 0$ q.e. on the coincidence set $\{u = \psi\}$. Thus $\pm\zeta \in T_u(K_\psi)$. Furthermore $y = u - \psi$ satisfies the variational inequality (see (48) with $t = 0$)

$$\langle \tilde{\mathcal{A}}(y), \varphi - y\rangle \geq 0 \quad \text{for all } \varphi \in H^1(\Omega) \text{ and } \varphi \leq 0 \text{ a.e. in } \Omega.$$

From the $H^2(\Omega)$-regularity of u and ψ we deduce that (in a pointwise formulation) $\tilde{\mathcal{A}}(y) = 0$ a.e. in $\{x \in \Omega :\ u(x) < \psi(x)\}$. In particular we see that

$$\langle \tilde{\mathcal{A}}(y), \varphi\rangle = 0 \text{ for all } \varphi \in H^1(\Omega) \text{ with } \{x \in \Omega :\ \varphi(x) = 0\} \supseteq \{x \in \Omega :\ u(x) = \psi(x)\} \text{ a.e.}$$

Testing with $\varphi = \pm\zeta$ yields $\pm\zeta \in kern(\tilde{\mathcal{A}}(y)) = kern(\mathcal{A}(u))$.

Finally, $\zeta \in T_u(K_\psi) \cap kern(\mathcal{A}(u))$ implies $\zeta + S_u^X(K_\psi) \subseteq S_u^X(K_\psi)$, and $-\zeta \in T_u(K_\psi) \cap kern(\mathcal{A}(u))$ implies $\zeta + S_u^X(K_\psi) \supseteq S_u^X(K_\psi)$. □

In the following ψ_Ω is referred to as a static obstacle if there exists a fixed function $\psi \in H^2(D)$ such that $\psi_{\tilde{\Omega}} = \psi|_{\tilde{\Omega}}$ for all Lipschitz domains $\tilde{\Omega} \subseteq D$.

Remark 4.13. *Let $X \in C_c^1(D, \mathbb{R}^d)$. Suppose that ψ_Ω is a static obstacle, $u \in H^2(\Omega)$ and $\{X = \mathbf{0}\} \supseteq \{\tilde{u} = \tilde{\psi}_\Omega\}$ q.e. in $\overline{\Omega}$. Then $\dot{\psi}_X = \nabla\psi_\Omega \cdot X$ and the assumptions from Lemma 4.12 are satisfied for $\zeta = \dot{\psi}_X$ and we obtain*

$$\pm\dot{\psi}_X \in T_u(K_\psi) \cap kern(\mathcal{A}(u)).$$

In particular

$$S_u^X(K_\psi) = T_u(K_\psi) \cap kern(\mathcal{A}(u))$$

and

$$\varphi \in S_u^X(K_\psi) \quad \Leftrightarrow \quad \begin{cases} \varphi \in H^1(\Omega) \text{ with } \varphi \leq 0 \text{ q.e. on } \{u = \psi_\Omega\}, \\ \langle \mathcal{A}(u), \varphi\rangle = 0. \end{cases}$$

4.5 Limiting system for the state-shape derivative

The *state shape derivative* of u at Ω in direction $X \in C_c^1(D, \mathbb{R}^d)$ is defined by

$$u' = u'(X) := \dot{u} - \partial_X u \quad \text{on } \Omega \tag{62}$$

where u solves (40), $\dot{u}$ solves (60) and $\partial_X u := \nabla u \cdot X$. It is clear that $u' \in L_2(\Omega)$. Thus in general the state shape derivative is less regular than the material derivative. Another important observation is that the boundary conditions imposed on $\dot{u}$ on $\partial\Omega$ are not carried over to u'.

Lemma 4.14. *Let $X \in C_c^1(D, \mathbb{R}^d)$ be a vector field satisfying $X \cdot n = 0$ on $\partial\Omega$. Then the state shape derivative vanishes identically, that is, $u'(X) = 0$ a.e. on Ω.*

Proof. The X-flow Φ_t leaves the domain Ω unchanged, i.e., $\Phi_t(\Omega) = \Omega$ for all $t \in [0, \tau]$. Consequently, $u_t = u(\Omega_t) = u(\Omega) = u$ and thus $u^t = u_t \circ \Phi_t = u \circ \Phi_t$ for all $t \in [0, \tau]$. Hence by Lemma 2.9 (ii) we may calculate the material derivative $\dot{u}$ as

$$\frac{u^t - u}{t} = \frac{u \circ \Phi_t - u}{t} \to \partial_X u \quad \text{strongly in } L_2(\Omega).$$

Thus $\dot{u} = \partial_X u$ and consequently $u' = 0$. □

Now we are prepared to prove the main result of this section which gives a simplified variational inequality for the state-shape derivative u' under certain conditions. To derive this result we will assume the enhanced regularity $u \in H^2(\Omega)$. Preliminarily we observe from Corollary 4.11 and by using the relation (62) that

$$\begin{cases} u' \in \hat{S}_u^X(K_\psi) \text{ and } \forall \varphi \in \hat{S}_u^X(K_\psi) : \\ \mathfrak{a}(u', \varphi - u') + \int_\Omega \partial_y w(x,u) u'(\varphi - u') \, \mathrm{d}x \\ \quad \geq - \int_\Omega A'(0) \nabla u \cdot \nabla(\varphi - u') + \xi'(0)\big(\lambda u + w(x,u)\big)(\varphi - u') \, \mathrm{d}x \\ \quad\quad - \mathfrak{a}(\partial_X u, \varphi - u') - \int_\Omega \partial_y w(x,u) \partial_X u (\varphi - u') \, \mathrm{d}x, \end{cases} \tag{63}$$

where

$$\hat{S}_u^X(K_\psi) := T_u(K_\psi) \cap kern(\mathcal{A}(u)) + \dot{\psi}_X - \partial_X u.$$

We notice that in general the cone $\widehat{\mathcal{S}}_X(K)$ depend on the vector field X. In the case of a static obstacle problem (see Remark 4.1 (i)) we derive the following result:

Theorem 4.15. *Suppose that (A1)-(A3),* (59) *and* $u \in H^2(\Omega)$ *hold. Furthermore let* ψ_Ω *be a static obstacle function.*

Then $\hat{S}_u^X$ *is independent of* $X \in C_c^1(D, \mathbb{R}^d)$ *with*

$$\hat{S}_u^X(K_\psi) = T_u(K_\psi) \cap kern(\mathcal{A}(u)) =: S_u(K_\psi) \tag{64}$$

and the state shape derivative is the unique solution of

$$\begin{cases} u' \in S_u(K_\psi) \textit{ and } \forall \varphi \in S_u(K_\psi) : \\ \mathfrak{a}(u', \varphi - u') + \int_\Omega \partial_y w(x,u) u'(\varphi - u') \, \mathrm{d}x \\ \quad \geq \int_\Gamma \mathfrak{S}_1(x, u; \varphi - u') n \cdot n (X \cdot n) \, \mathrm{d}s, \end{cases} \tag{65}$$

where

$$\mathfrak{S}_1(x, u; \varphi) := \mathfrak{L}_1(x, u; \varphi) - \nabla u \otimes \nabla \varphi.$$

with $\mathfrak{L}_1$ *from Remark 4.10.*

Proof. By using the assumption $\dot{\psi}_X = \nabla \psi_\Omega \cdot X$ we find on the coincidence set $\{u = \psi_\Omega\}$ (here we resort to quasi-continuous representants):

$$\dot{\psi}_X - \partial_X u = \dot{\psi}_X - \partial_X \psi_\Omega = 0.$$

Lemma 4.12 applied to $\zeta = \dot{\psi}_X - \partial_X u$ yields $\pm(\dot{\psi}_X - \partial_X u) \in S_u(K_\psi)$ and therefore (64).

Furthermore by using the notation in Remark 4.10 and the identity (note that $u \in H^2(\Omega)$ by assumption)

$$\nabla(\partial_X u) = (\partial X)^T (\nabla u) + (\partial^2 u) X,$$

the variational inequality in (63) rewrites to $u' \in S_u(K_\psi)$ and for all $\varphi \in S_u(K_\psi)$:

$$\begin{aligned} \mathfrak{a}(u', \varphi - u') + &\int_\Omega \partial_y w(x,u) u'(\varphi - u') \,\mathrm{d}x \\ &\geq \int_\Omega \mathfrak{S}_1(x,u;\varphi - u') : \partial X + \mathfrak{S}_0(x,u;\varphi - u') \cdot X \,\mathrm{d}x, \end{aligned} \tag{66}$$

where

$$\begin{aligned} \mathfrak{S}_0(x,u,\varphi) &:= \mathfrak{L}_0(x,u,\varphi) - \partial_y w(x,u)\varphi \nabla u - (\partial^2 u)\nabla\varphi, \\ \mathfrak{S}_1(x,u,\varphi) &:= \mathfrak{L}_1(x,u,\varphi) - \nabla u \otimes \nabla \varphi. \end{aligned}$$

Picking any vector field $X \in C_c^1(D, \mathbb{R}^d)$ with $X \cdot n = 0$ on Γ we know from Lemma 4.14 that $u'(\pm X) = 0$ and it follows from (66)

$$\int_\Omega \mathfrak{S}_1(x,u;\tilde{\varphi}) : \partial X + \mathfrak{S}_0(x,u;\tilde{\varphi}) \cdot X \,\mathrm{d}x = 0 \tag{67}$$

for all $\tilde{\varphi} \in S_u(K_\psi)$. Then integrating by parts in (67) shows the pointwise identity

$$-\mathrm{div}(\mathfrak{S}_1(x,u(x);\tilde{\varphi}(x))) + \mathfrak{S}_0(x,u(x);\tilde{\varphi}(x)) = 0 \quad \text{ a.e. on } \Omega. \tag{68}$$

Now for an arbitrary $X \in C_c^1(D, \mathbb{R}^d)$ and $\tilde{\varphi} \in S_u(K_\psi)$ we consider the additive splitting $X = X_n + X_T$ for $X_n, X_T \in C_c^1(D, \mathbb{R}^d)$ such that $X_n = n(X \cdot n)$ and $X_T = X - n(X \cdot n)$ on Γ. Then $X_T \cdot n = 0$ on Γ and we get

$$\begin{aligned} &\int_\Omega \mathfrak{S}_1(x,u;\tilde{\varphi}) : \partial X + \mathfrak{S}_0(x,u;\tilde{\varphi}) \cdot X \,\mathrm{d}x \\ &= \underbrace{\int_\Omega \mathfrak{S}_1(x,u;\tilde{\varphi}) : \partial X_T + \mathfrak{S}_0(x,u;\tilde{\varphi}) \cdot X_T \,\mathrm{d}x}_{=0 \text{ by (67)}} \\ &\quad \underbrace{+ \int_\Omega \mathfrak{S}_1(x,u;\tilde{\varphi}) : \partial X_n + \mathfrak{S}_0(x,u;\tilde{\varphi}) \cdot X_n \,\mathrm{d}x}_{\text{partial integration and (68)}} \\ &= \int_\Gamma \mathfrak{S}_1(x,u;\tilde{\varphi}) n \cdot X_n \,\mathrm{d}s. \end{aligned} \tag{69}$$

We may test (69) with $\tilde{\varphi} = u'$ since $u' \in S_u(K_\psi)$. Then multiplying the resulting identity with -1 and exploiting linearity of $\mathfrak{S}_0$ and $\mathfrak{S}_1$ with respect to φ yields

$$\int_\Omega \mathfrak{S}_1(x,u;-u') : \partial X + \mathfrak{S}_0(x,u;-u') \cdot X \,\mathrm{d}x = \int_\Gamma \mathfrak{S}_1(x,u;-u') n \cdot X_n \,\mathrm{d}s. \tag{70}$$

Now we find by letting $\tilde{\varphi} = \varphi \in S_u(K_\psi)$ be arbitrary, adding (69) and (70), and again exploiting linearity

$$\int_\Omega \mathfrak{S}_1(x,u;\varphi - u') : \partial X + \mathfrak{S}_0(x,u;\varphi - u') \cdot X \,\mathrm{d}x = \int_\Gamma \mathfrak{S}_1(x,u;\varphi - u') n \cdot X_n \,\mathrm{d}s.$$

In combination with (66) we obtain (65). Uniqueness of u' is implied by uniqueness of $\dot{y}$ (see Theorem 4.8). □

It is readily checked that

$$\mathfrak{S}_1(x,u;\varphi)n\cdot n = -\nabla_\Gamma u\cdot\nabla_\Gamma\varphi - \big(\lambda u + w(x,u)\big)\varphi \quad \text{for all } \varphi\in S_u(K_\psi).$$

Thus we conclude this section with an explicit formula for the shape derivative in the case of a static obstacle.

Corollary 4.16. *Under the assumption of Theorem 4.15 the shape derivative u' is the unique solution of the following variational inequality:*

$$\begin{aligned} u'\in S_u(K_\psi),\quad \mathfrak{a}(u',\varphi-u') + \int_\Omega \partial_y w(x,u)u'(\varphi-u')\,\mathrm{d}x \\ \geq -\int_\Gamma \nabla_\Gamma u\cdot\nabla_\Gamma(\varphi-u') + \big(\lambda u + w(x,u)\big)(X\cdot n)(\varphi-u')\,\mathrm{d}s \end{aligned}$$

for all $\varphi\in S_u(K_\psi)$.

4.6 Eulerian semi-derivative of certain shape functions

We adopt the notation from Subsection 2.3 and denote by $J:\Xi\to\mathbb{R}$ a shape function. Application of Corollary 4.11, Lemma 2.8 and the chain rule yield the following result:

Corollary 4.17. *Let (A1)-(A3) be satisfied and let $\Omega\in\Xi$ be a Lipschitz domain, $X\in C_c^1(D,\mathbb{R}^d)$ and $\Phi_t:\Omega\to\Omega_t$ be the associated flow. Suppose that for all small $t>0$*

$$J(\Omega_t) = \mathfrak{J}(\Phi_t, u^t),$$

where

$$\mathfrak{J} = \mathfrak{J}(\Phi,u): C^{0,1}(\Omega;\mathbb{R}^d)\times H^1(\Omega)\to\mathbb{R}$$

is assumed to be a Fréchet differentiable functional and $u^t\in H^1(\Omega)$ the transported state $u^t = u_t\circ\Phi_t$ with the unique solution u_t of (40) on Ω_t.

Then the Eulerian semi-derivative exists and is given as

$$dJ(\Omega)(X) = \langle d_\Phi\mathfrak{J}(Id,u^0), X\rangle_{C^{0,1}(\Omega;\mathbb{R}^d)} + \langle d_u\mathfrak{J}(Id,u^0), \dot{u}_X\rangle_{H^1(\Omega)},$$

where $\dot{u}_X$ denotes the unique solution of (60).

In particular $dJ(\Omega)(\cdot)$ is positively 1-homogeneous.

5 Applications to damage phase field models

In this section we investigate shape optimization problems for a coupled inclusion/pde system describing damage processes in linear elastic materials. Our aim is to apply the abstract results from Section 4 designed for semilinear variational inequalities with dynamic obstacles to such concrete application scenarios. In this way we demonstrate how necessary optimality conditions for shape problems can be derived for relevant engineering tasks.

5.1 Physical model

The physical model under consideration was derived in [9] and is described in the time-continuous setting by the following relations:

$$\mathbf{u}_{tt} - \operatorname{div}\big(\mathbb{C}(\chi)\varepsilon(\mathbf{u})\big) = \boldsymbol{\ell}, \tag{71a}$$

$$0 \in \partial I_{(-\infty,0]}(\chi_t) + \chi_t - \Delta\chi + \frac{1}{2}\mathbb{C}'(\chi)\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) + f'(\chi), \tag{71b}$$

with the damage-dependent stiffness tensor $\mathbb{C}$ and the damage potential function f. The variable $\mathbf{u}$ denotes the displacement field, $\varepsilon(\mathbf{u}) := \frac{1}{2}(\partial\mathbf{u} + (\partial\mathbf{u})^T)$ the linearised strain tensor and χ is an internal variable (a so-called phase field variable) indicating the degree of damage. In terms of damage mechanics χ is interpreted as the density of micro-defects and is therefore valued in the unit interval (cf. [17]). In this spirit we may use the following interpretation:

$$\chi(x) = \begin{cases} 1 & \leftrightarrow \text{ no damage in } x, \\ \in (0,1) & \leftrightarrow \text{ partial damage in } x, \\ 0 & \leftrightarrow \text{ maximal damage in } x. \end{cases}$$

The system is supplemented with initial-time values for χ, $\mathbf{u}$ and $\mathbf{u}_t$, Dirichlet boundary condition for $\mathbf{u}$ and homogeneous Neumann boundary condition for χ. The governing state system (71) can be derived by balance equations and suitable constitutive relations such that the laws of thermodynamics from continuum physics are fulfilled. We refer to [9] for more details on the derivation of the model.

A main feature of the evolution system (71) is the uni-directionality constraint $\chi_t \leq 0$ enforced by the subdifferential $\partial I_{(-\infty,0]}(\chi_t)$. This leads to non-smooth/switching behaviour of the evolution law by noticing that (71b) rewrites as

$$\chi_t = \begin{cases} d, & \text{if } d \leq 0, \\ 0, & \text{if } d > 0 \end{cases} \quad \text{with the driving force} \quad d = \Delta\chi - \frac{1}{2}\mathbb{C}'(\chi)\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) - g'(\chi).$$

A weak formulation of (71) and existence of weak solution can be found in [11] with minor adaption. Existence and uniqueness results for strong solutions for the above system with higher-order viscous terms are established in [7]. For the analysis of quasi-linear variants of (71) and for rate-independent as well as rate-dependent cases, we refer to [16] and the references therein.

The following remark justifies that the phase field variable χ takes only admissible values provided $H^1(0,T;H^1(\Omega))$-regularity and mild growth assumptions on $\mathbb{C}$ and g. In that case it is not necessary to include a second sub-differential of the type $\partial I_{[0,1]}(\chi)$ in (71b) in order to force χ to be bounded in the unit interval. The precise assumptions for $\mathbb{C}$ and g will be stated in (D1) below. At this point they are assumed to be continuously differentiable.

Remark 5.1 (Maximum principle)**.** *Suppose that $\mathbb{C}'(x) = \mathbf{0}$ and $g'(x) = 0$ for all $x < 0$. Then a weak solution $\chi \in H^1(0,T;H^1(\Omega))$ of* (71b) *is always bounded in the unit interval as long as the initial-time value $\chi(0) = \chi^0$ is.*

Proof of Remark 5.1. Because of $\chi_t(t) \leq 0$ for all times $t \in [0,T]$ and $\chi(0) \in [0,1]$ we obtain $\chi(t) \leq 1$. It remains to show $\chi(t) \geq 0$.

Please notice that we cannot directly test (71b) with $(\chi^-)_t$ since $\chi^- := \min\{0,\chi\}$ is not necessarily in $H^1(0,T;H^1(\Omega))$ even for smooth χ. Instead, we test the inclusion (71b) with $(m_\epsilon(\chi))_t$ where m_ϵ denotes the following concave $C^{1,1}$-approximation of $\min\{0,\cdot\}$

$$m_\epsilon(x) = \begin{cases} x, & \text{if } x \in (-\infty,-\epsilon], \\ -\frac{1}{16\epsilon}(x-3\epsilon)^2, & \text{if } x \in (-\epsilon, 3\epsilon], \\ 0, & \text{if } x \in (3\epsilon,+\infty), \end{cases}$$

we obtain by simple rewriting

$$\begin{aligned} &\iint |\chi_t|^2 m_\epsilon'(\chi) + \nabla m_\epsilon(\chi)\cdot\nabla(m_\epsilon(\chi))_t + \big(\nabla\chi - \nabla m_\epsilon(\chi)\big)\cdot\nabla(m_\epsilon(\chi))_t \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \iint \Big(\frac{1}{2}\mathbb{C}'(\chi)\varepsilon(\mathbf{u}):\varepsilon(\mathbf{u}) + g'(\chi) + \xi\Big) m_\epsilon'(\chi)\chi_t \,\mathrm{d}x\,\mathrm{d}t = 0, \end{aligned}$$

where the function ξ satisfies $\xi \in \partial I_{(-\infty,0]}(\chi_t)$ pointwise. We obtain by noticing that $m_\epsilon(\cdot) \to (\cdot)^- := \min\{\cdot,0\}$ strongly in $H^1(\mathbb{R})$ and weakly-star in $W^1_\infty(\mathbb{R})$ as $\epsilon \searrow 0$:

$$\begin{aligned} &\underbrace{\iint |\chi_t|^2 m_\epsilon'(\chi)\,\mathrm{d}x\,\mathrm{d}t}_{\to\iint |(\chi^-)_t|^2\,\mathrm{d}x\,\mathrm{d}t} + \underbrace{\frac{1}{2}\int_\Omega |\nabla m_\epsilon(\chi(t))|^2 - |\nabla m_\epsilon(\chi(0))|^2\,\mathrm{d}x}_{\to\frac{1}{2}\int_\Omega |\nabla\chi^-(t)|^2 - |\nabla\chi^-(0)|^2\,\mathrm{d}x} \\ &+ \underbrace{\iint \big(\nabla\chi\cdot\nabla\chi_t\big)(1-m_\epsilon'(\chi))m_\epsilon'(\chi)\,\mathrm{d}x\,\mathrm{d}t}_{\to 0} + \iint \underbrace{|\nabla\chi|^2(1-m_\epsilon'(\chi)m_\epsilon''(\chi)\chi_t}_{\geq 0 \text{ due to } m_\epsilon''\leq 0,\,\chi_t\leq 0,\,m_\epsilon'\in[0,1]}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \underbrace{\iint \Big(\frac{1}{2}\mathbb{C}'(\chi)\varepsilon(\mathbf{u}):\varepsilon(\mathbf{u}) + g'(\chi) + \xi\Big) m_\epsilon'(\chi)\chi_t\,\mathrm{d}x\,\mathrm{d}t}_{\to\iint\left(\frac{1}{2}\mathbb{C}'(\chi^-)\varepsilon(\mathbf{u}):\varepsilon(\mathbf{u}) + g'(\chi^-)+\xi\right)\chi_t^-\,\mathrm{d}x\,\mathrm{d}t} = 0. \end{aligned}$$

We have by assumption $\mathbb{C}'(\chi^-) = \mathbf{0}$ and $g'(\chi^-) = 0$. Furthermore $\xi \times (\chi^-)_t = 0$ since $\xi = 0$ as long as $\chi_t < 0$. All in all we find by passing to $\epsilon \searrow 0$

$$\frac{1}{2}\int_\Omega |\nabla\chi^-(t)|^2 - |\nabla\chi^-(0)|^2\,\mathrm{d}x + \iint |(\chi^-)_t|^2\,\mathrm{d}x\,\mathrm{d}t \leq 0.$$

Since $\chi^-(0) = 0$ in Ω we find $\chi^-(t) = 0$ in Ω for all times $t \in [0,T]$. □

In the next section we will consider a time-discrete version of (71) where such a maximum principle can also be obtained.

5.2 Setting up time-discretisation scheme and shape optimisation problem

The shape optimization problems will be performed on a time-discrete version of (71) and for two spatial dimensions. Let $\{0,\tau,2\tau,\ldots,\tau N\}$ be an equidistant partition of $[0,T]$. The positive parameter $\tau > 0$ denotes the time step size. In the remaining part of this work we make use of the following assumptions:

Assumption (D1)

(i) $d = 2$;

(ii) *The damage-dependent stiffness tensor satisfies* $\mathbb{C}(\cdot) = \mathsf{c}(\cdot)\mathbf{C}$, *where the coefficient function* c *is assumed to be of the form*

$$\mathsf{c} = \mathsf{c}_1 + \mathsf{c}_2 \text{ where } \mathsf{c}_1 \in C^2(\mathbb{R}) \text{ is convex and } \mathsf{c}_2 \in C^2(\mathbb{R}) \text{ is concave.}$$

Moreover, we assume that $\mathsf{c}, \mathsf{c}_1', \mathsf{c}_1'', \mathsf{c}_2', \mathsf{c}_2''$ *are bounded and as well as*

$$\mathsf{c}(x) \geq \eta \qquad \text{for all } x \in \mathbb{R}.$$

with constant $\eta > 0$. *The* 4^{th} *order stiffness tensor* $\mathbf{C} \in \mathcal{L}(\mathbb{R}^{n\times n}_{\text{sym}}; \mathbb{R}^{n\times n}_{\text{sym}})$ *is assumed to be symmetric and positive definite, i.e.*

$$\mathbf{C}_{ijlk} = \mathbf{C}_{jilk} = \mathbf{C}_{lkij} \text{ and } e : \mathbf{C}e \geq \eta|e|^2 \text{ for all } e \in \mathbb{R}^{n\times n}_{\text{sym}};$$

(iii) g *is assumed to be of the form*

$$g = g_1 + g_2 \text{ where } g_1 \in C^2(\mathbb{R}) \text{ is convex and } g_2 \in C^2(\mathbb{R}) \text{ is concave.}$$

Moreover we assume g_1' *and* g_2' *to be Lipschitz continuous;*

(iv) $\boldsymbol{\ell}^k \in L_2(D;\mathbb{R}^2)$ *for all* $k = 0,\dots,N$;

(v) $\mathbf{d}^k \in H^2(D;\mathbb{R}^2)$ *for all* $k = 0,\dots,N$;

(vi) *initial values:* $\mathbf{u}^0, \mathbf{v}^0 \in H^2(D;\mathbb{R}^2)$ *and* $\chi^0 \in H^2(D)$.

Let $\Omega \subseteq D$ be a given Lipschitz domain. In this section a time-discrete model to (71) will be investigated in a *thermodynamically consistent* scheme (in this context it indicates that the time-discrete energy-dissipation inequality is satisfied). A related time-discretisation scheme has been used in [7]. For all $k \in \{1,\dots,N\}$ we are looking for a weak solution of

$$\frac{\mathbf{u}^k - 2\mathbf{u}^{k-1} + \mathbf{u}^{k-2}}{\tau^2} - \operatorname{div}\big(\mathbb{C}(\chi^k)\varepsilon(\mathbf{u}^k)\big) = \boldsymbol{\ell}^k, \tag{72a}$$

$$\begin{aligned} 0 \in \partial I_{(-\infty,0]}\Big(\frac{\chi^k - \chi^{k-1}}{\tau}\Big) + \frac{\chi^k - \chi^{k-1}}{\tau} - \Delta\chi^k + g_1'(\chi^k) + g_2'(\chi^{k-1}) \\ + \frac{1}{2}\big(\mathsf{c}_1'(\chi^k) + \mathsf{c}_2'(\chi^{k-1})\big)\mathbf{C}\varepsilon(\mathbf{u}^{k-1}) : \varepsilon(\mathbf{u}^{k-1}). \end{aligned} \tag{72b}$$

In accordance with the time-continuous model from the previous section χ^0, $\mathbf{u}^0$ and $\mathbf{u}^{-1} := \mathbf{u}^0 + \tau\mathbf{v}^0$ are the initial values and the boundary conditions are chosen as

$$\nabla\chi\cdot\nu = 0, \quad \mathbf{u}^k = \mathbf{d}^k \quad \text{ on } \partial\Omega. \tag{73}$$

For notational convenience we will write $\mathbf{z} = \{\mathbf{u}^k, \chi^k\}_{k=0}^N$.

Remark 5.2. *(i) Existence of weak solutions for* (72) *can be obtained by alternate minimisation for each time step by firstly solving* (72b) *and then solving* (72a). *In particular the solution* χ^k *from* (72b) *is the unique minimiser of the strictly convex potential*

$$F(\chi) = \int_\Omega \frac{1}{2}|\nabla\chi|^2 + \frac{\tau}{2}\Big|\frac{\chi - \chi^{k-1}}{\tau}\Big|^2 + \frac{1}{2}\big(\mathsf{c}_1(\chi) + \mathsf{c}_2'(\chi^{k-1})\chi\big)\mathbf{C}\varepsilon(\mathbf{u}^{k-1}) : \varepsilon(\mathbf{u}^{k-1})\,\mathrm{d}x$$

$$+\int_\Omega g_1(\chi) + g_2'(\chi^{k-1})\chi \,\mathrm{d}x.$$

over the convex set

$$K^{k-1} := \{\chi \in H^1(\Omega) \,:\, \chi \le \chi^{k-1} \text{ a.e. in } \Omega\}.$$

As we point out later a higher integrability result from [8] yields $\varepsilon(\mathbf{u}) \in L_p(\Omega)$ *for a* $p > 2$. *In combination with the embedding* $H^1(\Omega) \hookrightarrow L_q(\Omega)$ *for every* $q \ge 1$ *valid for* $d = 2$ *and Assumption (D1) (ii), the potential term* $\int_\Omega \frac{1}{2}\big(\mathsf{c}_1(\chi) + \mathsf{c}_2'(\chi^{k-1})\chi\big)\mathbf{C}\varepsilon(\mathbf{u}^{k-1}) : \varepsilon(\mathbf{u}^{k-1})\,\mathrm{d}x$ *in* F *is well-defined.*

(ii) Under the additional assumptions that $\mathsf{c}_1(x) \ge \mathsf{c}_1(0)$ *and* $g_1(x) \ge g_1(0)$ *for all* $x \le 0$ *as well as* $\mathsf{c}_2'(x) \le 0$ *and* $g_2'(x) \le 0$ *for all* $x \in [0,1]$ *we obtain that* $F(\max\{\chi^k, 0\}) \le F(\chi^k)$ *(cf. [16, Proposition 4.1]). Thus* χ^k *is bounded in the unit interval as long as* χ^{k-1} *is.*

(iii) The discretisation scheme above is motivated by the fact that the associated time-discrete energy-dissipation inequality is obtained by testing (72a) *with* $\mathbf{u}^k - \mathbf{u}^{k-1} - (\mathbf{d}^k - \mathbf{d}^{k-1})$ *and* (72b) *with* $\chi^k - \chi^{k-1}$, *adding and using convexity and concavity estimates (cf. [7, Lemma 2.9]).*

For the shape optimisation problem it is convenient to rewrite the pde/inclusion system (72b) as

$$\left.\begin{aligned} &\mathbf{u}^k \in \mathbf{d}^k + \mathring{H}^1(\Omega;\mathbb{R}^2), \text{ and } \forall \varphi \in \mathring{H}^1(\Omega;\mathbb{R}^2) : \\ &\int_\Omega \frac{\mathbf{u}^k - 2\mathbf{u}^{k-1} + \mathbf{u}^{k-2}}{\tau^2}\varphi + \mathbb{C}(\chi^k)\varepsilon(\mathbf{u}^k) : \varepsilon(\varphi)\,\mathrm{d}x = \int_\Omega \boldsymbol{\ell}^k \cdot \varphi \,\mathrm{d}x \end{aligned}\right\} \tag{74a}$$

$$\left.\begin{aligned} &\chi^k \in K^{k-1} \text{ and } \forall \varphi \in K^{k-1} : \\ &\int_\Omega \nabla\chi^k \cdot \nabla(\varphi - \chi^k) + \frac{\chi^k - \chi^{k-1}}{\tau}(\varphi - \chi^k) + \big(g_1'(\chi^k) + g_2'(\chi^{k-1})\big)(\varphi - \chi^k)\,\mathrm{d}x \\ &+ \int_\Omega \frac{1}{2}\big(\mathsf{c}_1'(\chi^k) + \mathsf{c}_2'(\chi^{k-1})\big)\mathbf{C}\varepsilon(\mathbf{u}^{k-1}) : \varepsilon(\mathbf{u}^{k-1})(\varphi - \chi^k)\,\mathrm{d}x \ge 0. \end{aligned}\right\} \tag{74b}$$

In other words the state system is given by coupled variational inequalities with dynamic obstacles for the N time steps. The obstacles are determined as the solutions of the damage variational inequality from the previous time step.

Statement of the shape optimization problem

Our aim is to determine an optimal shape $\Omega \in \Xi$ from a suitable class of domains such that a tracking type cost functional

$$J(\Omega, \mathbf{z}(\Omega)) = \frac{\lambda_{\mathbf{u}}}{2}\sum_{k=1}^N \|\mathbf{u}^k(\Omega) - \mathbf{u}_r^k\|^2_{L_2(\Omega;\mathbb{R}^2)} + \frac{\lambda_\chi}{2}\sum_{k=1}^N \|\chi^k(\Omega) - \chi_r^k\|^2_{L_2(\Omega)} \tag{75}$$

is minimised under the constraint that

$$\mathbf{z}(\Omega) \text{ solve (74) on } \Omega \text{ for all } k \in \{1,\dots,N\}. \tag{76}$$

The functions $\mathbf{u}_r^k \in L_2(D;\mathbb{R}^2)$ and $\chi_r^k \in L_2(D)$ for $k = 1,\dots,N$ are prescribed displacements and damage patterns. Since the state $\mathbf{z}(\Omega)$ is uniquely determined by Ω we may equivalently say that we aim to minimise the shape function

$$J(\Omega) := J(\Omega, \mathbf{z}(\Omega)). \tag{77}$$

5.3 Material derivative and necessary optimality system

Let us fix a vector field $X \in C_c^1(D, \mathbb{R}^2)$. In accordance with Section 4 the associated perturbed solutions of (72a)-(72b) on $\Omega_t := \Phi_t(\Omega)$ are denoted by $\mathbf{z}_t = \{\mathbf{u}_t^k, \chi_t^k\}_{k=0}^N$ whereas the transported perturbed solutions are indicated by $\mathbf{z}^t = \{\mathbf{u}^{k,t}, \chi^{k,t}\}_{k=0}^N$. Note that $\mathbf{z}^0 = \mathbf{z}$.

We proceed inductively over $k = 1, \ldots, N$ and assume that the strong material derivatives at the time steps $k-1$ and $k-2$ exist, i.e. for a subsequence $t \searrow 0$

$$\frac{\mathbf{u}^{k-1,t} - \mathbf{u}^{k-1,0}}{t} \to \dot{\mathbf{u}}^{k-1} \quad \text{strongly in } H^1(\Omega; \mathbb{R}^2), \tag{78a}$$

$$\frac{\mathbf{u}^{k-2,t} - \mathbf{u}^{k-2,0}}{t} \to \dot{\mathbf{u}}^{k-2} \quad \text{strongly in } H^1(\Omega; \mathbb{R}^2), \tag{78b}$$

$$\frac{\chi^{k-1,t} - \chi^{k-1,0}}{t} \to \dot{\chi}^{k-1} \quad \text{strongly in } H^1(\Omega). \tag{78c}$$

Material derivative for the $\boldsymbol{\chi^k}$-variable
We want to apply Corollary 4.11 which is based on Theorem 4.8 to establish the material derivative for the χ^k-variable and its variational inequality.

To check that the Assumptions (A1)-(A3) are fulfilled we require higher integrability estimates for $\mathbf{u}^{k-1,t}$. Note that $\mathbf{u}^{k-1,t}$ satisfies equation (84) below for $k-1$ which is the unique minimiser of

$$U(\mathbf{u}) := \int_\Omega \xi(t)\mathbb{C}(\chi^{k-1,t})\varepsilon^t(\mathbf{u}) : \varepsilon^t(\mathbf{u}) + \xi(t)\frac{\mathbf{u} - 2\mathbf{u}^{k-2,t} + \mathbf{u}^{k-3,t}}{\tau^2} \cdot \mathbf{u} - \xi(t)\boldsymbol{\ell}^{k-1,t} \cdot \mathbf{u} \, \mathrm{d}x.$$

over $\mathbf{u} \in \mathbf{d}^{k-1} + \mathring{H}^1(\Omega; \mathbb{R}^2)$, where

$$\varepsilon^t(\mathbf{u}) := \frac{1}{2}\Big((\partial \mathbf{u})(\partial \Phi_t)^{-1} + \big((\partial \mathbf{u})(\partial \Phi_t)^{-1}\big)^T\Big). \tag{79}$$

By using the calculation (here $S(\mathbf{A}) := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ and $\mathbf{B} := \partial \Phi_t$)

$$\begin{aligned} &\mathbf{C}\varepsilon^t(\mathbf{u}) : \varepsilon^t(\mathbf{u}) \\ &\quad = \mathbf{C}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) - \mathbf{C}S((\partial \mathbf{u})(1 - B)) : S(\partial \mathbf{u}) - \mathbf{C}S((\partial \mathbf{u})\mathbf{B}) : S((\partial \mathbf{u})(1 - \mathbf{B})) \\ &\quad \geq \mathbf{C}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) - c|\partial \mathbf{u}|^2 |1 - \mathbf{B}| - c|\partial \mathbf{u}|^2 |\mathbf{B}||1 - \mathbf{B}| \end{aligned}$$

and Korn's inequality, we find a $t^* > 0$ and constants $c_0, c_1 > 0$ such that for all $t \in [0, t^*]$ and $\mathbf{u} \in \mathbf{d}^{k-1} + \mathring{H}^1(\Omega; \mathbb{R}^2)$

$$\int_\Omega \xi(t)\mathbb{C}(\chi^{k-1,t})\varepsilon^t(\mathbf{u}) : \varepsilon^t(\mathbf{u}) \, \mathrm{d}x \geq c_0 \|\partial \mathbf{u}\|_{L_2}^2 - c_1 \|\mathbf{u}\|_{L_2}^2.$$

Then the higher integrability result from [8] shows that there exists a constant $p > 2$ independent of t such that $\mathbf{u}^{k-1,t} \in W_p^1(\Omega; \mathbb{R}^2)$ and $\|\mathbf{u}^{k-1,t}\|_{W_p^1(\Omega; \mathbb{R}^2)}$ is uniformly bounded in $t \in [0, t^*]$. In combination with (78a) we see that

$$\mathbf{u}^{k-1,t} \to \mathbf{u}^{k-1,0} \quad \text{strongly in } W_q^1(\Omega; \mathbb{R}^2) \text{ as } t \searrow 0 \text{ for all } q \in [2, p). \tag{80}$$

Furthermore we deduce from (78c) by the Sobolev embeddings in 2D

$$\frac{\chi^{k-1,t} - \chi^{k-1,0}}{t} \to \dot{\chi}^{k-1} \quad \text{strongly in } L_q(\Omega) \text{ for all } q \in [1, \infty). \tag{81}$$

and from (78a)

$$\frac{\varepsilon^t(\mathbf{u}^{k-1,t}) - \varepsilon(\mathbf{u}^{k-1,0})}{t} \to \varepsilon(\dot{\mathbf{u}}^{k-1}) - \frac{1}{2}\big((\partial\mathbf{u}^{k-1,0})(\partial X) + ((\partial\mathbf{u}^{k-1,0})(\partial X))^T\big) =: \dot{\varepsilon}_X(\dot{\mathbf{u}}^{k-1}) \quad \text{strongly in } L_2(\Omega;\mathbb{R}^{2\times 2}). \tag{82}$$

The damage variational inequality (74b) can now be rewritten in the abstract form (40) by setting W_Ω in the energy (38) as follows

$$\begin{aligned} W_\Omega(x,y) :=& -\frac{1}{\tau}\chi^{k-1}(x)y + \frac{1}{2}\big(\mathsf{c}_1(y) + \mathsf{c}_2'(\chi^{k-1}(x))y\big)\mathbf{C}\varepsilon(\mathbf{u}^{k-1}(x)) : \varepsilon(\mathbf{u}^{k-1}(x)) \\ &+ g_1(y) + g_2'(\chi^{k-1}(x))y. \end{aligned}$$

Note that $W_\Omega(x,\cdot)$ is convex in our discretisation scheme and that

$$\begin{aligned} w_X^t(x,y) =& -\frac{1}{\tau}\chi^{k-1,t}(x) + \frac{1}{2}\big(\mathsf{c}_1'(y) + \mathsf{c}_2'(\chi^{k-1,t}(x))\big)\mathbf{C}\varepsilon^t(\mathbf{u}^{k-1,t}(x)) : \varepsilon^t(\mathbf{u}^{k-1,t}(x)) \\ &+ g_1'(y) + g_2'(\chi^{k-1,t}(x)). \end{aligned}$$

Recall that $w_X^0(x,y) = w(x,y) = \partial_y W_\Omega(x,y)$.

With the help of the convergence properties (78a)-(78c), (80), (82) and (81), we see that Assumptions (A1)-(A3) are fulfilled with

$$\begin{aligned} \partial_y w(x,y) =& \frac{1}{2}\mathsf{c}_1''(y)\mathbf{C}\varepsilon(\mathbf{u}^{k-1}(x)) : \varepsilon(\mathbf{u}^{k-1}(x)) + g_1''(y), \\ \dot{w}_X(x,y) =& -\frac{1}{\tau}\dot{\chi}^{k-1}(x) + \frac{1}{2}\mathsf{c}_2''(\chi^{k-1}(x))\dot{\chi}^{k-1}(x)\mathbf{C}\varepsilon(\mathbf{u}^{k-1}(x)) : \varepsilon(\mathbf{u}^{k-1}(x)) \\ &+ \mathsf{c}_2'(\chi^{k-1}(x))\mathbf{C}\varepsilon(\mathbf{u}^{k-1}(x)) : \dot{\varepsilon}_X(\dot{\mathbf{u}}^{k-1}(x)) + g_2''(\chi^{k-1}(x))\dot{\chi}^{k-1}(x). \end{aligned}$$

Applying Corollary 4.11 yields existence of the strong material derivative $\dot{\chi}^k$ which satisfies the following variational inequality:

$$\left.\begin{aligned} &\dot{\chi}^k \in S^k \text{ and } \forall\varphi \in S^k: \\ &\int_\Omega \nabla\dot{\chi}^k \cdot \nabla(\varphi - \dot{\chi}^k) + \frac{1}{\tau}\dot{\chi}^k(\varphi - \dot{\chi}^k) + \partial_y w(x,\chi^k)\dot{\chi}^k(\varphi - \dot{\chi}^k)\,\mathrm{d}x \\ &\quad \geq -\int_\Omega A'(0)\nabla\chi^k \cdot \nabla(\varphi - \dot{\chi}^k) + \xi'(0)\Big(\frac{\chi^k}{\tau} + w(x,\chi^k)\Big)(\varphi - \dot{\chi}^k)\,\mathrm{d}x \\ &\qquad -\int_\Omega \dot{w}_X(x,\chi^k)(\varphi - \dot{\chi}^k)\,\mathrm{d}x. \end{aligned}\right\} \tag{83}$$

with

$$S^k := T_{\chi^k}(K^{k-1}) \cap kern(\mathcal{A}(\chi^k)) + \dot{\chi}^{k-1}$$

and $A'(0)$ and $\xi'(0)$ are given in Lemma 4.4 and $\mathcal{A} = \mathcal{A}_0$ is defined in (46).

Material derivative for the $\mathbf{u}^k$-variable

We only sketch the proof of the strong material derivative $\dot{\mathbf{u}}^k$ in the following and make use of standard calculations. The main ingredient will be the uniform boundedness of $\|\mathbf{u}^{k,t}\|_{W_p^1(\Omega;\mathbb{R}^2)}$ with respect to t and for some fixed $p > 2$.

The perturbed and transported equation to (74a) is given by

$$\int_\Omega \xi(t)\frac{\mathbf{u}^{k,t}-2\mathbf{u}^{k-1,t}+\mathbf{u}^{k-2,t}}{\tau^2}\varphi + \xi(t)\mathbb{C}(\chi^{k,t})\varepsilon^t(\mathbf{u}^{k,t}) : \varepsilon^t(\varphi)\,\mathrm{d}x = \int_\Omega \xi(t)\boldsymbol{\ell}^{k,t}\cdot\varphi\,\mathrm{d}x \tag{84}$$

for all $\varphi \in \mathring{H}^1(\Omega;\mathbb{R}^2)$, where $\xi(t)$ is defined in (43) and ε^t is defined in (79). Therefore by testing (84) and testing (74a) with $\varphi = \mathbf{u}^{k,t} - \mathbf{u}^{k,0} - (\mathbf{d}^{k,t} - \mathbf{d}^{k,0})$ and subtracting the result, we obtain the sensitvity estimate

$$\|\mathbf{u}^{k,t} - \mathbf{u}^{k,0}\|_{H^1(\Omega;\mathbb{R}^2)} \le ct.$$

Thus we may choose a weak cluster point $\dot{\mathbf{u}}^k \in H^1(\Omega;\mathbb{R}^2)$ such that for a subsequence

$$\frac{\mathbf{u}^{k,t}-\mathbf{u}^{k,0}}{t} \rightharpoonup \dot{\mathbf{u}}^k \quad \text{weakly in } H^1(\Omega;\mathbb{R}^2).$$

Considering difference quotient of (84) and passing to the limit shows that $\dot{\mathbf{u}}^k$ is the weak solution of the following pde:

$$\begin{aligned}
&\int_\Omega \xi'(0)\frac{\mathbf{u}^k-2\mathbf{u}^{k-1}+\mathbf{u}^{k-2}}{\tau^2}\varphi + \frac{\dot{\mathbf{u}}^k-2\dot{\mathbf{u}}^{k-1}+\dot{\mathbf{u}}^{k-2}}{\tau^2}\varphi\,\mathrm{d}x \\
&+\int_\Omega \Big(\xi'(0)\mathbb{C}(\chi^k)\varepsilon(\mathbf{u}^k) + \mathbb{C}'(\chi^k)\dot{\chi}^k\varepsilon(\mathbf{u}^k) + \mathbb{C}(\chi^k)\dot{\varepsilon}_X(\dot{\mathbf{u}}^k)\Big) : \varepsilon(\varphi) + \mathbb{C}(\chi^k)\varepsilon(\mathbf{u}^k) : \dot{\varepsilon}_X(\varphi)\,\mathrm{d}x \\
&= \int_\Omega \xi'(0)\mathbf{f}^k\cdot\varphi + \dot{\mathbf{f}}^k\cdot\varphi\,\mathrm{d}x
\end{aligned} \tag{85}$$

for all $\varphi \in \mathring{H}^1(\Omega;\mathbb{R}^2)$ and $\dot{\mathbf{u}}^k = \dot{\mathbf{d}}^k$ on $\partial\Omega$ where $\dot{\mathbf{d}}^k = \partial_X\mathbf{d}^k$ and $\dot{\mathbf{f}}^k = \partial_X\mathbf{f}^k$. Here, $\dot{\varepsilon}_X$ is defined in (82).

Furthermore, it is not hard to see that the solution $\dot{\mathbf{u}}^k$ is unique for given functions $\mathbf{u}^k$, $\mathbf{u}^{k-1}$, $\mathbf{u}^{k-2}$, $\dot{\mathbf{u}}^{k-1}$, $\dot{\mathbf{u}}^{k-2}$, χ^k, $\dot{\chi}^k$, $\mathbf{f}^k$, $\dot{\mathbf{f}}^k$ and $\dot{\mathbf{d}}^k$. Indeed, given to weak solutions $\dot{\mathbf{u}}_1^k$ and $\dot{\mathbf{u}}_2^k$ of (85) we find after subtraction

$$\int_\Omega \frac{1}{\tau^2}(\dot{\mathbf{u}}_1^k - \dot{\mathbf{u}}_2^k) + \mathbb{C}(\chi^k)\varepsilon(\dot{\mathbf{u}}_1^k - \dot{\mathbf{u}}_2^k) : \varepsilon(\varphi)\,\mathrm{d}x = 0$$

Testing with $\varphi = \dot{\mathbf{u}}_1^k - \dot{\mathbf{u}}_2^k$ yields uniqueness.

Finally, subtracting from the difference quotient taken from (84) the equation (85) and testing with $\varphi = \frac{\mathbf{u}^{k,t}-\mathbf{u}^{k,0}}{t} - \dot{\mathbf{u}}^k - \Big(\frac{\mathbf{d}^{k,t}-\mathbf{d}^{k,0}}{t} - \dot{\mathbf{d}}^k\Big)$ (the $\mathbf{d}$-terms are necessary to achieve 0-boundary conditions for the test-function), we find via a limit passage

$$\frac{\mathbf{u}^{k,t}-\mathbf{u}^{k,0}}{t} - \dot{\mathbf{u}}^k \to 0 \quad \text{strongly in } H^1(\Omega;\mathbb{R}^2) \text{ as } t \searrow 0.$$

Optimality system

We conclude with a necessary optimality system. Let a Lipschitz domain $\Omega \subseteq D$ with its state $\mathbf{z} = \{\mathbf{u}^k, \chi^k\}_{k=0}^N$ be a minimiser of J from (77). Given an arbitrary vector field $X \in C_c^1(D,\mathbb{R}^d)$ we obtain the associated flow Φ_t, the perturbed domain $\Omega_t := \Phi_t(\Omega)$, the transported perturbed solution $\mathbf{z}^t = \{\mathbf{u}^{k,t}, \chi^{k,t}\}_{k=0}^N$ and

$$J(\Omega_t) = \frac{\lambda_\mathbf{u}}{2}\sum_{k=1}^N \int_\Omega \xi(t)|\mathbf{u}^{k,t} - \mathbf{u}_r^k\circ\Phi_t|^2\,\mathrm{d}x + \frac{\lambda_\chi}{2}\sum_{k=1}^N \int_\Omega \xi(t)|\chi^{k,t} - \chi_r^k\circ\Phi_t|^2\,\mathrm{d}x.$$

By calculating the Euler semi-derivative of J and using the relations for the material derivatives above, we have proven the following results:

Proposition 5.3. *Under the assumption (D1) the optimality condition* $dJ(\Omega)(X) = 0$ *is given in the volume expression of the shape derivative by*

$$
\begin{aligned}
0 = &\frac{\lambda_{\mathbf{u}}}{2} \sum_{k=1}^{N} \int_{\Omega} \xi'(0) |\mathbf{u}^k - \mathbf{u}_r^k|^2 \,\mathrm{d}x + \frac{\lambda_\chi}{2} \sum_{k=1}^{N} \int_{\Omega} \xi'(0) |\chi^k - \chi_r^k|^2 \,\mathrm{d}x \\
&+ \lambda_{\mathbf{u}} \sum_{k=1}^{N} \int_{\Omega} (\mathbf{u}^k - \mathbf{u}_r^k) \cdot (\dot{\mathbf{u}}^k - \partial_X \mathbf{u}_r^k) \,\mathrm{d}x + \lambda_\chi \sum_{k=1}^{N} \int_{\Omega} (\chi^k - \chi_r^k)(\dot{\chi}^k - \partial_X \chi_r^k) \,\mathrm{d}x,
\end{aligned}
$$

where for all $k = 1, \ldots, K$*:*

$\mathbf{u}^k$ *fulfills* (74a) *with* $\mathbf{u}^k = \mathbf{d}^k$ *on* $\partial\Omega$, χ^k *fulfills* (74b),

$\dot{\mathbf{u}}^k$ *fulfills* (85) *with* $\dot{\mathbf{u}}^k = \dot{\mathbf{d}}^k$ *on* $\partial\Omega$, $\dot{\chi}^k$ *fulfills* (83).

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