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Systems describing electrothermal effects with $p(x)$-Laplacian like structure for discontinuous variable exponents

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Abstract

We consider a coupled system of two elliptic PDEs, where the elliptic term in the first equation shares the properties of the $p(x)$-Laplacian with discontinuous exponent, while in the second equation we have to deal with an a priori L^1 term on the right hand side. Such a system of equations is suitable for the description of various electrothermal effects, in particular those, where the non-Ohmic behavior can change dramatically with respect to the spatial variable. We prove the existence of a weak solution under very weak assumptions on the data and also under general structural assumptions on the constitutive equations of the model. The main difficulty consists in the fact that we have to overcome simultaneously two obstacles - the discontinuous variable exponent (which limits the use of standard methods) and the L^1 right hand side of the heat equation. Our existence proof based on Galerkin approximation is highly constructive and therefore seems to be suitable also for numerical purposes.

1 Introduction

In this paper, we prove the existence of a weak solution to a system of two elliptic equations. In the first equation the growth condition for the elliptic term depends on the spatial variable, and the second one has a right hand side in L^1. To state the problem more precisely, we consider an open bounded set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary, and we aim to find weak solutions $u, T : \Omega \to \mathbb{R}$ to the following problem

$$-\operatorname{div} \mathsf{S}\big(x, T(x), \nabla u(x)\big) = 0, \tag{1.1a}$$
$$-\operatorname{div}\big(\mathsf{A}(x)\nabla T(x)\big) = f\big(x, T(x), \nabla u(x)\big), \tag{1.1b}$$

which is equipped with the following Dirichlet, Neumann, and Newton boundary conditions

$$u = u^D \quad \text{on } \Gamma_D \quad \text{and} \quad \mathsf{S}(x, T(x), \nabla u(x)) \cdot \nu = 0 \quad \text{on } \Gamma_N, \tag{1.2a}$$
$$(\mathsf{A}(x)\nabla T) \cdot \nu + \kappa(x)(T - T_\Gamma) = 0 \quad \text{on } \partial\Omega, \tag{1.2b}$$

where $\Gamma_D, \Gamma_N \subset \partial\Omega$ are relatively open subsets (with respect to $\partial\Omega$) and satisfy $\Gamma_D \cap \Gamma_N = \emptyset$ as well as $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$. For simplicity, we consider only the case $\operatorname{mes}(\Gamma_D) > 0$ in this paper, but it will be evident from the proof that all results can be extended also to the case of pure Neumann boundary conditions, where one just needs to fix a mean value of u a priori.

The main difficulty in proving the existence of a solution to (1.1)–(1.2) consists in a very specific form of S and f. We present the precise assumptions on them in the next section but in order to outline the main purpose of the paper we briefly sketch the typical structures considered here. For S we consider the case

$$\mathsf{S}(x, T, z) \sim \sigma(x, T)|z|^{p(x)-2}z,$$

while for f we always set

$$f(x, T, z) := \eta(x)\mathsf{S}(x, T, z) \cdot z. \tag{1.3}$$

In this setting it is evident from (1.1) that f will be a priori a L^1 quantity, which is the first source of difficulties. The second one is that we consider here the situation that the function $x \mapsto p(x)$ is measurable and discontinuous with respect to x, however fulfills $1 < p_- < p(x) < p_+ < \infty$ for almost all $x \in \Omega$ (see the next section for the detailed assumptions).

The above system of equations is not academic but usually appears in the modeling of materials conducting both heat and electrical current and for which the electrical conductivity strongly depends on the temperature. Devices of this type are called thermistors (see [1, 2]). In this setting, u denotes the electrostatic potential and T the temperature in the device. Correspondingly, the first equation in (1.1) describes the current flow in the structure and the second equation is the heat equation with Joule heat term on the right hand side with the spatially dependent nonnegative coefficient η describing how much of the electrical power is converted into heat.

Recently, systems of the form (1.1) were also introduced in [3] to describe electrothermal effects, such as self-heating and inhomogeneous current distributions, in organic, i.e. carbon-based, semiconductor devices, see also [4, 5] and the discussion in Section 4 of the present paper. For example, organic light-emitting diodes are thin-film heterostructures based on organic molecules or polymers, where each functional layer has, in general, its own current-voltage characteristics and material parameters. In particular, the exponent $p(x)$, which describes the non-Ohmic behavior of each layer, changes abruptly from one material to another. In electrodes, the typically used parameter is $p(x) = 2$, while organic layers feature significantly larger values, e.g. $p(x) \approx 9$ (see [5]).

The existence of solutions in the setting of organic semiconductor devices (the system of the form (1.1)) was investigated in [6] in the two-dimensional case for $p_- \geq 2$. The existence proof given in [6] is based on Schauder's fixed-point theorem and improved integrability of the gradient of the electrostatic potential, i.e. $\nabla u \in L^{sp(\cdot)}(\Omega)$ for some $s > 1$. Such improvement significantly helps to deal with right hand side f in the heat equation in the existence proof, since then we have a priori control of f in $L^s(\Omega)$. Especially, one does not need to face the problem with possible concentration effects and correspondingly the presence of a singular measure. All these techniques however heavily rely on the use of the Poincaré inequality, which does in general not hold for discontinuous exponents p, see [7, Sec. 8.2]. This lack of regularity was overcome in [6] by considering only the two dimensional setting and $p(x) \geq p_- \geq 2$. Notice that this setting allows one to use the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$. Although these improved integrability estimates are not established for a higher dimensional setting, it seems to be clear that the regularity methods from [6] can be adapted to all cases when $\mathrm{W}^{1,\mathrm{p}_-}(\Omega)$ is compactly embedded into $L^{p_+}(\Omega)$, i.e., the case $p_+ < dp_-/(d - p_-)$, or $p_- \geq d$. However, it is totally unclear whether these techniques, i.e., the Caccioppoli-type estimate, the generalized Poincaré inequality and the Gehring-type lemma, can be extended beyond the range of parameters when the above mentioned embedding hold. In particular notice here that in the case of electrodes ($p(x) = 2$) and organic layers ($p(x) \approx 9$) the required embedding is not valid in the three dimensional setting. Thus the method developed in [6] seems not to be suitable for establishing weak solutions for systems of the form (1.1) by means of higher regularity.

Therefore in this paper, we follow a different approach. We do not use Schauder's fixed point theorem but rather investigate a very constructive technique based on regularization and Galerkin approximation. More precisely, we consider a regularized version of the system in (1.1), where $\mathbf{S}$ is replaced by a strictly monotone operator and f is approximated so that it remains bounded, see Section 3.1. The existence of solutions to the regularized problem follows then from Galerkin approximations. By using suitably chosen test functions in the weak formulation of the regularized version of (1.1) we can derive uniform estimates for u and T independent of the regularization parameter $\varepsilon > 0$. In particular, these estimates allow us to pass to the limit $\varepsilon \to 0$ and to obtain weak solutions of (1.1). In addition, we show that T is an entropy solution to (1.1b). We refer to [8] and [9], where the existence and uniqueness of entropy solutions is discussed in the case of Dirichlet boundary conditions for nonlinear elliptic and parabolic equations, respectively. Moreover, the method developed in this paper allows to replace (1.1b) by a nonlinear version, i.e., instead of the linear operator on the left hand side for T one could consider a general monotone operator. But since we have in mind the application to the heat transfer, when the heat flux is usually considered to be linear with respect to the temperature gradient, we do not include such a generalization here but rather refer the interested reader to [10], where the very general case is treated.

The paper is organized as follows. In Section 2, we first precisely state all assumptions on data and then formulate the main result of the paper. Section 3 is devoted to the proof of the main theorem and in the final Section 4, we give a physical example of a model for the electrothermal behavior of organic light-emitting diodes that fits into the assumptions of the main result of the paper and thus widely extends the results of [6].

2 Notation, assumptions, and main result

In this section we introduce the assumptions, the function spaces, and the basic notation used in the paper. We denote the set of all measurable functions $p : \Omega \to [1, \infty]$ by $\mathcal{P}(\Omega)$, and call a function $p \in \mathcal{P}(\Omega)$ a variable exponent. For $p \in \mathcal{P}(\Omega)$, we set

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

Next, we specify the requirements on the data.

(i) The variable exponent $p \in \mathcal{P}(\Omega)$ satisfies $1 < p_- \le p_+ < \infty$.

(ii) For a given $\underline{T_0}$, the mapping $\mathbf{S} : \Omega \times [\underline{T_0}, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is Carathéodory and we assume that for almost all $x \in \Omega$, all $T \in [\underline{T_0}, \infty)$ and all $z_1, z_2 \in \mathbb{R}^d$ we have

$$\big(\mathbf{S}(x, T, z_1) - \mathbf{S}(x, T, z_2)\big) \cdot (z_1 - z_2) \ge 0 \quad \text{and} \quad \mathbf{S}(x, T, z_1) \cdot z_1 \ge 0. \tag{2.1}$$

Moreover, we assume that $\mathbf{S}$ satisfies $p(x)$-coercivity and $p(x)-1$ growth, i.e., there exist constants $\sigma_1 > 0$, $\sigma_2 \ge 0$ and $\sigma_3 > 0$ such that for almost all $x \in \Omega$, all $T \in [\underline{T_0}, \infty)$ and all $z \in \mathbb{R}^d$ there holds

$$\mathbf{S}(x, T, z) \cdot z \ge \sigma_1 |z|^{p(x)} - \sigma_2 \quad \text{and} \quad |\mathbf{S}(x, T, z)| \le \sigma_3 (1 + |z|)^{p(x)-1}. \tag{2.2}$$

(iii) The function $\eta \in L^\infty(\Omega)$ is nonnegative.

(iv) The function $\kappa \in L^\infty(\partial\Omega)$ is nonnegative and satisfies $\int_{\partial\Omega} \kappa(x) \, \mathrm{d}\Gamma > 0$.

(v) The coefficient matrix A is measurable, elliptic and essentially bounded, i.e., there exist $\alpha_1, \alpha_2 > 0$ such that for almost all $x \in \Omega$ and all $z \in \mathbb{R}^d$

$$\mathsf{A}(x)z \cdot z \geq \alpha_1 |z|^2 \quad \text{and} \quad |\mathsf{A}(x)| \leq \alpha_2.$$

(vi) We assume that the Dirichlet data $u^D \in W^{1,p_-}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u^D(x)|^{p(x)} \, \mathrm{d}x < \infty.$$

(vii) The parameter $T_\Gamma \in \mathbb{R}$ satisfies $T_\Gamma \geq \underline{T_0}$.

For a given variable exponent $p \in \mathcal{P}(\Omega)$, we consider the standard variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$, which consists of all measurable functions v for which the modular

$$m_{p(\cdot)}(v) := \int_\Omega |v(x)|^{p(x)} \, \mathrm{d}x$$

is finite, see [11, 12, 7]. We equip this space with the Luxemburg norm

$$\|v\|_{L^{p(\cdot)}} = \|v\|_{p(\cdot)} := \inf \left\{ \tau > 0 : \, m_{p(\cdot)} \left(\frac{v}{\tau} \right) \leq 1 \right\},$$

for which $L^{p(\cdot)}(\Omega)$ becomes a Banach space. In addition, we have that $m_{p(\cdot)}(v) \leq 1$ if and only if $\|v\|_{L^{p(\cdot)}(\Omega)} \leq 1$. Moreover, all $f \in L^{p(\cdot)}(\Omega)$ satisfy the following inequality (see [7, Lemma 3.2.5])

$$\min \left\{ m_{p(\cdot)}(v)^{\frac{1}{p_-}}, m_{p(\cdot)}(v)^{\frac{1}{p_+}} \right\} \leq \|v\|_{L^{p(\cdot)}} \leq \max \left\{ m_{p(\cdot)}(v)^{\frac{1}{p_-}}, m_{p(\cdot)}(v)^{\frac{1}{p_+}} \right\}. \tag{2.3}$$

Furthermore, if $p_+ < \infty$ then $m_{p(\cdot)}(v_n) \to 0$ if and only if $\|v_n\|_{L^{p(\cdot)}} \to 0$ (see [11, Eqn. (2.28)]).

Next, we focus on a proper definition of generalized Sobolev spaces that will be appropriate for our problem. We would like to emphasize here, that the spaces introduced here are not necessarily equivalent to the standard Sobolev spaces with the variable exponent. The reason for such a generalization is that we do not have the proper Poincaré inequality in case that p is not continuous and therefore we will not be able to control the $L^{p(\cdot)}$ norm of u. Thus, for a given $p \in \mathcal{P}(\Omega)$ we introduce the generalized Sobolev space

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in W^{1,p_-}(\Omega) : \, \int_\Omega |\nabla u(x)|^{p(x)} \, \mathrm{d}x < \infty \right\},$$

which we equip with the following norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{1,p_-} + \|\nabla u\|_{p(\cdot)}.$$

It is easy to see that in the case $1 < p_- \leq p_+ < \infty$ the space $W^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space, since $L^{p(\cdot)}$ has the same properties. Second, we introduce the subspace

$$W^{1,p(\cdot)}_{\Gamma_D}(\Omega) := \{ u \in W^{1,p(\cdot)}(\Omega) : \, u = 0 \text{ on } \Gamma_D \}.$$

Since we assume that Γ_D is of positive $(d-1)$-dimensional measure, this space can be equipped with the equivalent norm, as follows

$$C_1\|u\|_{1,p(\cdot)} \le \|\nabla u\|_{p(\cdot)} \le C_2\|u\|_{1,p(\cdot)}.$$

Indeed, we can use the facts that the classical Sobolev space $W^{1,p_-}_{\Gamma_D}(\Omega)$ satisfies the Poincaré inequality and that the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is continuously embedded into the Lebesgue space $L^{p_-}(\Omega)$ to obtain for arbitrary $u \in W^{1,p(\cdot)}_{\Gamma_D}(\Omega)$

$$\begin{aligned}\|u\|_{1,p(\cdot)} &= \|u\|_{p_-} + \|\nabla u\|_{p_-} + \|\nabla u\|_{p(\cdot)}\\ &\le c(\|\nabla u\|_{p_-} + \|\nabla u\|_{p(\cdot)}) \le c\|\nabla u\|_{p(\cdot)} \le c\|u\|_{1,p(\cdot)}.\end{aligned}$$

Furthermore, we denote by $H^1(\Omega)$ the usual Hilbert space, which by means of the assumption (iv) can be equipped with the equivalent norm

$$\|T\|_{1,2}^2 = \int_\Omega |\nabla T|^2 \, \mathrm{d}x + \int_{\partial\Omega} \kappa T^2 \, \mathrm{d}\Gamma.$$

Since we also work with entropy solutions, we define for arbitrary $k \ge 0$ the function β_k as

$$\beta_k(s) := \min(|s|, k)\, \mathrm{sgn}\, s. \tag{2.4}$$

Finally, we state the main result of this paper.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^d$ with $d \ge 2$ be a bounded open set with Lipschitz boundary. Assume that the data satisfy (i)–(vii).*

1. Then there exists $u \in W^{1,p(\cdot)}(\Omega)$ and $T \in W^{1,q}(\Omega)$ for all $1 \le q < d/(d-1)$, solving the problem (1.1) *in a weak sense and satisfying $T \ge T_\Gamma$ almost everywhere in Ω.*

2. For all $k \ge 0$ we have $\beta_k(T) \in H^1(\Omega)$ and T is an entropy solution to (1.1b)*, i.e., for all $w \in H^1(\Omega) \cap L^\infty(\Omega)$ and all $k \ge 0$ there holds*

$$\int_\Omega \mathbf{A}\nabla T \cdot \nabla \beta_k(T-w) \, \mathrm{d}x + \int_{\partial\Omega} \kappa(T-T_\Gamma)\beta_k(T-w) \, \mathrm{d}\Gamma \le \int_\Omega f(\cdot, T, \nabla u)\beta_k(T-w) \, \mathrm{d}x. \tag{2.5}$$

3 Proof of the main result

3.1 The regularized problem

In the following we will use the truncation at $\underline{T_0}$, which is defined for a function $x \mapsto T(x)$ via $T^{\min}(x) := \max\{T(x), \underline{T_0}\}$. Next, for arbitrary $\varepsilon > 0$ we define the regularized quantities

$$\mathbf{S}_\varepsilon(x, T, z) := \varepsilon|z|^{p(x)-2}z + \mathbf{S}(x, T^{\min}, z), \tag{3.1a}$$

$$f_\varepsilon(x, T, z) := \eta(x)\frac{\mathbf{S}(x, T^{\min}, z)\cdot z}{1 + \varepsilon|\mathbf{S}(x, T^{\min}, z)\cdot z|}. \tag{3.1b}$$

It is evident that $f_\varepsilon(x, T, z) \le \|\eta\|_{L^\infty}/\varepsilon$ and also that $\mathbf{S}_\varepsilon$ is strictly monotone, i.e., the first inequality in (2.1) holds with strict inequality sign whenever $z_1 \ne z_2$. With $\mathbf{S}_\varepsilon$ and f_ε we introduce the following regularized problem

$$-\operatorname{div} \mathbf{S}_\varepsilon\big(x, T(x), \nabla u(x)\big) = 0, \tag{3.2a}$$

$$-\operatorname{div}\big(\mathbf{A}(x)\nabla T(x)\big) = f_\varepsilon\big(x, T(x), u(x)\big), \tag{3.2b}$$

completed by the boundary conditions (1.2). For this problem we have the following result.

Lemma 3.1 *For each $\varepsilon > 0$, the regularized problem* (3.2) *and* (1.2) *has a weak solution* $(u_\varepsilon, T_\varepsilon)$ *with* $u_\varepsilon - u^D \in W^{1,p(\cdot)}_{\Gamma_D}(\Omega)$ *and* $T_\varepsilon \in H^1(\Omega)$*. Moreover,* T_ε *satisfies* $T_\varepsilon \geq T_\Gamma$ *almost everywhere on* Ω*.*

Proof. 1. Galerkin approximation. Due to the separability of $W^{1,p(\cdot)}_{\Gamma_D}(\Omega)$ and $H^1(\Omega)$, we can find families $\{\tilde{w}_i\}_{i=1}^\infty \subset W^{1,p(\cdot)}_{\Gamma_D}(\Omega)$ and $\{\tilde{\theta}_i\}_{i=1}^\infty \subset H^1(\Omega)$, which are dense in $W^{1,p(\cdot)}_{\Gamma_D}(\Omega)$ and $H^1(\Omega)$, respectively. Moreover, we can easily find linearly independent families $\{w_i\}_{i=1}^\infty$ and $\{\theta_i\}_{i=1}^\infty$, whose linear hull is dense in the corresponding spaces. Next, we introduce a Galerkin approximation for the regularized problem (3.2). Namely for each $n \in \mathbb{N}$ we look for (u_n, T_n) given as

$$u_n(x) := u^D(x) + \sum_{i=1}^n a_i^n w_i(x) \quad \text{and} \quad T_n(x) := \sum_{i=1}^n b_i^n \theta_i(x)$$

solving the following system of algebraic equations for the tuples $\boldsymbol{a}_n = (a_1^n, \dots, a_n^n) \in \mathbb{R}^n$ and $\boldsymbol{b}_n = (b_1^n, \dots, b_n^n) \in \mathbb{R}^n$ for all $i = 1, \dots, n$ (we omit writing the variable x)

$$g_i(\boldsymbol{a}_n, \boldsymbol{b}_n) := \int_\Omega \big(\varepsilon |\nabla u_n|^{p(\cdot)-2} \nabla u_n + \mathbf{S}(\cdot, T_n^{\min}, \nabla u_n)\big) \cdot \nabla w_i \, \mathrm{d}x = 0, \tag{3.3a}$$

$$h_i(\boldsymbol{a}_n, \boldsymbol{b}_n) := \int_\Omega \mathbf{A} \nabla T_n \cdot \nabla \theta_i - f_\varepsilon(\cdot, T_n^{\min}, \nabla u_n)\theta_i \, \mathrm{d}x + \int_{\partial\Omega} \kappa(T_n - T_\Gamma)\theta_i \, \mathrm{d}\Gamma = 0. \tag{3.3b}$$

2. Existence of approximate solutions. The existence of a solution $(\boldsymbol{a}_n, \boldsymbol{b}_n)$ of (3.3) is guaranteed by a Corollary of Brouwer's fixed point theorem. We argue as follows: On $\mathbb{R}^{2n}$ we use an equivalent norm related to the function spaces of the approximated solutions given via $|(\boldsymbol{a}, \boldsymbol{b})| := \max\{\|\nabla(u_n - u^D)\|_{p(\cdot)}, \|T_n\|_{1,2}\}$. Then, g_i and h_i are continuous functions of $(\boldsymbol{a}, \boldsymbol{b})$. Due to the growth condition in (2.2) and the estimates for the $p(\cdot)$-modular in [7, Lemma 3.2.5] we have for all $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{2n}$

$$\begin{aligned}
\sum_{i=1}^n g_i(\boldsymbol{a}, \boldsymbol{b}) a_i &= \int_\Omega \big(\varepsilon |\nabla u_n|^{p(\cdot)-2} \nabla u_n + \mathbf{S}(\cdot, T_n^{\min}, \nabla u_n)\big) \cdot \nabla(u_n - u^D) \, \mathrm{d}x \\
&\geq \int_\Omega (\varepsilon + \sigma_1) |\nabla u_n|^{p(\cdot)} - \sigma_2 - \varepsilon |\nabla u_n|^{p(\cdot)-1} |\nabla u^D| - \sigma_3 (1 + |\nabla u_n|)^{p(\cdot)-1} |\nabla u^D| \, \mathrm{d}x \\
&\geq c_1 \int_\Omega |\nabla(u_n - u^D)|^{p(x)} \, \mathrm{d}x - c_2 \\
&\geq c_1 \min\{\|\nabla(u_n - u^D)\|_{p(\cdot)}^{p_-}, \|\nabla(u_n - u^D)\|_{p(\cdot)}^{p_+}\} - c_2,
\end{aligned}$$

where we used (2.3) for the last inequality. For (3.3b) we proceed similar and obtain

$$\begin{aligned}
\sum_{i=1}^n h_i(\boldsymbol{a}, \boldsymbol{b}) b_i &= \int_\Omega \mathbf{A} \nabla T_n \cdot \nabla T_n \, \mathrm{d}x + \int_{\partial\Omega} \kappa(T_n - T_\Gamma) T_n \, \mathrm{d}\Gamma - \int_\Omega f_\varepsilon(\cdot, T_n^{\min}, \nabla u_n) T_n \, \mathrm{d}x \\
&\geq c \|T_n\|_{1,2}^2 - \big(\tilde{c} + \tilde{c}(\varepsilon)\big) \|T_n\|_{1,2} \geq c_3 \|T_n\|_{1,2}^2 - c_4.
\end{aligned}$$

Hence, adding both inequalities we can find a constant $R > 0$ such that

$$\sum_{i=1}^{n} \big(g_i(\boldsymbol{a},\boldsymbol{b})a_i + h_i(\boldsymbol{a},\boldsymbol{b})b_i\big) \geq 0 \quad \text{for all } (\boldsymbol{a},\boldsymbol{b}) \text{ with } |(\boldsymbol{a},\boldsymbol{b})| = R.$$

Then according to Lemma 2.26 in [13, Chap. 1] there exists $(\boldsymbol{a}_n, \boldsymbol{b}_n)$ solving the Galerkin approximation scheme.

3. Uniform estimates. In this step, we derive estimates for the approximate solutions (u_n, T_n) which are independent of n. In what follows, we will denote by $C > 0$ some generic constant depending only on the data of the problem. If it depends on ε, it will be explicitly mentioned.

Multiplying (3.3a) by a_i^n and summing over $i = 1, \ldots, n$ we get the identity.

$$\begin{aligned} &\int_\Omega \big(\varepsilon|\nabla u_n|^{p(\cdot)-2}\nabla u_n + \mathbf{S}(\cdot, T_n^{\min}, \nabla u_n)\big) \cdot \nabla u_n \,\mathrm{d}x \\ &= \int_\Omega \big(\varepsilon|\nabla u_n|^{p(\cdot)-2}\nabla u_n + \mathbf{S}(\cdot, T_n^{\min}, \nabla u_n)\big) \cdot \nabla u^D \,\mathrm{d}x. \end{aligned} \tag{3.4}$$

For ε sufficiently small, we can use the assumption (2.2) and the Young inequality to obtain the estimate

$$\int_\Omega |\nabla u_n(x)|^{p(x)} \,\mathrm{d}x \leq C(\Omega, \sigma_1, \sigma_2, \sigma_3) \left(1 + \int_\Omega |\nabla u^D(x)|^{p(x)} \,\mathrm{d}x\right) \leq C. \tag{3.5}$$

Similarly, multiplying (3.3b) by b_i^n and summing over $i = 1, \ldots, n$ we get

$$\int_\Omega \mathbf{A}\nabla T_n \cdot \nabla T_n \,\mathrm{d}x + \int_{\partial\Omega} \kappa(T_n - T_\Gamma)T_n \,\mathrm{d}\Gamma = \int_\Omega \frac{\eta\mathbf{S}(\cdot, T_n^{\min}, \nabla u_n) \cdot \nabla u_n}{1 + \varepsilon|\mathbf{S}(\cdot, T_n^{\min}, \nabla u_n) \cdot \nabla u_n|} T_n \,\mathrm{d}x. \tag{3.6}$$

Hence, using the assumptions on $\mathbf{A}$, κ, and η, the Young inequality and also the regularization of f on the right hand side, we arrive at the estimate

$$\begin{aligned} \int_\Omega |\nabla T_n|^2 \,\mathrm{d}x + \int_{\partial\Omega} \kappa|T_n|^2 \,\mathrm{d}\Gamma &\leq C(\varepsilon, \alpha_1, \|\eta\|_\infty) \left(\int_\Omega |T_n| \,\mathrm{d}x + \int_{\partial\Omega} \kappa|T_\Gamma|^2 \,\mathrm{d}\Gamma\right) \\ &\leq C(\varepsilon). \end{aligned} \tag{3.7}$$

Next, it follows from the estimate in (3.5) that (using the equivalence of norms for functions having zero trace on Γ_D)

$$\begin{aligned} \|u_n\|_{1,p(\cdot)} &\leq \|u_n - u^D\|_{1,p(\cdot)} + \|u^D\|_{1,p(\cdot)} \leq C\|\nabla(u_n - u^D)\|_{p(\cdot)} + \|u^D\|_{1,p(\cdot)} \\ &\leq C(\|\nabla u_n\|_{p(\cdot)} + \|u^D\|_{1,p(\cdot)}) \leq C. \end{aligned} \tag{3.8}$$

Note, that from the assumption (2.2) we immediately obtain that

$$\big\|\varepsilon|\nabla u_n|^{p(\cdot)-2}\nabla u_n + \mathbf{S}(\cdot, T_n^{\min}, \nabla u_n)\big\|_{p'(\cdot)} \leq C, \tag{3.9}$$

where p' denotes the dual exponent to p, i.e., $1/p(x) + 1/p'(x) = 1$ for almost all $x \in \Omega$. Similarly, since κ is positive on a set of positive measure, it follows from (3.7) that

$$\|T_n\|_{1,2} \leq C\big(\|\nabla T_n\|_2 + \|\sqrt{\kappa}T_n\|_{L^2(\partial\Omega)}\big) \leq C(\varepsilon). \tag{3.10}$$

4. Limit passage $n \to \infty$. Using the reflexivity of all spaces we can find a subsequence that we do not relabel such that

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p(\cdot)}(\Omega) \quad \text{and} \quad T_n \rightharpoonup T \quad \text{weakly in } H^1(\Omega), \tag{3.11a}$$

Moreover, we find limits $\overline{\mathbf{S}}_\varepsilon \in L^{p'(\cdot)}(\Omega;\mathbb{R}^d)$ and $\overline{f}_\varepsilon \in L^\infty(\Omega)$ (note that ε is still fixed) such that

$$\mathbf{S}_\varepsilon(\cdot, T_n^{\min}, \nabla u_n) \rightharpoonup \overline{\mathbf{S}}_\varepsilon \qquad \text{weakly in } L^{p'(\cdot)}(\Omega;\mathbb{R}^d), \tag{3.11b}$$

$$f_\varepsilon(\cdot, T_n^{\min}, \nabla u_n) \rightharpoonup^* \overline{f}_\varepsilon \qquad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{3.11c}$$

Having these convergence results, we can pass to the limit $n \to \infty$ in (3.3a)–(3.3b) and since the linear hull of $\{\theta_i\}_{i=1}^\infty$ and $\{w_i\}_{i=1}^\infty$ is dense in the corresponding spaces we obtain

$$\int_\Omega \overline{\mathbf{S}}_\varepsilon \cdot \nabla w \,\mathrm{d}x = 0 \qquad \forall\, w \in W^{1,p(\cdot)}_{\Gamma_D}(\Omega), \tag{3.12a}$$

$$\int_\Omega \mathbf{A}\nabla T \cdot \nabla\theta \,\mathrm{d}x + \int_{\partial\Omega} \kappa(T - T_\Gamma)\theta \,\mathrm{d}\Gamma = \int_\Omega \overline{f}_\varepsilon \theta \,\mathrm{d}x \qquad \forall\, \theta \in H^1(\Omega). \tag{3.12b}$$

5. Identification of the limits. Finally, we show that almost everywhere in Ω we have the identities

$$\overline{\mathbf{S}}_\varepsilon = \mathbf{S}_\varepsilon(\cdot, T, \nabla u) \quad \text{and} \quad \overline{f}_\varepsilon = f_\varepsilon(\cdot, T, \nabla u), \tag{3.13}$$

from which the statement of the proposition follows.

First, we show the lower bound $T \geq T_\Gamma$ almost everywhere in Ω. Indeed, due to the nonnegativity $f_\varepsilon(x, T_n^{\min}(x), \nabla u_n(x)) \geq 0$ for almost all $x \in \Omega$, we also get that $\overline{f}_\varepsilon \geq 0$ almost everywhere in Ω. Thus, setting $\theta = \min\{0, (T - T_\Gamma)\} := (T - T_\Gamma)_-$, we obtain

$$\int_\Omega \mathbf{A}\nabla(T - T_\Gamma)_- \cdot \nabla(T - T_\Gamma)_- \,\mathrm{d}x + \int_{\partial\Omega} \kappa(T - T_\Gamma)_-(T - T_\Gamma)_- \,\mathrm{d}\Gamma \leq 0.$$

Thus, due to the ellipticity of $\mathbf{A}$ and the equivalence of norms, we deduce that $(T - T_\Gamma)_- \equiv 0$, which in turn implies $T \geq T_\Gamma$ almost everywhere in Ω. Thus, it follows from the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, the assumption $T_\Gamma \geq \underline{T_0}$ and the definition of $T_n^{\min}$ that (for a subsequence)

$$T_n^{\min} \to T \quad \text{strongly in } L^2(\Omega). \tag{3.14}$$

Next, setting $w := u - u^D \in W^{1,p(\cdot)}_{\Gamma_D}(\Omega)$ in (3.12a) yields the identity

$$\int_\Omega \overline{\mathbf{S}}_\varepsilon \cdot \nabla u \,\mathrm{d}x = \int_\Omega \overline{\mathbf{S}}_\varepsilon \cdot \nabla u^D \,\mathrm{d}x. \tag{3.15}$$

Then, letting $n \to \infty$ in (3.4) and using the convergence in (3.11b) we deduce

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \mathbf{S}_\varepsilon(\cdot, T_n^{\min}, \nabla u_n) \cdot \nabla u_n \,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega \mathbf{S}_\varepsilon(\cdot, T_n^{\min}, \nabla u_n) \cdot \nabla u^D \,\mathrm{d}x \\ &= \int_\Omega \overline{\mathbf{S}}_\varepsilon \cdot \nabla u^D \,\mathrm{d}x. \end{aligned} \tag{3.16}$$

Thus, comparing both identities in (3.15) and (3.16) we observe that

$$\lim_{n\to\infty}\int_\Omega \mathbf{S}_\varepsilon(\cdot,T_n^{\min},\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x=\int_\Omega \overline{\mathbf{S}}_\varepsilon\cdot\nabla u\,\mathrm{d}x. \tag{3.17}$$

Finally, using the monotonicity of the $p(\cdot)$-Laplacian and the assumption (2.1), we arrive at the point-wise estimate

$$\begin{aligned}0&\le\varepsilon\big(|\nabla u_n|^{p(\cdot)-2}\nabla u_n-|\nabla u|^{p(\cdot)-2}\nabla u\big)\cdot\nabla(u_n-u)\\&\le\big(\mathbf{S}_\varepsilon(\cdot,T_n^{\min},\nabla u_n)-\mathbf{S}_\varepsilon(\cdot,T_n^{\min},\nabla u)\big)\cdot\nabla(u_n-u).\end{aligned}$$

Integrating this inequality over Ω and using that u_n solves (3.3a), we deduce that

$$\begin{aligned}0&\le\lim_{n\to\infty}\varepsilon\int_\Omega\big|\big(|\nabla u_n|^{p(\cdot)-2}\nabla u_n-|\nabla u|^{p(\cdot)-2}\nabla u\big)\cdot\nabla(u_n-u)\big|\,\mathrm{d}x\\&\le\lim_{n\to\infty}\int_\Omega\mathbf{S}_\varepsilon(\cdot,T_n^{\min},\nabla u)\cdot\nabla(u-u_n)\,\mathrm{d}x\\&=\lim_{n\to\infty}\int_\Omega\mathbf{S}_\varepsilon(\cdot,T,\nabla u)\cdot\nabla(u-u_n)\,\mathrm{d}x\\&\quad+\lim_{n\to\infty}\int_\Omega\big(\mathbf{S}_\varepsilon(\cdot,T_n^{\min},\nabla u)-\mathbf{S}_\varepsilon(\cdot,T,\nabla u)\big)\cdot\nabla(u-u_n)\,\mathrm{d}x.\end{aligned}\tag{3.18}$$

Due to the assumptions on the function $\mathbf{S}$, namely (2.2), we have that $\mathbf{S}_\varepsilon(\cdot,T,\nabla u)\in L^{p'(\cdot)}(\Omega;\mathbb{R}^d)$. Thus, it follows from (3.11a) that

$$\lim_{n\to\infty}\int_\Omega\mathbf{S}_\varepsilon(\cdot,T,\nabla u)\cdot\nabla(u-u_n)\,\mathrm{d}x=0.$$

Moreover, using the strong convergence of T_n in (3.14) and the continuity of $\mathbf{S}$ we have (for a subsequence) that $\mathbf{S}_\varepsilon(x,T_n^{\min}(x),\nabla u(x))$ converges to $\mathbf{S}_\varepsilon(x,T(x),\nabla u(x))$ for almost all $x\in\Omega$. From the growth condition in (2.2) we deduce

$$\big|\mathbf{S}_\varepsilon\big(x,T_n^{\min}(x),\nabla u(x)\big)-\mathbf{S}_\varepsilon\big(x,T(x),\nabla u(x)\big)\big|^{p'(x)}\le C\big(1+|\nabla u(x)|^{p(x)}\big),$$

where the right hand side is in $L^1(\Omega)$. Therefore, we can apply the Lebesgue dominated convergence theorem to conclude

$$\lim_{n\to\infty}\int_\Omega|\mathbf{S}_\varepsilon(\cdot,T_n^{\min},\nabla u)-\mathbf{S}_\varepsilon(\cdot,T,\nabla u)|^{p'(\cdot)}\,\mathrm{d}x=0.$$

Hence, by using also (3.11a) it is evident that the integral on the last line of (3.18) vanishes, and we obtain

$$\lim_{n\to\infty}\varepsilon\int_\Omega\big|\big(|\nabla u_n|^{p(\cdot)-2}\nabla u_n-|\nabla u|^{p(\cdot)-2}\nabla u\big)\cdot\nabla(u_n-u)\big|\,\mathrm{d}x=0,$$

which due to the strict monotonicity leads to $\nabla u_n\to\nabla u$ a.e. in Ω. Combination of this result with (3.14) immediately gives the desired identities in (3.13).

$\square$

3.2 Limit of vanishing regularization parameter ε

Lemma 3.2 *For $\varepsilon > 0$ sufficiently small the weak solutions $(u_\varepsilon, T_\varepsilon)$ to the regularized problem* (3.2) *satisfy for all $k \geq 0$ and all $q \in [1, d/(d-1))$ the following uniform bounds*

$$\|u_\varepsilon\|_{1,p(\cdot)} \leq C, \quad \|\beta_k(T_\varepsilon)\|_{1,2} \leq C(k) \quad \textit{and} \quad \|T_\varepsilon\|_{1,q} \leq C(q),$$

where β_k is defined in (2.4)*. Moreover, we have that*

$$\|\mathbf{S}_\varepsilon(\cdot, T_\varepsilon, \nabla u_\varepsilon)\|_{p'(\cdot)} \leq C \quad \textit{and} \quad \|f_\varepsilon(\cdot, T_\varepsilon, \nabla u_\varepsilon)\|_1 \leq C.$$

Proof. For brevity, we introduce $\overline{\mathbf{S}}_\varepsilon \in L^{p'(\cdot)}(\Omega; \mathbb{R}^d)$, where $1/p(x) + 1/p'(x) = 1$, by setting

$$\overline{\mathbf{S}}_\varepsilon(x) := \varepsilon|\nabla u_\varepsilon(x)|^{p(x)-2}\nabla u_\varepsilon(x) + \mathbf{S}\big(x, T_\varepsilon(x), \nabla u_\varepsilon(x)\big).$$

Repeating the same procedure as in the preceding subsection, we obtain the uniform estimate

$$\big\|\overline{\mathbf{S}}_\varepsilon\big\|_{p'(\cdot)} + \|u_\varepsilon\|_{1,p(\cdot)} \leq C. \tag{3.19}$$

From this it follows for $\overline{f}_\varepsilon(x) := f_\varepsilon(x, T_\varepsilon(x), \nabla u_\varepsilon(x))$ that

$$\int_\Omega |\overline{f}_\varepsilon| \, \mathrm{d}x \leq C. \tag{3.20}$$

Thus, taking the test function $\theta \equiv 1$ in the weak formulation of (3.2b) leads to

$$\int_{\partial\Omega} \kappa(T_\varepsilon - T_\Gamma) \, \mathrm{d}\Gamma = \int_\Omega \overline{f}_\varepsilon \, \mathrm{d}x \leq C$$

and consequently, since $T_\varepsilon \geq T_\Gamma$ we have

$$\|\kappa T_\varepsilon\|_{L^1(\partial\Omega)} \leq C. \tag{3.21}$$

Next, we consider the test function $\theta := T_\varepsilon^{-\lambda}$, where $\lambda \in (0,1)$ is arbitrary. Employing θ in the weak formulation of (3.2b) and using the fact that $T_\varepsilon \geq T_\Gamma > 0$ we get

$$\int_\Omega \frac{\overline{f}_\varepsilon}{T_\varepsilon^\lambda} + \lambda \frac{\mathbf{A}\nabla T_\varepsilon \cdot \nabla T_\varepsilon}{T_\varepsilon^{\lambda+1}} \, \mathrm{d}x = \int_{\partial\Omega} \frac{\kappa(T_\varepsilon - T_\Gamma)}{T_\varepsilon^\lambda} \, \mathrm{d}\Gamma \leq \frac{1}{T_\Gamma^\lambda} \int_{\partial\Omega} \kappa T_\varepsilon \, \mathrm{d}\Gamma \leq C,$$

where the last estimate follows from (3.21).

Thus, using the assumptions on $\mathbf{A}$ and the nonnegativity of $\overline{f}_\varepsilon$ we get from the above estimate (recall that $\lambda \in (0,1)$)

$$\int_\Omega \big|\nabla\big(T_\varepsilon^{\frac{1-\lambda}{2}}\big)\big|^2 \, \mathrm{d}x = \frac{(1-\lambda)^2}{4} \int_\Omega \frac{|\nabla T_\varepsilon|^2}{T_\varepsilon^{1+\lambda}} \, \mathrm{d}x \leq C \int_\Omega \frac{\mathbf{A}\nabla T_\varepsilon \cdot \nabla T_\varepsilon}{T_\varepsilon^{1+\lambda}} \, \mathrm{d}x \leq \frac{C}{\lambda}. \tag{3.22}$$

Hence, together with the estimate for the boundary term in (3.21) we deduce

$$\big\|T_\varepsilon^{\frac{1-\lambda}{2}}\big\|_{1,2}^2 \leq C\Big(\big\|\nabla\big(T_\varepsilon^{\frac{1-\lambda}{2}}\big)\big\|_2^2 + \big\|\sqrt{\kappa}T_\varepsilon^{\frac{1-\lambda}{2}}\big\|_{L^2(\partial\Omega)}^2\Big) \leq \frac{C}{\lambda}. \tag{3.23}$$

Consequently, using the Sobolev embedding theorem $H^1(\Omega) \subset L^{2d/(d-2)}(\Omega)$ and (3.23) with $\lambda = 1 - r(d-2)/d$, we get for arbitrary $r < \frac{d}{d-2}$ that

$$\|T_\varepsilon\|_r^r = \left\| T_\varepsilon^{\frac{r(d-2)}{2d}} \right\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} \leq C \left\| T_\varepsilon^{\frac{1-(1-\frac{r(d-2)}{d})}{2}} \right\|_{1,2}^{\frac{2d}{d-2}} \leq C(r) \qquad \text{for all } r \in \Big[1, \frac{d}{d-2}\Big). \tag{3.24}$$

Note that in the case $d = 2$ we can choose any finite r.

Finally, we combine this estimate with (3.22) and exploit Hölder's inequality to obtain for all $1 < q < 2$

$$\begin{aligned} \int_\Omega |\nabla T_\varepsilon|^q \,\mathrm{d}x &= \int_\Omega \left(\frac{|\nabla T_\varepsilon|^2}{T_\varepsilon^{1+\lambda}} \right)^{\frac{q}{2}} T_\varepsilon^{\frac{q(1+\lambda)}{2}} \,\mathrm{d}x \\ &\leq \left(\int_\Omega \frac{|\nabla T_\varepsilon|^2}{T_\varepsilon^{1+\lambda}} \,\mathrm{d}x \right)^{\frac{q}{2}} \left(\int_\Omega T_\varepsilon^{\frac{q(1+\lambda)}{2-q}} \,\mathrm{d}x \right)^{\frac{2-q}{2}} \leq C(\lambda), \end{aligned} \tag{3.25}$$

which directly gives the result for $d = 2$ and ensures a bound uniform with respect to ε in the case $d > 2$, provided that we are able to find $\lambda \in (0,1)$ such that $\frac{q(1+\lambda)}{2-q} < \frac{d}{d-2}$. However, this is possible if $q < \frac{d}{d-1}$, and (3.25) reduces to

$$\int_\Omega |\nabla T_\varepsilon|^q \,\mathrm{d}x \leq C(q) \qquad \text{for all } q \in \Big[1, \frac{d}{d-1}\Big). \tag{3.26}$$

Moreover, it directly follows from (3.23) and the fact $T_\varepsilon \geq T_\Gamma > 0$ that for all $k \geq 0$ and β_k as in (2.4) we have

$$\|\beta_k(T_\varepsilon)\|_{1,2} \leq C(k), \tag{3.27}$$

which finishes the proof. □

Proof. [Proof of the main result.] (1) By Lemma 3.2 and the reflexivity of corresponding spaces we can extract a subsequence, which we denote again by ε, such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,p(\cdot)}(\Omega) \quad \text{and} \tag{3.28}$$

$$T_\varepsilon \rightharpoonup T \quad \text{weakly in } W^{1,q}(\Omega) \text{ for all } q \in \Big[1, \frac{d}{d-1}\Big). \tag{3.29}$$

In particular, due to the compact embedding we also have that $T_\varepsilon \to T$ strongly in $L^r(\Omega)$ for all $r \in [1, \frac{d}{d-2})$, and, by possibly passing to a further non-relabeled subsequence, we can assume that $T_\varepsilon(x) \to T(x)$ for almost all $x \in \Omega$. Consequently, by (3.27) and $\|\beta_k(T_\varepsilon)\|_\infty \leq C(k)$ we can also deduce that

$$\beta_k(T_\varepsilon) \rightharpoonup \beta_k(T) \quad \text{weakly in } H^1(\Omega) \text{ for all } k \geq 0, \tag{3.30}$$

$$\beta_k(T_\varepsilon) \to \beta_k(T) \quad \text{strongly in } L^q(\Omega) \text{ for all } k \geq 0 \text{ and all } q \in [1, \infty). \tag{3.31}$$

Moreover, using the uniform boundedness of $\overline{\mathbf{S}}_\varepsilon := \mathbf{S}_\varepsilon(\cdot, T_\varepsilon, \nabla u_\varepsilon)$ in $L^{p(\cdot)}(\Omega; \mathbb{R}^d)$ and $\overline{f}_\varepsilon := f_\varepsilon(\cdot, T_\varepsilon, \nabla u_\varepsilon)$ in $L^1(\Omega)$ we get a further subsequence, that we do not relabel, such that

$$\overline{\mathbf{S}}_\varepsilon \rightharpoonup \overline{\mathbf{S}} \text{ weakly in } L^{p'(\cdot)}(\Omega; \mathbb{R}^d) \quad \text{and} \quad \overline{f}_\varepsilon \rightharpoonup^* \overline{F} \text{ weakly}^* \text{ in } \mathcal{M}(\overline{\Omega}), \tag{3.32}$$

where $\mathcal{M}(\overline{\Omega})$ is the space of Radon measures and weak*-convergence means

$$\forall\, \theta \in C(\overline{\Omega}) : \quad \lim_{\varepsilon\to 0} \int_\Omega \overline{f}_\varepsilon \theta \,\mathrm{d}x = \langle \overline{F}, \theta\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})}.$$

Having these convergence results, it is straightforward to pass to the limit $\varepsilon \to 0$ in the weak formulation of the system in (3.2). Noting that $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ for all $q > d$ we obtain

$$\begin{aligned}
\int_\Omega \overline{\mathbf{S}} \cdot \nabla w \;\,\mathrm{d}x = 0 \qquad\qquad & \forall\, w \in W^{1,p(\cdot)}_{\Gamma_D}(\Omega),\\
\int_\Omega \mathbf{A}\nabla T \cdot \nabla\theta \,\mathrm{d}x + \int_{\partial\Omega} \kappa(T - T_\Gamma)\theta \,\mathrm{d}\Gamma = \langle \overline{F}, \theta\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \quad & \forall\, \theta \in W^{1,q}(\Omega) \text{ with } q > d.
\end{aligned}$$

Similarly to the previous subsection, our goal is to show that $\overline{F}$ is absolutely continuous with respect to the Lebesgue measure with density $\overline{f}$ and that almost everywhere in Ω the following identities are satisfied

$$\overline{\mathbf{S}} = \mathbf{S}(\cdot, T, \nabla u), \quad \text{and} \quad \overline{f} = f(\cdot, T, \nabla u), \tag{3.33}$$

where f is given in (1.3).

Exactly in the same way as in the previous subsection (comp. with arguments leading to (3.17) in the proof of Lemma 3.1), we can deduce that

$$\lim_{\varepsilon\to 0} \int_\Omega \mathbf{S}_\varepsilon(\cdot, T_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \,\mathrm{d}x = \int_\Omega \overline{\mathbf{S}} \cdot \nabla u \,\mathrm{d}x. \tag{3.34}$$

Moreover, using the monotonicity condition for $\mathbf{S}$ in (2.1), we obtain for an arbitrary $Z \in L^{p(\cdot)}(\Omega; \mathbb{R}^d)$ and almost all $x \in \Omega$

$$\begin{aligned}
0 &\le \big(\mathbf{S}(x, T_\varepsilon(x), \nabla u_\varepsilon(x)) - \mathbf{S}(x, T_\varepsilon(x), Z(x))\big) \cdot \big(\nabla u_\varepsilon(x) - Z(x)\big)\\
&= \big(\mathbf{S}_\varepsilon(x, T_\varepsilon(x), \nabla u_\varepsilon(x)) - \mathbf{S}(x, T(x), Z(x))\big) \cdot \big(\nabla u_\varepsilon(x) - Z(x)\big)\\
&\quad - \varepsilon|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \cdot \big(\nabla u_\varepsilon(x) - Z(x)\big)\\
&\quad + \big(\mathbf{S}(x, T(x), Z(x)) - \mathbf{S}(x, T_\varepsilon(x), Z(x))\big) \cdot \big(\nabla u_\varepsilon(x) - Z(x)\big).
\end{aligned} \tag{3.35}$$

We will estimate all terms on the right hand side. First, we consider the term on the third line of (3.35). Using the fact that $Z \in L^{p(\cdot)}(\Omega; \mathbb{R}^d)$, the uniform boundedness of u_ε in $W^{1,p(\cdot)}(\Omega)$ and Hölder's inequality we deduce

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_\Omega -\varepsilon|\nabla u_\varepsilon|^{p(\cdot)-2}\nabla u_\varepsilon \cdot (\nabla u_\varepsilon - Z) \,\mathrm{d}x\\
&\quad \le \lim_{\varepsilon\to 0} \int_\Omega \varepsilon|\nabla u_\varepsilon|^{p(\cdot)-2}\nabla u_\varepsilon \cdot Z \,\mathrm{d}x \le \lim_{\varepsilon\to 0} C\varepsilon = 0.
\end{aligned} \tag{3.36}$$

Next, for the term on the last line in (3.35), we exploit that T_ε converges almost everywhere in Ω to infer

$$\mathbf{S}(x, T_\varepsilon(x), Z(x)) \to \mathbf{S}(x, T(x), Z(x)) \text{ a.e. in } \Omega.$$

Moreover, using the growth condition for $\mathbf{S}$ in (2.2), we obtain

$$\big|\mathbf{S}(x, T_\varepsilon(x), Z(x)) - \mathbf{S}(x, T(x), Z(x))\big|^{p'(x)} \le C(1 + |Z(x)|^{p(x)}),$$

where the right hand side is in $L^1(\Omega)$. Therefore, it follows from Lebesgue's dominated convergence theorem that

$$\lim_{\varepsilon\to 0}\int_\Omega |\mathbf{S}(\cdot,T_\varepsilon,Z)-\mathbf{S}(\cdot,T,Z)|^{p'(\cdot)}\,\mathrm{d}x = 0. \tag{3.37}$$

Consequently, using that ∇u_ε is uniformly bounded and applying Hölder's inequality again leads to

$$\lim_{\varepsilon\to 0}\int_\Omega \big(\mathbf{S}(\cdot,T,Z)-\mathbf{S}(\cdot,T_\varepsilon,Z)\big)\cdot(\nabla u_\varepsilon - Z)\,\mathrm{d}x = 0. \tag{3.38}$$

Finally, we consider the term on the second line in (3.35). The weak convergence of u_ε in $W^{1,p(\cdot)}(\Omega)$ and of $\mathbf{S}_\varepsilon(\cdot,T_\varepsilon,\nabla u_\varepsilon)$ in $L^{p'(\cdot)}(\Omega;\mathbb{R}^d)$ (see (3.28) and (3.32)), as well as (3.34) yield

$$\lim_{\varepsilon\to 0}\int_\Omega \big(\mathbf{S}_\varepsilon(\cdot,T_\varepsilon,\nabla u_\varepsilon)-\mathbf{S}(\cdot,T,Z)\big)\cdot(\nabla u_\varepsilon - Z)\,\mathrm{d}x = \int_\Omega \big(\overline{\mathbf{S}}-\mathbf{S}(\cdot,T,Z)\big)\cdot(\nabla u - Z)\,\mathrm{d}x. \tag{3.39}$$

Hence, integrating the inequality (3.35) over Ω and exploiting the convergences in (3.36), (3.38) and (3.39), we obtain for all $Z\in L^{p(\cdot)}(\Omega;\mathbb{R}^d)$

$$\begin{aligned} 0 &\le \limsup_{\varepsilon\to 0}\int_\Omega \big|\big(\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)-\mathbf{S}(\cdot,T_\varepsilon,Z)\big)\cdot(\nabla u_\varepsilon - Z)\big|\,\mathrm{d}x \\ &\le \int_\Omega \big(\overline{\mathbf{S}}-\mathbf{S}(\cdot,T,Z)\big)\cdot(\nabla u - Z)\,\mathrm{d}x. \end{aligned} \tag{3.40}$$

Since this estimate holds for all $Z\in L^{p(\cdot)}(\Omega;\mathbb{R}^d)$ we can choose $Z:=\nabla u \pm hW$ with an arbitrary $W\in L^{p(\cdot)}(\Omega;\mathbb{R}^d)$ and $h>0$. Dividing by h and using the continuity of $\mathbf{S}$ as well as (2.2) we obtain, after passing to the limit $h\to 0$

$$\int_\Omega \big(\overline{\mathbf{S}}-\mathbf{S}(\cdot,T,\nabla u)\big)\cdot W\,\mathrm{d}x = 0.$$

Since W was arbitrary, this proves the first identity in (3.33), i.e., $\overline{\mathbf{S}}=\mathbf{S}(\cdot,T,\nabla u)$.

In order to prove the second identity in (3.33), namely $\overline{F}=f(\cdot,T,\nabla u)\mathcal{L}^d|_\Omega$, we first set $Z:=\nabla u$ in the estimate (3.40) to obtain

$$\limsup_{\varepsilon\to 0}\int_\Omega \big|\big(\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)-\mathbf{S}(\cdot,T_\varepsilon,\nabla u)\big)\cdot\nabla(u_\varepsilon - u)\big|\,\mathrm{d}x = 0,$$

which in particular implies

$$\big(\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)-\mathbf{S}(\cdot,T_\varepsilon,\nabla u)\big)\cdot\nabla(u_\varepsilon - u)\to 0 \qquad \text{strongly in } L^1(\Omega). \tag{3.41}$$

Next, it follows from the weak convergence of u_ε in $W^{1,p(\cdot)}(\Omega)$ in (3.28) and the strong convergence of $\mathbf{S}(\cdot,T_\varepsilon,\nabla u)$ in $L^{p'(\cdot)}(\Omega;\mathbb{R}^d)$ (see (3.37) with $Z:=\nabla u$) that $\mathbf{S}(\cdot,T_\varepsilon,\nabla u))\cdot\nabla(u_\varepsilon - u)\rightharpoonup 0$ weakly in $L^1(\Omega)$. Therefore, it follows from (3.41) that

$$\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(u_\varepsilon - u)\rightharpoonup 0 \qquad \text{weakly in } L^1(\Omega). \tag{3.42}$$

Moreover, using the above established weak convergence of $\mathbf{S}_\varepsilon(\cdot,T_\varepsilon,\nabla u_\varepsilon)$ to $\mathbf{S}(\cdot,T,\nabla u)$ in the space $L^{p'(\cdot)}(\Omega;\mathbb{R}^d)$, we also deduce that $\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u$ converges weakly to $\mathbf{S}(\cdot,T,\nabla u)\cdot\nabla u$ in $L^1(\Omega)$. In particular, using this in (3.42) we have the crucial convergence

$$\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon \rightharpoonup \mathbf{S}(\cdot,T,\nabla u)\cdot\nabla u \quad \text{weakly in } L^1(\Omega). \tag{3.43}$$

In addition, using the characterization of weakly convergent sequences in L^1, we know that for all $\rho > 0$ there exists $\delta > 0$ such that for all measurable $U \subset \Omega$ fulfilling $\mathrm{mes}(U) \leq \delta$ and all ε, we have

$$\int_U \left|\mathbf{S}(\cdot, T_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon\right| \,\mathrm{d}x \leq \rho. \tag{3.44}$$

Now we have prepared everything to prove the second identity in (3.33). Let $\theta \in L^\infty(\Omega)$ be arbitrary. Using the definition of f_ε (see (3.1b)) and abbreviating $\widehat{\mathbf{S}}_\varepsilon := \mathbf{S}(\cdot, T_\varepsilon, \nabla u_\varepsilon)$, we have that

$$\int_\Omega f_\varepsilon(\cdot, T_\varepsilon, \nabla u_\varepsilon)\theta \,\mathrm{d}x = \int_\Omega \eta\theta\, \widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon \,\mathrm{d}x - \int_\Omega \eta\theta\, \widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon \frac{\varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|}{1 + \varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|} \,\mathrm{d}x. \tag{3.45}$$

Considering the first term on the right hand side, it directly follows from (3.43) that

$$\lim_{\varepsilon\to 0} \int_\Omega \eta\theta \mathbf{S}(\cdot, T_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \,\mathrm{d}x = \int_\Omega \eta\theta \mathbf{S}(\cdot, T, \nabla u) \cdot \nabla u \,\mathrm{d}x. \tag{3.46}$$

For the second term we consider an arbitrary $\lambda > 0$ and introduce the sets

$$U_\varepsilon^\lambda := \{x \in \Omega : |\nabla u_\varepsilon(x)| \geq \lambda\} \quad \text{and} \quad V_\varepsilon^\lambda := \{x \in \Omega : |\nabla u_\varepsilon(x)| \leq \lambda\}.$$

We estimate as follows

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \left| \int_\Omega \eta\theta\, \widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon \frac{\varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|}{1 + \varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|} \,\mathrm{d}x \right| \leq C \lim_{\varepsilon\to 0} \int_\Omega |\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon| \frac{\varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|}{1 + \varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|} \,\mathrm{d}x \\
&\leq C \lim_{\varepsilon\to 0} \int_{U_\varepsilon^\lambda} |\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon| \frac{\varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|}{1 + \varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|} \,\mathrm{d}x + C \lim_{\varepsilon\to 0} \int_{V_\varepsilon^\lambda} |\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon| \frac{\varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|}{1 + \varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|} \,\mathrm{d}x \\
&\leq C \lim_{\varepsilon\to 0} \int_{U_\varepsilon^\lambda} |\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon| \,\mathrm{d}x + C \lim_{\varepsilon\to 0} \varepsilon \int_\Omega (1+\lambda)^{2p(\cdot)} \,\mathrm{d}x \\
&= C \lim_{\varepsilon\to 0} \int_{U_\varepsilon^\lambda} |\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon| \,\mathrm{d}x.
\end{aligned} \tag{3.47}$$

Using the uniform boundedness of u_ε in $W^{1,p(\cdot)}(\Omega)$ we have that $\mathrm{mes}(U_\varepsilon^\lambda) \leq \frac{C}{\lambda}$. Consequently, for any $\rho > 0$ we can find $\lambda > 0$ such that $\mathrm{mes}(\{x \in \Omega : |\nabla u_\varepsilon(x)| \geq \lambda\}) \leq \delta$. Therefore, with the weak convergence in (3.43) we deduce from (3.47)

$$\lim_{\varepsilon\to 0} \left| \int_\Omega \eta\theta\, \widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon \frac{\varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|}{1 + \varepsilon|\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon|} \,\mathrm{d}x \right| \leq C\rho. \tag{3.48}$$

Finally, combining (3.45)–(3.48) we get for all $\theta \in L^\infty(\Omega)$

$$\lim_{\varepsilon\to 0} \int_\Omega f_\varepsilon(\cdot, T_\varepsilon, \nabla u_\varepsilon)\theta \,\mathrm{d}x = \int_\Omega \theta f(\cdot, T, \nabla u) \,\mathrm{d}x$$

and consequently, the second identity in (3.33) follows. Moreover, we find

$$\langle \overline{F}, \theta \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} = \int_\Omega \theta f(\cdot, T, \nabla u) \,\mathrm{d}x.$$

(2) To finish the proof, it remains to show (2.5). Hence, let $w \in H^1(\Omega) \cap L^\infty(\Omega)$ be arbitrary. With β_k as in (2.4), it follows from Lemma 3.2 that

$$\|\beta_k(T_\varepsilon - w)\|_{1,2} \le \|\beta_{k+\|w\|_\infty}(T_\varepsilon)\|_{1,2} + \|w\|_{1,2} \le C(w,k).$$

Therefore, using the point-wise convergence of T_ε, we can find a subsequence that we again do not relabel such that for all $k \ge 0$

$$\beta_k(T_\varepsilon - w) \rightharpoonup \beta_k(T - w) \qquad \text{weakly in } W^{1,2}(\Omega), \tag{3.49}$$

$$\beta_k(T_\varepsilon - w) \to \beta_k(T - w) \qquad \text{almost everywhere in } \Omega, \tag{3.50}$$

$$\beta_k(T_\varepsilon - w) \rightharpoonup^* \beta_k(T - w) \qquad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{3.51}$$

Next, setting $\theta := \beta_k(T_\varepsilon - w)$ in (3.12b) we obtain

$$\int_\Omega \mathbf{A}\nabla T_\varepsilon \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x + \int_{\partial\Omega} \kappa(T_\varepsilon - T_\Gamma)\beta_k(T_\varepsilon - w)\,\mathrm{d}\Gamma = \int_\Omega \overline{f}_\varepsilon \beta_k(T_\varepsilon - w)\,\mathrm{d}x. \tag{3.52}$$

Our goal is to identify the limit in all terms in (3.52). First, for the boundary integral, we use (3.29), the compactness of the trace operator and the fact that β_k is bounded, to easily deduce that

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega} \kappa(T_\varepsilon - T_\Gamma)\beta_k(T_\varepsilon - w)\,\mathrm{d}\Gamma = \int_{\partial\Omega} \kappa(T - T_\Gamma)\beta_k(T - w)\,\mathrm{d}\Gamma. \tag{3.53}$$

For the first integral on the left hand side of (3.52) we use the identity

$$\begin{aligned}
&\int_\Omega \mathbf{A}\nabla T_\varepsilon \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x \\
&\quad = \int_\Omega \mathbf{A}\nabla(T_\varepsilon - w) \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x + \int_\Omega \mathbf{A}\nabla w \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x \\
&\quad = \int_\Omega \mathbf{A}\nabla\beta_k(T_\varepsilon - w) \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x + \int_\Omega \mathbf{A}\nabla w \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x \\
&\quad = \int_\Omega \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)\nabla\beta_k(T_\varepsilon - w) \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x + \int_\Omega \mathbf{A}\nabla w \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x,
\end{aligned}$$

where $\mathbf{A}^\top$ denotes the transpose of $\mathbf{A}$. Due to the weak convergence (3.49), we directly obtain

$$\lim_{\varepsilon\to 0} \int_\Omega \mathbf{A}\nabla w \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x = \int_\Omega \mathbf{A}\nabla w \cdot \nabla\beta_k(T - w)\,\mathrm{d}x. \tag{3.54}$$

Similarly, using again (3.49) and the weak lower semicontinuity of convex functionals (note that here we again use the ellipticity of the matrix $\mathbf{A}$), we have that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} &\int_\Omega \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)\nabla\beta_k(T_\varepsilon - w) \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x \\
&\ge \int_\Omega \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)\nabla\beta_k(T - w) \cdot \nabla\beta_k(T - w)\,\mathrm{d}x.
\end{aligned} \tag{3.55}$$

Combining (3.54) and (3.55), we therefore obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int_\Omega \mathbf{A}\nabla T_\varepsilon \cdot \nabla\beta_k(T_\varepsilon - w)\,\mathrm{d}x &\ge \int_\Omega \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)\nabla\beta_k(T - w) \cdot \nabla\beta_k(T - w)\,\mathrm{d}x \\
&\quad + \int_\Omega \mathbf{A}\nabla w \cdot \nabla\beta_k(T - w)\,\mathrm{d}x \\
&= \int_\Omega \mathbf{A}\nabla T \cdot \nabla\beta_k(T - w)\,\mathrm{d}x.
\end{aligned} \tag{3.56}$$

Finally, we focus on the term on the right hand side of (3.52). Following (3.45), we have

$$\begin{aligned}\int_\Omega f_\varepsilon(\cdot,T_\varepsilon,\nabla u_\varepsilon)\beta_k(T_\varepsilon - w)\,\mathrm{d}x = &\int_\Omega \eta\beta_k(T_\varepsilon - w)\widehat{\mathbf{S}}_\varepsilon\cdot\nabla u_\varepsilon\,\mathrm{d}x\\ &-\int_\Omega \eta\beta_k(T_\varepsilon - w)\widehat{\mathbf{S}}_\varepsilon\cdot\nabla u_\varepsilon\frac{\varepsilon|\widehat{\mathbf{S}}_\varepsilon\cdot\nabla u_\varepsilon|}{1+\varepsilon|\widehat{\mathbf{S}}_\varepsilon\cdot\nabla u_\varepsilon|}\,\mathrm{d}x.\end{aligned} \tag{3.57}$$

Since β_k is bounded, we can follow the procedure in (3.47) and (3.48) to conclude

$$\lim_{\varepsilon\to0}\int_\Omega \eta\beta_k(T_\varepsilon - w)\widehat{\mathbf{S}}_\varepsilon\cdot\nabla u_\varepsilon\frac{\varepsilon|\widehat{\mathbf{S}}_\varepsilon\cdot\nabla u_\varepsilon|}{1+\varepsilon|\widehat{\mathbf{S}}_\varepsilon\cdot\nabla u_\varepsilon|}\,\mathrm{d}x = 0. \tag{3.58}$$

Thus, it remains to identify the limit of the first term on the right hand side of (3.57). We rewrite this term into the following form

$$\begin{aligned}\int_\Omega \eta\beta_k(T_\varepsilon - w)\widehat{\mathbf{S}}_\varepsilon\cdot\nabla u_\varepsilon\,\mathrm{d}x = &\int_\Omega \eta\beta_k(T_\varepsilon - w)\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u\,\mathrm{d}x\\ &+\int_\Omega \eta\beta_k(T_\varepsilon - w)\big(\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)-\mathbf{S}(\cdot,T_\varepsilon,\nabla u)\big)\cdot\nabla(u_\varepsilon - u)\,\mathrm{d}x\\ &+\int_\Omega \eta\beta_k(T_\varepsilon - w)\mathbf{S}(\cdot,T_\varepsilon,\nabla u)\cdot\nabla(u_\varepsilon - u)\,\mathrm{d}x.\end{aligned} \tag{3.59}$$

Now we identify all limits in the right hand side of (3.59). First, it trivially follows from (3.19), (3.41) and the fact that η and β_k are bounded that

$$\lim_{\varepsilon\to0}\int_\Omega \eta\beta_k(T_\varepsilon - w)\big(\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)-\mathbf{S}(\cdot,T_\varepsilon,\nabla u)\big)\cdot\nabla(u_\varepsilon - u)\,\mathrm{d}x = 0. \tag{3.60}$$

Then, using the point-wise convergence of T_ε, the boundedness of η and β_k, we can apply Lebesgue's dominated convergence theorem to deduce (compare with (3.37))

$$\lim_{\varepsilon\to0}\int_\Omega |\eta\beta_k(T_\varepsilon - w)\mathbf{S}(\cdot,T_\varepsilon,\nabla u)-\eta\beta_k(T-w)\mathbf{S}(\cdot,T,\nabla u)|^{p'(\cdot)}\,\mathrm{d}x = 0.$$

Consequently, using the convergence result (3.28), we have that

$$\lim_{\varepsilon\to0}\int_\Omega \eta\beta_k(T_\varepsilon - w)\mathbf{S}(\cdot,T_\varepsilon,\nabla u)\cdot\nabla(u_\varepsilon - u)\,\mathrm{d}x = 0. \tag{3.61}$$

Now, we discuss the properties of the first integral on the right hand side of (3.59). Again from the point-wise convergence of T_ε, the boundedness of η and β_k and the Lebesgue convergence dominated theorem, we have

$$\lim_{\varepsilon\to0}\int_\Omega |\eta\beta_k(T_\varepsilon - w)\nabla u-\eta\beta_k(T-w)\nabla u|^{p(\cdot)}\,\mathrm{d}x = 0.$$

Therefore, using (3.19), (3.32) and (3.33) we get

$$\lim_{\varepsilon\to0}\int_\Omega \eta\beta_k(T_\varepsilon - w)\mathbf{S}(\cdot,T_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u\,\mathrm{d}x = \int_\Omega \eta\beta_k(T-w)\mathbf{S}(\cdot,T,\nabla u)\cdot\nabla u\,\mathrm{d}x. \tag{3.62}$$

Hence, letting $\varepsilon \to 0$ in (3.59), using (3.60), (3.61) and (3.62), we have that

$$\lim_{\varepsilon\to 0}\int_\Omega \eta\beta_k(T_\varepsilon - w)\widehat{\mathbf{S}}_\varepsilon \cdot \nabla u_\varepsilon \,\mathrm{d}x = \int_\Omega \beta_k(T-w)f(\cdot, T, \nabla u)\,\mathrm{d}x. \tag{3.63}$$

Clearly, it the follows from (3.57), (3.58) and (3.63) that

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon(\cdot, T_\varepsilon, \nabla u_\varepsilon)\beta_k(T_\varepsilon - w)\,\mathrm{d}x = \int_\Omega f(\cdot, T, \nabla u)\beta_k(T-w)\,\mathrm{d}x. \tag{3.64}$$

Finally, letting $\varepsilon \to 0$ in (3.52) and substituting the corresponding terms by (3.53), (3.56) and (3.64), we deduce (2.5). This finishes the proof of the main result. □

4 Application to organic semiconductor devices

In [3] and [6] a special case of the system in (1.1)–(1.2) was considered to describe self-heating effects in organic semiconductor devices such as organic light-emitting diodes. Therein, the proposed current-density function, in the non-dimensionalized case, is of the special form

$$\mathbf{S}(x, T, \nabla u) = \sigma_0(x)F(x,T)|\nabla u|^{p(x)-2}\nabla u, \tag{4.1}$$

where the temperature dependence is given via an Arrhenius-type factor of the form

$$F(x,T) = \exp\Big[-\gamma(x)\Big(\frac{1}{T} - \frac{1}{T_\Gamma}\Big)\Big].$$

The coefficient $\sigma_0 \in L^\infty_+(\Omega)$ describes the effective electrical conductivity and satisfies the usual bounds $0 < \underline{\sigma_0} \le \sigma_0(x) \le \overline{\sigma_0}$ for almost all $x \in \Omega$. Moreover, $\gamma \in L^\infty_+(\Omega)$ is related to the so-called activation energy of each material layer.

We easily check that for this special application from the field of organic semiconductor devices the assumptions (i)–(vi) in Section 2 are satisfied (comp. [3]). Hence, Theorem 2.1 guarantees the existence of weak solutions u and T, where T is to be interpreted as entropy solution to the heat equation with a Joule heat term describing the self-heating of the organic device due to positive feedback in the electric conductivity. Note, however, that in this case $\mathbf{S}$ is already strictly monotone, and hence a regularization of $\mathbf{S}$, as in Section 3.1 (compare (3.1a)), is not necessary.

We highlight that the positivity of γ in the Arrhenius law gives rise to a feedback loop in the device: At constant voltage, the electric current leads to an increase of the temperature, which in turn leads to an improved conductivity and therefore higher currents. The structure continuously heats up, often leading to the destruction of the device by thermal breakdown if the heat cannot be dispersed into the environment, see [14].

A thermistor-like behavior of organic semiconductors induced by self-heating has been demonstrated for the organic semiconductor C_{60} in [4] and for organic materials used as active layers in OLEDs (organic light-emitting diodes) in [5]. Moreover, in large-area OLEDs self-heating leads to spatially inhomogeneous current and temperature distributions resulting in inhomogeneities in the luminance for higher light intensities. Especially, in lighting panels the area becomes spotty, see [15, 16].

Equations of the form in (1.1) together with (4.1) model this interplay between current flow and heat conduction and therefore help to understand and improve large-area OLEDs.

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