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Fast computations with the harmonic Poincaré–Steklov operators on nested refined meshes

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Abstract

In this paper we develop asymptotically optimal algorithms for fast computations with the discrete harmonic Poincaré-Steklov operators in presence of nested mesh refinement. For both interior and exterior problems the matrix-vector multiplication for the finite element approximations to the Poincaré-Steklov operators is shown to have a complexity of the order $O(N_{ref} \log^3 N)$ where N_{ref} is the number of degrees of freedom on the polygonal boundary under consideration and $N = 2^{-p_0} \cdot N_{ref}$, $p_0 \geq 1$, is the dimension of a finest quasi-uniform level. The corresponding memory needs are estimated by $O(N_{ref} \log^2 N)$. The approach is based on the multilevel interface solver (as in the case of quasi-uniform meshes, see [20]) applied to the Schur complement reduction onto the nested refined interface associated with nonmatching decomposition of a polygon by rectangular substructures.

1 Introduction

Asymptotically optimal algorithms for solving boundary or interface reductions of elliptic boundary value problems have been under a forced attention during the last few years. Main results have been developed for the case of quasi-uniform meshes generating boundary element spaces. We refer to [2], [10]–[14], [27, 28, 29] for recent results on wavelet approximation in classical BEM yielding algorithms of the complexity $O(N \log^q N)$, $q \geq 1$, with respect to the number N of degrees of freedom on the underlying boundary. The similar asymptotical performance is achieved for panel clustering techniques in BEM [17]. Efficient matrix compression techniques of optimal complexity for the interior Poincaré-Steklov (PS) operators associated with several classes of the second and fourth order elliptic operators have been recently developed [19, 20, 23].

In this paper we propose and analyze asymptotically optimal algorithms for the treatment of the finite element (FE) discretizations to the harmonic *interior* and *exterior* PS operators on polygons in presence of nested mesh refinements near the corner points. For both interior and exterior problems the proposed technique is shown to have a complexity of the order $O(N_{ref} \log^3 N)$ where N_{ref} is the number of degrees of freedom on the underlying boundary and $N = 2^{-p_0} N_{ref}$, $p_0 \geq 1$, denotes the boundary dimension of a finest quasi-uniform level. The corresponding memory needs are estimated by $O(N_{ref} \log^q N)$ where $q = 2$ or $q = 3$ for the interior or exterior problems, respectively. The approach is based on the multilevel interface solver (as in the case of quasi-uniform meshes, see [20]) applied to the Schur complement reduction onto the nested refined interface associated with a nonmatching decomposition of the underlying polygon by rectangular substructures.

The remainder of the paper is organized as follows. In Section 2 we derive the FE approximation to harmonic PS operators on the polygonal boundary and provide the corresponding error analysis. The general framework for the multilevel additive Schwartz method is overviewed. Section 3 deals with the multilevel scheme on the nested refined interface for a treatment of the interior PS operator in the case of mesh refinement by nested selection. In Section 4, a similar approach is adapted to the case of exterior PS operators based on the non conforming geometrical decomposition of the artificial annual-type domain with enlargement of the coarse mesh to infinity. The FE approximation of the PS operators with slave nodes on the interface defining the decomposition is applied. We conclude in Section 5 with a brief discussion on applications of the proposed techniques for solving classical boundary integral equations.

2 Preliminaries

Let $\Omega \subset R^2$ be a polygonal domain with the boundary $\Gamma = \cup_{j=1}^{N_0} \overline{\Gamma}_j$ where Γ_j is an open edge and $\omega_j \in (0, 2\pi)$ is the interior angle at $s_j = \overline{\Gamma}_j \cap \overline{\Gamma}_{j+1}$, $j = 1, \ldots, N_0$ with the convention $\Gamma_{N_0+1} = \Gamma_1$. We denote by $C\Omega := R^2 \backslash \Omega$ the exterior domain. Let n be the unit outward normal vector on Γ. Introduce the trace spaces on Γ by

$$H^s(\Gamma) := \begin{cases} \gamma_0 u : \ u \in H^{s+1/2}(\Omega) , & 0 < s < 3/2 \\ L^2(\Gamma) , & s = 0 \end{cases}$$

equipped with the norm [16, 24]

$$\|v\|_{H^s(\Gamma)} := \inf_{z\in H^{s+1/2}(\Omega):\gamma_0 z=v} \|z\|_{H^{s+1/2}(\Omega)} .$$

The trace operator

$$\gamma_0 : H^{s+1/2}_{loc}(R^2) \to H^s(\Gamma), \quad 0 < s < 3/2$$

is continuous and has a continuous right inverse. The space $H^s(\Gamma) := (H^{-s}(\Gamma))'$, $-3/2 < s < 0$ is defined by $L^2(\Gamma)$-duality. The generalized normal derivative operator

$$\gamma_1 \ : \ H^1(\Omega_1, \Delta) \to H^{-1/2}(\Gamma)$$

is continuous, surjective (see [8]) and coincides with the operator $\frac{\partial u}{\partial n}_{|\Gamma} = \partial_n u$ for $u \in H^2_{loc}(R^2)$.

Consider the interior and exterior boundary value problems (BVPs) for the Laplacian

$$\Delta \overline{u} = 0 \ \text{ in } \ \Omega_i, \quad \overline{u} \in H^1_{loc}(\Omega_i), \ i = 1, 2 \tag{2.1}$$

subject to the Dirichlet boundary condition

$$\gamma_0 \overline{u} = u \in H^{1/2}(\Gamma) \ \text{ on } \ \Gamma \tag{2.2}$$

or Neumann boundary condition

$$\gamma_1 \overline{u} = v \in H^{-1/2}(\Gamma) \ \text{ on } \ \Gamma, \quad (v, 1)_\Gamma = 0 , \tag{2.3}$$

where the notations $\Omega = \Omega_1$ and $C\Omega = \Omega_2$ are temporarily used. For the exterior problem in Ω_2, we additionally require the "radiation conditions"

$$\overline{u}(x) = c_\infty + O\Big(\frac{1}{|x|}\Big), \quad |x| \to \infty . \tag{2.4}$$

Given $s \in [-1, 1]$ and $f \in (H^s(\Gamma))'$, introduce the subspace

$$H^s_f(\Gamma) := \{u \in H^s(\Gamma) : \ (u, f)_{L^2} = 0\} .$$

Since the Dirichlet problems are uniquely solvable for any $u \in H^{1/2}(\Gamma)$ with a continuous solution operator

$$M_i u := \overline{u} : H^{1/2}(\Gamma) \to H^1_{loc}(\Omega_i, \Delta) , \quad i = 1, 2$$

one can introduce the continuous Dirichlet-Neumann mappings

$$T_1 : H^{1/2}(\Gamma) \ \to \ H^{-1/2}(\Gamma),$$
$$T_2 : H^{1/2}_{g_0}(\Gamma) \ \to \ H_1^{-1/2}(\Gamma)$$

related to the interior and exterior problems, respectively, which map the trace $u = \gamma_0 \overline{u}$ into the generalized normal derivative $\gamma_1 \overline{u}$ of a harmonic function $\overline{u}$ and possess $Ker\, T_i = span\{1\}$, $i = 1, 2$. There exist continuous pseudoinverse operators $S_i : H_1^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $i = 1, 2$, satisfying $T_i S_i T_i = T_i$ which are called Poincaré-Steklov operators.

Our goal is to construct asymptotically optimal interface solvers for the natural boundary reductions of the Dirichlet and Neumann problems based on the FE approximations to Poincaré-Steklov operators. We consider both interior and exterior BVPs in presence of nested mesh refinement near the corner points.
The mapping properties of the operators T_i and S_i , $i = 1, 2$ are well known [1, 8, 18].

Lemma 2.1 *For $s \in [-1/2, 1/2]$ the following operators are symmetric and continuous*

$$T_i : H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \quad i = 1, 2,$$
$$S_1 : H_1^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma),$$
$$S_2 : H_{g_0}^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$$

where $g_0 \in H^{-1/2}(\Gamma)$ is the Robin potential on Γ. The operators S_i are positive definite on their domains of definition while the operators T_i are positive definite on $H_1^{1/2}(\Gamma)$ with $Ker\, T_i = span\{1\}$, $i = 1, 2$. ∎

We deal with FE approximations of the operators T_1 and T_2. In the case of the interior operator we assume that Ω is composed of M_R rectangles Ω_i, $i \in I_R$ and M_T regular right triangles Ω_i, $i \in I_T$ (without overlapping), such that

$$\overline{\Omega} = \cup_{i \in I_R \cup I_T} \overline{\Omega}_i \,. \tag{2.5}$$

Let we are given the subspace $\overline{W}_h \subset H^1(\Omega)$ of piecewise linear C^0-finite elements related to the quasi-uniform triangulation τ_h of Ω such that the restriction $\tau_{h_{|\Omega_i}}$ on any subdomain Ω_i is defined by a tensor product of uniform meshes. The original decomposition is associated with the coarse triangulation τ_0 being quasi-uniform and regular with the mesh size $h_0 \sim O(1) \sim diam\Omega$. The mesh parameter of the finest level $\overline{W}_h$ is supposed to be of the order $h = h_p \sim 2^{-p}$, $p \in \mathbb{N}$. Let $W_p \subset \overline{W}_h$ be the space admitting the sequence of nested subspaces

$$W_0 \subset W_1 \subset \ldots W_p \tag{2.6}$$

where W_0 is the FE space with respect to τ_0. The choice of W_l, $l = 1, \ldots, p$ is subject to some nested selection strategy producing a conformal mesh refinement (with slave nodes) near the corner points. It will be specified in Section 3. Denote

$$X_p = W_{p_{|\Gamma}} \subset H^1(\Gamma), \quad W_{0p} = W_p \cap H_0^1(\Omega) \,.$$

The FE approximation $T_{1p} : X_p \to X_p'$ of the operator T_1 under consideration is defined by

$$(T_{1p} u, \gamma_0 v)_{L^2(\Gamma)} = \int_\Omega \nabla \overline{u}_h \nabla v dx \,, \qquad \forall v \in W_p \tag{2.7}$$

where $\overline{u}_h \in W_p$ satisfies $\gamma_0 \overline{u}_h = u \in X_p$ and

$$\int_\Omega \nabla \overline{u}_h \nabla z dx = 0 \,, \qquad \forall z \in W_{0p} \,. \tag{2.8}$$

The operator T_{1p} implements the discrete Dirichlet-Neumann mapping related to the Galerkin approximation to the weak solution $\overline{u}$ of the interior problem (2.1), (2.2). With given $u \in H^{1/2}(\Gamma)$ from (2.2), the approximate Neumann data $\gamma_{1h}\overline{u}$ now takes the form

$$\gamma_{1h}\overline{u} := T_{1p} Q_2 u \qquad \text{with } \overline{u}_h = \mathcal{P}_h \overline{u} \tag{2.9}$$

where $Q_2 : L_2(\Gamma) \to X_p$ is the L^2-orthoprojection onto X_p and $\mathcal{P}_h : H^1(\Omega) \to W_p$ is the FE Ritz projection defined for any $z \in H^1(\Omega)$ by

$$\begin{cases} \int_\Omega \nabla(z - \mathcal{P}_h z)\, \nabla \eta dx = 0, \qquad \forall \eta \in W_{0p} \\ \gamma_0 \mathcal{P}_h z = Q_2 \gamma_0 z \,. \end{cases}$$

The error estimate for the mapping γ_{1h} is given by the following Lemma.

Lemma 2.2 *Let u_h be the FE approximation of $\overline{u}$ defined by $u_h = \mathcal{P}_h\overline{u}$. There exists a constant C, independent of h, such that*

$$\|Q_2T_1u - \gamma_{1h}\overline{u}\|_{H^{-1/2}(\Gamma)} \le Ch^{s-1/2}\|u\|_{H^s(\Gamma)}, \quad 1/2 \le s \le s_0, \tag{2.10}$$

where $s_0 = 3/2$ in the case of a convex polygon and $s_0 < 3/2$ in the general case depending on the elliptic regularity (in sense of (3.19)) of the problem (2.1), (2.2).

Proof. Consider for brevity the case of a convex polygon. Applying the approximation properties of the Ritz projection $\mathcal{P}_h$, see [6],

$$\|\overline{u} - \mathcal{P}_h\overline{u}\|_{H^\alpha(\Omega)} \le ch^{s+1/2-\alpha}\|u\|_{H^s(\Gamma)}, \qquad \alpha - \frac{1}{2} \le s \le 3/2\,, \tag{2.11}$$

with $\alpha = 0, 1$ one obtains the error estimate in the $H^{-1/2}(\Gamma)$–norm for the FE approximation T_{1p} to the PS operator T_1

$$\sup_{v\in X_p, v\neq 0} \frac{(Q_2T_1u - T_{1p}Q_2u, v)}{\|v\|_{H^{1/2}(\Gamma)}} =$$
$$= \sup_{v\in X_p, v\neq 0} \|v\|^{-1}_{H^{1/2}(\Gamma)} \int_\Omega \nabla(\overline{u} - \mathcal{P}_h\overline{u}) \cdot \nabla\mathcal{P}_hM_1v dx \le$$
$$\le c\|\overline{u} - \mathcal{P}_h\overline{u}\|_{H^1(\Omega)} \cdot \frac{\|\mathcal{P}_hM_1v\|_{H^1(\Omega)}}{\|v\|_{H^{1/2}(\Gamma)}}.$$

which yields (2.10) due to (2.11). Lemma 2.2 is proven.

Remark 2.1 *The estimate* (2.10) *immediately implies*

$$\|T_1u - \gamma_{1h}\overline{u}\|_{H^{-1/2}(\Gamma)} \le c_1h^{s-1/2}\|u\|_{H^s(\Gamma)} + c_2h^{\sigma+1/2}\|T_1u\|_{H^\sigma(\Gamma)}\,,$$
$$1/2 \le s \le s_0, \quad -1/2 \le \sigma \le 1/2$$

when taking into account the approximation property of the L^2-projection, see [6],

$$\|z - Q_2z\|_{H^{-1/2}(\Gamma)} \le ch^{1/2+s}\|z\|_{H^s(\Gamma)}, \quad -1/2 \le s \le 3/2\,. \qquad \blacksquare$$

We first construct an efficient compression technique for the stiffness matrix $\mathcal{T}_{1p}$ (that is, in fact, the Schur complement related to (2.8)) of the operator T_{1p} which admits a matrix-vector multiplication of the complexity $O(N\log^3 N)$ up to the approximation error of the order $O(N^{-\alpha})$, $\alpha > 0$, where $N = dim X_p$. The approach is based on the multilevel interface solver (in the framework of the cascadic CG method, see [31]) applied to the Schur complement reduction onto the nested refined interface $\Gamma_0 = \cup_{i\in I_R\cup I_T}\partial\Omega_i\backslash\Gamma$ aligned with the nested refined coarse mesh decomposition.

We then extend the proposed technique to the case of the exterior PS operator T_2. To that end we first approximate the exact "radiation condition" (2.4) by homogeneous Dirichlet or Neumann conditions on the boundary Γ_∞ of the artificial rectangle $\Pi_\infty \supset \Omega$ with $diam\Pi_\infty \sim N^\alpha$, $\alpha > 0$. Then we introduce a non conforming geometrical decomposition of the domain $\Omega_\infty = \Pi_\infty\backslash\overline{\Omega}$ which produces the coarse mesh space with enlargement

to infinity and with the number of subdomains of the order $O(\log N)$. Then the multilevel additive Schwarz algorithm (in particular, the BPX scheme [7]) on the refined skeleton may be designed along the same line as in the case of a bounded domain providing again the complexity $O(N \log^3 N)$, where $N = dim\, X_p$ is the number of degrees of freedom on the underlying boundary Γ, as above.

Note that in the case of nested refined meshes on a rectangular boundary the treatment of the interior PS operator remains to be of the complexity $O(N \log^2 N)$ while for exterior problems we arrive at the complexity $O(N \log^3 N)$ even for this particular geometry. This is due to the decomposition of Ω_∞ by $O(\log N)$ substructures.

The proposed construction is substantially based on the multilevel additive Schwarz (MAS) method for fast solving the interface equation on the refined skeleton. The analysis of the MAS method under consideration appears to be a particular case of the general theory for $H^{1/2}$-setting. For the sake of completeness, we give here a brief description of the MAS method for a strongly elliptic symmetric variational problem, see e.g. [15].

Let V be a finite dimensional Hilbert space with scalar product $(\cdot,\cdot)_V$ and related norm $\|\cdot\|_V$. Consider the following V-elliptic symmetric variational problem.

$$\text{Find } u \in V : a(u,v) = f(v), \quad \forall v \in V \tag{2.12}$$

with the given continuous functional $f \in V'$ and with a symmetric bilinear form $a : V \times V \to R$ providing the norm equivalence

$$a(u,u) \cong \|u\|_V^2, \qquad \forall u \in V\,. \tag{2.13}$$

Assume we are given the decomposition

$$V = \sum_{j=0}^{p} V_j, \qquad V_j \subset V \tag{2.14}$$

onto a finite number of subspaces V_j equipped with scalar products $(\cdot,\cdot)_{V_j}$ and related norms which admit the following equivalence relation

$$b_j(u,u) \cong \|u\|_{V_j}^2, \qquad \forall u \in V_j \tag{2.15}$$

where $b_j : V_j \times V_j \to R$ is an auxiliary symmetric bilinear form for $j = 0, 1, \ldots, p$.
The theory of MAS methods is based on the stability property of the splitting (2.14), namely,

$$\inf_{u=\sum_j v_j : v_j \in V_j} \sum_{j=0}^{p} \|v_j\|_{V_j}^2 \cong \|u\|_V^2, \quad \forall u \in V \tag{2.16}$$

uniformly (or with a moderate growth of the related condition number) with respect to the number of levels p. The MAS method is defined in terms of the projections $P_{V_j} : V \to V_j$ and the elements $f_j \in V_j$ determined by

$$\begin{aligned} b_j(P_{V_j}u, v) &= a(u,v), \quad \forall u \in V\,, \\ b_j(f_j, v) &= f(v), \qquad \forall v \in V_j\,. \end{aligned}$$

Lemma 2.3 [15]. *Assume the norm equivalences* (2.13), (2.15) *and* (2.16) *to be valid. Then the operator equation*

$$P_V u = F := \sum_j f_j \tag{2.17}$$

with $P_V = \sum_{j=0}^{p} P_{V_j}$ *is equivalent to the variational problem* (2.12). *The relation* $\kappa(P_V) = O(1)$ *holds uniformly with respect to* p *and* $N_V = dim\, V$. ∎

In our context the stability (2.16) is derived in a standard way for the nested sequence of subspaces

$$V_0 \subset V_1 \subset \ldots \subset V_p = V \tag{2.18}$$

as a consequence of the stable splitting of the $H^{1/2}(\Gamma_p)$-norm on the nested refined skeleton. It will be shown that from the computational point of view the coarse mesh enlargement introduced for the proper interface reduction of the exterior problem has just the same data structure as the reduction onto the nested refined skeleton in the situation of local mesh refinement. We now conclude with the observation that the computing complexity of a treatment of the *exterior* harmonic PS operator for both quasi-uniform and refined meshes appears to be of just the same order as in the case of *interior* ones in presence of nested geometrical refinement near the corner point.

3 Local mesh refinement by nested selection

We proceed in the situation of Section 2 and suppose that the decomposition (2.5) of the polygon Ω is given, see Fig. 1, associated with the coarse triangulation τ_0 with mesh parameter $h_0 = O(1)$, where $diam\Omega = O(1)$. Let

$$\tau_0 \subset \tau_1 \subset \ldots \subset \tau_p \tag{3.1}$$

be the nested sequence of regular triangulations with mesh size $h_j \sim 2^{-j}$, $j = 0, 1, \ldots, p$, obtained by a successive dyadic refinement of the initial triangulation τ_0. The correspond-

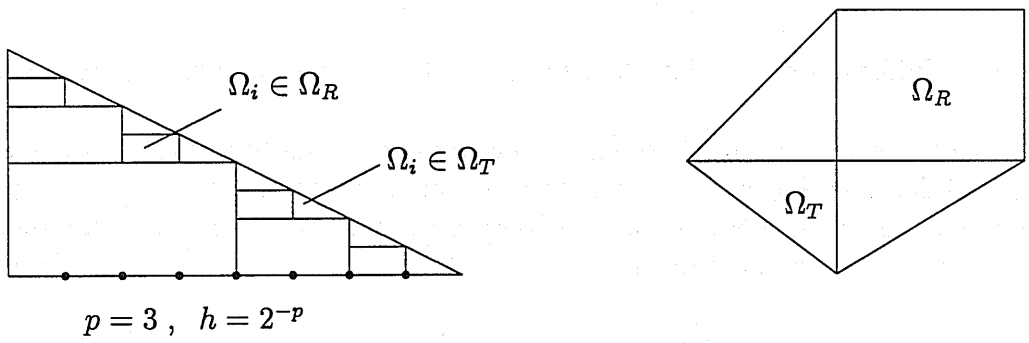

Figure 1: Decompositions of right triangular and polygonal domains.

ing nested sequence

$$\overline{W}_0 \subset \overline{W}_1 \subset \ldots \subset \overline{W}_p \equiv \overline{W}_h$$

of C^0-piecewise linear FE spaces with respect to the quasi-uniform triangulation $\{\tau_j\}$ will be used now to construct the sequence (2.6) of nested subspaces W_j, $j = 0, \ldots, p$. To that end, following [25, 7], we choose some index $0 < p_0 < p$ and define

$$W_j := \overline{W}_j \quad \text{for } j = 0, 1, \ldots, p_0 \,. \tag{3.2}$$

Remark 3.1 *In the case of quasi-uniform meshes the MAS method applied to the interface problems on the uniformly refined skeleton (see Fig. 2 where dots '•' denote the mesh nodes on the interface) has been developed in* [20] *which provides an algorithm of the complexity* $O(N \log^3 N)$ *for the treatment of interior PS operators. This case corresponds to the choice* $p_0 = p$.

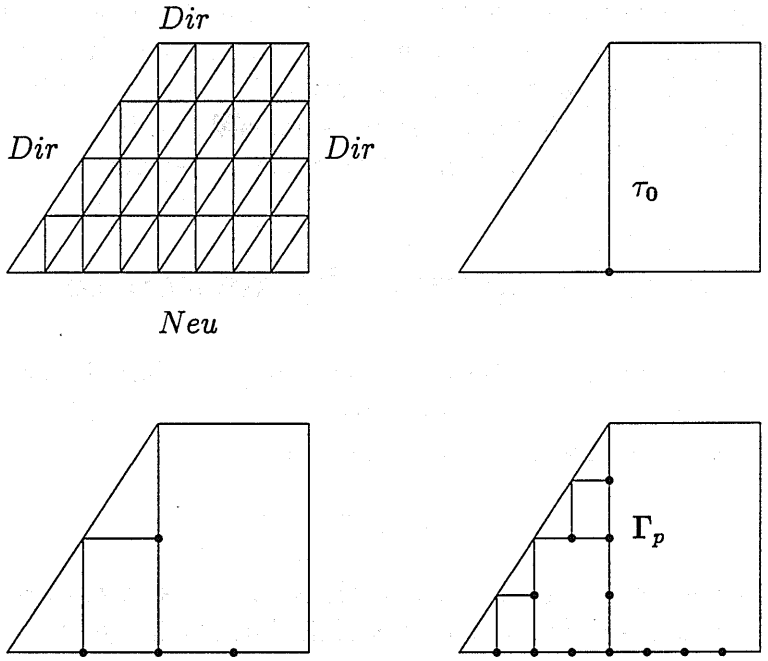

Figure 2: Uniformly refined interface for $p = 0, 1, 2$ (mixed boundary conditions).

For the sake of clarity we further restrict our domain to the couple of rectangular and triangular subdomains Ω_R and Ω_T, respectively, adjacent to the corner point where a mesh refinement is given, see Fig. 3a). Adapting the nested refinement techniques, see [7, 25, 9, 5], we introduce the sequence

$$\Omega_p \subset \Omega_{p-1} \subset \ldots \subset \Omega_0 = \Omega$$

of imbedded subdomains such that $\Omega_k = \Omega$ for $k = 0, 1, \ldots, p_0$ and $\Omega_{k+1} \subset \Omega_k$ for $k \geq p_0$. Let Ω_{p_0+l} be composed of the upper-left quarter Q_l^1 of the rectangle $Q_{l-1}^1 = \cup_{i=1}^4 Q_l^i$ with $Q_0^1 = \Omega_R$ and of the upper quarter Q_l, $l = 1$ of the triangle Ω_T as indicated on Fig.

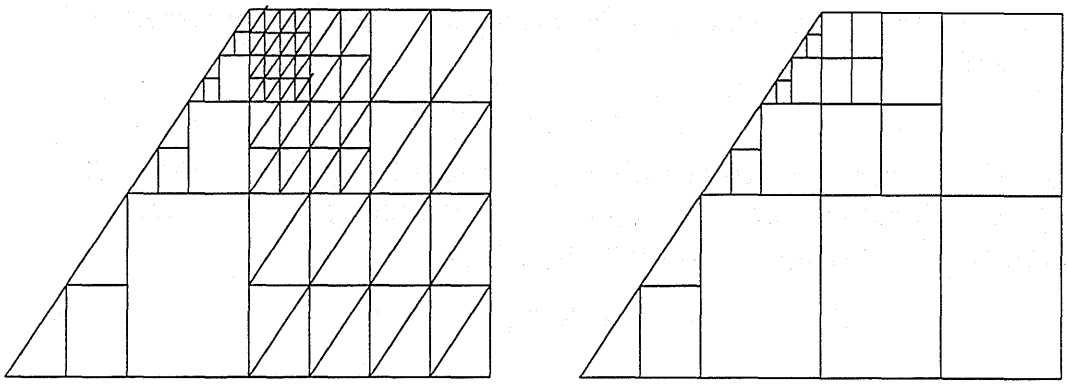

Figure 3: Nested refined interface b) for $p = 4$, $p_0 = 2$ and related triangulation a).

3a). We then proceed with such a recurrent procedure for $l = 2, \ldots, p - p_0$. Let $\overline{N}_{j,x}$ be the nodal basis functions of $\overline{W}_j$. To determine the corresponding nested sequence of subspaces we set

$$W_j = W_{j-1} + \widetilde{W}_j, \qquad j = p_0 + 1, \ldots p$$

where

$$\widetilde{W}_j = span\{\overline{N}_{j,x} : supp\overline{N}_{j,x} \subset \Omega_j\}, \quad j > p_0 \, .$$

If we put $\widetilde{W}_j = W_j$ for $j = 0, \ldots, p_0$ then the final computational space W_p may be introduced by

$$W_p = \sum_{j=0}^{p} \widetilde{W}_j, \qquad \widetilde{W}_j \subset \overline{W}_j \tag{3.3}$$

implying the required nestedness $W_0 \subset W_1 \subset \ldots \subset W_p$.

The nested refined interface Γ_p related to W_p may be introduced by assembling the corresponding ones associated with the subspaces W_j for $j = p_0, p_0 + 1, \ldots, p$. In fact, following [20], let us introduce the uniformly refined interface $\widehat{\Gamma}_{p_0+l}$ associated with the triangle Q_l and with the mesh size $h_{p_0+l} = 2^{-(p_0+l)}$ for $l = 1$ and define the contribution from the level $p_0 + 1$ by

$$\widetilde{\Gamma}_{p_0+1} = (\cup_{i=1}^{4} Q_1^i \backslash \partial\Omega_{p_0}) \cup \widehat{\Gamma}_{p_0+1} \, .$$

Then define recurrently

$$\widetilde{\Gamma}_{p_0+l} = (\cup_{i=1}^{4} Q_l^i \backslash \partial\Omega_{p_0+l-1}) \cup \widehat{\Gamma}_{p_0+l}$$

for $l = 1, 2, \ldots, l_0 = p - p_0$. Now the resultant refined interface Γ_p corresponding to the computational space W_p is defined by

$$\Gamma_p = \Gamma_{p_0} \cup_{l=1}^{l_0} \widetilde{\Gamma}_{p_0+l} \, . \tag{3.4}$$

The particular case with $p = 4$ and $p_0 = 2$ is shown on Fig 3.b).

Now we are in a position to define the FE approximation T_{1p} of the interior Poincaré-Steklov operator T_1 by (2.7), (2.8) and then substitute instead of the W_p-harmonic function $\overline{u}$ in (2.8) the corresponding Schur complement reduction to the skeleton Γ_p from (3.4). To that end we first introduce the trace spaces $V_j := \gamma\widetilde{W}_j$ and $V_{0j} := \gamma\widetilde{W}_{0j}$, $j = 0, 1, \ldots p$ on Γ_p where $\gamma : W_p \to C(\Gamma_p)$ is the usual trace operator with respect to Γ_p. We equip V_j with the norm

$$\|u\|_{V_j} := \inf_{\overline{u}\in W_j : \gamma\overline{u}=u} \|\overline{u}\|_{H^1(\Omega)}$$

providing a Hilbert space structure with $H^{1/2}$-setting. Let $\langle\cdot,\cdot\rangle_{\Gamma_p}$ be the duality with respect to the L^2-inner product $(\cdot,\cdot)_{L^2(\Gamma_p)}$ on Γ_p. Then the operator T_{1p} from (2.7) and (2.8) admits the following equivalent definition (cf. [19, 21])

$$(T_{1p}u, v)_{L^2(\Gamma)} = \langle A_{\Gamma_p}\overline{u}, \overline{v}\rangle_{\Gamma_p}\,, \qquad \forall \overline{v} \in V_p \tag{3.5}$$

where $\gamma_0\overline{v} = v$ and $\overline{u} \in V_p$ satisfies

$$\langle A_{\Gamma_p}\overline{u}, z\rangle_{\Gamma_p} = 0, \quad \forall z \in V_{0p}\,; \quad \gamma_0\overline{u} = u\,. \tag{3.6}$$

Here the SPD interface operator $A_{\Gamma_p} : V_p \to V_p'$ is defined by

$$\langle A_{\Gamma_p}u, v\rangle_{\Gamma_p} := \sum_{\mathbf{k}} (T_{1\mathbf{k}}u_{\mathbf{k}}, v_{\mathbf{k}})_{L^2(\Gamma_{\mathbf{k}})}, \qquad \forall u, v \in V_p\,, \tag{3.7}$$

where $T_{1\mathbf{k}}$ is the FE approximation of the interior PS operator related to any subdomain $\Omega_{\mathbf{k}}$ (most of which are rectangles) generated by the skeleton Γ_p and $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ are the traces of u and v onto $\partial\Omega_{\mathbf{k}}$. The operator T_{1p} from (3.5) admits a quasioptimal error estimate like (2.10).

The interface equation (3.6) to be solved may be easily transformed (by subtraction of a particular solution) to the variational form (2.12) with $V = V_{0p}$ and

$$a(u, v) := \langle A_{\Gamma_p}u, v\rangle_{\Gamma_p}\,, \tag{3.8}$$

$$f(v) := \langle \Psi, v\rangle_{\Gamma_p} = \sum_{\mathbf{k}} (\Psi_{\mathbf{k}}, v)_{\partial\Omega_{\mathbf{k}}}\,, \quad \forall u, v \in V_{0p}$$

with some given $\Psi_{\mathbf{k}} \in (V_{|\Gamma_{\mathbf{k}}})'$. To apply the MAS method to the equation (2.12)–(3.6)–(3.8) we have to check the stability property (2.16) with respect to the splitting

$$V_{0p} = \sum_{j=0}^{p} V_{0j}\,, \quad V_{0j} = \gamma W_{0j}\,. \tag{3.9}$$

Lemma 3.1 *For every $u \in V_{0p}$ the norm equivalences*

$$\langle A_{\Gamma_p}u, u\rangle_{\Gamma_p} \cong \|u\|_V^2\,, \tag{3.10}$$

$$\|u\|_V^2 \cong \inf_{u=\sum_j v_j : v_j \in V_{0j}} \sum_{j=0}^{p} 2^j \|v_j\|^2_{L^2(\Gamma_p)} \tag{3.11}$$

hold uniformly with respect to p_0 and p.

Proof. The relation (3.10) follows from

$$\|u\|_V = \inf_{\overline{u}\in W_{0p}:\gamma\overline{u}=u} \|\overline{u}\|_{H^1(\Omega)} \cong \inf_{\overline{u}\in W_{0p}:\gamma\overline{u}=u} |\overline{u}|_{H^1(\Omega)} = \langle A_{\Gamma_p} u, u\rangle^{1/2}_{\Gamma_p} .$$

The equivalence (3.11) is the consequence of stability results for H^1-setting

$$\|u\|^2_{H^1(\Omega)} \cong \inf_{u=\sum\limits_j u_j:u_j\in W_{0j}} \sum_{j=0}^{p} 2^{2j}\|u_j\|^2_{L^2(\Omega)} , \quad \forall u \in W_{0p} \tag{3.12}$$

with respect to the corresponding domain splitting $W_{0p} = \sum\limits_{j=0}^{p} W_{0j}$ in the case of nested mesh refinement involved. The lower estimate in (3.12) is trivial. To prove the upper estimate a special bounded projection $\tilde{Q}_j : L^2(\Omega) \to W_j$ is constructed in [25] to keep the properties $\tilde{Q}_j(\widetilde{W}_j) \subset \widetilde{W}_j$, $j \le p$ and $\tilde{Q}_j u_j = u_j$, $\forall u_j \in \overline{W}_j$. Then a particular representation

$$u = \tilde{Q}_0 u + \sum_{j=1}^{p} (\tilde{Q}_j u - \tilde{Q}_{j-1} u), \qquad (\tilde{Q}_j - \tilde{Q}_{j-1})u \in \widetilde{W}_j$$

related to the splitting (3.3) does a job, see [25] for more details. Now (3.11) may be obtained from (3.12) by passing to the interface Γ_p and adapting the arguments similar to [25, 9] applied there in the case of the single boundary $\Gamma = \partial\Omega$. In fact, it may be easily checked that the local property

$$2^{-j}\|h\|^2_{L^2(\partial\Omega_{\mathbf{k}})} \cong \inf_{g\in W_{0j}:\gamma_{\mathbf{k}} g=h} \|g\|^2_{L^2(\Omega_{\mathbf{k}})} \tag{3.13}$$

is valid for any subdomain $\Omega_{\mathbf{k}}$. Now, similarly to the case of a uniformly refined interface [20], one derives from (3.13)

$$\begin{aligned}
\|u\|^2_V &= \inf_{\overline{u}\in W_{0p}:\gamma\overline{u}=u} \|\overline{u}\|^2_{H^1(\Omega)} \\
&\cong \inf_{\overline{u}\in W_{0p}:\gamma\overline{u}=u} \left\{ \inf_{\overline{u}=\sum\limits_{j=0}^{p} u_j:u_j\in W_{0j}} \sum_{\mathbf{k}}\sum_{j=0}^{p} 2^{2j}\|\gamma_{\mathbf{k}} u_j\|^2_{L^2(\Omega_{\mathbf{k}})} \right\} \qquad (3.14) \\
&\cong \inf_{u=\sum\limits_j \gamma u_j:u_j\in W_{0j}} \sum_{\mathbf{k}}\sum_{j=0}^{p} 2^{2j}\|\gamma_{\mathbf{k}} u_j\|^2_{L^2(\Omega_{\mathbf{k}})} \\
&\cong \inf_{u=\sum\limits_{j=0}^{p} v_j:\, v_j\in V_{0j}} \sum_{j=0}^{p} 2^{j}\|v_j\|^2_{L^2(\Gamma_p)}, \quad \forall u \in V_{0p} .
\end{aligned}$$

This completes our proof. ∎

Starting with the decomposition (3.9), consider a more refined splitting

$$V = \sum_{j=0}^{p} \sum_{m=1}^{dim V_{0j}} V_{j,m} \tag{3.15}$$

based on the decomposition

$$V_{0j} = \sum_m V_{j,m} \tag{3.16}$$

into one-dimensional subspaces $V_{j,m} = span\{\varphi_{j,m}\}$ where the nodal basis functions $\varphi_{j,m} = \gamma N_{j,m}$ of V_{0j} are chosen in such a way that $suppN_{j,m} \cap \Gamma_p \neq \emptyset$. Here $N_{j,m}$ is the nodal basis function of W_{0j}.
As in the case of quasi-uniform meshes, for the geometrical refinement chosen the splitting (3.16) appears to be stable, i.e.,

$$\|v_j\|^2_{L^2(\Gamma_p)} \cong h_j \sum_m c^2_{j,m}, \qquad \forall v_j = \sum_m c_{j,m}\varphi_{j,m} \in V_{0j}$$

with certain scaling constants h_j independent of v_j. With the bilinear form $a(\cdot,\cdot)$ given by (3.8) we now introduce the MAS operator $P_{BPX} : V \to V$ with respect to the choice

$$\begin{aligned} b_j(u,v) := (u,v)_{V_{j,m}} &= 2^j(u,v)_{L^2(\Gamma_p)}, \quad \forall u,v \in V_{j,m}, \\ (P_{V_{j,m}}u, v)_{V_{j,m}} &= \langle A_{\Gamma_p}u, v\rangle_{\Gamma_p}, \quad \forall v \in V_{j,m}, \ \forall u \in V. \end{aligned}$$

The SPD operator P_{BPX} takes the form

$$P_{BPX}u := \sum_{j=0}^{p} \sum_{\mathbf{k}\in\mathbf{k}(j)} \sum_{m=1}^{dim\gamma_{\mathbf{k}}V_{0j}} \frac{(T_{1\mathbf{k}}u, \varphi_{j,m})}{2^j(\varphi_{j,m},\varphi_{j,m})_{L^2(\Gamma_p)}} \cdot \varphi_{j,m} \tag{3.17}$$

where the index $\mathbf{k}$ runs the subset

$$\mathbf{k}(j) = \begin{cases} \mathbf{k} : \partial\Omega_{\mathbf{k}} \in \Gamma_{p_0}, & j \le p_0 \\ \mathbf{k} : \partial\Omega_{\mathbf{k}} \in \tilde{\Gamma}_j, & j > p_0. \end{cases}$$

The right hand side F from (2.17) is defined by substituting in (3.17) the elements $\Psi_{\mathbf{k}}$ from (3.8) instead of $T_{1\mathbf{k}}u$. The operator P_{BPX} corresponds to the BPX-scheme, see [7], applied now to the interface problem with the operator A_{Γ_p}.

As a direct consequence of Lemmas 2.3 and 3.1 we arrive at the following statement.

Theorem 3.1 *The operator equation*

$$P_{BPX}u = F \in V \tag{3.18}$$

is equivalent to the original interface problem

$$\langle A_{\Gamma_p}u, v\rangle_{\Gamma_p} = \langle \Psi, v\rangle_{L^2(\Gamma_p)}, \qquad \forall v \in V_{0p},$$

and $\kappa(P_{BPX}) = O(1)$ holds uniformly with respect to p and p_0. The computation of $P_{BPX}u$, $u \in V_{0p}$, has the complexity $O(N_{ref}\log^3 N)$ with the memory needs of the order $O(N_{ref}\log^2 N)$ where $N = 2^{-p_0}$ and $N_{ref} = (p - p_0 + 1)N$. The solution of (3.18) on a sequence of grids (say, by cascadic (C) CG method) up to the approximation error $\varepsilon = \varepsilon_{tol}N^{-\alpha}$, $\alpha > 0$, has the complexity $\log\varepsilon_{tol}^{-1} \cdot O(N_{ref}\log^3 N)$ providing the same cost estimate for the matrix-vector multiplication with the stiffness matrix $\mathcal{T}_{1p}$. ∎

Proof. The first part of the theorem follows from Lemmas 2.3 and 3.1. To estimate the complexity of the residual computation $P_{BPX}u$, $u \in V_{0p}$, we note that any of the skeletons Γ_{p_0}, $\widetilde{\Gamma}_{p_0+l}$, $l = 1, \ldots, p_0 - p$ is *uniformly* refined and, thus, the corresponding contributions from those pieces of Γ_p into the sum determining $P_{BPX}u$ have the expense $O(N \log^3 N)$, see [20], resulting in the overall complexity $O(N_{ref} \log^3 N)$. Since the memory needs in the case of a uniformly refined skeleton are of the order $O(N \log^2 N)$ we finally arrive at the desired estimate $O(N_{ref} \log^2 N)$. In the case of quasi-uniform meshes an optimal convergence of the CCG method is achieved, see [4, 30], assuming the H^{1+s}-regularity of the underlying interface problem (3.6), that means for the case of the Dirichlet problem (2.1), (2.2)

$$\|\overline{u}\|_{H^{1+s}(\Omega)} \leq c\|\gamma_0\overline{u}\|_{H^{1/2+s}(\Gamma)}, \qquad s \in (0,1] \tag{3.19}$$

with s depending on the interior angles ω_j, $j = 1, \ldots, N_0$ and on the type of boundary conditions on Γ involved. However, the results from [30] has been recently extended in [31] to the case of nested refined meshes. This completes our proof.

Remark 3.2 *To fit the typical singularities near the corner point one should implement the mesh grading of the order $h_{ref} \sim h_{p_0}^{\alpha}$, $\alpha > 1$, which yields $p - p_0 \sim O(\log N)$. In this case we get $N_{ref} \sim \alpha N \log N$ indicating that the underlying nested refinement strategy has no redundancy (in asymptotical sense) in compare with the quasi-optimal exponential mesh grading since the latter also produces $O(N \log N)$ mesh points.*

To conclude this Section we note that a more parallel version of the PBX scheme may be involved if we choose the coarse mesh space by $V_0 := V_{0j}$ with some $0 < j \leq p_0$.

4 Exterior Poincaré-Steklov operators

Consider the exterior Dirichlet problem (2.1), (2.2) in the polygonal domain $C\Omega$ with $diam\Omega = O(1)$ subject to the "radiation condition" of the form

$$|u(x)| = O(|x|^{-\nu}), \quad \text{as } |x| \to \infty, \ \nu \geq 1. \tag{4.1}$$

To approximate the condition (4.1) we introduce, following [22, 19], an artificial rectangular domain $\Pi_\infty \supset \Omega$ with $diam\,\Pi_\infty = O(N^q)$, see Fig. 4a) presenting one quarter of Π_∞, where N would be the number of degrees of freedom on Γ and $q = q(\nu) > 0$ will be specified later on. Here '$*$' marks the slave nodes. Consider the approximate equation to (2.1), (2.2)

stated on the domain $\Omega_\infty = \Pi_\infty \backslash \overline{\Omega}$

$$\begin{cases} \Delta\overline{u} = 0 & \text{in } \Omega_\infty \\ \gamma_0\overline{u} = u & \text{on } \Gamma \\ B_\infty\overline{u} = 0 & \text{on } \Gamma_\infty = \partial\Pi_\infty \end{cases} \tag{4.2}$$

where the boundary operator B_∞ keeps either Dirichlet or Neumann conditions on Γ_∞. We set $q = \frac{2}{\nu}$ in the case of Dirichlet conditions and $q = \frac{2}{\nu+1}$ in the case of Neumann ones. For the sake of clarity we further choose the Dirichlet condition on Γ_∞, i.e., $B_\infty = \gamma_{0,\Gamma_\infty}$ and assume Ω to be a unit square.

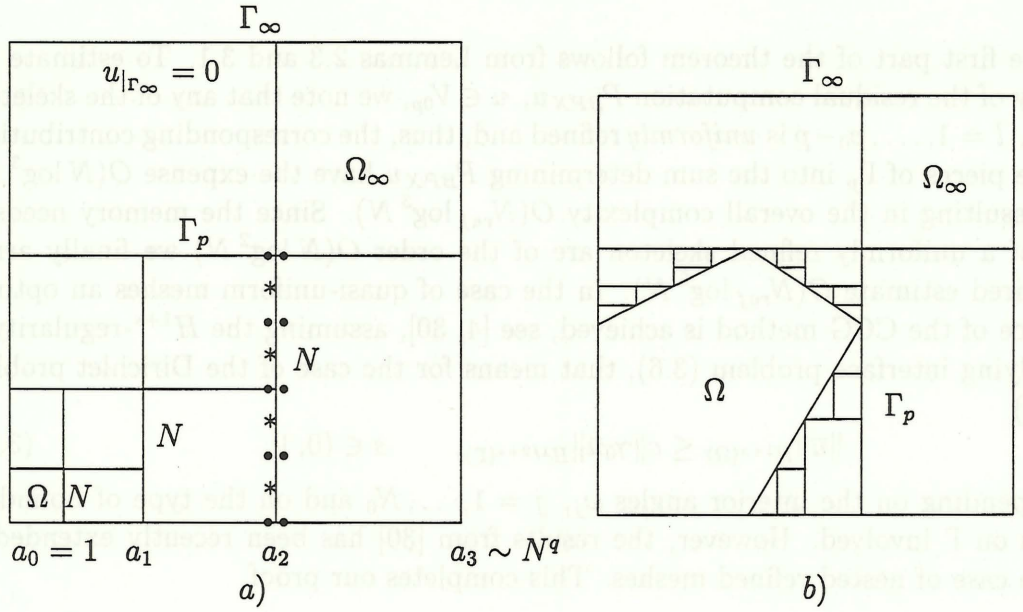

Figure 4: Refined interface for domains exterior to a rectangle (a) and a polygon (b).

Note that the domain decomposition method on the space extensive domain Ω_∞ to approximate the exterior Poincaré-Steklov and boundary integral operators has been introduced in [22, 19].
Define the non conforming geometrical decomposition of Ω_∞ with a coarsening to "infinity", taking into account the estimate

$$|\frac{\partial^l u}{\partial x_1^k \partial x_2^{l-k}}| \leq c|x|^{-\nu-l}, \quad l = 1, 2, \ldots, \quad 0 \leq k \leq l \tag{4.3}$$

for the solution u of the equations (2.1), (2.2) and (4.1). This decomposition is associated with a tensor product mesh defined by the coarse mesh points $a_1 = 1, a_2, \ldots, a_m = N^q$ for some fixed $m > 1$, see Fig. 4a), to be specified.

Related to the above coarse mesh decomposition define the sequence of L-shaped subdomains $\mathcal{D}_i$, $i = 0, 1, \ldots, m$ by

$$\mathcal{D}_i = [0, a_i] \times [0, a_i] \backslash \mathcal{D}_{i-1}, \qquad \mathcal{D}_0 = \Omega$$

such that $\overline{\Omega}_\infty = \cup_{i=1}^m \overline{\mathcal{D}}_i$. Let $N = 2^{p_0}$, $p_0 \geq 1$. Introduce the piecewise uniform mesh δ_i defined by $N+1$ equidistant mesh points on any interval $[a_i, a_{i-1}]$, $i = 1, \ldots, m$, $a_{-1} = 0$. Another set $\overline{\delta}_i$ of the considered uniform meshes will correspond to the intervals $[0, a_{i-1}]$. For any subdomain $\mathcal{D}_i$, $i = 1, \ldots, m$ introduce the space $\widehat{W}_i$ of C^0 linear elements on the triangulation associated with the product meshes $(\delta_i \cup \overline{\delta}_i) \times (\delta_i \cup \overline{\delta}_i)$ and subject to the constrain $\gamma_{0,\Gamma_\infty} u = 0$. Assume that coarse mesh nodes $a_1, \ldots, a_m$ are chosen in such a way that $\widehat{W}_{i|_{\widehat{\Gamma}_i}} \subset \widehat{W}_{i-1|_{\widehat{\Gamma}_i}}$, $\widehat{\Gamma}_i = \partial\mathcal{D}_i \cap \partial\mathcal{D}_{i-1}$ and, thus, by introducing slave nodes on $\widehat{\Gamma}_i$, $i = 2, \ldots, m$ one obtains the computational space $W_p \subset H^1(\Omega_\infty)$ with $p = p_0 + m$. This space satisfies the relation $W_{p_{|\mathcal{D}_i}} \subset \widehat{W}_i$ by definition and admits the splitting of the type (3.3) with the nested sequence of subspaces $W_0 \subset W_1 \subset \ldots \subset W_p$. Let $X_p := W_{p_{|\Gamma}}$ and $W_{0p} = W_p \cap H_0^1(\Omega_\infty)$. The FE approximation $T_{2p} : X_p \to X_p'$ of the operator T_2 is defined

by

$$(T_{2p}u, \gamma_0 v)_{L^2(\Gamma)} = \int_{\Omega_\infty} \nabla \overline{u}_h \nabla v dx, \qquad \forall v \in W_p \tag{4.4}$$

where $\overline{u}_h \in W_p$ satisfies $\gamma_0 \overline{u}_h = u \in X_p$ and

$$\int_{\Omega_\infty} \nabla \overline{u}_h \nabla z dx = 0, \qquad \forall z \in W_{0p}\,. \tag{4.5}$$

The interface representation of T_{2p} may be given along the same line as for the scheme behind (3.5). The nested refined interface Γ_p is now defined by the nonmatching decomposition with respect to the sequence $a_1, a_2, \ldots, a_m$, see Fig. 4a), and we fall in the situation of Section 3 with the interface reduction (3.5), (3.6) and (3.7). The corresponding analogue of the Theorem 3.1 remains valid.
To estimate the complexity of the algorithm we should choose the sequence $a_1, \ldots, a_m$ to keep the approximation error of the order $O(N^{-\alpha})$, $\alpha \geq 1$, and to minimize the number M_∞ of subdomains equals to $3 \cdot m$.
To simplify the exposition we assume the following hypothesis.

Hypothesis 1. *For $u \in H^{1+\alpha}(\Omega_\infty)$ the estimate*

$$\inf_{v \in W_p} \|u - v\|_{H^1(\Omega_\infty)} \leq c \sum_{i=1}^{m} \inf_{v_i \in W_{p|_{\mathcal{D}_i}}} \|u_{|_{\mathcal{D}_i}} - v_i\|_{H^1(\mathcal{D}_i)}$$

holds where c does not depend on u and m.

The fine mesh size h_i on any interval $[a_i, a_{i-1}]$ is defined by $h_i = N^{-1} H_i$, $H_i = a_i - a_{i-1}$, $i = 0, 1, \ldots, m$. To determine the values H_i we assume the *balancing equation*

$$h_0 \cdot |u|_{H^1(\Omega)} \sim h_i |u|_{H^1(\mathcal{D}_i)}, \qquad i = 1, \ldots, m \tag{4.6}$$

to be hold. Due to the asymptotics (4.3) this equation may be rewritten in the form

$$h_0 = N^{-1}, \qquad h_i = \frac{c_0}{N} a_{i-1}^{\nu}, \qquad i = 1, ..., m\,. \tag{4.7}$$

Lemma 4.1 *Let Hypothesis* 1 *hold and h_i be chosen by* (4.7). *If $a_m = c \cdot N^q$, then*

$$m \leq \begin{cases} c \log N, & \nu = 1 \\ c \log(\log N), & \nu \geq 2\,. \end{cases} \tag{4.8}$$

■

Proof. Equation (4.7) yields the recurrence relations

$$a_i = a_{i-1} + c_0 a_{i-1}^{\nu}, \qquad i = 1, \ldots, m$$

which imply the estimate

$$a_m \cong \begin{cases} (1 + c_0)^{m-1}, & \nu = 1 \\ (1 + c_0)^{\nu^{m-1}}, & \nu \geq 2\,. \end{cases}$$

Now (4.8) follows.

Corollary 4.1 *For the number of subdomains M_∞ the estimate*

$$M_\infty = \begin{cases} O(\log N), & \nu = 1 \\ O(\log\log N), & \nu \geq 2 \end{cases}$$

holds. ∎

The mesh enlargement factor for the choice (4.7) is given by the equations

$$\begin{aligned} h_{i+1} &= (1+c_0)\, h_i\,, & \nu = 1 \\ h_{i+1} &= a_i\, h_i\,, & \nu \geq 2\,. \end{aligned}$$

Note that in the case of $\nu \geq 2$ the quasi-optimal choice (4.7) of the mesh sizes h_i seems to be not promising since $a_i \sim O(N^q)$ for large indices i and, thus, the conformal nested spaces W_j, $j = 0, 1, \ldots p = p_0 + m$ are hardly available. In this case the Lagrange multipliers method may be applied.
Thus the conformal approximation of the exterior interface problem under consideration appears to be efficient with the choice $\nu = 1$ in (4.7) even for $\nu \geq 2$ in the "radiation condition" (4.1). The complexity of the matrix vector multiplication for the operator A_{Γ_p} is estimated now by

$$Q(A_{\Gamma_p}) \sim M_\infty \cdot O(N \log^2 N), \qquad M_\infty \sim O(\log N)\,.$$

In the case of a polygonal boundary the refined interface is defined to patch the boundary Γ with the closest rectangular domain, see Fig 4b), and then we proceed with the non-matching decomposition as on Fig. 4a). The number of right triangles involved is of the same order as the number N_0 of the edges on Γ. Thus, in the general case we obtain

$$Q(A_{\Gamma_p}) = N_0 \cdot O(N \log^3 N) + M_\infty \cdot O(N \log^2 N) \sim O(N \log^3 N)\,.$$

Theorem 4.1 *The MAS method related to the interface reduction of the exterior problem* (4.5) *has the overall complexity and memory needs $O(N \log^3 N)$. Thus, the matrix-vector multiplication of the Schur complement matrix $\mathcal{T}_{2p}$ related to T_{2p} has the same cost estimate. The approximate Neumann data $\gamma_{1h}\overline{u} = T_{2p} Q_2 u$ on Γ admits the error estimate*

$$\|Q_2 T_2 u - \gamma_{1h}\overline{u}\|_{H^{-1/2}(\Gamma)} \leq c \|\overline{u} - \mathcal{P}_h \overline{u}\|_{H^1(\Omega_\infty)}$$

where $\overline{u}$ solves the problem (2.1), (2.2), (4.1) *and $\mathcal{P}_h$ is the Ritz projection defined for W_p.*

Remark 4.1 *The implementation of the exterior PS operators in presence of a nested mesh refinement near the corner points s_j, $j = 0, 1, \ldots, N_0$ may be designed along the same line as in the case of interior problems providing the complexity $O(N \log^3 N)$.*

5 Application in BEM and concluding remarks

Due to Theorems 3.1 and 4.1 the proposed compression scheme provides the complexity $O(N \log^3 N)$ of the matrix-vector multiplication for the stiffness matrices $\mathcal{T}_{1p}$ and $\mathcal{T}_{2p}$

related to the FE approximations T_{1p} and T_{2p} of interior and exterior Poincaré-Steklov operators, respectively. The memory needs are estimated by $O(N \log^2 N)$ and $O(N \log^3 N)$, correspondingly. This leads to the interface solvers (on a nested refined interface) for the Dirichlet, Neumann and some mixed BVPs in both bounded and unbounded domains with polygonal boundaries.

The above results may be easily applied to the construction of asymptotically optimal compression schemes for solving the classical boundary integral equations involving weakly singular, hypersingular and double layer harmonic potential operators V, D and K, respectively. The approach is based on the representation of the inverse to the above mentioned boundary operators in terms of interior and exterior harmonic Poincaré-Steklov operators proposed in [19]. In fact, the following theorem holds.

Theorem 5.1 [19]. *The operator* $V^{-1} : H^{1/2}_{g_0}(\Gamma) \to H^{-1/2}_{-1}(\Gamma)$ *has the representation*

$$V^{-1} = \frac{1}{2}(T_1 + T_2)\,. \tag{5.9}$$

The following formulae

$$(E - K)^{-1} z = \frac{1}{2}(E + S_2 \cdot T_1) z, \qquad \forall\, z \in H^{1/2}(\Gamma) \tag{5.10}$$

$$(E + K)^{-1} z = \frac{1}{2}(E + S_1 \cdot T_2) z\,, \qquad \forall\, z \in H^{1/2}_{g_0}(\Gamma) \tag{5.11}$$

hold. The operator $D^{-1} : H^{-1/2}_{-1}(\Gamma) \to H^{1/2}_{g_0}(\Gamma)$ *has the representation*

$$D^{-1} = \frac{1}{2}(S_1 + S_2) \qquad \square \tag{5.12}$$

Substituting into the above formulae the developed FE approximations of the operators S_i and T_i with $i = 1,\ 2$ one obtains the FE approximations (with respect to h-harmonic extensions of the boundary data) of the inverse to harmonic boundary integral operators. These approximations admit an efficient matrix compression providing a matrix-vector multiplication of the complexity $O(N \log^3 N)$. Here N is the number of degrees of freedom on the underlying polygonal boundary. This approach is well suited for both quasi-uniform and nested refined meshes. Numerical examples for the particular case of step-type boundaries and quasi-uniform meshes may be found in [19].

The extension of the proposed techniques to the case of three-dimensional problems and more general boundary conditions seems to be rather straightforward.

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