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Mean-field interaction of Brownian occupation measures. II: A rigorous construction of the Pekar process

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1. MOTIVATION AND INTRODUCTION

1.1 Motvation. Questions on path measures pertaining to self-attractive random motions, or Gibbs measures on interacting random paths, are often motivated by the important rôle they play in quantum statistical mechanics. A problem similar in spirit to these considerations is connected with the *polaron problem*. The physical question arises from the description of the slow movement of a charged particle, e.g. an electron, in a crystal whose lattice sites are polarized by this motion, influencing the behavior of the electron and determining its *effective behavior*. For the physical relevance of this model, we refer to the lectures by Feynman [F72]. The mathematical layout of this problem was also founded by Feynman. Indeed, he introduced a path integral formulation of this problem and pointed out that the aforementioned effective behavior can be studied via studying a certain path measure. This measure is written in terms of a three dimensional Brownian motion acting under a self-attractive Coulomb interaction:

$$\widehat{\mathbb{P}}_{\lambda,t}(\mathrm{d}\omega) = \frac{1}{Z_{\lambda,t}} \exp\left\{\lambda \int_0^t \int_0^t \mathrm{d}\sigma \mathrm{d}s \frac{\mathrm{e}^{-\lambda|\sigma-s|}}{|\omega_\sigma - \omega_s|}\right\} \mathbb{P}(\mathrm{d}\omega),$$

where $\lambda > 0$ is a parameter, $\mathbb{P}$ refers to the three dimensional Wiener measure and $Z_{\lambda,t}$ is the normalization constant or partition function. One calls $\alpha = 1/\sqrt{\lambda}$ the *coupling parameter*. The physically relevant regime is the *strong coupling limit* as $\alpha \to \infty$, i.e., $\lambda \to 0$.

We remark that the above interaction is *self-attractive*, as the asymptotic behavior of the path measure is essentially determined by those paths which make $|\omega_\sigma - \omega_s|$ small, when $|\sigma - s|$ is also small. In other words, these paths tend to clump together on short time scales.

The asymptotic behavior of the partition function $Z_{\lambda,t}$ in the limit $t \to \infty$, followed by $\lambda \to 0$, was rigorously studied by Donsker and Varadhan ([DV83]). The intuition says that, for small λ, the interaction should get more and more smeared out and should approach the *mean-field interaction*

$$\frac{1}{t} \int_0^t \int_0^t \mathrm{d}\sigma \mathrm{d}s \, \frac{1}{|\omega_\sigma - \omega_s|}.$$

Donsker and Varadhan proved this intuition on the level of the logarithmic large t-asymptotics for the partition function on base of explicit variational formulas. More precisely, they showed that that the large-t asymptotics of $Z_{\lambda,t}$ coincides in the limit $\lambda \to 0$ with the large-t asymptotics of the *mean field partition function*

$$Z_t = \mathbb{E}\left[\exp\left\{\frac{1}{t} \int_0^t \int_0^t \mathrm{d}\sigma \mathrm{d}s \, \frac{1}{|W_\sigma - W_s|}\right\}\right], \tag{1.1}$$

where $(W_s)_{s\in[0,\infty)}$ denotes a three-dimensional standard Brownian motion. In other words, they proved *Pekar's conjecture* ([P49]) and derived the following variational formula for the free energy:

$$\begin{aligned} \lim_{\lambda\to 0} \lim_{t\to\infty} \frac{1}{t} \log Z_{\lambda,t} &= \lim_{t\to\infty} \frac{1}{t} \log Z_t \\ &= \sup_{\substack{\psi\in H^1(\mathbb{R}^3)\\ \|\psi\|_2=1}} \left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{d}x\mathrm{d}y \, \frac{\psi^2(x)\psi^2(y)}{|x-y|} - \frac{1}{2}\big\|\nabla\psi\big\|_2^2\right\}, \end{aligned} \tag{1.2}$$

with $H^1(\mathbb{R}^3)$ denoting the usual Sobolev space of square integrable functions with square integrable gradient. The variational formula in (1.2) was analyzed by Lieb ([L76]) with the result that there is a rotationally symmetric maximizer ψ_0, which is unique modulo spatial shifts.

Given (1.2), it is natural to guess that the polaron path measure should somehow be related to the mean-field path measure, given by

$$\widehat{\mathbb{P}}_t(\mathrm{d}\omega) = \frac{1}{Z_t}\, \exp\left\{\frac{1}{t}\int_0^t\int_0^t \mathrm{d}\sigma\mathrm{d}s\, \frac{1}{\big|\omega_\sigma - \omega_s\big|}\right\}\mathbb{P}(\mathrm{d}\omega). \tag{1.3}$$

However, even a clear formulation of such a relation is by far not obvious. Spohn ([Sp87]) presented a heuristic analysis of the effective behavior of the polaron measure $\widehat{\mathbb{P}}_{\lambda,t}$ for $\lambda \sim 0$, whose rigorous asymptotic analysis remains open. The heuristic discussion in [Sp87] is based on the idea that, for the strong coupling regime (i.e, $\lambda \to 0$), on time scales of order λ^{-1}, the polaron measure $\widehat{\mathbb{P}}_{\lambda,t}$ should resemble a process, named as the *Pekar process*, whose empirical distribution can be guessed from the limiting asymptotic behavior of $\widehat{\mathbb{P}}_t \circ L_t^{-1}$, the distributions of the normalized Brownian occupation measures $L_t = \frac{1}{t}\int_0^t \mathrm{d}s\, \delta_{W_s}$, under the mean-field transformations $\widehat{\mathbb{P}}_t$, whose rigorous analysis was also left open.

This paper is devoted to a precise analysis of the limiting behavior of the distributions of mean-field path measures $\widehat{\mathbb{P}}_t \circ L_t^{-1}$ and a rigorous construction of the law of the aforementioned Pekar process. Hence, it is a contribution to the understanding of the mean-field approximation of the polaron problem on the level of path measures.

1.2 The model and the problem.

We turn to a precise formulation and the mathematical layout of the problem. We start with the Wiener measure $\mathbb{P}$ on $\Omega = C([0,\infty),\mathbb{R}^3)$ corresponding to a 3-dimensional Brownian motion $W = (W_t)_{t\geq 0}$ starting from the origin. Let

$$L_t = \frac{1}{t}\int_0^t \mathrm{d}s\, \delta_{W_s} \tag{1.4}$$

be the normalized occupation measure of W until time t. This is a random element of $\mathcal{M}_1(\mathbb{R}^3)$, the space of probability measures on $\mathbb{R}^3$. Then the mean-field path measure $\widehat{\mathbb{P}}_t$ defined in (1.3) can be written as

$$\widehat{\mathbb{P}}_t(A) = \frac{1}{Z_t}\,\mathbb{E}\big[1\!\!1_A\, \exp\big\{tH(L_t)\big\}\big], \qquad A \subset \Omega,$$

where Z_t is the normalizing constant defined in (1.1) and

$$H(\mu) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\mu(\mathrm{d}x)\,\mu(\mathrm{d}y)}{|x-y|}, \qquad \mu \in \mathcal{M}_1(\mathbb{R}^3), \tag{1.5}$$

denotes the *Coulomb potential energy functional* of μ, or the Hamiltonian. It is the goal of the present paper to analyze and identify the limiting distribution of L_t under $\widehat{\mathbb{P}}_t$.

We recall that the partition function $Z_t = \mathbb{E}\big[\exp\big\{tH(L_t)\big\}\big]$, which is finite in $\mathbb{R}^3$, was analyzed by Donsker and Varadhan [DV83] resulting in the variational formula (1.2). For future reference, let us write

$$\rho = \sup_{\mu\in\mathcal{M}_1(\mathbb{R}^3)}\Big\{H(\mu) - I(\mu)\Big\} = \sup_{\substack{\psi\in H^1(\mathbb{R}^3)\\ \|\psi\|_2=1}}\Bigg\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{d}x\mathrm{d}y\,\frac{\psi^2(x)\psi^2(y)}{|x-y|} - \frac{1}{2}\big\|\nabla\psi\big\|_2^2\Bigg\}, \tag{1.6}$$

where we introduced the functional

$$I(\mu) = \frac{1}{2}\|\nabla\psi\|_2^2 \tag{1.7}$$

if μ has a density ψ^2 with $\psi \in H^1(\mathbb{R}^3)$, and $I(\mu) = \infty$ otherwise. Note that this is the rate function for the classical weak large deviation theory for the distribution of L_t under $\mathbb{P}$, developed by Donsker and Varadhan ([DV75-83]). We remark that both H and I are shift-invariant functionals, i.e., $H(\mu) = H(\mu \star \delta_x)$ and $I(\mu) = I(\mu\star\delta_x)$ for any $x \in \mathbb{R}^3$.

We also recall ([L76]) that the variational formula (1.6) possesses a smooth, rotationally symmetric and centered maximizer ψ_0, which is unique modulo spatial translations. In other words, if $\mathfrak{m}$ denotes the set of maximizing densities, then

$$\mathfrak{m} = \big\{\mu_0 \star \delta_x \colon x \in \mathbb{R}^3\big\}, \tag{1.8}$$

where μ_0 is the probability measure with density ψ_0^2. We will often write $\mu_x = \mu_0 \star \delta_x$ and write ψ_x^2 for its density.

Note that given (1.2) and (1.8), we expect the distribution of L_t under the transformed measure $\widehat{\mathbb{P}}_t$ to concentrate around $\mathfrak{m}$ and, even more, to converge towards a mixture of spatial shifts of μ_0, thanks to the uniqueness statement (1.8) for the free energy variational problem (1.6). Such a precise analysis was carried out by Bolthausen and Schmock [BS97] for a spatially discrete version of $\widehat{\mathbb{P}}_t$, i.e., for the continuous-time simple random walk on $\mathbb{Z}^d$ instead of Brownian motion and a bounded interaction potential $V\colon \mathbb{Z}^d \to [0,\infty)$ with finite support instead of the singular Coulomb potential $x \mapsto 1/|x|$. A first key step in [BS97] was to show that, under the transformed measure, the probability of the local times falling outside any neighborhood of the maximizers decays exponentially. For its proof, the lack of a strong large deviation principle ([DV75-83]) for the local times was handled by an extended version of a standard periodization procedure by folding the random walk into some large torus. Combined with this, an explicit tightness property of the distributions of the local times led to an identification of the limiting distribution.

However, in the continuous setting with a singular Coulomb interaction, the aforementioned periodization technique or any standard compactification procedure does not work well to circumvent the lack of a strong large deviation principle. An investigation of $\widehat{\mathbb{P}}_t \circ L_t^{-1}$, the distribution of L_t under $\widehat{\mathbb{P}}_t$, remained open until a recent result [MV14] rigorously justified the above heuristics, leading to the statement

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t\big\{L_t \notin U(\mathfrak{m})\big\} < 0, \tag{1.9}$$

where $U(\mathfrak{m})$ is any neighborhood of $\mathfrak{m}$ in the weak topology induced by the Prohorov metric, the metric that is induced by all the integrals against continuous bounded test functions. (1.9) implies that the distribution of L_t under $\widehat{\mathbb{P}}_t$ is asymptotically concentrated around $\mathfrak{m}$. Since a one-dimensional picture of $\mathfrak{m}$ reflects an infinite line, its neighborhood resembles an infinite tube. Therefore, assertions similar to (1.9) are referred to as *tube properties*.

It is worth pointing out that although (1.9) requires only the weak topology in the statement, its proof is crucially based on a robust theory of *compactification* $\widetilde{\mathcal{X}}$ of the quotient space

$$\widetilde{\mathcal{M}}_1(\mathbb{R}^d) \hookrightarrow \widetilde{\mathcal{X}}$$

of orbits $\widetilde{\mu} = \{\mu \star \delta_x \colon x \in \mathbb{R}^3\}$ of probability measures μ on $\mathbb{R}^d$ under translations and a full large deviation principle for the distributions of $\widetilde{L}_t \in \widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ embedded in the compactification. In particular, this is based on a topology induced by a different metric in the compactfication $\widetilde{\mathcal{X}}$. However, the statement (1.9) simply drops out from this abstract set up, thanks to the shift invariance of the Hamiltonian $H(\mu)$ as well as the rate function $I(\mu)$.

Note that, even after proving (1.9), since $\mathfrak{m}$ and hence any neighborhood of this shift-invariant set is non-compact, the occupation measures L_t^{-1} under $\widehat{\mathbb{P}}_t$ could still fluctuate wildly in the infinite tube. The crucial result of the present article says that this can happen only with small $\widehat{\mathbb{P}}_t$-probability and that the distributions $\widehat{\mathbb{P}}_t \circ L_t^{-1}$ converge weakly to an explicit mixture of the elements in $\mathfrak{m}$. The process corresponding to this mixture is the aforementioned Pekar process.

2. Main results

We turn to the statements of our main results.

2.1 Convergence of $\widehat{\mathbb{P}}_t \circ L_t^{-1}$ and identification of the limit.

Recall that ψ_0 is the unique radially symmetric and centered maximizer of the Pekar variational problem (1.6) and that μ_x denotes the probability measure with density $\psi_x^2 = \psi_0^2 \star \delta_x$. Here is the main result of the present paper.

Theorem 2.1. *Let $\widehat{\mathbb{Q}}_t$ denote the distribution of L_t under $\widehat{\mathbb{P}}_t$. Then,*

$$\lim_{t\to\infty} \widehat{\mathbb{Q}}_t = \frac{\int_{\mathbb{R}^3} \mathrm{d}x \; \psi_0(x)\delta_{\mu_x}}{\int_{\mathbb{R}^3} \mathrm{d}x \; \psi_0(x)}, \tag{2.1}$$

weakly as probability measures on $\mathcal{M}_1(\mathbb{R}^3)$.

In words, the distribution of the occupation measures under $\widehat{\mathbb{P}}_t$ converges to a random spatial shift of the maximizer ψ_0^2, and the distribution of this shift is $\psi_0(x)\,\mathrm{d}x$, properly normalized. The proof of Theorem 2.1 can be found in Section 4. For a heuristic sketch, we refer to Section 2.4. We also remark that, following exactly the same line of arguments appearing in the proof of Theorem 2.1, one can derive the asymptotic behavior of the distribution $\widehat{\mathbb{P}}_t \circ W_t^{-1}$ of the end point of the path W_t under $\widehat{\mathbb{P}}_t$. The limiting distribution is equal to $(\psi_0 \star \psi_0)(x)\,\mathrm{d}x / \int_{\mathbb{R}^3} \mathrm{d}y\,(\psi_0 \star \psi_0)(y)$.

Let us now briefly comment on the limiting behavior of $\widehat{\mathbb{P}}_t$ itself, which drops out from the main steps for the proof of Theorem 2.1, see Section 2.4 for the main arguments in terms of a heuristic proof. This limiting assertion is directly related to the interpretation of a process, the aforementioned Pekar process, corresponding to this infinite volume limit of the Gibbs measures $\widehat{\mathbb{P}}_t$ as $t \to \infty$. Let us first remark on the heuristic definition of the Pekar process set forth in [Sp87]. For any $\mu \in \mathcal{M}_1(\mathbb{R}^3)$, let

$$\big(\Lambda\mu\big)(x) = \bigg(\mu \star \frac{1}{|\cdot|}\bigg)(x) = \int_{\mathbb{R}^3} \frac{\mu(\mathrm{d}y)}{|x-y|}$$

be the smooth *Coulomb functional* of μ. Then via the Feynman-Kac formula corresponding to the semigroup of $\frac12\Delta + \Lambda\psi_0^2$, one can construct the measures

$$\frac{1}{Z_t^{(\psi_0)}} \exp\bigg\{\int_0^t \mathrm{d}s\,(\Lambda\psi_0^2)(W_s)\bigg\}\,\mathrm{d}\mathbb{P}. \tag{2.2}$$

These probability measures then should converge, as $t \to \infty$, towards a measure which governs the law of a stationary diffusion process $(X_s^{(\mathrm{Pek})})_{s\in[0,\infty)}$, driven by the stochastic differential equation

$$\mathrm{d}X_t^{(\mathrm{Pek})} = \mathrm{d}W_t + \bigg(\frac{\nabla\psi_0}{\psi_0}\bigg)(W_t)\,\mathrm{d}t.$$

Note that ψ_0 is centered and the drift points towards the origin and suppresses large fluctuations. Furthermore, the process is ergodic with invariant measure ψ_0^2.

In the light of the above heuristic discussion, our main result has an interesting consequence on a rigorous level, as the proof of Theorem 2.1 reveals what the precise infinite volume limit of the path measures $\widehat{\mathbb{P}}_t$ itself should be. Indeed, this is a spatially inhomogeneous mixture of Markovian path measures with generators

$$\frac12\Delta + \frac{\nabla\psi_x}{\psi_x}\cdot\nabla, \qquad x \in \mathbb{R}^3,$$

with the spatial mixture being taken w.r.t. the measure $\psi_0(x)\,\mathrm{d}x / \int \mathrm{d}y\,\psi_0(y)$. This assertion actually carries the flavor of the classical Gibbs conditioning principle. Given the salient features constituting the proof of Theorem 2.1 (see Section 2.4 for a heuristic sketch), a complete proof of this assertion therefore follows a routine way. To avoid repetition, we refrain from spelling out the details and content ourselves only with its statement, which clearly underlines the rigorous interpretation of the law of the Pekar process as the aforementioned spatial mixture of Markovian path measures and justifies the heuristic discussion in [Sp87].

2.2 Earlier results: Tube properties under $\widehat{\mathbb{P}}_t$.

We collect here some crucial results from [MV14] and [KM15] that will be used in the sequel.

As mentioned before, a crucial first step is to prove that, under $\widehat{\mathbb{P}}_t$, the occupation measures L_t concentrate with high probability in any neighborhood of $\mathfrak{m}$. This has been rigorously justified in Theorem 5.1 in [MV14]:

Theorem 2.2 (Tube property under the weak topology)**.** *For any neighborhood $U(\mathfrak{m})$ of $\mathfrak{m}$ in the weak topology in $\mathcal{M}_1(\mathbb{R}^3)$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t\{L_t \notin U(\mathfrak{m})\} < 0.$$

In [KM15], this tube property in the weak topology has been strengthened in a “functional formïn the strong topology under the uniform metric. Note that $H(\mu) = \big\langle \mu, \Lambda\mu \big\rangle = \int (\Lambda\mu)(x)\, \mu(\mathrm{d}x)$, where we recall the Coulomb functional $\Lambda(\mu)$ of μ. We remark that the Coulomb energy of the Brownian occupation measure,

$$\Lambda_t(x) = \big(\Lambda L_t\big)(x) = \int_{\mathbb{R}^3} \frac{L_t(\mathrm{d}y)}{|x-y|} = \frac{1}{t}\int_0^t \frac{\mathrm{d}s}{|W_s - x|}, \tag{2.3}$$

is almost surely finite in $\mathbb{R}^3$.

Let us write $\Lambda(\psi^2)(x) = \int \mathrm{d}y\, \frac{\psi^2(y)}{|x-y|}$ for functions ψ^2, and recall that $\psi_w^2 = \psi_0^2 \star \delta_w$ denotes the shift of the maximizer ψ_0^2 of the second variational formula (1.6) by $w \in \mathbb{R}^3$. The following theorem was coined as the tube property for Λ_t in the uniform metric (see Theorem 1.1, [KM15]).

Theorem 2.3 (Tube property under the uniform metric)**.** *For any $\varepsilon > 0$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t\bigg\{ \inf_{w\in\mathbb{R}^3} \big\|\Lambda_t - \Lambda\psi_w^2\big\|_\infty > \varepsilon \bigg\} < 0. \tag{2.4}$$

As a consequence of Theorem 2.3, the Hamiltonian $H(L_t) = \langle L_t, \Lambda L_t\rangle$ converges in distribution towards the common Coulomb energy of any member of $\mathfrak{m}$:

Corollary 2.4. *Under $\widehat{\mathbb{P}}_t$, the distributions of $H(L_t)$ converge weakly to the Dirac measure at*

$$H(\psi_0^2) = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\psi_0^2(x)\psi_0^2(y)}{|x-y|}\, \mathrm{d}x\mathrm{d}y.$$

Finally, we state an interesting fact concerning the exponential decay of the probability of deviations of the uniform norm of Λ_t from zero. This fact followed from the exponential regularity estimates for Λ_t in the uniform norm derived in [KM15], see Theorem 1.3 and Corollary 1.4 in [KM15]. The following proposition will be used often in the sequel.

Proposition 2.5. *For any $a > 0$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big\{\|\Lambda_t\|_\infty > a\big\} < 0.$$

2.3 Tightness of the distributions of L_t under $\widehat{\mathbb{P}}_t$.

We now turn to formulations and brief proof outlines of main steps constituting the proof of Theorem 2.1.

Even knowing the localization properties stated in Section 2.2, the occupation measures L_t could *a priori* still float around freely in the infinite tubular neighborhood of $\mathfrak{m}$. The next main step is to justify that this can happen only with small $\widehat{\mathbb{P}}_t$ probability, and this is our second main result.

Theorem 2.6 (Tightness)**.** *For any $\eta > 0$, there exists $k(\eta) > 0$ such that, for any neighborhood U of the set $\{\mu_x : x \in \mathbb{R}^3, |x| \le k(\eta)\}$,*

$$\limsup_{t\to\infty} \widehat{\mathbb{P}}_t\{L_t \notin U\} < \eta.$$

We will actually prove a slightly weaker version of Theorem 2.6. Combined with Theorem 1.9, this result will clearly imply Theorem 2.6.

Theorem 2.7. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$ and for all $\eta > 0$, there exist $k(\varepsilon,\eta) > 0$ and $u(\varepsilon,\eta) > 0$ so that*

$$\widehat{\mathbb{P}}_t\bigg\{ \bigcup_{|x|\ge k(\varepsilon,\eta)} \{L_t \in U_\varepsilon(\mu_x)\}\bigg\} < \eta,$$

for all $t \ge u(\varepsilon,\eta)$.

Section 3 is devoted to the proof of Theorem 2.7. The final step to prove Theorem 2.1 is then to justify the weak convergence of the measures $\widehat{\mathbb{Q}}_t$ to the limiting measure appearing on (2.1). This follows from Theorem 1.9 and Theorem 2.6. Section 4 constitutes the proof of Theorem 2.1. Let us now heuristically sketch the central idea behind the proof of Theorem 2.7, which contains the heart of the argument for Theorem 2.1.

2.4 Heuristic proof of Theorem 2.7 and Theorem 2.1: Partial path exchange.

We now heuristically justify that, under $\widehat{\mathbb{P}}_t$, L_t can not build up its mass over a long time close to some μ_x if x is far away from the starting point of the path. To estimate this event, we study the ratio

$$\frac{\widehat{\mathbb{P}}_t(L_t \approx \mu_x)}{\widehat{\mathbb{P}}_t(L_t \approx \mu_0)} = \frac{\mathbb{E}\big\{ \exp\{tH(L_t)\}\, 1\!\!1_{\{L_t\approx\mu_x\}}\big\}}{\mathbb{E}\big\{ \exp\big\{tH(L_t)\big\}\, 1\!\!1_{\{L_t\approx\mu_0\}}\big\}}, \tag{2.5}$$

and show that this is small as $t \to \infty$. For this, we emulate an approach similar to [BS97], which resembles the Peierls argument in the Ising model.

It is reasonable to expect, that for the event $\{L_t \approx \mu_x\}$ to happen, the path, starting from the origin, reaches a neighborhood of x relatively quickly, say by time t_0, and concentrates in that neighborhood for the remaining time time $t - t_0$. This leads to a splitting of the occupation measure

$$L_t = \frac{t_0}{t} L_{t_0} + \frac{t-t_0}{t} L_{t_0,t}$$

where $L_{t_0,t}$ denotes the normalized occupation measure of the path from t_0 to t and L_{t_0} is the normalized occupation times of a path that runs relatively quickly from the origin to x. Although, $t_0 \ll t$, note that the time t_0 should also get large as $|x|$ gets large too. Note that this splitting the occupation measure L_t also leads to the splitting of the Hamiltonian

$$tH(L_t) = \frac{t_0^2}{t} H(L_{t_0}) + 2\frac{t_0(t-t_0)}{t}\big\langle L_{t_0}, \Lambda_{t_0,t}\big\rangle + \frac{(t-t_0)^2}{t} H(L_{t_0,t}), \tag{2.6}$$

where $\Lambda_{t_0,t}(x) = \int_{\mathbb{R}^3} \frac{L_{t_0,t}(\mathrm{d}y)}{|x-y|}$ is the Coulomb functional of $L_{t_0,t}$. Since $t_0 \ll t$, the first term on the right-hand side should be negligible.

Let us turn to the second term, which makes the difference between $L_t \approx \mu_x$ and $L_t \approx \mu_0$, and makes the ratio (2.5) small as follows. Here, Theorem 2.3 plays an important rôle. Note that with high probability, on $\{L_t \approx \mu_x\}$, we expect that $L_{t_0,t} \approx \mu_x$ in the weak topology. Hence, $\Lambda_{t_0,t} \approx \Lambda(\mu_x)$ with high $\widehat{\mathbb{P}}_t$-probability in the uniform strong topology, and the second term can essentially be replaced by $2t_0\big\langle L_{t_0}, \Lambda(\mu_x)\big\rangle$. But $\Lambda(\mu_x)$, being concentrated around x, has only vanishing interaction with L_{t_0} as $|x|$ is large. On the contrary, on the event $\{L_{t_0} \approx \mu_0\}$, we have $\Lambda_{t_0,t} \approx \Lambda(\mu_0)$, which has a non-trivial interaction with L_{t_0}. Hence, the difference made by the second term appearing in the numerator and the denominator in (2.5) can be quantified as $-Ct_0$ with some C that depends only on ψ_0, and t_0 is large if $|x|$ is large.

The third term on the right hand side of (2.6) involves only the path on the time interval $[t_0, t]$ when it hangs around x (for the numerator) respectively around 0 (for the denominator). To compare these two scenarios, we just shift the path in the numerator (when it is close to x) after time t_0 by the amount of $-x$. The crucial upshot is, due to the shift-invariance of the Hamiltonian $H(L_{t_0,t})$, which is being exploited heavily, this *path exchange argument* does not cost anything to the main bulk of the path in the long time interval $[t_0, t]$. If we call the occupation measure of the shifted path $L_t^{(\mathrm{shift})}$, the only feature that essentially distinguishes $H(L_t)$ from $H(L_t^{(\mathrm{shift})})$ is that the former has essentially no contribution from the interaction between the path on $[0, t_0]$ with the path on $[t_0, t]$, where the latter has. For our purposes, we need to quantify this interaction, for which we switch from our original measure $\mathbb{P}$ to the ergodic Markov process with generator

$$\frac{1}{2}\Delta + \left(\frac{\nabla\psi_0}{\psi_0}\right).\nabla$$

starting from 0 with invariant measure μ_0 and density ψ_0^2. Then the Girsanov transformation and the ergodic theorem imply that, for t_0 large,

$$\mathbb{E}\left\{\mathrm{e}^{t_0 H(L_{t_0}\otimes\mu_0)}\mathbb{1}\{W_{t_0}\in \mathrm{d}y\}\right\} \approx \mathrm{e}^{\rho t_0}\psi_0(0)\psi_0(y), \tag{2.7}$$

where ρ is the variational formula defined in (1.6). Summarizing the contributions in the splitting (2.6), it turns out that, on the event $\{L_t \approx x\}$,

$$H(L_t^{(\mathrm{shift})}) \approx H(L_t) + Ct_0.$$

Substituting this in (2.5), we obtain

$$\mathbb{E}\left\{\exp\{tH(L_t)\}\,\mathbb{1}_{\{L_t\approx\mu_x\}}\right\} \lesssim \mathrm{e}^{-Ct_0}\,\mathbb{E}\left\{\exp\{tH(L_t)\}\,\mathbb{1}_{\{L_t\approx\mu_0\}}\right\}.$$

Recall that t_0 is large, since $|x|$ is large. Hence, the ratio (2.5) gets small uniformly in large t. This implies that under $\widehat{\mathbb{P}}_t$, L_t must have its main weight close to the starting point. This ends our survey on the proof of the tightness in Theorem 2.7.

The additional argument for the proof of Theorem 2.1 involves combining (2.7) with a similar statement with the rôles of x and 0 interchanged. Combining it with the aforementioned tightness argument then leads to the conclusion

$$\lim_{t\to\infty}\frac{\widehat{\mathbb{P}}_t(L_t\approx\mu_x)}{\widehat{\mathbb{P}}_t(L_t\approx\mu_0)} = \frac{\psi_x(0)}{\psi_0(0)} = \frac{\psi_0(-x)}{\psi_0(0)}$$

The above statement then implies that,

$$\lim_{t\to\infty}\widehat{\mathbb{P}}_t(L_t\approx\mu_x) = \frac{\psi_0(-x)}{\int_{\mathbb{R}^3}\mathrm{d}y\,\psi_0(y)} = \frac{\psi_0(x)}{\int_{\mathbb{R}^3}\mathrm{d}y\,\psi_0(y)}.$$

Combined with the tube property stated in Theorem 2.2, Theorem 2.1 then follows. Justifying the above heuristic idea will be the content of the rest of the article.

3. Tightness: Proof of Theorem 2.6

In this section we will prove Theorem 2.7. Theorem 3.1 contains the main argument.

Let $B_r(x)$ denote the ball of radius $r > 0$ around $x \in \mathbb{R}^3$ and

$$\tau_r(x) = \inf\{s > 0\colon \big|W_s - x\big| \le r\}$$

be the first hitting time of $B_r(x)$. We also denote by $\xi_r(x)$ the time the Brownian path spends in $B_1\big(W_{\tau_r(x)}\big)$ after time $\tau_r(x)$, before exiting this ball for the first time.

Given any $\varepsilon > 0$, we choose a radius $r_\varepsilon > 0$ so that

$$r_\varepsilon/2 \ge 1/\varepsilon, \qquad \mu_0(B_{r_\varepsilon/2}(0)^{\mathrm{c}}) \le \varepsilon, \qquad \psi_0^2(\cdot) \le \varepsilon \text{ on } B_{r_\varepsilon/2}(0)^{\mathrm{c}}. \tag{3.1}$$

We will denote by d the Prohorov metric on the set of probability measures on $\mathbb{R}^3$. This metric induces the weak topology, which is governed by test integrals against continuous and bounded functions. For any two probability measures μ, ν on $\mathbb{R}^3$, we will also denote by $\|\mu - \nu\|_{\mathrm{TV}}$ the total-variation distance between μ and ν.

Theorem 3.1. *If $\varepsilon > 0$ is chosen small enough, there exists $M(\varepsilon) > 0$ such that for $\min\{|x|, t, u\} > M(\varepsilon)$ and any $\theta \in (0, 1]$,*

$$\widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_x), \xi_{r_\varepsilon}(x) > \theta, \tau_{r_\varepsilon} > u\Big\} \le \frac{C}{\theta}\mathrm{e}^{-u/C} \tag{3.2}$$

for some universal constant $C > 0$. Here $U_\varepsilon(\mu)$ denotes the ε-neighborhood of μ in the Prohorov metric.

Proof. We will prove this theorem in several steps. To abbreviate notation, we will write

$$\tau = \tau_{r_\varepsilon}(x), \quad \xi = \xi_{r_\varepsilon}(x), \quad B_1 = B_1(W_\tau).$$

Along the way, we will introduce positive constants $c_1, c_2, c_3, \dots$, which do not depend on t nor on ε nor on randomness, whose values will not alter after being introduced. However, there will be a universal constant C whose value may and will change from appearance to appearance.

STEP 1: Let us first note that, on the event $\big\{L_t \in U_\varepsilon(\mu_x)\big\}$, for some constant $c_1 > 0$,

$$\tau \le c_1\varepsilon t < t.$$

Indeed, let us choose $M(\varepsilon) > r_\varepsilon$. Since we are interested in the region $|x| > M(\varepsilon)$, we have $0 \notin B_{r_\varepsilon}(x)$. Furthermore, since $|x| > r_\varepsilon$ and $L_t \in U_\varepsilon(\mu_x)$, the time the Brownian motion spends outside $B_{r_\varepsilon}(x)$ is less than a proportion of εt. Hence, on the event $\{L_t \in U_\varepsilon(\mu_x)\}$, $\tau \le c_1\varepsilon t \le t$, for some constant $c_1 > 0$.

Now, let us estimate

$$\begin{aligned}
&\mathbb{E}\Big\{\mathrm{e}^{tH(L_t)}\,\mathbb{1}_{\big\{L_t\in U_\varepsilon(\mu_x),\, \xi>\theta,\, \tau>u\big\}}\Big\} \\
&\le \mathbb{E}\Big\{\mathrm{e}^{tH(L_t)}\,\mathbb{1}_{\big\{L_t\in U_\varepsilon(\mu_x),\, \xi>\theta,\, c_1\varepsilon t\ge\tau>u\big\}}\Big\} \\
&\le \frac{1}{\theta}\int_u^{c_1\varepsilon t+\theta} \mathrm{d}t_0\ \mathbb{E}\Big\{\mathrm{e}^{tH(L_t)}\,\mathbb{1}_{\big\{L_t\in U_\varepsilon(\mu_x),\, \xi>\theta,\, t_0\ge\tau>t_0-\theta\big\}}\Big\} \\
&\le \frac{1}{\theta}\int_u^{c_1\varepsilon t+1} \mathrm{d}t_0\ \mathbb{E}\Big\{\mathrm{e}^{tH(L_t)}\,\mathbb{1}_{\big\{L_t\in U_\varepsilon(\mu_x),\, W_{t_0}\in B_1,\, \tau>t_0-\theta\big\}}\Big\}.
\end{aligned} \tag{3.3}$$

In the last step we have argued that on the event $\{\xi > \theta, t_0 - \theta < \tau \le t_0\}$, the path is the on the ball B_1 of radius 1 around W_τ. Furthermore, we chose $\varepsilon > 0$ small enough so that $c_1\varepsilon t + 1 \le t$ for all sufficiently large t, such that $t_0 < t$ inside the integrand. Also, we will assume that t is so large that $t_0 \le c_1\varepsilon t + 1 \le C\varepsilon t$ for some C.

STEP 2: Le us fix $t_0 \in [u, c_1\varepsilon t + 1]$ and write the convex combination

$$L_t = \frac{t_0}{t}L_{t_0} + \frac{(t - t_0)}{t}L_{t_0,t} \tag{3.4}$$

with

$$L_{t_0,t} = \frac{1}{t - t_0}\int_{t_0}^t \delta_{W_s}\mathrm{d}s$$

denoting the normalized occupation measure of the Brownian path in the time interval $[t_0, t]$. Since,

$$\|L_t - L_{t_0,t}\|_{\mathrm{TV}} \le 2t_0/t,$$

we clearly have $\mathrm{d}\big(L_t, L_{t-t_0}\big) \leq c_2\varepsilon$ for some constant c_2 and all sufficiently small ε. On the event $\{L_t \in U_\varepsilon(\mu_x)\}$, we have $L_{t-t_0} \in U_{c_2\varepsilon}(\mu_x)$. Moreover, let us also denote by

$$\Lambda_{t_0,t}(y) = \int \frac{L_{t_0,t}(\mathrm{d}z)}{|z-y|}$$

the Coulomb functional of $L_{t_0,t}$.

Then, for some constant $c_3 > 0$ to be chosen later,

$$\begin{aligned}
&\mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{L_t\in U_\varepsilon(\mu_x),\ W_{t_0}\in B_1,\ \tau>t_0-\theta\big\}}\bigg\} \\
&\leq \mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{L_{t_0,t}\in U_{c_2\varepsilon}(\mu_x),\ W_{t_0}\in B_1,\ \tau>t_0-\theta,\ \|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty\leq c_3\sqrt{\varepsilon}\big\}}\bigg\} \\
&\qquad + \mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{L_t\in U_\varepsilon(\mu_x),\ \|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty>c_3\sqrt{\varepsilon}\big\}}\bigg\}.
\end{aligned} \tag{3.5}$$

The second term on the right hand side above can be rewritten as, for some constant $c_4 < c_3$,

$$\begin{aligned}
&\mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{L_t\in U_\varepsilon(\mu_x),\ \|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty>c_3\sqrt{\varepsilon}\big\}}\bigg\} \\
&= \mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{L_t\in U_\varepsilon(\mu_x),\ \|\Lambda_t-\Lambda\psi_x^2\|_\infty>c_4\sqrt{\varepsilon}\big\}}\bigg\} \\
&\qquad + \mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{\|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty>c_3\sqrt{\varepsilon},\ \|\Lambda_t-\Lambda\psi_x^2\|_\infty\leq c_4\sqrt{\varepsilon}\big\}}\bigg\}
\end{aligned} \tag{3.6}$$

Let us estimate the second term on the right hand side above. Note that

$$\|\Lambda_t - \Lambda_{t_0,t}\|_\infty = \frac{t_0}{t}\|\Lambda_{t_0} - \Lambda_{t_0,t}\|_\infty \leq C\varepsilon\|\Lambda_{t_0} - \Lambda_{t_0,t}\|_\infty \leq C\varepsilon\big(\|\Lambda_{t_0}\|_\infty + \|\Lambda_{t_0,t}\|_\infty\big).$$

On the prescribed events on the second term on the right hand side of (3.6),

$$\|\Lambda_t - \Lambda_{t_0,t}\|_\infty \geq \|\Lambda_{t_0,t} - \Lambda\psi_x^2\|_\infty - \|\Lambda_t - \Lambda\psi_x^2\|_\infty \geq (c_3 - c_4)\sqrt{\varepsilon}.$$

Hence, we have the estimate

$$\begin{aligned}
&\mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{\|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty>c_3\sqrt{\varepsilon},\ \|\Lambda_t-\Lambda\psi_x^2\|_\infty\leq c_4\sqrt{\varepsilon}\big\}}\bigg\} \\
&\leq \mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{\|\Lambda_{t_0}\|_\infty\geq a_\varepsilon\big\}}\bigg\} + \mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\, 1\!\!1_{\big\{|\Lambda_{t_0,t}\|_\infty\geq b_\varepsilon\big\}}\bigg\}
\end{aligned}$$

where $a_\varepsilon = a\varepsilon^{-1/2}$, $b_\varepsilon = b\varepsilon^{-1/2}$ for some $a, b > 0$. Estimating the right hand side will be the task of the next step.

STEP 3: In this step, we show the following important lemma.

Lemma 3.2. *For any $a, b > 0$ large enough,*

$$\begin{aligned}
&\limsup_{t\to\infty}\ \sup_{t_0\leq t}\ \frac{1}{t_0}\log\widehat{\mathbb{P}}_t\bigg\{\|\Lambda_{t_0}\|_\infty > a\bigg\} < 0, \\
&\limsup_{t\to\infty}\ \sup_{t_0\leq t}\ \frac{1}{t-t_0}\log\widehat{\mathbb{P}}_t\bigg\{\|\Lambda_{t_0,t}\|_\infty > b\bigg\} < 0.
\end{aligned} \tag{3.7}$$

Proof. Let us prove the first statement in (3.7) and denote by $A_{t_0} = \{\|\Lambda_{t_0}\|_\infty > a\}$, and note that, for any $\sigma > 0$, $H(L_\sigma) = \langle\Lambda_\sigma, L_\sigma\rangle \leq \|\Lambda_\sigma\|_\infty$. We recall the convex decomposition (3.4). Then the Hamiltonian

$H(L_t)$ also decomposes accordingly and can be estimated as

$$
\begin{aligned}
tH(L_t) &= \frac{t_0^2}{t}\big\langle \Lambda_{t_0}, L_{t_0}\big\rangle + 2\frac{t_0(t-t_0)}{t}\big\langle \Lambda_{t_0}, L_{t_0,t}\big\rangle + \frac{(t-t_0)^2}{t}H(L_{t_0,t}) \\
&\le t_0\|\Lambda_{t_0}\|_\infty + 2t_0\|\Lambda_{t_0}\|_\infty + \frac{(t-t_0)^2}{t}H(L_{t_0,t})
\end{aligned}
\tag{3.8}
$$

since $t_0 \le t$. Then, by the Markov property at time t_0,

$$
\begin{aligned}
\mathbb{E}\Big\{ \mathrm{e}^{tH(L_t)}\, 1\!\!1_{A_{t_0}} \Big\} &\le \mathbb{E}\bigg\{ \Big(\mathrm{e}^{Ct_0\|\Lambda_{t_0}\|_\infty}\, 1\!\!1_{A_{t_0}} \Big)\, \mathrm{e}^{\frac{(t-t_0)^2}{t}H(L_{t_0,t})} \bigg\} \\
&= \mathbb{E}\bigg[\Big\{ \mathrm{e}^{Ct_0\|\Lambda_{t_0}\|_\infty}\, 1\!\!1_{A_{t_0}} \Big\}\ \mathbb{E}_{W_{t_0}}\Big\{ \mathrm{e}^{\frac{(t-t_0)^2}{t}H(L_{t-t_0})} \Big\} \bigg] \\
&= \mathbb{E}\Big\{ \mathrm{e}^{Ct_0\|\Lambda_{t_0}\|_\infty}\, 1\!\!1_{A_{t_0}} \Big\}\, \mathbb{E}\Big\{ \mathrm{e}^{\frac{(t-t_0)^2}{t}H(L_{t-t_0})} \Big\}.
\end{aligned}
$$

In the last identity above we used the shift-invariance of H. On the other hand, by the above decomposition of $tH(L_t)$, we have a lower bound for the partition function,

$$
Z_t = \mathbb{E}\big\{\mathrm{e}^{tH(L_t)}\big\} \ge \mathbb{E}\big\{\mathrm{e}^{\frac{(t-t_0)^2}{t}H(L_{t_0,t})}\big\} = \mathbb{E}\big\{\mathrm{e}^{\frac{(t-t_0)^2}{t}H(L_{t-t_0})}\big\}.
$$

Then, for $t_0 \le t$,

$$
\widehat{\mathbb{P}}_t\big\{A_{t_0}\big\} \le \mathbb{E}\big\{\mathrm{e}^{Ct_0\|\Lambda_{t_0}\|_\infty}\, 1\!\!1_{A_{t_0}}\big\} \le \mathbb{E}\big\{\mathrm{e}^{2Ct_0\|\Lambda_{t_0}\|_\infty}\big\}^{1/2}\mathbb{P}(A_{t_0})^{1/2}.
$$

By Proposition 2.5, for any $a > 0$,

$$
\limsup_{t_0\to\infty} \frac{1}{2t_0}\log \mathbb{P}\{A_{t_0}\} = \limsup_{t_0\to\infty} \frac{1}{2t_0}\log \mathbb{P}\big\{\|\Lambda_{t_0}\|_\infty > a\big\} < 0.
$$

The proof of Corollary 1.4 in [KM15] reveals that the above negative exponential rate can be made as large as needed if we chose $a > 0$ large enough. Hence, the first assertion in (3.7) is proved, once we justify, for any $C > 0$,

$$
\limsup_{t_0\to\infty} \frac{1}{2t_0}\log \mathbb{E}\big\{\mathrm{e}^{Ct_0\|\Lambda_{t_0}\|_\infty}\big\} < \infty. \tag{3.9}
$$

This assertion will follow from the regularity properties of the random function $x \mapsto \Lambda_1(x)$, derived in in [KM15]. Indeed, note that, by successive conditioning, the Markov property and the shift-invariance of $\|\Lambda_1\|_\infty$, it is enough to justify that some exponential moment of $\|\Lambda_1\|_\infty$ is finite. Note that, we can write, for any $\delta > 0$,

$$
\|\Lambda_1\|_\infty \le \sup_{x_1,x_2\in\mathbb{R}^3:\, |x_1-x_2|\le\delta} \big|\Lambda_1(x_1) - \Lambda_1(x_2)\big| + \sup_{x\in\delta\mathbb{Z}^3} \int_0^1 \frac{\mathrm{d}s}{|W_s - x|}. \tag{3.10}
$$

Let us now handle the first summand. In Lemma 2.4 in [KM15] we proved that, for any $\eta \in (\frac{1}{3}, \frac{1}{2})$, if

$$
M = \int\int_{|x_1-x_2|\le 1} \mathrm{d}x_1\mathrm{d}x_2 \left[\exp\left\{ \beta\Big(\frac{|\Lambda_1(x_1) - \Lambda_1(x_2)|}{|x_1-x_2|^a} \Big)^\rho \right\} - 1 \right],
$$

where $\rho = \frac{1}{1-\eta} > 1$ and $a = 1 - 2\eta > 0$, then, for some $\beta > 0$,

$$
\mathbb{E}(M) < \infty. \tag{3.11}
$$

Using the Garsia-Rodemich-Rumsey estimate, we also proved that (see page 13-14 in [KM15]), for some fixed constant $\gamma > 0$,

$$
\sup_{|x_1-x_2|\le\delta} \big|\Lambda_1(x_1) - \Lambda_1(x_2)\big| \le \frac{1-2\eta}{\beta^{1/\rho}} \int_0^\delta \log\left(1 + \frac{M}{\gamma u^6}\right)^{1/\rho} u^{-2\varepsilon}\mathrm{d}u.
$$

Now if we choose δ small enough, then the right hand side above is smaller than

$$\frac{1-2\eta}{\beta^{1/\rho}}C(\delta)\log\big(M\vee 1\big)^{1/\rho}$$

for some constant $C(\delta)$ which goes to 0 as $\delta\to 0$. Hence, for any $C>0$, by (3.11), we have

$$\mathbb{E}\bigg\{\mathrm{e}^{C\sup_{|x_1-x_2|\le\delta}\left|\Lambda_1(x_1)-\Lambda_1(x_2)\right|}\bigg\}<\infty.$$

Let us turn to the second term on the right hand side of (3.10). Since we are interested in the behavior of the path in the time horizon $[0,1]$, it is enough to estimate the supremum in a bounded box. We will show that, for any fixed $\delta>0$ and any $C>0$,

$$\mathbb{E}\bigg[\sup_{\substack{x\in\delta\mathbb{Z}^3\\|x|\le 2}}\exp\bigg\{C\int_0^1\frac{\mathrm{d}s}{|W_s-x|}\bigg\}\bigg]\le(2/\delta)^3\mathbb{E}\bigg[\exp\bigg\{C\int_0^1\frac{\mathrm{d}s}{|W_s|}\bigg\}\bigg]<\infty. \tag{3.12}$$

For any $\eta>0$, we can write $1/|x|=V_\eta(x)+Y_\eta(x)$ for $V_\eta(x)=1/(|x|^2+\eta^2)^{1/2}$. Since, for any fixed $\eta>0$, V_η is a bounded function, the above claim holds with $V_\eta(W_s)$ replacing $1/|W_s|$. Hence, (by Cauchy-Schwarz inequality, for instance), it suffices to check the above statement with the difference $Y_\eta(W_s)$, which can be written as

$$\begin{aligned}Y_\eta(x)=\frac{1}{|x|}-\frac{1}{\sqrt{\eta^2+|x|^2}}&=\frac{\sqrt{\eta^2+|x|^2}-|x|}{|x|\sqrt{\varepsilon^2+|x|^2}}=\frac{\eta^2}{|x|+\sqrt{\eta^2+|x|^2}}\,\frac{1}{\sqrt{\eta^2+|x|^2}}\,\frac{1}{|x|}\\&=\eta^{-1}\phi\bigg(\frac{x}{\eta}\bigg),\end{aligned}$$

with

$$\phi(x)=\frac{1}{|x|}\,\frac{1}{\sqrt{1+|x|^2}}\,\frac{1}{|x|+\sqrt{1+|x|^2}}.$$

One can bound $\phi(x)$ by $\frac{b}{|x|^{\frac32}}$, since it behaves like $\frac{1}{|x|}$ near 0 and like $\frac{1}{|x|^3}$ near ∞. In particular

$$Y_\eta(x)\le\frac{b\sqrt{\eta}}{|x|^{\frac32}}.$$

Hence, for (3.12), it suffices to show, for $\eta>0$ small enough and any $C>0$,

$$\mathbb{E}\bigg[\exp\bigg\{Cb\sqrt{\eta}\int_0^1\frac{\mathrm{d}s}{|W_s|^{3/2}}\bigg\}\bigg]<\infty. \tag{3.13}$$

For this, we appeal to Portenko's lemma (see [P76]), which states that, if for a Markov process $\{\mathbb{P}^{(x)}\}$ and for a function $\widetilde{V}\ge 0$

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}^{(x)}\bigg\{\int_0^1\widetilde{V}(W_s)\mathrm{d}s\bigg\}\le\gamma<1$$

then

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}^{(x)}\bigg\{\exp\bigg\{\int_0^1\widetilde{V}(W_s)\mathrm{d}s\bigg\}\bigg\}\le\frac{\gamma}{1-\gamma}<\infty.$$

Hence, to prove (3.13), we need to verify that

$$\sup_{x\in\mathbb{R}^3}\mathbb{E}^{(x)}\bigg\{\int_0^1\frac{\mathrm{d}\sigma}{|W_\sigma|^{\frac32}}\bigg\}=\sup_{x\in\mathbb{R}^3}\int_0^1\mathrm{d}\sigma\int_{\mathbb{R}^3}\mathrm{d}y\,\frac{1}{|y|^{\frac32}}\frac{1}{(2\pi\sigma)^{\frac32}}\exp\bigg\{-\frac{(y-x)^2}{2\sigma}\bigg\}<\infty.$$

One can see that

$$\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{d}y\frac{1}{|y|^{\frac32}}\frac{1}{(2\pi\sigma)^{\frac32}}\exp\bigg\{-\frac{(y-x)^2}{2\sigma}\bigg\}$$

is attained at $x = 0$ because we can rewrite the integral by Parseval's identity as

$$c\int_{\mathbb{R}^3} \exp\bigg\{-\frac{\sigma|\xi|^2}{2} + i\langle x,\xi\rangle\bigg\}\frac{1}{|\xi|^{\frac{3}{2}}}d\xi,$$

where $c > 0$ is a constant. When $x = 0$, the integral reduces to $\int_0^1 \sigma^{-3/4}\,\mathrm{d}\sigma$, which is finite. This finishes the proof of the first assertion in (3.7). The second assertion follows essentially the same arguments, if we upper estimate by Markov property,

$$\mathbb{E}\bigg\{\mathrm{e}^{tH(L_t)}\,\mathbb{1}_{\{\|\Lambda_{t_0,t}\|_\infty > b\varepsilon^{-1/2}\}}\bigg\} \le \mathbb{E}\bigg\{\mathrm{e}^{\frac{t_0^2}{t}H(L_{t_0})}\bigg\}\mathbb{E}\bigg\{\mathrm{e}^{C(t-t_0)\|\Lambda_{t-t_0}\|_\infty}\,\mathbb{1}_{\{\|\Lambda_{t-t_0}\|_\infty > b\varepsilon^{-1/2}\}}\bigg\}$$

and lower estimate

$$Z_t = \mathbb{E}\big\{\mathrm{e}^{tH(L_t)}\big\} \ge \mathbb{E}\big\{\mathrm{e}^{\frac{t_0^2}{t}H(L_{t_0})}\big\}.$$

Lemma 3.2 is proved. □

STEP 4: We continue with the proof of Theorem 3.1. Let us pick up from (3.6) and turn to the first term on the right hand side. We show that, for x chosen as before (in the statement of Theorem 3.1) and for small enough $\varepsilon > 0$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\widehat{\mathbb{P}}_t\bigg\{L_t \in U_\varepsilon(\mu_x),\quad \|\Lambda_t - \Lambda\psi_x^2\|_\infty > c_4\sqrt{\varepsilon}\bigg\} < 0. \tag{3.14}$$

By Theorem 2.3 it is enough to justify,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\widehat{\mathbb{P}}_t\bigg\{L_t \in U_\varepsilon(\mu_x),\quad \|\Lambda_t - \Lambda\psi_x^2\|_\infty > c_4\sqrt{\varepsilon},\\ \inf_{y\in\mathbb{R}^3}\|\Lambda_t - \Lambda\psi_y^2\|_\infty \le c_4\sqrt{\varepsilon}\bigg\} < 0.\end{aligned} \tag{3.15}$$

Suppose $y \in \mathbb{R}^3$ be such that $|y - x| \le \sqrt{\varepsilon}$. Since ψ_0^2 is a smooth function vanishing at infinity (see [L76]) and the Coulomb function $x \mapsto 1/|x|$ lies in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$, the function $\Lambda\psi_0^2 = \psi_0^2 \star 1/|\cdot|$ is smooth and hence a Lipschitz function. Hence, for some c_5, we have

$$\big|\Lambda\psi_x^2(x) - \Lambda\psi_y^2(x)\big| = \big|\Lambda\psi_0^2(0) - \Lambda\psi_0^2(y-x)\big| \le c_5\sqrt{\varepsilon}.$$

Then, if we chose $c_4 > c_5$, on the event, $\|\Lambda_t - \Lambda\psi_x^2\|_\infty > c_4\sqrt{\varepsilon}$, for $|y - x| \le \varepsilon$,

$$\|\Lambda_t - \Lambda\psi_y^2\|_\infty \ge (c_4 - c_5)\sqrt{\varepsilon}. \tag{3.16}$$

On the other hand, since ψ_0^2 is concentrated at 0, a simple argument using polar coordinates and triangle inequality shows that, if $|y - x| \ge \sqrt{\varepsilon}$,

$$\big|\Lambda\psi_x^2(x) - \Lambda\psi_y^2(x)\big| = \big|\Lambda\psi_0^2(0) - \Lambda\psi_0^2(y-x)\big| \ge c_6\sqrt{\varepsilon}.$$

Hence, for $|y - x| \ge \varepsilon$ and for any $\eta > 0$,

$$\begin{aligned}\|\Lambda_t - \Lambda\psi_y^2\|_\infty &= \sup_{w\in\mathbb{R}^3}\big|\Lambda_t(w) - \Lambda\psi_y^2(w)\big|\\ &\ge \big|\Lambda\psi_x^2(x) - \Lambda\psi_y^2(x)\big| - \bigg|\int_{B_\eta(x)}\frac{L_t(\mathrm{d}z) - \psi_x^2(z)\mathrm{d}z}{|z-x|} + \int_{B_\eta(x)^c}\frac{L_t(\mathrm{d}z) - \psi_x^2(z)\mathrm{d}z}{|z-x|}\bigg|\\ &\ge c_6\sqrt{\varepsilon} - \bigg[\int_{B_\eta(x)}\frac{L_t(\mathrm{d}z)}{|z-x|} + \int_{B_\eta(x)}\frac{\psi_x^2(z)\mathrm{d}z}{|z-x|} + \frac{1}{\eta}\bigg\langle\frac{\eta}{|\cdot - x|}\wedge 1, L_t - \psi_x^2\bigg\rangle\bigg]\\ &\ge c_6\sqrt{\varepsilon} - \bigg[\int_{B_\eta(0)}\frac{L_t(\mathrm{d}z)}{|z|} + \int_{B_\eta(0)}\frac{\psi_0^2(z)\mathrm{d}z}{|z-x|} + \frac{1}{\eta}\mathrm{d}\big(L_t, \psi_x^2\big)\bigg]\\ &\ge c_6\sqrt{\varepsilon} - \bigg[\int_{B_\eta(0)}\frac{L_t(\mathrm{d}z)}{|z|} + \eta^2 + \frac{\varepsilon}{\eta}\bigg],\end{aligned}$$

since $L_t \in U_\varepsilon(\psi_x^2)$. Let us chose $\eta = \sqrt{\varepsilon}$. Then above estimate and (3.17) imply that, for (3.15), it is enough to derive, for some constant $c_7 > 0$ and ε small enough,

$$\limsup_{t\to\infty} \frac{1}{t} \log \widehat{\mathbb{P}}_t \bigg\{ \int_{B_{\sqrt{\varepsilon}}(0)} \frac{L_t(\mathrm{d}z)}{|z|} > c_7\sqrt{\varepsilon} \bigg\} < 0.$$

But the above fact follows from [KM15] (see the proof of Eq.(3.6), p.15, [KM15]). Hence, (3.15) is proved.

STEP 5: We return to (3.5). The arguments of Step 3 and Step 4 above imply that for our purposes, it is enough to study the first term on the right hand side of (3.5). Handling this term will be the goal for the rest of the proof. First, let us write,

$$\begin{aligned} &\mathbb{E}\bigg\{ \mathrm{e}^{tH(L_t)} \, 1\!\!1_{\big\{L_{t_0,t}\in W_{c_2\varepsilon}(\mu_x),\, W_{t_0}\in B_1,\, \tau>t_0-\theta,\, \|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty \le c_3\sqrt{\varepsilon}\big\}} \bigg\} \\ &\le \mathbb{E}\bigg\{ \mathrm{e}^{tH(L_t)} \, 1\!\!1_{\big\{W_{t_0}\in B_1,\, \tau>t_0-\theta,\, \|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty \le c_3\sqrt{\varepsilon},\, \|\Lambda_{t_0}\|_\infty\le c_8\varepsilon^{-1/2}\big\}} \bigg\} \\ &\qquad + \mathbb{E}\bigg\{ \mathrm{e}^{tH(L_t)} 1\!\!1\{\|\Lambda_{t_0}\|_\infty > c_8\varepsilon^{-1/2}\} \bigg\}. \end{aligned} \tag{3.17}$$

Again the argument of Step 3 implies that, for t chosen large enough, the second summand above is exponentially small in t_0 if $\varepsilon > 0$ is chosen small enough and t_0 is chosen large enough, see (3.7). Hence, we turn to the first summand on the right hand side.

Let us rewrite the decomposition of the Hamiltonian in the first line of (3.8)

$$tH(L_t) = \frac{t_0^2}{t}\big\langle \Lambda_{t_0}, L_{t_0}\big\rangle + 2\frac{t_0(t-t_0)}{t}\big\langle \Lambda_{t_0}, L_{t_0,t}\big\rangle + \frac{(t-t_0)^2}{t} H(L_{t_0,t}). \tag{3.18}$$

We will handle the three contributions on the right-hand side separately. The first term is relatively easy to handle. Recall that we assumed that $t_0 \le C\varepsilon t$. Hence $\frac{t_0^2}{t} \le C\varepsilon t_0$. Hence,

$$\frac{t_0^2}{t} H(L_{t_0}) \le C\varepsilon t_0 H(L_{t_0}) = C\varepsilon t_0 \big\langle \Lambda_{t_0}, L_{t_0}\big\rangle \le C\varepsilon t_0 \|\Lambda_{t_0}\|_\infty \le C\sqrt{\varepsilon} t_0, \tag{3.19}$$

on the event $\{\|\Lambda_{t_0}\|_\infty \le c_8\varepsilon^{-1/2}\}$.

The second term is estimated on the prescribed event $\big\{\|\Lambda_{t_0,t} - \Lambda\psi_x^2\|_\infty \le c_3\varepsilon, \tau > t_0 - \theta\big\}$ as follows:

$$\begin{aligned} 2\frac{t_0(t-t_0)}{t}\big\langle \Lambda_{t_0,t}, L_{t_0}\big\rangle &\le t_0 C\sqrt{\varepsilon} + \ \ t_0\big\langle \Lambda\psi_x^2, L_{t_0}\big\rangle \\ &\le t_0 C\sqrt{\varepsilon} + t_0\big\langle \Lambda\psi_x^2, L_{t_0-\theta}\big\rangle + \big\langle \Lambda\psi_x^2, \big\|L_{t_0} - L_{t_0-\theta}\big\|_{\mathrm{TV}}\big\rangle \\ &\le t_0 C\sqrt{\varepsilon} + t_0\big\langle \Lambda\psi_x^2, L_{t_0-\theta}\big\rangle, \end{aligned} \tag{3.20}$$

In the last line we used the fact that

$$\big\|L_{t_0} - L_{t_0-\theta}\big\|_{\mathrm{TV}} \le \frac{2\theta}{t_0} < \frac{2}{u} \le C\sqrt{\varepsilon}.$$

Let us now estimate $\langle \Lambda\psi_x^2, L_{t_0-\theta}\rangle$ on the event $\{\tau > t_0 - \theta\}$, for which we want to use the fact that ψ_x^2 puts most of its mass around $B_{r_\varepsilon/2}(x)$, while on the event under interest, the Brownian path until time $t_0 - \theta$

has not yet touched $B_{r_\varepsilon}(x)$, recall the requirement (3.1). Then,

$$\begin{aligned}
&\big\langle \Lambda\psi_x^2, L_{t_0-\theta}\big\rangle \;\mathbb{1}_{\{\tau>t_0-\theta\}}\\
&= \int_{B_{r_\varepsilon/2}(x)} L_{t_0-\theta}(\mathrm{d}z)\int_{B_{r_\varepsilon/2}(x)^c} \frac{\psi_x^2(y)L_{t_0-\theta}}{|y-z|}\,\mathrm{d}y + \int_{B_{r_\varepsilon/2}(x)^c} L_{t_0-\theta}(\mathrm{d}z)\int_{B_{r_\varepsilon/2}(x)^c\cap B_1(z)} \frac{\psi_x^2(y)}{|y-z|}\,\mathrm{d}y\\
&\quad + \int_{B_{r_\varepsilon/2}(x)^c} L_{t_0-\theta}(\mathrm{d}z)\int_{B_{r_\varepsilon/2}(x)^c\cap B_1(z)^c} \frac{\psi_x^2(y)}{|y-z|}\,\mathrm{d}y\\
&\le \frac{2}{r_\varepsilon} + \varepsilon\int L_{t_0-\theta}(\mathrm{d}z)\int_{B_1(0)}\frac{\mathrm{d}y}{|y|} + \varepsilon\int L_{t_0-\theta}(\mathrm{d}z)\\
&\le C\varepsilon,
\end{aligned} \tag{3.21}$$

where we used that $|y-z|\ge r_\varepsilon/2$ in the first integral, while $\psi_x^2(\cdot)\le\varepsilon$ on $B_{r_\varepsilon/2}(x)^c$ for the second and third integral and that $|y-z|\ge 1$ in the third integral. Furthermore, recall that $2/r_\varepsilon\le\varepsilon$. If we combine (3.20), (3.22) and (3.18), we have an estimate for the first term on the right hand side of (3.17):

$$\begin{aligned}
&\mathbb{E}\Big\{\mathrm{e}^{tH(L_t)}\cdot\mathbb{1}_{\big\{W_{t_0}\in B_1,\ \tau>t_0-\theta,\ \|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty\le c_3\sqrt{\varepsilon},\|\Lambda\|_\infty\le c_9\varepsilon^{-1/2}\big\}}\Big\}\\
&\quad\le \mathrm{e}^{C\sqrt{\varepsilon}t_0}\;\mathbb{E}\Big\{\mathrm{e}^{\frac{(t-t_0)^2}{t}H(L_{t_0,t})}\;\mathbb{1}_{\|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty\le c_3\sqrt{\varepsilon}\;W_{t_0}\in B_1\}}\Big\}.
\end{aligned} \tag{3.22}$$

Let us denote by $\mathcal{F}_{t_0,t}$ the canonical σ-field generated by $(W_s)_{s\in[t_0,t]}$. We claim that, on the event $\{\|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty\le c_3\sqrt{\varepsilon}, W_{t_0}\in B_1\}$, we have

$$\mathbb{E}_x\Big[\mathrm{e}^{tH(L_t)}\big|\mathcal{F}_{t_0,t}\Big]\ge\exp\Big\{c_9t_0+\frac{(t-t_0)^2}{t}H(L_{t_0,t})\Big\}. \tag{3.23}$$

Let us first finish the proof of Theorem 3.1 with this estimate. Note that,

$$\begin{aligned}
Z_t=\mathbb{E}_x\Big\{\mathbb{E}_x\big(\mathrm{e}^{tH(L_t)}\big|\mathcal{F}_{t_0,t}\big)\Big\}&\ge\int_{B_1}\mathbb{E}_x\Big\{\mathbb{E}_x\{\mathrm{e}^{tH(L_t)}\big|\mathcal{F}_{t_0,t}\}\,\mathbb{1}\big\{\|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty\le c_3\sqrt{\varepsilon},W_{t_0}\in\mathrm{d}y\big\}\Big\}\\
&\ge\int_{B_1}\frac{p_{t_0}(x,y)}{p_{t_0}(y,0)}\,\mathbb{E}_0\Big\{\mathbb{E}_x\{\mathrm{e}^{tH(L_t)}\big|\mathcal{F}_{t_0,t}\};\;\mathbb{1}\big\{\|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty\le c_3\sqrt{\varepsilon},W_{t_0}\in\mathrm{d}y\big\}\Big\}
\end{aligned}$$

Note that, if $M(\varepsilon)$ is chosen large enough, then for $y\in B_1=B_1(W_\tau)$, $t\ge u>M(\varepsilon)$ and $|x|>M(\varepsilon)$, we have $p_{t_0}(x,y)\ge p_{t_0}(y,0)$. Then,

$$\begin{aligned}
Z_t&\ge\int_{B_1}\mathbb{E}_0\Big\{\mathbb{E}_x\{\mathrm{e}^{tH(L_t)}\big|\mathcal{F}_{t_0,t}\}\,\mathbb{1}\big\{\|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty\le c\varepsilon,W_{t_0}\in\mathrm{d}y\big\}\Big\}\\
&=\mathrm{e}^{c_9t_0}\mathbb{E}_0\Big\{\exp\Big\{\frac{(t-t_0)^2}{t}H(L_{t_0,t})\Big\}\,\mathbb{1}\big\{\|\Lambda_{t_0,t}-\Lambda\psi_x^2\|_\infty\le c\varepsilon,W_{t_0}\in B_1\big\}\Big\}
\end{aligned}$$

Let us combine the above estimate with (3.22) and recall the conclusions of Step 3 and Step 4. Then, if we go back to (3.3), we conclude that

$$\begin{aligned}
\widehat{\mathbb{P}}_t\Big\{L_t\in U_\varepsilon(\mu_x),\xi>\theta,\tau>u\Big\}&\le\frac{1}{\theta}\int_u^{c_1\varepsilon t+1}\mathrm{d}t_0\;\mathrm{e}^{C\sqrt{\varepsilon}t_0}\,\mathrm{e}^{-c_9t_0}\\
&\le\frac{C}{\theta}\mathrm{e}^{-u/C}.
\end{aligned}$$

This proves Theorem 3.1.

STEP 6: It remains to prove (3.23). Let us again recall the splitting introduced in (3.18). Then, we have a lower bound

$$
\begin{aligned}
tH(L_t) &\geq \frac{(t-t_0)^2}{t} H(L_{t_0,t}) + 2t_0 \frac{t-t_0}{t} \big\langle \Lambda_{t_0,t}, L_{t_0} \big\rangle \\
&\geq \frac{(t-t_0)^2}{t} H(L_{t_0,t}) + Ct_0 \Big[\big\langle \Lambda\psi_x^2, L_{t_0} \big\rangle - \big\langle L_{t_0}, \Lambda_{t_0,t} - \Lambda\psi_x^2 \big\rangle \Big] \\
&\geq \frac{(t-t_0)^2}{t} H(L_{t_0,t}) + Ct_0 \Big[\big\langle \Lambda\psi_x^2, L_{t_0} \big\rangle - \big\| \Lambda_{t_0,t} - \Lambda\psi_x^2 \big\|_\infty \Big] \\
&\geq \frac{(t-t_0)^2}{t} H(L_{t_0,t}) + Ct_0 \Big[\big\langle \Lambda\psi_x^2, L_{t_0} \big\rangle - c\sqrt{\varepsilon} \Big]
\end{aligned} \tag{3.24}
$$

on the event $\|\Lambda_{t_0,t} - \Lambda\psi_x^2\|_\infty \leq c\sqrt{\varepsilon}$. Hence, we infer, on $\{\|\Lambda_{t_0,t} - \Lambda\psi_x^2\|_\infty \leq c\sqrt{\varepsilon}, W_{t_0} \in B_1\}$,

$$
\begin{aligned}
&\mathbb{E}_x \Big[\mathrm{e}^{tH(L_t)} \big| \mathcal{F}_{t_0,t} \Big] \\
&\geq e^{-C\sqrt{\varepsilon} t_0} \exp\Big\{ \frac{(t-t_0)^2}{t} H(L_{t_0,t}) \Big\} \mathbb{E}_x \Big\{ \exp\big\{ t_0 H(L_{t_0} \otimes \mu_x) \big\} \big| W_{t_0} \in B_1 \Big\}.
\end{aligned} \tag{3.25}
$$

Now we only need to handle the expectation on the right hand side.

We consider a diffusion with generator

$$
\mathfrak{L}^{(\psi_x)} = \frac{1}{2}\Delta + \Big(\frac{\nabla\psi_x}{\psi_x} \Big) \cdot \nabla
$$

corresponding to an ergodic Markov process $\mathbb{P}_x^{(\psi_x)}$ starting from x with invariant density $\psi_x^2(\cdot)$. From the underlying expectation $\mathbb{E}_x$ on the right hand side of (3.25), we want to switch to the corresponding expectation $\mathbb{E}_x^{(\psi_x)}$. By the Cameron-Martin-Girsanov formula ([SV79]),

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{P}_x}{\mathrm{d}\mathbb{P}_x^{(\psi_x)}}(\omega)\bigg|_{\mathcal{F}_{t_0}} &= \exp\Big[-\int_0^{t_0} \frac{\nabla\psi_x(\omega_s)}{\psi_x(\omega_s)} \mathrm{d}W_s + \frac{1}{2}\int_0^{t_0} \Big| \frac{\nabla\psi_x(\omega_s)}{\psi_x(\omega_s)} \Big|^2 \mathrm{d}s \Big] \\
&= \exp\Big[\log\psi_x(\omega_0) - \log\psi_x(\omega_{t_0}) + \frac{1}{2}\int_0^{t_0} \frac{\Delta\psi_x(\omega_s)}{\psi_x(\omega_s)} \mathrm{d}s \Big] \\
&= \frac{\psi_x(x)}{\psi_x(\omega_{t_0})} \exp\Big[\frac{1}{2}\int_0^{t_0} \frac{\Delta\psi_x(\omega_s)}{\psi_x(\omega_s)} \mathrm{d}s \Big].
\end{aligned} \tag{3.26}
$$

Let us recall the variational formula (1.6). A simple perturbation argument shows that the maximizing function $\psi_0 \in H^1(\mathbb{R}^3)$ satisfies the Euler-Lagrange equation ([L76])

$$
\Big(\frac{1}{2}\Delta + \int_{\mathbb{R}^3} \frac{\psi_0^2(y)}{|x-y|} \mathrm{d}y \Big) \psi_0(x) = \rho\psi_0(x). \tag{3.27}
$$

We multiply (3.27) on both sides by $\psi_0(x)$, integrate over $\mathbb{R}^3$ and recall that $\int_{\mathbb{R}^3} \psi_0^2 = 1$ to see that

$$
\rho = \Big[\int\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\psi_0^2(x)\psi_0^2(y)}{|x-y|} - \frac{1}{2}\|\nabla\psi_0\|_2^2 \Big] > 0.
$$

Now we divide (3.27) by $\psi_0(x)$, plug in $x = W(s)$ and integrate on the time interval $[0, t_0]$ to get

$$
\begin{aligned}
\frac{1}{2}\int_0^{t_0} \frac{\Delta\psi_0(W_s)}{\psi_0(W_s)} \mathrm{d}s + \int_0^{t_0} \int_{\mathbb{R}^3} \frac{\psi_0^2(y)\mathrm{d}y}{|W_s - y|} &= \frac{1}{2}\int_0^{t_0} \frac{\Delta\psi_0(W_s)}{\psi_0(W_s)} \mathrm{d}s + t_0 H(L_{t_0} \otimes \mu_0) \\
&= \rho t_0.
\end{aligned}
$$

Repeating the same argument for $\psi_x^2 = \psi_0^2 \star \delta_x$ we get

$$\frac{1}{2}\int_0^{t_0} \frac{\Delta\psi_x(W_s)}{\psi_x(W_s)}\,\mathrm{d}s + t_0 H(L_{t_0}\otimes\mu_x) = \rho t_0. \tag{3.28}$$

We plug in this identity in (3.26) and perform a change of measure in the expectation of the right hand side of (3.25). Then,

$$\begin{aligned}\mathbb{E}_x\bigg\{\exp\big\{t_0 H(L_{t_0}\otimes\mu_x)\big\}\,\big|\,W_{t_0}\in B_1\bigg\} &\geq \int_{B_1}\mathbb{E}_x\bigg[\exp\big\{t_0 H(L_{t_0}\times\mu_x)\big\}\,\mathbb{1}\big\{W_{t_0}\in\mathrm{d}y\big\}\bigg] \\ &= \mathrm{e}^{\rho t_0}\int_{B_1} E_x^{(\psi_x)}\bigg[\frac{\psi_x(x)}{\psi_x(y)}\,\mathbb{1}\big\{W_{t_0}\in\mathrm{d}y\big\}\bigg] \\ &= \mathrm{e}^{\rho t_0}\int_{B_1}\frac{\psi_0(0)}{\psi_0(y-x)}\,\mathbb{P}_x^{(\psi_x)}\big\{W_{t_0}\in\mathrm{d}y\big\}.\end{aligned}$$

Recall that $\mathbb{P}_x^{(\psi_x)}$ is ergodic with invariant measure $\mu_x(\mathrm{d}y) = \psi_x^2(y)\,\mathrm{d}y = \psi_0^2(y-x)\mathrm{d}y$. Hence, by the ergodic theorem,

$$\begin{aligned}&\liminf_{t_0\to\infty}\mathrm{e}^{-\rho t_0}\,\mathbb{E}_x\bigg\{\exp\big\{t_0 H(L_{t_0}\otimes\mu_x)\big\}\,\big|\,W_{t_0}\in B_1\bigg\} \\ &\qquad\geq \int_{B_1}\frac{\psi_0(0)}{\psi_0(y-x)}\,\psi_0^2(y-x)\mathrm{d}y = \psi_0(0)\int_{B_1}\psi_0(y-x)\mathrm{d}y.\end{aligned}$$

We choose, $\sigma_0(\varepsilon)$ such that, for $t_0 \geq \sigma_0(\varepsilon)$,

$$\mathbb{E}_x\bigg\{\exp\big\{t_0 H(L_{t_0}\otimes\mu_x)\big\}\,\big|\,W_{t_0}\in B_1\bigg\} \geq \mathrm{e}^{\rho t_0/2}.$$

Since we are interested in the regime $|x| > M(\varepsilon)$, we need to pick $M(\varepsilon) \geq \sigma_0(\varepsilon)$. We combine this estimate with (3.25) to prove (3.23). Theorem 3.1 is proved. □

To finish the proof of Theorem 2.7, we need two more technical estimates.

Recall that for any $r > 0$, $\tau = \tau_r(x)$ denotes the first hitting time of the ball $B_r(x)$ and $\xi = \xi_r(x)$ denotes the time the Brownian path spends in $B_1\big(W_{\tau_r(x)}\big)$ after time $\tau_r(x)$, before exiting this ball for the first time. It is well known that, for any $\theta > 0$,

$$\mathbb{P}\bigg\{\xi\leq\theta\bigg\} \leq C\exp\bigg\{-\frac{1}{C\theta}\bigg\} \tag{3.29}$$

for some $C > 0$.

Lemma 3.3. *Uniformly in $t > 0, r > 0, x \in \mathbb{R}^3$,*

$$\lim_{\theta\to 0}\widehat{\mathbb{P}}_t\bigg\{\xi_r(x)\leq\theta,\ \tau_r(x)\leq t\bigg\} = 0.$$

Proof. We now split at two time horizons τ and $\tau+\xi$:

$$L_t = \frac{\tau}{t}L_\tau + \frac{\xi}{t}L_{\tau,\tau+\xi} + \frac{t-\tau-\xi}{t}L_{\tau+\xi,t}. \tag{3.30}$$

This also leads to a similar decomposition of Λ_t.

We also write

$$\begin{aligned}L_t' &= \frac{\tau}{t}L_\tau + \frac{t-\tau-\xi}{t}L_{\tau+\xi,t}, \\ \Lambda_t' &= \frac{\tau}{t}\Lambda_\tau + \frac{t-\tau-\xi}{t}\Lambda_{\tau+\xi,t}\end{aligned}$$

and

$$L_t'' = \frac{\tau}{t} L_\tau + \frac{t-\tau-\xi}{t} L_{\tau+\xi,t+\xi}$$
$$\Lambda_t'' = \frac{\tau}{t} \Lambda_\tau + \frac{t-\tau-\xi}{t} \Lambda_{\tau+\xi,t+\xi}.$$

This leads to one crucial upshot. Consider the process $\{Y_s\}_{0\le s\le t}$, defined on $\{\tau < t\}$ by

$$Y_s = \begin{cases} W_s & \text{for } s < t \\ W_{s+\xi} & \text{for } s \in [\tau, t]\,, \end{cases}$$

which jumps at time τ from W_τ to the boundary of $B_1(W_\tau)$. On $\{\tau < t\}$ consider also $Z_s = W_{\tau+s} - W_\tau$ for $s \le \xi$. This is a process starting at 0 observed until the first time it hits the boundary of $B_1(0)$. We consider $\{Z_s\}_{s\le\xi}$ process modulo rotations, i.e. we write

$$\Xi := \left[(W_{\tau+s} - W_\tau)_{0\le s\le\xi}\right]$$

where $[\cdot]$ denotes the equivalence class under the action of rotational group on the whole path in $\mathbb{R}^3$. Since, the distribution of a Brownian motion on the boundary of a ball (when started at the centre of the ball) is the uniform harmonic measure on the sphere, the process $\{Y_s\}$ and (ξ, Ξ) are independent under $\mathbb{P}$.

The splitting of the Hamiltonian according to (3.30) is:

$$tH(L_t) = t\,\langle L_t', \Lambda_t'\rangle + \xi\,\langle L_{\tau,\tau+\xi}, \Lambda_t'\rangle + \xi\,\langle \Lambda_{\tau,\tau+\xi}, L_t'\rangle\,.$$

We fix some constant $a > 0$. Then on the events $\Big\{\,\|\Lambda_t'\|_\infty \le a\Big\}$, $\Big\{\,\|\Lambda_{\tau,\tau+\xi}\|_\infty \le a\Big\}$ and $\Big\{\xi \le \theta \le 1\Big\}$, we have

$$tH(L_t) \le t\,\langle L_t', \Lambda_t'\rangle + 2a \le t\,\langle L_t'', \Lambda_t''\rangle + 2a. \tag{3.31}$$

Note that $\|\Lambda_t'\|_\infty > a$ implies $\|\Lambda_t\|_\infty > a$. Therefore,

$$\begin{aligned} \widehat{\mathbb{P}}_t\Big\{\xi \le \theta,\ \tau+\xi \le t\Big\} \le\ & \widehat{\mathbb{P}}_t\Big\{\xi \le \theta,\ \tau+\xi \le t,\ \|\Lambda_t'\|_\infty \le a,\ \|\Lambda_{\tau,\tau+\xi}\|_\infty \le a\Big\} \\ & + \widehat{\mathbb{P}}_t\Big\{\,\|\Lambda_t\|_\infty > a\Big\} + \widehat{\mathbb{P}}_t\Big\{\,\|\Lambda_{\tau,\tau+\xi}\|_\infty > a,\ \xi \le 1\Big\} \end{aligned} \tag{3.32}$$

We can estimate the first probability on the right hand side above, since by (3.31),

$$\begin{aligned} & \mathbb{E}\Big\{\,\exp\left[tH(L_t)\right]\ \mathbb{1}\big\{\xi \le \theta,\ \tau+\xi \le t,\ \|\Lambda_t'\|_\infty \le a,\ \|\Lambda_{\tau,\tau+\xi}\|_\infty \le a\big\}\Big\} \\ & \le \mathrm{e}^{2a}\ \mathbb{E}\Big\{\,\exp\left[t\,\langle L_t'', \Lambda_t''\rangle\right]\ \mathbb{1}\{\xi \le \theta\}\Big\}. \end{aligned} \tag{3.33}$$

Furthermore, since the Hamiltonian $t\,\langle L_t'', \Lambda_t''\rangle$ is independent of ξ under $\mathbb{P}$, we have,

$$\tag{3.34}$$

Also, since,

$$t\,\langle L_{t+1}, \Lambda_{t+1}\rangle \le 2\,\|\Lambda_1\|_\infty + \|\Lambda_{1,t+1}\|_\infty + t\,\langle L_{1,t+1}, \Lambda_{1.t+1}\rangle\,,$$

we have the estimate,

$$\mathbb{E}\Big\{\,\exp\left[t\,\langle L_{t+1}, \Lambda_{t+1}\rangle\right]\Big\} \le \mathbb{E}\Big\{\mathrm{e}^{2\|\Lambda_1\|_\infty}\Big\}\mathbb{E}\Big\{\mathrm{e}^{\|\Lambda_t\|_\infty}\mathrm{e}^{tH(L_t)}\Big\}.$$

Summarizing (3.32)-(3.34), we have,

$$\begin{aligned}\widehat{\mathbb{P}}_t\Big\{\xi \le \theta,\ \tau+\xi \le t\Big\} \ &\le\ \frac{\mathbb{P}\,(\xi \le \theta)}{\mathbb{P}\,(\xi \le 1)}\mathbb{E}\Big\{\mathrm{e}^{2\|\Lambda_1\|_\infty}\Big\}\ \mathrm{e}^{2a}\ \widehat{\mathbb{E}}_t\Big\{\mathrm{e}^{t\|\Lambda_t\|_\infty}\Big\} \\ &\quad +\widehat{\mathbb{P}}_t\Big\{\ \|\Lambda_t\|_\infty > a\Big\} + \widehat{\mathbb{P}}_t\Big\{\ \|\Lambda_{\tau,\tau+\xi}\|_\infty > a,\ \xi \le 1\Big\}.\end{aligned}$$

By (3.30), we then have,

$$\lim_{\theta\to 0}\sup_{t,x}\widehat{\mathbb{P}}_t\Big\{\xi \le \theta,\ \tau+\xi \le t\Big\} \le \sup_t \widehat{\mathbb{P}}_t\Big\{\ \|\Lambda_t\|_\infty > a\Big\} + \sup_{t,x}\widehat{\mathbb{P}}_t\Big\{\ \|\Lambda_{\tau,\tau+\xi}\|_\infty > a,\ \xi \le 1\Big\}.$$

Since by Corollary 2.5, $\sup_t \widehat{\mathbb{P}}_t\big\{\ \|\Lambda_t\|_\infty > a\big\} \to 0$ for $a \to \infty$, we only have to prove that

$$\lim_{a\to\infty}\sup_{t,x}\widehat{\mathbb{P}}_t\Big\{\ \|\Lambda_{\tau,\tau+\xi}\|_\infty > a,\ \xi \le 1\Big\} = 0.$$

This can again be proved in the same way exploiting that $\|\Lambda_{\tau,\tau+\xi}\|_\infty$ is a function of the equivalence class Ξ and invoking the above independence argument. We drop the details to avoid repetition. $\square$

Lemma 3.4. *For every $\eta > 0$ there exists $r_0 = r_0(\eta) \in \mathbb{N}$ such that,*

$$\sup_{t\ge 1}\sum_{r\ge r_0}\widehat{\mathbb{P}}_t\Big\{\tau_r(0) \le \sqrt{r}\Big\} \le \eta. \tag{3.35}$$

Proof. For any $r > 0$, we again use the splitting of the Hamiltonian as before and use the estimates obtained in Lemma 3.3 to get,

$$\big|H(L_t) - H(L_{\sqrt{r},t+\sqrt{r}})\big| \le \frac{C}{t}\sqrt{r}\|\Lambda_{\sqrt{r}}\|_\infty.$$

Then,

$$\begin{aligned}\mathbb{E}\Big\{\mathrm{e}^{tH(L_t)}\,1\!\!1\{\tau_r \le \sqrt{r}\}\Big\} &\le \mathbb{E}\Big\{\mathrm{e}^{C\sqrt{r}\|\Lambda_{\sqrt{r}}\|_\infty}\,1\!\!1\{\tau_r(0) \le \sqrt{r}\}\Big\}\ \mathbb{E}\Big\{\mathrm{e}^{tH(L_{\sqrt{r},t+\sqrt{r}})}\Big\} \\ &= \mathbb{E}\Big\{\mathrm{e}^{2C\sqrt{r}\|\Lambda_{\sqrt{r}}\|_\infty}\Big\}^{1/2}\mathbb{P}\Big\{\tau_r(0) \le \sqrt{r}\Big\}^{1/2}\mathbb{E}\Big\{\mathrm{e}^{tH(L_t)}\Big\}.\end{aligned}$$

The first expectation grows like $\exp\{C\sqrt{r}\}$ by (3.9) and the probability $\mathbb{P}\{\tau_r(0) \le \sqrt{r}\}$ decays like $\exp\{-Cr^{3/2}\}$. This proves the lemma. $\square$

Finally we are ready to prove Theorem 2.7.

Proof of Theorem 2.7: Given any $\eta > 0$, by Lemma 3.3, we choose $\theta(\eta) > 0$ so that

$$\widehat{\mathbb{P}}_t\Big\{\xi_r(x) \le \theta(\eta)\Big\} \le \eta/3. \tag{3.36}$$

for any t, r, x. From lemma 3.4 we choose $r_0(\eta/3)$ so that

$$\widehat{\mathbb{P}}_t\Big\{\tau_r(0) \le \sqrt{r}\ \text{for some}\ r \ge r_0(\eta)\Big\} \le \eta/3. \tag{3.37}$$

For any given $\varepsilon > 0$, we pick $u(\varepsilon,\eta) > 0$ so that $u(\varepsilon,\eta) \ge \max\big\{M(\varepsilon), r_0(\eta/3)^2\big\}$ such that

$$\frac{C}{\theta(\eta)}\exp\big\{-u(\varepsilon,\eta)/C\big\} \le \eta/3, \tag{3.38}$$

where C is the constant coming from Theorem 3.1 and this does not depend on ε or η. Now, let us choose the radius r_ε as required in (3.1). Then, for

$$|x| \geq k(\varepsilon, \eta) := \max\Big\{M(\varepsilon) + r_\varepsilon, u(\varepsilon, \eta)^2\Big\}, \tag{3.39}$$

on the complement of the event in (3.37), we have $\tau_{r_\varepsilon}(x) \geq \tau^2_{u(\varepsilon,\eta)}(0)$. Hence, according to Theorem 3.1, we have

$$\widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_x),\, \xi_{r_\varepsilon}(x) > \theta(\eta),\, \tau_{r_\varepsilon}(x) > u(\varepsilon, \eta)\Big\} \leq \eta/3. \tag{3.40}$$

Let us combine (3.36), (3.37) (3.40) and stare at the requirement (3.39). We have proved Theorem 2.7. □

4. Identification of the limiting distribution: Proof of theorem 2.1

In this section we will finish the proof of Theorem 2.1. Note that, by the tube property in Theorem 2.2, for the statement in Theorem 2.1, it is enough to prove, for any $x \in \mathbb{R}^3$,

$$\lim_{t\to\infty} \widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_x)\Big\} = \frac{\psi_0(x)}{\int_{\mathbb{R}^3} \mathrm{d}y\, \psi_0(y)}, \tag{4.1}$$

for any $\varepsilon > 0$ small enough. Given the estimates developed in Section 3, the proof of the above claim, modulo slight modification, follows the same spirit of arguments as in [BS97] for random walks on a lattice with a bounded interaction.

For our purposes, in addition to the estimates in Section 3, we will need another technical fact.

$$\lim_{\ell\to\infty} \limsup_{t\to\infty} \sup_{t_0 \leq t} \widehat{\mathbb{P}}_t\Big\{\big|W_{t_0}\big| \geq \ell\Big\} = 0. \tag{4.2}$$

The proof of the above fact is essentially a repetition of the argument of the Proof of Theorem 2.6 and we drop the details. Then, combined with Theorem 2.6, the above fact implies that, for any $\varepsilon > 0$ small enough and any $\eta > 0$, there exists $k(\eta)$ and $\ell(\eta)$ (both independent of ε) such that, for all $t \geq t^\star(\varepsilon, \eta)$,

$$\sup_{t_0\leq t} \widehat{\mathbb{P}}_t\Big\{\bigcup_{|x|\leq k(\eta)} U_\varepsilon(\mu_x),\, \big|W_{t_0}\big| \leq \ell(\eta)\Big\} \geq 1 - \eta. \tag{4.3}$$

Hence, to derive (4.1), we fix $x \in \mathbb{R}^3$ and $\eta > 0$ so that, $|x| \leq k(\eta) \leq \ell(\eta)$. We will choose some $t_0 = t_0(\eta)$ later, which will not depend on ε. For such t_0, let us again invoke the splitting (3.18). Then, by (4.3), for $t > t(\varepsilon, \eta)$,

$$\begin{aligned}\widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_x)\Big\} = \frac{1}{Z_t}\mathbb{E}\bigg[\exp\bigg\{\frac{t_0^2}{t}H(L_{t_0}) + \frac{2t_0(t-t_0}{t}\big\langle L_{t_0}, \Lambda_{t_0,t}\big\rangle + \frac{(t-t_0)^2}{t}H(L_{t_0,t})\bigg\} \\ .\mathbb{1}\big\{L_t \in U_\varepsilon(\mu_x),\, \big|W_{t_0}\big| \leq \ell(\eta)\big\}\bigg] + O(\eta).\end{aligned} \tag{4.4}$$

Since $t \geq t^\star(\varepsilon, \eta)$, if $t^\star(\varepsilon, \eta)$ is appropriately chosen, then

$$\big\{L_t \in U_{\varepsilon/2}(\mu_x)\big\} \subset \big\{L_{t_0,t} \in U_\varepsilon(\mu_x)\big\} \subset \big\{L_t \in U_{2\varepsilon}(\mu_x)\big\}. \tag{4.5}$$

Furthermore, since Theorem 2.2 holds for any $\varepsilon > 0$,

$$\limsup_{t\to\infty} \Big[\widehat{\mathbb{P}}_t\big\{L_t \in U_{2\varepsilon}(\mu_x)\big\} - \widehat{\mathbb{P}}_t\big\{L_t \in U_{\varepsilon/2}(\mu_x)\big\}\Big] = 0. \tag{4.6}$$

Hence, for t large as before,

$$\widehat{\mathbb{P}}_t\Big\{L_t\in U_\varepsilon(\mu_x)\Big\}=\frac{1}{Z_t}\mathbb{E}\Bigg[\exp\bigg\{\frac{t_0^2}{t}H(L_{t_0})+\frac{2t_0(t-t_0}{t}\big\langle L_{t_0},\Lambda_{t_0,t}\big\rangle+\frac{(t-t_0)^2}{t}H(L_{t_0,t})\bigg\}$$
$$.\mathbb{1}\big\{L_{t_0,t}\in U_\varepsilon(\mu_x),\ \big|W_{t_0}\big|\le\ell(\eta)\big\}\Bigg]+O(\eta). \tag{4.7}$$

Now, along a similar line of arguments presented in Step 2, Step 3 and Step 4 in Section 3, we want to make some replacements in the first two summands in the exponential above. Indeed, again, for $t\ge t^\star(\varepsilon,\eta)$ chosen appropriately and for $t_0=t_0(\eta)$ large enough (whose choice, as mentioned, will be made later), we can estimate the first summand

$$\frac{t_0^2}{t}H(L_{t_0})\le C\varepsilon t_0\|\Lambda_{t_0}\|_\infty\le C\sqrt{\varepsilon}t_0+C\varepsilon t_0\|\Lambda_{t_0}\|_\infty\mathbb{1}\big\{\|\Lambda_{t_0}\|_\infty>c\varepsilon^{-1/2}\big\}.$$

This causes an error $O(t_0\sqrt{\varepsilon})$ in the exponent and an additive error $O(\eta)$ (coming from the exponentially small error in $t_0=t_0(\eta)$. We refer to (3.7) for details. Similarly, in the second summand in the exponential in (4.7), we can replace $\Lambda_{t_0,t}$ by $\Lambda\psi_x^2$ by making an exactly similar error, see (3.14) for details. Summarizing, for $\sqrt{\varepsilon}t_0<1$, we have

$$\begin{aligned}
&\widehat{\mathbb{P}}_t\Big\{L_t\in U_\varepsilon(\mu_x)\Big\}\\
&=\frac{1}{Z_t}\mathbb{E}\Bigg[\exp\bigg\{2t_0H(L_{t_0}\otimes\mu_x)+\frac{(t-t_0)^2}{t}H(L_{t_0,t})\bigg\}\mathbb{1}\big\{F_{t_0,t}(x),\ \big|W_{t_0}\big|\le\ell(\eta)\big\}\Bigg]\Big(1+O(\sqrt{\varepsilon}t_0)\Big)+O(\eta)\\
&=\frac{1}{Z_t}\int_{B_{\ell(\eta)}(0)}\mathbb{E}\Bigg[\exp\bigg\{2t_0H(L_{t_0}\otimes\mu_x)\bigg\}\Bigg|W_{t_0}\in\mathrm{d}y\Bigg]\\
&\qquad\mathbb{E}_y\Bigg[\exp\bigg\{\frac{(t-t_0)^2}{t}H(L_{t-t_0})\bigg\}.\mathbb{1}\{F_{t-t_0}(x)\}\Bigg]\Big(1+O(\sqrt{\varepsilon}t_0)\Big)+O(\eta),
\end{aligned} \tag{4.8}$$

where we denoted the event

$$F_{t-t_0}(x)=\big\{L_{t-t_0}\in U_\varepsilon(\mu_x),\ \|\Lambda_{t-t_0}-\Lambda\psi_x^2\|_\infty\le c\sqrt{\varepsilon}\big\}. \tag{4.9}$$

At this point, we will work with an $\varepsilon=\varepsilon(\eta)$ dependent on η so that

$$\sqrt{\varepsilon}t_0=\sqrt{\varepsilon}t_0(\eta)\le\eta,$$

after choosing $t_0=t_0(\eta)$, whose choice will be made precise below. Hence, for $t\ge t^\star(\varepsilon(\eta),\eta)$, the error $O(\sqrt{\varepsilon}t_0)$ can be absorbed in the summand $O(\eta)$.

Recall the measure tilting argument (3.26). Then (3.28) and the ergodic theorem for the measure $\mathbb{P}_x^{(\psi_x)}$ with invariant density ψ_x^2 again implies, as before,

$$\begin{aligned}
\mathbb{E}\Bigg[\exp\bigg\{2t_0H(L_{t_0}\otimes\mu_x)\bigg\}\Bigg|W_{t_0}\in\mathrm{d}y\Bigg]&=\bigg(\frac{\psi_x(0)}{\psi_x(y)}\mathrm{e}^{\rho t_0/2}\bigg)\frac{\mathbb{P}_0^{(\psi_x)}(W_{t_0}\in\mathrm{d}y)}{\mathbb{P}_x(W_{t_0}\in\mathrm{d}y)}\\
&=\Big(\psi_x(0)\psi_x(y)\mathrm{e}^{\rho t_0/2}t^{d/2}\Big)\Big(1+O(\eta)\Big),
\end{aligned} \tag{4.10}$$

uniformly in $|y|\le 2\ell(\eta)$ and for $t_0\ge t_0(\eta,\ell(\eta))$. Here we have also used that,

$$\mathbb{P}_x(W_{t_0}\in\mathrm{d}y)=p_{t_0}(x,y)\mathrm{d}y=t_0^{-d/2}\ \big(1+o(1)\big),$$

uniformly in $|y| \le 2\ell(\eta)$ for the above choice of t_0. Hence, for $t_0 \ge t_0(\eta, \ell(\eta))$ we have,

$$
\begin{aligned}
&\widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_x)\Big\} \\
&\le \frac{\mathrm{e}^{\frac{\rho t_0}{2}} t_0^{d/2} \psi_x(0)}{Z_t} \int_{B_{\ell(\eta)}(0)} \mathrm{d}y\ \psi_x(y) \mathbb{E}_y\bigg[\exp\bigg\{\frac{(t-t_0)^2}{t} H(L_{t-t_0})\bigg\}\ \mathbb{1}_{F_{t-t_0}(x)}\bigg] + O(\eta) \\
&= \frac{\mathrm{e}^{\frac{\rho t_0}{2}} t_0^{d/2} \psi_x(0)}{\psi_0(0) Z_t} \int_{B_{\ell(\eta)}(0)} \mathrm{d}y\ \psi_0(0)\psi_0(y-x) \mathbb{E}_{y-x}\bigg[\exp\bigg\{\frac{(t-t_0)^2}{t} H(L_{t-t_0})\bigg\}\ \mathbb{1}_{F_{t-t_0}(0)}\bigg] + O(\eta) \\
&\le \frac{\mathrm{e}^{\frac{\rho t_0}{2}} t_0^{d/2} \psi_x(0)}{\psi_0(0) Z_t} \int_{B_{2\ell(\eta)}(0)} \mathrm{d}y\ \psi_0(0)\psi_0(y) \mathbb{E}_{y}\bigg[\exp\bigg\{\frac{(t-t_0)^2}{t} H(L_{t-t_0})\bigg\}\ \mathbb{1}_{F_{t-t_0}(0)}\bigg] + O(\eta) \\
&\le \frac{\psi_x(0)}{\psi_0(0)} \frac{1}{Z_t} \int_{B_{2\ell(\eta)}(0)} \mathbb{E}_y\bigg[\exp\Big\{tH(L_t)\Big\}\ \mathbb{1}_{L_{t-t_0} \in U_\varepsilon(\mu_0)}\bigg] + O(\eta).
\end{aligned}
$$

Let us explain the last three steps: In the equality above we used the definition of $F_{t-t_0}(0)$ from (4.9), while in the following estimate, we have used that $|x| \le k(\eta) \le \ell(\eta)$. In the last estimate, we again invoked (4.10) and the bound (3.24).

Furthermore, by using (4.5) and (4.6), we conclude, for $t \ge t^\star(\varepsilon(\eta), \eta)$,

$$
\begin{aligned}
\widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_x)\Big\} &\le \frac{\psi_x(0)}{\psi_0(0)} \widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_0),\ |W_{t_0}| \le 2\ell_\eta\Big\} + O(\eta) \\
&\le \frac{\psi_x(0)}{\psi_0(0)} \widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_0)\Big\} + O(\eta).
\end{aligned}
$$

The above estimate holds true for $\varepsilon = \varepsilon(\eta)$ chosen as before. However, we can combine the above statement with the tube property in Theorem 2.2, which holds for *any* $\varepsilon > 0$. Hence, the above estimate holds for any fixed ε which is small enough (but independent of η), if t is chosen large. We conclude,

$$
\limsup_{t\to\infty} \widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_x)\Big\} \le \frac{\psi_x(0)}{\psi_0(0)} \liminf_{t\to\infty} \widehat{\mathbb{P}}_t\Big\{L_t \in U_\varepsilon(\mu_0)\Big\}.
$$

Since the rôles of x and 0 can be interchanged in the above display, we have proved

$$
\lim_{t\to\infty} \frac{\widehat{\mathbb{P}}_t\{L_t \in U_\varepsilon(\mu_x)\}}{\widehat{\mathbb{P}}_t\{L_t \in U_\varepsilon(\mu_0)\}} = \frac{\psi_x(0)}{\psi_0(0)}.
$$

This proves the desired claim (4.1) and finishes the proof of Theorem 2.1. □

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