On a system of nonlinear PDE's with temperature-dependent hysteresis in one-dimensional thermoplasticity

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1991 Mathematics Subject Classification. 35G25, 73B30, 73E60, 73B05.
Keywords. Thermoplasticity, hysteresis, Prandtl-Ishlinskii operator, weak solutions, PDE's with hysteresis.

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Abstract

In this paper, we develop a thermodynamically consistent description of the uniaxial behaviour of thermoelastoplastic materials that are characterized by a constitutive law of the form $\sigma(x,t) = \mathcal{P}[\varepsilon, \theta(x,t)](x,t)$, where $\varepsilon, \sigma, \theta$ denote the fields of strain, stress and absolute temperature, respectively, and where $\{\mathcal{P}[\cdot, \theta]\}_{\theta>0}$ denotes a family of (rate-independent) hysteresis operators of Prandtl-Ishlinskii type, parametrized by the absolute temperature. The system of state equations governing the space-time evolution of the material are derived. It turns out that the resulting system of two nonlinearly coupled partial differential equations involves partial derivatives of hysteretic nonlinearities at different places. It is shown that an initial-boundary value problem for this system admits a global weak solution. The paper can be regarded as a first step towards a thermodynamic theory of rate-independent hysteresis operators depending on temperature.

1 Introduction

For many materials the stress-strain (σ-ε) relations measured in uniaxial load-deformation experiments strongly depend on the absolute (Kelvin) temperature θ and, at the same time, exhibit a strong elastoplasticity that is witnessed by the occurrence of *hysteresis loops* that are *rate-independent*, i.e. independent of the speed with which there are traversed. Due to the hysteresis, which reflects the presence of a *rate-independent memory* in the material, the stress-strain relation can no longer be expressed in terms of a simple single-valued function. Among the materials showing such very strong temperature-dependent and rate-independent hysteretic effects are the so-called *shape memory alloys* (see, for instance, Chapter 5 in [1]); but even quite ordinary steels are well-known to exhibit this kind of behaviour (cf. [22]), although to a smaller extent.

A classical approach that has been used repeatedly to model temperature-dependent hysteretic stress-strain relations is the following: one first tries to construct a free energy density $F(\varepsilon, \varepsilon_x, \theta)$ of Landau-Ginzburg type in such a way that the observed stress-strain phenomena are matched using the relation

$$\sigma = \frac{\partial F}{\partial \varepsilon}(\varepsilon, \varepsilon_x, \theta)\,, \tag{1.1}$$

and then determines the field equations governing the space-time evolution from the balance laws of linear momentum and of internal energy. In order that a hysteresis be modeled by (1.1), the free energy density $F(\cdot, \varepsilon_x, \theta)$ needs to be non-convex in the range of interesting temperatures.

A typical example for such an approach is the model introduced by F. Falk (cf. [3-5]) to explain the hysteresis phenomena in shape memory alloys. In Falk's model, a Landau-Ginzburg free energy density of Devonshire form,

$$F(\varepsilon, \varepsilon_x, \theta) \;=\; F_0(\theta) + \alpha_1 (\theta - \theta_c)\, \varepsilon^2 - \alpha_2\, \varepsilon^4 + \alpha_3\, \varepsilon^6 + \frac{\gamma}{2} \varepsilon_x^2\,, \tag{1.2}$$

$$F_0(\theta) \;=\; \rho \Big(- C_V\, \theta \ln(\theta/\tilde{\theta}) + C_V\, \theta + C_0 \Big)\,, \tag{1.3}$$

with positive physical constants α_1 , α_2 , α_3 , γ, θ_c , $\tilde{\theta}$, C_V , C_0 , has been assumed. The model leads to a system of nonlinear partial differential equations that has recently been studied in a number of papers (see [8, 18, 20, 21, 25], for both stress- and temperature-controlled experiments, and [2], for deformation-controlled experiments).

The above approach has several disadvantages. At first, the use of a non-convex free energy does not guarantee that a hysteresis actually occurs; for example, in the case of deformation-controlled experiments in shape memory alloys there are strong indications (see [2, 16, 17]) that the occurrence of hysteretic effects is rather due to the presence of an interfacial energy than to a non-convexity of the free energy density. In addition, simple functional relations like (1.1) are certainly not able to give a correct account of the inherent memory structures that are responsible for the complicated loopings in the interior of the hysteresis loops that are observed in experiments.

To avoid these difficulties, we propose a different approach to thermoelastoplastic hysteresis in this paper. For this purpose, we employ the notion of *rate-independent hysteresis operators* introduced by the Russian group around M. A. Krasnoselskii in the seventies. We express the temperature-dependent stress-strain relation in the form of an *operator equation*,

$$\sigma = \mathcal{P}[\varepsilon, \theta] \,, \tag{1.4}$$

where, for every fixed temperature θ, $\mathcal{P}[\cdot, \theta]$ denotes a rate-independent hysteresis operator acting on the set of strain fields ε.

The advantage of this approach is that an operator equation like (1.4) is suited much better than a simple relation like (1.1) to keep track of the memory effects imprinted on the material in the past history. In fact, the output at any time $t \in [0, T]$ may depend on the whole evolution of the input in the time interval $[0, t]$. Observe that the rate-independence implies that $\mathcal{P}$ cannot be expressed in terms of an integral operator of convolution type, i. e. we are not dealing with a model with fading memory.

Unfortunately, there are two disadvantages: the input-output behaviour of rate-independent hysteresis operators usually cannot be described explicitly, and they have, as a rule, only very restricted smoothness properties. Both these facts render the analysis of partial differential equations involving such hysteresis operators difficult.

For the isothermal case, i. e. if $\mathcal{P}$ is independent of θ, a one-dimensional approach to elastoplasticity using rate-independent hysteresis operators has been carried out by P. Krejčí in a series of papers (cf. [10, 11, 13]). In this case, the field equation governing the space-time evolution is the equation of motion which takes the form

$$\rho\, u_{tt}(x,t) \;-\; \Big(\frac{\partial}{\partial x}\, \mathcal{P}[u_x]\Big)(x,t) = f(x,t) \,, \tag{1.5}$$

where ρ and u denote mass density and displacement, respectively.

The non-isothermal case considered in this paper is more complicated. Indeed, the equation of motion has to be supplemented by a field equation representing the balance law of internal energy, and the second principle of thermodynamics in form of the Clausius-Duhem inequality must be obeyed. It is, however, not obvious how the correct expressions for thermodynamic state functions like the densities of free energy, internal energy and entropy, should look like in a situation with a constitutive law of the form (1.5). In the following section, we carry out a corresponding construction for the case when $\mathcal{P}[\cdot, \theta]$ is a family of hysteresis operators of Prandtl-Ishlinskii type, parametrized by the absolute temperature θ. More precisely, we consider stress-strain relations of the form

$$\sigma = \int_0^\infty \varphi(r, \theta)\, \mathfrak{s}_r[\varepsilon]\, dr \,, \tag{1.6}$$

where $\varphi = \varphi(r, \theta)$ is some density function, and where $\mathfrak{s}_r$ denotes the so-called *stop operator* or *elastic-plastic element*. Note that this class of operators is already rather general in the framework of rate-independent elastoplasticity. It will turn out that in our setting it is convenient to regard the densities of free energy, internal energy and entropy as *operators* instead of as *functions*.

The remainder of the paper is organized as follows. In Section 2, we derive the field equations governing the space-time evolution in thermoelastoplastic materials with the constitutive law (1.6). In Section 3, we study an approximating system for which global a priori estimates are derived. Section 4 discusses the passage to the limit and ultimately results in the proof of the existence of a weak solution.

2 Derivation of the Model

The aim of this section is to give a thermodynamically consistent description of the dynamical behaviour of a thermoelastoplastic material characterized by the constitutive law (1.6).

A. Hysteresis Constitutive Operators of Elastoplasticity
L. Prandtl's normalized elastic-perfectly plastic model, corresponding to the rheological combination in series of one elastic (with elasticity modulus 1) and one rigid-perfectly plastic element, provides the simplest example for a hysteresis constitutive operator. It can formally be described as follows.

Let $r > 0$ (the yield limit) and $\sigma_r^0 \in [-r, r]$ (the initial stress) be given numbers. For any input function $\varepsilon \in W^{1,1}(0,T)$, we define the output $\sigma_r \in W^{1,1}(0,T)$ as the solution to the variational inequality (the superimposed dot denotes the time derivative)

$$\sigma_r(t) \in [-r, r] \quad \forall\, t \in [0,T]\,, \tag{2.1}$$

$$(\dot\varepsilon(t) - \dot\sigma_r(t))\,(\sigma_r(t) - \tilde\sigma) \geq 0 \quad \forall\, \tilde\sigma \in [-r, r]\,, \quad \text{a.e. in } (0,T)\,, \tag{2.2}$$

$$\sigma_r(0) = \sigma_r^0\,. \tag{2.3}$$

In Fig. 1, the typical input-output behaviour is depicted.

Fig. 1. Prandtl's normalized elastic-perfectly plastic element.

It can easily be proved that the problem (2.1)-(2.3) admits a unique solution $\sigma_r \in W^{1,1}(0,T)$ for every $\varepsilon \in W^{1,1}(0,T)$ and $\sigma_r^0 \in [-r, r]$ (even in the multi-dimensional case, see [12, 13, 23, 24]). The solution operator

$$\mathfrak{s}_r : [-r, r] \times W^{1,1}(0,T) \to W^{1,1}(0,T) : (\sigma_r^0\,, \varepsilon) \mapsto \sigma_r$$

is called *stop operator* (cf. [9]).

It is immediately seen that for piecewise monotone inputs ε the output $\mathfrak{s}_r[\sigma_r^0\,, \varepsilon]$ can be explicitly described in each monotonicity interval $[t_0, t_1] \subset [0,T]$. Indeed, we can infer that

$$\mathfrak{s}_r[\sigma_r^0, \varepsilon](t) = \begin{cases} \min\{r, \mathfrak{s}_r[\sigma_r^0, \varepsilon](t_0) + \varepsilon(t) - \varepsilon(t_0)\}\,, & t \in [t_0, t_1]\,, \\ \qquad \text{if } \varepsilon \text{ is non-decreasing in } [t_0, t_1]\,, & \\[2ex] \max\{-r, \mathfrak{s}_r[\sigma_r^0, \varepsilon](t_0) + \varepsilon(t) - \varepsilon(t_0)\}\,, & t \in [t_0, t_1]\,, \\ \qquad \text{if } \varepsilon \text{ is non-increasing in } [t_0, t_1]\,. & \end{cases} \tag{2.4}$$

The stop operator has the following properties (For a proof, see [1, 13]).

Proposition 2.1

(i) *Let* $\sigma_r^0 \in [-r,r]$ *and* $\varepsilon \in W^{1,1}(0,T)$ *be given, and let* $\sigma_r := \mathfrak{s}_r[\sigma_r^0,\varepsilon]$. *Then*

$$\sigma_r(t)\ (\dot\varepsilon(t) - \dot\sigma_r(t)) \geq 0\,, \quad a.e.\ in\ (0,T)\,, \tag{2.5}$$

$$(\dot\sigma_r(t))^2 = \dot\varepsilon(t)\,\dot\sigma_r(t)\,, \quad a.e.\ in\ (0,T)\,. \tag{2.6}$$

(ii) *For every* $\sigma_r^{01}, \sigma_r^{02} \in [-r,r]$, $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$, *and* $\sigma_r^i := \mathfrak{s}_r[\sigma_r^{0i},\varepsilon_i]$, $i = 1,2$, *it holds*

$$\int_0^T |\dot\sigma_r^1(t) - \dot\sigma_r^2(t)|\,dt \leq |\sigma_r^{01} - \sigma_r^{02}| + 2\int_0^T |\dot\varepsilon_1(t) - \dot\varepsilon_2(t)|\,dt\,, \tag{2.7}$$

$$|\sigma_r^1(t) - \sigma_r^2(t)| \leq |\sigma_r^{01} - \sigma_r^{02}| + 2\max_{0\leq\tau\leq t}|\varepsilon_1(\tau) - \varepsilon_2(\tau)|\,, \quad \forall\, t \in [0,T]\,. \tag{2.8}$$

Notice that the inequality (2.7) enables us to extend the domain of definition of the stop operator to the whole space $C[0,T]$ and to consider $\mathfrak{s}_r : [-r,r] \times C[0,T] \to C[0,T]$ as a Lipschitz continuous operator.

For functions ε of two variables, $\varepsilon : \mathcal{I} \times [0,T] \to \mathbb{R}$, where $\mathcal{I} \subset \mathbb{R}$ is an interval such that $\varepsilon(x,\cdot) \in C[0,T]$ for almost every $x \in \mathcal{I}$, we define the output of the stop operator with initial configuration $|\sigma_r^0(x)| \leq r$, $x \in \mathcal{I}$, through the formula

$$\mathfrak{s}_r[\sigma_r^0,\varepsilon](x,t) = \mathfrak{s}_r[\sigma_r^0(x),\varepsilon(x,\cdot)](t)\,, \quad (x,t) \in \mathcal{I} \times [0,T]\,. \tag{2.9}$$

Here, we have used the same symbol $\mathfrak{s}_r$ since there is no risk of confusion.

The following properties of the operator defined in (2.9) follow directly from Proposition 2.1.

Proposition 2.2

(i) *The operator* $\mathfrak{s}_r : C(\bar{\mathcal{I}};[-r,r]) \times C(\bar{\mathcal{I}}\times[0,T]) \to C(\bar{\mathcal{I}}\times[0,T])$ *defined in (2.9) is Lipschitz continuous with respect to the supremum-norm.*

(ii) *For any* $\sigma_r^0 \in W^{1,1}(\mathcal{I};[-r,r])$ *and any* $\varepsilon \in W^{1,1}(\mathcal{I}\times(0,T))$ *with* $\varepsilon_x \in L^1(\mathcal{I};L^\infty(0,T))$, *it holds with* $\sigma_r = \mathfrak{s}_r[\sigma_r^0,\varepsilon]$ *that*

$$|(\sigma_r)_x(x,t)| \leq |(\sigma_r^0)_x(x)| + 2\sup_{0\leq\tau\leq t}|\varepsilon_x(x,\tau)|\,, \quad for\ a.e.\ t \in (0,T)\,. \tag{2.10}$$

Following the approach of Prandtl [19] and Ishlinskii [7], we now consider the parallel rheological combination of simple elasto-plastic elements defined by the operator $\mathfrak{s}_r$ with a density function φ which, in our case, *is assumed to depend also on the temperature* θ; that is, we consider a stress-strain relation of the form

$$\sigma(x,t) = \int_0^\infty \varphi(r,\theta(x,t))\,\mathfrak{s}_r[\sigma_r^0,\varepsilon](x,t)\,dr\,. \tag{2.11}$$

For the sake of simplicity, we assume that the initial stress configuration $\sigma_r^0 = \sigma_r^0(x)$ is of the form

$$\sigma_r^0(x) = \operatorname{sign}\varepsilon(x,0)\min\{r,|\varepsilon(x,0)|\}\,, \quad x \in \mathcal{I}\,, \tag{2.12}$$

which characterizes the state without initial memory.

For this choice of σ_r^0, we may simply write $\mathfrak{s}_r[\varepsilon]$ instead of $\mathfrak{s}_r[\sigma_r^0, \varepsilon]$, and the operator $\mathfrak{s}_r : C(\bar{\mathcal{I}} \times [0,T]) \to C(\bar{\mathcal{I}} \times [0,T])$ thus defined is still Lipschitz continuous. This enables us to rewrite the constitutive equation (2.11) in the simpler operator form

$$\sigma = \mathcal{P}[\varepsilon, \theta] := \int_0^\infty \varphi(r, \theta) \mathfrak{s}_r[\varepsilon]\, dr\,, \tag{2.13}$$

with a given non-negative function φ whose properties will be specified below in the hypothesis **(H1)**.

We remark that for each constant temperature θ the function $\varphi(\cdot, \theta)$ can be identified from the initial loading curve $\sigma = \Phi(\varepsilon, \theta)$ which is obtained by plotting the value of σ against a monotonically increasing value of ε from the imperturbed state $\varepsilon(0) = \sigma_r^0 = 0$ for every $r > 0$. We then have (see [1, 10, 13]) $\varphi(r, \theta) = -\Phi_{rr}(r, \theta)$ for constant θ, and the branches of the hysteresis loops are given by the functions $C \pm 2\,\Phi(\frac{1}{2}|\varepsilon - \varepsilon_0|, \theta)$ (see Fig. 2).

Fig. 2. Hysteresis diagram of the operator (2.13) at constant θ.

Similarly to [5, 18, 21], we consider the equation of motion in the form (the density is supposed to be constant and normalized to unity, i.e. $\rho \equiv 1$)

$$u_{tt} - \sigma_x + \mu_{xx} = f(x,t)\,, \tag{2.14}$$

where $\mu = \gamma\, \varepsilon_x\,, \gamma > 0$ given, denotes the *couple stress*, $\varepsilon = u_x\,, \sigma$ has the form (2.13), and f is a given function. We need to couple (2.14) with the balance law for the density of internal energy.

B. The Balance Law of Internal Energy

We now construct an *internal energy operator* $\mathcal{U} = \mathcal{U}[\varepsilon, \theta]$ and an *entropy operator* $\mathcal{S} = \mathcal{S}[\varepsilon, \theta]$ that assign to each pair (ε, θ) of functions the densities of internal energy and of entropy, respectively. For thermodynamic consistency, the first and second laws of thermodynamics, expressed by the balance of internal energy,

$$\mathcal{U}_t = \sigma\, \varepsilon_t + \mu\, \varepsilon_{xt} + g - q_x\,, \tag{2.15}$$

and by the Clausius-Duhem inequality,

$$\mathcal{S}_t \geq \frac{g}{\theta} - \left(\frac{q}{\theta}\right)_x\,, \tag{2.16}$$

respectively, must hold almost everywhere for all functions $\varepsilon, \theta \in L^1(\mathcal{I} \times (0,T))$ satisfying $\varepsilon_t\,, \varepsilon_{xt}\,, \theta_t \in L^1(\mathcal{I} \times (0,T))$. Here, q denotes the heat flux and g is a given heat source density.

Moreover, we have used the abbreviations

$$\mathcal{U}_t = \frac{\partial}{\partial t}\mathcal{U}[\varepsilon,\theta]\,, \quad \mathcal{S}_t = \frac{\partial}{\partial t}\mathcal{S}[\varepsilon,\theta]\,. \tag{2.17}$$

For the heat flux q, we assume Fourier's law

$$q = -\kappa\,\theta_x\,, \tag{2.18}$$

with a constant heat conductivity $\kappa > 0$; hence, assuming $\theta > 0$ (this will have to be verified later on), we can rewrite (2.16) as

$$\mathcal{U}_t - \theta\,\mathcal{S}_t - \sigma\,\varepsilon_t - \mu\,\varepsilon_{xt} \le \frac{\kappa}{\theta}\,\theta_x^2\,, \tag{2.19}$$

or, introducing the *free energy operator* $\mathcal{F} := \mathcal{U} - \theta\mathcal{S}$, as

$$\mathcal{F}_t - \sigma\,\varepsilon_t - \mu\,\varepsilon_{xt} + \theta_t\,\mathcal{S} \le \frac{\kappa}{\theta}\,\theta_x^2\,. \tag{2.20}$$

Here, we have used the abbreviation

$$\mathcal{F}_t = \frac{\partial}{\partial t}\mathcal{F}[\varepsilon,\theta] = \frac{\partial}{\partial t}\left(\mathcal{U}[\varepsilon,\theta] - \theta\,\mathcal{S}[\varepsilon,\theta]\right)\,. \tag{2.21}$$

In order to find a suitable expression for $\mathcal{F}$, we now make use of the well-known fact that the inequality (2.5) can be interpreted as an energy inequality for the individual stop operator $\mathfrak{s}_r$; in addition, the operator $\frac{1}{2}\mathfrak{s}_r^2$ is known to be a *hysteresis potential* for $\mathfrak{s}_r$ (cf. Section 2.5 in [1]). This observation suggests to define the free energy operator $\mathcal{F}$ in the form

$$\mathcal{F}[\varepsilon,\theta] := F_0(\theta) + \frac{\gamma}{2}\varepsilon_x^2 + \frac{1}{2}\int_0^\infty \varphi(r,\theta)\,\mathfrak{s}_r^2[\varepsilon]\,dr\,, \tag{2.22}$$

where F_0 is defined in (1.3). With this choice of $\mathcal{F}$, (2.20) becomes (at least formally)

$$\theta_t\left(\mathcal{S}[\varepsilon,\theta] + F_0'(\theta) + \frac{1}{2}\int_0^\infty \varphi_\theta(r,\theta)\,\mathfrak{s}_r^2[\varepsilon]\,dr\right) \le \frac{\kappa}{\theta}\,\theta_x^2 + \int_0^\infty \varphi(r,\theta)\,\mathfrak{s}_r[\varepsilon]\,(\varepsilon_t - (\mathfrak{s}_r[\varepsilon])_t)\,dr\,. \tag{2.23}$$

Taking (2.5) into account, we see that (2.20) is satisfied provided the density φ is non-negative and the entropy operator $\mathcal{S}$ is defined as

$$\mathcal{S}[\varepsilon,\theta] := -F_0'(\theta) - \frac{1}{2}\int_0^\infty \varphi_\theta(r,\theta)\,\mathfrak{s}_r^2[\varepsilon]\,dr\,. \tag{2.24}$$

Note that then the classical thermodynamic relation between free energy and entropy becomes

$$\mathcal{S} = \mathcal{S}[\varepsilon,\theta] = -\frac{\partial}{\partial\theta}\mathcal{F}[\varepsilon,\theta] = -\frac{\partial\mathcal{F}}{\partial\theta}\,. \tag{2.25}$$

In addition,

$$\mathcal{U}[\varepsilon,\theta] = \mathcal{F}[\varepsilon,\theta] + \theta\,\mathcal{S}[\varepsilon,\theta] = F_0(\theta) - \theta\,F_0'(\theta) + \frac{\gamma}{2}\varepsilon_x^2 + \frac{1}{2}\int_0^\infty (\varphi(r,\theta) - \theta\,\varphi_\theta(r,\theta))\,\mathfrak{s}_r^2[\varepsilon]\,dr\,. \tag{2.26}$$

Consequently, the balance of internal energy (2.15) takes the form

$$\left(C_V\,\theta + \mathcal{V}[\varepsilon,\theta]\right)_t - \kappa\,\theta_{xx} = g + \int_0^\infty \varphi(r,\theta)\,\mathfrak{s}_r[\varepsilon]\,dr\,\varepsilon_t\,, \tag{2.27}$$

where

$$\mathcal{V}[\varepsilon,\theta] \;:=\; \frac{1}{2}\int_0^\infty \left(\varphi(r,\theta) \;-\; \theta\,\varphi_\theta(r,\theta)\right)\mathfrak{s}_r^2[\varepsilon]\,dr\,. \tag{2.28}$$

The analysis of the above equation is independent of the concrete value of the positive constant C_V; we therefore assume in the sequel $C_V = 1$.

C. Statement of the Problem

Since $\varepsilon = u_x$, the equation of motion (2.14) and the balance law of internal energy (2.27), (2.28) constitute a coupled system of nonlinear partial differential equations of the form

$$u_{tt} \,+\, \gamma\,u_{xxxx} \,-\, \left(\mathcal{P}[u_x,\theta]\right)_x \,=\, f(x,t)\,, \tag{2.29}$$

$$\left(\theta \,+\, \mathcal{V}[u_x,\theta]\right)_t \,-\, \kappa\,\theta_{xx} \,=\, g(x,t) \,+\, \mathcal{P}[u_x,\theta]\,u_{xt}\,, \tag{2.30}$$

to be satisfied in $\mathcal{I}\times(0,T)$, where, for the sake of convenience, we assume that $\mathcal{I} = (0,\pi)$. The functions f and g are given data, and $\mathcal{V}[u_x,\theta]$ is defined by (2.28). We complement (2.29), (2.30) by the initial and boundary conditions

$$u(0,t) \,=\, u(\pi,t) \,=\, u_{xx}(0,t) \,=\, u_{xx}(\pi,t) \,=\, 0\,, \quad t\in[0,T]\,, \tag{2.31}$$

$$\theta_x(0,t) \,=\, \theta_x(\pi,t) \,=\, 0\,, \quad t\in[0,T]\,, \tag{2.32}$$

$$u(x,0) \,=\, u^0(x)\,, \quad u_t(x,0) \,=\, v^0(x)\,, \quad \theta(x,0) \,=\, \theta^0(x)\,, \quad x\in[0,\pi]\,. \tag{2.33}$$

We make the following general assumptions on the data of our problem.

(H1)

(i) *$f,g \in L^2\left((0,\pi)\times(0,T)\right)$, $f_x \in L^1\left(0,T;L^2(0,\pi)\right)$, $f(0,t) = f(\pi,t)$ for almost every $t\in(0,T)$, $g(x,t)\geq 0$ for almost every $(x,t)\in(0,\pi)\times(0,T)$.*

(ii) *$u^0 \in W^{3,2}(0,\pi)$, $v^0 \in W^{2,2}(0,\pi)$, $\theta^0 \in W^{1,2}(0,\pi)$, $\theta^0(x)\geq 0$ on $[0,\pi]$, $u^0(0) = u^0(\pi) = u^0_{xx}(0) = u^0_{xx}(\pi) = 0$, $v^0(0) = v^0(\pi) = 0$.*

(iii) *$\varphi_{\theta\theta} \in L^\infty\left((0,\infty),\mathbb{R}\right)$, and there exists some non-negative function $\lambda\in L^1(0,\infty)$, such that for every $\theta\in\mathbb{R}$ and almost every $r>0$ it holds*

$$0 \,\leq\, \varphi(r,\theta) \,\leq\, \lambda(r)\,, \tag{2.34}$$

$$0 \,\leq\, \varphi(r,\theta) \,-\, \theta\,\varphi_\theta(r,\theta) \,\leq\, \lambda(r)\,, \tag{2.35}$$

$$|\varphi_\theta(r,\theta)| \,\leq\, \lambda(r)\,, \tag{2.36}$$

$$|\theta\,\varphi_{\theta\theta}(r,\theta)| \,\leq\, \frac{1}{L\,r^2}\lambda(r)\,, \tag{2.37}$$

where $L := \int_0^\infty \lambda(r)\,dr$.

Remark 1. A non-trivial example for a function φ satisfying **(H1), (iii)** is given by

$$\varphi(r,\theta) \,:=\, \lambda(r)\left(\frac{1}{2} \,+\, \gamma(r)\arctan\left(\frac{\theta}{k(r)}\right)\right)\,, \tag{2.38}$$

provided that $\lambda \in L^1(0,\infty)$ is non-negative, $\gamma \in L^\infty(0,\infty)$ and

$$\sup_{r\geq 0} |\gamma(r)| < \frac{1}{\pi}, \quad k(r) \geq |\gamma(r)| \max\left\{1, \frac{L\,r^2}{2}\right\} \text{ a.e. on } (0,\infty)\,. \tag{2.39}$$

We now state the main result of this paper.

Theorem 2.3 *Suppose that* **(H1)** *holds. Then there exist functions* $u, \theta \in C([0,\pi]\times[0,T])$ *with* $u_{xt}, u_{xxx} \in L^\infty\left(0,T; L^2(0,\pi)\right)$, $u_{tt} \in L^2\left(0,T; W^{-1,2}(0,\pi)\right)$, $\theta_t, \theta_{xx} \in L^2\left((0,\pi)\times(0,T)\right)$, *such that the initial and boundary conditions (2.31)-(2.33) are satisfied and such that the following conditions hold.*

$$\begin{aligned} &\int_0^\pi \int_0^T \Big(-u_t(x,t)\,w(x)\,\psi'(t) + \Big(-\gamma\,u_{xxx} + \mathcal{P}[u_x,\theta]\Big)(x,t)\,w'(x)\,\psi(t)\Big)\,dt\,dx \\ = &\int_0^\pi \int_0^T f(x,t)\,w(x)\,\psi(t)\,dt\,dx\,, \quad \forall\, w \in \overset{o}{W}{}^{1,2}(0,\pi)\,, \quad \forall\, \psi \in \mathcal{D}(0,T)\,, \end{aligned} \tag{2.40}$$

$$(\theta + \mathcal{V}[u_x,\theta])_t - \kappa\,\theta_{xx} = g(x,t) + \mathcal{P}[u_x,\theta]\,u_{xt}\,, \quad \textit{for a.e. } (x,t) \in (0,\pi)\times(0,T)\,, \tag{2.41}$$

with the operator $\mathcal{V}$ *defined in (2.28). In addition,* θ *is non-negative on* $[0,\pi]\times[0,T]$.

Remark 2.
1. Any pair (u,θ) having the properties stated in Theorem 2.3 is called a *weak solution* to the system (2.29)-(2.33).
2. The uniqueness of the solution is an open problem.
3. The equations (2.40), (2.41) are meaningful, since u_x is continuous on $[0,\pi]\times[0,T]$ if u_{xt}, u_{xxx} have the required regularity.

3 Approximation and A Priori Estimates

To establish the existence result, we employ an approximation. For this purpose, we approximate for any $R > 0$ the operator $\mathcal{P}$ by the truncated operator

$$\mathcal{P}_R[\varepsilon,\theta] := \int_0^R \varphi(r,\theta)\,\mathfrak{s}_r[\varepsilon]\,dr + \mathfrak{s}_R[\varepsilon]\int_R^\infty \varphi(r,\theta)\,dr\,, \tag{3.1}$$

and we replace the equations (2.40), (2.41) by a regularized system with parameters $R > 0$ and $\alpha \in (0,1)$, namely

$$\begin{aligned} &\int_0^\pi \int_0^T \Big(-u_t\,w(x)\,\psi'(t) - \alpha\,u_{xt}\,w'(x)\,\psi'(t) + (-\gamma\,u_{xxx} \\ &+ \mathcal{P}_R[u_x,\theta])\,w'(x)\,\psi(t)\Big)\,dt\,dx = \int_0^\pi \int_0^T f\,w(x)\,\psi(t)\,dt\,dx\,, \\ &\forall\, w \in \overset{o}{W}{}^{1,2}(0,\pi)\,, \quad \forall\, \psi \in \mathcal{D}(0,T)\,, \end{aligned} \tag{3.2}$$

$$(\theta + \mathcal{V}_R[u_x,\theta])_t - \kappa\,\theta_{xx} = g + \mathcal{P}_R[u_x,\theta]\,u_{xt}\,, \quad \text{a.e. in } (0,\pi)\times(0,T)\,, \tag{3.3}$$

with the truncated operator

$$\mathcal{V}_R[\varepsilon,\theta] := \frac{1}{2}\int_0^R \Big(\varphi(r,\theta) - \theta\,\varphi_\theta(r,\theta)\Big)\,\mathfrak{s}_r^2[\varepsilon]\,dr + \frac{1}{2}\mathfrak{s}_R^2[\varepsilon]\int_R^\infty \Big(\varphi(r,\theta) - \theta\,\varphi_\theta(r,\theta)\Big)\,dr\,. \tag{3.4}$$

Our intention is to let $\alpha \searrow 0$ and $R \nearrow \infty$.

For every fixed $n \in \mathbb{N}$, we replace the system (2.31)-(2.33), (3.1)-(3.4) by Galerkin approximations: we consider the system

$$(1 + \alpha k^2)\ddot{u}_k + \gamma k^4 u_k + a_k \int_0^\pi \mathcal{P}_R[u_x^{(n)}, \theta^{(n)}] k \cos(kx)\, dx = a_k \int_0^\pi f(x,t) \sin(kx)\, dx\,, \tag{3.5}$$

$$\begin{aligned} \frac{1}{n}\ddot{\theta}_k + a_k \int_0^\pi \left(\theta^{(n)} + \mathcal{V}_R[u_x^{(n)}, \theta^{(n)}]\right)_t \cos(kx)\, dx + \kappa k^2 \theta_k \\ = a_k \int_0^\pi \left(g(x,t) + \mathcal{P}_R[u_x^{(n)}, \theta^{(n)}]\, u_{xt}^{(n)}\right) \cos(kx)\, dx\,, \end{aligned} \tag{3.6}$$

for $k = 0, ..., n$, where $a_0 := \sqrt{\frac{1}{\pi}}$, $a_k := \sqrt{\frac{2}{\pi}}$ for $k \geq 1$,

$$u^{(n)}(x,t) := \sum_{k=1}^n u_k(t)\, a_k \sin(kx)\,, \tag{3.7}$$

$$\theta^{(n)}(x,t) := \sum_{k=0}^n \theta_k(t)\, a_k \cos(kx)\,, \tag{3.8}$$

and where the unknown functions $u_1, ..., u_n, \theta_0, ..., \theta_n$ satisfy the initial conditions

$$u_k(0) = a_k \int_0^\pi u^0(x) \sin(kx)\, dx\,, \quad \dot{u}_k(0) = a_k \int_0^\pi v^0(x) \sin(kx)\, dx\,,$$

$$\theta_k(0) = a_k \int_0^\pi \theta^0(x) \cos(kx)\, dx\,, \quad \dot{\theta}_k(0) = 0\,. \tag{3.9}$$

We rewrite (3.5), (3.6) as a first order system

$$\dot{u}_k = \frac{1}{1 + \alpha k^2} v_k\,, \tag{3.10}$$

$$\dot{v}_k = -\gamma k^4 u_k + a_k \int_0^\pi \left(f \sin(kx) - \mathcal{P}_R[u_x^{(n)}, \theta^{(n)}] k \cos(kx)\right) dx\,, \tag{3.11}$$

$$\frac{1}{n}\dot{\theta}_k = \zeta_k - a_k \int_0^\pi (\theta^{(n)} + \mathcal{V}_R[u_x^{(n)}, \theta^{(n)}]) \cos(kx)\, dx\,, \tag{3.12}$$

$$\dot{\zeta}_k = -\kappa k^2 \theta_k + a_k \int_0^\pi \left(g + \mathcal{P}[u_x^{(n)}, \theta^{(n)}]\, w^{(n)}\right) \cos(kx)\, dx\,, \tag{3.13}$$

with

$$w^{(n)}(x,t) := \sum_{k=1}^n \frac{k\, a_k}{1 + \alpha k^2} v_k(t) \cos(kx)\,, \tag{3.14}$$

and where v_k, ζ_k satisfy

$$v_k(0) = a_k(1 + \alpha k^2) \int_0^\pi v^0(x) \sin(kx)\, dx\,, \tag{3.15}$$

$$\zeta_k(0) = a_k \int_0^\pi \left(\theta^0(x) + \mathcal{V}_R[u_x^{(n)}, \theta^{(n)}](x,0)\right) \cos(kx)\, dx\,. \tag{3.16}$$

Obviously, the system (3.10)-(3.16) is of the form

$$\dot{W} = G(W) + b\,, \quad W(0) = W_0\,, \tag{3.17}$$

where W is a function with values in $\mathbb{R}^{4n+2}$ having the components u_k, v_k, θ_k, ζ_k, and where G is an operator in $C([0,T];\mathbb{R}^{4n+2})$ which is Lipschitz continuous on bounded subsets of $C([0,T];\mathbb{R}^{4n+2})$. The vector function $b \in L^2(0,T;\mathbb{R}^{4n+2})$ is given in terms of f and g. Hence, using standard arguments (successive approximations, say), the system (3.17) has a unique solution $W \in W^{1,2}(0,T_n;\mathbb{R}^{4n+2})$ for some maximal $T_n \in (0,T]$.

We will now derive some a priori estimates that will guarantee that $T_n = T$, for all $n \in \mathbb{N}$, and that will enable us to pass to the limit as $n \to \infty$. In the sequel, C and C_i, $i \in \mathbb{N}$, will always denote positive constants that may depend on the given data, but not on n, R, α. To reduce the notation effort, we will occasionally omit the arguments of functions, and we will denote the $L^2(0,\pi)$- norm by $\|\cdot\|$.

Lemma 3.1 *There is some $C>0$ such that*

$$\|u_{xt}^{(n)}(t)\|^2 + \alpha\|u_{xxt}^{(n)}(t)\|^2 + \gamma\|u_{xxx}^{(n)}(t)\|^2 + \|\theta_x^{(n)}(t)\|^2 + \frac{1}{n}\|\theta_t^{(n)}(t)\|^2 + \int_0^t \|\theta_t^{(n)}(\tau)\|^2 d\tau \le C e^{C(1+R^2T)}, \tag{3.18}$$

for all $n \in \mathbb{N}$, $\alpha \in (0,1)$, $R>0$, and $t \in [0,T_n)$.

Proof. For the sake of brevity, we suppress the index n, and we write $\mathcal{P}_R = \mathcal{P}_R[u_x,\theta]$. Let $t \in [0,T_n)$ be arbitrary. At first, we multiply (3.5) by $k^2\,\dot{u}_k$, sum over k and integrate over $[0,t]$ and by parts to obtain

$$\begin{aligned}
&\frac{1}{2}\Big(\|u_{xt}(t)\|^2 + \alpha\|u_{xxt}(t)\|^2 + \gamma\|u_{xxx}(t)\|^2\Big) - \int_0^\pi \mathcal{P}_R(t)\,u_{xxx}(t)\,dx \\
= \;& \frac{1}{2}\Big(\|u_{xt}(0)\|^2 + \alpha\|u_{xxt}(0)\|^2 + \gamma\|u_{xxx}(0)\|^2\Big) - \int_0^\pi \mathcal{P}_R(0)\,u_{xxx}(0)\,dx \\
&- \int_0^t\!\int_0^\pi u_{xxx}\,(\mathcal{P}_R)_t\,dx\,d\tau + \int_0^t\!\int_0^\pi f_x\,u_{xt}\,dx\,d\tau\,.
\end{aligned} \tag{3.19}$$

From **(H1), (iii)** and (2.6) we infer that almost everywhere in $(0,\pi)\times(0,T)$ it holds

$$|\mathcal{P}_R[u_x,\theta]| \le L\,R, \quad |(\mathcal{P}_R[u_x,\theta])_t| \le L(R|\theta_t| + |u_{xt}|)\,. \tag{3.20}$$

In addition, by Young's inequality and **(H1), (i)**,

$$\int_0^t\!\int_0^\pi f_x\,u_{xt}\,dx\,dt \le \int_0^t \|f_x(\tau)\|\,\|u_{xt}(\tau)\|\,d\tau \le \frac{1}{4}\sup_{0\le\tau\le t}\|u_{xt}(\tau)\|^2 + C_1\,. \tag{3.21}$$

Therefore, using **(H1), (ii)**, and Young's inequality, we can easily see that

$$\begin{aligned}
&\|u_{xt}(t)\|^2 + \alpha\|u_{xxt}(t)\|^2 + \frac{\gamma}{2}\|u_{xxx}(t)\| \\
\le\;& \frac{1}{2}\sup_{0\le\tau\le t}\|u_{xt}(\tau)\|^2 + \frac{1}{4}\int_0^t \|\theta_t(\tau)\|^2\,d\tau \\
&+ C_2(1+R^2)\Big(1 + \int_0^t \Big(\|u_{xt}(\tau)\|^2 + \|u_{xxx}(\tau)\|^2\Big)\,d\tau\Big)\,.
\end{aligned} \tag{3.22}$$

Next, we multiply (3.6) by $\dot{\theta}_k$, sum over k and integrate over $[0,t]$ and by parts to obtain

$$\begin{aligned}
&\frac{1}{2n}\Big(\|\theta_t(t)\|^2 - \|\theta_t(0)\|^2\Big) + \frac{\kappa}{2}\Big(\|\theta_x(t)\|^2 - \|\theta_x(0)\|^2\Big) \\
&+ \int_0^t\!\int_0^\pi |\theta_t|^2\left[1 - \frac{1}{2}\int_0^R \theta\,\varphi_{\theta\theta}(r,\theta)\,\mathfrak{s}_r^2[u_x]\,dr - \frac{1}{2}\,\mathfrak{s}_R^2[u_x]\int_R^\infty \theta\,\varphi_{\theta\theta}(r,\theta)\,dr\right]dx\,d\tau \\
=\;& \int_0^t\!\int_0^\pi \theta_t\left[g + \mathcal{P}_R[u_x]\,u_{xt} - \int_0^R \Big(\varphi(r,\theta) - \theta\,\varphi_\theta(r,\theta)\Big)\,\mathfrak{s}_r[u_x]\,(\mathfrak{s}_r[u_x])_t\,dr\right. \\
&\left. - \,\mathfrak{s}_R[u_x]\,(\mathfrak{s}_R[u_x])_t\int_R^\infty \Big(\varphi(r,\theta) - \theta\,\varphi_\theta(r,\theta)\Big)\,dr\right]dx\,d\tau\,.
\end{aligned} \tag{3.23}$$

Now observe that **(H1), (iii)** implies that almost everywhere on $(0,\pi)\times(0,T)$ it holds

$$\left| \int_0^R \theta\, \varphi_{\theta\theta}(r,\theta)\, \mathfrak{s}_r^2[u_x]\, dr \,+\, \mathfrak{s}_R^2[u_x] \int_R^\infty \theta\, \varphi_{\theta\theta}(r,\theta)\, dr \right| \,\le\, 1\,, \tag{3.24}$$

and, owing to (2.6) and **(H1), (iii)**, the expression in the brackets in the integrand on the right-hand side of (3.23) is bounded from above by $|g| \,+\, 2LR|u_{xt}|$, a.e. on $(0,\pi)\times(0,T)$.

Consequently, we obtain from (3.23) via Young's inequality that

$$\frac{1}{n}\|\theta_t(t)\|^2 \,+\, \kappa\|\theta_x(t)\|^2 \,+\, \frac{1}{2}\int_0^t \|\theta_t(\tau)\|^2\, d\tau \,\le\, C_3(1\,+\,R^2)\Big(1\,+\,\int_0^t \|u_{xt}(\tau)\|^2\, d\tau\Big)\,. \tag{3.25}$$

Adding the inequalities (3.22) and (3.25), taking the supremum with respect to t on both sides and applying Gronwall's inequality, we obtain (3.18). This concludes the proof of the lemma. □

Lemma 3.1 implies, in particular, that the unknowns u_k, v_k, θ_k, ζ_k in the system (3.10)-(3.13) remain bounded in $[0,T_n)$, and therefore we have $T_n \,=\, T$ for all $n\in\mathbb{N}$. Observe also that the bound on the right-hand side of (3.18) is independent of α.

Lemma 3.2 *There is some* $C>0$ *such that*

$$\int_0^T \Big(\|u_{tt}^{(n)}(t)\|^2 \,+\, \alpha\|u_{xtt}^{(n)}(t)\|^2\Big)\, dt \,\le\, \frac{C}{\alpha}\, e^{C(1\,+\,R^2)T}\,, \tag{3.26}$$

for every $n\in\mathbb{N}$, $\alpha\in(0,1)$ *and* $R>0$.

Proof. Again, we omit the index n. Let $t\in[0,T]$ be arbitrary. Multiplying (3.5) by $\ddot{u}_k$, summing over k and integrating over $[0,t]$, we find that

$$\int_0^t \Big(\|u_{tt}(\tau)\|^2 \,+\, \alpha\|u_{xtt}(\tau)\|^2\Big)\, d\tau \,\le\, \int_0^t\int_0^\pi \Big(\gamma\,|u_{xxx}|\,|u_{xtt}| \,+\, LR\,|u_{xtt}| \,+\, |f|\,|u_{tt}|\Big)\, dx\, d\tau, \tag{3.27}$$

whence, using (3.18) and Young's inequality, the assertion easily follows. □

4 Passage to the Limit

In this section, we finish the proof of Theorem 2.3 using compactness arguments and a passage-to-the-limit procedure. We first verify that for fixed α and R the approximate solutions $(u^{(n)},\theta^{(n)})$ have a limit point $(u^{\alpha,R},\theta^{\alpha,R})$ satisfying the system (2.31)-(2.33), (3.1)-(3.4). Then we let α tend to 0 and check that there exists some $R_0>0$, independent of α, such that for every $R\ge R_0$ the limit functions fulfil the conditions of Theorem 2.3.

Step 1: Passage to the limit as $n\to\infty$.

For fixed $\alpha\in(0,1)$ and $R>0$ we can by Lemmas 3.1 and 3.2 extract from $\{(u^{(n)},\theta^{(n)})\}$ a subsequence (still denoted $\{(u^{(n)},\theta^{(n)})\}$), such that there exist $u^{\alpha,R},\theta^{\alpha,R}$ in appropriate function spaces satisfying

$$u_{xxx}^{(n)} \to u_{xxx}^{\alpha,R}\,, \quad u_{xxt}^{(n)} \to u_{xxt}^{\alpha,R}\,, \quad \theta_x^{(n)} \to \theta_x^{\alpha,R}\,, \quad \text{all weakly-star in } L^\infty(0,T;L^2(0,\pi))\,, \tag{4.1}$$

$$u_{tt}^{(n)} \to u_{tt}^{\alpha,R}\,, \quad u_{xtt}^{(n)} \to u_{xtt}^{\alpha,R}\,, \quad \theta_t^{(n)} \to \theta_t^{\alpha,R}\,, \quad \text{all weakly in } L^2((0,\pi)\times(0,T))\,, \tag{4.2}$$

and, by compact imbedding,

$$u^{(n)} \to u^{\alpha,R}\,, \quad u_t^{(n)} \to u_t^{\alpha,R}\,, \quad u_x^{(n)} \to u_x^{\alpha,R}\,, \quad u_{xx}^{(n)} \to u_{xx}^{\alpha,R}\,,$$

$$u_{xt}^{(n)} \to u_{xt}^{\alpha,R}\,, \quad \theta^{(n)} \to \theta^{\alpha,R}\,, \quad \text{all strongly in } C([0,\pi]\times[0,T])\,. \tag{4.3}$$

Let now $N \in \mathbb{N}$ be fixed, and let $\psi \in \mathcal{D}(0,T)$, $w \in \operatorname{span}\{\sin(kx) \mid k = 1,...,N\}$, $z \in \operatorname{span}\{\cos(kx) \mid k = 0,...,N\}$ be arbitrary test functions. For every $n \geq N$ the functions $u^{(n)}, \theta^{(n)}$ satisfy the system

$$\begin{aligned}&\int_0^T\int_0^\pi \Big[-u_t^{(n)}\,w(x)\,\psi'(t) - \alpha\,u_{xt}^{(n)}\,w'(x)\,\psi'(t) + \Big(-\gamma\,u_{xxx}^{(n)} + \mathcal{P}_R[u_x^{(n)},\theta^{(n)}]\Big)\,w'(x)\,\psi(t)\Big]\,dx\,dt\\ &= \int_0^T\int_0^\pi f\,w(x)\,\psi(t)\,dx\,dt\,,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\int_0^T\int_0^\pi \Big[-\frac{1}{n}\theta_t^{(n)}\,z(x)\,\psi'(t) + \theta_t^{(n)}\,z(x)\,\psi(t) + \kappa\,\theta_x^{(n)}\,z'(x)\,\psi(t) - \mathcal{V}_R[u_x^{(n)},\theta^{(n)}]\,z(x)\,\psi'(t)\Big]\,dx\,dt\\ &= \int_0^T\int_0^\pi \Big(g + u_{xt}^{(n)}\,\mathcal{P}_R[u_x^{(n)},\theta^{(n)}]\Big)\,z(x)\,\psi(t)\,dx\,dt\,.\end{aligned} \tag{4.5}$$

Using (4.1)-(4.3), we can pass to the limit as $n \to \infty$ in (4.4), (4.5). The continuity of the stop operator with respect to the uniform convergence stated in Propositions 2.1 and 2.2, and Lebesgue's Theorem of Dominated Convergence yield that $(u^{\alpha,R},\theta^{\alpha,R})$ satisfies (3.2), as well as the integral identity

$$\begin{aligned}&\int_0^T\int_0^\pi \Big[\Big(\theta^{\alpha,R} + \mathcal{V}_R[u_x^{\alpha,R},\theta^{\alpha,R}]\Big)_t - g - u_{xt}^{\alpha,R}\,\mathcal{P}_R[u_x^{\alpha,R},\theta^{\alpha,R}]\Big]\,z(x)\,\psi(t)\,dx\,dt\\ &= -\kappa\int_0^T\int_0^\pi \theta_x^{\alpha,R}\,z'(x)\,\psi(t)\,dx\,dt\,,\end{aligned} \tag{4.6}$$

for any $z \in W^{1,2}(0,\pi)$ and $\psi \in \mathcal{D}(0,T)$. This implies, in particular, that $\theta_{xx}^{\alpha,R} \in L^2((0,\pi)\times(0,T))$ and that $\theta_x^{\alpha,R}(0) = \theta_x^{\alpha,R}(\pi) = 0$. Hence, $(u^{\alpha,R},\theta^{\alpha,R})$ satisfy (3.3) almost everywhere, as well as the initial and boundary conditions (2.31)-(2.33). We now collect some properties of the functions $(u^{\alpha,R},\theta^{\alpha,R})$.

Lemma 4.1 *For any $\alpha \in (0,1)$ and $R > 0$, there holds*

$$\theta^{\alpha,R}(x,t) \geq 0\,, \quad \textit{on } [0,\pi]\times[0,T]\,, \tag{4.7}$$

and, for every $t \in [0,T]$,

$$\|u_{xt}^{\alpha,R}(t)\|^2 + \gamma\|u_{xxx}^{\alpha,R}(t)\|^2 + \|\theta_x^{\alpha,R}(t)\|^2 + \int_0^t \|\theta_t^{\alpha,R}(\tau)\|^2\,d\tau \leq C\,e^{C(1+R^2)T}\,, \tag{4.8}$$

$$\int_0^\pi \mathcal{V}_R[u_x^{\alpha,R},\theta^{\alpha,R}](t)\,dx + \|\theta^{\alpha,R}(t)\|_{L^1(0,\pi)} + \|u_t^{\alpha,R}(t)\|^2 + \|u_{xx}^{\alpha,R}(t)\|^2 + \|u_x^{\alpha,R}(t)\|_{L^\infty(0,\pi)} \leq C\,. \tag{4.9}$$

Proof. (4.8) is an immediate consequence of Lemma 3.1. Next, observe that equation (3.3) with $u = u^{\alpha,R}$, $\theta = \theta^{\alpha,R}$, can be rewritten as

$$b(x,t)\,\theta_t^{\alpha,R} - \kappa\,\theta_{xx}^{\alpha,R} - a(x,t)\,\theta^{\alpha,R} = \tilde{g}(x,t)\,, \tag{4.10}$$

where

$$b(x,t) := 1 - \frac{1}{2}\int_0^R \theta^{\alpha,R}\,\varphi_{\theta\theta}(r,\theta^{\alpha,R})\,\mathfrak{s}_r^2[u_x^{\alpha,R}]\,dr - \frac{1}{2}\,\mathfrak{s}_R^2[u_x^{\alpha,R}]\int_R^\infty \theta^{\alpha,R}\,\varphi_{\theta\theta}(r,\theta^{\alpha,R})\,dr \ge \frac{1}{2}, \tag{4.11}$$

$$a(x,t) := \int_0^R \varphi_\theta(r,\theta^{\alpha,R})\,\mathfrak{s}_r[u_x^{\alpha,R}]\left(\mathfrak{s}_r[u_x^{\alpha,R}]\right)_t dr + \mathfrak{s}_R[u_x^{\alpha,R}]\left(\mathfrak{s}_R[u_x^{\alpha,R}]\right)_t\int_R^\infty \varphi_\theta(r,\theta^{\alpha,R})\,dr\,, \tag{4.12}$$

$$\begin{aligned}\tilde g(x,t) \ &:= \ g(x,t) + \int_0^R \varphi(r,\theta^{\alpha,R})\,\mathfrak{s}_r[u_x^{\alpha,R}]\Big(u_{xt}^{\alpha,R} - (\mathfrak{s}_r[u_x^{\alpha,R}])_t\Big)\,dr \\ &\quad + \mathfrak{s}_R[u_x^{\alpha,R}]\Big(u_{xt}^{\alpha,R} - (\mathfrak{s}_R[u_x^{\alpha,R}])_t\Big)\int_R^\infty \varphi(r,\theta^{\alpha,R})\,dr\,. \end{aligned}\tag{4.13}$$

Owing to (2.5) and hypothesis **(H1), (i)**, we see that $\tilde g(x,t) \ge 0$ almost everywhere. In addition, (2.6), (2.36) and (4.3) entail that $a \in L^\infty((0,\pi)\times(0,T))$. Therefore, thanks to the classical theory of linear parabolic equations (cf., for instance, [14]), (4.7) is satisfied.

To confirm the validity of (4.9), we multiply (3.5) by $\dot u_k$, and sum over k to arrive at

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}\|u_t^{(n)}(t)\|^2 + \frac{\alpha}{2}\|u_{xt}^{(n)}(t)\|^2 + \frac{\gamma}{2}\|u_{xx}^{(n)}(t)\|^2\Big) \\ &= -\int_0^\pi \mathcal{P}_R[u_x^{(n)},\theta^{(n)}](t)\,u_{xt}^{(n)}(t)\,dx + \int_0^\pi f(t)\,u_t^{(n)}(t)\,dx\,,\end{aligned}\tag{4.14}$$

whence, integrating over $[0,t]$ and then letting $n\to\infty$,

$$\begin{aligned}&\frac{1}{2}\|u_t^{\alpha,R}(t)\|^2 + \frac{\alpha}{2}\|u_{xt}^{\alpha,R}(t)\|^2 + \frac{\gamma}{2}\|u_{xx}^{\alpha,R}(t)\|^2 \\ &\le C_1 - \int_0^t\!\!\int_0^\pi \mathcal{P}_R[u_x^{\alpha,R},\theta^{\alpha,R}]\,u_{xt}^{\alpha,R}\,dx\,d\tau + \int_0^t\!\!\int_0^\pi f\,u_t^{\alpha,R}\,dx\,d\tau\,.\end{aligned}\tag{4.15}$$

Integrating (3.3) for $u = u^{\alpha,R}$, $\theta = \theta^{\alpha,R}$, over $[0,\pi]\times[0,t]$ and adding the result to (4.15), we find that

$$\begin{aligned}&\frac{1}{2}\|u_t^{\alpha,R}(t)\|^2 + \frac{\alpha}{2}\|u_{xt}^{\alpha,R}(t)\|^2 + \frac{\gamma}{2}\|u_{xx}^{\alpha,R}(t)\|^2 \\ &+ \int_0^\pi\Big(\theta^{\alpha,R}(t) + \mathcal{V}_R[u_x^{\alpha,R},\theta^{\alpha,R}](t)\Big)\,dx \le C_2 + \int_0^t\!\!\int_0^\pi f\,u_t^{\alpha,R}\,dx\,d\tau\,.\end{aligned}\tag{4.16}$$

By (2.35), $\mathcal{V}_R[u_x,\theta]$ is non-negative. Using (4.7) and Young's inequality, we obtain (4.9) by an application of Gronwall's lemma. □

Remark 3. Note that the bound established in (4.9) is independent of R. It could be derived from the fact that $\theta^{\alpha,R}$ is non-negative. In fact, we have introduced the α-approximation exactly with the purpose to get $u_{xt}^{\alpha,R}$ bounded in L^∞ in order to apply the maximum principle for $\theta^{\alpha,R}$. Of course, (4.9) means that the energy is globally bounded.

Step 2: Passage to the limit as $\alpha \to 0+$.

Since the bounds established in (4.8), (4.9) are independent of α, there exist functions u^R, θ^R in appropriate function spaces such that, possibly after selecting a subsequence, we have for $\alpha\to 0+$,

$$\begin{aligned}&u_{xt}^{\alpha,R} \to u_{xt}^R, \quad u_{xxx}^{\alpha,R} \to u_{xxx}^R, \quad \theta_x^{\alpha,R} \to \theta_x^R, \quad u_t^{\alpha,R} \to u_t^R, \\ &u_{xx}^{\alpha,R} \to u_{xx}^R, \quad \text{all weakly-star in } L^\infty(0,T;L^2(0,\pi))\,,\end{aligned}\tag{4.17}$$

$$\theta_t^{\alpha,R} \to \theta_t^R, \quad \text{weakly in } L^2((0,\pi)\times(0,T))\,, \tag{4.18}$$

$$\theta^{\alpha,R} \to \theta^R, \quad u^{\alpha,R} \to u^R, \quad u_x^{\alpha,R} \to u_x^R, \quad \text{all strongly in } C([0,\pi]\times[0,T])\,. \tag{4.19}$$

In addition, (4.9) implies that

$$\|u_x^{\alpha,R}\|_{L^\infty((0,\pi)\times(0,T))} \le C^* , \tag{4.20}$$

with C^* independent of α and $\mathcal{R}$. Similarly as in Step 1, we may now pass to the limit as $\alpha \to 0+$ in (3.2) and (4.6) to see that (u^R, θ^R) satisfies (3.2) and (3.3), as well as the boundary conditions (2.31) and (2.32), with $\alpha = 0$. By (4.19), also $u^R(x,0) = u^0(x)$, $\theta^R(x,0) = \theta^0(x)$, on $[0,\pi]$. To confirm that also $u_t^R(x,0) = v^0(x)$ almost everywhere on $(0,\pi)$, we note that equation (3.5) and the estimate (3.18) imply that the sequence $\{u_{tt}^{(n)}\}$ is bounded in $L^2(0,T;W^{-1,2}(0,\pi))$, independently of n and α. Hence, $\{u_{tt}^{\alpha,R}\}$ is bounded in $L^2(0,T;W^{-1,2}(0,\pi))$, independently of α. We may therefore assume that, for $\alpha \to 0+$,

$$u_{tt}^{\alpha,R} \to u_{tt}^R, \quad \text{weakly in } L^2(0,T;W^{-1,2}(0,\pi)). \tag{4.21}$$

In addition, since $W^{1,2}(0,T;W^{-1,2}(0,\pi)) \cap L^\infty(0,T;\overset{o}{W}{}^{1,2}(0,\pi))$ is by a classical compactness result (cf. [15]) compactly imbedded in $C([0,T];L^2(0,\pi))$, we may select the sequence $\alpha \to 0+$ in such a way that

$$u_t^{\alpha,R} \to u_t^R, \quad \text{strongly in } C([0,T];L^2(0,\pi)), \tag{4.22}$$

so that $u_t^R(\cdot,0) = v^0$ in the sense of $L^2(0,\pi)$.

STEP 3. PASSAGE TO THE LIMIT AS $R \to \infty$.

Recalling the definition (2.1)-(2.3) of the stop operator with initial condition given by (2.12), we see that the implication

$$|\varepsilon(t)| < r \quad \forall\, t \in [0,T] \Rightarrow \mathfrak{s}_r[\varepsilon](t) = \varepsilon(t) \quad \forall\, t \in [0,T] \tag{4.23}$$

is valid for every $r > 0$ and every $\varepsilon \in W^{1,1}(0,T)$ (and hence, by Proposition 2.1, also for every $\varepsilon \in C[0,T]$). In particular, given any function $\varepsilon \in C[0,T]$, we have $\mathfrak{s}_r[\varepsilon] = \varepsilon$ provided that $r > \|\varepsilon\|_{L^\infty(0,T)}$.

Now fix some $R > C^*$, where C^* is the constant defined in (4.20). Then $\mathfrak{s}_R[u_x^R] = u_x^R = \mathfrak{s}_r[u_x^R]$ for all $r \ge R$. Consequently, for $\alpha = 0$ the systems (3.2), (3.3) and (2.40), (2.41) coincide. Hence $(u,\theta) = (u^R, \theta^R)$ is a weak solution to the system (2.29)-(2.33). Finally, (4.7) and (4.19) imply that θ is non-negative on $[0,\pi]\times[0,T]$, which concludes the proof of Theorem 2.3. $\square$

Remark 4.

1. It follows from the maximum principle that θ is everywhere positive on $[0,\pi]\times[0,T]$ if $\theta^0(x) > 0 \quad \forall\, x \in [0,\pi]$.
2. The fact that we cannot prove the uniqueness of solutions has nothing to do with the hysteresis branching. In PDE's with hysteresis which are linear with respect to the hysteretic term there exist techniques for proving the uniqueness (cf. [6, 11, 24]). The difficulty here consists in the complicated nonlinear coupling in (2.29), (2.30) and in a loss of regularity in the hysteretic terms.

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