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# Optimal control of doubly nonlinear evolution equations governed by subdifferentials without uniqueness of solutions

M. Hassan Farshbaf-Shaker[1], Noriaki Yamazaki[2]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Hassan.Farshbaf-Shaker@wias-berlin.de

[2] Kanagawa University
Department of Mathematics
3-27-1 Rokkakubashi
Kanagawa-ku
Yokohama, 221-8686
Japan
E-Mail: noriaki@kanagawa-u.ac.jp

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

**Abstract**

In this paper we study an optimal control problem for a doubly nonlinear evolution equation governed by time-dependent subdifferentials. We prove the existence of solutions to our equation. Also, we consider an optimal control problem without uniqueness of solutions to the state system. Then, we prove the existence of an optimal control which minimizes the nonlinear cost functional. Moreover, we apply our general result to some model problem.

# 1 Introduction

The present paper is concerned with an optimal control problem without uniqueness of solutions to a doubly nonlinear evolution equation governed by time-dependent subdifferentials in a real Hilbert space $H$.

In our optimal control problem, for each control $f$, the state system (P; $f$) is as follows:

**State system (P; $f$):**

$$\text{(P; } f) \begin{cases} \partial\psi^t(u'(t)) + \partial\varphi(u(t)) + g(u(t)) \ni f(t) \ \text{ in } H \ \text{ for a.e. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } H, \end{cases} \tag{1.1}$$

where $0 < T < \infty$, $u' = du/dt$ in $H$, $\psi^t : H \to \mathbb{R} \cup \{\infty\}$ is a time-dependent proper, l.s.c. (lower semi-continuous), convex function for each $t \in [0,T]$, $\varphi : H \to \mathbb{R} \cup \{\infty\}$ is a time-independent proper, l.s.c., convex function, $\partial\psi^t$ and $\partial\varphi$ are their subdifferential in $H$, $g(\cdot)$ is a single-valued Lipschitz operator in $H$, $f$ is a given $H$-valued control function and $u_0 \in H$ is a given initial data.

In this present paper, we consider the optimal control problem without uniqueness of solutions to the state system (P; $f$). To this end, let $V$ be a real Hilbert space such that the embedding $V \hookrightarrow H$ is dense and compact. Then, we study the following optimal control problem without uniqueness of solutions to (P; $f$):

**Problem (OP):** Find the optimal control $f^* \in \mathcal{F}$ such that

$$J(f^*) = \inf_{f \in \mathcal{F}} J(f).$$

Here $\mathcal{F} := W^{1,2}(0,T;H) \cap L^2(0,T;V)$ is the control space and $J(f)$ is the cost functional defined by

$$J(f) := \inf_{u \in \mathcal{S}(f)} \pi_f(u), \tag{1.2}$$

where $f \in \mathcal{F}$ is the control, $S(f)$ is the set of all solutions to (P; $f$) with the control function $f$. Also, $u$ is a solution to the state system (P; $f$) and $\pi_f(u)$ is its functional defined by

$$\pi_f(u) := \frac{1}{2}\int_0^T |u(t) - u_{ad}|_H^2 dt + \frac{1}{2}\int_0^T |f(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_t(t)|_H^2 dt, \tag{1.3}$$

where $u_{ad} \in L^2(0,T;H)$ is a given target profile and $|\cdot|_H$ (resp. $|\cdot|_V$) is the norm of $H$ (resp. $V$).

There is vast literature on optimal control problems to (parabolic or elliptic) variational inequalities. For instance, we refer to [5, 10, 11, 17, 18, 19, 23]. In particular, Lions [18] and Neittaanmäki et al. [19, Section 3.1.3.1] discussed the singular control problems, which is the class of control problems characterized by not well-posed state systems.

The theory of nonlinear evolution equations are useful in the systematic study of variational inequalities. For instance, many mathematicians studied the nonlinear evolution equation of the form:

$$u'(t) + \partial\varphi^t(u(t)) \ni f(t) \text{ in } H \text{ for a.e. } t \in (0,T), \tag{1.4}$$

where $\varphi^t(\cdot) : H \to \mathbb{R} \cup \{\infty\}$ is a proper, l.s.c. and convex function. For various aspects of (1.4), we refer to [11, 14, 20, 22]. In particular, Hu–Papageorgiou [11] studied the optimal control problems to (1.4).

Also, doubly nonlinear evolution equations were studied. For instance, Kenmochi–Pawlow [15] studied the following type of doubly nonlinear evolution equations:

$$\frac{d}{dt}\partial\psi(u(t)) + \partial\varphi^t(u(t)) \ni f(t) \text{ in } H \text{ for a.e. } t \in (0,T), \tag{1.5}$$

where $\psi(\cdot) : H \to \mathbb{R}\cup\{\infty\}$ is a proper, l.s.c. and convex function. The abstract results of doubly nonlinear evolution equations (1.5) can be applied to elliptic-parabolic equations. Therefore, from the view point of (1.5), Hoffmann et al. [10] studied optimal control problems for quasi-linear elliptic-parabolic variational inequalities with time-dependent constraints. Also, Kadoya–Kenmochi [12] studied the optimal sharp design of elliptic-parabolic equations.

On the other hand, Akagi [1], Arai [2], Aso et al. [3, 4], Colli [8], Colli–Visintin [9], Senba [21] investigated the following type of doubly nonlinear evolution equations (cf. (1.1)):

$$\partial\psi^t(u'(t)) + \partial\varphi(u(t)) \ni f(t) \text{ in } H \text{ for a.e. } t \in (0,T). \tag{1.6}$$

However, there was no result of optimal control for (1.1) and (1.6) since (1.1) and (1.6) are not well-posed state systems, in general. Therefore, by arguments similar to Kadoya et al. [13], more precisely, using the cost functional defined by (1.2) and (1.3), we establish the abstract theory of the optimal control problem (OP) without uniqueness of solutions to the state system (1.1).

The plan of this paper is as follows. In the next Section 2, we state the main result in this paper. In Section 3, we first give the sketch of the proof of solvability for (1.1). Also, we prove the convergence result (Proposition 3.1) of solutions to (P; $f$). Moreover, we prove the main result (Theorem 2.1) on the existence of the optimal control to (OP). In the final Section 4, we apply our abstract result to a parabolic PDE with Neumann boundary condition.

## Notations

Throughout this paper, let $H$ be a real Hilbert space with the inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$, respectively. Also, let $V$ be a real Hilbert space with the norm $|\cdot|_V$ such that the embedding $V \hookrightarrow H$ is dense and compact.

Let us here prepare some notations and definitions of subdifferential of convex functions. To this end, let $E$ be a real Hilbert space with the inner product $(\cdot,\cdot)_E$. Then, for a proper (i.e., not identically equal to infinity), l.s.c. and convex function $\phi : E \to \mathbb{R} \cup \{\infty\}$, the effective domain $D(\phi)$ is defined by

$$D(\phi) := \{z \in E;\ \phi(z) < \infty\}.$$

The subdifferential of $\phi$ is a possibly multi-valued operator in $E$ and is defined by $z^* \in \partial\phi(z)$ if and only if

$$z \in D(\phi) \quad \text{and} \quad (z^*, y - z)_E \le \phi(y) - \phi(z) \quad \text{for all } y \in E.$$

The next proposition is concerned with the closedness of maximal monotone operator $\partial\phi$ in $E$.

**Proposition 1.1 (cf. [7, Lemma 1.2])** *Let $E$ be a real Hilbert space with the inner product $(\cdot,\cdot)_E$. Let $\phi : E \to \mathbb{R}\cup\{\infty\}$ be a proper, l.s.c. and convex function. Also, let $[z_n, z_n^*] \in \partial\phi$ and $[z, z^*] \in E\times E$ be such that*

$$z_n \to z \text{ weakly in } E \quad \text{and} \quad z_n^* \to z^* \text{ weakly in } E \quad \text{as } n \to \infty.$$

*Suppose that*

$$\limsup_{n\to\infty} (z_n, z_n^*)_E \le (z, z^*)_E.$$

*Then, it follows that $[z, z^*] \in \partial\phi$ and $\lim_{n\to\infty}(z_n, z_n^*)_E = (z, z^*)_E$.*

For various properties and related notions of the proper, l.s.c., convex function $\phi$ and its subdifferential $\partial\phi$, we refer to a monograph by Brézis [6].

## 2 Main Theorem

We begin by defining the notion of a solution to (P; $f$).

**Definition 2.1** *Given $f \in L^2(0,T;H)$ and $u_0 \in H$, the function $u : [0,T] \to H$ is called a solution to (P;$f$) on $[0,T]$, if the following conditions are satisfied:*

(i) $u \in W^{1,2}(0,T;H)$.

(ii) *There exist functions $\xi \in L^2(0,T;H)$ and $\zeta \in L^2(0,T;H)$ such that*

$$\xi(t) \in \partial\psi^t(u'(t)) \ \text{ in } H \quad \text{for a.e. } t \in (0,T),$$

$$\zeta(t) \in \partial\varphi(u(t)) \ \text{ in } H \quad \text{for a.e. } t \in (0,T)$$

*and*

$$\xi(t) + \zeta(t) + g(u(t)) = f(t) \ \text{ in } H \quad \text{for a.e. } t \in (0,T).$$

(iii) $u(0) = u_0$ *in $H$.*

Now, we give the assumptions on $\psi^t$, $\varphi$ and $g$.

(A1) For each $t \in [0,T]$, $\psi^t(\cdot) : H \to \mathbb{R}\cup\{\infty\}$ is a proper, l.s.c. and convex function. Also, $\varphi(\cdot) : H \to \mathbb{R}\cup\{\infty\}$ is a proper, l.s.c. and convex function.

(A2) There exists a positive constant $C_1 > 0$ such that

$$\psi^t(z) \ge C_1|z|_H^2, \quad \forall t \in [0,T],\ \forall z \in D(\psi^t).$$

(A3) There exists a positive constant $C_2 > 0$ such that

$$|z^*|_H^2 \le C_2(\psi^t(z) + 1), \quad \forall [z, z^*] \in \partial\psi^t,\ \forall t \in [0,T].$$

(A4) There are functions $\alpha \in W^{1,2}(0,T)$ and $\beta \in W^{1,1}(0,T)$ satisfying the following property: for any $s, t \in [0,T]$ with $s \le t$ and $z \in D(\psi^s)$, there exists $\tilde{z} \in D(\psi^t)$ such that

$$|\tilde{z} - z|_H \le |\alpha(t) - \alpha(s)| \left(1 + \psi^s(z)^{\frac{1}{2}}\right),$$
$$\psi^t(\tilde{z}) - \psi^s(z) \le |\beta(t) - \beta(s)| \left(1 + \psi^s(z)\right).$$

(A5) There exists a positive constant $C_3 > 0$ such that

$$\varphi(z) \ge C_3 |z|_H^2, \quad \forall z \in D(\varphi).$$

(A6) For each $r > 0$, the level set $\{z \in H; \varphi(z) \le r\}$ is compact in $H$.

(A7) $g : H \to H$ is a single-valued Lipschitz operator. Namely, there is a positive constant $L_g > 0$ such that

$$|g(z_1) - g(z_2)|_H \le L_g |z_1 - z_2|_H, \quad \forall z_i \in H \ (i = 1, 2).$$

**Remark 2.1** *The assumption (A4) is the standard time-dependence condition of convex functions (cf. [14, 20, 22]).*

By a slight modification of [1, 3], we can prove the following existence result for problem (P;$f$). We give a sketch of its proof in Section 3.

**Proposition 2.1 (cf. [1, Theorem 3.2], [3, Theorem 2.1])** *Assume* (A1)–(A7)*. Then, for each $u_0 \in D(\varphi)$ and $f \in L^2(0,T;H)$, there exists at least one solution $u$ to* (P;$f$) *on $[0,T]$. Moreover, there exists a positive constant $N_0 > 0$, independent of $u_0$, such that*

$$\int_0^T \psi^t(u'(t))dt + \sup_{t \in [0,T]} \varphi(u(t)) \le N_0 \left(\varphi(u_0) + |f|^2_{L^2(0,T;H)} + 1\right). \tag{2.1}$$

**Remark 2.2** *Colli [8, Theorem 5] and Colli–Visintin [9, Remark 2.5] showed several criteria for the uniqueness of solutions to the following type of doubly nonlinear evolution equations:*

$$\partial\psi(u'(t)) + \partial\varphi(u(t)) \ni f(t) \ \text{ in } H \ \text{ for a.e. } t \in (0,T). \tag{2.2}$$

*For instance, if $\partial\varphi$ is linear, positive, self-adjoint in $H$ and $\partial\psi$ is strictly monotone in $H$, we can show the uniqueness of solutions to* (2.2)*. However, $\partial\psi^t$ and $\partial\varphi$ in* (1.1) *are nonlinear and not self-adjoint, and hence, the uniqueness question to* (1.1) *is still open.*

Now, we state the main result of this paper, which is directed to the existence of an optimal control to (OP) without uniqueness of solutions to (P;$f$).

**Theorem 2.1** *Assume* (A1)–(A7) *and $u_0 \in D(\varphi)$. Let $u_{ad}$ be an element in $L^2(0,T;H)$. Then,* (OP) *has at least one optimal control $f^* \in \mathcal{F}$ such that*

$$J(f^*) = \inf_{f \in \mathcal{F}} J(f).$$

## 3 Proof of Main Theorem 2.1

In this section, we give the sketch of the proof of Proposition 2.1 by arguments similar to Akagi [1] and Aso et al. [3]. Moreover, we prove Theorem 2.1.

Throughout this section, we suppose that all the assumptions of Theorem 2.1 hold.

*Sketch of the proof of Proposition 2.1.*
By arguments similar to Akagi [1] and Aso et al. [3], we can prove Proposition 2.1. In fact, for each $\lambda \in (0,1]$, we consider the following approximate problem for (P;$f$), denoted by (P;$f$)$_\lambda$:

$$(\mathrm{P}; f)_\lambda \begin{cases} \lambda u_\lambda'(t) + \partial\psi^t(u_\lambda'(t)) + \partial\varphi_\lambda(u_\lambda(t)) + g(J_\lambda^\varphi u_\lambda(t)) \ni f(t) \ \text{ in } H \\ \hfill \text{for a.e. } t \in (0,T), \\ u_\lambda(0) = u_0 \ \text{ in } H, \end{cases}$$

where $\partial\varphi_\lambda$ and $J_\lambda^\varphi := (I + \lambda\partial\varphi)^{-1}$ denote the Yosida approximation and the resolvent of $\partial\varphi$, respectively.

By Cauchy–Lipschitz–Picard's existence theorem and the fixed point argument for compact operators (e.g. the Schauder's fixed point theorem), we can prove the existence of solutions $u_\lambda$ to (P; $f$)$_\lambda$ on $[0,T]$.

From the standard calculation, we can establish a priori estimate (cf. (2.1)) of solutions $u_\lambda$ to (P; $f$)$_\lambda$ with respect to $\lambda \in (0,1]$. Therefore, by the limiting procedure of solutions $u_\lambda$ to (P; $f$)$_\lambda$ as $\lambda \to 0$, we can construct the solution to (P; $f$) on $[0,T]$ satisfying the boundedness estimate (2.1). For a detailed argument, see [1, Sections 4 and 5] or [3, Sections 3 and 4], for instance. □

Here, let us mention the result of the convergence of solutions to (P; $f$), which is a key proposition to proving Theorem 2.1.

**Proposition 3.1** *Assume* (A1)–(A7). *Let* $\{f_n\} \subset L^2(0,T;H)$, $\{u_{0,n}\} \subset D(\varphi)$, $f \in L^2(0,T;H)$ *and* $u_0 \in D(\varphi)$. *Assume that*

$$f_n \to f \text{ strongly in } L^2(0,T;H), \tag{3.1}$$

$$u_{0,n} \to u_0 \text{ in } H \quad \text{and} \quad \varphi(u_{0,n}) \to \varphi(u_0) \tag{3.2}$$

*as* $n \to \infty$. *Let* $u_n$ *be a solution to* (P;$f_n$) *on* $[0,T]$ *with initial data* $u_{0,n}$. *Then, there exist a subsequence* $\{n_k\} \subset \{n\}$ *and a function* $u \in W^{1,2}(0,T;H)$ *such that* $u$ *is a solution to* (P;$f$) *on* $[0,T]$ *with initial data* $u_0$ *and*

$$u_{n_k} \to u \ \text{ in } C([0,T];H) \ \text{ as } k \to \infty. \tag{3.3}$$

*Proof.* From the bounded estimate (2.1), (A2), (A5) and the level set compactness of $\varphi$ (cf. (A6)), we derive that there are a subsequence $\{n_k\}$ of $\{n\}$ and a function $u \in W^{1,2}(0,T;H)$ such that $n_k \to \infty$,

$$\left.\begin{array}{ll} u_{n_k} \to u & \text{weakly in } W^{1,2}(0,T;H), \\ & \text{in } C([0,T];H), \\ & \text{weakly-}* \ \text{ in } L^\infty(0,T;H) \end{array}\right\} \tag{3.4}$$

as $k \to \infty$. Hence, we observe from (3.2) and (3.4) that $u(0) = u_0$ in $H$.

Now, let us show that $u$ is a solution of (P; $f$) on $[0,T]$ with initial data $u_0$. Since $u_{n_k}$ is a solution of (P; $f_{n_k}$) on $[0,T]$ with initial data $u_{0,n_k}$, there exist functions $\xi_{n_k} \in L^2(0,T;H)$ and $\zeta_{n_k} \in L^2(0,T;H)$ such that

$$\xi_{n_k}(t) \in \partial\psi^t(u_{n_k}'(t)) \ \text{ in } H \quad \text{for a.e. } t \in (0,T), \tag{3.5}$$

$$\zeta_{n_k}(t) \in \partial\varphi(u_{n_k}(t)) \text{ in } H \quad \text{for a.e. } t \in (0,T), \tag{3.6}$$

$$\xi_{n_k}(t) + \zeta_{n_k}(t) + g(u_{n_k}(t)) = f_{n_k}(t) \text{ in } H \quad \text{for a.e. } t \in (0,T). \tag{3.7}$$

Then, it follows from (2.1) and (A3) that

$$\{\xi_{n_k}\} \text{ is bounded in } L^2(0,T;H). \tag{3.8}$$

Therefore, taking a subsequence if necessary (still denote it by $\{n_k\}$), we observe that:

$$\xi_{n_k} \to \xi \text{ weakly in } L^2(0,T;H) \text{ for some } \xi \in L^2(0,T;H) \text{ as } k \to \infty. \tag{3.9}$$

Also, it follows from (A7) and (3.4) and that

$$g(u_{n_k}) \to g(u) \text{ in } C([0,T];H) \text{ as } k \to \infty. \tag{3.10}$$

Therefore, we infer from (3.1), (3.7), (3.8) and (3.10) that

$$\{\zeta_{n_k}\} \text{ is bounded in } L^2(0,T;H).$$

Hence, taking a subsequence if necessary (still denote it by $\{n_k\}$), we observe that:

$$\zeta_{n_k} \to \zeta \text{ weakly in } L^2(0,T;H) \text{ for some } \zeta \in L^2(0,T;H) \text{ as } k \to \infty. \tag{3.11}$$

Thus, we infer from (3.1), (3.7), (3.9), (3.10) and (3.11) that:

$$\xi + \zeta + g(u) = f \text{ in } L^2(0,T;H). \tag{3.12}$$

Also, from (3.4), (3.6), (3.11) and the demi-closedness of maximal monotone operator $\partial\varphi$ (cf. Proposition 1.1), we infer that

$$\zeta \in \partial\varphi(u) \text{ in } L^2(0,T;H), \tag{3.13}$$

which implies that $\zeta(t) \in \partial\varphi(u(t))$ in $H$ for a.e. $t \in (0,T)$.

Now, we show that

$$\xi(t) \in \partial\psi^t(u'(t)) \text{ in } H \quad \text{for a.e. } t \in (0,T). \tag{3.14}$$

From (3.1), (3.2) and (3.4)–(3.13) we observe that

$$\begin{aligned}
& \limsup_{k\to\infty} \int_0^T (\xi_{n_k}(t), u'_{n_k}(t))dt \\
= \ & \limsup_{k\to\infty} \int_0^T (f_{n_k}(t) - \zeta_{n_k}(t) - g(u_{n_k}(t)), u'_{n_k}(t))dt \\
= \ & \limsup_{k\to\infty} \left[ \int_0^T (f_{n_k}(t) - g(u_{n_k}(t)), u'_{n_k}(t))dt - \int_0^T \frac{d}{ds}\varphi(u_{n_k}(s))ds \right] \\
\le \ & \int_0^T (f(t) - g(u(t)), u'(t))dt + \limsup_{k\to\infty} \left(-\varphi(u_{n_k}(T)) + \varphi(u_{0,n_k})\right) \\
\le \ & \int_0^T (f(t) - g(u(t)), u'(t))dt - \varphi(u(T)) + \varphi(u_0) \\
= \ & \int_0^T (f(t) - g(u(t)) - \zeta(t), u'(t))dt \\
= \ & \int_0^T (\xi(t), u'(t))dt,
\end{aligned}$$

thus, we observe from Proposition 1.1, namely, the closedness of maximal monotone operator $\partial\psi^t$ that

$$\xi \in \partial\psi^t(u') \ \text{ in } L^2(0,T;H),$$

which implies that (3.14) holds. Therefore, we observe that $u$ is a solution of (P; $f$) on $[0,T]$ with initial data $u_0$. Thus, the proof of this proposition has been completed. □

Now, let us prove the main Theorem 2.1 in our paper, which is the existence of an optimal control to (OP).

*Proof of Theorem 2.1.*
Note that we show the existence of an optimal control to (OP) without uniqueness of solutions to state problem (P; $f$)

Also note from (1.2) and (1.3) that $J(f) \geq 0$ for all $f \in \mathcal{F}$. Let $\{f_n\} \subset \mathcal{F}$ be a minimizing sequence such that

$$d^* := \inf_{f\in\mathcal{F}} J(f) = \lim_{n\to\infty} J(f_n).$$

Then, we observe that $\{J(f_n)\}$ is bounded. Therefore, by the definition (1.2) of $J(f_n)$, for each $n$ there is a solution $u_n \in \mathcal{S}(f_n)$ such that

$$\pi_{f_n}(u_n) < J(f_n) + \frac{1}{n}.$$

Hence, we observe that $\{\pi_{f_n}(u_n)\}$ is bounded. Thus, by the definition of $\pi_{f_n}(u_n)$ (cf. (1.3)) and by the Aubin's compactness theorem (cf. [16, Chapter1, Section 5]), there are a subsequence $\{n_k\} \subset \{n\}$ and a function $f^* \in \mathcal{F}$ such that

$$\left.\begin{array}{ll} f_{n_k} \to f^* & \text{weakly in } W^{1,2}(0,T;H), \\ & \text{weakly in } L^2(0,T;V), \\ & \text{in } L^2(0,T;H) \end{array}\right\} \tag{3.15}$$

as $k \to \infty$,

Now, taking a subsequence if necessary, we infer from Proposition 3.1 that there is a solution $u^*$ to (P;$f^*$) on $[0,T]$ with initial data $u_0$ satisfying

$$u_{n_k} \to u^* \text{ in } C([0,T];H) \ \text{ as } k \to \infty. \tag{3.16}$$

Therefore, it follows from (3.15)–(3.16), $u^* \in \mathcal{S}(f^*)$ and the weak lower semicontinuity of $L^2$–norm that

$$\begin{aligned} d^* &= \inf_{f\in\mathcal{F}} J(f) \leq J(f^*) = \inf_{u\in\mathcal{S}(f^*)} \pi_{f^*}(u) \\ &\leq \pi_{f^*}(u^*) = \frac{1}{2}\int_0^T |u^*(t) - u_{ad}|_H^2 dt + \frac{1}{2}\int_0^T |f^*(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_t^*(t)|_H^2 dt \\ &\leq \liminf_{k\to\infty} \pi_{f_{n_k}}(u_{n_k}) \\ &\leq \liminf_{k\to\infty} \left\{ J(f_{n_k}) + \frac{1}{n_k} \right\} \\ &= \lim_{k\to\infty} J(f_{n_k}) = d^*. \end{aligned}$$

Hence, we have $d^* = \inf_{f\in\mathcal{F}} J(f) = J(f^*)$, which implies that $f^* \in \mathcal{F}$ is an optimal control to (OP). Thus, the proof of Theorem 2.1 has been completed. □

# 4 Application

In this section, we apply Theorem 2.1 to the simple model problem as follows:

$$
(\mathrm{SMP})_p \begin{cases} A(t,u_t) - \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + g(u) \ni f(t) \ \text{ in } Q := (0,T)\times\Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \Sigma := (0,T)\times\Gamma, \\ u(0) = u_0 \ \text{ a.e. in } \Omega, \end{cases}
$$

where $0 < T < \infty$, $\Omega$ is a bounded domain in $\mathbb{R}^N$ $(1 \le N < \infty)$, the boundary $\Gamma := \partial\Omega$ of $\Omega$ is smooth if $N > 1$, $g$ is Lipschitz on $\mathbb{R}$, $p$ is a positive number with $p \ge 2$, $\nu$ is an outward normal vector on $\Gamma$ and $u_0$ is a given initial data. Also, $A(t,\cdot)$ is the given time-dependent function defined by

$$
A(t,z) := \begin{cases} z - c(t), & \text{if } z - c(t) \ge 1, \\ 1, & \text{if } 0 < z - c(t) < 1, \\ [-1,1], & \text{if } z = c(t), \\ -1, & \text{if } -1 < z - c(t) < 0, \\ z - c(t), & \text{if } z - c(t) \le -1, \end{cases}
$$

where $c(\cdot)$ is a given smooth function on $[0,T]$.

To apply the abstract result to (P;$f$), we put $H := L^2(\Omega)$ and $V := H^1(\Omega)$ with usual real Hilbert space structures. Define a function $\varphi$ on $H$ by

$$
\varphi(z) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla z(x)|^p dx + C_4, & \text{if } z \in W^{1,p}(\Omega), \\ \infty, & \text{otherwise }, \end{cases}
$$

Also, for each $t \in [0,T]$, define a function $\psi^t$ on $H$ by

$$
\psi^t(z) := \int_\Omega \hat{A}(t, z(x))dx \quad \text{for all } z \in H := L^2(\Omega),
$$

where $\hat{A}(t,\cdot)$ is the primitive of $A(t,\cdot)$ such that $\hat{A}(t,\cdot) \ge 0$ for all $t \in [0,T]$.

It is not difficult to show that the assumptions (A1)–(A7) are satisfied. For instance, put $\tilde{z} = z - c(s) + c(t)$, $\alpha(t) := \int_0^t |c'(\tau)|d\tau$ and $\beta(t) \equiv 0$ for (A4) (cf. [14, Chapter 3]). Therefore, by applying Theorem 2.1, we can consider the control problem (OP) without uniqueness of solutions to $(\mathrm{SMP})_p$.

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