On the evolution by fractional mean curvature

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submitted: November 25, 2015

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No. 2183
Berlin 2016

2010 Mathematics Subject Classification. 35R11, 35K93, 53A10.

Key words and phrases. Nonlocal mean curvature, geometric motions, evolving surfaces.

This work has been supported by NSF grant DMS-1160802, the Alexander von Humboldt Foundation, the ERC grant 277749 E.P.S.I.L.O.N. “Elliptic Pde’s and Symmetry of Interfaces and Layers for Odd Nonlinearities”, and the PRIN grant 201274FYK7 “Aspetti variazionali e perturbativi nei problemi differenziali nonlineari.”
Abstract

In this paper we study smooth solutions to a fractional mean curvature flow equation. We establish a comparison principle and consequences such as uniqueness and finite extinction time for compact solutions. We also establish evolutions equations for fractional geometric quantities that yield preservation of certain quantities (such as positive fractional curvature) and smoothness of graphical evolutions.

1. Introduction

In the recent literature, an intense study has been performed on some fractional counterparts of the classical perimeter and of the motion by mean curvature. The interest in this kind of topics comes from several considerations. First of all, from the theoretical point of view, the analysis of nonlocal and fractional operators has an ancient tradition, which have been vividly renovated recently by new exciting discoveries. In particular, a notion of fractional perimeter has been introduced in [7] and its relation with a fractional mean curvature flow was discussed in detail in [15, 8].

Roughly speaking, given $s \in (0, 1)$ the fractional perimeter in the whole of $\mathbb{R}^n$ of a bounded set $E$ may be seen as the seminorm in $H^{s/2}(\mathbb{R}^n)$ of the characteristic function of $E$ (and this notion may be also localized inside a bounded domain $\Omega \subset \mathbb{R}^n$). The first variation of the fractional perimeter functional may be seen as a fractional counterpart of the mean curvature. As $s \to 1$, these notions approach the classical objects in different senses (see e.g. [5, 19, 40, 9, 2, 10] for details). The limit as $s \to 0$ has also been taken into account under various circumstances (see e.g. [33, 22]).

These fractional theories of geometric type found very often concrete applications in real-world problems. For instance, fractional perimeter functionals naturally appear in the large-scale description of interfaces of nonlocal phase transitions (see [37, 38]). A very natural application arises also in computer science: indeed, the a square pixels of small side $\epsilon$ produce, along the diagonal, an error of order one for the classical perimeter, but an error of order only $\epsilon^{1-s}$ for the fractional perimeter. In this sense, fractional objects are very useful to “average out” the errors caused by the possible fine anisotropic structure of the media.

Many results of great interest about the fractional mean curvature flow have been recently obtained in [12, 13, 14]. See also [1] for a detailed study of the fractional mean curvature, with analogies and important differences with respect to the classical case. The question of the regularity of the minimal surfaces corresponding to the fractional perimeter has been investigated in [7, 36, 10, 28, 3, 11], several connections with the isoperimetric problems have been studied in [29, 27, 21] and remarkable examples of surfaces corresponding to the fractional perimeter has been investigated in [7, 36, 10, 28, 3, 11], several important differences with respect to the classical case. The question of the regularity of the minimal surfaces corresponding to the fractional perimeter has been investigated in [7, 36, 10, 28, 3, 11], several connections with the isoperimetric problems have been studied in [29, 27, 21] and remarkable examples of surfaces corresponding to the fractional perimeter has been investigated in [7, 36, 10, 28, 3, 11].

In this work we are interested in studying classical solutions to the $L^2$-gradient flow associated to the fractional perimeter. More precisely, we consider a set $E_0$ and we are interested in a family $E_t$ that satisfies for every $x \in \partial E_t$ the law of motion

$$\partial_t x \cdot \nu = -H_s,$$

(1.1)

where $\nu$ is the outer normal to $E_t$ and the quantity $H_s$ is the fractional mean curvature defined by

$$H_s(x, E) := \lim_{\delta \searrow 0} s(1-s) \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{x - y}{|x - y|^{n+s}} dy.$$

(1.2)

Here above and in the sequel we use the notation

$$\tilde{\chi}_E(y) := \chi_{\mathbb{R}^n \setminus E}(y) - \chi_E(y),$$

while $\chi_E$ is the classical indicator function of $E$, that is 1 on $E$ and 0 on $\mathbb{R}^n \setminus E$. We also assume that the parameter $s$ belongs to the interval $(0, 1)$. Notice that with this convention the mean curvature of a sphere is positive (more details on this case will be given in the forthcoming Section 2.1.1). Moreover, under this convention, the $s$-perimeter of solutions to (1.1) decreases in the fastest direction; In fact it holds that (see Theorem 14)

$$\partial_t P_s(E_t) = -\int_{\partial E_t} H_s^2(\omega) dH^{n-1}(y) \leq 0.$$

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The flow described by equation (1.1) is the natural analog of the mean curvature flow, which has been studied largely in the literature (see for instance [23], [24], [30], [31], [32], [35] and references therein). The mean curvature flow has been used in several contexts that range from modeling interface transition [34] to obtain topological classification of certain surfaces [31, 4].

The mean curvature flow is a quasilinear geometric equation of parabolic nature that has regularizing effects as long as the mean curvature remains bounded (i.e., solutions are $C^\infty$ in space and time while the mean curvature is bounded), but it may form singularities in finite time. One of the main topics within the subject is the study of singularity formation during the evolution. The first important result in this direction is due to G. Huisken [30] who showed that convexity is preserved by the flow and that singularities only form at an extinction time at which the surface collapses to a “round point”, that is after appropriate rescaling convex surfaces are asymptotic to spheres. Later on, it has been proved that in fact the flow preserves $k$-convexity for any $1 \leq k \leq n - 1$ ([31]) and that homothetic solutions play an important role in the understanding of singularity formation. In this paper we show that $H_s$-convexity is preserved by the fractional flow (see Section 5) and we observe that in fact spheres are self-similar solutions to the flow (see Section 2.1.1).

Another important classical example of evolution by mean curvature flow is the evolution of entire graphs with linear growth. In [24] it is shown that in that case the evolution exists and it is smooth for all times. The estimates of that work were later localized in [25] to obtain short time estimates for any evolution. Other graphical evolutions have been studied in [35]. In Section 6 we show that graphical solutions to (1.1) have bounded $H_s$-curvature for all times and are in fact $C^\infty$. A key element of the proof is the preserved quantity $(\nu \cdot e_n)^{-1}$ which is known as the height function. On the other hand, star-shaped surfaces also have a preserved quantity and we briefly address this case in Section 7.

Other results that we present here are a comparison principle, the preservation of the positivity of $H_s$ and some estimates for entire graphs.

The organization of the paper is as follows: Section 2 is devoted to formulate Equation (1.1) for star-shaped surfaces and entire graphs. We compute in particular the example of an evolving sphere. In Section 3 we show that a comparison principle holds for the flow and as a corollary we find bounds on the maximal existence time and uniqueness of smooth solutions. Section 4 is devoted to compute the evolution of local and non-local geometric quantities. Of particular interest is the evolution equation of $H_s$ that is given by

\[
\frac{\partial_t H_s}{2s(1-s)}(x) = \text{P.V.} \int_{\partial E_t} \frac{H_s(y) - H_s(x)}{|x-y|^{n+s}} dy + H_s(x) \text{P.V.} \int_{\partial E_t} \frac{1 - \nu(x) \cdot \nu(y)}{|x-y|^{n+s}} dy.
\]

This equation implies that if the initial condition satisfies $H_s > 0$ then this is preserved by the flow. This result is proved in Section 5. In Section 6 we prove bounds for graphical solutions and in Section 7 that star-shapedness is preserved by the flow as long as the fractional curvature remains bounded.

2. Some special cases

In this section, we consider some particular forms of the fractional mean curvature motion, namely the cases in which the evolving surface is the boundary of a star-shaped domain or it is a graph in a given direction. A simple and concrete example of fractional mean curvature evolution for star-shaped surfaces is given by the spheres, in which the equation can be explicitly solved by scale invariance. On the other hand, planes are trivial examples of graphical evolutions.

### 2.1. Evolution of star-shaped surfaces

In this subsection we assume that the initial set is of the form

\[ E_0 = \{ \rho \omega, \ \omega \in S^{n-1}, \ \rho \in [0, f_0(\omega)] \} \]

with $\nu(p) \cdot p \geq 0$ for any $p \in \partial E_0$, where $\nu(p)$ is the outer unit normal at $p$.

We deal with the motion of $\partial E_0$ by its fractional mean curvature. We assume that this evolution is regular and star-shaped around the origin for all times $t \in [0, T)$ That is, we consider

\[ E_t = \{ \rho \omega, \ \omega \in S^{n-1}, \ \rho \in [0, f(\omega, t)] \} \]

and \[ \partial E_t = \{ f(\omega, t)\omega, \ \omega \in S^{n-1} \} \]
with \( f \in C^2(S^{n-1} \times (0, +\infty), [0, +\infty)) \cap C^0(S^{n-1} \times [0, +\infty), [0, +\infty)) \) and \( f > 0 \).

In order to write (1.1) more explicitly in dependence of \( f \) we extend the function \( f = f(\cdot, t) \), that was originally defined on \( S^{n-1} \), to the whole of \( \mathbb{R}^n \setminus \{0\} \) by homogeneity, namely we suppose, without loss of generality, that \( f : \mathbb{R}^n \setminus \{0\} \to [0, +\infty) \), with

\[
(2.1) \quad f(x) = f \left( \frac{x}{|x|} \right) \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\}.
\]

Notice that we omitted, for simplicity, the dependence on the time \( t \) in the notation above. Similarly, given \( \omega \in S^{n-1} \), unless otherwise specified, we denote by \( \nu \) the exterior normal at the point \( f(\omega) \). Hence we have:

**Lemma 1.** The external normal \( \nu \) of \( E \) can be expressed in terms of \( f \) by

\[
(2.2) \quad \nu = \frac{f\omega - \nabla f}{\sqrt{|\nabla f|^2 + f^2}}.
\]

Also, given any \( \omega \in S^{n-1} \), for any \( \eta \in \mathbb{R}^n \) orthogonal to \( \omega \) we have that

\[
(2.3) \quad (\nabla f(\omega) \cdot \eta) (\omega \cdot \nu) + f(\omega) \eta \cdot \nu = 0.
\]

Finally, (1.1) is equivalent to

\[
(2.4) \quad \begin{cases}
\partial_t f(\omega, t) = -H_\nu(\star, E_t) \frac{\sqrt{|\nabla f|^2 + f^2}}{f}, & \text{for every } \omega \in S^{n-1} \text{ and } t > 0, \\
f(\omega, 0) = f_0(\omega), & \text{for every } \omega \in S^{n-1},
\end{cases}
\]

where \( \star = f(\omega, t) \).

**Proof.** First we point out that, by (2.1),

\[
(2.5) \quad \nabla f(\omega) \cdot \omega = \frac{d}{d\tau} f(\tau \omega) \bigg|_{\tau = 1} = \frac{d}{d\tau} f(\omega) \bigg|_{\tau = 1} = 0
\]

for any \( \omega \in S^{n-1} \). Also, if \( \tau \mapsto \omega(\tau) \) is a curve on \( S^{n-1} \), we have that

\[
(2.6) \quad \omega \cdot \dot{\omega} = \frac{d}{d\tau} |\omega|^2 = \frac{d}{d\tau} 1 = 0
\]

and a generic tangent vector at \( \partial E \) is

\[
T := \frac{d}{d\tau}(f(\omega)) = (\nabla f \cdot \dot{\omega}) \omega + f \dot{\omega}.
\]

We observe that

\[
(f\omega - \nabla f) \cdot T = f (\nabla f \cdot \dot{\omega}) + f^2 \dot{\omega} \cdot \omega - (\nabla f \cdot \dot{\omega})(\nabla f \cdot \omega) - f (\nabla f \cdot \dot{\omega}) = 0,
\]

thanks to (2.5) and (2.6). This shows that the vector \( f\omega - \nabla f \) is normal to \( \partial E \). Also, by (2.5), the component of \( f\omega - \nabla f \) in direction \( \omega \) is \( f \), which is positive: accordingly, this normal vector points outwards and this completes the proof of (2.2).

Using (2.5) and (2.2), we also obtain that

\[
(2.7) \quad \omega \cdot \nu = \frac{f}{\sqrt{|\nabla f|^2 + f^2}},
\]

and this shows that (1.1) and (2.4) are equivalent (recall indeed that \( x = f(\omega) \)).
It remains to prove (2.3). For this, we take $\eta$ orthogonal to $\omega$ and we use (2.2) and (2.7) to compute

$$(\nabla f \cdot \eta) (\omega \cdot \nu) + f \eta \cdot \nu = f (\nabla f \cdot \eta) \sqrt{|\nabla f|^2 + f^2} + f^2 \eta \cdot \omega - f (\eta \cdot \nabla f) \sqrt{|\nabla f|^2 + f^2}$$

that clearly equals to zero and proves (2.3). \(\square\)

For the analogue of (2.4) in the classical mean curvature flow see, e.g., formula (2.8) in [39].

As a matter of fact, from Lemma 1, we can easily present an explicit derivation of (1.1) in terms of the prescribed normal velocity (we refer to Section 2 of [39] for a similar argument in the classical case).

Indeed, suppose that a smooth, compact hypersurface of $\mathbb{R}^n$ is defined by an embedding $X : S^{n-1} \rightarrow \mathbb{R}^n$, and consider the evolution equation in which the normal velocity is some prescribed $v$ (in our case, we will take $v$ to be $-H_s$, but the argument is general). We then obtain the equation

$$\partial_t X(\zeta, t) = v(X(\zeta, t), t) \nu(X(\zeta, t), t),$$

for any $\zeta \in S^{n-1}$. A multiplication by the normal vector then yields

$$(2.8) \quad \partial_t X(\zeta, t) \cdot \nu(X(\zeta, t), t) = v(X(\zeta, t), t).$$

If the region enclosed by the manifold is star-shaped (say, with respect to the origin), one writes $X = f\omega$, i.e. one considers the diffeomorphism $\omega(\cdot, t) : S^{n-1} \rightarrow S^{n-1}$,

$$\omega(\zeta, t) := \frac{X(\zeta, t)}{|X(\zeta, t)|},$$

with inverse mapping $\zeta(\omega, t)$, and defines

$$f(\omega, t) := |X(\zeta(\omega, t), t)|.$$

We remark that $|\omega(\zeta, t)| = 1$, therefore

$$\omega(\zeta, t) \cdot \partial_t \omega(\zeta, t) = \partial_t \frac{|\omega(\zeta, t)|^2}{2} = \partial_t \frac{1}{2} = 0.$$

Therefore we can apply (2.3) and conclude that

$$(2.9) \quad (\nabla f \cdot \partial_t \omega) (\omega \cdot \nu) + f \partial_t \omega \cdot \nu = 0.$$

On the other hand

$$\partial_t X = \partial_t (f\omega) = (\nabla f \cdot \partial_t \omega + \partial_t f)\omega + f \partial_t \omega.$$

Thus, by (2.9),

$$\partial_t X \cdot \nu = (\nabla f \cdot \partial_t \omega)(\omega \cdot \nu) + \partial_t f(\omega \cdot \nu) + f \partial_t \omega \cdot \omega$$

$$= \partial_t f(\omega \cdot \nu)$$

By substituting this into (2.8), we obtain

$$\partial_t f(\omega \cdot \nu) = v.$$

Then, (1.1) is simply the particular case in which the normal velocity is the fractional mean curvature, pointing inwards.
2.1.1. A concrete example: The evolution of spheres. In this section we compute the example of a concrete evolution, namely we show that the spheres shrink self-similarly in finite time. We think it is a very interesting open problem to determine whether or not these are the only embedded self-similar shrinking solutions of (1.1).

Lemma 2. The fractional mean curvature of the ball of radius $R$ is equal, up to dimensional constants, to $R^{-s}$. More explicitly, for any $x \in \partial B_1(0)$,

$$H_s(x, B_1(0)) = \varpi$$

for some $\varpi > 0$, and, for any $x \in \partial B_R(0)$,

$$H_s(x, B_R(0)) = \varpi R^{-s}.$$

Proof. By rotational invariance of the integrals, we have that $H_s(x_1, B_1(0)) = H_s(x_2, B_1(0))$ for every $x_1, x_2 \in \partial B_1(0)$, thus showing (2.10). Moreover, if $\omega \in S^{n-1}$ and $x = R \omega$, by changing variable $\tilde{y} := Ry$, we see that

$$H_s(x, B_R(0)) = \lim_{\delta \searrow 0} s(1-s) \int_{\mathbb{R}^n \setminus B_s(x)} \frac{\tilde{X}_{B_R(0)}(\tilde{y})}{R \omega - \tilde{y} | \omega |^{n+s}} d\tilde{y}$$

$$= R^s \lim_{\delta \searrow 0} s(1-s) \int_{\mathbb{R}^n \setminus B_{R^{n-1}s}(x)} \frac{\tilde{X}_{B_R(0)}(Ry)}{R \omega - Ry | \omega |^{n+s}} dy$$

$$= R^{-s} \lim_{\delta \searrow 0} s(1-s) \int_{\mathbb{R}^n \setminus B_s(x)} \frac{\tilde{X}_{B_1(0)}(y)}{| \omega - y |^{n+s}} dy$$

$$= R^{-s} H_s(\omega, B_1(0)).$$

This, together with (2.10), proves (2.11). \qed

Corollary 3. Let $\varpi$ be as in (2.10) and $C_0 := \varpi (s + 1)$. Let $R(t) := (R_0^{s+1} - C_0 t)^{\frac{1}{s+1}}$. Then $B_{R(t)}(0)$ is a star-shaped solution to fractional mean curvature flow with initial condition $B_{R_0}(0)$ and it collapses to the origin in the finite time $\frac{R_0^{s+1}}{C_0}$.

Proof. We only need to show that (2.4) is satisfied with $f(\omega, t) := R(t)$ and $f_0(\omega) := R_0$. For this, we use Lemma 2 to compute

$$\partial_t f + H_s \frac{\sqrt{\nabla f^2 + f^2}}{f} = -\frac{C_0}{s+1} (R_0^{s+1} - C_0 t)^{\frac{-s}{s+1}} + H_s = \varpi (R_0^{s+1} - C_0 t)^{\frac{-s}{s+1}} + \varpi R^{-s} = 0,$$

that shows the validity of (2.4). \qed

From the results in Section 3, we will see that the one provided in Corollary 3 is indeed the unique smooth solution of the fractional mean curvature flow with spherical initial datum.

It is also easy to check that a similar computation yields an analogous result for the evolution of cylinders.

2.2. Evolution of graphical surfaces. In this subsection we assume that the initial set is of the form

$$E_0 = \{(x, z), x \in \mathbb{R}^{n-1}, z \in [-\infty, u(x)]\}.$$

The appropriate choice of normal in this situation is given by

$$\nu(x, u(x)) = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

We assume that

$$E_t = \{(x, z), x \in \mathbb{R}^{n-1}, z \in (-\infty, u(x, t))\}$$

and

$$\partial E_t = \{(x, u(x, t)), x \in \mathbb{R}^{n-1}\}$$
with \( u \in C^2(\mathbb{R}^{n-1} \times (0, +\infty), [0, +\infty)) \cap C^0(\mathbb{R}^{n-1} \times [0, +\infty), [0, +\infty)) \). In this setting, the geometric flow in (1.1) is equivalent to
\[
\begin{aligned}
\partial_t u(x, t) &= -H_s(x, E_t) \sqrt{\nabla u}^2 + 1, & \text{for every } x \in \mathbb{R}^{n-1} \text{ and } t > 0, \\
\quad u(x, 0) &= u_0(x), & \text{for every } x \in \mathbb{R}^{n-1},
\end{aligned}
\]
\hspace{1cm} (2.13)

A concrete example in this case is any linear \( u \), which has fractional mean curvature equal to 0.

**Remark 4.** Equations (2.4) and (2.13) are well posed imposing weaker regularity conditions on \( f \) and \( u \) respectively.

## 3. Comparison principle

In this section we show that two surfaces evolving under fractional mean curvature flow that are initially nested remain nested while the evolution is smooth. More precisely, we have the following comparison result:

**Theorem 5.** Let \( E_t \) and \( F_t \) be two smooth solutions to (1.1) in \([0, \omega)\) such that \( E_0 \subseteq F_0 \). Assume additionally that \( \partial_x x(\cdot, t), \partial_t y(\cdot, t) \) are continuous in \([0, T)\) for \( x(\cdot, t) \in \partial E_t \) and \( y(\cdot, t) \in \partial F_t \). Then \( E_t \subseteq F_t \).

**Proof.** We first assume that \( E_0 \) is strictly contained in \( F_0 \) and suppose that there is a time \( t_0 \) and a point \( x_t \) at which \( E_{t_0} \) and \( \partial F_{t_0} \) touch for the first time and the normal velocity of \( E_{t_0} \) at \( x_t \) is bigger than the normal velocity of \( \partial F_{t_0} \) at that point (i.e. the boundaries cross at point of space time). Since \( \partial E_{t_0} \) and \( \partial F_{t_0} \) are tangential at \( x_t \) the normal vectors agree at that point. Then we have
\[
0 \geq (\partial_t x_{E_t} - \partial_t x_{E_0}) \cdot \nu_{E}(x_t) = H_s(E_t, x_t) - H_s(F_t, x_t)
\]

Moreover, since \( E_{t_0} \subseteq F_{t_0} \) we have \( H_s(E_{t_0}, x_t) \geq H_s(F_{t_0}, x_t) \), which yields a contradiction.

If \( E_0 \) is not strictly contained in \( F_0 \), then we can proceed as before by observing that the equation holds in the limit as \( t \to 0 \).

The previous theorem implies a more general result, as stated here below:

**Corollary 6.** Let \( E_t \) and \( F_t \) be two smooth solutions to (1.1) in \([0, \omega)\) such that \( \partial E_0 \cap \partial F_0 = \emptyset \). Then \( \partial E_t \cap \partial F_t = \emptyset \).

**Proof.** By noticing that the evolution of \( E_0^c \) equals the complement of the evolution of \( E_0 \) we have that if \( F_0 \subset E_0^c \). Theorem 5 implies \( F_t \subset (E_t)^c \). Since \( \partial E_0 \cap \partial F_0 = \emptyset \) implies that either \( F_0 \subset E_0 \) or \( F_0 \subset E_0^c \), the conclusion follows.

**Theorem 5** implies uniqueness of smooth solutions to (1.1)

**Corollary 7.** There is at most one smooth solution to (1.1)

**Proof.** Assume that \( E_0 = F_0 \). By Theorem 5, we have that \( F_t \subseteq E_t \) and \( E_t \subseteq F_t \).

By trapping the solution between balls, we obtain estimates about the evolution of the fractional mean curvature and the extinction time:

**Corollary 8.** Let \( R > \delta > 0 \) and \( E_t \) a solution to (1.1) such that there are \( x_\delta \) and \( x_R \) that satisfy \( B_\delta(x_\delta) \subseteq E_0 \subseteq B_R(x_R) \), then \( B_{\delta^{s+1} - C_0 t}(x_\delta) \subseteq E_t \subseteq B_{(R^{s+1} - C_0 t)}(x_R) \).

In particular, if \( f \in C^1(S^{n-1} \times (0, T)) \cap C^0(S^{n-1} \times [0, T]) \) is a solution of (2.4), with \( f(\omega, t) > 0 \) for every \( (\omega, t) \in S^{n-1} \times [0, T] \), that satisfies \( \delta < f(\omega, 0) < R \), for every \( \omega \in S^{n-1} \). then
\[
(\delta^{s+1} - C_0 t) \leq f(\omega, t) \leq (R^{s+1} - C_0 t)\frac{1}{t^{\frac{1}{s+1}}}.
\]

Moreover, the maximal existence time is bounded from above by \( \frac{R^{s+1}}{C_0} \).

**Proof.** The result follows directly from Theorem 5.
4. The evolution of the geometric quantities

In this section we study the evolution of local and nonlocal geometric quantities.

We first remark that equation (1.1) is invariant under reparameterization: Suppose that \( x \) satisfies (1.1) and consider a reparameterization \( \varphi(\omega, t) \). Then we have that \( \tilde{x} = x(\varphi(\omega, t), t) \) satisfies
\[
\partial_t \tilde{x} \cdot \tilde{v} = (Dx(\partial_t \varphi) + \partial_t x) \cdot \tilde{v} = -H_s(\tilde{x}) .
\]

Moreover, by reparameterizing the smooth surface with a time dependent parameter it is possible to obtain an evolution equation that has tangent velocity equal to 0.

**Theorem 9.** Suppose that \( E_t \) is smooth and satisfies the evolution equation (1.1). Then, there is a parameterization of \( \partial E_t \) such that
\[
\partial E_t = -H_s(x(t), E_t) \nu ,
\]
for \( x \in \partial E_t \).

**Proof.** We follow the analogous proof for other geometric flows (see [23] for instance).

Assume that \( \partial E_t \) is parameterized by spatial coordinates \( (\omega_1, \ldots, \omega_{n-1}) \in U \subset \mathbb{R}^{n-1} \). Then we have that \( x(\omega, t) \in \partial E_t \) satisfies (1.1). We want to reparameterize \( \omega \) in term of new time-dependent local coordinates. Hence, we assume that the coordinates \( (\omega_1, \ldots, \omega_{n-1}) \) are parameterized by a spatial parameter \( \Theta = (\theta_1, \ldots, \theta_{n-1}) \) and time \( t \). Then we define
\[
\Gamma(\Theta, t) = x(\omega(\Theta, t), t)
\]
We have
\[
\partial_t \Gamma = \sum_i \partial_{\omega_i} x(\omega(\Theta, t), t) \partial_{\omega_i} \omega + \partial_t x(\omega, t)|_{q=\omega(\Theta, t)}
\]
\[
= -H_s(\Gamma(\Theta, t)) \nu + (\tau \partial_{\omega_i} \omega + (\partial_t x)^T)|_{q=\omega(\Theta, t)} ,
\]
where \( \tau \) is the tangential vector \( \partial_{\omega_i} x(\omega(\Theta, t), t) \) and \( (\partial_t x)^T = \partial_t x - (\partial_t x \cdot \nu) \nu \) is the tangential part of \( \partial_t x \).

Standard ODE theory implies the existence of a solution to
\[
\partial_t \omega_i(\Theta, t) = (\partial_t x)^T g^{ij} \omega^T \cdot \tau_j ,
\]
with \( \omega(\Theta, t_0) = \omega \) (the original parameterization at time \( t_0 \)).

Hence, the surface \( \Gamma(\Theta, t) \) satisfies (4.1) for time close to \( t_0 \). \( \square \)

In the next subsection we assume that \( \Gamma(t) \) is the reparameterization of \( \partial E_t \) described by Theorem 9. For simplicity, we still denote the spatial parameter as \( \omega \in U \subset \mathbb{R}^{n-1} \) or \( x \in \mathbb{R}^{n-1} \).

4.1. Evolution of local quantities. In this subsection we consider the evolution of some geometric quantities associated to \( \partial E_t \). We assume that the \( \partial E_t \) is smooth.

Consider \( \Gamma(t) \) satisfying (4.1). We start by recalling the definition of the metric \( g_{ij} \), the second fundamental form \( a_{ij} \) and the square of its norm \( |A|^2 \). Here we denote by \( (m_{ij}) \) the matrix of components \( m_{ij} \) and we use Einstein’s summation convention whenever repeated indices occur. We denote the inverse of the metric as \( g^{ij} \) and we raise indices of matrices to indicate contraction by this matrix (e.g. \( m^j_i = g^{ij} m_{ij} \)).

In this setting, we have:
\[
g_{ij} = \partial_{\omega_i} \Gamma \cdot \partial_{\omega_j} \Gamma ,
\]
\[
(g^{ij}) = (g_{ij})^{-1} ,
\]
\[
a_{ij} = \partial_{\omega_i} \nu \cdot \partial_{\omega_j} \Gamma = -\nu \cdot \partial_{\omega_j} \partial_{\omega_i} \Gamma = \partial_{\omega_j} \nu \cdot \partial_{\omega_i} \Gamma ,
\]
\[
|A|^2 = g^{ij} a_{ik} g^{kl} a_{jl} .
\]

We also denote
\[
\nabla^\Gamma F = g^{ij} \partial_{\omega_j} F \partial_{\omega_i} \Gamma ,
\]
which correspond to projecting the gradient of \( F \) on the tangent space (for a globally defined function) and
\[
\nabla^\Gamma_i X^j = \partial_{\omega_i} X^j + C^j_{ik} X^k ,
\]
where $C^j_{ik}$ are the Christoffel symbols on the surface.

**Theorem 10.** Assume that $\Gamma(\Theta,t) = \partial E_t$ is parameterized such that it satisfies (4.1). Then we have that

$$
\begin{align}
\partial_t g_{ij} &= -2H_s a_{ij}, \\
\partial_t g^{ij} &= 2H_s a^{ij}, \\
\partial_t \nu &= \nabla^\Gamma H_s, \\
\partial_t a_{ij} &= \nabla^\Gamma_j \nabla^\Gamma_i H_s - H_s a_{ik} a^k_j, \\
\partial_t |A|^2 &= 2a^{ij} \nabla^\Gamma_i \nabla^\Gamma_j H_s + 2H_s a_{ik} a^k_j a^{ij}.
\end{align}
$$

Proof. The proofs are similar to the local case (see [23] for instance). First, we prove (4.3) by computing the evolution of the metric: we recall that $\partial_\omega_i \Gamma$ is a tangent vector, thus

$$
\partial_\omega_i \Gamma = \nu = 0.
$$

Also $\Gamma$ satisfies (4.1), and so $\partial_t \Gamma = -H_s \nu$. As a consequence,

$$
\begin{align}
\partial_t g_{ij} = \partial_{\omega_i} (\partial_t \Gamma) \cdot \omega_j \Gamma + \omega_i \Gamma \cdot \partial_{\omega_j} (\partial_t \Gamma) &= \partial_{\omega_i} (-H_s \nu) \cdot \omega_j \Gamma + \omega_i \Gamma \cdot \partial_{\omega_j} (-H_s \nu) \\
&= -2H_s a_{ij}
\end{align}
$$

and so we obtain (4.3).

Now, since $g_{ij} g^{jk} = \delta^k_i$ (here we are adding on the repeated index $j$), using (4.3) we have that

$$
0 = \partial_t \delta^k_i = \partial_t g_{ij} g^{jk} + g_{ij} \partial_t g^{jk} = -2H_s a_{ij} g^{jk} + g_{ij} \partial_t g^{jk},
$$

which gives (4.4).

Also, using that $\nu \cdot \nu = 1$ and (4.8), we have that

$$
\partial_t \nu \cdot \nu = 0,
$$

that

$$
\partial_\omega_i \nu \cdot \nu = 0
$$

and

$$
\partial_t \nu \cdot \partial_\omega_i \Gamma = -\nu \cdot \partial_\omega_i (\partial_t \Gamma) = \nu \cdot \partial_\omega_i (H_s \nu) = \partial_\omega_i H_s.
$$

Hence, decomposing $\partial_t \nu$ along the orthogonal directions $\{\nu, \partial_\omega_1 \Gamma, \ldots, \partial_\omega_n \Gamma\}$, we conclude that

$$
\partial_t \nu = g^{ij} \partial_\omega_j H_s \partial_\omega_i \Gamma = \nabla^\Gamma H_s.
$$

This completes the proof of (4.5).

Now we use (4.2) and (4.5) and we obtain that

$$
\begin{align}
\partial_t a_{ij} &= -\partial_t \nu \cdot \partial_\omega_j \partial_\omega_i \Gamma + \nu \cdot \partial_\omega_j \partial_\omega_i (H_s \nu) \\
&= -\nabla^\Gamma H_s \cdot \partial_\omega_j \partial_\omega_i \Gamma + \partial_\omega_j \partial_\omega_i H_s + H_s \nu \cdot \partial_\omega_j \partial_\omega_i \nu.
\end{align}
$$

Moreover,

$$
0 = \frac{1}{2} \partial_{\omega_j} \partial_{\omega_i} (\nu \cdot \nu) = \partial_{\omega_j} (\nu \cdot \partial_\omega_i \nu) = \nu \cdot \partial_\omega_j \partial_\omega_i \nu + \partial_\omega_i \nu \cdot \partial_\omega_j \nu
$$

and so we see that

$$
\partial_t a_{ij} = -\nabla^\Gamma H_s \cdot \partial_\omega_j \partial_\omega_i \Gamma + \partial_\omega_j \partial_\omega_i H_s - H_s \partial_\omega_j \nu \cdot \partial_\omega_i \nu.
$$

(4.9)

Now we assume that we have normal coordinates at $x_t$. Then at $x_t$ the metric $g_{ij}$ equals to $\delta_{ij}$ and the Christoffel symbols are 0. In particular, formula (4.9) reduces to

$$
\begin{align}
\partial_t a_{ij} &= \partial_{\omega_j} \partial_\omega_i H_s - H_s \partial_\omega_i \nu \cdot \partial_\omega_j \nu \\
&= \partial_{\omega_j} \partial_\omega_i H_s - H_s a_{ik} a^k_j.
\end{align}
$$

Since in normal coordinates $\partial_\omega_j \partial_\omega_i H_s = \nabla^\Gamma_i \nabla^\Gamma_j H_s$ and the latter is a coordinate invariant quantity, this establishes (4.6).
Now we prove (4.7). For this, we use (4.4) and (4.6), and we see that
\[
\partial_t(g^{ij}a_{ik}) = \partial_t g^{ij}a_{ik} + g^{ij}\partial_t a_{ik} \\
= 2H_s a^{ij}a_{ik} + g^{ij}(\nabla^T_i \nabla^T_k H_s - H_s a_{im} a^m_k) \\
= 2H_s a^{ij}a_{ik} + g^{ij}\nabla^T_i \nabla^T_k H_s - H_s a_{im} a^m_k.
\]
Therefore
\[
\partial_t(g^{ij}a_{ik})(g^{kl}a_{jl}) = 2H_s a^{ij}a_{ik}g^{kl}a_{jl} + g^{ij}g^{kl}a_{jl}\nabla^T_i \nabla^T_k H_s - H_s a_{im} a^m_k g^{kl}a_{jl} \\
= 2H_s a^{ij}a_{ik}g^{kl}a_{jl} + a^{ik}\nabla^T_i \nabla^T_k H_s - H_s a_{im} a^m_l a_{jl} \\
= H_s a^{ij}a_{ik}g^{kl}a_{jl} + a^{ik}\nabla^T_i \nabla^T_k H_s.
\]
This and the fact that (recall (4.2))
\[
\partial_t|A|^2 = \partial_t(g^{ij}a_{ik}g^{kl}a_{jl}) = \partial_t(g^{ij}a_{ik})g^{kl}a_{jl} + \partial_t(g^{kl}a_{jl})g^{ij}a_{ik} \\
= 2\partial_t(g^{ij}a_{ik})g^{kl}a_{jl}
\]
implies (4.7). \qed

For further reference, we also point out the following computation in local coordinates:

**Lemma 11.** For local coordinates \(\{\omega_1, \ldots, \omega_{n-1}\}\) we have that
\[
\partial_t(\partial_{\omega_i} \Gamma) = -H_s a^{ij}_{12}\partial_{\omega_j} \Gamma - \partial_{\omega_i} H_s \nu.
\]

**Proof.** Since \(\Gamma\) satisfies (4.1),
\[
\partial_t(\partial_{\omega_i} \Gamma) = \partial_{\omega_i}(\partial_t \Gamma) = \partial_{\omega_i}(-H_s \nu) = -\partial_{\omega_i} H_s \nu - H_s \partial_{\omega_i} \nu.
\]
On the other hand, by definition
\[
\partial_{\omega_i} \nu = a^i_j \partial_{\omega_j} \Gamma,
\]
which implies the result. \qed

### 4.2. Evolution of non-local quantities

In this subsection we will analyze the evolution of the perimeter, the fractional mean curvature and their first order spatial derivatives. In order to simplify the notation we write the point \(x(t) \in \partial E_t\) and the unit normal vector \(\nu(x(t))\) to \(\partial E_t\) at \(x(t)\) as
\[
x_t := x(t) \quad \text{and} \quad \nu_t := \nu(x(t)).
\]
We remark that when we integrate on the surface \(\partial E_t\) the integration variable, that we usually denote by \(y\), depends on \(t\), but we do not make explicit this dependence. Note additionally that \(v \cdot w\) denotes the standard dot product on \(\mathbb{R}^n\) between the vectors \(v\) and \(w\).

We observe that the integrand in (1.2) carries a singular kernel, therefore it is convenient to remove such singularity by using a cancellation. We perform these computations here, and we will use them in the forthcoming Section 5 to show that the positivity of the fractional mean curvature is preserved by the geometric flow.

To this goal, we write
\[
H_s(x_t, E_t) = H^\text{reg}_s(x_t, E_t) + H^\text{sing}_s(x_t, E_t),
\]
with
\[
H^\text{sing}_s(x_t, E_t) = \lim_{\delta \searrow 0} s(1 - s) \int_{C^\text{reg}_R(x_t) \setminus B_\delta(x_t)} \frac{\tilde{\chi}_{E_t}(y)}{|x_t - y|^{n+s}} dy, \quad \text{and}
\]
\[
H^\text{reg}_s(x_t, E_t) = s(1 - s) \int_{\mathbb{R}^n \setminus C^\text{reg}_R(x_t)} \frac{\tilde{\chi}_{E_t}(y)}{|x_t - y|^{n+s}} dy,
\]
where \( C^R_{\nu_t}(x_t) \) is a fixed cylinder centered at \( x_t \) with flat direction parallel to the normal of the surface at \( x_t \), namely

\[
C^R_{\nu_t}(x_t) := \{ x \in \mathbb{R}^n \text{ s.t. } x = x_t + y \text{ with } |y \cdot \nu(x_t)| < R \text{ and } |y - (y \cdot \nu(x_t))\nu(x_t)| < R \}.
\]

In what follows, we denote the surface \( \partial E_t \) as \( \Gamma(\omega, t) \) and we assume that is parameterized such that (4.1) holds. Consider \( x_t \in \Gamma \) and the epigraph of the tangent plane \( \Pi \) at \( x_t \) given by

\[
\Pi(x_t, E_t) := \{ \xi \in \mathbb{R}^n \text{ s.t. } \nu_t \cdot (\xi - x_t) \geq 0 \},
\]

where \( \nu_t \) is the unit normal to \( \Gamma(t) \) at the point \( x_t \).

Note that for \( R \) small enough, \( \Gamma(t) \) can be written as a graph over the tangent plane at \( x_t \in \Gamma(t) \). More precisely, \( \nu_t \) be the normal vector at \( x_t \) and let us parameterize \( \partial \Pi \) (or equivalently, the linear space perpendicular to \( \nu_t \)) in appropriate polar coordinates \( (r, \varphi) \in [0, R] \times S^{n-2} \). Then using the implicit function theorem, near \( x_t \) we may define a function \( h \) such that

\[
\Gamma(\omega, t) = x_t + \rho M_{x_t} \varphi + \rho h(\rho, \varphi) \nu_t.
\]

Here \( \rho \) is the distance to \( x_t \) on \( \partial \Pi \) and \( M_{x_t} \varphi \in \partial \Pi \) is defined as follows:

Assume that \( x_t = \Gamma(\bar{\omega}, t) \). Consider an orthonormal frame \( \{ v_j \} \) on \( \partial \Pi(x_t, E_t) \). Since \( \{ \partial_{x_i} \Gamma \}_{i=1, \ldots, n-1} \) span \( \partial \Pi(x_t, E_t) \), there are \( c^j(t_0) \) that satisfy

\[
v_j = c^j(t_0) \partial_{\omega_j} \Gamma.
\]

We define \( c^j(t) \) for \( t \leq t_0 \) as solutions to the ODE system

\[
\begin{align*}
\partial_t c^j &- c^j a^i_x(\bar{\omega}, t) H_s(\Gamma(\bar{\omega}, t)) = 0 \\
c^j(t)|_{t=t_0} &= c^j(t_0).
\end{align*}
\]

Notice that, for technical convenience, we are taking here the backward ODE flow from time \( t_0 \). Then for \( t \leq t_0 \) we define

\[
v_j(\bar{\omega}, t) = c^j(t) \partial_{\omega_j} \Gamma(\bar{\omega}, t).
\]

We note that \( v_j(\bar{\omega}, t_0) = v_j \) and \( \{ v_j(t) \} \subset \partial \Pi(x_t, E_t) \), where \( x_t = \Gamma(\bar{\omega}, t) \) and \( \partial \Pi(x_t, E_t) \) is the tangent plane of \( \Gamma(\bar{\omega}, t) \).

From (4.14) and Lemma 11

\[
\partial_t v_j = \partial_t c^j(t) \partial_{\omega_j} \Gamma(\bar{\omega}, t) + c^j(t) \partial_t (\partial_{\omega_j} \Gamma(\bar{\omega}, t))
\]

\[
= - (\nabla^\Gamma H_s \cdot v_j) \nu_t.
\]

Moreover,

\[
\partial_t (v_j \cdot v_i) = - (\nabla^\Gamma H_s \cdot v_j) (\nu_t \cdot v_i) - (\nabla^\Gamma H_s \cdot v_i) (v_j \cdot \nu_t) = 0.
\]

Hence, \( \{ v_j \} \) remains an orthonormal base of \( \Pi(x_t, E_t) \).

Now we define

\[
M_{x_t} \varphi = \varphi' v_t, \text{ where } \varphi \in S^{n-2}.
\]

In particular, if we denote \( x_t = \Gamma(\bar{\omega}, t) \), from (4.17) we have

\[
\partial_t M_{x_t} \varphi = - (\nabla^\Gamma H_s \cdot M_{x_t} \varphi) \nu_t.
\]

We also note that, from equation (4.13) and the quadratic separation of the smooth surfaces from their tangent planes, it follows that \( h(0, \varphi) = 0 \).

Notice also that by symmetry, for \( \Pi_t := \Pi(x_t, E_t) \) and any \( R > \delta > 0 \)

\[
\int_{C^R_{\nu_t}(x_t) \setminus B_\delta(x_t)} \frac{\hat{x}_{nu}(y)}{|x_t - y|^{n+s}} dy = 0.
\]
Then, parameterizing $C^\nu_R(x_t)$ as $x_t + \rho M_{x_t} \varphi + \rho z \nu_t$ with $\rho \in [0, R]$, $\varphi \in S^{n-2}$ and $z \in [-R, R]$, due to cancellations we have that
\[
H^{\text{sing}}_s(x_t, E) = \lim_{\delta \searrow 0} s(1 - s) \int_{C^\nu_R(x_t) \setminus B_s(x_t)} \frac{\tilde{X}_E(y) + \tilde{X}_L(y)}{|x_t - y|^{n+s}} \, dy
\]
(4.21)
\[
= s(1 - s) \int_{S^{n-2}} \left[ \int_0^R \rho^{-1-s} \left( \int_h^{(\rho, \varphi)} \frac{1}{(\rho^2 + 1)^{n/2}} \, d\rho \right) \, d\varphi \right] \, d\varphi,
\]
where $\Pi_t = \Pi_t(x_t, E_t)$. We now compute the derivatives of $h$.

**Proposition 12.** For a given time $t$, consider a point $x_t = \Gamma(\bar{\omega}, t)$ and $\nu_t$ the normal vector to $\Gamma$ at $x_t$. Let $h$ be given by (4.13) where $x_t$ is fixed as above. Then denoting by $\nu$ the normal to $\Gamma(\omega, t)$, we have that
\[
\partial h(\rho, \varphi) = \frac{1}{\rho} \left( H_s(x_t) - H_s(\Gamma) \nu \cdot \nu_t \right) + (\nabla^T H_s(x_t) \cdot M_{x_t} \varphi + \frac{1}{\rho} (\nu_t \cdot D_\omega \Gamma(\omega, t)) \partial_h \omega,
\]
\[
\partial \omega_j = \left( (g^{ij}(x_t) + O(\rho)) \left( H_s(\Gamma(D_\omega \Gamma(\bar{\omega}, t)))^T \nu - \rho h(\rho, \varphi)(D_\omega \Gamma(\bar{\omega}, t))^T \nabla^T H(x_t) \right) \right)
\sim H_s(\Gamma) \left( O(\rho) + O(\rho^2) \right)
\]

**Proof.** First, we note that from (4.13), $\omega$ becomes implicitly a function of $\varphi$ and $\rho$, but also of $x_t$, hence it does depend implicitly on $t$. Hence, taking derivatives on equation (4.13) we have
(4.22)
\[
D_\omega \Gamma(\omega, t) \partial_t \omega + \partial_t \Gamma = \partial_t x_t + \rho \partial_t M_{x_t} \varphi + \rho \partial_t h(\rho, \varphi) \nu_t + \rho h(\rho, \varphi) \partial_t \nu_t.
\]
Note that
(4.23)
\[
\partial_t \Gamma \cdot \nu_t = -H_s(\Gamma) \nu \cdot \nu_t \quad \text{and} \quad \partial_t x_t \cdot \nu_t = -H_s(x_t) \nu_t \cdot \nu_t = -H_s(x_t).
\]
Moreover, since $M_{x_t} \varphi$ is a tangential vector at $x_t$, we have that $M_{x_t} \varphi \cdot \nu_t = 0$, thus
(4.24)
\[
-\partial_t M_{x_t} \varphi \cdot \nu_t = M_{x_t} \varphi \cdot \partial_t \nu_t = M_{x_t} \varphi \cdot \nabla^T H_s(x_t),
\]
where the latter identity follows from (4.5). Then, using (4.1) and taking dot product with $\nu_t$ (recall also that $\partial_t \nu_t \cdot \nu_t$), we have
\[
\partial_t h(\rho, \varphi) = \frac{1}{\rho} \left( H_s(x_t) - H_s(\Gamma) \nu \cdot \nu_t \right) + \nabla^T H_s(x_t) \cdot M_{x_t} \varphi + \frac{1}{\rho} (\nu_t \cdot D_\omega \Gamma(\omega, t)) \partial_t \omega.
\]
Now we are left to compute $\partial_t \omega$. To this end, we multiply equation (4.22) by $D_\omega \Gamma(\bar{\omega}, t)^T$, we exploit (4.23) and (4.24) and we obtain
\[
(D_\omega \Gamma(\bar{\omega}, t)^T D_\omega \Gamma(\omega, t)) \partial_t \omega - H_s(\Gamma) D_\omega \Gamma(\bar{\omega}, t)^T \nu = \rho h(\rho, \varphi)(D_\omega \Gamma(\bar{\omega}, t)^T \nabla^T H(x_t).
\]
Since $(D_\omega \Gamma(\bar{\omega}, t)^T D_\omega \Gamma(\omega, t) = (g^{ij}(x_t)))$, we have that the first matrix is $(g^{ij}(x_t) + O(\rho))$. Similarly, since $D_\omega \Gamma(\bar{\omega}, t)^T \nu_t = 0$, the second term is like $H_s(\Gamma)O(\rho)$. Hence
\[
\partial_t \omega = \left( g^{ij}(x_t) + O(\rho) \right) \left( H_s(\Gamma)(D_\omega \Gamma(\bar{\omega}, t))^T \nu + \rho h(\rho, \varphi)(D_\omega \Gamma(\bar{\omega}, t))^T \nabla^T H(x_t) \right)
\sim H_s(\Gamma) \left( O(\rho) + O(\rho^2) \right),
\]
as desired. \qed

We will also use a rotation that aligns the cylinder $C^\nu_R(x_t)$ with $C^\nu_R(x_r)$. We remark that since the vectors $\{v_i(t) : i = \ldots n-1\} \cup \{\nu_t\}$ are an orthonormal basis of $\mathbb{R}^n$ we may define for $y = y^i v_i(t) + y^n \nu_t$ the following rotation
(4.25)
\[
R_{t, \tau} y = y^i v_i(\tau) + y^n \nu_\tau.
\]
Notice that in particular \( y^i = y \cdot v_i(t) \) and \( y^a = y \cdot \nu_t \).

Then it is direct to show that

**Proposition 13.** Consider \( \mathcal{R}_{t, \tau} \) given by (4.25) and denote \( \nabla^R H_s(\tau) \) the tangential gradient of \( H_s(x_\tau) \). Then it holds that

1. \( \mathcal{R}_{t, \tau} = \text{Id.} \)
2. \( \partial_{\nu_t} \mathcal{R}_{t, \tau} y = [(y \cdot v_1(\tau_1))\partial_t v_i(t) + (y \cdot \nu_{\tau_1})\partial_t v_i] |_{\tau_1 = \tau_2} = -(y \cdot v_1(\tau_1)) v_i(\tau_2) \cdot \nabla^R H_s(\tau_2) + (y \cdot \nu_{\tau_1}) \nabla^R H_s(\tau_2). \)

Now we study the evolution of the \( s \)-perimeter \( P_s \) and of the \( s \)-mean curvature.

**Theorem 14.** Let \( x_t, v_t \) and \( h \) be as in (4.13). We have the following equations:

\[
\frac{\partial}{\partial t} P_s(x_t) = - \int_{\partial x_t} H^2_s(\omega) \frac{dH^{n-1}(y)}{s(1-s)} \leq 0, \tag{4.26}
\]

\[
\frac{\partial h}{\partial t} = \frac{1}{s(1-s)} \int_{\partial x_t} \left[ \int_0^R \frac{\rho^{-1-s}\partial h(\rho, \varphi)}{(1 + h^2(\rho, \varphi))^{n+2}} d\rho \right] d\varphi, \tag{4.27}
\]

\[
\frac{\partial (H^a_{\text{sing}})(x_t)}{s(1-s)} = 2 \int_{\partial x_t \cap C^a_R(x_t)} \frac{(\partial_t x_t - \partial_t y + (y - x_t) \cdot \nabla^R H_s(\nu_t) - (y - x_t) \cdot \nu_t \nabla^R H_s(x_t)) \cdot \nu}{|x_t - y|^{n+s}} dy \tag{4.28}
\]

\[
= 2 \int_{\partial x_t \cap C^a_R(x_t)} \frac{(\partial_t x_t - \partial_t y) \cdot \nu}{|x_t - y|^{n+s}} dy + P.V. \int_{\partial x_t} \frac{\partial_t x_t - \partial_t y}{|x_t - y|^{n+s}} dy + H_s(x_t) P.V. \int_{\partial x_t} \frac{1 - \nu(x) \cdot \nu(y)}{|x_t - y|^{n+s}} dy. \tag{4.29}
\]

Also,

\[
\text{the function } (0, R) \times S^{n-2} \ni (\rho, \varphi) \mapsto \frac{\rho^{-1-s}\partial h(\rho, \varphi)}{(1 + h^2(\rho, \varphi))^{n+2}} \text{ is integrable,} \tag{4.30}
\]

\[
\frac{\partial h}{\partial t} = O(R^{1-s}) \text{ and} \tag{4.31}
\]

\[
\int_{S^{n-1}} \int_0^1 (\chi_{E_t} + \chi_{\Pi_t})(x_t + R\mu_t, \omega + Rz\nu_t(x_t)) \frac{z M_{x_t}\omega \cdot \nabla^R H_s(x_t)}{(1 + z^2)^{\frac{n+s}{2}}} dz d\omega = O(R). \tag{4.32}
\]

**Proof.** Formula (4.26) follows from Theorem 6.1 in [27] and (4.27) from (4.21).

To compute the derivative of the regular part we need to compute

\[
\lim_{h \to 0} \frac{H^s_{\text{reg}}(x_t(t), E_t) - H^s_{\text{reg}}(x_t(t-h), E_{t-h})}{h} = \frac{1}{h} \left( \int_{\mathbb{R}^n \setminus C^a_R(x_t)} \frac{\tilde{\chi}_{E_t}(y)}{|x_t - y|^{n+s}} - \int_{\mathbb{R}^n \setminus C^a_R(x_{t-h})} \frac{\tilde{\chi}_{E_{t-h}}(y)}{|x_{t-h} - y|^{n+s}} \right). \tag{4.33}
\]

We divide the computation as follows:

\[
I_h = \frac{1}{h} \left( \int_{\mathbb{R}^n \setminus C^a_R(x_t)} \frac{\tilde{\chi}_{E_t}(y)}{|x_t - y|^{n+s}} - \int_{\mathbb{R}^n \setminus C^a_R(x_{t-h})} \frac{\tilde{\chi}_{E_{t-h}}(y)}{|x_{t-h} - y|^{n+s}} \right) \quad \text{and} \quad \tag{4.34}
\]

\[
II_h = \frac{1}{h} \left( \int_{\mathbb{R}^n \setminus C^a_R(x_{t-h})} \frac{\tilde{\chi}_{E_{t-h}}(y)}{|x_{t-h} - y|^{n+s}} - \int_{\mathbb{R}^n \setminus C^a_R(x_{t-h})} \frac{\tilde{\chi}_{E_{t-h}}(y)}{|x_{t-h} - y|^{n+s}} \right). \tag{4.35}
\]
For the first integral we consider a function $\phi^\epsilon \in C^\infty_0$ that approximates $\tilde{\chi}_{Et}$. Then,

$$I_h = \lim_{\epsilon \to 0} I_h^\epsilon,$$

with

$$I_h^\epsilon := \frac{1}{h} \left( \int_{\mathbb{R}^n \setminus C_R^n(0)} \frac{\phi^\epsilon(y)}{|y|^{n+s}} - \int_{\mathbb{R}^n \setminus C_R^{n-h}(x_{t-h})} \frac{\phi^\epsilon(y)}{|x_{t-h} - y|^{n+s}} \right)$$

$$= \frac{1}{h} \int_{\mathbb{R}^n \setminus C_R^n(0)} \frac{\phi^\epsilon(y + x_{t}) - \phi^\epsilon(R_{t-h}y + x_{t-h})}{|y|^{n+s}} dy$$

$$= \int_{\mathbb{R}^n \setminus C_R^n(0)} \left[ \int_0^1 \nabla \phi^\epsilon(y_{h,l}) \cdot \delta_h \right] dy,$$

where

$$\delta_h := \frac{x_t - x_{t-h} + \partial_t R_{t-h}y}{h}.$$  

$$R_{t,\tau}$$ is given by (4.25)

and $y_{h,l} = R_{t-h}(1-\epsilon)h+y_t + h(x_t - x_{t-h})$.

From Proposition 13 we have $\partial_t R_{t-h(1-\epsilon)h}y = h [(y \cdot v_t) \partial_x v_t (\tau) + (y \cdot \nu_t) \partial_x \nu_t]_{\tau = t-h}.$

Moreover, if we denote by $R_{T(t-1-\epsilon)h}$ the inverse of the transpose of $R_{t-h(1-\epsilon)h}$ we have

$$\text{div}_y \left( \frac{\phi^\epsilon(y_{h,l})R_{T(t-h(1-\epsilon)h)} \delta_h}{|y|^{n+s}} \right)$$

$$= \frac{R_{T(t-h(1-\epsilon)h)} \nabla \phi^\epsilon(y_{h,l}) \cdot R_{T(t-h(1-\epsilon)h)} \delta_h}{|y|^{n+s}} + \phi^\epsilon(y_{h,l}) \text{div}_y \left( \frac{\delta_h}{|y|^{n+s}} \right),$$

and so the divergence theorem gives that

$$\int_{\partial C_R^n(0)} \frac{\phi^\epsilon(y_{h,l})R_{T(t-h(1-\epsilon)h)} \delta_h}{|y|^{n+s}} \cdot \nu_{C_R^n(0)} d\mathcal{H}^{n-1}(y)$$

$$= \int_{\mathbb{R}^n \setminus C_R^n(0)} \nabla \phi^\epsilon(y_{h,l}) \cdot \delta_h dy + \int_{\mathbb{R}^n \setminus C_R^n(0)} \phi^\epsilon(y_{h,l}) \text{div}_y \left( \frac{\delta_h}{|y|^{n+s}} \right) dy.$$

We insert this information into (4.34) and we obtain that

$$I_h^\epsilon = - \int_0^1 \left[ \int_{\mathbb{R}^n \setminus C_R^n(0)} \phi^\epsilon(y_{h,l}) \text{div}_y \left( \frac{\delta_h}{|y|^{n+s}} \right) dy \right] dl$$

$$+ \int_0^1 \left[ \int_{\partial C_R^n(0)} \frac{\phi^\epsilon(y_{h,l})R_{T(t-h(1-\epsilon)h)} \delta_h}{|y|^{n+s}} \cdot \nu_{C_R^n(0)} d\mathcal{H}^{n-1}(y) \right] dl.$$
where \( \nu_{C_R}^{\tau}(0) \) is the unit normal to the cylinder at \( y \). Now we observe that

\[
\chi_E(\mathcal{R}_{t,t-(1-l)h} y + x_{t-h} + \ell(x_t - x_{t-h})) - \chi_E(\chi_{[y+O(h)} - \chi_{E}(y + x_{t-h})) = \chi_{E}(\chi_{[y+O(h)} - \chi_{E}(y + x_{t-h})),
\]

so this function is supported in a neighborhood of size \( O(h) \) of a smooth surface. This fact, (4.36) and the integrability of the kernel \( |y|^{-n-s} \) at infinity give that

\[
I_h = - \int_0^1 \left[ \int_{\mathbb{R}^n \setminus C_R^\tau(0)} \chi_{E}(y + x_{t-h}) \operatorname{div}_y \left( \frac{\delta_h}{|y|^{n+s}} \right) dy \right] d\ell \\
+ \int_0^1 \left[ \int_{\partial C_R^\tau(0)} \chi_{E}(y + x_{t-h}) \frac{\delta_h}{|y|^{n+s}} \cdot \nu_{C_R^\tau(0)} d\mathcal{H}^{n-1}(y) \right] d\ell + o(1),
\]
as \( h \to 0 \). Recalling (4.35) and Proposition 13, we have for \( \tau = t - (1-l)h \) that

\[
\operatorname{div}_y \left( \frac{\delta_h}{|y|^{n+s}} \right) = v_i(t) \cdot \partial_t v_i(t) + \nu_i \cdot \partial_t \nu_i - (n+s) \frac{y \cdot (x_t - x_{t-h} + \partial_t \mathcal{R}_{t,t-(1-l)h} y)}{|y|^{n+s+2} h} \\
\to -(n+s) \frac{y \cdot (\partial_t x_t + (y \cdot v_i(t)) \partial_t v_i(t) + (y \cdot \nu_i) \partial_t \nu_i)}{|y|^{n+s+2}} \text{ as } h \to 0,
\]

and \( \mathcal{R}_{t-(1-l)h,t}^T x_t - x_{t-h} + \partial_t \mathcal{R}_{t,t-(1-l)h} y \)

\[
\to \partial_t x_t + (y \cdot v_i(t)) \partial_t v_i(t) + (y \cdot \nu_i) \partial_t \nu_i \text{ as } h \to 0.
\]

Additionally, from (4.17) and (4.5), we have that

\[
y \cdot [(y \cdot v_i(t)) \partial_t v_i(t) + (y \cdot \nu_i) \partial_t \nu_i] = -(y \cdot v_i)(\nabla^T H_s \cdot v_i(t)) (y \cdot \nu_i) + (y \cdot \nu_i)(y \cdot \nabla^T H_s) \\
= -(\nabla^T H_s \cdot y^T)(y \cdot \nu_i) + (y \cdot \nu_i)(y \cdot \nabla^T H_s) \\
= 0.
\]

Hence,

\[
(4.37) \quad \lim_{h \to 0} I_h = (n+s) \int_{\mathbb{R}^n \setminus C_R^\tau(0)} \tilde{\chi}_{E}(y + x_t) \frac{y \cdot \partial_t x}{|y|^{n+s+2}} dy \\
+ \int_{\partial C_R^\tau(0)} \tilde{\chi}_{E}(y + x_t) \frac{(\partial_t x_t + (y \cdot v_i(t)) \partial_t v_i(t) + (y \cdot \nu_i) \partial_t \nu_i) \cdot \nu_{C_R^\tau(0)}}{|y|^{n+s}} d\mathcal{H}^{n-1}(y).
\]

Now we notice that

\[
-(n+s) \frac{y \cdot \partial_t x}{|y|^{n+s+2}} = \operatorname{div}_y \left( \frac{\partial_t x}{|y|^{n+s}} \right).
\]

Then using the divergence theorem we have

\[
(n+s) \int_{\mathbb{R}^n \setminus C_R^\tau(0)} \tilde{\chi}_{E}(y + x_t) \frac{y \cdot \partial_t x}{|y|^{n+s+2}} dy = \\
\int_{\mathcal{E} \setminus C_R^\tau(x_t)} \operatorname{div}_y \left( \frac{\partial_t x}{|y - x_t|^{n+s}} \right) dy - \int_{\mathcal{E} \setminus C_R^\tau(x_t)} \operatorname{div}_y \left( \frac{\partial_t x}{|y - x_t|^{n+s}} \right) dy \\
= 2 \int_{\partial \mathcal{E} \setminus C_R^\tau(x_t)} \nu_{\partial \mathcal{E}}(y) \cdot \partial_t x \ d\mathcal{H}^{n-1}(y) - \int_{\partial \mathcal{E} \setminus C_R^\tau(x_t)} \chi_E(y) \nu_{\partial \mathcal{E}^\tau(0)}(y) \cdot \partial_t x \ d\mathcal{H}^{n-1}(y),
\]

where \( \nu_{\partial \mathcal{E}}(y) \) denotes the unit normal to \( \partial \mathcal{E}_t \) at \( y \).
Plugging this into (4.37) we obtain

\[ \lim_{h \to 0} I_h = 2 \int_{\partial E_i \setminus C_R(0)} \frac{\nu_{\partial E_i}(y) \cdot \partial x}{|y - x_t|^{n+s}} dH^{n-1}(y) \]

\[ - \int_{\partial C_R^i(x_t)} \chi_{E_i}(y) \frac{\nu_{\partial E_i}(y) \cdot ((y - x_t)^i \partial v_i(t) + (y - x)^n \partial \nu(x))}{|y - x_t|^{n+s}} dH^{n-1}(y). \]

Now we notice that, from the definition of \( C_R(0) \), the normal \( \nu_{C_R(0)} \) is either on the tangent plane at \( x_t \) (for the sides of the cylinder) or it is parallel to the normal at \( x_t \) (at the top and the bottom of the cylinder). Hence, at the top and bottom of the cylinder we have \( \pm \nu_{\partial C_R^i}(y) \cdot \partial v_i(t) = -\nabla^T H_s \cdot v_i(t) \) and \( \nu_{\partial C_R^i}(y) \cdot \partial \nu_t = 0 \), while along the sides of the cylinder \( \nu_{\partial C_R^i}(y) \cdot \partial v_i(t) = 0 \) and \( \nu_{\partial C_R^i}(y) \cdot \partial \nu_t = \frac{(y - x_t)^i}{|y - x_t|^2} \cdot \nabla^T H_s \). In addition, \( \tilde{\chi}_{E_i} = -1 \) on the bottom of the cylinder and \( \tilde{\chi}_{E_i} = 1 \) on the top. As a consequence,

\[ \int_{\partial C_R^i(x_t)} \chi_{E_i}(y) \frac{\nu_{\partial E_i}(y) \cdot ((y - x_t)^i \partial v_i(t) + (y - x)^n \partial \nu_t)}{|y - x_t|^{n+s}} dH^{n-1}(y) = \]

\[ - 2 \int_{S^{n-2}} \int_0^1 R^{n-s} \rho^{-2} \nabla^T H_s(t) \cdot \omega \frac{\rho^{n-2} \nabla^T H_s(t) \cdot \omega}{(\rho^2 + 1)^{\frac{n+s}{2}}} d\rho d\omega \]

\[ + \int_{S^{n-2}} \int_{-1}^1 \chi_{E_i}(x_t + RMz, \omega + Rz \nu_t) R^{n-s} \frac{zM_{x_t, \omega} \cdot \nabla^T H_s(x_t)}{(1 + z^2)^{\frac{n+s}{2}}} d\omega dz. \]

By symmetry the first term is 0 and

\[ \int_{S^{n-2}} \int_{-1}^1 \chi_{E_i}(x_t + RMz, \omega + Rz \nu_t(x_t)) R^{n-s} \frac{zM_{x_t, \omega} \cdot \nabla^T H_s(x_t)}{(1 + z^2)^{\frac{n+s}{2}}} d\omega dz = 0. \]

we obtain the first equality of (4.28).

The second equality may be obtained observing that

\[ -(n + s) \frac{y \cdot (\partial x + y \cdot v_i(t) \partial v_i(t) + y \cdot \nu_t \partial \nu_t)}{|y|^n + 2} = \text{div}_y \left( \frac{\partial x + (y - x_n)^i \partial v_i(t) + (y - x)^n \partial \nu_t}{|y|^{n+s}} \right). \]

For the integral defining \( II_h \), we have

\[ II_h = \frac{1}{h} \int_{\mathbb{R}^n \setminus C_R(0)} \frac{\tilde{\chi}_{E_i}(y + x_t - h) - \tilde{\chi}_{E_i}(y + x_t - h)}{|y|^{n+s}} dy. \]

Notice that the integrand is not 0 for \( y + x_t \in E_i \setminus E_{t-h} \). Since we assume that \( \partial E_i \) is smooth, we may parameterize this neighborhood as \( y = y_t + z \nu_{\partial E_i}(y_t) \) where \( y_t \in \partial E_i \). Since we assume that the sets \( E_i \) are continuous in \( t \), for \( h \) small enough, \( E_i \setminus E_{t-h} \) is contained in this tubular neighborhood. Moreover, a Taylor expansion in \( t \) yields that

\[ y_t - h = y_t - h \partial y_t + O(h^2) \] and \( (y_t - h) \cdot \nu_{\partial E_i}(y) = -h \partial y_t \cdot \nu_{\partial E_i}(y) + O(h^2) \).

Then we have

\[ II_h = \frac{1}{h} \int_{\partial E_i \setminus C_R(0)} \int_0^1 -h \partial y_t \cdot \nu_{\partial E_i}(y + O(h^2)) \frac{2}{|y - x_t - h|^{n+s}} dz dH^{n-1}(y) \]

\[ - 2 \int_{\partial E_i \setminus C_R(0)} \frac{\partial y_t \cdot \nu_{\partial E_i}(y)}{|y - x_t - h|^{n+s}} dH^{n-1}(y) \quad \text{as } h \to 0. \]

This, together with (4.38), proves (4.28).

From Proposition 12, we have that

\[ \frac{\rho^{1-s} \partial h(\rho, \varphi)}{(1 + h^2(\rho, \varphi))^{\frac{n+s}{2}}} = O(\rho^{-s}), \]
which is integrable, thus (4.31) follows directly from (4.21). Similarly, we observe that
\[
\int_{S^{n-1}} \int_{-1}^{1} (\chi_{E_t} + \chi_{M})(x_t + RM_{x_t} + Rz\nu_t(x_t)) \frac{z M_{x_t} \cdot \nabla H_s(x_t)}{(1 + z^2)^{\frac{n+s}{2}}} dz d\omega
\]
= \int_{S^{n-1}} \int_{\min(h(R\omega),1)}^{0} \frac{z M_{x_t} \cdot \nabla H_s(x_t)}{(1 + z^2)^{\frac{n+s}{2}}} dz d\omega
\]
and equation (4.32) follows from the fact \( h(0) = 0 \).

Finally, equation (4.29) follows from [6] and the fact that \( \partial_\omega x \) is tangential.

Equation (4.30) follows now by combining (4.27) and (4.28) and taking \( R \to 0 \) (another proof of (4.30) can be obtained using formula (B.2) of [20]; using Lemmata A.2 and A.4 there, one also obtains an expansion of the quantity in (4.30) as \( s \) approaches 1). \( \square \)

Remark 15. An equation analogous to (4.30) was obtained in [20] in a different context. Their results imply that

\[
s(1 - s)P.V. \int_{\partial E_t} \frac{H_s(y) - H_s(x)}{|x - y|^{n+s}} dy \to \Delta_{\partial E_t} H \quad \text{as} \quad s \to 1
\]
\[
s(1 - s)P.V. \int_{\partial E_t} \frac{1 - \nu(y) \cdot \nu(x)}{|x - y|^{n+s}} dy \to |A|^2 \quad \text{as} \quad s \to 1,
\]
which recovers the classical evolution for the mean curvature \( H \) under evolution by mean curvature flow.

5. Preservation of the fractional mean curvature

In this section we show that the geometric flow preserves the positivity of the fractional mean curvature. We need the following lemma that excludes the possibility of compact hypersurfaces with fractional mean curvature equal to zero (we state the result for smooth sets for the sake of simplicity):

Lemma 16. Let \( E \subset \mathbb{R}^n \) be a set with \( C^2 \)-boundary and such that \( H_s(x, E) = 0 \) for any \( x \in \partial E \). Assume that \( E \) is bounded in one direction, i.e. there exist \( \omega \in S^{n-1} \) and \( M \in \mathbb{R} \) such that
\[
E \subset \{ x \in \mathbb{R}^n, \; x \cdot \omega < M \}.
\]
Then \( E \) is a halfspace (unless it is empty).

In particular, there exists no compact hypersurface with vanishing fractional mean curvature.

Proof. The proof is based on a sliding method. Roughly speaking, we take a plane of normal direction \( \omega \) and we slide it from infinity till it touches \( E \), and then we compare the fractional mean curvatures at a touching point to obtain the desired result. The details of the proof go as follows. We suppose that
\[
E \neq \emptyset.
\]
Let
\[
\Pi_M := \{ x \in \mathbb{R}^n, \; x \cdot \omega < M \}
\]
and \( M_* := \inf\{M, \; E \subset \Pi_M\} \).

Notice that \( M_* \in \mathbb{R} \), thanks to (5.1) and (5.2). In addition, \( E \) is a subset of \( \Pi_{M_*} \) and there exists \( x_t \in (\partial E) \cap (\partial \Pi_{M_*}) \). We claim that \( E = \Pi_{M_*} \) (up to sets of measure zero, and this will end the proof of Lemma 16). Indeed, if not, the positivity set of the function
\[
\tilde{\chi}_E - \tilde{\chi}_{\Pi_{M_*}} = 2\chi_{\Pi_{M_*} \setminus E}
\]
would have positive measure and therefore
\[
0 < \int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(y) - \tilde{\chi}_{\Pi_{M_*}}(y)}{|x_t - y|^{n+s}} dy = H_s(x_t, E) - H_s(x_t, \Pi_{M_*}) = 0 - 0,
\]
and this is a contradiction. \( \square \)
Theorem 17. Let $E_t$ be a compact solution of (1.1). Assume that $H_s$ is differentiable and that $E_0$ has strictly positive fractional mean curvature. Then, $E_t$ has strictly positive fractional mean curvature for every $t \in (0, T)$.

Proof. Suppose the contrary. Then, if $E = E_t$ is the evolving surface, we have that $H_s(x, E_t) > 0$ for any $x \in \partial E_t$ and any $t \in (0, t)$, but

$$H_s(\bar{x}, E_t) = 0,$$

for some $\bar{x} \in \partial E_t$, with $\bar{t} \in (0, T)$.

Notice that $x_t \in \partial E_t$ and the function

$$t \mapsto H_s(x_t, E_t)$$

attains its minimum in the interval $[0, \bar{t}]$ and the endpoint $\bar{t}$ and therefore $\partial_t H_s(x_t, E_t)|_{t=\bar{t}} \leq 0$. Since it is also a spatial critical point for $H_s$, we have that $\nabla H_s(x, E_t)|_{t=\bar{t}} = 0$. From (4.30) in Theorem 14 and (1.1) we obtain that

$$\partial_t H_s(x, \bar{t}) = s(1-s) \int_{\partial E_t} \frac{H_s(y)}{|y-\bar{x}_t|^n + s} d\mathcal{H}^{n-1}(y) \geq 0.$$

However, since $\partial_t H_s(x_t, E_t)|_{t=\bar{t}} \leq 0$ we have that $H_s(y) \equiv 0$, which due to Lemma 16 contradicts the compactness of $E_t$. \qed

Following the same proof we can show for a non-compact solution that

Theorem 18. Let $E_t$ be a solution of (1.1). Assume that $H_s$ is differentiable, that $E_0$ has strictly positive fractional mean curvature and that $\partial E_t$ is uniformly spatially $C^2$ in $[0, T]$. Then, $E_t$ has strictly positive fractional mean curvature for every $t \in [0, T]$.

Proof. Proceeding as in the proof of the previous we can show that $E_t$ has strictly positive fractional mean curvature for every $t \in [0, T]$ or there is a $t_0$ such that $E_t$ has vanishing fractional mean curvature for every $t \geq t_0$.

Now we show that $H_s$ cannot become identically 0. For this, up to a dilation, we take a scale for which the evolving surface is locally a smooth graph in balls of radius 2 centered at the surface. Let $\phi$ be a nonnegative function supported in the unit ball $B_1$ and $\phi \equiv 1$ in $B_{\frac{3}{2}}$. Fix $x_t = x(0, t) \in \partial E_t$ and $\epsilon > 0$. Consider the function $v : \mathbb{R}^n \times [0, T)$ defined

$$v(y, t) = e^{C_1 t} \left( \frac{H_s(y)}{s(1-s)} + \epsilon \right) - \delta e^{-C_2 t} \phi(y-x_t),$$

where $\delta$ is chosen such that $v(y, 0) > 0$ and $C_1, C_2$ are real constant to be determined. Notice that $\delta$ can be chosen independently of $\epsilon > 0$.

Using equations (4.30) and (4.1), and denoting by $\nu_t$ the normal at $x_t$, we have, for $y \in \partial E_t$,

$$\partial_t v(y, t) = C_1 e^{C_1 t} \left( \frac{H_s(y)}{s(1-s)} + \epsilon \right) + e^{C_1 t} \left( 2 P.V. \int_{\partial E_t} \frac{H_s(z) - H_s(y)}{|z-y|^{n+s}} dz + 2 H_s(y) P.V. \int_{\partial E_t} 1 - \nu(z) \cdot \nu(y) |z-y|^{n+s} dz \right)$$

$$+ C_2 \delta e^{-C_2 t} \phi(y-x_t) + H_s(x_t) \delta e^{-C_2 t} \nu_t \cdot \nabla \phi(y-x_t)$$

$$= C_1 e^{C_1 t} \left( \frac{H_s(y)}{s(1-s)} + \epsilon \right) + 2s(1-s) \left( P.V. \int_{\partial E_t} \frac{v(z, t) - v(y, t)}{|z-y|^{n+s}} dz + e^{C_1 t} H_s(y) P.V. \int_{\partial E_t} 1 - \nu(z) \cdot \nu(y) |z-y|^{n+s} dz \right)$$

$$+ 2s(1-s) \delta e^{-C_2 t} P.V. \int_{\partial E_t} \frac{\phi(z-x_t) - \phi(y-x_t)}{|z-y|^{n+s}} dz + C_2 \delta e^{-C_2 t} \phi(y-x_t)$$

$$+ H_s(x_t) \delta e^{-C_2 t} \nu_t \cdot \nabla \phi(y-x_t).$$

Now we claim that

$$v(y, t) \geq 0.$$
Since this holds for \( t = 0 \) (as long as \( \delta \) is sufficiently small), to prove (5.4) we can argue by contradiction and assume that there is a first time \( \bar{t} \) and a point \( \bar{y} \) such that \( v(\bar{y}, \bar{t}) = 0 \). Such a point is a local minimum and it holds that

\[
\partial_t v(\bar{y}, \bar{t}) \leq 0,
\]

\[
P.V. \int_{\partial E_{\bar{t}}} \frac{v(z, \bar{t}) - v(\bar{y}, \bar{t})}{|z - \bar{y}|^{n+s}} dz \geq 0
\]

and

\[
e^{C\bar{t}} \left( \frac{H_s(\bar{y})}{s(1-s)} + \epsilon \right) = \delta e^{-C\bar{t}} \phi(\bar{y} - x_{\bar{t}}).
\]

Hence, we have

\[
0 \geq \partial_t v(\bar{y}, \bar{t}) \geq C_1 \delta e^{-C\bar{t}} \phi(\bar{y} - x_{\bar{t}}) + 2s (1-s) \delta e^{-C\bar{t}} P.V. \int_{\partial E_{\bar{t}}} \frac{\phi(z - x_{\bar{t}}) - \phi(\bar{y} - x_{\bar{t}})}{|z - \bar{y}|^{n+s}} dz
\]

\[
+ C_2 \delta e^{-C\bar{t}} \phi(\bar{y} - x_{\bar{t}}) + H_s(x_{\bar{t}}) \delta e^{-C\bar{t}} \nu_{\bar{t}} \cdot \nabla \phi(\bar{y} - x_{\bar{t}}).
\]

Now we claim that

\[
(5.5) \quad |\bar{y} - x_{\bar{t}}| < 1.
\]

To this end, we argue by contradiction and suppose that \( |\bar{y} - x_{\bar{t}}| \geq 1 \). Then, using (5.5) and the assumption on the support of \( \phi \), we find that

\[
0 \geq 2s (1-s) \delta e^{-C\bar{t}} P.V. \int_{\partial E_{\bar{t}}} \frac{\phi(z - x_{\bar{t}})}{|z - \bar{y}|^{n+s}} dz > 0.
\]

This is a contradiction and so (5.6) is proved.

Now we improve (5.6), by showing that there exists \( \epsilon_0 \in (0, 1) \) such that

\[
(5.7) \quad |\bar{y} - x_{\bar{t}}| < 1 - \epsilon_0.
\]

Again, we argue by contradiction and suppose that \( |\bar{y} - x_{\bar{t}}| \in [1 - \epsilon_0, 1] \). Since \( \phi \) is smooth and vanishes along \( \partial B_1 \), we have that \( \phi(x_{\bar{t}}) + |\nabla \phi(\bar{y} - x_{\bar{t}})| \leq C_{\epsilon_0} \), for some \( C > 0 \). Hence, using (5.5), and taking \( K > 0 \) such that

\[
(5.8) \quad H_s(x) \leq K \text{ for every } x \in \partial E_{\bar{t}},
\]

we see that

\[
0 \geq 2s (1-s) \delta e^{-C\bar{t}} P.V. \int_{\partial E_{\bar{t}}} \frac{\phi(z - x_{\bar{t}}) - C_{\epsilon_0}}{|z - \bar{y}|^{n+s}} dz - CK \delta e^{-C\bar{t}} \epsilon_0
\]

\[
\geq \delta e^{-C\bar{t}} \left[ 2s (1-s) P.V. \int_{(\partial E_{\bar{t}}) \cap B_{1/2}(x_{\bar{t}})} \frac{1 - C_{\epsilon_0}}{|z - \bar{y}|^{n+s}} dz - CK \epsilon_0 \right].
\]

So we multiply by \( \delta^{-1} e^{C\bar{t}} \) and, if \( \epsilon_0 \) is small enough, we find that

\[
0 \geq s (1-s) P.V. \int_{(\partial E_{\bar{t}}) \cap B_{1/2}(x_{\bar{t}})} \frac{dz}{|z - \bar{y}|^{n+s}} - CK \epsilon_0
\]

\[
\geq 2^{-n-s} (1-s) \mathcal{H}^{n-1}((\partial E_{\bar{t}}) \cap B_{1/2}(x_{\bar{t}})) - K \epsilon_0.
\]

The smoothness of the surface gives that

\[
\mathcal{H}^{n-1}((\partial E_{\bar{t}}) \cap B_{1/2}(x_{\bar{t}})) \geq c_0
\]

for some \( c_0 > 0 \). The last two inequalities easily give a contradiction if \( \epsilon_0 \) is small enough, and so we have established (5.7).

Now we set \( r_0 := 1 - \epsilon_0 \) and we choose \( C_1 \) large enough, such that

\[
(5.9) \quad C_1 \geq \frac{K \sup_{B_1} |\nabla \phi|}{\inf_{B_{r_0}} \phi},
\]

where \( K \) is as in (5.8).
Notice that, by (5.7) and (5.9),
\[ C_1 \delta e^{-C_2 t} \phi(\bar{y} - x_t) + H_s(x_t) \delta e^{-C_2 t} \nu \cdot \nabla \phi(\bar{y} - x_t) \geq 0. \]

Let also \( C_2 \) so large that
\[ C_2 \geq \sup_{y \in B_{r_0}(x_t)} \frac{2s(1-s)}{\phi(\bar{y} - x_t)} P.V. \int_{\partial E_t} \frac{\phi(z - x_t) - \phi(z - x_t)}{|z - \bar{y}|^{n+s}} dz. \]

In this way, and using again (5.7),
\[ 2s(1-s) \delta e^{-C_2 t} P.V. \int_{\partial E_t} \frac{\phi(z - x_t) - \phi(\bar{y} - x_t)}{|z - \bar{y}|^{n+s}} dz + C_2 \delta e^{-C_2 t} \phi(\bar{y} - x_t) \geq 0. \]

Then, we plug this information and (5.10) into (5.5) and we obtain a contradiction. This proves (5.4).

Then we take \( y = x_t \) and send \( \epsilon \to 0 \) in (5.4) and we obtain that \( H_s(x_t) \) remains positive. \( \square \)

### 6. Estimates for entire graphs

In this section we assume that the surface is an entire graph with linear growth. That is the surface can be parameterized by \((x, u(x, t))\) and \(|Du|(x, t) \to 0\) as \( |x| \to \infty\) is uniformly bounded for all times. Moreover, \( u \) satisfies
\[ \partial_t u = -\sqrt{1 + |Du|^2} H_s(E_u). \]

**Theorem 19.** Let \( \nu \) be the normal vector of a graphical surface evolving by (1.1) and \( e \) any fixed vector. Let \( v = (e \cdot \nu)^{-1} \), then
\[ v \leq \sup \{ v(\cdot, 0), C \}, \]
where \( C \) is such that \( \limsup_{|x| \to \infty} v(x, t) \leq C \).

**Proof.** Let us assume that the surface is parameterized according to (4.1) and \( \nu \) satisfies (4.5). Then
\[ v_t = -v^2(e \cdot \nabla^T H_s). \]

From Theorem 14 we have that
\[ e \cdot \nabla^T H_s = (n + s) (1 - s) P.V. \int_{\mathbb{R}^n} \tilde{\chi}_{E_t}(y) \frac{\langle y - x \rangle \cdot e^T}{|x - y|^{n+s+2}} dy, \]
where \( e^T \) is the tangential component of \( e \) at \( x_t \).

Noticing that \((n + s) \frac{\langle y - x \rangle \cdot e^T}{|x - y|^{n+s+2}} = -\text{div}_y \left( \frac{e^T}{|x - y|^{n+s+2}} \right)\), it follows from the divergence theorem that
\[ e \cdot \nabla^T H_s = 2s(1-s) \int_{\partial E_t} \frac{e^T \cdot \nu(y)}{|x - y|^{n+s}}. \]

Since \( e^T = e - v^{-1}(x) \nu(x) \), it holds that \( e^T \cdot \nu(y) = v^{-1}(y) - v^{-1}(x) \nu(x) \cdot \nu(y) \). Then, if \( v \) attains a maximum at \( x \), we have that \( e^T \cdot \nu(y) \geq 0 \) (and similarly \( e^T \cdot \nu(y) \leq 0 \) at minima). We may conclude from the maximum principle that \( v \) does not have interior maxima (resp. minima). \( \square \)

Noticing that for an evolving graph it holds that
\[ (e_n \cdot \nu)^{-1} = \sqrt{1 + |Du|^2}, \]
we have

**Corollary 20.** \(|Du|\) is uniformly bounded in time.

Also, we have the following regularity result:

**Corollary 21.** If \( u \) satisfies (2.13) then \( u \) is smooth.
Proof. Since $v$ is uniformly bounded above and below and
\[ v_t - 2s(1 - s)v^{-2} \int_{\partial \Omega_t} \frac{v^{-1}(y) - v^{-1}(x) \cdot \nu(y)}{|x - y|^{n+s}} dy = 0, \]
we have from [16] that $v$ is $C^\alpha$. Now following the proof in [3] we can conclude that $u \in C^\infty(\mathbb{R}^n \times [0, \infty))$. \qed

**Theorem 22.** Let $v = \sqrt{1 + |Du|^2}$, then the quantity $vH_s$ is uniformly bounded in terms of the initial condition.

**Proof.** Considering the set $\Pi$ as the epigraph of the plane $z = u(x_t, t) + \nabla u(x_t, t) \cdot (x - x_t) + u(x_t, t)$, we may write
\[ H_s(x_t, E) = s(1 - s) \int_{\mathbb{R}^n-1}^2 \frac{\nabla u(x_t, \xi) \cdot (x - x_t)}{|x - x_t|^{n+s-1}} \frac{dz}{(z - u(x_t, t))^2 + |x - x_t|^2} \]
\[ = s(1 - s) \int_{\mathbb{R}^n-1}^2 \frac{1}{|x - x_t|^{n+s-1}} \frac{\nabla u(x_t, \xi) \cdot (\frac{x - x_t}{|x - x_t|})}{|x - x_t|} \frac{dz}{(z^2 + 1) \frac{n+2}{2}}. \]

Let $z_m = \frac{u(x_t) - u(x_t, t)}{|x - x_t|}$ and $z_M = \nabla u(x_t, t) \cdot \frac{x - x_t}{|x - x_t|}$. Then
\[ \partial_t z_m = \frac{-H_s v(x_t, t) + H_s v(x_t, t)}{|x - x_t|}, \]
and
\[ \partial_t z_M = \frac{\nabla (-H_s v(x_t, t)) \cdot \frac{x - x_t}{|x - x_t|}}{|x - x_t|}. \]

As a consequence, we have
\[ \partial_t H_s(x_t, E) = s(1 - s) \int_{\mathbb{R}^n-1}^2 \frac{1}{|x - x_t|^{n+s-1}} \left( \frac{\partial_t z_M}{(z_M^2 + 1) \frac{n+2}{2}} - \frac{\partial_t z_m}{(z_m^2 + 1) \frac{n+2}{2}} \right), \]
and
\[ \partial_t (vH_s) = \frac{Du \cdot D(-H_s v)}{\sqrt{1 + |Du|^2}} H_s + v \partial_t H_s(x_t, E). \]

Assume that a maximum (resp. minimum) point of $vH_s$ is attained at $(x_t, t_0)$. Then $D(-H_s v) = 0$, $\partial_t z_M = 0$ and $\partial_t z_m \geq 0$ (resp. $\leq 0$) and only identically 0 if $vH_s$ is constant. Then we conclude that there are no interior maxima or minima for this quantity. \qed

**Remark 23.** The previous estimates imply that if there is decay at infinity $H_s$ remains bounded for all times.

7. Estimates for star-shaped surfaces

We show an estimate for star-shaped surfaces that is analog to Theorem 19

**Theorem 24.** Let $v = (x \cdot \nu)^{-1}$. Then there exists $T^* > 0$ such that $v(t) \leq C$ in $[0, T^*)$, where $C$ depends on $v(0)$ and $\sup |H_s|$.

**Proof.** We assume like in the proof of 19 that the surface is parameterized as in (4.1), then we have
\[ \partial_t v = -v^2(x_t \cdot \nu + x \cdot \nu_t) \]
\[ = v^2(H_s - x \cdot \nabla^T H_s) \]

Following the computations in the proof of Theorem 19 we have that
\[ x \cdot \nabla^T H_s = 2s(1 - s) \int_{\partial \Omega_t} \frac{x^T \cdot \nu(y)}{|x - y|^{n+s}} dy. \]

Since $x^T = x - x \cdot \nu(x)\nu(x) = (x - y) + (y - x \cdot \nu(x)\nu(x))$ we have
$$\frac{x \cdot \nabla^V H_s}{s(1-s)} = 2 \int_{\partial E_t} \frac{(x-y) \cdot \nu(y)}{|x-y|^{n+s}} dy + 2 \int_{\partial E_t} \frac{v^{-1}(y) - v^{-1}(x) \nu(x) \cdot \nu(y)}{|x-y|^{n+s}} dy$$

Accordingly, we have

$$\partial_t v = v^2 \left( (1-s^2(1-s)) H_s - 2s(1-s) \int_{\partial E_t} \frac{v^{-1}(y) - v^{-1}(x) \nu(x) \cdot \nu(y)}{|x-y|^{n+s}} dy \right).$$

Hence, at a spatial maximum of $v$ we have

$$\partial_t (\max_{\mathbb{S}^n} v(\cdot, t)) \leq (1-s^2(1-s)) \max_{\mathbb{S}^n} v(\cdot, t)^2 H_s.$$ 

Then we find that

$$\max_{\mathbb{S}^n} v(\cdot, t) \leq \frac{\max_{\mathbb{S}^n} v(\cdot, 0)}{1 - (1-s^2(1-s)) t \max_{\mathbb{S}^n} v(\cdot, 0) \sup_{S \times [0,T]} H_s}. $$

Notice that the bound can be extended as long as $H_s$ remains bounded.

The previous computation yields a gradient bound and that star-shapedness is preserved:

**Corollary 25.** Assume that $f$ satisfies (2.4). Then, if $H_s$ remains bounded, $|\nabla f|$ is bounded for a fixed time that depends of the initial condition and bounds of $H_s$. 

**Proof.** Notice that $x = f \omega$ and $\nu = \frac{f \omega - \nabla f}{\sqrt{f^2 + |\nabla f|^2}}.$ Then $x \cdot \nu = \frac{f^2}{\sqrt{f^2 + |\nabla f|^2}}.$ 

Then $v \leq C$ is equivalent to

$$\sqrt{f^2 + |\nabla f|^2} \leq C f^2 \leq C \max_{\mathbb{S}^n} f^2(\cdot, 0),$$

which gives the desired result. \qed

**Corollary 26.** Assume that $E_t$ is a solution to (1.1) and that $E_0$, then $E_t$ remains star-shaped.

**References**

[34] W. M. Mullins, Twodimensional motion of idealized grain boundaries, J. Appl. Phys. 27 (1956), 900904