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On the evolution by fractional mean curvature

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Abstract

In this paper we study smooth solutions to a fractional mean curvature flow equation. We establish a comparison principle and consequences such as uniqueness and finite extinction time for compact solutions. We also establish evolutions equations for fractional geometric quantities that yield preservation of certain quantities (such as positive fractional curvature) and smoothness of graphical evolutions.

1. Introduction

In the recent literature, an intense study has been performed on some fractional counterparts of the classical perimeter and of the motion by mean curvature. The interest in this kind of topics comes from several considerations. First of all, from the theoretical point of view, the analysis of nonlocal and fractional operators has an ancient tradition, which have been vividly renovated recently by new exciting discoveries. In particular, a notion of fractional perimeter has been introduced in [7] and its relation with a fractional mean curvature flow was discussed in detail in [15, 8].

Roughly speaking, given $s \in (0,1)$ the fractional perimeter in the whole of $\mathbb{R}^n$ of a bounded set E may be seen as the seminorm in $H^{s/2}(\mathbb{R}^n)$ of the characteristic function of E (and this notion may be also localized inside a bounded domain $\Omega \subset \mathbb{R}^n$). The first variation of the fractional perimeter functional may be seen as a fractional counterpart of the mean curvature. As $s \to 1$, these notions approach the classical objects in different senses (see e.g. [5, 19, 40, 9, 2, 10] for details). The limit as $s \to 0$ has also been taken into account under various circumstances (see e.g. [33, 22]).

These fractional theories of geometric type found very often concrete applications in real-world problems. For instance, fractional perimeter functionals naturally appear in the large-scale description of interfaces of nonlocal phase transitions (see [37, 38]). A very natural application arises also in computer science: indeed, the a square pixels of small side ϵ produce, along the diagonal, an error of order one for the classical perimeter, but an error of order only ϵ^{1-s} for the fractional perimeter. In this sense, fractional objects are very useful to "average out" the errors caused by the possible fine anisotropic structure of the media.

Many results of great interest about the fractional mean curvature flow have been recently obtained in [12, 13, 14]. See also [1] for a detailed study of the fractional mean curvature, with analogies and important differences with respect to the classical case. The question of the regularity of the minimal surfaces corresponding to the fractional perimeter has been investigated in [7, 36, 10, 28, 3, 11], several connections with the isoperimetric problems have been studied in [29, 27, 21] and remarkable examples of surfaces of vanishing and constant fractional mean curvature have been recently constructed in [20, 6].

In this work we are interested in studying classical solutions to the L^2-gradient flow associated to the fractional perimeter. More precisely, we consider a set E_0 and we are interested in a family E_t that satisfies for every $x \in \partial E_t$ the law of motion

$$\partial_t x \cdot \nu = -H_s, \tag{1.1}$$

where ν is the outer normal to E_t and the quantity H_s is the fractional mean curvature defined by

$$H_s(x,E) := \lim_{\delta \searrow 0} s(1-s) \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{\tilde{\chi}_E(y)}{|x-y|^{n+s}}\, dy. \tag{1.2}$$

Here above and in the sequel we use the notation

$$\tilde{\chi}_E(y) := \chi_{\mathbb{R}^n \setminus E}(y) - \chi_E(y),$$

while χ_E is the classical indicator function of E, that is 1 on E and 0 on $\mathbb{R}^n \setminus E$. We also assume that the parameter s belongs to the interval $(0,1)$. Notice that with this convention the mean curvature of a sphere is positive (more details on this case will be given in the forthcoming Section 2.1.1). Moreover, under this convention, the s-perimeter of solutions to (1.1) decreases in the fastest direction; In fact it holds that (see Theorem 14)

$$\partial_t P_s(E_t) = -\int_{\partial E_t} H_s^2(\omega)\, d\mathcal{H}^{n-1}(y) \leqslant 0.$$

The flow described by equation (1.1) is the natural analog of the mean curvature flow, which has been studied largely in the literature (see for instance [23], [24], [30], [31], [32] , [35] and references therein). The mean curvature flow has been used in several contexts that range from modeling interface transition [34] to obtain topological classification of certain surfaces [31, 4].

The mean curvature flow is a quasilinear geometric equation of parabolic nature that has regularizing effects as long as the mean curvature remains bounded (i.e. solutions are C^∞ in space and time while the mean curvature is bounded), but it may form singularities in finite time. One of the main topics within the subject is the study of singularity formation during the evolution. The first important result in this direction is due to G. Huisken [30] who showed that convexity is preserved by the flow and that singularities only form at an extinction time at which the surface collapses to a "round point", that is after appropriate rescaling convex surfaces are asymptotic to spheres. Later on, it has been proved that in fact the flow preserves k-convexity for any $1 \leqslant k \leqslant n-1$ ([31]) and that homothetic solutions play an important role in the understanding of singularity formation. In this paper we show that H_s-convexity is preserved by the fractional flow (see Section 5) and we observe that in fact spheres are self-similar solutions to the flow (see Section 2.1.1).

Another important classical example of evolution by mean curvature flow is the evolution of entire graphs with linear growth. In [24] it is shown that in that case the evolution exists and it is smooth for all times. The estimates of that work were later localized in [25] to obtain short time estimates for any evolution. Other graphical evolutions have been studied in [35]. In Section 6 we show that graphical solutions to (1.1) have bounded H_s- curvature for all times and are in fact C^∞. A key element of the proof is the preserved quantity $(\nu \cdot e_n)^{-1}$ which is known as the height function. On the other hand, star-shaped surfaces also have a preserved quantity and we briefly address this case in Section 7.

Other results that we present here are a comparison principle, the preservation of the positivity of H_s and some estimates for entire graphs.

The organization of the paper is as follows: Section 2 is devoted to formulate Equation (1.1) for star-shaped surfaces and entire graphs. We compute in particular the example of an evolving sphere. In Section 3 we show that a comparison principle holds for the flow and as a corollary we find bounds on the maximal existence time and uniqueness of smooth solutions. Section 4 is devoted to compute the evolution of local and non-local geometric quantities. Of particular interest is the evolution equation of H_s that is given by

$$\frac{\partial_t H_s}{2s(1-s)}(x) = \text{P.V.} \int_{\partial E_t} \frac{H_s(y) - H_s(x)}{|x_t - y|^{n+s}} dy + H_s(x) \text{P.V.} \int_{\partial E_t} \frac{1 - \nu(x) \cdot \nu(y)}{|x_t - y|^{n+s}} dy.$$

This equation implies that if the initial condition satisfies $H_s > 0$ then this is preserved by the flow. This result is proved in Section 5. In Section 6 we prove bounds for graphical solutions and in Section 7 that star-shapedness is preserved by the flow as long as the fractional curvature remains bounded.

2. Some special cases

In this section, we consider some particular forms of the fractional mean curvature motion, namely the cases in which the evolving surface is the boundary of a star-shaped domain or it is a graph in a given direction. A simple and concrete example of fractional mean curvature evolution for star-shaped surfaces is given by the spheres, in which the equation can be explicitly solved by scale invariance. On the other hand, planes are trivial examples of graphical evolutions.

2.1. **Evolution of star-shaped surfaces.** In this subsection we assume that the initial set is of the form

$$E_0 = \big\{\rho\omega,\ \omega \in S^{n-1},\ \rho \in [0, f_0(\omega)]\big\}$$

with $\nu(p) \cdot p \geqslant 0$ for any $p \in \partial E_0$, where $\nu(p)$ is the outer unit normal at p.

We deal with the motion of ∂E_0 by its fractional mean curvature. We assume that this evolution is regular and star-shaped around the origin for all times $t \in [0, T)$ That is, we consider

$$E_t = \big\{\rho\omega,\ \omega \in S^{n-1},\ \rho \in [0, f(\omega, t)]\big\}$$

$$\text{and} \qquad \partial E_t = \big\{f(\omega, t)\omega,\ \omega \in S^{n-1}\big\}$$

with $f \in C^2\big(S^{n-1} \times (0, +\infty), [0, +\infty)\big) \cap C^0\big(S^{n-1} \times [0, +\infty), [0, +\infty)\big)$ and $f > 0$.

In order to write (1.1) more explicitly in dependence of f we extend the function $f = f(\cdot, t)$, that was originally defined on S^{n-1}, to the whole of $\mathbb{R}^n \setminus \{0\}$ by homogeneity, namely we suppose, without loss of generality, that $f : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$, with

$$f(x) = f\left(\frac{x}{|x|}\right) \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\}. \tag{2.1}$$

Notice that we omitted, for simplicity, the dependence on the time t in the notation above. Similarly, given $\omega \in S^{n-1}$, unless otherwise specified, we denote by ν the exterior normal at the point $f(\omega)\omega$. Hence we have:

Lemma 1. *The external normal ν of E can be expressed in terms of f by*

$$\nu = \frac{f\omega - \nabla f}{\sqrt{|\nabla f|^2 + f^2}}. \tag{2.2}$$

Also, given any $\omega \in S^{n-1}$, for any $\eta \in \mathbb{R}^n$ orthogonal to ω we have that

$$\left(\nabla f(\omega) \cdot \eta\right)(\omega \cdot \nu) + f(\omega)\eta \cdot \nu = 0. \tag{2.3}$$

Finally, (1.1) *is equivalent to*

$$\begin{cases} \partial_t f(\omega, t) = -H_s(*, E_t)\dfrac{\sqrt{|\nabla f|^2 + f^2}}{f}, & \textit{for every } \omega \in S^{n-1} \textit{ and } t > 0, \\[2ex] f(\omega, 0) = f_0(\omega), & \textit{for every } \omega \in S^{n-1}, \end{cases} \tag{2.4}$$

where $* = f(\omega, t)\omega$.

Proof. First we point out that, by (2.1),

$$\nabla f(\omega) \cdot \omega = \frac{d}{d\tau} f(\tau\omega)\bigg|_{\tau=1} = \frac{d}{d\tau} f(\omega)\bigg|_{\tau=1} = 0 \tag{2.5}$$

for any $\omega \in S^{n-1}$. Also, if $\tau \mapsto \omega(\tau)$ is a curve on S^{n-1}, we have that

$$\omega \cdot \dot{\omega} = \frac{d}{d\tau}\frac{|\omega|^2}{2} = \frac{d}{d\tau}\frac{1}{2} = 0 \tag{2.6}$$

and a generic tangent vector at ∂E is

$$T := \frac{d}{d\tau}(f\omega) = (\nabla f \cdot \dot{\omega})\omega + f\dot{\omega}.$$

We observe that

$$(f\omega - \nabla f) \cdot T = f\left(\nabla f \cdot \dot{\omega}\right) + f^2 \dot{\omega} \cdot \omega - (\nabla f \cdot \dot{\omega})(\nabla f \cdot \omega) - f\left(\nabla f \cdot \dot{\omega}\right) = 0,$$

thanks to (2.5) and (2.6). This shows that the vector $f\omega - \nabla f$ is normal to ∂E. Also, by (2.5), the component of $f\omega - \nabla f$ in direction ω is f, which is positive: accordingly, this normal vector points outwards and this completes the proof of (2.2).

Using (2.5) and (2.2), we also obtain that

$$\omega \cdot \nu = \frac{f}{\sqrt{|\nabla f|^2 + f^2}}, \tag{2.7}$$

and this shows that (1.1) and (2.4) are equivalent (recall indeed that $x = f(\omega)\omega$).

It remains to prove (2.3). For this, we take η orthogonal to ω and we use (2.2) and (2.7) to compute

$$
\begin{aligned}
&(\nabla f \cdot \eta)\,(\omega \cdot \nu) + f\eta \cdot \nu \\
&\quad = \frac{f\,(\nabla f \cdot \eta)}{\sqrt{|\nabla f|^2 + f^2}} + \frac{f^2\eta \cdot \omega - f(\eta \cdot \nabla f)}{\sqrt{|\nabla f|^2 + f^2}} \\
&\quad = \frac{f\,(\nabla f \cdot \eta)}{\sqrt{|\nabla f|^2 + f^2}} + \frac{0 - f(\eta \cdot \nabla f)}{\sqrt{|\nabla f|^2 + f^2}}
\end{aligned}
$$

that clearly equals to zero and proves (2.3). □

For the analogue of (2.4) in the classical mean curvature flow see, e.g., formula (2.8) in [39].

As a matter of fact, from Lemma 1, we can easily present an explicit derivation of (1.1) in terms of the prescribed normal velocity (we refer to Section 2 of [39] for a similar argument in the classical case). Indeed, suppose that a smooth, compact hypersuperface of $\mathbb{R}^n$ is defined by an embedding $X : S^{n-1} \to \mathbb{R}^n$, and consider the evolution equation in which the normal velocity is some prescribed v (in our case, we will take v to be $-H_s$, but the argument is general). We then obtain the equation

$$\partial_t X(\zeta, t) = v(X(\zeta, t), t)\, \nu(X(\zeta, t), t),$$

for any $\zeta \in S^{n-1}$. A multiplication by the normal vector then yields

$$\partial_t X(\zeta, t) \cdot \nu(X(\zeta, t), t) = v(X(\zeta, t), t). \tag{2.8}$$

If the region enclosed by the manifold is star-shaped (say, with respect to the origin), one writes $X = f\omega$, i.e. one considers the diffeomorphism $\omega(\cdot, t) : S^{n-1} \to S^{n-1}$,

$$\omega(\zeta, t) := \frac{X(\zeta, t)}{|X(\zeta, t)|},$$

with inverse mapping $\zeta(\omega, t)$, and defines

$$f(\omega, t) := \big|X\big(\zeta(\omega, t), t\big)\big|.$$

We remark that $|\omega(\zeta, t)| = 1$, therefore

$$\omega(\zeta, t) \cdot \partial_t \omega(\zeta, t) = \partial_t \frac{|\omega(\zeta, t)|^2}{2} = \partial_t \frac{1}{2} = 0.$$

Therefore we can apply (2.3) and conclude that

$$(\nabla f \cdot \partial_t \omega)\,(\omega \cdot \nu) + f \partial_t \omega \cdot \nu = 0. \tag{2.9}$$

On the other hand

$$\partial_t X = \partial_t (f\omega) = (\nabla f \cdot \partial_t \omega + \partial_t f)\omega + f \partial_t \omega.$$

Thus, by (2.9),

$$
\begin{aligned}
\partial_t X \cdot \nu &= (\nabla f \cdot \partial_t \omega)(\omega \cdot \nu) + \partial_t f(\omega \cdot \nu) + f \partial_t \omega \cdot \omega \\
&= \partial_t f(\omega \cdot \nu)
\end{aligned}
$$

By substituting this into (2.8), we obtain

$$\partial_t f\,(\omega \cdot \nu) = v.$$

Then, (1.1) is simply the particular case in which the normal velocity is the fractional mean curvature, pointing inwards.

2.1.1. *A concrete example: The evolution of spheres.* In this section we compute the example of a concrete evolution, namely we show that the spheres shrink self-similarly in finite time. We think it is a very interesting open problem to determine whether or not these are the only embedded self-similar shrinking solutions of (1.1).

Lemma 2. *The fractional mean curvature of the ball of radius R is equal, up to dimensional constants, to R^{-s}. More explicitly, for any $x \in \partial B_1(0)$,*

$$H_s(x, B_1(0)) = \varpi \tag{2.10}$$

for some $\varpi > 0$, and, for any $x \in \partial B_R(0)$,

$$H_s(x, B_R(0)) = \varpi R^{-s}. \tag{2.11}$$

Proof. By rotational invariance of the integrals, we have that $H_s(x_1, B_1(0)) = H_s(x_2, B_1(0))$ for every $x_1, x_2 \in \partial B_1(0)$, thus showing (2.10). Moreover, if $\omega \in S^{n-1}$ and $x = R\omega$, by changing variable $\tilde{y} := Ry$, we see that

$$\begin{aligned} H_s(x, B_R(0)) =& \lim_{\delta \searrow 0} s(1-s) \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{\tilde{\chi}_{B_R(0)}(\tilde{y})}{|R\omega - \tilde{y}|^{n+s}} \, d\tilde{y} \\ =& R^n \lim_{\delta \searrow 0} s(1-s) \int_{\mathbb{R}^n \setminus B_{R^{-1}\delta}(x)} \frac{\tilde{\chi}_{B_R(0)}(Ry)}{|R\omega - Ry|^{n+s}} \, dy \\ =& R^{-s} \lim_{\delta \searrow 0} s(1-s) \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{\tilde{\chi}_{B_1(0)}(y)}{|\omega - y|^{n+s}} \, dy \\ =& R^{-s} H_s(\omega, B_1(0)). \end{aligned} \tag{2.12}$$

This, together with (2.10), proves (2.11). □

Corollary 3. *Let ϖ be as in (2.10) and $C_0 := \varpi\,(s+1)$. Let $R(t) := (R_0^{s+1} - C_0 t)^{\frac{1}{s+1}}$. Then $B_{R(t)}(0)$ is a star-shaped solution to fractional mean curvature flow with initial condition $B_{R_0}(0)$ and it collapses to the origin in the finite time $\frac{R_0^{s+1}}{C_0}$.*

Proof. We only need to show that (2.4) is satisfied with $f(\omega, t) := R(t)$ and $f_0(\omega) := R_0$. For this, we use Lemma 2 to compute

$$\partial_t f + H_s \, \frac{\sqrt{|\nabla f|^2 + f^2}}{f} = -\frac{C_0}{s+1} (R_0^{s+1} - C_0 t)^{\frac{-s}{s+1}} + H_s = \varpi\,(R_0^{s+1} - C_0 t)^{\frac{-s}{s+1}} + \varpi\, R^{-s} = 0,$$

that shows the validity of (2.4). □

From the results in Section 3, we will see that the one provided in Corollary 3 is indeed the unique smooth solution of the fractional mean curvature flow with spherical initial datum.

It is also easy to check that a similar computation yields an analogous result for the evolution of cylinders.

2.2. **Evolution of graphical surfaces.** In this subsection we assume that the initial set is of the form

$$E_0 = \big\{(x, z), x \in \mathbb{R}^{n-1}, z \in [-\infty, u(x)]\big\}.$$

The appropriate choice of normal in this situation is given by

$$\nu(x, u(x)) = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

We assume that

$$\begin{aligned} & E_t = \big\{(x, z), x \in \mathbb{R}^{n-1}, z \in (-\infty, u(x,t)]\big\} \\ \text{and} \qquad & \partial E_t = \big\{(x, u(x,t)),\, x \in \mathbb{R}^{n-1}\big\} \end{aligned}$$

with $u \in C^2\big(\mathbb{R}^{n-1} \times (0,+\infty), [0,+\infty)\big) \cap C^0\big(\mathbb{R}^{n-1} \times [0,+\infty), [0,+\infty)\big)$. In this setting, the geometric flow in (1.1) is equivalent to

$$\begin{cases} \partial_t u(x,t) = -H_s(x,E_t)\sqrt{|\nabla u|^2+1}, & \text{for every } x \in \mathbb{R}^{n-1} \text{ and } t>0, \\ u(x,0) = u_0(x), & \text{for every } x \in \mathbb{R}^{n-1}, \end{cases} \tag{2.13}$$

A concrete example in this case is any linear u, which has fractional mean curvature equal to 0.

Remark 4. *Equations* (2.4) *and* (2.13) *are well posed imposing weaker regularity conditions on f and u respectively*

3. Comparison principle

In this section we show that two surfaces evolving under fractional mean curvature flow that are initially nested remain nested while the evolution is smooth. More precisely, we have the following comparison result:

Theorem 5. *Let E_t and F_t be two smooth solutions to* (1.1) *in $[0,\omega)$ such that $E_0 \subseteq F_0$. Assume additionally that $\partial_t x(\cdot,t)$, $\partial_t y(\cdot,t)$ are continuous in $[0,T)$ for $x(\cdot,t) \in \partial E_t$ and $y(\cdot,t) \in \partial F_t$. Then $E_t \subseteq F_t$.*

Proof. We first assume that E_0 is strictly contained in F_0 and suppose that there is a time t_0 and a point x_t at which E_{t_0} and ∂F_{t_0} touch for the first time and the normal velocity of E_{t_0} at x_t is bigger than the normal velocity of ∂F_{t_0} at that point (i.e. the boundaries cross at point of space time). Since ∂E_{t_0} and ∂F_{t_0} are tangential at x_t the normal vectors agree at that point. Then we have

$$0 \geqslant (\partial_t x_{F_t} - \partial_t x_{E_t}) \cdot \nu_E(x_t) = H_s(E_t,x_t) - H_s(F_t,x_t)$$

Moreover, since $E_{t_0} \subset F_{t_0}$ we have $H_s(E_{t_0},x_t) \geqslant H_s(F_{t_0},x_t)$, which yields a contradiction.

If E_0 is not strictly contained in F_0, then we can proceed as before by observing that the equation holds in the limit as $t \to 0$. ☐

The previous theorem implies a more general result, as stated here below:

Corollary 6. *Let E_t and F_t be two smooth solutions to* (1.1) *in $[0,\omega)$ such that $\partial E_0 \cap \partial F_0 = \emptyset$. Then $\partial E_t \cap \partial F_t = \emptyset$.*

Proof. By noticing that the evolution of E_0^c equals the complement of the evolution of E_0 we have that if $F_0 \subset E_0^c$, Theorem 5 implies $F_t \subset (E_t)^c$. Since $\partial E_0 \cap \partial F_0 = \emptyset$ implies that either $F_0 \subset E_0$ or $F_0 \subset E_0^c$, the conclusion follows ☐

Theorem 5 implies uniqueness of smooth solutions to (1.1)

Corollary 7. *There is at most one smooth solution to* (1.1)

Proof. Assume that $E_0 = F_0$. By Theorem 5, we have that $F_t \subset E_t$ and $E_t \subset F_t$. ☐

By trapping the solution between balls, we obtain estimates about the evolution of the fractional mean curvature and the extinction time:

Corollary 8. *Let $R > \delta > 0$ and E_t a solution to* (1.1) *such that there are x_δ and x_R that satisfy $B_\delta(x_\delta) \subseteq E_0 \subseteq B_R(x_R)$, then $B_{(\delta^{s+1}-C_0t)^{\frac{1}{s+1}}}(x_\delta) \subseteq E_t \subset B_{(R^{s+1}-C_0t)^{\frac{1}{s+1}}}(x_R)$.*

In particular, if $f \in C^1\big(S^{n-1} \times (0,T)\big) \cap C^0\big(S^{n-1} \times [0,T]\big)$ is a solution of (2.4)*, with $f(\omega,t) > 0$ for every $(\omega,t) \in S^{n-1} \times [0,T]$, that satisfies $\delta < f(\omega,0) < R$, for every $\omega \in S^{n-1}$. then*

$$(\delta^{s+1} - C_0 t)^{\frac{1}{s+1}} \leqslant f(\omega,t) \leqslant (R^{s+1} - C_0 t)^{\frac{1}{s+1}}. \tag{3.1}$$

Moreover, the maximal existence time is bounded from above by $\frac{R^{s+1}}{C_0}$.

Proof. The result follows directly from Theorem 5. ☐

4. The evolution of the geometric quantities

In this section we study the evolution of local and nonlocal geometric quantities.

We first remark that equation (1.1) is invariant under reparameterizations: Suppose that x satisfies (1.1) and consider a reparameterization $\varphi(\omega,t)$. Then we have that $\tilde{x}=x(\varphi(\omega,t),t)$ satisfies

$$\partial_t\tilde{x}\cdot\tilde{\nu}=(Dx(\partial_t\varphi)+\partial_t x)\cdot\tilde{\nu}=-H_s(\tilde{x}).$$

Moreover, by reparameterizing the smooth surface with a time dependent parameter it is possible to obtain an evolution equation that has tangent velocity equal to 0.

Theorem 9. *Suppose that E_t is smooth and satisfies the evolution equation* (1.1). *Then, there is a parameterization of ∂E_t such that*

$$\partial_t x(t)=-H_s(x(t),E_t)\,\nu, \tag{4.1}$$

for $x\in\partial E_t$.

Proof. We follow the analogous proof for other geometric flows (see [23] for instance).

Assume that ∂E_t is parameterized by spatial coordinates $(\omega_1,\dots,\omega_{n-1})\in U\subset\mathbb{R}^{n-1}$. Then we have that $x(\omega,t)\in\partial E_t$ satisfies (1.1). We want to reparameterize ω in term of new time-dependent local coordinates. Hence, we assume that the coordinates $(\omega_1,\dots,\omega_{n-1})$ are parameterized by a spatial parameter $\Theta=(\theta_1,\dots,\theta_{n-1})$ and time t. Then we define

$$\Gamma(\Theta,t)=x(\omega(\Theta,t),t)$$

We have

$$\begin{aligned}\partial_t\Gamma=&\sum_i\partial_{\omega_i}x(\omega(\Theta,t),t)\partial_t\omega_i+\partial_t x(q,t)|_{q=\omega(\Theta,t)}\\=&-H_s(\Gamma(\Theta,t))\nu+\left.(\tau_i\partial_t\omega_i+(\partial_t x)^T)\right|_{q=\omega(\Theta,t)},\end{aligned}$$

where τ_i is the tangential vector $\partial_{\omega_i}x(\omega(\Theta,t),t)$ and $(\partial_t x)^T=\partial_t x-(\partial_t x\cdot\nu)\nu$ is the tangential part of $\partial_t x$.

Standard ODE theory implies the existence of a solution to

$$\partial_t\omega_i(\Theta,t)=(\partial_t x)^T g^{ij}\omega^T\cdot\tau_j,$$

with $\omega(\Theta,t_0)=\omega$ (the original parameterization at time t_0).

Hence, the surface $\Gamma(\Theta,t)$ satisfies (4.1) for time close to t_0. ☐

In the next subsection we assume that $\Gamma(t)$ is the reparameterization of ∂E_t described by Theorem 9. For simplicity, we still denote the spatial parameter as $\omega\in U\subset\mathbb{R}^{n-1}$ or $x\in\mathbb{R}^{n-1}$.

4.1. Evolution of local quantities.

In this subsection we consider the evolution of some geometric quantities associated to ∂E_t. We assume that the ∂E_t is smooth.

Consider $\Gamma(t)$ satisfying (4.1). We start by recalling the definition of the metric g_{ij}, the second fundamental form a_{ij} and the square of its norm $|A|^2$. Here we denote by (m_{ij}) the matrix of components m_{ij} and we use Einstein's summation convention whenever repeated indices occur. We denote the inverse of the metric as g^{ij} and we raise indices of matrices to indicate contraction by this matrix (e.g. $m^i_j=g^{ij}m_{ij}$). In this setting, we have:

$$\begin{aligned}g_{ij}=&\partial_{\omega_i}\Gamma\cdot\partial_{\omega_j}\Gamma,\\(g^{ij})=&(g_{ij})^{-1},\\a_{ij}=&\partial_{\omega_i}\nu\cdot\partial_{\omega_j}\Gamma=-\nu\cdot\partial_{\omega_j}\partial_{\omega_j}\Gamma=\partial_{\omega_j}\nu\cdot\partial_{\omega_i}\Gamma,\\|A|^2=&g^{ij}a_{ik}g^{kl}a_{jl}.\end{aligned}\tag{4.2}$$

We also denote

$$\nabla^\Gamma F=g^{ij}\partial_{\omega_j}F\partial_{\omega_i}\Gamma,$$

which correspond to projecting the gradient of F on the tangent space (for a globally defined function) and

$$\nabla^\Gamma_i X^j=\partial_{\omega_i}X^j+C^j_{ik}X^k,$$

where C_{ik}^j are the Christoffel symbols on the surface.

Theorem 10. *Assume that* $\Gamma(\Theta, t) = \partial E_t$ *is parameterized such that it satisfies* (4.1)*. Then we have that*

$$\partial_t g_{ij} = -2H_s a_{ij}, \tag{4.3}$$

$$\partial_t g^{ij} = 2H_s a^{ij}, \tag{4.4}$$

$$\partial_t \nu = \nabla^\Gamma H_s, \tag{4.5}$$

$$\partial_t a_{ij} = \nabla_i^\Gamma \nabla_j^\Gamma H_s - H_s a_{ik} a_j^k \tag{4.6}$$

$$\partial_t |A|^2 = 2a^{ij} \nabla_i^\Gamma \nabla_j^\Gamma H_s + 2H_s a_{ik} a_j^k a^{ij}. \tag{4.7}$$

Proof. The proofs are similar to the local case (see [23] for instance). First, we prove (4.3) by computing the evolution of the metric: we recall that $\partial_{\omega_i}\Gamma$ is a tangent vector, thus

$$\partial_{\omega_i}\Gamma \cdot \nu = 0. \tag{4.8}$$

Also Γ satisfies (4.1), and so $\partial_t \Gamma = -H_s \nu$. As a consequence,

$$\begin{aligned} \partial_t g_{ij} &= \partial_{\omega_i}(\partial_t \Gamma) \cdot \partial_{\omega_j}\Gamma + \partial_{\omega_i}\Gamma \cdot \partial_{\omega_j}(\partial_t \Gamma) \\ &= \partial_{\omega_i}(-H_s \nu) \cdot \partial_{\omega_j}\Gamma + \partial_{\omega_i}\Gamma \cdot \partial_{\omega_j}(-H_s \nu) \\ &= -2H_s a_{ij} \end{aligned}$$

and so we obtain (4.3).

Now, since $g_{ij}\, g^{jk} = \delta_i^k$ (here we are adding on the repeated index j), using (4.3) we have that

$$0 = \partial_t \delta_i^k = \partial_t g_{ij}\, g^{jk} + g_{ij}\, \partial_t g^{jk} = -2H_s\, a_{ij} g^{jk} + g_{ij}\, \partial_t g^{jk},$$

which gives (4.4).

Also, using that $\nu \cdot \nu = 1$ and (4.8), we have that

$$\partial_t \nu \cdot \nu = 0,$$

that

$$\partial_{\omega_i} \nu \cdot \nu = 0$$

and

$$\partial_t \nu \cdot \partial_{\omega_i}\Gamma = -\nu \cdot \partial_{\omega_i}(\partial_t \Gamma) = \nu \cdot \partial_{\omega_i}(H_s \nu) = \partial_{\omega_i} H_s.$$

Hence, decomposing $\partial_t \nu$ along the orthogonal directions $\{\nu, \partial_{\omega_1}\Gamma, \ldots, \partial_{\omega_{n-1}}\Gamma\}$, we conclude that

$$\partial_t \nu = g^{ij} \partial_{\omega_j} H_s \partial_{\omega_i}\Gamma = \nabla^\Gamma H_s.$$

This completes the proof of (4.5).

Now we use (4.2) and (4.5) and we obtain that

$$\begin{aligned} \partial_t a_{ij} &= -\partial_t \nu \cdot \partial_{\omega_j} \partial_{\omega_j} \Gamma + \nu \cdot \partial_{\omega_j} \partial_{\omega_j} (H_s \nu) \\ &= -\nabla^\Gamma H_s \cdot \partial_{\omega_j} \partial_{\omega_j} \Gamma + \partial_{\omega_j} \partial_{\omega_j} H_s + H_s \nu \cdot \partial_{\omega_j} \partial_{\omega_j} \nu. \end{aligned}$$

Moreover,

$$0 = \frac{1}{2} \partial_{\omega_j} \partial_{\omega_j} (\nu \cdot \nu) = \partial_{\omega_j} (\nu \cdot \partial_{\omega_j} \nu) = \nu \cdot \partial_{\omega_j} \partial_{\omega_j} \nu + \partial_{\omega_j} \nu \cdot \partial_{\omega_j} \nu$$

and so we see that

$$\partial_t a_{ij} = -\nabla^\Gamma H_s \cdot \partial_{\omega_j} \partial_{\omega_j} \Gamma + \partial_{\omega_j} \partial_{\omega_j} H_s - H_s \partial_{\omega_j} \nu \cdot \partial_{\omega_j} \nu. \tag{4.9}$$

Now we assume that we have normal coordinates at x_t. Then at x_t the metric g_{ij} equals to δ_{ij} and the Christoffel symbols are 0. In particular, formula (4.9) reduces to

$$\begin{aligned} \partial_t a_{ij} &= \partial_{\omega_j} \partial_{\omega_j} H_s - H_s \partial_{\omega_i} \nu \cdot \partial_{\omega_j} \nu \\ &= \partial_{\omega_j} \partial_{\omega_j} H_s - H_s a_{ik} a_j^k. \end{aligned}$$

Since in normal coordinates $\partial_{\omega_j} \partial_{\omega_j} H_s = \nabla_i^\Gamma \nabla_j^\Gamma H_s$ and the latter is a coordinate invariant quantity, this establishes (4.6).

Now we prove (4.7). For this, we use (4.4) and (4.6), and we see that

$$\begin{aligned}\partial_t(g^{ij}a_{ik}) &= \partial_t g^{ij}a_{ik} + g^{ij}\partial_t a_{ik}\\ &= 2H_s a^{ij}a_{ik} + g^{ij}(\nabla_i^\Gamma\nabla_k^\Gamma H_s - H_s a_{im}a_k^m)\\ &= 2H_s a^{ij}a_{ik} + g^{ij}\nabla_i^\Gamma\nabla_k^\Gamma H_s - H_s a_m^j a_k^m.\end{aligned}$$

Therefore

$$\begin{aligned}\partial_t(g^{ij}a_{ik})\,(g^{kl}a_{jl}) &= 2H_s a^{ij}a_{ik}g^{kl}a_{jl} + g^{ij}g^{kl}a_{jl}\nabla_i^\Gamma\nabla_k^\Gamma H_s - H_s a_m^j a_k^m g^{kl}a_{jl}\\ &= 2H_s a^{ij}a_i^l a_{jl} + a^{ik}\nabla_i^\Gamma\nabla_k^\Gamma H_s - H_s a_m^j a^{ml}a_{jl}\\ &= H_s a^{ij}a_i^l a_{jl} + a^{ik}\nabla_i^\Gamma\nabla_k^\Gamma H_s.\end{aligned}$$

This and the fact that (recall (4.2))

$$\begin{aligned}\partial_t|A|^2 &= \partial_t(g^{ij}a_{ik}g^{kl}a_{jl})\\ &= \partial_t(g^{ij}a_{ik})g^{kl}a_{jl} + \partial_t(g^{kl}a_{jl})g^{ij}a_{ik}\\ &= 2\partial_t(g^{ij}a_{ik})g^{kl}a_{jl}\end{aligned}$$

imply (4.7). □

For further reference, we also point out the following computation in local coordinates:

Lemma 11. *For local coordinates $\{\omega_1,\dots,\omega_{n-1}\}$ we have that*

$$\partial_t\left(\partial_{\omega_i}\Gamma\right) = -H_s a_i^j\partial_{\omega_j}\Gamma - \partial_{\omega_i}H_s\nu.$$

Proof. Since Γ satisfies (4.1),

$$\partial_t(\partial_{\omega_i}\Gamma) = \partial_{\omega_i}(\partial_t\Gamma) = \partial_{\omega_i}(-H_s\nu) = -\partial_{\omega_i}H_s\nu - H_s\partial_{\omega_i}\nu.$$

On the other hand, by definition

$$\partial_{\omega_i}\nu = a_i^j\partial_{\omega_j}\Gamma,$$

which implies the result. □

4.2. **Evolution of non-local quantities.** In this subsection we will analyze the evolution of the perimeter, the fractional mean curvature and their first order spatial derivatives. In order to simplify the notation we write the point $x(t)\in\partial E_t$ and the unit normal vector $\nu(x(t))$ to ∂E_t at $x(t)$ as

$$x_t := x(t) \text{ and } \nu_t := \nu(x(t)).$$

We remark that when we integrate on the surface ∂E_t the integration variable, that we usually denote by y, depends on t, but we do not make explicit this dependence. Note additionally that $v\cdot w$ denotes the standard dot product on $\mathbb{R}^n$ between the vectors v and w.

We observe that the integrand in (1.2) carries a singular kernel, therefore it is convenient to remove such singularity by using a cancellation. We perform these computations here, and we will use them in the forthcoming Section 5 to show that the positivity of the fractional mean curvature is preserved by the geometric flow.

To this goal, we write

$$H_s(x_t,E_t) = H_s^{\rm reg}(x_t,E_t) + H_s^{\rm sing}(x_t,E_t),$$

with

$$H_s^{\rm sing}(x_t,E_t) = \lim_{\delta\searrow 0} s(1-s)\int_{C_R^{\nu_t}(x_t)\setminus B_\delta(x_t)}\frac{\tilde\chi_{E_t}(y)}{|x_t-y|^{n+s}}\,dy, \text{ and} \tag{4.10}$$

$$H_s^{\rm reg}(x_t,E_t) = s(1-s)\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(x_t)}\frac{\tilde\chi_{E_t}(y)}{|x_t-y|^{n+s}}\,dy, \tag{4.11}$$

where $C_R^{\nu_t}(x_t)$ is a fixed cylinder centered at x_t with flat direction parallel to the normal of the surface at x_t, namely

$$C_R^{\nu_t}(x_t) := \Big\{ x \in \mathbb{R}^n \text{ s.t. } x = x_t + y \text{ with } |y \cdot \nu(x_t)| < R \text{ and } |y - (y \cdot \nu(x_t))\nu(x_t)| < R \Big\}.$$

In what follows, we denote the surface ∂E_t as $\Gamma(\omega, t)$ and we assume that is parameterized such that (4.1) holds. Consider $x_t \in \Gamma$ and the epigraph of the tangent plane Π at x_t given by

$$\Pi(x_t, E_t) := \{\xi \in \mathbb{R}^n \text{ s.t. } \nu_t \cdot (\xi - x_t) \geqslant 0\}, \tag{4.12}$$

where ν_t is the unit normal to $\Gamma(t)$ at the point x_t.

Note that for R small enough, $\Gamma(t)$ can be written as a graph over the tangent plane at $x_t \in \Gamma(t)$. More precisely, let ν_t be the normal vector at x_t and let us parameterize $\partial\Pi$ (or equivalently, the linear space perpendicular to ν_t) in appropriate polar coordinates $(r, \varphi) \in [0, R] \times S^{n-2}$. Then using the implicit function theorem, near x_t we may define a function h such that

$$\Gamma(\omega, t) = x_t + \rho M_{x_t}\varphi + \rho h(\rho, \varphi)\nu_t. \tag{4.13}$$

Here ρ is the distance to x_t on $\partial\Pi$ and $M_{x_t}\varphi \in \partial\Pi$ is defined as follows:

Assume that $x_t = \Gamma(\bar\omega, t)$. Consider an orthonormal frame $\{v_j\}$ on $\partial\Pi(x_t, E_t)$. Since $\{\partial_{\omega_i}\Gamma\}_{\{i=1,\dots n-1\}}$ span $\partial\Pi(x_t, E_t)$, there are $c^{ij}(t_0)$ that satisfy

$$v_j = c^{ji}(t_0)\partial_{\omega_i}\Gamma.$$

We define $c^{ji}(t)$ for $t \leqslant t_0$ as solutions to the ODE system

$$\partial_t c^{ij} - c^{rj} a_r^i(\bar\omega, t) H_s(\Gamma(\bar\omega, t)) = 0 \tag{4.14}$$

$$c^{ji}(t)|_{t=t_0} = c^{ji}(t_0). \tag{4.15}$$

Notice that, for technical convenience, we are taking here the backward ODE flow from time t_0. Then for $t \leqslant t_0$ we define

$$v_j(\bar\omega, t) = c^{ji}(t)\partial_{\omega_i}\Gamma(\bar\omega, t). \tag{4.16}$$

We note that $v_j(\bar\omega, t_0) = v_j$ and $\{v_j(t)\} \subset \partial\Pi(x_t, E_t)$, where $x_t = \Gamma(\bar\omega, t)$ and $\partial\Pi(x_t, E_t)$ is the tangent plane of $\Gamma(\bar\omega, t)$.

From (4.14) and Lemma 11

$$\begin{aligned} \partial_t v_j &= \partial_t c^{ji}(t)\partial_{\omega_i}\Gamma(\bar\omega, t) + c^{ji}(t)\partial_t(\partial_{\omega_i}\Gamma(\bar\omega, t)) \\ &= -(\nabla^\Gamma H_s \cdot v_j)\, \nu_t. \end{aligned} \tag{4.17}$$

Moreover,

$$\partial_t(v_j \cdot v_i) = -(\nabla^\Gamma H_s \cdot v_j)(\nu_t \cdot v_i) - (\nabla^\Gamma H_s \cdot v_i)(v_j \cdot \nu) = 0.$$

Hence, $\{v_j\}$ remains an orthonormal base of $\Pi(x_t, E_t)$.

Now we define

$$M_{x_t}\varphi = \varphi^i v_i, \text{ where } \varphi \in S^{n-2}. \tag{4.18}$$

In particular, if we denote $x_t = \Gamma(\bar\omega, t)$, from (4.17) we have

$$\partial_t M_{x_t}\varphi = -(\nabla^\Gamma H_s \cdot M_{x_t}\varphi)\, \nu_t. \tag{4.19}$$

We also note that, from equation (4.13) and the quadratic separation of the smooth surfaces from their tangent planes, it follows that $h(0, \varphi) = 0$.

Notice also that by symmetry, for $\Pi_t := \Pi(x_t, E_t)$ and any $R > \delta > 0$

$$\int_{C_R^{\nu_t}(x_t)\setminus B_\delta(x_t)} \frac{\tilde\chi_{\Pi_t}(y)}{|x_t - y|^{n+s}}\, dy = 0. \tag{4.20}$$

Then, parameterizing $C_R^{\nu_t}(x_t)$ as $x_t + \rho M_{x_t}\varphi + \rho z \nu_t$ with $\rho \in [0, R]$, $\varphi \in S^{n-2}$ and $z \in [-R, R]$, due to cancellations we have that

$$
\begin{aligned}
H_s^{\text{sing}}(x_t, E) &= \lim_{\delta \searrow 0} s(1-s) \int_{C_R^{\nu_t}(x_t)\setminus B_\delta(x_t)} \frac{\tilde{\chi}_{E_t}(y) + \tilde{\chi}_{\Pi_t}(y)}{|x_t - y|^{n+s}}\, dy \\
&= s(1-s) \int_{S^{n-2}} \left[\int_0^R \rho^{-1-s} \left(\int_{h(\rho,\varphi)}^0 \frac{1}{(z^2+1)^{\frac{n+s}{2}}} dz \right) d\rho \right] d\varphi,
\end{aligned}
\tag{4.21}
$$

where $\Pi_t = \Pi_t(x_t, E_t)$. We now compute the derivatives of h.

Proposition 12. *For a given time t, consider a point $x_t = \Gamma(\bar{\omega}, t)$ and ν_t the normal vector to Γ at x_t. Let h be given by (4.13) where x_t is fixed as above. Then denoting by ν the normal to $\Gamma(\omega, t)$, we have that*

$$
\begin{aligned}
\partial_t h(\rho, \varphi) =& \frac{1}{\rho}\Big(H_s(x_t) - H_s(\Gamma)\nu \cdot \nu_t \Big) + (\nabla^\Gamma H_s(x_t) \cdot M_{x_t}\varphi) + \frac{1}{\rho}(\nu_t \cdot D_\omega \Gamma(\omega, t)\partial_t \omega), \\
\partial_{\bar{\omega}_i} h(\rho, \varphi) =& \frac{\nu_t \cdot D_\omega \Gamma(\omega, t)\partial_{\bar{\omega}_i}\omega + A(M_{x_t}\varphi, \partial_{\bar{\omega}_i}\Gamma)}{\rho},
\end{aligned}
$$

where A denotes the second fundamental form of $\Gamma(t)$ at x_t and

$$
\begin{aligned}
\partial_t \omega_j &= \Big((g^{ij}(x_t) + O(\rho) \Big) \Big(H_s(\Gamma)(D_\omega \Gamma(\bar{\omega}, t))^T \nu - \rho h(\rho, \varphi)(D_\omega \Gamma(\bar{\omega}, t))^T \nabla^\Gamma H(x_t) \Big) \\
&\sim H_s(\Gamma)\Big(O(\rho) + O(\rho^2) \Big)
\end{aligned}
$$

Proof. First, we note that from (4.13), ω becomes implicitly a function of φ and ρ, but also of x_t, hence it does depend implicitly on t. Hence, taking derivatives on equation (4.13) we have

$$
D_\omega \Gamma(\omega, t)\partial_t \omega + \partial_t \Gamma = \partial_t x_t + \rho \partial_t M_{x_t}\varphi + \rho \partial_t h(\rho, \varphi)\nu_t + \rho h(\rho, \varphi)\partial_t \nu_t. \tag{4.22}
$$

Note that

$$
\partial_t \Gamma \cdot \nu_t = -H_s(\Gamma)\, \nu \cdot \nu_t \qquad \text{and} \qquad \partial_t x_t \cdot \nu_t = -H_s(x_t)\, \nu_t \cdot \nu_t = -H_s(x_t). \tag{4.23}
$$

Moreover, since $M_{x_t}\varphi$ is a tangential vector at x_t, we have that $M_{x_t}\varphi \cdot \nu_t = 0$, thus

$$
-\partial_t M_{x_t}\varphi \cdot \nu_t = M_{x_t}\varphi \cdot \partial_t \nu_t = M_{x_t}\varphi \cdot\ \nabla^\Gamma H_s(x_t), \tag{4.24}
$$

where the latter identity follows from (4.5). Then, using (4.1) and taking dot product with ν_t (recall also that $\partial_t \nu_t \cdot \nu_t$), we have

$$
\partial_t h(\rho, \varphi) = \frac{1}{\rho}\Big(H_s(x_t) - H_s(\Gamma)\nu \cdot \nu_t \Big) + \nabla^\Gamma H_s(x_t) \cdot M_{x_t}\varphi + \frac{1}{\rho}\nu_t \cdot D_\omega \Gamma(\omega, t)\partial_t \omega.
$$

Now we are left to compute $\partial_t \omega$. To this end, we multiply equation (4.22) by $D_\omega \Gamma(\bar{\omega}, t))^T$, we exploit (4.23) and (4.24) and we obtain

$$
(D_\omega \Gamma(\bar{\omega}, t))^T D_\omega \Gamma(\omega, t))\partial_t \omega - H_s(\Gamma) D_\omega \Gamma(\bar{\omega}, t))^T \nu = \rho h(\rho, \varphi)(D_\omega \Gamma(\bar{\omega}, t))^T \nabla^\Gamma H(x_t).
$$

Since $(D_\omega \Gamma(\bar{\omega}, t))^T D_\omega \Gamma(\bar{\omega}, t) = (g_{ij}(x_t))$, we have that the first matrix is $(g_{ij}(x_t) + O(\rho))$. Similarly, since $D_\omega \Gamma(\bar{\omega}, t))^T \nu_t = 0$, the second term is like $H_s(\Gamma)O(\rho)$. Hence

$$
\begin{aligned}
\partial_t \omega =& \Big(g^{ij}(x_t) + O(\rho) \Big) \Big(H_s(\Gamma)(D_\omega \Gamma(\bar{\omega}, t))^T \nu + \rho h(\rho, \varphi)(D_\omega \Gamma(\bar{\omega}, t))^T \nabla^\Gamma H(x_t) \Big) \\
&\sim H_s(\Gamma)\Big(O(\rho) + O(\rho^2) \Big),
\end{aligned}
$$

as desired. ☐

We will also use a rotation that aligns the cylinder $C_R^{\nu_t}(x_t)$ with $C_R^{\nu_\tau}(x_\tau)$. We remark that since the vectors $\{v_i(t) : i \dots n-1\} \cup \{\nu_t\}$ are an orthonormal basis of $\mathbb{R}^n$ we may define for $y = y^i v_i(t) + y^n \nu_t$ the following rotation

$$
\mathcal{R}_{t,\tau} y = y^i v_i(\tau) + y^n \nu_\tau. \tag{4.25}
$$

Notice that in particular $y^i = y \cdot v_i(t)$ and $y^n = y \cdot \nu_t$.

Then it is direct to show that

Proposition 13. *Consider $\mathcal{R}_{t,\tau}$ given by (4.25) and denote $\nabla^\Gamma H_s(\tau)$ the tangential gradient of $H_s(x_\tau)$. Then it holds that*

(1) $\mathcal{R}_{\tau,\tau} = Id.$

(2) $\partial_{\tau_2}\mathcal{R}_{\tau_1,\tau_2} y = [(y\cdot v_i(\tau_1))\partial_t v_i(t) + (y\cdot\nu_{\tau_1})\partial_t\nu_t)|_{t=\tau_2} = -(y\cdot v_i(\tau_1))(v_i(\tau_2)\cdot\nabla^\Gamma H_s(\tau_2))\,\nu_{\tau_2} + (y\cdot\nu_{\tau_1})\nabla^\Gamma H_s(\tau_2).$

Now we study the evolution of the s-perimeter P_s and of the s-mean curvature.

Theorem 14. *Let x_t, ν_t and h be as in (4.13). We have the following equations:*

$$\partial_t P_s(E_t) = -\int_{\partial E_t} H_s^2(\omega)\, d\mathcal{H}^{n-1}(y) \leqslant 0, \tag{4.26}$$

$$\frac{\partial_t(H_s^{sing})(x_t)}{s(1-s)} = -\int_{S^{n-2}}\left[\int_0^R \frac{\rho^{-1-s}\,\partial_t h(\rho,\varphi)}{(1+h^2(\rho,\varphi))^{\frac{n+s}{2}}}\,d\rho\right]\,d\varphi, \tag{4.27}$$

$$\begin{aligned}\frac{\partial_t(H_s^{reg})(x_t)}{s(1-s)} &= 2\int_{(\partial E_t)\setminus C_R^{\nu_t}(x_t)} \frac{(\partial_t x_t - \partial_t y + (y-x_t)\cdot\nabla^\Gamma H_s\,\nu_t - (y-x)\cdot\nu_t\,\nabla^\Gamma H_s(x_t))\cdot\nu}{|x_t-y|^{n+s}}dy \\ &= 2\int_{(\partial E_t)\setminus C_R^{\nu_t}(x_t)} \frac{(\partial_t x_t - \partial_t y)\cdot\nu}{|x_t-y|^{n+s}}dy \\ &\quad + R^{-s}\int_{S^{n-1}}\int_{-1}^1 (\chi_{E_t}+\chi_{\Pi_t})\big(x_t + RM_{x_t}\omega + Rz\nu_t(x_t)\big)\frac{zM_{x_t}\omega\cdot\nabla^\Gamma H_s(x_t)}{(1+z^2)^{\frac{n+s}{2}}}dzd\omega,\end{aligned} \tag{4.28}$$

where ν is the unit normal vector to ∂E_t at y and Π_t is defined as in (4.12)

$$\frac{\nabla_i^\Gamma H_s}{s(1-s)}(x) = (n+s)g^{ij}\left(P.V.\int_{\mathbb{R}^n}\frac{(y-x)\cdot\partial_{\omega_i}x}{|x-y|^{n+s+2}}dx\right)\partial_{\omega_j}x, \tag{4.29}$$

$$\begin{aligned}\frac{\partial_t H_s}{2s(1-s)} &= P.V.\int_{\partial E_t}\frac{(\partial_t x_t - \partial_t y)\cdot\nu(y)}{|x_t-y|^{n+s}}dy \\ &= P.V.\int_{\partial E_t}\frac{H_s(y)-H_s(x)}{|x_t-y|^{n+s}}dy + H_s(x)P.V.\int_{\partial E_t}\frac{1-\nu(x)\cdot\nu(y)}{|x_t-y|^{n+s}}dy.\end{aligned} \tag{4.30}$$

Also,

$$\text{the function } (0,R)\times S^{n-2}\ni(\rho,\varphi)\mapsto \frac{\rho^{-1-s}\,\partial_t h(\rho,\varphi)}{(1+h^2(\rho,\varphi))^{\frac{n+s}{2}}} \text{ is integrable,} \tag{4.31}$$

$$\partial_t(H_s^{sing}) = O(R^{1-s}) \text{ and}$$

$$\int_{S^{n-1}}\int_{-1}^1(\chi_{E_t}+\chi_{\Pi_t})\big(x_t + RM_{x_t}\omega + Rz\nu_t(x_t)\big)\frac{zM_{x_t}\omega\cdot\nabla^\Gamma H_s(x_t)}{(1+z^2)^{\frac{n+s}{2}}}dzd\omega = O(R). \tag{4.32}$$

Proof. Formula (4.26) follows from Theorem 6.1 in [27] and (4.27) from (4.21).

To compute the derivative of the regular part we need to compute

$$\lim_{h\to0}\frac{H_s^{\rm reg}(x_t(t),E_t) - H_s^{\rm reg}(x_t(t-h),E_{t-h})}{h} = \frac{1}{h}\left(\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(x_t)}\frac{\tilde\chi_{E_t}(y)}{|x_t-y|^{n+s}} - \int_{\mathbb{R}^n\setminus C_R^{\nu_{t-h}}(x_{t-h})}\frac{\tilde\chi_{E_{t-h}}(y)}{|x_{t-h}-y|^{n+s}}\right).$$

We divide the computation as follows:

$$\begin{aligned} I_h =& \frac{1}{h}\left(\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(x_t)}\frac{\tilde\chi_{E_t}(y)}{|x_t-y|^{n+s}} - \int_{\mathbb{R}^n\setminus C_R^{\nu_{t-h}}(x_{t-h})}\frac{\tilde\chi_{E_t}(y)}{|x_{t-h}-y|^{n+s}}\right) \text{ and} \\ II_h =& \frac{1}{h}\left(\int_{\mathbb{R}^n\setminus C_R^{\nu_{t-h}}(x_{t-h})}\frac{\tilde\chi_{E_t}(y)}{|x_{t-h}-y|^{n+s}} - \int_{\mathbb{R}^n\setminus C_R^{\nu_{t-h}}(x_{t-h})}\frac{\tilde\chi_{E_{t-h}}(y)}{|x_{t-h}-y|^{n+s}}\right).\end{aligned}$$

For the first integral we consider a function $\phi^\epsilon \in C_0^\infty$ that approximates $\tilde{\chi}_{E_t}$. Then,

$$I_h = \lim_{\epsilon \to 0} I_h^\epsilon, \tag{4.33}$$

with

$$\begin{aligned} I_h^\epsilon :=& \frac{1}{h}\left(\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(x_t)} \frac{\phi^\epsilon(y)}{|x_t - y|^{n+s}} - \int_{\mathbb{R}^n\setminus C_R^{\nu_{t-h}}(x_{t-h})} \frac{\phi^\epsilon(y)}{|x_{t-h} - y|^{n+s}}\right) \\ =& \frac{1}{h}\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \frac{\phi^\epsilon(y + x_t) - \phi^\epsilon(\mathcal{R}_{t,t-h}y + x_{t-h})}{|y|^{n+s}} dy \\ =& \int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \left[\int_0^1 \frac{\nabla\phi^\epsilon\big(y_{h,l}\big)\cdot\delta_h}{|y|^{n+s}}\, d\ell\right] dy, \end{aligned} \tag{4.34}$$

where

$$\begin{aligned} &\delta_h := \frac{x_t - x_{t-h} + \partial_\ell \mathcal{R}_{t,t-(1-\ell)h}y}{h}, \\ &\mathcal{R}_{t,\tau} \text{ is given by (4.25)} \\ &\qquad \text{and } y_{h,l} = \mathcal{R}_{t,t-(1-\ell)h}y + x_{t-h} + \ell(x_t - x_{t-h}). \end{aligned} \tag{4.35}$$

From Proposition 13 we have $\partial_\ell \mathcal{R}_{t,t-(1-\ell)h}y = h\,[(y\cdot v_i(t))\partial_\tau v_i(\tau) + (y\cdot\nu_t)\partial_\tau\nu_\tau]|_{\tau=t-(1-\ell)h}$

Moreover, if we denote by $\mathcal{R}^{-T}_{t-(1-\ell)h,t}$ the inverse of the transpose of $\mathcal{R}_{t,t-(1-\ell)h}$ we have

$$\begin{aligned} &\operatorname{div}_y\left(\frac{\phi^\epsilon(y_{h,l})\mathcal{R}^{-T}_{t-(1-\ell)h,t}\delta_h}{|y|^{n+s}}\right) \\ =& \quad \frac{\mathcal{R}_{t,t-(1-\ell)h}\nabla\phi^\epsilon\big(y_{h,l}\big)\cdot R^{-T}_{t-(1-\ell)h,t}\delta_h}{|y|^{n+s}} + \phi^\epsilon\big(y_{h,l}\big)\operatorname{div}_y\frac{\delta_h}{|y|^{n+s}} \\ =& \quad \frac{\nabla\phi^\epsilon\big(y_{h,l}\big)\cdot\delta_h}{|y|^{n+s}} + \phi^\epsilon\big(y_{h,l}\big)\operatorname{div}_y\left(\frac{\delta_h}{|y|^{n+s}}\right), \end{aligned}$$

and so the divergence theorem gives that

$$\begin{aligned} &\int_{\partial C_R^{\nu_t}(0)} \frac{\phi^\epsilon(y_{h,l})\mathcal{R}^{-T}_{t-(1-\ell)h,t}\delta_h}{|y|^{n+s}}\cdot\nu_{C_R^{\nu_t}(0)}\, d\mathcal{H}^{n-1}(y) \\ =& \int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \frac{\nabla\phi^\epsilon\big(y_{h,l}\big)\cdot\delta_h}{|y|^{n+s}}\, dy + \int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \phi^\epsilon\big(y_{h,l}\big)\operatorname{div}_y\left(\frac{\delta_h}{|y|^{n+s}}\right) dy. \end{aligned}$$

We insert this information into (4.34) and we obtain that

$$\begin{aligned} I_h^\epsilon \;=&\; -\int_0^1\left[\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \phi^\epsilon\big(y_{h,l}\big)\operatorname{div}_y\left(\frac{\delta_h}{|y|^{n+s}}\right) dy.\right] d\ell \\ &+\int_0^1\left[\int_{\partial C_R^{\nu_t}(0)} \frac{\phi^\epsilon(y_{h,l})\mathcal{R}^{-T}_{t-(1-\ell)h,t}\delta_h}{|y|^{n+s}}\cdot\nu_{C_R^{\nu_t}(0)}\, d\mathcal{H}^{n-1}(y))\right] d\ell. \end{aligned}$$

Thus, by (4.33),

$$\begin{aligned} I_h \;=& -\int_0^1\left[\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \chi_{E_t}\big(y_{h,l}\big)\operatorname{div}_y\left(\frac{\delta_h}{|y|^{n+s}}\right) dy.\right] d\ell \\ &+\int_0^1\left[\int_{\partial C_R^{\nu_t}(0)} \frac{\chi_{E_t}(y_{h,l})\mathcal{R}^{-T}_{t-(1-\ell)h,t}\delta_h}{|y|^{n+s}}\cdot\nu_{C_R^{\nu_t}(0)}\, d\mathcal{H}^{n-1}(y))\right] d\ell, \end{aligned} \tag{4.36}$$

where $\nu_{C_R^{\nu_t}(0)}$ is the unit normal to the cylinder at y. Now we observe that

$$\chi_{E_t}\big(\mathcal{R}_{t,t-(1-l)h}y + x_{t-h} + \ell(x_t - x_{t-h})\big) - \chi_{E_t}\big(y + x_{t-h}\big) = \chi_{E_t}\big(y + x_{t-h} + O(h)\big) - \chi_{E_t}\big(y + x_{t-h}\big),$$

so this function is supported in a neighborhood of size $O(h)$ of a smooth surface. This fact, (4.36) and the integrability of the kernel $|y|^{-n-s}$ at infinity give that

$$\begin{aligned} I_h \;=\; & -\int_0^1 \left[\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \chi_{E_t}\big(y + x_{t-h}\big)\,\mathrm{div}_y\left(\frac{\delta_h}{|y|^{n+s}}\right) dy.\right] d\ell \\ & + \int_0^1 \left[\int_{\partial C_R^{\nu_t}(0)} \frac{\chi_{E_t}(y + x_{t-h})\mathcal{R}^{-T}_{t-(1-\ell)h,t}\delta_h}{|y|^{n+s}} \cdot \nu_{C_R^{\nu_t}(0)}\, d\mathcal{H}^{n-1}(y))\right] d\ell + o(1), \end{aligned}$$

as $h \to 0$. Recalling (4.35) and Proposition 13, we have for $\tau = t - (1 - l)h$ that

$$\begin{aligned} \mathrm{div}_y\left(\frac{\delta_h}{|y|^{n+s}}\right) =& \frac{v_i(t)\cdot \partial_\tau v_i(\tau) + \nu_t \cdot \partial_\tau \nu_\tau}{|y|^{n+s+2}} - (n+s)\frac{y\cdot(x_t - x_{t-h} + \partial_\ell \mathcal{R}_{t,t-(1-\ell)h}y)}{|y|^{n+s+2}h} \\ \to& -(n+s)\frac{y\cdot(\partial_t x_t + (y\cdot v_i(t))\,\partial_t v_i(t) + (y\cdot \nu_t)\partial_t \nu_t)}{|y|^{n+s+2}} \text{ as } h \to 0, \\ \text{and } \mathcal{R}^{-T}_{t-(1-\ell)h,t}\,\delta_h =& \mathcal{R}^{-T}_{t-(1-\ell)h,t}\frac{x_t - x_{t-h} + \partial_\ell \mathcal{R}_{t,t-(1-\ell)h}y}{h} \\ \to& \partial_t x_t + (y\cdot v_i(t))\,\partial_t v_i(t) + (y\cdot\nu_t)\,\partial_t\nu_t \text{ as } h \to 0. \end{aligned}$$

Additionally, from (4.17) and (4.5) we have that

$$\begin{aligned} y\cdot[(y\cdot v_i(t))\,\partial_t v_i(t) + (y,\cdot\nu_t)\partial_t\nu_t] =& -(y\cdot v_i)(\nabla^\Gamma H_s\cdot v_i(t))\,(y\cdot\nu_t) + (y\cdot\nu_t)(y\cdot\nabla^\Gamma H_s) \\ =& -(\nabla^\Gamma H_s\cdot y^T)\,(y\cdot\nu_t) + (y\cdot\nu)(y\cdot\nabla^\Gamma H_s) \\ =& 0. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{h\to 0} I_h = (n+s)&\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \tilde\chi_{E_t}\big(y + x_t\big)\frac{y\cdot\partial_t x}{|y|^{n+s+2}}\,dy \\ &+\int_{\partial C_R^{\nu_t}(0)} \tilde\chi_{E_t}\big(y + x_t\big)\frac{(\partial_t x_t + (y\cdot v_i(t))\,\partial_t v_i(t) + (y,\cdot\nu_t)\partial_t\nu_t)\cdot\nu_{C_R^{\nu_t}(0)}}{|y|^{n+s}}\,d\mathcal{H}^{n-1}(y). \end{aligned} \tag{4.37}$$

Now we notice that

$$-(n+s)\frac{y\cdot\partial_t x}{|y|^{n+s+2}} = \mathrm{div}_y\left(\frac{\partial_t x}{|y|^{n+s}}\right).$$

Then using the divergence theorem we have

$$\begin{aligned} &(n+s)\int_{\mathbb{R}^n\setminus C_R^{\nu_t}(0)} \tilde\chi_{E_t}\big(y + x_t\big)\frac{y\cdot\partial_t x}{|y|^{n+s+2}}\,dy = \\ &\qquad\int_{E_t\setminus C_R^{\nu_t}(x_t)} \mathrm{div}_y\left(\frac{\partial_t x}{|y - x_t|^{n+s}}\right) dy - \int_{E_t^c\setminus C_R^{\nu_t}(x_t)} \mathrm{div}_y\left(\frac{\partial_t x}{|y - x_t|^{n+s}}\right) dy \\ &\qquad= 2\int_{\partial E_t\setminus C_R^{\nu_t}(x_t)} \frac{\nu_{\partial E_t}(y)\cdot\partial_t x}{|y - x_t|^{n+s}}\,d\mathcal{H}^{n-1}(y) - \int_{\partial C_R^{\nu_t}(x_t)} \chi_{E_t}\big(y\big)\frac{\nu_{\partial C_R^{\nu_t}}(y)\cdot\partial_t x}{|y - x_t|^{n+s}}\,d\mathcal{H}^{n-1}(y), \end{aligned}$$

where $\nu_{\partial E_t}(y)$ denotes the unit normal to ∂E_t at y.

Plugging this into (4.37) we obtain

$$
\begin{aligned}
\lim_{h\to 0} I_h =& 2\int_{\partial E_t\setminus C_R^{\nu_t}(x_t)} \frac{\nu_{\partial E_t}(y)\cdot \partial_t x}{|y-x_t|^{n+s}}\, d\mathcal{H}^{n-1}(y)\\
&-\int_{\partial C_R^{\nu_t}(x_t)} \chi_{E_t}(y)\frac{\nu_{\partial C_R^{\nu_t}}(y)\cdot((y-x_t)^i\partial_t v_i(t)+(y-x)^n\partial_t\nu(x))}{|y-x_t|^{n+s}}\, d\mathcal{H}^{n-1}(y).
\end{aligned} \tag{4.38}
$$

Now we notice that, from the definition of $C_R(0)$, the normal $\nu_{C_R(0)}$ is either on the tangent plane at x_t (for the sides of the cylinder) or it is parallel to the normal at x_t (at the top and the bottom of the cylinder). Hence, at the top and bottom of the cylinder we have $\pm\nu_{\partial C_R^{\nu_t}}(y)\cdot\partial_t v_i(t) = -\nabla^\Gamma H_s\cdot v_i(t)$ and $\nu_{\partial C_R^{\nu_t}}(y)\cdot\partial_t\nu_t = 0$, while along the sides of the cylinder $\nu_{\partial C_R^{\nu_t}}(y)\cdot\partial_t v_i(t) = 0$ and $\nu_{\partial C_R^{\nu_t}}(y)\cdot\partial_t\nu_t = \frac{(y-x_t)^T}{|(y-x_t)^T|}\cdot\nabla^\Gamma H_s$. In addition, $\tilde\chi_{E_t} = -1$ on the bottom of the cylinder and $\tilde\chi_{E_t} = 1$ on the top. As a consequence,

$$
\begin{aligned}
&\int_{\partial C_R^{\nu_t}(x_t)} \chi_{E_t}(y)\frac{\nu_{\partial C_R^{\nu_t}}(y)\cdot((y-x_t)^i\partial_t v_i(t)+(y-x)^n\partial_t\nu_t)}{|y-x_t|^{n+s}}\, d\mathcal{H}^{n-1}(y) =\\
&\qquad -2\int_{S^{n-2}}\int_0^1 R^{n-n-s}\frac{\rho^{n-2}\nabla^\Gamma H_s(t)\cdot\omega}{(\rho^2+1)^{\frac{n+s}{2}}}\, d\rho\, d\omega\\
&\qquad +\int_{S^{n-2}}\int_{-1}^1 \chi_{E_t}\big(x_t+RM_{x_t}\omega+Rz\nu_t\big)R^{n-n-s}\frac{zM_{x_t}\omega\cdot\nabla^\Gamma H_s(x_t)}{(1+z^2)^{\frac{n+s}{2}}}dz d\omega.
\end{aligned}
$$

By symmetry the first term is 0 and

$$
\int_{S^{n-2}}\int_{-1}^1 \chi_{\Pi_t}\big(x_t+RM_{x_t}\omega+Rz\nu_t(x_t)\big)R^{n-n-s}\frac{zM_{x_t}\omega\cdot\nabla^\Gamma H_s(x_t)}{(1+z^2)^{\frac{n+s}{2}}}dz d\omega = 0.
$$

we obtain the first equality of (4.28).

The second equality may be obtained observing that

$$
-(n+s)\frac{y\cdot(\partial_t x + y\cdot v_i(t)\partial_t v_i(t) + y\cdot\nu_t\partial_t\nu_t)}{|y|^{n+s+2}} = \operatorname{div}_y\left(\frac{\partial_t x + (y-x_t)^i\partial_t v_i(t) + (y-x)^n\partial_t\nu_t}{|y|^{n+s}}\right).
$$

For the integral defining II_h, we have

$$
II_h = \frac{1}{h}\int_{\mathbb{R}^n\setminus C_R(0)} \frac{\tilde\chi_{E_t}(y+x_{t-h}) - \tilde\chi_{E_{t-h}}(y+x_{t-h})}{|y|^{n+s}}dy.
$$

Notice that the integrand is not 0 for $y+x_{t+h}\in E_t\Delta E_{t-h}$. Since we assume that ∂E_t is smooth, we may parameterize this neighborhood as $y = y_t + z\nu_{\partial E_t}(y_t)$ where $y_t\in\partial E_t$. Since we assume that the sets E_t are continuous in t, for h small enough, $E_t\Delta E_{t-h}$ is contained in this tubular neighborhood. Moreover, a Taylor expansion in t yields that

$$
y_{t-h} = y_t - h\partial_t y_t + O(h^2) \text{ and } (y_{t-h}-y_t)\cdot\nu_{\partial E_t}(y) = -h\partial_t y_t\cdot\nu_{\partial E_t}(y) + O(h^2).
$$

Then we have

$$
\begin{aligned}
II_h =& \frac{1}{h}\int_{\partial E_t\setminus C_R(0)}\int_0^{-h\partial_t y_t\cdot\nu_{\partial E_t}(y)+O(h^2)} \frac{2}{|y-x_{t-h}|^{n+s}}\, dz\, d\mathcal{H}^{n-1}(y)\\
&\to -2\int_{\partial E_t\setminus C_R(0)} \frac{\partial_t y_t\cdot\nu_{\partial E_t}(y)}{|y-x_{t-h}|^{n+s}}\, d\mathcal{H}^{n-1}(y) \qquad \text{as } h\to 0
\end{aligned}
$$

This, together with (4.38), proves (4.28).

From Proposition 12, we have that

$$
\frac{\rho^{-1-s}\,\partial_t h(\rho,\varphi)}{(1+h^2(\rho,\varphi))^{\frac{n+s}{2}}} = O(\rho^{-s}),
$$

which is integrable, thus (4.31) follows directly from (4.21). Similarly, we observe that

$$\int_{S^{n-1}} \int_{-1}^{1} (\chi_{E_t} + \chi_{\Pi_t})\big(x_t + RM_{x_t}\omega + Rz\nu_t(x_t)\big) \frac{zM_{x_t}\omega \cdot \nabla^\Gamma H_s(x_t)}{(1+z^2)^{\frac{n+s}{2}}} dz d\omega$$
$$= \int_{S^{n-1}} \int_{\min(h(R\omega),1)}^{0} \frac{zM_{x_t}\omega \cdot \nabla^\Gamma H_s(x_t)}{(1+z^2)^{\frac{n+s}{2}}} dz d\omega$$

and equation (4.32) follows from the fact $h(0) = 0$.

Finally, equation (4.29) follows from [6] and the fact that $\partial_{\omega_i} x$ is tangential.

Equation (4.30) follows now by combining (4.27) and (4.28) and taking $R \to 0$ (another proof of (4.30) can be obtained using formula (B.2) of [20]; using Lemmata A.2 and A.4 there, one also obtains an expansion of the quantity in (4.30) as s approaches 1). □

Remark 15. *An equation analogous to* (4.30) *was obtained in* [20] *in a different context. Their results imply that*

$$s(1-s)P.V. \int_{\partial E_t} \frac{H_s(y) - H_s(x)}{|x-y|^{n+s}} dy \to \Delta_{\partial E_t} H \ \text{ as } s \to 1$$
$$s(1-s)P.V. \int_{\partial E_t} \frac{1 - \nu(y)\cdot\nu(x)}{|x-y|^{n+s}} dy \to |A|^2 \ \text{ as } s \to 1,$$

which recovers the classical evolution for the mean curvature H under evolution by mean curvature flow.

5. Preservation of the fractional mean curvature

In this section we show that the geometric flow preserves the positivity of the fractional mean curvature. We need the following lemma that excludes the possibility of compact hypersurfaces with fractional mean curvature equal to zero (we state the result for smooth sets for the sake of simplicity):

Lemma 16. *Let $E \subset \mathbb{R}^n$ be a set with C^2-boundary and such that $H_s(x,E) = 0$ for any $x \in \partial E$. Assume that E is bounded in one direction, i.e. there exist $\omega \in S^{n-1}$ and $M \in \mathbb{R}$ such that*

$$E \subset \{x \in \mathbb{R}^n,\ x\cdot\omega < M\}. \tag{5.1}$$

Then E is a halfspace (unless it is empty).

In particular, there exists no compact hypersurface with vanishing fractional mean curvature.

Proof. The proof is based on a sliding method. Roughly speaking, we take a plane of normal direction ω and we slide it from infinity till it touches E, and then we compare the fractional mean curvatures at a touching point to obtain the desired result. The details of the proof go as follows. We suppose that

$$E \neq \varnothing. \tag{5.2}$$

Let

$$\Pi_M := \{x \in \mathbb{R}^n,\ x\cdot\omega < M\}$$
$$\text{and} \qquad M_* := \inf\{M,\ E \subset \Pi_M\}.$$

Notice that $M_* \in \mathbb{R}$, thanks to (5.1) and (5.2). In addition, E is a subset of Π_{M_*} and there exists $x_t \in (\partial E) \cap (\partial \Pi_{M_*})$. We claim that $E = \Pi_{M_*}$ (up to sets of measure zero, and this will end the proof of Lemma 16). Indeed, if not, the positivity set of the function

$$\tilde\chi_E - \tilde\chi_{\Pi_{M_*}} = 2\chi_{\Pi_{M_*}\setminus E}$$

would have positive measure and therefore

$$0 < \int_{\mathbb{R}^n} \frac{\tilde\chi_E(y) - \tilde\chi_{\Pi_{M_*}}(y)}{|x_t - y|^{n+s}}\, dy = H_s(x_t, E) - H_s(x_t, \Pi_{M_*}) = 0 - 0,$$

and this is a contradiction. □

Theorem 17. *Let E_t be a compact solution of* (1.1). *Assume that H_s is differentiable and that E_0 has strictly positive fractional mean curvature. Then, E_t has strictly positive fractional mean curvature for every $t \in (0,T)$.*

Proof. Suppose the contrary. Then, if $E = E_t$ is the evolving surface, we have that $H_s(x, E_t) > 0$ for any $x \in \partial E_t$ and any $t \in (0, \bar{t})$, but

$$H_s(\bar{x}, E_{\bar{t}}) = 0, \tag{5.3}$$

for some $\bar{x} \in \partial E_{\bar{t}}$, with $\bar{t} \in (0,T)$.

Notice that $x_t \in \partial E_t$ and the function

$$t \mapsto H_s(x_t, E_t)$$

attains its minimum in the interval $[0, \bar{t}]$ and the endpoint $\bar{t}$ and therefore $\partial_t H_s(x_t, E_t)\big|_{t=\bar{t}} \leqslant 0$. Since it is also a spatial critical point for H_s, we have that $\nabla H_s(\bar{x}, E_t)\big|_{t=\bar{t}} = 0$. From (4.30) in Theorem 14 and (1.1) we obtain that

$$\partial_t H_s(\bar{x}, \bar{t}) = s(1-s) \int_{\partial E_t} \frac{H_s(y)}{|y - x_{t-h}|^{n+s}} \, d\mathcal{H}^{n-1}(y) \geqslant 0.$$

However, since $\partial_t H_s(x_t, E_t)\big|_{t=\bar{t}} \leqslant 0$ we have that $H_s(y) \equiv 0$, which due to Lemma 16 contradicts the compactness of E_t. □

Following the same proof we can show for a non-compact solution that

Theorem 18. *Let E_t be a solution of* (1.1). *Assume that H_s is differentiable, that E_0 has strictly positive fractional mean curvature and that ∂E_t is uniformly spatially C^2 in $[0,T]$. Then, E_t has strictly positive fractional mean curvature for every $t \in [0,T]$.*

Proof. Proceeding as in the proof of the previous we can show that E_t has strictly positive fractional mean curvature for every $t \in [0,T]$ or there is a t_0 such that E_t has vanishing fractional mean curvature for every $t \geqslant t_0$.

Now we show that H_s cannot become identically 0. For this, up to a dilation, we take a scale for which the evolving surface is locally a smooth graph in balls of radius 2 centered at the surface. Let ϕ be a nonnegative function supported in the unit ball B_1 and $\phi \equiv 1$ in $B_{\frac{1}{2}}$. Fix $x_t = x(0,t) \in \partial E_t$ and $\epsilon > 0$. Consider the function $v : \mathbb{R}^n \times [0,T)$ defined

$$v(y,t) = e^{C_1 t} \left(\frac{H_s(y)}{s\,(1-s)} + \epsilon \right) - \delta e^{-C_2 t} \phi(y - x_t),$$

where δ is chosen such that $v(y,0) > 0$ and C_1, C_2 are real constant to be determined. Notice that δ can be chosen independently of $\epsilon > 0$

Using equations (4.30) and (4.1), and denoting by ν_t the normal at x_t, we have, for $y \in \partial E_t$,

$$\begin{aligned}
&\partial_t v(y,t) \\
&= C_1 e^{C_1 t} \left(\frac{H_s(y)}{s\,(1-s)} + \epsilon \right) + e^{C_1 t} \left(2\text{P.V.} \int_{\partial E_t} \frac{H_s(z) - H_s(y)}{|z-y|^{n+s}} dz + 2H_s(y)\text{P.V.} \int_{\partial E_t} \frac{1 - \nu(z)\cdot\nu(y)}{|z-y|^{n+s}} dz \right) \\
&\qquad + C_2 \delta e^{-C_2 t} \phi(y - x_t) + H_s(x_t) \delta e^{-C_2 t} \nu_t \cdot \nabla \phi(y - x_t) \\
&= C_1 e^{C_1 t} \left(\frac{H_s(y)}{s\,(1-s)} + \epsilon \right) + 2s(1-s) \left(\text{P.V.} \int_{\partial E_t} \frac{v(z,t) - v(y,t)}{|z-y|^{n+s}} dz + e^{C_1 t} H_s(y) \text{P.V.} \int_{\partial E_t} \frac{1 - \nu(z)\cdot\nu(y)}{|z-y|^{n+s}} dz \right) \\
&\quad + 2s\,(1-s) \delta e^{-C_2 t} \text{P.V.} \int_{\partial E_t} \frac{\phi(z - x_t) - \phi(y - x_t)}{|z-y|^{n+s}} dz + C_2 \delta e^{-C_2 t} \phi(y - x_t) \\
&\quad + H_s(x_t) \delta e^{-C_2 t} \nu_t \cdot \nabla \phi(y - x_t).
\end{aligned}$$

Now we claim that

$$v(y,t) \geqslant 0. \tag{5.4}$$

Since this holds for $t=0$ (as long as δ is sufficiently small), to prove (5.4) we can argue by contradiction and assume that there is a first time $\bar t$ and a point $\bar y$ such that $v(\bar y,\bar t)=0$. Such a point is a local minimum and it holds that

$$\begin{aligned} &\partial_t v(\bar y,\bar t)\leqslant 0,\\ &\text{P.V.}\int_{\partial E_{\bar t}}\frac{v(z,\bar t)-v(\bar y,\bar t)}{|z-\bar y|^{n+s}}dz\geqslant 0\\ \text{and}\qquad &e^{C_1\bar t}\left(\frac{H_s(\bar y)}{s\,(1-s)}+\epsilon\right)=\delta e^{-C_2\bar t}\phi(\bar y-x_{\bar t}).\end{aligned}$$

Hence, we have

$$\begin{aligned} 0\geqslant\partial_t v(\bar y,\bar t)\geqslant C_1\delta e^{-C_2\bar t}\phi(\bar y-x_{\bar t})+2s\,(1-s)\delta e^{-C_2\bar t}\text{P.V.}\int_{\partial E_{\bar t}}\frac{\phi(z-x_{\bar t})-\phi(\bar y-x_{\bar t})}{|z-\bar y|^{n+s}}dz\\ +C_2\delta e^{-C_2\bar t}\phi(\bar y-x_{\bar t})+H_s(x_{\bar t})\delta e^{-C_2\bar t}\nu_{\bar t}\cdot\nabla\phi(\bar y-x_{\bar t}).\end{aligned}\tag{5.5}$$

Now we claim that

$$|\bar y-x_{\bar t}|<1.\tag{5.6}$$

To this end, we argue by contradiction and suppose that $|\bar y-x_t|\geqslant 1$. Then, using (5.5) and the assumption on the support of ϕ, we find that

$$0\geqslant 2s\,(1-s)\delta e^{-C_2\bar t}\text{P.V.}\int_{\partial E_{\bar t}}\frac{\phi(z-x_{\bar t})}{|z-\bar y|^{n+s}}dz>0.$$

This is a contradiction and so (5.6) is proved.

Now we improve (5.6), by showing that there exists $\epsilon_0\in(0,1)$ such that

$$|\bar y-x_{\bar t}|<1-\epsilon_0.\tag{5.7}$$

Again, we argue by contradiction and suppose that $|\bar y-x_{\bar t}|\in[1-\epsilon_0,1)$. Since ϕ is smooth and vanishes along ∂B_1, we have that $\phi(\bar y-x_{\bar t})+|\nabla\phi(\bar y-x_{\bar t})|\leqslant C\epsilon_0$, for some $C>0$. Hence, using (5.5), and taking $K>0$ such that

$$H_s(x)\leqslant K \text{ for every } x\in\partial E_t,\tag{5.8}$$

we see that

$$\begin{aligned} 0&\geqslant 2s\,(1-s)\delta e^{-C_2\bar t}\text{P.V.}\int_{\partial E_{\bar t}}\frac{\phi(z-x_{\bar t})-C\epsilon_0}{|z-\bar y|^{n+s}}dz-CK\delta e^{-C_2\bar t}\epsilon_0\\ &\geqslant \delta e^{-C_2\bar t}\left[2s\,(1-s)\text{P.V.}\int_{(\partial E_{\bar t})\cap B_{1/2}(x_{\bar t})}\frac{1-C\epsilon_0}{|z-\bar y|^{n+s}}dz-CK\epsilon_0\right].\end{aligned}$$

So we multiply by $\delta^{-1}e^{C_2\bar t}$ and, if ϵ_0 is small enough, we find that

$$\begin{aligned} 0&\geqslant s\,(1-s)\text{P.V.}\int_{(\partial E_{\bar t})\cap B_{1/2}(x_{\bar t})}\frac{dz}{|z-\bar y|^{n+s}}-CK\epsilon_0\\ &\geqslant 2^{-n-s}s\,(1-s)\mathcal{H}^{n-1}\big((\partial E_{\bar t})\cap B_{1/2}(x_{\bar t})\big)-K\epsilon_0.\end{aligned}$$

The smoothness of the surface gives that

$$\mathcal{H}^{n-1}\big((\partial E_{\bar t})\cap B_{1/2}(x_{\bar t})\big)\geqslant c_0$$

for some $c_0>0$. The last two inequalities easily give a contradiction if ϵ_0 is small enough, and so we have established (5.7).

Now we set $r_0:=1-\epsilon_0$ and we choose C_1 large enough, such that

$$C_1\geqslant\frac{K\sup_{B_1}|\nabla\phi|}{\inf_{B_{r_0}}\phi},\tag{5.9}$$

where K is as in (5.8).

Notice that, by (5.7) and (5.9),

$$C_1\delta e^{-C_2\bar{t}}\phi(\bar{y}-x_{\bar{t}})+H_s(x_{\bar{t}})\delta e^{-C_2\bar{t}}\nu_{\bar{t}}\cdot\nabla\phi(\bar{y}-x_{\bar{t}})\geqslant 0. \tag{5.10}$$

Let also C_2 so large that

$$C_2\geqslant\sup_{y\in B_{r_0}(x_{\bar{t}})}\frac{2s\,(1-s)\delta e^{-C_2\bar{t}}}{\phi(\bar{y}-x_{\bar{t}})}\text{P.V.}\int_{\partial E_{\bar{t}}}\frac{\phi(\bar{y}-x_{\bar{t}})-\phi(z-x_{\bar{t}})}{|z-\bar{y}|^{n+s}}dz.$$

In this way, and using again (5.7),

$$2s\,(1-s)\delta e^{-C_2\bar{t}}\text{P.V.}\int_{\partial E_{\bar{t}}}\frac{\phi(z-x_{\bar{t}})-\phi(\bar{y}-x_{\bar{t}})}{|z-\bar{y}|^{n+s}}dz+C_2\delta e^{-C_2\bar{t}}\phi(\bar{y}-x_{\bar{t}})\geqslant 0.$$

Then, we plug this information and (5.10) into (5.5) and we obtain a contradiction. This proves (5.4).

Then we take $y=x_t$ and send $\epsilon\to 0$ in (5.4) and we obtain that $H_s(x_t)$ remains positive. ☐

6. Estimates for entire graphs

In this section we assume that the surface is an entire graph with linear growth. That is the surface can be parameterized by $(x,u(x,t))$ and $|Du|(x,t)\to 0$ as $|x|\to\infty$ is uniformly bounded for all times. Moreover, u satisfies

$$\partial_t u=-\sqrt{1+|Du|^2}\;H_s(E_u).$$

Theorem 19. *Let ν be the normal vector of a graphical surface evolving by* (1.1) *and e any fixed vector. Let $v=(e\cdot\nu)^{-1}$, then*

$$v\leqslant\sup\{v(\cdot,0),C\},$$

where C is such that $\limsup_{|x|\to\infty}v(x,t)\leqslant C$

Proof. Let us assume that the surface is parameterized according to (4.1) and ν satisfies (4.5). Then

$$v_t=-v^2(e\cdot\nabla^\Gamma H_s).$$

From Theorem 14 we have that

$$e\cdot\nabla^\Gamma H_s=(n+s)s(1-s)\text{P.V.}\int_{\mathbb{R}^n}\tilde{\chi}_{E_t}(y)\frac{(y-x)\cdot e^T}{|x-y|^{n+s+2}}dy,$$

where e^T is the tangential component of e at x_t.

Noticing that $(n+s)\frac{(y-x)\cdot e^T}{|x-y|^{n+s+2}}=-\text{div}_y\left(\frac{e^T}{|x-y|^{n+s}}\right)$, it follows from the divergence theorem that

$$e\cdot\nabla^\Gamma H_s=2s(1-s)\int_{\partial E_t}\frac{e^T\cdot\nu(y)}{|x-y|^{n+s}}.$$

Since $e^T=e-v^{-1}(x)\nu(x)$, it holds that $e^T\cdot\nu(y)=v^{-1}(y)-v^{-1}(x)\nu(x)\cdot\nu(y)$. Then, if v attains a maximum at x, we have that $e^T\cdot\nu(y)\geqslant 0$ (and similarly $e^T\cdot\nu(y)\leqslant 0$ at minima). We may conclude from the maximum principle that v does not have interior maxima (resp. minima). ☐

Noticing that for an evolving graph it holds that

$$(e_n\cdot\nu)^{-1}=\sqrt{1+|Du|^2},$$

we have

Corollary 20. *$|Du|$ is uniformly bounded in time.*

Also, we have the following regularity result:

Corollary 21. *If u satisfies* (2.13) *then u is smooth.*

Proof. Since v is uniformly bounded above and below and

$$v_t - 2s(1-s)v^{-2}\int_{\partial E_t} \frac{v^{-1}(y) - v^{-1}(x)\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}dy = 0,$$

we have from [16] that v is C^α. Now following the proof in [3] we can conclude that $u \in C^\infty(\mathbb{R}^n \times [0,\infty))$. $\square$

Theorem 22. *Let $v = \sqrt{1+|Du|^2}$, then the quantity vH_s is uniformly bounded in terms of the initial condition.*

Proof. Considering the set Π as the epigraph of the plane $z = u(x_t,t) + \nabla u(x_t,t)\cdot(x-x_t) + u(x_t,t)$, we may write

$$\begin{aligned} H_s(x_t,E) =& s(1-s)\int_{\mathbb{R}^{n-1}} \int_{u(x,t)}^{\nabla u(x_t,t)\cdot(x-x_t)+u(x_t,t)} \frac{dz}{((z-u(x_t,t))^2+|x-x_t|^2)^{\frac{n+2}{2}}}dx \\ =& s(1-s)\int_{\mathbb{R}^{n-1}} \frac{1}{|x-x_t|^{n+s-1}} \int_{\frac{u(x,t)-u(x_t,t)}{|x-x_t|}}^{\nabla u(x_t,t)\cdot\left(\frac{x-x_t}{|x-x_t|}\right)} \frac{dz}{(z^2+1)^{\frac{n+s}{2}}}dx \end{aligned}$$

Let $z_m = \frac{u(x,t)-u(x_t,t)}{|x-x_t|}$ and $z_M = \nabla u(x_t,t)\cdot\frac{x-x_t}{|x-x_t|}$. Then

$$\partial_t z_m = \frac{-H_s v(x,t) + H_s v(x_t,t)}{|x-x_t|}$$

and

$$\partial_t z_M = \nabla(-H_s v(x_t,t))\cdot\frac{x-x_t}{|x-x_t|}.$$

As a consequence, we have

$$\begin{aligned} \partial_t H_s(x_t,E) =& s(1-s)\int_{\mathbb{R}^{n-1}} \frac{1}{|x-x_t|^{n+s-1}}\left(\frac{\partial_t z_M}{(z_M^2+1)^{\frac{n+s}{2}}} - \frac{\partial_t z_m}{(z_m^2+1)^{\frac{n+s}{2}}}\right), \text{ and} \\ \partial_t(vH_s) =& \frac{Du\cdot D(-H_s v)}{\sqrt{1+|Du|^2}}H_s + v\partial_t H_s(x_t,E). \end{aligned}$$

Assume that a maximum (resp. minimum) point of vH_s is attained at (x_t,t_0). Then $D(-H_s v) = 0$, $\partial_t z_M = 0$ and $\partial_t z_m \geqslant 0$ (resp. $\leqslant 0$) and only identically 0 if vH_s is constant. Then we conclude that there are no interior maxima or minima for this quantity. $\square$

Remark 23. *The previous estimates imply that if there is decay at infinity H_s remains bounded for all times.*

7. Estimates for star-shaped surfaces

We show an estimate for star-shaped surfaces that is analog to Theorem 19

Theorem 24. *Let $v = (x\cdot\nu)^{-1}$. Then there exists $T^* > 0$ such that $v(t) \leqslant C$ in $[0,T^*)$, where C depends on $v(0)$ and $\sup|H_s|$.*

Proof. We assume like in the proof of 19 that the surface is parameterized as in (4.1), then we have

$$\begin{aligned} \partial_t v =& -v^2(x_t\cdot\nu + x\cdot\nu_t) \\ =& v^2(H_s - x\cdot\nabla^\Gamma H_s) \end{aligned}$$

Following the computations in the proof of Theorem 19 we have that

$$x\cdot\nabla^\Gamma H_s = 2s(1-s)\int_{\partial E_t}\frac{x^T\cdot\nu(y)}{|x-y|^{n+s}}dy.$$

since $x^T = x - x\cdot\nu(x)\nu(x) = (x-y) + (y - x\cdot\nu(x)\nu(x))$ we have

$$\begin{aligned}\frac{x\cdot\nabla^\Gamma H_s}{s(1-s)} =&2\int_{\partial E_t}\frac{(x-y)\cdot\nu(y)}{|x-y|^{n+s}}dy+2\int_{\partial E_t}\frac{v^{-1}(y)-v^{-1}(x)\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}dy\\ =&sH_s+2\int_{\partial E_t}\frac{v^{-1}(y)-v^{-1}(x)\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}dy.\end{aligned}$$

Accordingly, we have

$$\partial_t v = v^2\left((1-s^2(1-s))H_s - 2s(1-s)\int_{\partial E_t}\frac{v^{-1}(y)-v^{-1}(x)\nu(x)\cdot\nu(y)}{|x-y|^{n+s}}dy\right).$$

Hence, at a spacial maximum of v we have

$$\partial_t(\max_{\mathbb{S}^n} v(\cdot,t)) \leqslant (1-s^2(1-s))\max_{\mathbb{S}^n} v(\cdot,t)^2 H_s.$$

Then we find that

$$\max_{\mathbb{S}^n} v(\cdot,t)\leqslant \frac{\max_{\mathbb{S}^n} v(\cdot,0)}{1-(1-s^2(1-s))t\max_{\mathbb{S}^n} v(\cdot,0)\sup_{S\times[0,T)} H_s}.$$

Notice that the bound can be extended as long as H_s remains bounded. $\square$

The previous computation yields a gradient bound and that star-shapedness is preserved:

Corollary 25. *Assume that f satisfies* (2.4)*. Then, if H_s remains bounded, $|\nabla f|$ is bounded for a fixed time that depends of the initial condition and bounds of H_s.*

Proof. Notice that $x = f\omega$ and $\nu = \frac{f\omega-\nabla f}{\sqrt{f^2+|\nabla f|^2}}$. Then $x\cdot\nu = \frac{f^2}{\sqrt{f^2+|\nabla f|^2}}$.

Then $v\leqslant C$ is equivalent to

$$\sqrt{f^2+|\nabla f|^2}\leqslant Cf^2\leqslant C\max f^2(\cdot,0),$$

which gives the desired result. $\square$

Corollary 26. *Assume that E_t is a solution to* (1.1) *and that E_0, then E_t remains star-shaped.*

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