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On an application of Tikhonov's fixed point theorem to a nonlocal Cahn–Hilliard type system modeling phase separation

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Abstract

This paper investigates a nonlocal version of a model for phase separation on an atomic lattice that was introduced by P. Podio-Guidugli in *Ric. Mat.* **55** (2006) 105-118. The model consists of an initial-boundary value problem for a nonlinearly coupled system of two partial differential equations governing the evolution of an order parameter ρ and the chemical potential μ. Singular contributions to the local free energy in the form of logarithmic or double-obstacle potentials are admitted. In contrast to the local model, which was studied by P. Podio-Guidugli and the present authors in a series of recent publications, in the nonlocal case the equation governing the evolution of the order parameter contains in place of the Laplacian a nonlocal expression that originates from nonlocal contributions to the free energy and accounts for possible long-range interactions between the atoms. It is shown that just as in the local case the model equations are well posed, where the technique of proving existence is entirely different: it is based on an application of Tikhonov's fixed point theorem in a rather unusual separable and reflexive Banach space.

1 Introduction

This paper deals with a nonlocal variant of a model for phase segregation through atom rearrangement on a lattice proposed in [36]. This model (see also [14] for a detailed derivation), which is a modification of the Fried–Gurtin approach to phase segregation processes (cf. [24], [32]), uses an order parameter ρ, which in many cases represents the (normalized) density of one of the phases and attains values in the interval $[-1, 1]$, and the chemical potential μ as unknowns. It is based on a local free energy density of the form

$$\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu\,\rho + F(\rho) + \frac{\sigma}{2}\,|\nabla\rho|^2, \tag{1.1}$$

where $\sigma > 0$ is a physical constant and F is a double-well potential, and leads to the evolutionary system

$$2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \Delta\mu = 0 \tag{1.2}$$

$$-\,\sigma\,\Delta\rho + F'(\rho) = \mu\,. \tag{1.3}$$

The above equations are assumed to hold in $Q := \Omega \times (0, T)$, where Ω is a three-dimensional domain and T is some given final time, and they are complemented with proper boundary and initial conditions. Typical examples for the double-well potential F are given by

$$F_{reg}(r) := \frac{1}{4}(r^2 - 1)^2\,, \quad r \in \mathbb{R} \tag{1.4}$$

$$F_{log}(r) := ((1 + r)\ln(1 + r) + (1 - r)\ln(1 - r)) - cr^2\,, \quad r \in (-1, 1), \tag{1.5}$$

where $c > 1$ in the latter case so that F_{log} is nonconvex. The potentials (1.4) and (1.5) are usually referred to as the *classical regular* and the *logarithmic double-well* potential, respectively. These potentials are smooth in their domains, where the derivative of the latter becomes singular at ± 1. However, one can even consider nondifferentiable potentials, where an important example is given by the so-called *double-obstacle* potential given by

$$F_{2obs}(r) := I(r) - cr^2\,, \quad r \in \mathbb{R}, \tag{1.6}$$

where $c > 0$ is a positive constant and $I : \mathbb{R} \to [0, +\infty]$ denotes the indicator function of $[-1, 1]$, i.e., we have $I(r) = 0$ if $|r| \leq 1$ and $I(r) = +\infty$ otherwise. In this case, the order parameter is subjected to the unilateral constraint $|\rho| \leq 1$ and (1.3) should be read as a differential inclusion with F' representing the subdifferential ∂I of I.

The system (1.2)-(1.3) constitutes a modification of the Cahn–Hilliard system originally introduced in [8] and first studied mathematically in the seminal paper [23] (for a large list of references on the original Cahn–Hilliard system, see [34]). It is ill-posed, in general. In fact, it was pointed out in [17] that an associated initial-boundary value problem with zero Neumann boundary conditions for both ρ and μ may have infinitely many smooth and even nonsmooth solutions. Therefore, two small regularizing parameters $\varepsilon > 0$ and $\delta > 0$ were introduced in [14], which led to the regularized model equations

$$\big(\varepsilon + 2\rho\big)\, \partial_t \mu + \mu\, \partial_t \rho - \Delta\mu = 0 \tag{1.7}$$

$$\delta\, \partial_t \rho - \sigma\, \Delta\rho + F'(\rho) = \mu\,. \tag{1.8}$$

The system (1.7)–(1.8), which constitutes a modification of the so-called *viscous* Cahn–Hilliard system (see [22]), was analyzed in the series of papers [14, 16, 18, 20, 11] concerning well-posedness, regularity, optimal control and numerical approximation. Later, the local free energy density (1.1) was generalized to the form

$$\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu\, g(\rho) + F(\rho) + \frac{\sigma}{2}\, |\nabla\rho|^2 \tag{1.9}$$

with a function g having suitable (see below) properties. If one puts, without loss of generality, $\varepsilon = \delta = 1$, then one obtains the more general system

$$\big(1 + 2g(\rho)\big)\, \partial_t \mu + \mu\, g'(\rho)\, \partial_t \rho - \Delta\mu = 0 \tag{1.10}$$

$$\partial_t \rho - \sigma\, \Delta\rho + F'(\rho) = \mu\, g'(\rho), \tag{1.11}$$

which was investigated in the papers [19, 15, 17, 12, 13].

In the present paper, we replace the local term $\frac{\sigma}{2}\, |\nabla\rho|^2$ in the local free energy density by a nonlocal expression. A prototypical case is to consider a total free energy functional of the form

$$\mathcal{F}_{tot}[\rho] = \int_\Omega \Big[-\mu(x)\, g(\rho(x)) + F(\rho(x))\Big]\, dx \; + \; \mathcal{Q}[\rho]\,, \tag{1.12}$$

where

$$\mathcal{Q}[\rho] := \int_\Omega \rho(x) \int_\Omega k(|y - x|)(1 - \rho(y))\, dy\, dx\,.$$

Employing the techniques described in, e. g., [14], we arrive with the variational derivative

$$B[\rho](x) = \int_\Omega k(|y - x|)\, (1 - 2\, \rho(y))\, dy, \quad x \in \Omega, \tag{1.13}$$

of the functional $\mathcal{Q}$ at the following nonlocal variant of the system (1.10)–(1.11):

$$\big(1+2g(\rho)\big)\,\partial_t\mu+\mu\,g'(\rho)\,\partial_t\rho-\Delta\mu=0 \tag{1.14}$$
$$\partial_t\rho+B[\rho]+F'(\rho)=\mu\,g'(\rho), \tag{1.15}$$

which is the system that we will investigate in the following. However, we do not restrict ourselves to operators B of the exact form given in (1.13). In fact, we consider general operators B acting on functions defined in Q that enjoy suitable properties. Very simple examples that satisfy the conditions specified below are given by time convolution operators of the form

$$B[\rho](x,t)=\int_0^t k(t-s)\,\rho(x,s)\,ds \tag{1.16}$$

and spatial convolutions of the form

$$B[\rho](x,t)=\int_\Omega k(|y-x|)\,\rho(y,t)\,dy \tag{1.17}$$

provided that the respective integral kernels k are smooth enough. For instance, the three-dimensional Newtonian potential will be admissible. However, we will not be able to include nonlocal-in-time nonlinearities of hysteresis type like the classical stop, play, Prandtl-Ishlinskii or Preisach operators (for the definitions of these hysteresis operators, see, e. g., [7]).

Free energies of the form (1.12) have been proposed in [30, 31] and rigorously justified as macroscopic limits of microscopic phase segregation models with particle conserving dynamics (see also [9]). In [30, 31], starting from a microscopic model, the authors derived a macroscopic equation for phase segregation phenomena that turns out to be a nonlocal version of the well-known Cahn–Hilliard equation. From the mathematical viewpoint, this nonlocal Cahn–Hilliard equation is simpler than our system (1.14)–(1.15) and has received a good deal of attention in the last decade (see, e.g., [4, 5, 21, 26, 28, 33, 35]). Most of the theoretical results are devoted to well-posedness and some are concerned with the long-time behavior of solutions. Well-posedness and regularity issues were analyzed for an equation with degenerate mobility and logarithmic potential in [28] (cf. also [21, 26, 27]). This fact required to show preliminarily that a solution stays eventually strictly away from the pure phases: the so-called separation property. For the constant mobility case and regular potentials, some existence, uniqueness and regularity results were obtained in [4, 5, 33]. Nonsmooth potentials are considered in [21]. The existence of a (connected) global attractor has been proven in [25] for constant mobility and singular potentials. This has been done by exploiting the energy identity obtained in [10] as a by-product of results related to a phase separation model in binary fluids. The question whether the global attractor has finite (fractal) dimension was examined in [29], where the authors proved the existence of an exponential attractor. In [1], an equation that is the conserved gradient flow of a nonlocal total free energy functional is considered: the functional is characterized by a Helmholtz free energy density, which can be of logarithmic type. We finally mention the paper [37], in which a distributed optimal control problem is studied for a nonlocal convective Cahn–Hilliard equation with degenerate mobility and singular potential in three dimensions of space.

The present paper is organized as follows. In the next section, we will list our assumptions, state the problem in a precise form and present our results. The corresponding proofs will be given in the last two sections. We remark at this place that the mathematical techniques employed here to prove existence differ significantly from those used in, e. g., [14] to handle the local case. Indeed, while in [14] a retarded argument method was utilized, we apply Tikhonov's fixed point theorem in a rather unusual functional analytic framework, namely in the space $L^2(0,T;H^1(\Omega))\cap L^{10/3}(Q)$.

Now, we list a number of tools and notations employed throughout the paper. We repeatedly use the Young inequalities

$$ab \le \delta a^2 + \frac{1}{4\delta}\, b^2 \quad \text{and} \quad ab \le \vartheta a^{\frac{1}{\vartheta}} + (1-\vartheta) b^{\frac{1}{1-\vartheta}} \quad \text{for every } a,b \ge 0,\ \delta > 0, \text{ and } \vartheta \in (0,1), \tag{1.18}$$

as well as the Hölder and Sobolev inequalities. In our three-dimensional framework, the latter read

$$H^1(\Omega) \subset L^p(\Omega) \quad \text{and} \quad \|v\|_p \le C_\Omega \|v\|_{H^1(\Omega)} \quad \text{for every } v \in H^1(\Omega) \text{ and } p \in [1,6], \tag{1.19}$$

where C_Ω depends only on Ω, and the embedding $H^1(\Omega) \subset L^p(\Omega)$ is compact if $p < 6$. We also recall the continuous embedding

$$\bigl(L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\bigr) \subset \bigl(L^{10/3}(Q) \cap L^{7/3}(0,T;L^{14/3}(\Omega))\bigr)\,, \tag{1.20}$$

which is a consequence of the Young, Sobolev and interpolation inequalities. In particular, there holds the inequality

$$\|v\|_{L^{10/3}(Q)\cap L^2(0,T;H^1(\Omega))} \le C_0 \max\{\|v\|_{L^\infty(0,T;L^2(\Omega))}, \|\nabla v\|_{L^2(Q)}\} \quad \text{for every } v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\,, \tag{1.21}$$

where C_0 depends only on Ω and T. Finally, in order to avoid a boring notation, we follow a general rule to denote constants. The small-case symbol c stands for different constants which depend only on Ω, on the final time T, the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements. A small-case symbol with a subscript like c_δ indicates that the constant might depend on the parameter δ, in addition. Hence, the meaning of c and c_δ might change from line to line and even in the same chain of equalities or inequalities. On the contrary, we mark precise constants that we can refer to by using different symbols, e.g., capital letters, like in (1.19). Also, for the sake of brevity again, we use the same symbol Φ to denote different continuous functions on $[0,+\infty)$ with the above dependence.

2 Statement of the problem and results

In this section, we describe the problem under study and give an outline of our results. As in the introduction, Ω is the body where the evolution takes place. We assume $\Omega \subset \mathbb{R}^3$ to be open, bounded, connected, and smooth, and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_ν stand for the boundary of Ω and the outward normal derivative, respectively. Now, we specify the assumptions on the structure of our system. We assume that

$$\beta : \mathbb{R} \to 2^{\mathbb{R}} \quad \text{is maximal monotone with} \quad 0 \in \beta(0) \tag{2.1}$$

$$\pi : \mathbb{R} \to \mathbb{R} \quad \text{is Lipschitz continuous} \tag{2.2}$$

$$g : \overline{D(\beta)} \to [0,+\infty) \quad \text{is } C^2\text{, bounded and concave, and } g' \text{ is bounded and Lipschitz continuous.} \tag{2.3}$$

In (2.3), $D(\beta)$ is the effective domain of β. For $r \in D(\beta)$, we also use the symbol $\beta^\circ(r)$ for the element of $\beta(r)$ having minimum modulus (see, e.g., [6, p. 28]). Notice that, in the notation used in the introduction, $F' = \beta + \pi$. Moreover, let us point out that, in the case when $D(\beta) = \mathbb{R}$, our assumption (2.3) necessarily implies that g is a constant function, so that our system (1.14)–(1.15) completely decouples; on the other hand, the significant physical case for our model (see [36, 14, 19]) corresponds to a bounded interval for $D(\beta)$ ($\subseteq [-1, 1]$, say) and in this framework g may be rather general.

Next, in order to list our assumptions on the nonlocal operator B and even for a future convenience, we set

$$V := H^1(\Omega), \quad H := L^2(\Omega) \quad \text{and} \quad W := \{v \in H^2(\Omega) : \ \partial_\nu v = 0\} \tag{2.4}$$

$$Q_t := \Omega \times (0, t) \quad \text{for } 0 < t \le T \quad \text{and} \quad Q := Q_T. \tag{2.5}$$

As for the nonlocal operator B, we assume that it maps $L^2(0, T; H) = L^2(Q)$ into itself, is causal, and enjoys the following properties:

$$B : L^2(0, T; H) \to L^2(0, T; H); \tag{2.6}$$

$$B[u]|_{Q_t} = B[v]|_{Q_t} \quad \text{whenever } u|_{Q_t} = v|_{Q_t}, \quad \text{for every } t \in (0, T]; \tag{2.7}$$

$$B(L^p(Q_t)) \subset L^p(Q_t) \quad \text{and} \quad \|B[v]\|_{L^p(Q_t)} \le C_{B,p}\big(1 + \|v\|_{L^p(Q_t)}\big)$$
$$\text{for every } v \in L^p(Q), t \in (0, T], \text{ and } p \in \big\{2, \tfrac{10}{3}, 6\big\}; \tag{2.8}$$

$$\|B[u] - B[v]\|_{L^2(Q_t)} \le C_B \|u - v\|_{L^2(Q_t)}$$
$$\text{for every } u, v \in L^2(Q) \text{ and } t \in (0, T]; \tag{2.9}$$

$$B(L^2(0, T; V)) \subset L^2(0, T; V) \text{ and, for every } v \in L^2(0, T; V) \text{ and } t \in (0, T],$$
$$\Big|\int_{Q_t} \nabla B[v] \cdot \nabla v \, dx \, ds\Big| \le C_B\big(1 + \int_{Q_t} (|v|^2 + |\nabla v|^2) \, dx \, ds\big) \, . \tag{2.10}$$

In the above formulas, $C_{B,p}$ and C_B are given structural constants, and, for any Banach space X, the symbol $\| \cdot \|_X$ denotes its norm. The same notation is then used also for powers of X. However, in the following we simply write $\| \cdot \|_p$ for the standard norm in $L^p(\Omega)$, for $1 \le p \le +\infty$.

Examples. It is obvious that convolution type integral operators of the form (1.17) or (1.16) satisfy the conditions (2.6)–(2.10) provided the kernel k is smooth. However, hysteresis operators like the classical stop, play, Prandtl-Ishlinskii or Preisach operators are not included. The reason for this is that these operators carry a nonlocal memory with respect to time. For instance, the one-dimensional stop operator $\mathcal{S}$ (to take the simplest of these four operators) only enjoys (cf. [7]) the *nonlocal* Lipschitz property

$$|\mathcal{S}[\rho_1](t) - \mathcal{S}[\rho_2](t)| \ \le \ 2 \max_{0 \le s \le t} |\rho_1(s) - \rho_2(s)|$$

for every $t \in [0, T]$ and every $\rho_1, \rho_2 \in C^0([0, T])$, and it is easily seen that the validity of the Lipschitz condition (2.9) cannot be guaranteed, in general.

As a further example for which the conditions can be verified, we consider the integral operator

$$K[\rho](x) = \int_\Omega k(|y - x|) \, \rho(y) \, dy \, , \tag{2.11}$$

which acts on functions defined in Ω, and its counterpart B acting on functions defined in Q, which is given by (1.17). We assume that $k \in C^0(0,+\infty)$ satisfies the condition

$$|k(r)| \le C_1 r^{-\alpha} \quad \forall\, r > 0 \quad \text{with some } C_1 > 0 \text{ and } \alpha < 3. \tag{2.12}$$

Such kernels belong to the class of *weakly singular* kernels. Obviously, (2.7) holds, and since Ω is a bounded domain, it is well known (see, e. g., [2, Sect. 8.10]) that, for any $p \in (1,+\infty)$ such that $\alpha < \frac{3}{q}$, where $\frac{1}{p}+\frac{1}{q} = 1$, the linear operator K maps $L^p(\Omega)$ continuously (even compactly) into $C^0(\overline{\Omega})$ and thus into $L^p(\Omega)$. It is then an easy exercise, using Hölder's inequality, to show that for $\alpha < \frac{3}{2}$ the corresponding operator B satisfies all of the conditions (2.6), (2.8) and (2.9).

In order to satisfy also (2.10), we need additional assumptions, for instance, that k is continuously differentiable on $(0,+\infty)$ with

$$|k'(r)| \le C_2 r^{-\beta} \quad \forall\, r > 0 \quad \text{with some } C_2 > 0 \text{ and } \beta < \tfrac{5}{2}. \tag{2.13}$$

Indeed, under this assumption we have for any $v \in L^2(0,T;V)$, using the continuity of the embedding $V \subset L^6(\Omega)$ and the fact that $\frac{6\beta}{5} < 3$,

$$\begin{aligned}
\left| \int_{Q_t} \nabla v \cdot \nabla B[v] \right| \quad & \le \int_{Q_t} |\nabla v|^2 + c \int_{Q_t} \left| \int_\Omega |y-x|^{-\beta} \, |v(y,s)| \, dy \right|^2 dx\, ds \\
& \le c \int_{Q_t} |\nabla v|^2 + c \int_{Q_t} \left[\int_\Omega \frac{dy}{|y-x|^{6\beta/5}} \right]^{5/3} \|v(s)\|_6^2 \, dx\, ds \\
& \le c \int_{Q_t} \left(|v|^2 + |\nabla v|^2 \right).
\end{aligned}$$

Finally, we observe that in the important case of the (long-range) three-dimensional Newtonian potential $k(r) = \frac{C}{r}$, for which we have $\alpha = 1$ and $\beta = 2$, both (2.12) and (2.13) are fulfilled.

At this point, we can describe the problem to be investigated. We assume that

$$\mu_0 \in V \quad \text{and} \quad \mu_0 \ge 0 \quad \text{a.e. in } \Omega \tag{2.14}$$

$$\rho_0 \in V, \quad \rho_0 \in D(\beta) \quad \text{a.e. in } \Omega \quad \text{and} \quad \rho_0 |\beta^\circ(\rho_0)|^{7/3} \in L^1(\Omega) \tag{2.15}$$

and look for a triplet (μ, ρ, ξ) satisfying

$$\mu \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,3/2}(\Omega)) \tag{2.16}$$

$$\mu \ge 0 \quad \text{a.e. in } Q \tag{2.17}$$

$$\rho \in L^\infty(0,T;V) \quad \text{and} \quad \partial_t \rho \in L^{10/3}(Q) \tag{2.18}$$

$$\xi \in L^2(0,T;H) \tag{2.19}$$

and solving the initial-boundary value problem

$$\big(1 + 2g(\rho)\big)\, \partial_t \mu + \mu\, g'(\rho)\, \partial_t \rho - \Delta \mu = 0 \quad \text{a.e. in } Q \tag{2.20}$$

$$\partial_t \rho + \xi + \pi(\rho) + B[\rho] = \mu\, g'(\rho) \quad \text{and} \quad \xi \in \beta(\rho) \quad \text{a.e. in } Q \tag{2.21}$$

$$\partial_\nu \mu = 0 \quad \text{a.e. on } \Sigma \tag{2.22}$$

$$\mu(0) = \mu_0 \quad \text{and} \quad \rho(0) = \rho_0\,, \tag{2.23}$$

where $\Sigma := \Gamma \times (0,T)$.

Here are our results.

Theorem 2.1. *With the assumptions and notations* (2.1)–(2.10) *on the structure, assume* (2.14)–(2.15) *on the initial data. Then, problem* (2.20)–(2.23) *has at least one solution satisfying* (2.16)–(2.19).

Theorem 2.2. *Under the assumptions of Theorem 2.1, suppose in addition that*

$$\mu_0 \in L^\infty(\Omega) \ \text{ and } \ \rho_0 \left(\beta^\circ(\rho_0)\right)^5 \in L^1(\Omega). \tag{2.24}$$

Then the solution to problem (2.20)–(2.23) *is unique and also satisfies*

$$\mu \in L^\infty(Q), \quad \partial_t \rho \in L^6(Q) \ \text{ and } \ \xi \in L^6(Q). \tag{2.25}$$

Remark 2.3. One can prove at least the existence of a solution to the more general problem obtained by replacing equation (2.20) by

$$\big(1 + 2g(\rho)\big)\, \partial_t \mu + \mu\, g'(\rho)\, \partial_t \rho - \operatorname{div}\big(\kappa(\mu) \nabla \mu\big) = 0, \tag{2.26}$$

where $\kappa : [0,+\infty) \to (0,+\infty)$ is a bounded continuous function such that $1/\kappa$ is also bounded (like the uniformly parabolic case discussed in [19, 17, 13], while the degenerate case also treated in [19] is more delicate). Moreover, one can insert a nonnegative source term u in the right-hand side of (2.26). The requirement $u \geq 0$ is needed to ensure that $\mu \geq 0$, as one can see by testing the equation by the negative part of μ (like in the proof of [19, Lemma 4.1]), and a sufficient condition that allows to generalize our results is $u \in L^\infty(Q)$. The introduction of such a source term would be necessary if a distributed control problem with the control u were to be studied. However, as the uniqueness of the solution would be needed in order to construct the control-to-state mapping, and since a continuous dependence result would have to be proved, one should consider the situation of [13] concerning the potential and other data (see, in particular, [13, formulas (2.9)–(2.12)]).

3 Existence

In this section, we prove Theorem 2.1. Our argument relies on a fixed point argument applied to a well-defined map $\mu \mapsto \rho \mapsto \mu$ involving equations (2.20) and (2.21), separately. In our construction, we will need two different extensions of the function g to the whole of $\mathbb{R}$. Although we will use the same notation in both cases, there will be no danger of confusion, since these extensions will be used in different steps. Furthermore, it will become evident that the constants related to these extensions, e.g., some Lipschitz constants, depend only on the corresponding constants related to the original map g.

The functional analytic framework. In order to make it precise, we first perform a formal estimate and construct a basic bound M_0. To this end, we formally multiply (2.20) by 2μ and observe that

$$\big\{\big(1 + 2g(\rho)\big)\, \partial_t \mu + \mu\, g'(\rho)\, \partial_t \rho\big\}\, 2\mu = \partial_t \big\{\big(1 + 2g(\rho)\big)\mu^2\big\}. \tag{3.1}$$

Hence, by integrating over Q_t with $t \in (0,T)$, we have

$$\int_\Omega \big(1+2g(\rho(t))\big)\,|\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2 = \int_\Omega \big(1+2g(\rho_0)\big)\,|\mu_0|^2.$$

The function g is nonnegative. However, for reasons that will become evident later on, we want to use just the inequality $g \geq -1/3$, i.e., $1+2g \geq 1/3$. We conclude that

$$\max\bigl\{\|\mu\|^2_{L^\infty(0,T;H)}, \|\nabla\mu\|^2_{L^2(0,T;H)}\bigr\} \leq 3(1+2\,\sup g)\,\|\mu_0\|^2_H. \tag{3.2}$$

Now, we owe to the embedding inequality (1.21) and deduce that

$$\|\mu\|_{L^{10/3}(Q)\cap L^2(0,T;V)} \leq M_0 := C_0\,(3+6\,\sup g)^{1/2}\,\|\mu_0\|_H\,. \tag{3.3}$$

Notice that the real number M_0 just defined depends only on Ω, T, g and μ_0. At this point, we can make the first choice we need and anticipate the next one. The used notation should help the reader, since $\mathcal{M}$ and $\mathcal{R}$ are the spaces in which μ and ρ are sought, respectively. We set

$$\mathcal{M} := L^{10/3}(Q) \cap L^2(0,T;V) \tag{3.4}$$
$$\mathcal{M}_0 := \{v \in \mathcal{M} :\ \|v\|_{\mathcal{M}} \leq M_0 \ \text{ and } \ v \geq 0 \text{ a.e. in } Q\} \tag{3.5}$$
$$\mathcal{R} := W^{1,10/3}(0,T;L^{10/3}(\Omega)) \cap L^\infty(0,T;V). \tag{3.6}$$

The next steps are devoted to the construction of the maps $\mathcal{F}_1 : \mathcal{M}_0 \to \mathcal{R}$ and $\mathcal{F}_2 : \mathcal{R} \to \mathcal{M}_0$. The fixed point argument will be performed on the map $\mathcal{F} := \mathcal{F}_2 \circ \mathcal{F}_1 : \mathcal{M}_0 \to \mathcal{M}_0$. The definition of $\mathcal{F}_1$ is based on the solution to the Cauchy problem for (2.21), for a given μ, i.e.,

$$\partial_t\rho + \xi + \pi(\rho) + B[\rho] = \mu\, g'(\rho) \quad \text{and} \quad \xi \in \beta(\rho) \quad \text{a.e. in } Q, \qquad \rho(0) = \rho_0\,. \tag{3.7}$$

We have to prove a well-posedness result.

The first approximating problem. In the following, we always assume that $\mu \in \mathcal{M}_0$, which implies, in particular, that $\mu \geq 0$ almost everywhere in Q. We introduce a proper approximating problem depending on a positive parameter ε. Namely, we replace β in (3.7) by its Yosida regularization β_ε at level ε. We recall that β_ε is monotone and Lipschitz continuous on $\mathbb{R}$ and that $|\beta_\varepsilon(r)| \leq |\beta^\circ(r)|$ for every $r \in D(\beta)$ (see, e.g., [6, p. 28]). Next, we replace μ on the right-hand side of (3.7) by $T_\varepsilon(\mu)$, where the truncation map $T_\varepsilon : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_\varepsilon(r) := \max\{-1/\varepsilon, \min\{1/\varepsilon, r\}\} \quad \text{for } r \in \mathbb{R}. \tag{3.8}$$

Finally, we temporarily extend g to the whole of $\mathbb{R}$ (still terming the extension g) in such a way that

$$g \text{ is a concave } C^2 \text{ function and } g' \text{ is bounded and Lipschitz continuous.} \tag{3.9}$$

We stress that we do not require g to be globally positive so that such an extension actually exists. At this point, we consider the problem of finding ρ_ε such that

$$\partial_t\rho_\varepsilon + \beta_\varepsilon(\rho_\varepsilon) + \pi(\rho_\varepsilon) + B[\rho_\varepsilon] = T_\varepsilon(\mu)\, g'(\rho_\varepsilon) \quad \text{a.e. in } Q \quad \text{and} \quad \rho_\varepsilon(0) = \rho_0\,. \tag{3.10}$$

As it is not completely obvious that such a problem has a unique solution (due to the presence of the nonlocal operator B), we give a proof of well-posedness. For a while, we do not stress the dependence on ε (which is fixed) and often avoid the subscript ε. Clearly, the solutions $\rho_\varepsilon \in H^1(0,T;H)$ of (3.10) are the fixed points (which necessarily belong to $H^1(0,T;H)$) of the nonlocal operator $\mathcal{S} : L^2(0,T;H) \to L^2(0,T;H)$ defined by

$$\mathcal{S}[v](t) := \rho_0 + \int_0^t \big(T_\varepsilon(\mu)\, g'(v) - \gamma_\varepsilon(v) - B[v]\big)(s)\, ds\,,$$

where, for brevity, we have set $\gamma_\varepsilon := \beta_\varepsilon + \pi$. In other words, for $u, v \in L^2(0,T;H)$, $u = \mathcal{S}v$ means that

$$u \in H^1(0,T;H), \quad \partial_t u = T_\varepsilon(\mu)\, g'(v) - \gamma_\varepsilon(v) - B[v] \quad \text{and} \quad u(0) = \rho_0\,. \tag{3.11}$$

We claim that some iterate $\mathcal{S}^m$ of $\mathcal{S}$ is a contraction. To this end, let $v_i \in L^2(0,T;H)$ be given, and set $u_i := \mathcal{S}[v_i]$ for $i = 1,2$. We immediately have, for every $t \in [0,T]$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u_1(t) - u_2(t)|^2 \le \frac{1}{2}\int_{Q_t} |u_1 - u_2|^2 \\
&\quad + \frac{1}{2}\int_{Q_t} \big|T_\varepsilon(\mu)\,\big(g'(v_1) - g'(v_2)\big) - \big(\gamma_\varepsilon(v_1) - \gamma_\varepsilon(v_2)\big) - \big(B[v_1] - B[v_2]\big)\big|^2.
\end{aligned}$$

Now, we recall that $0 \le T_\varepsilon(\mu) \le 1/\varepsilon$, that g' and γ_ε are Lipschitz continuous, and that (2.9) holds. Then, by using this and applying the Gronwall lemma, we obtain that

$$\|\mathcal{S}[v_1] - \mathcal{S}[v_2]\|^2_{L^\infty(0,t;H)} \le C\|v_1 - v_2\|^2_{L^2(0,t;H)} \quad \text{for every } t \in [0,T], \tag{3.12}$$

where we have marked the constant by using the capital letter C for future use. This inequality holds for every $v_1, v_2 \in L^2(0,T;H)$ and will be applied to different functions. We now aim to show that for arbitrary $v_1, v_2 \in L^2(0,T;H)$ and every positive integer m it holds

$$\|\mathcal{S}^m[v_1] - \mathcal{S}^m[v_2]\|^2_{L^\infty(0,t;H)} \le \frac{C^m t^{m-1}}{(m-1)!}\, \|v_1 - v_2\|^2_{L^2(0,t;H)} \quad \text{for every } t \in [0,T]. \tag{3.13}$$

Indeed, (3.13) with $m = 1$ concides with (3.12). By assuming that $m \ge 1$ and that (3.13) holds, and applying (3.12) to $\mathcal{S}^m[v_i]$ and (3.13) to v_i, we deduce that

$$\begin{aligned}
&\|\mathcal{S}^{m+1}[v_1] - \mathcal{S}^{m+1}[v_2]\|^2_{L^\infty(0,t;H)} = \|\mathcal{S}\,\mathcal{S}^m[v_1] - \mathcal{S}\,\mathcal{S}^m[v_2]\|^2_{L^\infty(0,t;H)} \\
&\le C\|\mathcal{S}^m[v_1] - \mathcal{S}^m[v_2]\|^2_{L^2(0,t;H)} = C\int_0^t \|(\mathcal{S}^m[v_1] - \mathcal{S}^m[v_2])(s)\|^2_H\, ds \\
&\le C\int_0^t \frac{C^m s^{m-1}}{(m-1)!}\, \|v_1 - v_2\|^2_{L^2(0,s;H)}\, ds \le \frac{C^{m+1} t^m}{m!}\, \|v_1 - v_2\|^2_{L^2(0,t;H)}.
\end{aligned}$$

Therefore, (3.13) holds for every m, whence $\mathcal{S}^m$ is a contraction in $L^2(0,T;H)$ if m is large enough. This proves that the approximating problem (3.10) has a unique solution $\rho_\varepsilon \in H^1(0,T;H)$.

Construction of the first map: existence. Next, we will derive some priori estimates and then let ε tend to zero. By testing the equation in (3.10) by ρ_ε, we obtain, for every $t \in [0,T]$,

$$\frac{1}{2}\int_\Omega |\rho_\varepsilon(t)|^2 + \int_{Q_t} \beta_\varepsilon(\rho_\varepsilon)\,\rho_\varepsilon = \frac{1}{2}\int_\Omega |\rho_0|^2 + \int_{Q_t} \big(T_\varepsilon(\mu) g'(\rho_\varepsilon) - \pi(\rho_\varepsilon) - B[\rho_\varepsilon]\big)\rho_\varepsilon\,.$$

The second term on the left-hand side is nonnegative since β_ε is monotone and $\beta_\varepsilon(0) = 0$ due to (2.1). As for the right-hand side, we owe to the definition of T_ε, the Lipschitz continuity of π and (2.8), and see that

$$\int_{Q_t} \big(T_\varepsilon(\mu)g'(\rho_\varepsilon) - \pi(\rho_\varepsilon) - B[\rho_\varepsilon]\big)\rho_\varepsilon \le c \int_{Q_t} \big(1 + |\rho_\varepsilon|^2 + |\mu|^2\big).$$

By applying the Gronwall lemma, we deduce that

$$\|\rho_\varepsilon\|_{L^\infty(0,T;H)} \le c\big(1 + \|\mu\|_{L^2(0,T;H)}\big) \le c\big(1 + \|\mu\|_{\mathcal{M}}\big). \tag{3.14}$$

Furthermore, as $\rho_0 \in V$ and (2.10) holds, one can prove that ρ_ε belongs to $L^2(0, T; V)$, so that (3.10) can be differentiated with respect to the space variables. Let us skip this and just derive a bound. We take the gradient of equation (3.10), multiply the resulting equality by $\nabla\rho_\varepsilon$ and integrate over Q_t. We obtain that

$$\frac{1}{2}\int_\Omega |\nabla\rho_\varepsilon(t)|^2 + \int_{Q_t} \beta_\varepsilon'(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2 = \frac{1}{2}\int_\Omega |\nabla\rho_0|^2$$
$$+ \int_{Q_t} \Big(T_\varepsilon'(\mu)g'(\rho_\varepsilon)\nabla\mu\cdot\nabla\rho_\varepsilon + T_\varepsilon(\mu)g''(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2 - \pi'(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2 - \nabla B[\rho_\varepsilon]\cdot\nabla\rho_\varepsilon\Big).$$

Both integrals on the left-hand side are nonnegative, and the second term in the volume integral on the right-hand side is nonpositive since $\mu \ge 0$ and $g'' \le 0$ (cf. (3.9)). Moreover, $0 \le T_\varepsilon' \le 1$, g' and π' are bounded and (2.10) holds. Hence, with the help of (3.14) we deduce that

$$\int_\Omega |\nabla\rho_\varepsilon(t)|^2 \le c + c\int_{Q_t} \big(1 + |\nabla\rho_\varepsilon|^2 + |\nabla\mu|^2\big).$$

Therefore, by applying the Gronwall lemma, we conclude that

$$\|\nabla\rho_\varepsilon\|_{L^\infty(0,T;H)} \le c\big(1 + \|\nabla\mu\|_{L^2(0,T;H)}\big) \le c\big(1 + \|\mu\|_{\mathcal{M}}\big). \tag{3.15}$$

Next, as $\mu \in L^{10/3}(Q)$, we derive an obvious bound for the family $\{T_\varepsilon(\mu)\}$ in $L^{10/3}(Q)$. Moreover, (3.14)–(3.15) and the embedding (1.20) imply that $\{\rho_\varepsilon\}$ is bounded in the same space, whence the same follows for $\{\pi(\rho_\varepsilon)\}$ and $\{B[\rho_\varepsilon]\}$ (see (2.8)). Since g' is bounded, we thus have

$$\|T_\varepsilon(\mu)g'(\rho_\varepsilon) - \pi(\rho_\varepsilon) - B[\rho_\varepsilon]\|_{L^{10/3}(Q)} \le c\big(1 + \|\mu\|_{\mathcal{M}}\big). \tag{3.16}$$

We term D the right-hand side of (3.16) and set $f_\varepsilon := T_\varepsilon(\mu)g'(\rho_\varepsilon) - \pi(\rho_\varepsilon) - B[\rho_\varepsilon]$, so that (3.16) becomes $\|f_\varepsilon\|_{L^{10/3}(Q)} \le D$.

We can derive a similar estimate for $\{\beta_\varepsilon(\rho_\varepsilon)\}$ using the following strategy. We set $v_\varepsilon := |\beta_\varepsilon(\rho_\varepsilon)|^{7/3} \operatorname{sign}\beta_\varepsilon(\rho_\varepsilon)$ (with $\operatorname{sign} 0 := 0$) and observe that $v_\varepsilon \in L^\infty(0, T; H)$ since β_ε is Lipschitz continuous, $\rho_\varepsilon \in L^\infty(0, T; V)$ and $V \subset L^{14/3}(\Omega)$ by (1.19). Then, we multiply (3.10) by v_ε and integrate over Q. We have

$$\int_\Omega \widetilde{\beta}_\varepsilon(\rho(T)) + \int_Q |\beta_\varepsilon(\rho)|^{10/3} = \int_\Omega \widetilde{\beta}_\varepsilon(\rho_0) + \int_Q f_\varepsilon|\beta_\varepsilon(\rho_\varepsilon)|^{7/3}\operatorname{sign}\beta_\varepsilon(\rho_\varepsilon),$$

where we have set

$$\widetilde{\beta}_\varepsilon(r) := \int_0^r |\beta_\varepsilon(s)|^{7/3}\operatorname{sign}\beta_\varepsilon(s)\, ds \quad \text{for } r \in \mathbb{R}.$$

Note that $\widetilde{\beta}_\varepsilon$ is nonnegative, $|\widetilde{\beta}_\varepsilon(r)| \le |r|\,|\beta_\varepsilon(r)|^{7/3} \le |r|\,|\beta^\circ(r)|^{7/3}$ for every $r \in D(\beta)$ and (2.15) holds. Then, by applying the second Young inequality (1.18) with $\vartheta = 3/10$, we deduce that

$$\int_Q |\beta_\varepsilon(\rho_\varepsilon)|^{10/3} \le \int_\Omega |\rho_0|\,|\beta^\circ(\rho_0)|^{7/3} + \frac{3}{10}\int_Q |f_\varepsilon|^{10/3} + \frac{7}{10}\int_Q |\beta_\varepsilon(\rho_\varepsilon)|^{10/3}$$
$$\le c + \frac{3}{10}\,D^{10/3} + \frac{7}{10}\int_Q |\beta_\varepsilon(\rho_\varepsilon)|^{10/3},$$

whence immediately

$$\int_Q |\beta_\varepsilon(\rho_\varepsilon)|^{10/3} \le c + D^{10/3}.$$

We conclude that

$$\|\beta_\varepsilon(\rho_\varepsilon)\|_{L^{10/3}(Q)} \le c\big(1 + \|\mu\|_{\mathcal{M}}\big)\,. \tag{3.17}$$

By comparison in (3.10) and thanks to our previous estimates, we easily infer that also

$$\|\partial_t\rho_\varepsilon\|_{L^{10/3}(Q)} \le c\big(1 + \|\mu\|_{\mathcal{M}}\big)\,. \tag{3.18}$$

At this point, it is straightforward to deduce that (for a subsequence)

$$\begin{aligned} \rho_\varepsilon \to \rho \qquad & \text{weakly star in } L^\infty(0,T;V) \\ \partial_t\rho_\varepsilon \to \partial_t\rho \qquad & \text{weakly in } L^{10/3}(Q) \\ \beta_\varepsilon(\rho_\varepsilon) \to \xi \qquad & \text{weakly in } L^{10/3}(Q). \end{aligned}$$

Moreover, $\{\rho_\varepsilon\}$ converges to ρ strongly in $C^0([0,T];L^p(\Omega))$ for $p < 6$, due to the compact embedding $V \subset L^p(\Omega)$ (see, e.g., [38, Sect. 8, Cor. 4]). In particular, $\rho(0) = \rho_0$. We also derive that $\{B[\rho_\varepsilon]\}$ converges to $B[\rho]$ strongly in $L^2(0,T;H)$ by (2.9), while $\{g'(\rho_\varepsilon)\}$ and $\{\pi(\rho_\varepsilon)\}$ converge to $g'(\rho)$ and to $\pi(\rho)$, respectively, strongly in $C^0([0,T];L^p(\Omega))$ by Lipschitz continuity.

Next, as $\mu \in L^{10/3}(Q)$, we see that $\{T_\varepsilon(\mu)\}$ converges strongly to μ in $L^q(Q)$ for $q < 10/3$, so that $\{T_\varepsilon(\mu)\,g'(\rho_\varepsilon)\}$ converges to $\mu g'(\rho)$ strongly in $L^2(0,T;H)$. Finally, since $\{\beta_\varepsilon(\rho_\varepsilon)\}$ converges to ξ weakly in $L^2(Q)$ and $\{\rho_\varepsilon\}$ converges to ρ strongly in $L^2(Q)$, we can apply, e.g., [3, Lemma 2.3, p. 38] to conclude that also $\rho \in D(\beta)$ and $\xi \in \beta(\rho)$ a.e. in Q (whence it follows that ρ takes its values in the domain of the original map g (cf. (2.3)). Therefore, (ρ,ξ) is a solution to the Cauchy problem (3.7) with the given μ. Notice that, just by semicontinuity, the a priori estimates (3.14), (3.15), (3.17), (3.18) are conserved in the limit, i.e.,

$$\|\rho\|_{\mathcal{R}} + \|\xi\|_{L^{10/3}(Q)} \le c\big(1 + \|\mu\|_{\mathcal{M}}\big) \quad \text{for every } \mu \in \mathcal{M}_0, \tag{3.19}$$

with obvious definition of $\|\cdot\|_{\mathcal{R}}$ by (3.6).

Construction of the first map: uniqueness. Let (ρ_i,ξ_i), $i = 1,2$, be two solutions to the Cauchy problem (3.7) for the same $\mu \in \mathcal{M}_0$. We write the equation for both of them and multiply the resulting equality by $\rho := \rho_1 - \rho_2$. Then, we integrate over Q_t. We obtain

$$\frac{1}{2}\int_\Omega |\rho(t)|^2 + \int_{Q_t} (\xi_1 - \xi_2)\rho$$
$$= \int_{Q_t} \mu\big(g'(\rho_1) - g'(\rho_2)\big)\rho - \int_{Q_t} \big(\pi(\rho_1) - \pi(\rho_2)\big)\rho - \int_{Q_t} \big(B[\rho_1] - B[\rho_2]\big)\rho.$$

The second integral on the left-hand side is nonnegative by monotonicity. The first one on the right-hand side is nonpositive since $\mu \geq 0$ and g' is nonincreasing by the concavity assumption (2.3) on g. By accounting for the Lipschitz continuity of π and (2.9), and using the Gronwall lemma, we conclude that $\rho = 0$, i.e., $\rho_1 = \rho_2$. By comparison in (3.7), we see that also $\xi_1 = \xi_2$.

At this point, we can define the first map $\mathcal{F}_1 : \mathcal{M}_0 \to \mathcal{R}$ as well as the auxiliary map $\mathcal{G}_1 : \mathcal{M}_0 \to \mathcal{R}$ as follows:

$$\text{for } \mu \in \mathcal{M}_0,\ \mathcal{F}_1(\mu) \text{ and } \mathcal{G}_1(\mu) \text{ are the components } \rho \text{ and } \xi \text{ of the unique solution to (3.7).} \tag{3.20}$$

By the definition of $\mathcal{F}_1$ and $\mathcal{G}_1$, (3.19) yields

$$\|\mathcal{F}_1(\mu)\|_{\mathcal{R}} + \|\mathcal{G}_1(\mu)\|_{L^{10/3}(Q)} \leq c\left(1 + \|\mu\|_{\mathcal{M}}\right) \leq c\,(1 + M_0) \quad \text{for every } \mu \in \mathcal{M}_0. \tag{3.21}$$

Construction of the second map: existence. Now, for a given $\rho \in \mathcal{R}$, we would like to consider the initial–boundary value problem given by (2.20), (2.22) and the first initial condition in (2.23). However, the terms $g(\rho)$ and $g'(\rho)$ might be meaningless since g is not necessarily everywhere defined (cf. (2.3)). Hence, we suitably extend g (in a different way with respect to the temporary (3.9), despite of the notation we are going to use) to a C^1 function defined in the whole real line $\mathbb{R}$ by preserving some of the properties postulated in (2.3). Namely, still writing g for this new extension for the remainder of the present section, we require that

$$g \text{ and } g' \text{ are bounded and Lipschitz continuous} \tag{3.22}$$
$$g(r) \geq -1/3, \quad \text{i.e.,} \quad 1 + 2g(r) \geq 1/3, \quad \text{for every } r \in \mathbb{R}. \tag{3.23}$$

Thus, the problem we consider is

$$\begin{aligned} \left(1 + 2g(\rho)\right) \partial_t \mu + \mu\, g'(\rho)\, \partial_t \rho - \Delta \mu = 0 \quad \text{a.e. in } Q, \\ \partial_\nu \mu = 0 \quad \text{a.e. on } \Sigma, \quad \mu(0) = \mu_0\,. \end{aligned} \tag{3.24}$$

The equation in (3.24) is linear, but its coefficients are not smooth. Therefore, we regularize them by introducing ρ_ε as smooth as needed and satisfying

$$\rho_\varepsilon \to \rho \qquad \text{strongly in } H^1(0,T;H) \text{ and weakly star in } L^\infty(0,T;V) \tag{3.25}$$
$$\partial_t \rho_\varepsilon \to \partial_t \rho \qquad \text{strongly in } L^{10/3}(Q). \tag{3.26}$$

Without loss of generality, we can also assume that

$$\|\rho_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq 1 + \|\rho\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \tag{3.27}$$
$$\|\partial_t \rho_\varepsilon\|_{L^{10/3}(Q)} \leq 1 + \|\partial_t \rho\|_{L^{10/3}(Q)}\,. \tag{3.28}$$

The approximating problem to be considered is then

$$\begin{aligned} \left(1 + 2g(\rho_\varepsilon)\right) \partial_t \mu_\varepsilon + \mu_\varepsilon\, g'(\rho_\varepsilon)\, \partial_t \rho_\varepsilon - \Delta \mu_\varepsilon = 0 \quad \text{a.e. in } Q, \\ \partial_\nu \mu_\varepsilon = 0 \quad \text{a.e. on } \Sigma, \quad \mu_\varepsilon(0) = \mu_0\,. \end{aligned} \tag{3.29}$$

It has a unique solution $\mu_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$ (for the definition of W, see (2.4)), thanks to the regularity of the coefficients and the uniform parabolicity ensured by (3.23). Moreover, the solution is nonnegative. Indeed, by testing the equation by $-2\mu_\varepsilon^-$, where μ_ε^- is the negative part of μ_ε, and using the identity

$$\big((1+2g(\rho_\varepsilon))\,\partial_t\mu_\varepsilon + \mu\, g'(\rho_\varepsilon)\,\partial_t\rho_\varepsilon\big)(-2\mu_\varepsilon^-) = \partial_t\big(1+2g(\rho_\varepsilon))|\mu_\varepsilon^-|^2\big),$$

we immediately obtain that

$$\int_\Omega \big(1+2g(\rho_\varepsilon(t))\big)|\mu_\varepsilon^-(t)|^2 + \int_{Q_t} |\nabla\mu_\varepsilon^-|^2 = 0\,,$$

whence (cf. (3.23)) we conclude that $\mu_\varepsilon^- = 0$, i.e., $\mu_\varepsilon \geq 0$.

At this point, we perform the estimate that formally led to (3.2) and was based on the inequality (3.23). Since here the argument uses μ_ε and ρ_ε, the calculation is completely justified. Hence, we obtain (cf. (3.3))

$$\|\mu_\varepsilon\|_{L^{10/3}(Q)\cap L^2(0,T;V)} \leq M_0\,. \tag{3.30}$$

Now, we estimate some norms of μ_ε in terms of suitable norms of ρ_ε. The symbol Φ denotes possibly different continuous functions, as explained at the end of Section 1. First, we test the equation in (3.29) by $\partial_t\mu_\varepsilon$. By accounting for the boundedness of g' and owing to the Hölder, Young and Sobolev inequalities, we have

$$\begin{aligned}
&\int_{Q_t} \big(1+2g(\rho_\varepsilon)\big)|\partial_t\mu_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla\mu_\varepsilon(t)|^2 \\
&\leq \frac{1}{2}\int_\Omega |\nabla\mu_0|^2 + c\int_{Q_t} \mu_\varepsilon\,|\partial_t\rho_\varepsilon|\,|\partial_t\mu_\varepsilon| \\
&\leq c + c\int_0^t \|\mu_\varepsilon(s)\|_6\,\|\partial_t\rho_\varepsilon(s)\|_3\,\|\partial_t\mu_\varepsilon(s)\|_2\,ds \\
&\leq c + \frac{1}{2}\int_{Q_t} |\partial_t\mu_\varepsilon|^2 + c\int_0^t \|\partial_t\rho_\varepsilon(s)\|_3^2\,\|\mu_\varepsilon(s)\|_V^2\,ds.
\end{aligned}$$

At this point, we recall (3.23) once more and observe that (3.28) implies

$$\int_0^T \|\partial_t\rho_\varepsilon(s)\|_3^2\,ds \leq c\,\|\partial_t\rho_\varepsilon\|_{L^{10/3}(Q)}^2 \leq c\big(1+\|\partial_t\rho\|_{L^{10/3}(Q)}^2\big).$$

Therefore, we can apply the Gronwall lemma and conclude that (see (3.6))

$$\|\mu_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq \Phi\big(\|\rho\|_{\mathcal{R}}\big). \tag{3.31}$$

Next, we estimate the first two terms of (3.29) by accounting for the Lipschitz continuity of g and g'. We also use the Hölder and Sobolev inequalities and owe to (3.27)–(3.28) and (3.31). We easily see that

$$\begin{aligned}
&\|(1+2g(\rho_\varepsilon))\,\partial_t\mu_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} \;\leq\; c\,\|(1+|\rho_\varepsilon|)\,\partial_t\mu_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} \\
&\leq\; c\,\big(1+\|\rho_\varepsilon\|_{L^\infty(0,T;L^6(\Omega))}\big)\,\|\partial_t\mu_\varepsilon\|_{L^2(0,T;H)} \;\leq\; \Phi\big(\|\rho\|_{\mathcal{R}}\big)\,,
\end{aligned}$$

$$\begin{aligned}
&\|g'(\rho_\varepsilon)\,\mu_\varepsilon\,\partial_t\rho_\varepsilon\|_{L^{10/3}(0,T;L^{15/7}(\Omega))} \;\leq c\,\|\mu_\varepsilon\,\partial_t\rho_\varepsilon\|_{L^{10/3}(0,T;L^{15/7}(\Omega))} \\
&\leq c\,\|\mu_\varepsilon\|_{L^\infty(0,T;L^6(\Omega))}\,\|\partial_t\rho_\varepsilon\|_{L^{10/3}(0,T;L^{10/3}(\Omega))} \;\leq\; \Phi\big(\|\rho\|_{\mathcal{R}}\big)\,.
\end{aligned}$$

As $10/3 > 2$ and $15/7 > 3/2$, by comparing the terms of the equation in (3.29), we deduce a similar bound for $\Delta\mu_\varepsilon$ in $L^2(0,T;L^{3/2}(\Omega))$, whence immediately

$$\|\mu_\varepsilon\|_{L^2(0,T;W^{2,3/2}(\Omega))} \le \Phi\big(\|\rho\|_{\mathcal{R}}\big) \tag{3.32}$$

by elliptic regularity. At this point, it is straightforward to see that we can let ε tend to zero to obtain a solution μ to the problem (3.24). Moreover, all of the uniform estimates shown above are preserved in the limit, so that we have

$$\mu \in \mathcal{M}_0 \quad \text{and} \quad \|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W^{2,3/2}(\Omega))} \le \Phi\big(\|\rho\|_{\mathcal{R}}\big). \tag{3.33}$$

Construction of the second map: uniqueness. Next, we prove that, for a given $\rho \in \mathcal{R}$, the solution μ to (3.24) is unique. We pick two solutions μ_i, $i = 1,2$, write the equation of (3.24) for both of them, multiply the difference by $\mu := \mu_1 - \mu_2$ and integrate over Q_t. Then, the identity (3.1) holds true for μ, and we have

$$\int_\Omega \big(1 + 2g(\rho(t))\big)\, |\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2 = 0.$$

Thus, by (3.23) we conclude that $\mu_1 = \mu_2$.

At this point, we can recall (3.33) and define $\mathcal{F}_2 : \mathcal{R} \to \mathcal{M}_0$ as follows:

$$\textit{for } \rho \in \mathcal{R},\ \mathcal{F}_2(\rho) \ \textit{ is the unique solution } \mu \textit{ to } (3.24). \tag{3.34}$$

We then define $\mathcal{F}$ by:

$$\mathcal{F} : \mathcal{M}_0 \to \mathcal{M}_0 \quad \textit{is given by} \quad \mathcal{F} := \mathcal{F}_2 \circ \mathcal{F}_1 \,. \tag{3.35}$$

The fixed point argument. We want to apply Tikhonov's fixed point theorem to $\mathcal{F}$. To this end, we observe that the Banach space $\mathcal{M}$ is both reflexive and separable and that $\mathcal{M}_0$ is a nonempty, bounded, closed and convex subset of $\mathcal{M}$. Hence, if we endow $\mathcal{M}$ with its weak topology, then $\mathcal{M}_0$ is compact, and the topology induced on it by the weak topology of $\mathcal{M}$ is associated to a metric. Therefore, in order to apply Tikhonov's theorem, we only need to show that $\mathcal{F}$ is sequentially continuous with respect to the weak topology of $\mathcal{M}$. This is equivalent to showing that, for every $\overline{\mu} \in \mathcal{M}_0$ and every sequence $\{\overline{\mu}_n\}$ of elements of $\mathcal{M}_0$ converging to $\overline{\mu}$ weakly in $\mathcal{M}$, the sequence $\{\mathcal{F}(\overline{\mu}_n)\}$ converges to $\mathcal{F}(\overline{\mu})$ weakly in $\mathcal{M}$.

To this end, let $\overline{\mu}_n, \overline{\mu} \in \mathcal{M}_0$ be such that $\overline{\mu}_n \to \overline{\mu}$ weakly in $\mathcal{M}$, and set $\rho_n := \mathcal{F}_1(\overline{\mu}_n)$, $\xi_n := \mathcal{G}_1(\overline{\mu}_n)$, and $\mu_n := \mathcal{F}(\overline{\mu}_n) = \mathcal{F}_2(\rho_n)$. Thus, we have

$$\begin{aligned} &\partial_t\rho_n + \xi_n + \pi(\rho_n) + B[\rho_n] = \overline{\mu}_n\, g'(\rho_n) \\ &\quad \text{and} \quad \xi_n \in \beta(\rho_n) \quad \text{a.e. in } Q, \quad \rho_n(0) = \rho_0 \end{aligned} \tag{3.36}$$

and we observe that the estimate (3.19) for ρ_n and ξ_n becomes

$$\|\rho_n\|_{\mathcal{R}} + \|\xi_n\|_{L^{10/3}(Q)} \le c\big(1 + \|\overline{\mu}_n\|_{\mathcal{M}}\big) \le c\,.$$

Therefore, we have

$$\rho_n \to \rho \qquad \text{weakly star in } \mathcal{R} \text{ and strongly in } C^0([0,T];H) \tag{3.37}$$

$$\xi_n \to \xi \qquad \text{weakly in } L^{10/3}(Q) \tag{3.38}$$

for some ρ and ξ in the above spaces, at least for a subsequence (which is still indexed by $n \in \mathbb{N}$). Now, we show that $\rho = \mathcal{F}_1(\overline{\mu})$ and $\xi = \mathcal{G}_1(\overline{\mu})$, i.e.,

$$\partial_t \rho + \xi + \pi(\rho) + B[\rho] = \overline{\mu}\, g'(\rho) \quad \text{and} \quad \xi \in \beta(\rho) \quad \text{a.e. in } Q, \quad \rho(0) = \rho_0 \,. \tag{3.39}$$

Indeed, the above strong convergence for $\{\rho_n\}$ implies both the Cauchy condition $\rho(0) = \rho_0$ and the strong convergence in $L^2(Q)$ of $\{\pi(\rho_n)\}$ and $\{B[\rho_n]\}$ to $\pi(\rho)$ and $B[\rho]$, respectively, thanks to assumptions (2.2) and (2.9). Furthermore, we also have $\xi \in \beta(\rho)$ by, e.g., [3, Lemma 2.3, p. 38]. Finally, $\{g'(\rho_n)\}$ converges to $g'(\rho)$ strongly in $C^0([0,T];H)$, and $\{\overline{\mu}_n\}$ converges to $\overline{\mu}$ weakly in $L^2(0,T;H)$, whence it readily follows that $\{\overline{\mu}_n\, g'(\rho_n)\}$ converges to $\overline{\mu}\, g'(\rho)$ weakly in $L^1(Q)$. Therefore, the pair (ρ, ξ) is the unique solution to (3.39), and thus $\rho = \mathcal{F}_1(\overline{\mu})$. Moreover, the limit of $\{\rho_n\}$ is uniquely determined, from which we may conclude that all of the above convergences, in particular (3.37) and (3.38), which were initially proved to be valid only for suitable subsequences, hold in fact true for the entire sequences.

At this point, by setting for convenience $\mu_n := \mathcal{F}_2(\rho_n)$, we have

$$\begin{aligned} \big(1 + 2g(\rho_n)\big)\, \partial_t \mu_n + \mu_n\, g'(\rho_n)\, \partial_t \rho_n - \Delta \mu_n = 0 \quad \text{a.e. in } Q, \\ \partial_\nu \mu_n = 0 \quad \text{a.e. on } \Sigma, \quad \mu_n(0) = \mu_0 \end{aligned} \tag{3.40}$$

and (3.33) for μ_n becomes

$$\mu_n \in \mathcal{M}_0 \quad \text{and} \quad \|\mu_n\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,3/2}(\Omega))} \le \Phi\big(\|\rho_n\|_{\mathcal{R}}\big).$$

As $\{\rho_n\}$ converges to ρ weakly star in $\mathcal{R}$, $\{\mu_n\}$ is bounded in the above norm. Thus, we have

$$\mu_n \to \mu \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,3/2}(\Omega)) \tag{3.41}$$

for some $\mu \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W^{2,3/2}(\Omega))$, at least for a subsequence (which is still indexed by $n \in \mathbb{N}$). We prove that $\mu = \mathcal{F}_2(\rho)$, i.e., μ solves (3.24). Indeed, since $\{\rho_n\}$ converges to ρ strongly in $C^0([0,T];H)$, $\{g(\rho_n)\}$ and $\{g'(\rho_n)\}$ converge in the same space to $g(\rho)$ and $g'(\rho)$, respectively, just by Lipschitz continuity. Furthermore, $\{\partial_t \mu_n\}$ and $\{\partial_t \rho_n\}$ converge to $\partial_t \mu$ and $\partial_t \rho$ at least weakly in $L^2(0,T;H)$. Hence, we can pass to the limit in the equation of (3.40) and deduce the first equality in (3.24). On the other hand, it is clear that both the boundary condition and the initial condition in (3.24) follow from the convergence of $\{\mu_n\}$ to μ. We conclude that $\mu = \mathcal{F}_2(\rho)$, that is, μ is the unique solution to (3.24). In view of the uniqueness, we may infer that (3.41) holds true for the entire sequence.

Finally, we recall that $\rho = \mathcal{F}_1(\overline{\mu})$. Hence, we have proved that $\mu = \mathcal{F}(\overline{\mu})$. In conclusion, Tikhonov's theorem can be applied, and $\mathcal{F}$ has at least one fixed point $\mu \in \mathcal{M}_0$. If we consider any such μ and the corresponding pair (ρ, ξ) given by $\rho := \mathcal{F}_1(\mu)$ and $\xi := \mathcal{G}_1(\mu)$, then the estimates (3.19) and (3.33) are valid, so that the triplet (μ, ρ, ξ) is a solution to problem (2.20)–(2.23) satisfying the regularity conditions (2.16)–(2.19).

4 Uniqueness and regularity

In this section, we prove Theorem 2.2. More precisely, we first derive (2.25) for every solution and then show that the solution is unique.

So, we fix any solution (μ, ρ, ξ) to problem (2.20)–(2.23) satisfying (2.16)–(2.19). In order to prove the regularity part of the statement, we would like to test (2.21) by ξ^5. As no further summability of ξ besides (2.19) is known, we approximate ρ and ξ as follows.

First auxiliary problem. We observe that $B[\rho] \in L^2(Q)$ and consider the problem of finding $(\overline{\rho}, \overline{\xi})$ satisfying (2.18)–(2.19) and

$$\partial_t \overline{\rho} + \overline{\xi} - \pi(\overline{\rho}) - \mu g'(\overline{\rho}) = -B[\rho] \quad \text{and} \quad \overline{\xi} \in \beta(\overline{\rho}) \quad \text{a.e. in } Q \tag{4.1}$$

$$\overline{\rho}(0) = \rho_0 \,. \tag{4.2}$$

Obviously, (ρ, ξ) is a solution satisfying the regularity conditions (2.18)–(2.19). We claim that there cannot exist another such solution. To this end, let (ρ_i, ξ_i), $i = 1, 2$, be two solutions satisfying (2.18)–(2.19). We write (4.1) for both of them, multiply the difference by $\rho_1 - \rho_2$, and integrate over Q_t to obtain

$$\frac{1}{2} \int_\Omega |\rho_1(t) - \rho_2(t)|^2 + \int_{Q_t} (\xi_1 - \xi_2)(\rho_1 - \rho_2) + \int_{Q_t} (-\mu)\big(g'(\rho_1) - g'(\rho_2)\big)(\rho_1 - \rho_2)$$
$$= - \int_{Q_t} \big(\pi(\rho_1) - \pi(\rho_2)\big)(\rho_1 - \rho_2) \,.$$

The second and third integrals on the left-hand side are nonnegative since β is monotone, μ is nonnegative, and g' is nonincreasing (see (2.3)). Thus, by accounting for the Lipschitz continuity of π and applying Gronwall's lemma, we obtain $\rho_1 = \rho_2$, which proves the claim.

Second auxiliary problem. Now, we choose $\mu_\varepsilon \in L^\infty(0,T;V) \cap L^\infty(Q)$ with $\mu_\varepsilon \geq 0$ such that

$$\mu_\varepsilon \to \mu \quad \text{strongly in } L^\infty(0,T;V) \tag{4.3}$$

and consider the Cauchy problem

$$\partial_t \rho_\varepsilon + \beta_\varepsilon(\rho_\varepsilon) + \pi(\rho_\varepsilon) - \mu_\varepsilon g'(\rho_\varepsilon) = -B[\rho] \quad \text{a.e. in } Q \quad \text{and} \quad \rho_\varepsilon(0) = \rho_0, \tag{4.4}$$

where β_ε is the Yosida regularization of β and where g denotes the extension of g to the whole real line $\mathbb{R}$ which was introduced in Section 3 and has the properties listed in (3.9).

Since all of the nonlinearities on the left-hand side are Lipschitz continuous (uniformly with respect to both space and time, since μ_ε is bounded) and $B[\rho] \in L^2(Q)$, problem (4.4) has a unique solution $\rho_\varepsilon \in H^1(0,T;H)$. Moreover, since $B[\rho] \in L^2(0,T;V)$ by (2.10), one can easily prove that ρ_ε, $\partial_t \rho_\varepsilon \in L^2(0,T;V)$ and that the equations can be differentiated with respect to the space variables. Thus, we can argue as we did for the proof of (3.15) (in particular, using $\mu_\varepsilon \geq 0$ and $g'' \leq 0$) and derive a bound for the family $\{\rho_\varepsilon\}$ in $L^\infty(0,T;V)$. At this point, it is straightforward to show that $\{(\rho_\varepsilon, \beta_\varepsilon(\rho_\varepsilon))\}$ converges to some $(\overline{\rho}, \overline{\xi})$ weakly in $H^1(0,T;H) \times L^2(0,T;H)$ (as ε tends to zero, at least for a subsequence) and that $(\overline{\rho}, \overline{\xi})$ is a solution to problem (4.1)–(4.2). But, as shown in the previous step, (ρ, ξ) is the unique solution to this problem. Therefore, we have proved that

$$\big(\rho_\varepsilon, \beta_\varepsilon(\rho_\varepsilon)\big) \to (\rho, \xi) \quad \text{weakly in } H^1(0,T;H) \times L^2(0,T;H) \tag{4.5}$$

and that the convergence holds true for the whole family.

Regularity. Next, we prove that $\xi \in L^6(Q)$ and $\partial_t\rho \in L^6(Q)$. To this end, we consider the solution ρ_ε to (4.4) and first show that the family $\{\xi_\varepsilon := \beta_\varepsilon(\rho_\varepsilon)\}$ is bounded in $L^6(Q)$. We write the equation in (4.4) in the form

$$\partial_t\rho_\varepsilon + \xi_\varepsilon = f_\varepsilon := \mu_\varepsilon g'(\rho_\varepsilon) - \pi(\rho_\varepsilon) - B[\rho] \quad \text{and} \quad \xi_\varepsilon = \beta_\varepsilon(\rho_\varepsilon). \tag{4.6}$$

By (4.3) and the Sobolev inequality, $\{\mu_\varepsilon\}$ is bounded in $L^\infty(0,T;L^6(\Omega))$ and thus also in $L^6(Q)$. Since g' is bounded, also $\{\mu_\varepsilon g'(\rho_\varepsilon)\}$ is bounded in $L^6(Q)$. Moreover, $\{\pi(\rho_\varepsilon)\}$ is bounded in $L^6(Q)$, since π is Lipschitz continuous and $\{\rho_\epsilon\}$ is known to be bounded in $L^\infty(0,T;V)$. Finally, as (2.8) holds with $p = 6$, we derive that $B[\rho] \in L^6(Q)$. Thus, $f_\varepsilon \in L^6(Q)$ and $\{f_\varepsilon\}$ is bounded in $L^6(Q)$. We skip the simple proof that $\xi_\varepsilon \in L^6(Q)$ for $\varepsilon > 0$ and just derive the bound we are interested in. We multiply (4.6) by $\xi_\varepsilon^5 \in L^{6/5}(Q)$ and integrate over Q. By noting that $\partial_t\rho_\varepsilon\, \xi_\varepsilon^5 = \partial_t\widetilde{\beta}_\varepsilon(\rho_\varepsilon)$, where

$$\widetilde{\beta}_\varepsilon(r) := \int_0^r \big(\beta_\varepsilon(s)\big)^5\, ds \quad \text{for } r \in \mathbb{R},$$

we obtain

$$\int_\Omega \widetilde{\beta}_\varepsilon(\rho(T)) + \int_Q \xi_\varepsilon^6 = \int_\Omega \widetilde{\beta}_\varepsilon(\rho_0) + \int_Q f_\varepsilon\, \xi_\varepsilon^5 \le \int_\Omega |\rho_0|\, |\beta_\varepsilon(\rho_0)|^5 + \int_Q |f_\varepsilon|\, |\xi_\varepsilon|^5.$$

As $\widetilde{\beta}_\varepsilon$ is nonnegative by (2.1), $|\beta_\varepsilon(r)| \le |\beta^\circ(r)|$ for every $r \in D(\beta)$ (see, e.g., [6, p. 28]), and thanks to the second condition in (2.24), we can owe to the Hölder and Young inequalities in the last term and deduce that $\{\xi_\varepsilon\}$ is bounded in $L^6(Q)$. By comparison in (4.6), it turns out that also $\{\partial_t\rho_\varepsilon\}$ is bounded in $L^6(Q)$. On account of (4.5), we deduce that ξ and $\partial_t\rho$ belong to $L^6(Q)$, i.e., the second and third assertions in (2.25) are proved.

In order to complete the proof of (2.25), we observe that

$$\partial_t\rho \in L^{7/3}(0,T;L^{14/3}(\Omega)),$$

thus we can account for the assumption $\mu_0 \in L^\infty(\Omega)$ to infer that $\mu \in L^\infty(Q)$, i.e., the validity of the first assertion in (2.25), by repeating the argument developed in the proof of [14, Thm. 2.3], which is based on the above summability of $\partial_t\rho$. We should remark that the quoted proof is performed with $g(r) = r$; however, only minor changes are sufficient to arrive at the same conclusion in the present situation (see also the proof of the analogous [19, Thm. 3.7] in an even more complicated case).

Uniqueness. We closely follow the proof of [12, Thm. 2.6] and adapt the argument developed there to our situation, also giving the details for the reader's convenience. Indeed, on the one hand, some of the estimates have to be changed due to the presence of the nonlocal operator B; on the other hand, it has to be clear that the further assumptions that were made in [12] in order to prove a more complicated statement are not used here.

To begin with, we pick two solutions (μ_i, ρ_i, ξ_i), $i = 1, 2$, recalling that $\mu_i \in L^\infty(Q)$ by the above proof. We write (2.21) for both of them in the form

$$\partial_t\rho_i + \xi_i = w_i, \tag{4.7}$$

where

$$w_i := \mu_i\, g'(\rho_i) - \pi(\rho_i) - B[\rho_i]. \tag{4.8}$$

We infer that

$$(\partial_t\rho_1 - \partial_t\rho_2) + (\xi_1 - \xi_2) = w_1 - w_2 \quad \text{a.e. in } Q \tag{4.9}$$

$$\partial_t|\rho_1 - \rho_2| + |\xi_1 - \xi_2| \le |w_1 - w_2| \quad \text{a.e. in } Q\,. \tag{4.10}$$

The equality (4.9) is an obvious consequence of (4.7), while (4.10) can be proved by pointwise multiplication of (4.9) by $\operatorname{sign}(\xi_1 - \xi_2)$ in the set where $\xi_1 \ne \xi_2$ (since either $\rho_1 \ne \rho_2$ and $\operatorname{sign}(\rho_1 - \rho_2) = \operatorname{sign}(\xi_1 - \xi_2)$ or $\partial_t\rho_1 = \partial_t\rho_2$) and by $\operatorname{sign}(\rho_1 - \rho_2)$ (with $\operatorname{sign} 0 = 0$) in the set where $\xi_1 = \xi_2$. From (4.9) we obtain that for a.a. $(x,t) \in Q$ it holds (where we avoid writing the x variable for brevity)

$$\int_0^t |\partial_t\rho_1(s) - \partial_t\rho_2(s)|\,ds \le \int_0^t |\xi_1(s) - \xi_2(s)|\,ds + \int_0^t |w_1(s) - w_2(s)|\,ds\,,$$

while (4.10) yields that

$$|\rho_1(t) - \rho_2(t)| + \int_0^t |\xi_1(s) - \xi_2(s)|\,ds \le \int_0^t |w_1(s) - w_2(s)|\,ds.$$

By addition, we deduce that

$$|\rho_1(t) - \rho_2(t)| + \int_0^t |\partial_t\rho_1(s) - \partial_t\rho_2(s)|\,ds \le 2\int_0^t |w_1(s) - w_2(s)|\,ds.$$

At this point, we recall (4.8) and infer that

$$|\rho_1(t) - \rho_2(t)| + \int_0^t |\partial_t\rho_1(s) - \partial_t\rho_2(s)|\,ds \le c\int_0^t f(s)\,ds\,, \quad \text{where}$$
$$f := |\rho_1 - \rho_2| + |\mu_1 - \mu_2| + |B[\rho_1] - B[\rho_2]|\,. \tag{4.11}$$

Here, and in the remainder of the proof, c depends also on $\|\mu_i\|_{L^\infty(Q)},\ i = 1,2$. We deduce that

$$|\rho_1(t) - \rho_2(t)|^2 \le c\Big|\int_0^t f(s)\,ds\Big|^2 \quad \text{and} \quad \Big|\int_0^t |\partial_t\rho_1(s) - \partial_t\rho_2(s)|\,ds\Big|^2 \le c\Big|\int_0^t f(s)\,ds\Big|^2,$$

whence also (by integrating over Ω and using Schwarz's inequality)

$$\int_\Omega |\rho_1(t) - \rho_2(t)|^2 \le c\int_{Q_t} |f|^2 \quad \text{and} \quad \int_\Omega \Big|\int_0^t |\partial_t\rho_1(s) - \partial_t\rho_2(s)|\,ds\Big|^2 \le c\int_{Q_t} |f|^2.$$

Now, we have that

$$\int_{Q_t} |f|^2 \le c\int_{Q_t} \big(|\rho_1 - \rho_2|^2 + |\mu_1 - \mu_2|^2\big)\,,$$

by the definition of f and (2.9). Therefore, we conclude that

$$\int_\Omega |\rho_1(t) - \rho_2(t)|^2 \le D\int_{Q_t} \big(|\rho_1 - \rho_2|^2 + |\mu_1 - \mu_2|^2\big) \tag{4.12}$$

$$\int_\Omega \Big|\int_0^t |\partial_t\rho_1(s) - \partial_t\rho_2(s)|\,ds\Big|^2 \le c\int_{Q_t} \big(|\rho_1 - \rho_2|^2 + |\mu_1 - \mu_2|^2\big)\,, \tag{4.13}$$

where we have marked the constant in (4.12) for future use by using the capital letter D.

At this point, we turn our interest to the first equation of our system. We write it in the form

$$\partial_t u_i - \Delta\mu_i = \mu_i g'(\rho_i)\partial_t\rho_i, \tag{4.14}$$

where $u_i := \big(1+2g(\rho_i)\big)\mu_i$, for $i=1,2$. Then, take the difference and integrate with respect to time. With the general notation

$$(1*v)(t) := \int_0^t v(s)\,ds\,, \quad t\in[0,T],$$

we have

$$(u_1-u_2) - 1*\Delta(\mu_1-\mu_2) = 1*\big(\mu_1 g'(\rho_1)\partial_t\rho_1 - \mu_2 g'(\rho_2)\partial_t\rho_2\big)\,. \tag{4.15}$$

Then, we multiply (4.15) by $\mu_1-\mu_2$ and integrate over Q_t. The contribution arising from the Laplacian is nonnegative. Now, we owe to the assumptions (2.3) on g' and to Young's inequality to obtain that

$$(u_1-u_2)(\mu_1-\mu_2) \geq \frac{1}{2}\,|\mu_1-\mu_2|^2 - c\,|\rho_1-\rho_2|^2.$$

Hence, we have

$$\begin{aligned}
&\frac{1}{2}\int_{Q_t}|\mu_1-\mu_2|^2 \\
&\leq c\int_{Q_t}|\rho_1-\rho_2|^2 + \int_{Q_t}\big(1*\big(\mu_1 g'(\rho_1)\partial_t\rho_1 - \mu_2 g'(\rho_2)\partial_t\rho_2\big)\big)(\mu_1-\mu_2) \\
&\leq c\int_{Q_t}|\rho_1-\rho_2|^2 + \frac{1}{4}\int_{Q_t}|\mu_1-\mu_2|^2 \\
&\quad + c\int_{Q_t}\Big(\int_0^s\big(|\mu_1-\mu_2| + |\rho_1-\rho_2| + |\partial_t\rho_1-\partial_t\rho_2|\big)(\tau)\,d\tau\Big)^2 \\
&\leq c\int_{Q_t}|\rho_1-\rho_2|^2 + \frac{1}{4}\int_{Q_t}|\mu_1-\mu_2|^2 \\
&\quad + c\int_{Q_t}\int_0^s|(\mu_1-\mu_2)(\tau)|^2\,d\tau + c\int_{Q_t}\Big|\int_0^s|\partial_t\rho_1-\partial_t\rho_2|(\tau)\,d\tau\Big|^2 \\
&= c\int_{Q_t}|\rho_1-\rho_2|^2 + \frac{1}{4}\int_{Q_t}|\mu_1-\mu_2|^2 \\
&\quad + c\int_0^t\Big(\int_{Q_s}|(\mu_1-\mu_2)|^2\Big)\,ds + c\int_{Q_t}\Big|\int_0^s|\partial_t\rho_1-\partial_t\rho_2|(\tau)\,d\tau\Big|^2.
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
&\int_{Q_t}|\mu_1-\mu_2|^2 \leq c\int_{Q_t}|\rho_1-\rho_2|^2 \\
&\quad + c\int_0^t\Big(\int_{Q_s}|(\mu_1-\mu_2)|^2\Big)\,ds + c\int_{Q_t}\Big|\int_0^s|\partial_t\rho_1-\partial_t\rho_2|(\tau)\,d\tau\Big|^2.
\end{aligned}$$

On the other hand, an integration of (4.13) over $(0,t)$ yields the estimate

$$\int_{Q_t} \Big| \int_0^s |\partial_t \rho_1(s) - \partial_t \rho_2(\tau)| \, d\tau \Big|^2 \le c \int_0^t \int_{Q_s} \big(|\rho_1 - \rho_2|^2 + |\mu_1 - \mu_2|^2 \big)$$
$$\le c \int_{Q_t} |\rho_1 - \rho_2|^2 + c \int_0^t \Big(\int_{Q_s} |\mu_1 - \mu_2|^2 \Big) \, ds.$$

Hence, we obtain that

$$(D+1) \int_{Q_t} |\mu_1 - \mu_2|^2 \le c \int_{Q_t} |\rho_1 - \rho_2|^2 + c \int_0^t \Big(\int_{Q_s} |(\mu_1 - \mu_2)|^2 \Big) \, ds \,, \tag{4.16}$$

where D is the constant appearing in (4.12). At this point, we take the sum of (4.16) and (4.12) to arrive at the estimate

$$\int_\Omega |\rho_1(t) - \rho_2(t)|^2 + \int_{Q_t} |\mu_1 - \mu_2|^2 \le c \int_{Q_t} |\rho_1 - \rho_2|^2 + c \int_0^t \Big(\int_{Q_s} |(\mu_1 - \mu_2)|^2 \Big) \, ds.$$

Applying Gronwall's lemma, we conclude that $\rho_1 = \rho_2$ and $\mu_1 = \mu_2$. Then, a comparison in (4.7) yields $\xi_1 = \xi_2$, and the proof is complete.

References

[1] H. Abels, S. Bosia, M. Grasselli, Cahn–Hilliard equation with nonlocal singular free energies, *Ann. Mat. Pura Appl. (4)* **194** (2015) 1071-1106.

[2] H. W. Alt, "Lineare Funktionalanalysis: Eine anwendungsorientierte Einführung", Springer-Verlag, Berlin–Heidelberg, 1985.

[3] V. Barbu, "Nonlinear semigroups and differential equations in Banach spaces", Noordhoff, Leyden, 1976.

[4] P. W. Bates, J. Han, The Neumann boundary problem for a nonlocal Cahn-Hilliard equation, *J. Differential Equations* **212** (2005) 235-277.

[5] P. W. Bates, J. Han, The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation, *J. Math. Anal. Appl.* **311** (2005) 289-312.

[6] H. Brezis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert", North-Holland Math. Stud. **5**, North-Holland, Amsterdam, 1973.

[7] M. Brokate, J. Sprekels, "Hysteresis and phase transitions", Applied Mathematical Sciences **121**, Springer, New York, 1996.

[8] J. W. Cahn, J. E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, *J. Chem. Phys.* **2** (1958) 258-267.

[9] C. K. Chen, P. C. Fife, Nonlocal models of phase transitions in solids, *Adv. Math. Sci. Appl.* **10** (2000) 821-849.

[10] P. Colli, S. Frigeri, M. Grasselli, Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system, *J. Math. Anal. Appl.* 386 (2012) 428-444.

[11] P. Colli, G. Gilardi, P. Krejčí, P. Podio-Guidugli, J. Sprekels, Analysis of a time discretization scheme for a nonstandard viscous Cahn-Hilliard system, *ESAIM Math. Model. Numer. Anal.* **48** (2014) 1061-1087.

[12] P. Colli, G. Gilardi, P. Krejčí, J. Sprekels, A vanishing diffusion limit in a nonstandard system of phase field equations, *Evol. Equ. Control Theory* **3** (2014) 257-275.

[13] P. Colli, G. Gilardi, P. Krejčí, J. Sprekels, A continuous dependence result for a nonstandard system of phase field equations, *Math. Methods Appl. Sci.* **37** (2014) 1318-1324.

[14] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, Well-posedness and long-time behaviour for a nonstandard viscous Cahn-Hilliard system, *SIAM J. Appl. Math.* **71** (2011) 1849-1870.

[15] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, Global existence for a strongly coupled Cahn-Hilliard system with viscosity, *Boll. Unione Mat. Ital. (9)* **5** (2012) 495-513.

[16] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, Distributed optimal control of a nonstandard system of phase field equations, *Contin. Mech. Thermodyn.* **24** (2012) 437-459.

[17] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, Continuous dependence for a nonstandard Cahn-Hilliard system with nonlinear atom mobility, *Rend. Sem. Mat. Univ. Pol. Torino* **70** (2012) 27-52.

[18] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, An asymptotic analysis for a nonstandard Cahn-Hilliard system with viscosity, *Discrete Contin. Dyn. Syst. Ser. S* **6** (2013) 353-368.

[19] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, Global existence and uniqueness for a singular/degenerate Cahn-Hilliard system with viscosity, *J. Differential Equations* **254** (2013) 4217-4244.

[20] P. Colli, G. Gilardi, J. Sprekels, Analysis and optimal boundary control of a nonstandard system of phase field equations, *Milan J. Math.* **80** (2012) 119-149.

[21] P. Colli, P. Krejčí, E. Rocca, J. Sprekels, Nonlinear evolution inclusions arising from phase change models, *Czechoslovak Math. J.* **57** (2007) 1067-1098.

[22] C. M. Elliott, A. M. Stuart, Viscous Cahn–Hilliard equation. II. Analysis, *J. Differential Equations* **128** (1996) 387-414.

[23] C. M. Elliott, S. Zheng, On the Cahn–Hilliard equation, *Arch. Rational Mech. Anal.* **96** (1986) 339-357.

[24] E. Fried, M. E. Gurtin, Continuum theory of thermally induced phase transitions based on an order parameter, *Phys. D* **68** (1993) 326-343.

[25] S. Frigeri, M. Grasselli, Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials, *Dyn. Partial Differ. Equ.* 9 (2012) 273ÃŘ304.

[26] H. Gajewski, On a nonlocal model of non-isothermal phase separation, *Adv. Math. Sci. Appl.* **12** (2002) 569-586.

[27] H. Gajewski, J. A. Griepentrog, A descent method for the free energy of multicomponent systems, *Discrete Contin. Dyn. Syst.* 15 (2006) 505-528.

[28] H. Gajewski, K. Zacharias, On a nonlocal phase separation model, *J. Math. Anal. Appl.* **286** (2003) 11-31.

[29] C. G. Gal, M. Grasselli, Longtime behavior of nonlocal Cahn-Hilliard equations, *Discrete Contin. Dyn. Syst.* 34 (2014) 145-179.

[30] G. Giacomin, J. L. Lebowitz, Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits, *J. Statist. Phys.* **87** (1997) 37-61.

[31] G. Giacomin, J. L. Lebowitz, Phase segregation dynamics in particle systems with long range interactions. II. Phase motion, *SIAM J. Appl. Math.* **58** (1998) 1707-1729.

[32] M. E. Gurtin, Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance, *Phys. D* **92** (1996) 178-192.

[33] J. Han, The Cauchy problem and steady state solutions for a nonlocal Cahn-Hilliard equation, *Electron. J. Differential Equations* **113** (2004), 9 pp.

[34] M. Heida, Existence of solutions for two types of generalized versions of the Cahn–Hilliard equation, *Appl. Math.* **60** (2015) 51-90.

[35] S.-O. Londen, H. Petzeltová, Convergence of solutions of a non-local phase-field system, *Discrete Contin. Dyn. Syst. Ser. S* **4** (2011) 653-670.

[36] P. Podio-Guidugli, Models of phase segregation and diffusion of atomic species on a lattice, *Ric. Mat.* **55** (2006) 105-118.

[37] E. Rocca, J. Sprekels, Optimal distributed control of a nonlocal convective Cahn–Hilliard equation by the velocity in three dimensions, *SIAM J. Control Optim.* **53** (2015) 1654-1680.

[38] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl. (4)* **146** (1987) 65-96.