The lifting line equation for a curved wing in oscillatory motion

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Zusammenfassung

Vorgestellt wird eine Methode der nicht stationären linearen tragenden Linie zur Bestimmung der Zirkulation und der Auftriebsverteilung entlang der Spannweite eines gekrümmten Tragflügels unter den Bedingungen harmonischer Schwingungen kleiner Amplitude. Die Methode beruht auf der sogenannten Dreiviertelpunktmethode von Pistolesi-Weissinger und deren Kopplung mit der nicht stationären Theorie von Possio für die Bewegung von Tragflächen. Dies führt auf eine singuläre Integro-Differentialgleichung vom modifizierten Prandtl'schen Typ mit der Zirkulation als gesuchte Lösung. Diese Gleichung wird eingehend analysiert, insbesondere werden alle Singularitäten der verschiedenen Kernfunktionen, die für die Lösung der Gleichung von entscheidender Bedeutung sind, untersucht. Die numerische Lösungsmethode ist eine auf Tschebyscheff-scher polynomialer Approximation für die gesuchte Lösung beruhende Kollokation unter Benutzung der Gauss'schen Quadraturtechnik. Die Methode wird getestet durch Vergleich der numerischen Lösungen für verschiedene Tragflügelkonfigurationen mit bekannten Lösungen, die auf der Theorie der tragenden Fläche basieren.

Abstract

An unsteady linear lifting line method for the determination of the circulation and lift distribution along the span of a curved wing subject to harmonic small amplitude oscillations is presented. The method relies on the Pistolesi-Weissinger 3/4-chord steady lifting line theory and couples it to the unsteady theory developed by Possio for the motion of lifting surfaces. It leads to an integro-differential equation of a modified Prandtl's type, where the unknown is the circulation. This equation has been carefully analysed in order to evidence all the singularities and to treat them in the most convenient way. The numerical procedure consists of a gaussian quadrature technique based on Chebyshev's polynomial approximation of the unknown function. The method has been appraised through the comparison of a number of solutions, pertaining to different wing configurations, with existing solutions based on lifting surface theory.

List of symbols

a	= wing sweep at tips [m]
$A\!\!\!R$	= wing aspect ratio, $A\!\!\!R = (2b)^2/S$
b	= wing semispan [m]
C_L	= wing lift coefficient
E	= spanwise integration variable [m]
h	= sectional wing deflection (referred to b)
h_e	= wing tip deflection (referred to b)
$l(Y)$	= wing chord [m]
l_o	= wing chord at midspan [m]
l_e	= tip wing chord [m]
L	= wing overall lift [N]
p	= pressure [Pa]

S	=	area of wing planform [m^2]
t	=	time [s]
U	=	streamwise velocity perturbation [m/s]
$\mathbf{V}$	=	flow velocity [m/s]
X	=	streamwise coordinate [m]
x	=	like X, but referred to b
$x_A(y)$	=	reference line for wing motions (referred to b)
$X_l(Y)$	=	lifting line coordinate [m]
$x_l(y)$	=	like $X_l(Y)$, but referred to b
$x_r(y)$	=	coordinate of the 3/4-chord line (referred to b)
Y	=	spanwise coordinate [m]
y	=	like Y, but referred to b
W	=	wing induced downwash [m/s]
w	=	like W, but referred to V_∞
Z	=	coordinate normal to the XY-plane [m]
z	=	reduced integration variable, $z = y - \eta$
α_e	=	wing tip rotation [rad]
Γ	=	like Υ, but referred to $V_\infty b$
Δx_r	=	$x_r - x_A$ (at fixed y)
ζ	=	reduced integration variable, $\zeta = \xi - x_l(\eta)$
η	=	like E, but referred to b
Ξ	=	streamwise integration variable [m]
ξ	=	like Ξ, but referred to b
ρ	=	fluid density [kg/m^3]. Also Jacobi weight function, see Eq.32
σ	=	reduced streamwise coordinate, $\sigma(y,\eta) = x(y) - x_l(\eta)$
σ_o	=	value assumed by σ when $y = \eta$
Υ	=	sectional wing circulation [m/s]
Φ	=	phase angle [deg, rad]
Ψ	=	acceleration potential [m^2/s^2]
ω	=	circular frequency [rad/s]
$\omega^\star$	=	reduced frequency, $\omega^\star = \omega b/V_\infty$
$\omega_o^\star$	=	like $\omega^\star$, but referred to l_o

Subscripts

s	steady
u	unsteady
∞	undisturbed free stream

An overbar always denotes a complex amplitude.

1 Introduction

This work presents a theoretical study on the determination of the circulation and lift distributions along the span of a curved wing subject to unsteady harmonic motions of small amplitudes in an otherwise uniform free stream. According to the classical model of Prandtl [1] the wing is substituted by a single bound vortex of variable intensity followed by a wake, i.e. a sheet of trailing vortices whose mean shape in a linear approximation can be assumed as planar. Introduced to deal with steady flows, this scheme has been very successful, leading to satisfactory results even for aspect ratios $AR = (2b)^2/S$ unexpectedly lower than those for which the theory was originally proposed. Mathematically, the problem reduces to a singular integro-differential equation in the unknown circulation distribution $\Gamma(y)$ displaying the finite Hilbert transform of the derivative $\mathrm{d}\Gamma/\mathrm{d}y$ as singular term.

However, this method fails for oblique or swept wings because of the appearance of a new (logarithmic) singular term, a problem overcome by Pistolesi's [2] and Weissinger's [3] "3/4-chord" theory, where the bound vortex is assumed to lie on the 1/4-chord line of the wing and the induced velocities are computed on the 3/4-chord line instead of on the vortex itself. So doing the wing-stream tangency condition can be explicitely enforced on the 3/4-chord line, differently as in Prandtl's model; more important, the additional singularity is avoided and also the circulation distribution on swept wings can be predicted by solving an equation of the same type as Prandtl's one. An extension of the 3/4-chord theory to curved wing in steady flow (with lunate or crescent-moon planforms) has been recently proposed by Prößdorf and Tordella [4] together with an efficient mathematical procedure to treat the singularity of the equation (basically still of Prandtl's type) and to solve it. The extension of that procedure to the equation describing the more complex unsteady case is one of the objects of the present work.

Beginning from the mid-thirties many research articles studied the flow field on a wing in accelerated motion. Unsteady simplified lifting-line theories for straight wings were proposed by Cicala [5] [6] and Borbély [7]; a later theory, due to Reissner [8] [9], starts form a lifting-surface assumption but the equation is reduced to the lifting-line form thanks to the adoption of suitable chord integrated quantities. A similar reduction was operated by Possio [10] [11] [12] [13] [14] in his works where, however, the derivation of the starting equation followed an indipendent path. Among the unsteady lifting surface theories, the most remarkable ones are that by Küssner [15], which holds for compressible subsonic flows as well and provided the basis for the later procedure by Laschka [16], and that by Krienes and Schade [17] [18]. The latter is confined to the circular planform but up to now it is the only one able to lead to fully analytical computations of the forces acting on a wing in harmonic motion. The related results are of great value as benchmarks to test more general theories; unfortunately the excessively low aspect ratio makes them basically unsuited to the comparison with the results of lifting-line theories.

As a consequence of the increasing computing capabilities the rather simple lifting-line models were progressively abandoned and replaced with lifting-surface theories like the well-known doublet-lattice method of Albano and Rodden [19]. However, the implementation of the related procedures is expensive and the results, deprived of analytical nature, may be more attractive for applied engineers than for theoretical researchers. A trend inversion made itself noticeable only during the last decades, when articles progressively appeared [20] [21] [22] [23] [24] [25] in which Prandtl's lifting-line scheme was regarded as an "outer solution" of the general problem to be asymptotically matched to an "inner solution", i.e. the two-dimensional flow field around each wing section. The method is powerful, providing not only the spanwise circulation distribution, but also the load distribution along each chord: the latter results from the two-dimensional one corrected through the matching to the outer solution. Furthermore, such theories are able to highlight the several interactions between the inner and the outer fields. Nevertheless these interactions greatly contribute to their complexity, being often concealed in matching terms at first neglected and only discovered in later works.

In the context of the aforementioned theories, the 3/4-chord method is neither an outer nor an inner approximation, even if it is certainly nearer to the former. As a matter of fact in a matched asymptotic expansion procedure the far-field boundary conditions are enforced on the outer solution and the wall-stream tangency condition on the inner one. All free and unknown terms are therafter determined through the matching. In Weissinger's scheme all conditions are directly enforced on the outer solution, though in an approximate way, and matching is no more needed. One should here remark that the inner solution disappears (therefore no load distributions along the different chords can be computed); however, all of the unsteady influences from the whole outer solution to the local wing sections are taken into account. This is the case, for instance, of the gust term detected by Ahmadi and Widnall [24], which is implicitely considered in the present theory because the unsteady lifting line field on the 3/4-chord line is exactly computed. From this standpoint Weissinger's method can be regarded as an approximate closure of the overall problem as regard to the inner boundary conditions, rather then an 'incomplete calculation of the outer solution' as stated by Cheng and Murillo [23].

The 3/4-chord theory in unsteady flow has been questioned also from another, more physical point of view by Van Holten [22] observing that Pistolesi's theorem does not hold even in the two-dimensional case if the flow is not steady and therefore it cannot be extended to the spatial one. This is true. However, one of the present authors [26] assessed the error implicit in the use of the 3/4-chord assumption in the plane flow around a flat plate in harmonic motion and found that, in spite of an expected increase with the reduced frequency $\omega_o^\star$ (based on the chord), it can be tolerated up to $\omega_o^\star \approx 1$. Moreover, a remarkable improvement can be obtained by using two bound vortices together with their wakes instead of only one: this could be done also for a finite span wing through a simple extension of the present theory. An inherent restriction of the one-lifting-line theory, namely its inadequacy to predict also the aerodynamic pitch moment distribution, could then be removed and the model could find application also for preliminary aeroelastic computations.

Being basically an outer solution based on the wing semispan b as length scale, the governing parameter of the present theory is the reduced frequency $\omega^\star$ based on b. As in the plane case considered in [26], it must be expected that the quality of the results deteriorates when the distance from the steady conditions increases, i.e. with increasing values of $\omega^\star$. Indeed, the computed cases display acceptable results (with regard to the simplicity of the physical model) up to $\omega^\star \approx 1$. It is important here to emphasize that this boundary is by no means implicit in the mathematical model, being never introduced in it: as a consequence all computations are stable also for very high values of $\omega^\star$. The limitation is rather of pure physical nature, in that the model assumptions fail to reflect reality if one moves too far away from the steady conditions. On the other hand it can be remarked that $\omega^\star \leq 1$ (i.e. the very-low and low frequency domains among the five identified by Cheng [27]) is also the validity range of the more complete theories above quoted, with the exception of [25]: their results in fact diverge from the reference ones (generally obtained through lifting-surface models) for $\omega^\star > 1$ (for $\omega^\star < 1$ they are of course more exact than those here presented). As a consequence of stating an upper limit to $\omega^\star$ instead of ω and b separately, a moderate value of $\omega^\star$ can be obtained either through a large aspect ratio coupled to a low mechanical frequency or through the reverse: therefore acceptable results for higher frequency can still be obtained, provided the aspect ratio is hold as low as allowed by the lifting line model (actually the 3/4-chord theory extends the validity range of Prandtl's original scheme down to aspect ratios as low as 2).

Apart from the adoption of the 3/4-chord model, two other significant points characterize this work. First, the unsteady lifting-line equation for a curved wing has been derived from the steady one by applying the ingenious procedure due to Possio already referred to as "an indipendent path". So doing, the deduction of the unsteady equation becomes a rather formal matter and errors are avoided which could easily occur when dealing with curved wings. Secondly, the obtained integro-differential equation is no more of Prandtl's type, because it displays two additional singular terms besides the finite Hilbert transform of the derivative of $\Gamma(y)$: the finite transforms with logarithmic kernel of $\Gamma(y)$ and that of its derivative. To isolate these singularities a careful analysis of the terms of the main equation has been performed. A special procedure which extends that of [4] has been developed to numerically solve the problem by means of a gaussian quadrature method where the unknown function $\Gamma(y)$ is represented through a sum of Chebyshev polynomials of the second kind. After assessing the convergence of the method, circulation distributions and overall lift coefficients have been computed for some of the geometrical and kinematical configurations already studied by Laschka [16] and results are compared.

Finally the method is applied to present results for a crescent-moon wing, a geometry for which no chordwise integrated circulation distributions along the span could be found in the literature and therefore no direct comparison is possible. This set of results, like those obtainable for analogous wing planforms, may find application in the physics of the animal locomotion (bird flight and fish propulsion), where the parameter $\omega^\star$ effectively does not exceed unity.

2 Brief outline of Possio's theory and derivation of the lifting line equation

Startpoint for Possio's theory was the quantity Ψ introduced by L.Prandtl [28] in 1936 to express through a gradient the acceleration of a fluid particle:

$$\operatorname{grad} \Psi = \frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}$$

and therefore called acceleration potential (here $\mathrm{D}/\,\mathrm{D}t$ is the material derivative operator). If perturbations are small, Ψ satisfies the Laplace equation

$$\Delta\Psi = 0 \tag{1}$$

exactly as the velocity potential, but with the important advantage of being everywhere continuous in a subsonic flow field: in fact, there exists a direct relation between Ψ and the pressure disturbance, reducing to

$$\Psi = -\frac{p - p_\infty}{\rho} \tag{2}$$

if the flow is incompressible, or compressible but only weakly disturbed. In this case also the operator $\mathrm{D}/\mathrm{D}t$ can be linearized; then the Z-component of grad Ψ gives

$$\frac{\partial \Psi}{\partial Z} = \frac{\partial W}{\partial t} + V_\infty \frac{\partial W}{\partial X}$$

a relation which can be inverted to obtain the downwash W from a known Ψ:

$$W(X, Y, Z, t) = \frac{1}{V_\infty} \int_{-\infty}^{X} \frac{\partial \Psi}{\partial Z} \left(\Xi, Y, Z, t - \frac{X - \Xi}{V_\infty} \right) \mathrm{d}\Xi \tag{3}$$

In 1938 C.Possio [10] [11] [12] [13] [14] observed that no time derivatives appear in Eq.1, neither the boundary conditions for Ψ depend from the motion being steady or unsteady. In other words, a solution $\Psi(X, Y, Z, t)$ representing the instantaneous picture of the unsteady field at time t, describes as well a steady field having the same configuration of the former at that instant. Then, if Ψ_s is the acceleration potential of the steady field

$$\Psi(X, Y, Z, t) = \Psi_s(X, Y, Z; t)$$

where the argument t in Ψ_s has been retained to remember that Ψ_s has been obtained from the steady motion formulae, but with all boundaries at the position they assume in the unsteady field at time t (for such a field the name quasi-steady is often used). For Ψ_s all properties of the steady fields hold: in particular, the linearized Bernoulli theorem gives $p - p_\infty = -\rho V_\infty U_s$ where U_s is the X-component of the velocity perturbation. Then Eq.2 becomes $\Psi_s = V_\infty U_s$ in the steady field, where also the irrotationality condition $\partial U_s / \partial Z = \partial W_s / \partial X$ holds (note that here W_s is the steady flow downwash). From these relations:

$$\frac{\partial \Psi_s}{\partial Z} = V_\infty \frac{\partial U_s}{\partial Z} = V_\infty \frac{\partial W_s}{\partial X}$$

Looking back now to Eq.3 it appears that the derivative of Ψ with respect to Z, there required, can be substituted by that of Ψ_s, provided the latter is evaluated at the same (retarded) time $\tau = t - (X - \Xi)/V_\infty$ at which both fields coincide. Thanks to the last relation above, Eq.3 assumes then a form in which the unsteady downwash is related only to the steady one and any reference to the acceleration potential disappears:

$$W(X, Y, Z, t) = \int_{-\infty}^{X} \frac{\partial W_s}{\partial \Xi} (\Xi, Y, Z, \tau) \, \mathrm{d}\Xi$$

By further observing that

$$\frac{\partial W_s}{\partial \Xi} (\Xi, Y, Z, \tau) = \frac{\mathrm{d} W_s}{\mathrm{d}\Xi} - \frac{1}{V_\infty} \frac{\partial W_s}{\partial \tau}$$

Eq.3 takes finally the form

$$W(X, Y, Z, t) = W_s(X, Y, Z; t) - \frac{1}{V_\infty} \int_{-\infty}^{X} \frac{\partial W_s}{\partial \tau} (\Xi, Y, Z; \tau) \, \mathrm{d}\Xi$$

When the motion is harmonic the time dependence is known

$$W(X,Y,Z,t) = \overline{W}(X,Y,Z)\,\mathrm{e}^{\mathrm{i}\omega t}; \qquad W_s(X,Y,Z;\tau) = \overline{W}_s(X,Y,Z)\,\mathrm{e}^{\mathrm{i}\omega\tau}$$

where W_s differs from W only for being obtained from the steady motion theory with solid boundaries at the position of instant τ: hence the presence of $\exp(\mathrm{i}\omega\tau)$ in the 'steady' downwash. The last equation reduces then to one relating the amplitudes

$$\overline{W}(X,Y,Z) = \overline{W}_s(X,Y,Z) - \frac{\mathrm{i}\omega}{V_\infty}\int_{-\infty}^{X} \overline{W}_s(\Xi,Y,Z)\,\mathrm{e}^{\mathrm{i}\omega\frac{X-\Xi}{V_\infty}}\,\mathrm{d}\Xi \tag{4}$$

This is main Possio's result: it is noteworthy that it alone does not solve the problem of finding the forces acting on a solid immersed in a fluid in unsteady motion. Eq.4 gives rather in simple form the unsteady velocity if the steady one on a 'frozen' configuration of the same field is known; the unsteady boundary conditions are still to be imposed on the velocity distribution so determined. Pressures and forces, however, can be computed with reference to the steady flow fields which at every instant coincide with the successive configurations of the real unsteady field: but this is an easier task than the original one. An example was provided by Possio himself in [10], where the unsteady downwash generated by a pulsating vortex of circulation $\overline{K}\exp\mathrm{i}\omega t$ located on the axes origin in a uniform stream V_∞ was computed. In this case $\overline{W}_s(X,0,0) = -\overline{K}/2\pi X$ and the integral term of Eq.4 completely describes the contribution to W of the vorticity which has been shed into the wake in order to satisfy Lagrange's theorem. Imposition of a value for W in one field point would determine $\overline{K}$: the unsteady force acting on the pulsating vortex could then be simply computed from the steady Kutta - Joukowsky formula $\rho V_\infty \overline{K}$.

In the present work Possio's theory has been used to derive the unsteady donwwash past a curved wing of finite aspect ratio in oscillatory motion. The configuration is represented in Fig.1: the lifting line is placed on the first quarter chord locus of the wing along the curve of equation

$$X_l(Y) = a\left(\frac{Y}{b}\right)^n \tag{5}$$

where a is the tip sweep ($Y = b$) and n an exponent governing the curvature distribution along the span: if for $n = 2$ the lifting line is parabolic and the wing shape resembles that of a swallow, while a higher value of n makes the maximum sweep region move outboard and the resulting planform remind of that of sea-gull. The wing itself is supposed to be planar and without thickness, but the extension to the case of a thin slightly cambered wing would be immediate. The spanwise chord distribution is assigned through the law

$$\frac{l(Y)}{l_o} = \left[1 + \left(\frac{l_e}{l_o}\frac{|Y|}{b}\right)^{1/q} - \left(\frac{|Y|}{b}\right)^p\right]^q \tag{6}$$

which is rather general: for instance all rectangular, trapezoidal and triangular chord distributions are described by $p = q = 1$ and by different ratios of the tip chord l_e to the midspan one l_o; alternatively, the popular elliptical distribution is given by $p = 2$, $q = 0.5$ and $l_e = 0$ (actually a zero value for l_e should be avoided because it would imply the computation of the induction of the bound vortex on itself at wing tip; a sufficiently small value satisfies the same practical requirements).

A second important curve is the locus of the three quarter chord points, given by $X_r(Y) = X_l(Y) + l(Y)/2$: in the Pistolesi - Weissinger model developed for the steady case this is the line along which the flow - wall tangency condition is enforced, so determining the bound circulation per unit span $\Upsilon(Y)$. Thanks to Possio's theory the results of this steady model can be exploited to obtain the unsteady equations. For the described wing the steady case has been studied in [4], where the expression of the downwash can be found:

$$
\begin{aligned}
W_s(X,Y) \;&=\; \frac{1}{4\pi}\int_{-b}^{b}\frac{X-X_l(E)+X_l'(E)(E-Y)}{\{[X-X_l(E)]^2+(Y-E)^2\}^{3/2}}\,\Upsilon(E)\,\mathrm{d}E\,+ \\
&+\; \frac{1}{4\pi}\int_{-b}^{b}\left[1+\frac{X-X_l(E)}{\sqrt{[X-X_l(E)]^2+(Y-E)^2}}\right]\frac{1}{Y-E}\,\frac{\mathrm{d}\Upsilon}{\mathrm{d}E}\,\mathrm{d}E
\end{aligned}
\tag{7}
$$

being $X_l' = \mathrm{d}X_l/\mathrm{d}Y$. In this equation the first integral gives the velocity induced on any point of the $Z=0$ plane by the curved bound vortex, while the second one describes the contribution of the trailing wake vortices originated because of the spanwise variation of $\Upsilon(Y)$. If the motion is unsteady W_s and Υ depend also on time; furthermore, the 'true' unsteady downwash W makes its appearance. Considering the harmonic case:

$$
\begin{aligned}
W(X,Y,t) \;&=\; \overline{W}(X,Y)\,\mathrm{e}^{\mathrm{i}\omega t}; \qquad W_s(X,Y,t)=\overline{W}_s(X,Y)\,\mathrm{e}^{\mathrm{i}\omega t} \\
&\quad\;\Upsilon(Y,t)=\overline{\Upsilon}(Y)\,\mathrm{e}^{\mathrm{i}\omega t}
\end{aligned}
\tag{8}
$$

Eq.4 can then be immediately applied to give

$$
\overline{W}(X,Y)=\overline{W}_s(X,Y)-\frac{\mathrm{i}\omega}{V_\infty}\int_{-\infty}^{X}\overline{W}_s(\Xi,Y)\,\mathrm{e}^{-\mathrm{i}\omega\frac{X-\Xi}{V_\infty}}\,\mathrm{d}\Xi \tag{9}
$$

where using complex amplitudes means that phase shifts between the bound circulation $\Upsilon(Y,t)$ and all other field quantities are taken into account.

Imposing the tangency condition at $X_r(Y)$ on the downwash obtained from Eq.9 leads to an equation in $\overline{\Upsilon}(Y)$ whose manipulation and solution will be the object of the remainder of this work. Important is here that Possio's theory provides a simple relation between the spanwise lift distribution and that of bound circulation: as already stated, in fact, force computations can be performed in the istantaneous field configurations by means of steady formulae. This allows to use the steady Kutta - Joukowsky theorem and to write for the complex amplitude of the lift $\partial L(Y,t)/\partial Y$ per unit span:

$$
\frac{\mathrm{d}\overline{L}}{\mathrm{d}Y}=\rho V_\infty\overline{\Upsilon}(Y) \tag{10}
$$

3 Mathematical formulation

Let us introduce the following nondimensional variables

$$
\begin{aligned}
y,\eta \;&=\; \frac{Y}{b},\frac{E}{b}; \qquad & x,x_l,\xi &= \frac{X}{b},\frac{X_l}{b},\frac{\Xi}{b}; \\
w,w_s \;&=\; \frac{W}{V_\infty},\frac{W_s}{V_\infty}; \qquad & \overline{w},\overline{w}_s &= \frac{\overline{W}}{V_\infty};\frac{\overline{W}_s}{V_\infty}, \\
\overline{\Gamma} \;&=\; \frac{\overline{\Upsilon}}{V_\infty b}; \qquad & \omega^\star &= \frac{\omega b}{V_\infty}
\end{aligned}
\tag{11}
$$

where the semispan b and the free stream velocity V_∞ have been selected as reference quantities. As a consequence the reduced frequency $\omega^\star$ spontaneously arising from the equations is defined with respect to b and not to the midspan chord l_o as in most classical works on this topic. However, also the more traditional form $\omega_o^\star = \omega^\star l_o/b$ will later be used to present the numerical results.

Eq.9 shows that the overall induced downwash in the unsteady case can be represented as the superposition of a steady part $\overline{W}_s$ to a totally unsteady one given by the integral term. In normalized form:

$$\overline{w}(x,y) = \overline{w}_s(x,y) + \overline{w}_u(x,y) \tag{12}$$

with $\overline{w}_s$ and $\overline{w}_u$ given by Eqs.7 and 9, respectively,

$$\begin{aligned}
\overline{w}_s(x,y) &= \frac{1}{4\pi}\int_{-1}^{1} \frac{x - x_l(\eta) + x_l'(\eta)(\eta - y)}{\{[x - x_l(\eta)]^2 + (y-\eta)^2\}^{3/2}}\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta\, + \\
&+ \frac{1}{4\pi}\int_{-1}^{1} \left[1 + \frac{x - x_l(\eta)}{\sqrt{[x - x_l(\eta)]^2 + (y-\eta)^2}}\right] \frac{1}{y-\eta}\,\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta
\end{aligned} \tag{13}$$

$$\overline{w}_u(x,y) = -\mathrm{i}\omega^\star\,\mathrm{e}^{-\mathrm{i}\omega^\star x}\int_{-\infty}^{x} \overline{w}_s(\xi,y)\,\mathrm{e}^{\mathrm{i}\omega^\star \xi}\,\mathrm{d}\xi \tag{14}$$

being now $x_l' = \mathrm{d}x_l/\,\mathrm{d}y$ (an analogous but not identical splitting can be found in [25]). All this can be written in more compact form after introducing the following reduced variables:

$$\begin{aligned}
z &= y - \eta \\
\zeta &= \xi - x_l(\eta) \\
\sigma(y,\eta) &= x(y) - x_l(\eta)
\end{aligned} \tag{15}$$

where σ isn't but the nondimensional downstream coordinate measured from a lifting line point lying a distance z aside. Its special value for $z = 0$ is hereafter called σ_o:

$$\sigma_o(y) = x(y) - x_l(y)$$

Eqs.13 and 14 assume then the form:

$$\begin{aligned}
\overline{w}_s(x,y) &= \frac{1}{4\pi}\int_{-1}^{1} \frac{\sigma - x_l' z}{(\sigma^2 + z^2)^{3/2}}\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta\, + \\
&+ \frac{1}{4\pi}\int_{-1}^{1} \left[1 + \frac{\sigma}{\sqrt{\sigma^2 + z^2}}\right] \frac{1}{z}\,\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta \\
\overline{w}_u(x,y) &= -\frac{\mathrm{i}\omega^\star}{4\pi}\int_{-1}^{1} \left[\mathrm{e}^{-\mathrm{i}\omega^\star \sigma}\int_{-\infty}^{\sigma} \frac{\zeta - x_l' z}{(\zeta^2 + z^2)^{3/2}}\,\mathrm{e}^{\mathrm{i}\omega^\star \zeta}\,\mathrm{d}\zeta\right] \overline{\Gamma}(\eta)\,\mathrm{d}\eta\, - \\
&- \frac{\mathrm{i}\omega^\star}{4\pi}\int_{-1}^{1} \left[\mathrm{e}^{-\mathrm{i}\omega^\star \sigma}\int_{-\infty}^{\sigma} \left(1 + \frac{\zeta}{\sqrt{\zeta^2 + z^2}}\right) \mathrm{e}^{\mathrm{i}\omega^\star \zeta}\,\mathrm{d}\zeta\right] \frac{1}{z}\,\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta
\end{aligned} \tag{16}$$

Both Eqs.16 are singular for $z = 0$ and must be modified before attempting their numerical solution. This can be done in a uniform way, first by evidencing their singular term

$$4\overline{w}_s(x,y) = (1 + \mathrm{sgn}\sigma_o)\,\mathrm{H}\left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\right) + \frac{1}{\pi}\int_{-1}^{1} \mathrm{L}_s(\sigma,z)\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta\, +$$

$$
\begin{aligned}
& + \frac{1}{\pi}\int_{-1}^{1} \mathrm{M}_s(\sigma,z)\,\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta \\
4\overline{w}_u(x,y) &= (1+\mathrm{sgn}\sigma_o)(\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}-1)\,\mathrm{H}\left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\right) + \\
& + \frac{1}{\pi}\int_{-1}^{1} \mathrm{L}_u(\sigma,z)\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta + \frac{1}{\pi}\int_{-1}^{1} \mathrm{M}_u(\sigma,z)\,\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta
\end{aligned} \tag{17}
$$

where the operator H is given by the integral

$$
\mathrm{H}f(y) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(\eta)}{y-\eta}\,\mathrm{d}\eta \tag{18}
$$

to be interpreted as its Cauchy principal value and is called the finite Hilbert transform of the function f. The other functions are so defined:

$$
\begin{aligned}
\mathrm{L}_s(\sigma,z) &= \frac{\sigma - x_l' z}{(\sigma^2+z^2)^{3/2}} \\
\mathrm{L}_u(\sigma,z) &= -\mathrm{i}\omega^\star\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma}\int_{-\infty}^{\sigma} \frac{\zeta - x_l' z}{(\zeta^2+z^2)^{3/2}}\,\mathrm{e}^{\mathrm{i}\omega^\star\zeta}\,\mathrm{d}\zeta \\
\mathrm{M}_s(\sigma,z) &= \frac{1}{z}\left(-\mathrm{sgn}\sigma_o + \frac{\sigma}{\sqrt{\sigma^2+z^2}}\right) \\
\mathrm{M}_u(\sigma,z) &= -\frac{1}{z}\left[(1+\mathrm{sgn}\sigma_o)\left(\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}-1\right) + \mathrm{i}\omega^\star\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma}\,\mathrm{S}(\sigma,z)\right]
\end{aligned}
$$

where S is the integral:

$$
\mathrm{S}(\sigma,z) = \int_{-\infty}^{\sigma}\left(1+\frac{\zeta}{\sqrt{\zeta^2+z^2}}\right)\,\mathrm{e}^{\mathrm{i}\omega^\star\zeta}\,\mathrm{d}\zeta
$$

Then let us perform by parts the third integrals of both Eqs.17. After recalling that $\overline{\Gamma}(-1) = \overline{\Gamma}(1) = 0$ there results:

$$
\begin{aligned}
\int_{-1}^{1} \mathrm{M}_s(\sigma,z)\,\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta &= -\int_{-1}^{1} \frac{\partial \mathrm{M}_s}{\partial\eta}\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta \\
\int_{-1}^{1} \mathrm{M}_u(\sigma,z)\,\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta &= -\int_{-1}^{1} \frac{\partial \mathrm{M}_u}{\partial\eta}\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta
\end{aligned}
$$

But $\partial z/\partial\eta = -1$ and $\partial\sigma/\partial\eta = -x_l'(\eta)$; furthermore, as it can be easily verified:

$$
\frac{\partial \mathrm{S}}{\partial\eta} = \frac{\mathrm{i}z}{\omega^\star}\,\mathrm{e}^{\mathrm{i}\omega^\star\sigma}\mathrm{L}_u(\sigma,z) - \mathrm{i}\omega^\star x_l'(\eta)\,\mathrm{S}(\sigma,z)
$$

Then the derivatives with respect to η of M_s and M_u are

$$
\begin{aligned}
\frac{\partial \mathrm{M}_s}{\partial\eta} &= -\frac{1}{\sqrt{\sigma^2+z^2}\,(\sigma+\mathrm{sgn}\sigma_o\sqrt{\sigma^2+z^2})} + \mathrm{L}_s(\sigma,z) \\
\frac{\partial \mathrm{M}_u}{\partial\eta} &= -(1+\mathrm{sgn}\sigma_o)\frac{\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}-1}{z^2} - \frac{\mathrm{i}\omega^\star}{z^2}\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma}\mathrm{S}(\sigma,z) + \mathrm{L}_u(\sigma,z)
\end{aligned}
$$

and consequentely Eqs.17 assume the forms:

$$
\begin{aligned}
4\overline{w}_s(x,y) &= (1+\mathrm{sgn}\sigma_o)\,\mathrm{H}\left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\right) + \frac{1}{\pi}\int_{-1}^{1}\mathrm{T}_s(\sigma,z)\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta \\
4\overline{w}_u(x,y) &= (1+\mathrm{sgn}\sigma_o)\left(\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}-1\right)\,\mathrm{H}\left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\right) + \frac{1}{\pi}\int_{-1}^{1}\mathrm{T}_u^\star(\sigma,z)\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta
\end{aligned}
\tag{19}
$$

Two new functions have been here introduced:

$$
\mathrm{T}_s(\sigma,z) = \frac{1}{\sqrt{\sigma^2+z^2}\,(\sigma+\mathrm{sgn}\sigma_o\sqrt{\sigma^2+z^2})} \tag{20}
$$

$$
\begin{aligned}
\mathrm{T}_u^\star(\sigma,z) &= \frac{1}{z^2}\Big\{(1+\mathrm{sgn}\sigma_o)\left(\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}-1\right)+ \\
&+ 1+\mathrm{i}\omega^\star\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma}\int_0^\sigma\frac{\zeta\,\mathrm{e}^{\mathrm{i}\omega^\star\zeta}}{\sqrt{\zeta^2+z^2}}\,\mathrm{d}\zeta - \mathrm{i}\omega^\star|z|\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma}\cdot \\
&\cdot \left[\frac{\pi}{2}\left[\mathrm{I}_1(\omega^\star|z|)-\mathrm{L}_{-1}(\omega^\star|z|)\right]-\mathrm{i}\mathrm{K}_1(\omega^\star|z|)\right]\Big\}
\end{aligned}
\tag{21}
$$

In the last relation I_1, K_1 and L_{-1} are modified Bessel and Struve functions and the integral $\mathrm{S}(\sigma,z)$ has been splitted in a first part running from $-\infty$ to 0 and a second one running from 0 to σ. While the evaluation of the second part can occur only numerically, the first integral has been analytically performed (see Appendix A for details). So far as the fuctions T_s and $\mathrm{T}_u^\star$ are concerned, the first one is the same already arising from the study of the steady case [4] and is fully regular for $z=0$ (i.e. $\eta=y$), where it takes the value

$$
\mathrm{T}_s(\sigma,0) = \frac{\mathrm{sgn}\sigma_o}{2\sigma_o^2}
$$

(actually this result is singular for $\sigma_o=0$, but this occurrence is excluded by the Pistolesi - Weissinger wing theory, which never requires the induced velocities to be computed on the lifting line). The computation of $\mathrm{T}_u^\star$, conversely, indicates that its values become unbounded in the neighbourhood of $z=0$ and that therefore some other singularities exist, which have still to be isolated. This can be done by expanding the quantity in curly brackets of Eq.21 in the neighbourood of $|z|=0$ up to the order of z^2. As a result two new singular terms appear, respectively of order z^{-1} and $\ln|z|$, allowing Eq.21 to be rewritten as

$$
\mathrm{T}_u^\star(\sigma,z) = \mathrm{T}_u(\sigma,z) + (1+\mathrm{sgn}\sigma_o)\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}\left[x_l'(y)\frac{\mathrm{i}\omega^\star}{z} - \frac{\omega^{\star 2}}{2}\ln|z|\right]
$$

where T_u is the function regular for $z=0$:

$$
\begin{aligned}
\mathrm{T}_u(\sigma,z) &= \frac{1}{z^2}\Big\{-\mathrm{sgn}\sigma_o+ \\
&+ (1+\mathrm{sgn}\sigma_o)\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}\left[1-\mathrm{i}\omega^\star x_l'(y)z+\frac{\omega^{\star 2}}{2}z^2\ln|z|\right]+ \\
&+ \mathrm{e}^{-\mathrm{i}\omega^\star\sigma}\Big[\mathrm{i}\omega^\star\int_0^\sigma\frac{\zeta\,\mathrm{e}^{\mathrm{i}\omega^\star\zeta}}{\sqrt{\zeta^2+z^2}}\,\mathrm{d}\zeta- \\
&- \mathrm{i}\frac{\pi}{2}\omega^\star|z|\left[\mathrm{I}_1(\omega^\star|z|)-\mathrm{L}_{-1}(\omega^\star|z|)\right]-\omega^\star|z|\mathrm{K}_1(\omega^\star|z|)\Big]\Big\}
\end{aligned}
\tag{22}
$$

Inserting $\mathrm{T}_u^\star$ in the second of Eqs.19 leads to a form for the unsteady downwash equation in which all singularities are properly isolated:

$$
\begin{aligned}
4\overline{w}_u(x,y) \;&=\; (1+\mathrm{sgn}\sigma_o)\left[\left(\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}-1\right)\,\mathrm{H}\!\left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\right)+\right.\\
&+\;\left.\mathrm{i}\omega^\star x_l'(y)\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}\mathrm{H}(\overline{\Gamma})-\frac{\omega^{\star 2}}{2}\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}\frac{1}{\pi}\int_{-1}^{1}\ln|z|\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta\right]+\\
&+\;\frac{1}{\pi}\int_{-1}^{1}\mathrm{T}_u(\sigma,z)\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta
\end{aligned}
\tag{23}
$$

Although regular, the function $\mathrm{T}_u(\sigma,z)$ as given by Eq.22 is difficult to evaluate for small z, because it takes there the form $(0/0)^2$. For this reason the expansion of the term in curly brackets of Eq.21 has been extended up to the order of z^4: so doing, a second order approximation of $\mathrm{T}_u(\sigma,z)$ to be used in the computations at small $|z|$ has been obtained. This is:

$$
\mathrm{T}_u(\sigma,z)=T_o+T_1 z+T_2 z^2+L_o z\ln|z|+\mathcal{O}(z^3,z^2\ln|z|)
\tag{24}
$$

where the coefficients T_i and L_o are given by

$$
\begin{aligned}
T_o =\;& \frac{\mathrm{i}\omega^\star}{2}\frac{\mathrm{sgn}\sigma_o}{\sigma_o}+\\
&+\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}\left\{\mathrm{sgn}\sigma_o\frac{\omega^{\star 2}}{2}\left[\Omega_1+\mathrm{Ci}(\omega^\star|\sigma_o|)-\ln\gamma\omega^\star+\mathrm{i}\mathrm{Si}(\omega^\star\sigma_o)\right]+\right.\\
&+\left.\frac{\omega^{\star 2}}{2}\left(\frac{1}{2}+\ln 2-\ln\omega^\star\gamma-\mathrm{i}\frac{\pi}{2}\right)+\frac{\omega^\star}{2}(1+\mathrm{sgn}\sigma_o)(\omega^\star x_l'^2-\mathrm{i}x_l'')\right\}\\
T_1 =\;& \omega^\star\left(\omega^\star-\frac{\mathrm{i}}{\sigma_o}\right)\frac{\mathrm{sgn}\sigma_o}{2\sigma_o}\,x_l'-\\
&-\mathrm{i}\omega^\star\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}\left\{\frac{\omega^{\star 2}}{2}\left(\frac{1}{2}+\ln 2-\ln\omega^\star\gamma-\mathrm{i}\frac{\pi}{2}\right)x_l'+\right.\\
&+\mathrm{sgn}\sigma_o\frac{\omega^{\star 2}}{2}\left[\Omega_1+\mathrm{Ci}(\omega^\star|\sigma_o|)-\ln\gamma\omega^\star+\mathrm{i}\mathrm{Si}(\omega^\star\sigma_o)\right]x_l'\\
&+\left.\frac{1}{2}(1+\mathrm{sgn}\sigma_o)\left(\frac{x_l'^3}{3}-\frac{x_l''}{2}-\mathrm{i}\omega^\star x_l'x_l''\right)\right\}\\
T_2 =\;& \mathrm{i}\omega^\star\frac{\mathrm{sgn}\sigma_o}{4\sigma_o}\left[\frac{1}{\sigma_o^2}\left(2x_l'^2-\frac{1}{2}\right)+\frac{1}{2\sigma_o}\left(2x_l''-\frac{\mathrm{i}\omega^\star}{2}\right)-\left(2x_l'^2-\frac{1}{4}\right)\omega^{\star 2}\right]-\\
&-\mathrm{i}\omega^\star\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}\left\{\frac{\mathrm{i}\omega^{\star 3}}{16}\left[\frac{8\Omega_2}{3}\mathrm{sgn}\sigma_o+\frac{3}{4}+\mathrm{Ci}(\omega^\star|\sigma_o|)+\mathrm{i}\left(\mathrm{Si}(\omega^\star\sigma_o)-\frac{\pi}{2}\right)\right]\right.\\
&-\frac{\mathrm{i}\omega^{\star 2}}{4}(\omega^\star x_l'^2-\mathrm{i}x_l'')\left[\mathrm{sgn}\sigma_o\left(\frac{\Omega_1}{2}+\mathrm{Ci}(\omega^\star|\sigma_o|)-\ln\gamma\omega^\star+\mathrm{i}\mathrm{Si}(\omega^\star\sigma_o)\right)\right.\\
&-\left.\frac{1}{2}-\ln 2+\ln\omega^\star\gamma-\mathrm{i}\frac{\pi}{2}\right]-\frac{\omega^\star}{4\sigma_o^2}\mathrm{sgn}\sigma_o(\omega^\star\sigma_o-\mathrm{i})\,x_l'^2-\frac{\mathrm{i}\omega^\star}{4\sigma_o}x_l''-\\
&-\left.(1+\mathrm{sgn}\sigma_o)\left[\frac{\mathrm{i}\omega^{\star 3}}{24}x_l'^4+\frac{\omega^{\star 2}}{4}x_l'^2x_l''-\frac{\mathrm{i}\omega^\star}{4}\left(\frac{x_l''^2}{2}+\frac{2}{3}x_l'x_l'''\right)-\frac{x_l''''}{24}\right]\right\}\\
L_o =\;& \frac{\mathrm{i}\omega^{\star 3}}{2}\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}(1+\mathrm{sgn}\sigma_o)\,x_l'
\end{aligned}
$$

In these relations all derivatives of x_l are to be evaluated at the constant abscissa y, while $\gamma = 1.7810724\ldots$ is Euler's constant and Ω_1 and Ω_2 are given by

$$\begin{aligned}\Omega_1 &= \frac{23}{15} - 2\sum_{k=2}^{\infty}\binom{-1/2}{k}\frac{4k+1}{(2k+3)(2k-2)} = 1.1931471\ldots\\ \Omega_2 &= \frac{1441}{1680} + \sum_{k=3}^{\infty}\binom{-1/2}{k}\frac{4k+1}{(2k+5)(2k-4)} = 0.7286802\ldots\end{aligned}$$

A plot of the regular function $\mathrm{T}_u(\sigma, z)$ for the fixed position $y = 0$ and for the midspan region is in Fig.2 for a curved wing of constant chord where also the expansion Eq.24 is shown for comparison. It appears that the latter is necessary to compute $\mathrm{T}_u(\sigma, z)$ for $z \approx 10^{-3}$, where Eq.22 leads to unreliable results .

Finally, after summing Eq.23 to the first of Eqs.19 and recalling Eq.12 an equation for the overall induced velocity $\overline{w}(x, y)$ is obtained. By doing this it is convenient to restate the singular term expressing the finite Hilbert transform of $\overline{\Gamma}(\eta)$ in the form [29]:

$$\mathrm{H}(\overline{\Gamma}) \equiv \frac{1}{\pi}\int_{-1}^{1}\frac{\overline{\Gamma}(\eta)}{y-\eta}\,\mathrm{d}\eta = \frac{1}{\pi}\int_{-1}^{1}\ln|\eta - y|\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta$$

and to introduce the total regular function $\mathrm{T} = \mathrm{T}_s + \mathrm{T}_u$, sum of its steady and unsteady parts. Then the equation is obtained:

$$\begin{aligned}4\overline{w}(x,y) = (1+\mathrm{sgn}\sigma_o)\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_o}\Bigg[\mathrm{H}\left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\right) +\\ + \mathrm{i}\omega^\star x_l'(y)\,\frac{1}{\pi}\int_{-1}^{1}\ln|\eta-y|\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\,\mathrm{d}\eta - \frac{\omega^{\star 2}}{2}\frac{1}{\pi}\int_{-1}^{1}\ln|y-\eta|\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta\Bigg] +\\ +\frac{1}{\pi}\int_{-1}^{1}\mathrm{T}(\sigma, y-\eta)\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta \end{aligned} \qquad (25)$$

From this equation it appears that the singular terms only exist if $\sigma_o > 0$, i.e. if the induced velocities are computed on the wake plane downstream of the lifting line. Since the Pistolesi - Weissinger model assumes the tangency condition to be enforced on the three-quarter point of each chord, all singular terms must be accounted for: Eq.25 is therefore a complex integro-differential equation with one Hilbert and two logarithmically singular terms to be solved subject to the conditions $\overline{\Gamma}(-1) = \overline{\Gamma}(1) = 0$.

4 Numerical procedure

The procedure to solve Eq.25 extends the one introduced in [4] to study the steady case. An approximate solution $\overline{\Gamma}_N(\eta)$ is sought in the form:

$$\overline{\Gamma}_N(\eta) = \sqrt{1-\eta^2}\sum_{j=1}^{N}\overline{\gamma}_j\,\frac{\mathrm{U}_N(\eta)}{(\eta-\eta_j)\mathrm{U}_N'(\eta_j)} \qquad (26)$$

Here $\mathrm{U}_N(\eta)$ are the Chebyshev polynomials of the second kind

$$\mathrm{U}_N(\eta) = \frac{\sin[(N+1)\arccos\eta]}{\sqrt{1-\eta^2}} = \frac{\sin[(N+1)\psi]}{\sin\psi} \qquad (27)$$

being ψ a new variable ($0 \le \psi \le \pi$) such that $\eta = \cos\psi$. Eq.26 defines the complex quantities $\overline{\gamma}_j$ in such a way that $\overline{\Gamma}_N(\eta_j) = \overline{\gamma}_j\,\sqrt{1-\eta_j^2}$ when $\eta = \eta_j$.

In Eq.25 also the derivative of $\overline{\Gamma}_N(\eta)$ with respect to η is needed. This can be written [30]:

$$\overline{\Gamma}'_N(\eta) = -\frac{2}{N+1}\frac{1}{\sqrt{1-\eta^2}}\sum_{j=1}^{N}\overline{\gamma}_j\sin\psi_j\sum_{m=1}^{N} m\sin m\psi_j\cos m\psi \tag{28}$$

Inserting Eqs.26 and 28 into Eq.25 and collocating the latter on the points of abscissa $y = y_k$ given by:

$$y_k = \cos\psi_k; \qquad \psi_k = \frac{k\pi}{N+1}, \tag{29}$$

i.e. on the zeroes of $\mathrm{U}_N(y)$, leads to the complex linear algebraic system

$$\sum_{j=1}^{N} E_{kj}\overline{\gamma}_j = 4\overline{w}(x, y_k) \tag{30}$$

The coefficient matrix $\mathbf{E}$ is

$$\mathbf{E} = \mathbf{S}\left(\mathbf{A} + \mathrm{i}\omega^\star\mathbf{X}\mathbf{B} - \frac{\omega^{\star 2}}{2}\mathbf{C}\right) + \mathbf{D}$$

where $\mathbf{S}$ and $\mathbf{X}$ are diagonal matrices whose elements are respectively given by

$$\begin{aligned} S_k &= (1 + \mathrm{sgn}\sigma_{ok})\,\mathrm{e}^{-\mathrm{i}\omega^\star\sigma_{ok}} \\ x'_k &= x'_l(y_k) \end{aligned}$$

being $\sigma_{ok} = x(y_k) - x_l(y_k)$. The four matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ correspond to the four integrals of the r.h.s. of Eq.25. In the following these matrices are separately illustrated; the integrals are evaluated with respect to the variable ψ partitioned as in Eq.29 (with y substituted through η and k through j).

a) The matrix $\mathbf{A}$ is related to the Hilbert term

$$\mathrm{H}\left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\eta}\right) \simeq \mathrm{H}(\overline{\Gamma}'_N) \simeq \sum_{j=1}^{N} A_{kj}\overline{\gamma}_j$$

and is the same of the steady case. Its elements can be found in [4]:

$$A_{kj} = \begin{cases} (N+1)/2 & \text{if } k = j, \\ 0 & \text{if } |k-j| \text{ is even}, \\ \frac{\sin\psi_j}{2(N+1)\sin\psi_k}\left[\left(\tan\frac{\psi_j+\psi_k}{2}\right)^{-2} - \left(\tan\frac{\psi_j-\psi_k}{2}\right)^{-2}\right] & \\ & \text{if } |k-j| \text{ is odd} \end{cases}$$

b) The matrix $\mathbf{B}$ is related to the integral

$$\frac{1}{\pi}\int_{-1}^{1}\ln|\eta - y|\,\overline{\Gamma}'(\eta)\,\mathrm{d}\eta \simeq \frac{1}{\pi}\int_{-1}^{1}\ln|\eta - y_k|\,\overline{\Gamma}'_N(\eta)\,\mathrm{d}\eta \simeq \sum_{j=1}^{N} B_{kj}\overline{\gamma}_j$$

By noting that
$$\cos m\psi \equiv \cos(m \arccos \eta) = \mathrm{T}_m(\eta)$$
where T_m is the Chebyshev polynomial of the first kind, the relation holds [31]:
$$\frac{1}{\pi}\int_{-1}^{1} \ln|\eta - y_k| \frac{\mathrm{T}_m(\eta)}{\sqrt{1-\eta^2}}\,\mathrm{d}\eta = -\frac{\mathrm{T}_m(y_k)}{m} = -\frac{1}{m}\cos\left(\frac{mk\pi}{N+1}\right)$$
giving ultimately, after introduction of Eq.28:
$$B_{kj} = \frac{2}{N+1}\sqrt{1-\eta_j^2}\sum_{m=1}^{N} \sin\left(\frac{mj\pi}{N+1}\right)\cos\left(\frac{mk\pi}{N+1}\right)$$

c) The matrix $\mathbf{C}$ is related to the integral
$$\frac{1}{\pi}\int_{-1}^{1} \ln|y-\eta|\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta \simeq \frac{1}{\pi}\int_{-1}^{1} \ln|y_k-\eta|\,\overline{\Gamma}_N(\eta)\,\mathrm{d}\eta \simeq \sum_{j=1}^{N} C_{kj}\overline{\gamma}_j$$
It can be evaluated by recurring to the formulae (the proof is given in Appendix B):
$$\frac{1}{\pi}\int_{-1}^{1} \ln|\eta - y_k|\sqrt{1-\eta^2}\,\mathrm{U}_m(\eta)\,\mathrm{d}\eta = \tag{31}$$
$$= \begin{cases} -\frac{1}{2}\ln 2 + \frac{1}{4}\mathrm{T}_2(y_k) & \text{if } m=0 \\ -\frac{1}{2m}\mathrm{T}_m(y_k) + \frac{1}{2(m+2)}\mathrm{T}_{m+2}(y_k) & \text{if } m \geq 1 \end{cases}$$
and to the representation of the U_N's in terms of the fundamental Lagrangian polynomials:
$$l_{j,N-1}(\eta) := \frac{\mathrm{U}_N(\eta)}{(\eta-\eta_j)\mathrm{U}'_N(\eta_j)} = \sum_{m=0}^{N-1} c_{mj}\mathrm{U}_m(\eta)$$
where the coefficients c_{mj} are uniquely determined from the solution of the N linear systems:
$$l_{j,N-1}(\eta_i) = \sum_{m=0}^{N-1} c_{mj}\mathrm{U}_m(\eta_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$
Hence:
$$C_{kj} = c_{0j}\left[-\frac{\ln 2}{2} + \frac{\mathrm{T}_2(y_k)}{4}\right] + \sum_{m=1}^{N-1} c_{mj}\left[-\frac{\mathrm{T}_m(y_k)}{2m} + \frac{\mathrm{T}_{m+2}(y_k)}{2(m+2)}\right]$$

d) The matrix $\mathbf{D}$ is related to the regular integral
$$\frac{1}{\pi}\int_{-1}^{1} \mathrm{T}(\sigma, y-\eta)\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta \simeq \frac{1}{\pi}\int_{-1}^{1} \mathrm{T}(y_k,\eta)\,\overline{\Gamma}_N(\eta)\,\mathrm{d}\eta \simeq \sum_{j=1}^{N} D_{kj}\overline{\gamma}_j$$
Applying the Gauss-Chebyshev quadrature formula to the regular function $\mathrm{T}(y_k,\eta)\,\overline{\Gamma}_N(\eta)$ the integral is evaluated as
$$\frac{1}{N+1}\sum_{m=1}^{N}(1-\eta_m^2)\,\mathrm{T}(y_k,\eta_m)\sum_{j=1}^{N}\frac{\mathrm{U}_N(\eta_m)}{(\eta_m-\eta_j)\mathrm{U}'_N(\eta_j)}\overline{\gamma}_j =$$
$$= \frac{1}{N+1}\sum_{j=1}^{N}(1-\eta_j^2)\,\mathrm{T}(y_k,\eta_j)\,\overline{\gamma}_j$$

because only the terms with $j = m$ contribute to the sum. Hence:

$$D_{kj} = \frac{1-\eta_j^2}{N+1}\,\mathrm{T}(y_k, \eta_j)$$

It can be remarked that matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are all real. The overall coefficient matrix $\mathbf{E}$, however, is complex because such are the matrices $\mathbf{D}, \mathbf{S}$ and $\mathrm{i}\omega^{\star}\mathbf{X}$ contained in it.

The rigid wing geometry is in general symmetrical across midspan, but its motion, described by the quantity $\overline{w}(x, y_k)$ in Eq.30 evaluated on the 3/4-chord line $x_r = x(y_k)$, may be of any type. However, symmetrical and antisymmetrical motions are the most important ones: in these cases also the unknown $\overline{\gamma}_j$ vector is symmetrical or antisymmetrical, according to $\overline{w}$. To save computer storage and to reduce execution times it is then expedient to solve only for one semispan: so doing the order of the problem reduces to $N/2$ (for even N: note that in this case the used partition Eq.29 excludes the central point). The coefficient matrix $\mathbf{E}$ must be replaced through two reduced matrices, say $\mathbf{E}^s$ and $\mathbf{E}^a$ for the symmetrical and antisymmetrical cases respectively, whose elements are related to those of $\mathbf{E}$ by:

$$E^s_{kj} = E_{kj} + E_{k,N+1-j}; \qquad E^a_{kj} = E_{kj} - E_{k,N+1-j}$$

with $k, j = 1, N/2$.

In order to give error estimates for the outlined procedure, weighted Sobolev-like norms are introduced as follows. Let ρ be the Jacobi weight function (for usage reasons the same symbol as for the fluid density has been here adopted):

$$\rho(y) = (1-y)^{\alpha}(1+y)^{\beta}, \qquad y \in (-1, 1), \quad \alpha, \beta > -1$$

and $L^2_\rho = L^2_\rho(-1,1)$ be the Hilbert space of all square integrable functions on the interval $(-1, 1)$ with respect to the weight $\rho(y)$, endowed with the scalar product

$$(f, g)_\rho := \frac{1}{\pi}\int_{-1}^{1} f(y)\overline{g(y)}\rho(y)\,\mathrm{d}y \tag{32}$$

and with the norm $\|f\|_\rho := \sqrt{(f,f)_\rho}$. Let $p_n \equiv \mathrm{P}_n^{(\alpha,\beta)}(y)$, $n = 0, 1, \ldots$ denote the Jacobi polynomial of degree n, orthonormal with respect to the scalar product Eq.32 and with positive leading coefficients. Furthermore, for any real number $s \geq 0$, the subspace $L^2_{\rho,s} = L^2_{\rho,s}(-1,1)$ of L^2_ρ is defined as

$$L^2_{\rho,s} := \{f \in L^2_\rho : \|f\|_{\rho,s} < \infty\}$$

where $\|f\|_{\rho,s} := \sqrt{(f,f)_{\rho,s}}$ and

$$(f, g)_{\rho,s} := \sum_{m=0}^{\infty}(1+m)^{2s}(f, p_m)_\rho\overline{(g, p_m)_\rho}$$

Then $L^2_{\rho,s}$ is again a Hilbert space, where $L^2_{\rho,0} = L^2_\rho$, and the set Π of algebraic polynomials is dense in $L^2_{\rho,s}$. Moreover, it is well known and easy to see that

$$L^2_{\rho,s} \subseteq L^2_{\rho,t} \quad \text{and} \quad \|f\|_{\rho,t} \leq \|f\|_{\rho,s}$$

for $f \in L^2_{\rho,s}$ and all $0 \le t \le s$. For $0 \le t < s$, the space $L^2_{\rho,s}$ is compactly embedded in $L^2_{\rho,t}$.

In particular, let us consider the weights $\nu(y) = \sqrt{1-y^2}$ and $\mu(y) = 1/\nu(y)$. Notice that, via $\overline{\Gamma}(-1) = \overline{\Gamma}(1) = 0$, the last integral in Eq.25 can also be rewritten as

$$\frac{1}{\pi}\int_{-1}^{1} \mathrm{T}(\sigma, y-\eta)\,\overline{\Gamma}(\eta)\,\mathrm{d}\eta = \frac{1}{\pi}\int_{-1}^{1} \tilde{N}(y,\eta)\,\frac{\mathrm{d}\overline{\Gamma}(\eta)}{\mathrm{d}\eta}\,\mathrm{d}\eta$$

where

$$\tilde{N}(y,\eta) := \int_{\eta}^{1} \mathrm{T}(\sigma, y-\tau)\,\mathrm{d}\tau.$$

In what follows we make assumptions about the smoothness of the kernel function $\tilde{N}(y,\eta)$. Suppose

$$\tilde{N}(\cdot,\eta) \in L^2_{\nu,s}, \qquad \text{uniformly with respect to } \eta \in [-1,1], \text{ and}$$

$$\tilde{N}(y,\cdot) \in L^2_{\mu,r}, \qquad \text{uniformly with respect to } y \in [-1,1],$$

with $s > \frac{1}{2}$ and $r \ge s + \frac{1}{2}$. More precisely, we assume that there are certain positive constants C_1 and C_2 independent of both η and y such that

$$\|\tilde{N}(\cdot,\eta)\|_{\nu,s} \le C_1 \quad \text{and} \quad \|\tilde{N}(y,\cdot)\|_{\mu,r} \le C_2,$$

for all $y, \eta \in [-1,1]$. Note that the latter conditions are certainly satisfied if the function $\tilde{N}$ possesses continuous partial derivatives up to the order r. As visible from Eqs.20 and 24 where it is always $\sigma_o > 0$, the order of r is 2 or more if the coefficient L_o vanishes, whereas it cannot exceed 1 if $L_o \neq 0$ because of the term $z\ln|z|$ in expansion Eq.24. Being L_o proportional to $x_l'(y)$, the first case is an important one because it relates to a straight wing; also a lunate wing, however, may display a more or less large midspan region in which $x_l'(y)$ and L_o are very small and r can be greater than 1. Let now the homogeneous Eq.25 have only the trivial solution in the subspace $L^{2,0}_{\nu}$ of all functions $u \in L^2_{\nu}$ satisfying

$$\int_{-1}^{1} u(y)\,\mathrm{d}y = 0.$$

Moreover, assume $\overline{w}(x,\cdot) \in L^2_{\nu,s}$. Then the linear algebraic system Eq.30 is uniquely solvable for all sufficiently large N. For the solution $\overline{\Gamma}$ of Eq.25 and the approximate solution Eq.26 the error estimates hold:

$$\left\| \frac{\mathrm{d}\overline{\Gamma}(\eta)}{\mathrm{d}\eta} - \frac{\mathrm{d}\overline{\Gamma_N}(\eta)}{\mathrm{d}\eta} \right\|_{\nu,t} \le \text{const} \cdot N^{t-s} \left\| \frac{\mathrm{d}\overline{\Gamma}(\eta)}{\mathrm{d}\eta} \right\|_{\nu,s} \tag{33}$$

for $0 \le t < s$ and

$$|\overline{\Gamma}(\eta) - \overline{\Gamma}_N(\eta)| + \sqrt{1-\eta^2}\left| \frac{\mathrm{d}\overline{\Gamma}(\eta)}{\mathrm{d}\eta} - \frac{\mathrm{d}\overline{\Gamma_N}(\eta)}{\mathrm{d}\eta} \right| \le \text{const} \cdot N^{t-s} \left\| \frac{\mathrm{d}\overline{\Gamma}(\eta)}{\mathrm{d}\eta} \right\|_{\nu,s} \tag{34}$$

if $\frac{1}{2} < t < s$ and $-1 \le \eta \le 1$.

The proof of the estimate Eq.33 is given in Appendix C. The inequality Eq.34 is an immediate consequence of Eq.33, of the estimate

$$\max_{-1\le\eta\le1} \sqrt{1-\eta^2}\,|f(\eta)| \le \text{const} \cdot \|f\|_{\nu,s} \tag{35}$$

which is true provided $f \in L^2_{\nu,s}$ with $s > 1/2$ (the proof is given in Appendix D), and of the obvious inequality

$$|\overline{\Gamma}(\eta) - \overline{\Gamma}_N(\eta)| \leq \text{const} \cdot \left\| \frac{d\overline{\Gamma}(\eta)}{d\eta} - \frac{d\overline{\Gamma}_N(\eta)}{d\eta} \right\|_{\nu,0} .$$

5 Method appraisal and discussion

To evaluate the capability and discuss the performances of this method the rigid motion simulating the fundamental modes of pitching and heaving have been computed for a few representative geometrical configurations.

If $x_A(y)$ is the equation of a reference line for the vertical displacement $h(y,t)$ and for the sectional rotation $\alpha(y,t)$ then the deflection of any point $P(x,y)$ of the wing surface is $h(y,t) - \Delta x(y)\,\alpha(y,t)$, being $\Delta x(y)$ the downstream distance between P and the reference line at fixed y (see Fig.3). In the present model quantities $h(y,t)$ and $\alpha(y,t)$ have been given the form

$$h(y,t) = \overline{h}_e f(y)\, \mathrm{e}^{\mathrm{i}\omega t}; \qquad \alpha(y,t) = \overline{\alpha}_e f(y)\, \mathrm{e}^{\mathrm{i}\omega t}$$

where $\overline{h}_e$ and $\overline{\alpha}_e$ are the complex amplitudes of the harmonic heave and pitch motions at wing tips and $f(y)$ is a shape function describing their spanwise distributions. Rigid wing motions are represented by $f = 1$ or $f = y$, whereas $f = \sin(\pi y/2)$ describes a possible sinusoidal mode.

To solve Eq.30 in the framework of the Pistolesi - Weissinger theory, the downwash has to be evaluated on the 3/4-chord line of the wing described by the equation $x = x_r(y)$. This leads to

$$\overline{w} = \left\{ \mathrm{i}\omega^\star \overline{h}_e - [1 + \mathrm{i}\omega^\star \Delta x_r(y)]\; \overline{\alpha}_e \right\} f(y)$$

being now $\Delta x_r(y) = x_r(y) - x_A(y)$. In the following computations the reference curve $x_A(y)$ has been supposed to coincide with the midchord line: therefore $\Delta x_r(y)$ always amounts to one quarter of the local chord. As for the amplitudes $\overline{h}_e$ and $\overline{\alpha}_e$, the first one has been given zero reference phase: consequently its values are always real. The amplitude $\overline{\alpha}_e$ of the pitch motion, conversely, is generally complex, allowing for in phase as well as out of phase components with respect to the heave motion.

The numerical procedure to solve, described in section 4, allows for the analytical deduction of error estimates in terms of Sololev-like norms of the first derivative of the circulation distribution, see formula 33 and the inequality Eq.34. More physically, we prefer here to present examples of convergence based on a global result such as the complex lift coefficient, that is an integral operator of the unknown function. Its definition is the following:

$$C_L = \frac{\rho V_\infty \int_{-b}^{b} \Upsilon(Y)dY}{\frac{1}{2}\rho V_\infty^2 S} = \frac{2b^2}{S} \int_{-1}^{1} \Gamma(y)dy$$

Applying the Gauss-Chebyshev quadrature formula to the function $\Gamma(y)$, see formula 26, the amplitude of the lift coefficient is given by:

N	Straight wing $AR = 2$		Curved wing $AR = 5.1$	
	$\Re(\overline{C_L})$	$\Im(\overline{C_L})$	$\Re(\overline{C_L})$	$\Im(\overline{C_L})$
2	3.385702	2.192085E-01	3.794706	1.064616E-01
4	3.880250	2.391228E-01	4.246472	1.028718E-01
8	4.040998	2.355483E-01	4.217864	1.027298E-01
16	4.078930	2.305990E-01	4.181500	1.027043E-01
32	4.095923	2.278647E-01	4.194034	1.011701E-01
64	4.104604	2.264649E-01	4.200689	1.005210E-01
128	4.108951	2.257468E-01	4.204317	1.001976E-01
256	4.110828	2.254405E-01	4.205995	1.000935E-01
512	4.111004	2.255132E-01	4.206315	1.002114E-01

Table 1: Results of the convergence tests

$$\overline{C}_L \simeq \frac{2b^2}{S}\frac{\pi}{N+1}\sum_{j=1}^{N}(1-\eta_j^2)\overline{\gamma}_j$$

In table 1 a simple precision convergence test for a straigth and a curved wing configuration is presented. The number of collocation points N refers to a wing semispan, as explained in section 4. All numerical solutions used to assess the method and described in the following have been obtained with N=128, a value arising from table 1 as a good compromise between precision and computing time.

The results of the computations are shown through Figs.4-8. To discuss their propriety we chosed as reference data the body of results produced by Laschka [16] in 1963. The reason of this choise is manyfold. First they are presented in a suitable way for the kind of outcomes we obtain from the present method, because they provide the distribution of $\overline{\Gamma}$ along the span. In references [19] to [25] we find instead the, more detailed, plots of the pressure distributions along each wing section profile. To obtain $\overline{\Gamma(y)}$ it would then be necessary to integrate the functions represented in these plots, an intrinsically inaccurate procedure which would not provide reliable reference data. Secondly, Laschka's results cover a broad class of exemplar wings and are relatively recent, having been published sixteen years after the other well known comprehensive set of computations by Reissner [8] and by Reissner and Stevens [9]. Thirdly, Laschka's results are analytically well based and come about a basically more correct lifting-surface theory.

As already pointed out in the introduction a global feature of this simplified theory, as based on the 3/4-chord Pistolesi-Weissinger method and on the unsteady Possio theory for the motion of lifting surfaces, is that its accuracy lowers with increasing reduced frequency $\omega^\star$. Nevertheless, results display generally a satisfying agreement with the reference data by Laschka up to $\omega^\star \sim 1$, a boundary which is also found in the planar case [26]. Note that this limit value on the $\omega^\star$ does not imply an equivalent limit on the pulsation ω, because $\omega^\star \leq 1$ actually means that the product $\omega b \leq V_\infty$. Therefore in case of a low aspect ratio wing the pulsation migth assume a relatively high value and vice versa.

Figs.4 and 5 refer to a straight rectangular wing of aspect ratio $AR = 2$. Fig.4 shows flapping motion results for reduced frequencies up to $\omega^\star = 2$. The quality of the computations turns out to be about uniform in the above range of reduced frequencies, a fact that is beyond expectation. In fact, relative errors are at most about 4%, see part c of Fig.4 pertaining to the global complex lift coefficient computation. For the same wing the case of the pitch motion has been also computed (see Fig.5). In this situation the agreement is still good qualitatively as far as the circulation distribution shape is involved, but the numerical values of $\Re(\overline{\Gamma})$ and $\Im(\overline{\Gamma})$, compared to those by Laschka, are less accurate with respect to the flapping situation. Anyway they are still loosely acceptable up to $\omega^\star \sim 1$, where the maximum relative error rise to about 15% for $\Re(\overline{\Gamma})$ and to about 30% for $\Im(\overline{\Gamma})$. For values of $\omega^\star$ of the order of 2 our numerical outcomes must be disregarded, at least in so far as the wing bound vorticity distribution remains concentrated in one lumped curved vortex placed at the forward quarter of the chord of each wing section. As suggested in [26], for the planar case, the ratio between $\Re(\overline{\Gamma})$ and $\Im(\overline{\Gamma})$, for a wing subject to pitch oscillations, may be improved by optimizing the position of the bound vortex to respect to the chord. For instance in [26] the optimal position has been recognized to be equal to the 15% of the chord length (of course, in so doing the aerodynamic moment will no more assume the correct value for $\omega^\star = 0$). We did not carry this parametric study because it would have been beyond the scope of the present work. Moreover, this would not be the only way to raise the general performances of the method: for instance, distributing the bound vorticity on two or more discrete vortices, toghether with their wakes, instead of on only one, would surely improve results. We did not take advantage of this issue. Here we want just to underline that the method still allows for further improvements.

The ensemble of test cases computed by Laschka includes also a delta wing ($A\!\!\!R$=4) and a straight 45° swept wing ($A\!\!\!R$=2) with constant chord. The circulation distributions relevant to this two wings in flapping motion have been computed and the results are in Figs.6 and 7. The agreement with Laschka outcomes appears to be excellent for the delta wing, Fig.6. Such a result is at a first glance beyond expectation; however, a possible explanation arises if one considers that the delta configuration allows for a higher aspect ratio ($A\!\!\!R$=4) with the same semispan value (b=2) of the previous wing and consequently with the same values of $\omega^\star$. Therefore the delta configuration mets better the lifting line model requests. The confront of the 45° swept wings, Fig.7, is very good as the module of $\overline{\Gamma}$ is concerned and moderately good as its phase Φ is concerned (in this case the maximum relative error is about 13%).

To conclude with a different wing configuration, we show also the results for a curved elliptic wing of aspect ratio 5.1 and a aft-swept of the tips equal to 2 undergoing a flapping motion, see Fig.8. This wing has a lunate shape and could be considered of interest in the context of animal locomotion. For this very reason and to illustrate the capability of the model, we decided to include this case even if we could not find in literature a similar numerical solution to compare with our results.

6 Conclusions

This theoretical study allows the determination in a conceptually simple way of the circulation distribution along the span of a curved wing exercising small amplitude harmonic oscillations. It presents an unsteady lifting line theory based on the Pistolesi-Weissinger 3/4-chord method coupled with the Possio theory for the unsteady motion of a lifting surface. It provides an extension to non-steady harmonic conditions of the steady lifting line model for a curved wing proposed in [4].

The mathematical formulation is developed rigorously and leads to an integro-differential equation that may be loosely classified as of modified Prandtl's type, because, besides the finite Hilbert transform of the derivative of the circulation, it displays other singular terms represented by finite logarithmic transforms of the circulation and its first derivative. These singularities have been accurately analysed and treated. A detailed numerical procedure of resolution of that equation, plus relevant boundary conditions, have been elaborated, under similar guidelines as those used in [4], through the application of the Gauss quadrature method based on Chebyshev second kind polynomial approximation of the unknown function. The convergence characteristics are assessed.

Numerical solutions for several wing configurations and oscillatory motions have been carried out and compared to corresponding results by Laschka [16]. The confront is positive, above all in the case of flapping harmonic oscillations. It is noticeable that the behaviour of the method reflects the physical restrictions of the theory on which it is founded and is not subject to any limitation of analytical or numerical nature. Results are good for slow and moderately fast pulsating motions, such that the reduced frequency based on the wing semispan be less or equal to 1. Remarkably this value, that actually turns out to be also a limit of validity for other investigation based on the method of asymptotic matched expansions, see [20] to [24], does not appear to be an intrinsic property of the mathematical treatment used here. As a consequence the numerical solutions converge for any value of the reduced frequency.

This method is capable of improvement by searching for the optimal values of a few key parameters in order to best satisfy the physical constraints relevant to the kind of motion and configuration under simulation. Increasing the number of discrete vortices used to represent the bound vorticity would be a further possible development able to ameliorate the global performances of the method. In such a way, for instance, it would be possible to predict also the aerodynamic pitching moment and the method could find application in preliminary estimates of wing aeroelastic behaviour. In the present form, however, the prevailing application field seems to be the naturalistic one. Flapping motions with reduced frequencies well in the validity range of the model are in fact a common occurrence in many phases of birds flight.

Appendix A - Evaluation of the integral $\mathbf{S}(\sigma, z)$

First $\mathrm{S}(\sigma, z)$ is splitted in two integrals S_1 and S_2 running respectively from $-\infty$ to 0 and from 0 to σ. Then the integration boundaries of the former are changed from $(-\infty, 0)$ to $(0, +\infty)$ by substituting the inner variable ζ through $-\zeta$; then S_1 assumes the form:

$$\mathrm{S}_1(\sigma, z) = \int_0^\infty \left(1 - \frac{\zeta}{\sqrt{\zeta^2 + z^2}}\right) \mathrm{e}^{-\mathrm{i}\omega^\star \zeta}\, \mathrm{d}\zeta$$

To its evaluation it is expedient to start from the integral

$$\Lambda^\star(z) = \int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}\omega^\star \zeta}}{(\zeta^2 + z^2)^{3/2}}\, \mathrm{d}\zeta$$

whose analytical form is known (see [32] p.376 and 498 for separate real and imaginary parts):

$$\Lambda^\star(z) = \frac{\omega^\star}{|z|} \left\{ \mathrm{K}_1(\omega^\star |z|) - \mathrm{i}\frac{\pi}{2} \left[\mathrm{I}_1(\omega^\star |z|) - \mathrm{L}_{-1}(\omega^\star |z|)\right] \right\}$$

Here I_1 is the modified Bessel functions of first kind and order 1, K_1 that of second kind and order 1 and L_{-1} the modified Struve function of order -1. Integrating $\Lambda^\star$ by parts gives

$$\begin{aligned}\Lambda^\star(z) &= \left[\mathrm{e}^{-\mathrm{i}\omega^\star\zeta}\left(\frac{\zeta}{z^2\sqrt{\zeta^2+z^2}}+C(z)\right)\right]_0^\infty + \\ &+ \mathrm{i}\omega^\star\int_0^\infty \mathrm{e}^{-\mathrm{i}\omega^\star\zeta}\left(\frac{\zeta}{z^2\sqrt{\zeta^2+z^2}}+C(z)\right)\mathrm{d}\zeta\end{aligned}$$

The arbitrary function $C(z)$ has been introduced in order to make the result bounded. Indeed, by choosing $C(z)=-1/z^2$ one obtains:

$$\Lambda^\star(z)=\frac{1}{z^2}-\frac{\mathrm{i}\omega^\star}{z^2}\mathrm{S}_1(z)$$

From this relation S_1 can be obtained. After summing it to the integral S_2 rewritten as:

$$\mathrm{S}_2(\sigma,z)=\frac{\mathrm{i}}{\omega^\star}\left(1-\mathrm{e}^{\mathrm{i}\omega^\star\sigma}\right)+\int_0^\sigma\frac{\zeta\,\mathrm{e}^{\mathrm{i}\omega^\star\zeta}}{\sqrt{\zeta^2+z^2}}\mathrm{d}\zeta$$

the expression for $\mathrm{S}(\sigma,z)$ finally arises

$$\begin{aligned}\mathrm{S}(\sigma,z) &= -\frac{\mathrm{i}}{\omega^\star}\mathrm{e}^{\mathrm{i}\omega^\star\sigma}+\int_0^\sigma\frac{\zeta\,\mathrm{e}^{\mathrm{i}\omega^\star\zeta}}{\sqrt{\zeta^2+z^2}}\mathrm{d}\zeta - \\ &- |z|\left\{\frac{\pi}{2}\left[\mathrm{I}_1(\omega^\star|z|)-\mathrm{L}_{-1}(\omega^\star|z|)\right]-\mathrm{i}\mathrm{K}_1(\omega^\star|z|)\right\}\end{aligned}$$

The integral

$$\Lambda(\sigma,z)=\int_0^\sigma\frac{\zeta\,\mathrm{e}^{\mathrm{i}\omega^\star\zeta}}{\sqrt{\zeta^2+z^2}}\mathrm{d}\zeta$$

can be evaluated only numerically. In the present work this has been done by means of Filon's integration formula (see [32] pp.890-891); difficulties arised only at very high values of $\omega^\star$ because of numerical instability, but it was possible to overcome them by using the following expansion valid for $\omega^\star \gg 1$ and $z \neq 0$:

$$\begin{aligned}\Lambda(\sigma,z) &\approx \frac{\sigma\,\mathrm{e}^{\mathrm{i}\omega^\star\sigma}}{\mathrm{i}\omega^\star\sqrt{\sigma^2+z^2}}+\frac{1}{\omega^{\star 2}}\left[\frac{z^2\,\mathrm{e}^{\mathrm{i}\omega^\star\sigma}}{(\sigma^2+z^2)^{3/2}}-\frac{1}{|z|}\right]+ \\ &+ \frac{3z^2\sigma\,\mathrm{e}^{\mathrm{i}\omega^\star\sigma}}{\mathrm{i}\omega^{\star 3}(\sigma^2+z^2)^{5/2}}-\frac{3}{\omega^{\star 4}}\left[\frac{z^2(z^2-4\sigma^2)\,\mathrm{e}^{\mathrm{i}\omega^\star\sigma}}{(\sigma^2+z^2)^{7/2}}-\frac{1}{|z|^3}\right]+ \\ &+ \frac{15z^2\sigma(3z^2-4\sigma^2)\,\mathrm{e}^{\mathrm{i}\omega^\star\sigma}}{\mathrm{i}\omega^{\star 5}(\sigma^2+z^2)^{9/2}}+\ldots \qquad \text{(A.1)}\end{aligned}$$

In Fig.9 the values of $\Lambda(z)$ given by Filon's integration formula are compared to those obtained from expansion (A.1) for a curved wing with constant chord and for $\omega_o^\star=2$ (i.e. $\omega^\star=10$, being $b/l_o=5$: this is a sufficiently high value for Eq.A.1 to hold). Plotted is also the fourth order expansion of $\Lambda(z)$ for small z which had to be derived as an intermediate step to expand the full function T_u (see Eqs.22 and 24). The excellent agreement between these expansions and the results of Filon's formula indicate the good degree of confidence with which the latter has been used.

Appendix B - Proof of Eqs.31

Proposition. *The Chebyshev polynomial of first and second kind* $\mathrm{T}_k(x)$ *and* $\mathrm{U}_k(x)$ *respectively fulfil the relations:*

$$\frac{1}{\pi}\int_{-1}^{1} \ln|t-s|\,\sqrt{1-t^2}\,\mathrm{U}_k(t)\,\mathrm{d}t = \tag{B.1}$$

$$= \begin{cases} -\frac{1}{2}\ln 2 + \frac{1}{4}\mathrm{T}_2(s) & \text{if } k=0 \\ \\ -\frac{1}{2k}\mathrm{T}_k(s) + \frac{1}{2(k+2)}\mathrm{T}_{k+2}(s) & \text{if } k \geq 1 \end{cases}$$

Proof. First, notice that ([33], (4.1.7))

$$\mathrm{U}_k(x) = \frac{1}{2d_{k+1}}\mathrm{P}_k^{\left(\frac{1}{2},\frac{1}{2}\right)}(x), \qquad d_k = \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}$$

where $\mathrm{P}_k^{(\alpha,\beta)}(x)$, $\alpha,\beta > -1$, denote the Jacobi polynomials orthogonal on $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, see [33], Chapt.4. Using the relation [33] (4.10.1):

$$\sqrt{1-t^2}\,\mathrm{P}_k^{\left(\frac{1}{2},\frac{1}{2}\right)}(t) = -\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)^{3/2}\,\mathrm{P}_{k-1}^{\left(\frac{3}{2},\frac{3}{2}\right)}(t)\right]$$

and integrating by parts ([29], Corollary 6.1), we deduce

$$\frac{1}{\pi}\int_{-1}^{1}\ln|t-s|\,\sqrt{1-t^2}\,\mathrm{P}_k^{\left(\frac{1}{2},\frac{1}{2}\right)}(t)\,\mathrm{d}t = \frac{1}{2k}\frac{1}{\pi}\int_{-1}^{1}(1-t^2)^{3/2}\,\frac{\mathrm{P}_{k-1}^{\left(\frac{3}{2},\frac{3}{2}\right)}(t)}{t-s}\,\mathrm{d}t$$

Combining these relations with the one quoted in [33], (4.5.5):

$$(1-t^2)\,\mathrm{P}_{k-1}^{\left(\frac{3}{2},\frac{3}{2}\right)}(t) = \frac{2k+1}{2(k+1)}\mathrm{P}_{k-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(t) - \frac{2k}{2k+3}\mathrm{P}_{k+1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(t)$$

and [30]

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,\frac{\mathrm{U}_k(t)}{t-s}\,\mathrm{d}t = -\mathrm{T}_{k+1}(t)$$

the proposition can be deduced for $k \geq 1$.

If $k=0$ a separate evaluation is needed. In this case $\mathrm{U}_0(t)=1$ and the integrand on the l.h.s. of Eq.B.1 reduces to $\ln|t-s|\,\sqrt{1-t^2}$. The expansion (see [31], p.311)

$$\ln|t-s| = -\ln 2 - \sum_{m=1}^{\infty}\frac{2}{m}\mathrm{T}_m(s)\mathrm{T}_m(t)$$

can then be introduced to obtain

$$\frac{1}{\pi}\int_{-1}^{1}\ln|t-s|\,\sqrt{1-t^2}\,\mathrm{d}t =$$

$$-\frac{\ln 2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,\mathrm{d}t - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\mathrm{T}_m(s)}{m}\int_{-1}^{1}\mathrm{T}_m(t)\,\sqrt{1-t^2}\,\mathrm{d}t$$

The value of the first integral is $\pi/2$; for the one included in the sum we write $t = \cos\psi$ and therefore $\mathrm{T}_m(t) = \cos(m\psi)$. Then

$$\int_{-1}^{1} \mathrm{T}_m(t)\,\sqrt{1-t^2}\,\mathrm{d}t = \int_{0}^{\pi} \cos(m\psi)\,\sin^2\psi\,\mathrm{d}\psi$$

The last integral is different from zero only if $m = 2$ ([36], p.372 no.7 and p.373 no.12). In this case the value is $-\pi/4$ and therefore the proposition is verified also for $k = 0$.

Appendix C - Proof of Eq.33

Rewrite Eq.25 in the equivalent operator form

$$(\mathrm{H} + \mathrm{L}_1 + \mathrm{L}_2 + \tilde{\mathrm{T}})u = \tilde{w} \tag{C.1}$$

where $u = \mathrm{d}\overline{\Gamma}/\,\mathrm{d}\eta \in L^{2,0}_{\nu}$ is the unknown solution,

$$\begin{aligned}
\mathrm{L}_1 u(y) &= i\omega^* x_l'(y)\frac{1}{\pi}\int_{-1}^{1} \ln|\eta - y|\,u(\eta)\,\mathrm{d}\eta,\\
\mathrm{L}_2 u(y) &= -\frac{\omega^{*2}}{2\pi}\int_{-1}^{1} \ln|\eta - y|\int_{-1}^{\eta} u(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta,\\
\tilde{\mathrm{T}}u(y) &= \frac{m(y)}{\pi}\int_{-1}^{1} \tilde{N}(y,\eta)\,u(\eta)\,\mathrm{d}\eta,\\
\tilde{w} &= 4m\overline{w}, \qquad \text{and} \quad m(y) = (1 + \mathrm{sgn}\sigma_o)^{-1} e^{\mathrm{i}\omega^*\sigma_o}.
\end{aligned}$$

Let P_N denote the Lagrangian interpolation operator with respect to the Chebyshev polynomials of the second kind defined by

$$\mathrm{P}_N f(y) = \sum_{k=1}^{N} f(y_k)\frac{\mathrm{U}_N(y)}{(y - y_k)\,\mathrm{U}_N'(y_k)}$$

with $y_k = \cos k\pi/(N+1) = \eta_k$.

As can be easily seen from the formulas used in the subsections a) throughout d) of Section 4, Eq.30 is equivalent to the operator equation

$$\mathrm{H}u_N + \mathrm{P}_N(\mathrm{L}_1 + \mathrm{L}_2 + \tilde{\mathrm{T}}_N)\,u_N = \mathrm{P}_N\tilde{w}, \tag{C.2}$$

where $u_N = v_N/\nu$ being v_N a polynomial ($v_N \in \Pi_N$). Here $\tilde{\mathrm{T}}_N u$ is defined by the Gauss-Chebyshev quadrature formula applied to the integral

$$\frac{m(y)}{\pi}\int_{-1}^{1} \mathrm{T}(\sigma, y - \eta)\,w(\eta)\,\mathrm{d}\eta$$

with $w(\eta) := \int_{-1}^{\eta} u(\xi)\,\mathrm{d}\xi$, i.e.

$$\tilde{\mathrm{T}}_N u(y) = m(y)\sum_{k=1}^{N} \mathrm{T}(y, \eta_k)\,\frac{w(\eta_k)}{\nu(\eta_k)}\,\lambda_k$$

for $u \in L^{2,0}_{\nu}$, where

$$\lambda_k = \frac{\pi}{N+1}(1 - \eta_k^2) = \frac{\pi}{N+1}\sin^2\frac{k\pi}{N+1}$$

are the Christoffel numbers of the Chebyshev polynomial U_N.

It is well known that the Hilbert transform $\mathrm{H} : L^{2,0}_{\nu,t} \to L^2_{\nu,t}$ is a bounded and invertible operator with the bounded inverse $\mathrm{H}^{-1} = \hat{\mathrm{H}}$ defined by $\hat{\mathrm{H}}v = -\frac{1}{\nu}\mathrm{H}\nu v$ for all $v \in L^2_{\nu,t}$ and $0 \le t < s$, where $L^{2,0}_{\nu,t} = L^2_{\nu,t} \cap L^{2,0}_{\nu}$ (see, e.g., [29, 34]).

Since the operators $\mathrm{L} = \mathrm{L}_1 + \mathrm{L}_2$ and $\tilde{\mathrm{T}}$ are compact operators in this pair of spaces (see [34]) and, by assumption, the equation $(\mathrm{H}+\mathrm{L}+\tilde{\mathrm{T}})u = 0$ has only the trivial solution in $L^{2,0}_{\nu}$ we conclude that $\mathrm{A} = \mathrm{H}+\mathrm{L}+\tilde{\mathrm{T}} : L^{2,0}_{\nu,t} \to L^2_{\nu,t}$ is a bounded and invertible operator for all $t,\ 0 \le t < s$. We show that this assertion is true for the operators $\mathrm{A}_N = \mathrm{H}+\mathrm{P}_N(\mathrm{L}+\tilde{\mathrm{T}}_N)$, too, provided N is large enough. Indeed, let $u \in L^2_{\nu,t}$ and $v \in L^2_{\nu,t_1}$ be arbitrary functions with $0 \le t < t_1$. From the well known estimates

$$\|(\tilde{\mathrm{T}} - \mathrm{P}_N\tilde{\mathrm{T}}_N)u\|_{\nu,t} \le cN^{t-s}\|u\|_{\nu,t} \tag{C.3}$$

(see [34], Lemma 4.4) and

$$\|v - \mathrm{P}_N\, v\|_{\nu,t} \le cN^{t-t_1}\|v\|_{\nu,t_1} \tag{C.4}$$

(see [34], Theorem 3.4 (ii)) we derive

$$\begin{aligned}
&\|(\mathrm{H}+\mathrm{L}+\tilde{\mathrm{T}})\,u - (\mathrm{H}+\mathrm{P}_N(\mathrm{L}+\tilde{\mathrm{T}}_N))\,u\|_{\nu,t} \le \\
&\|(\tilde{\mathrm{T}} - \mathrm{P}_N\tilde{\mathrm{T}}_N)\,u\|_{\nu,t} + \|\mathrm{L}u - \mathrm{P}_N\mathrm{L}u\|_{\nu,t} \le \\
&cN^{t-s}\|u\|_{\nu,t} + cN^{t-(t+1)}\|\mathrm{L}u\|_{\nu,t+1} \le \\
&c_1N^{-\min\{s-t,1\}}\|u\|_{\nu,t}
\end{aligned}$$

since $L : L^2_{\nu,t} \to L^2_{\nu,t+1}$ is a bounded operator (see [34], Lemma 5.1 (iv)). Consequently, the norm of the operator $\mathrm{A} - \mathrm{A}_N$ in the space $L^2_{\nu,t}$ does not exceed $c_1N^{-\min\{s-t,1\}}$ which tends to zero as $N \to \infty$. Hence, for N large enough, the operators $\mathrm{A}_N : L^{2,0}_{\nu,t} \to L^2_{\nu,t}$ are invertible and the norms of their inverses A_N^{-1} are uniformly bounded with respect to N. Thus Eq. (C.2) is uniquely solvable. Its solution u_N is of the form $u_N = v_N/\nu$ with $v_N \in \Pi_N$. This follows from

$$u_N = \mathrm{H}^{-1}\mathrm{P}_N(\tilde{w} - \mathrm{L}u_N - \tilde{\mathrm{T}}_N\, u_N)$$

and the well known formula (see [30])

$$\mathrm{H}(\nu\mathrm{U}_{N-1}) = \mathrm{T}_N$$

for the action of the Hilbert transform on the Chebyshev polynomials. Therefore, by (C.3) and (C.4) and the relation

$$u_N - u = \mathrm{A}_N^{-1}(\mathrm{P}_N\tilde{w} - \mathrm{A}_N\, u)$$

we obtain

$$\begin{aligned}
&\|u_N - u\|_{\nu,t} \le c_2(\|\tilde{w} - \mathrm{P}_N\tilde{w}\|_{\nu,t} + \\
&\|\mathrm{L}u - \mathrm{P}_N\,\mathrm{L}u\|_{\nu,t} + \|\tilde{\mathrm{T}}u - \mathrm{P}_N\tilde{\mathrm{T}}_N u\|_{\nu,t}) \le \\
&c_3(N^{t-s}\|\tilde{w}\|_{\nu,s} + N^{t-s-1}\|\mathrm{L}u\|_{\nu,s+1} + N^{t-s}\|u\|_{\nu,t}) \le \\
&c_4N^{t-s}\|u\|_{\nu,s}
\end{aligned}$$

for all $t,\ 0 \le t < s$, provided $\tilde{w} \in L^2_{\nu,s}$. The proof of Eq.33 is complete.

Appendix D - Proof of Eq.35

The proof of Eq.35 runs similarly to the proof of Theorem 7 in [35].
Assume $f \in L^2_{\nu,s}$ with $s > 1/2$ and $\nu(y) = \sqrt{1-y^2}$. Setting

$$\tilde{f}(y) = \nu(y) f(y), \quad \tilde{p}_n(y) = \nu(y)\, p_n(y), \quad \mu = 1/\nu,$$

where $p_n = \mathrm{P}_n^{\left(\frac{1}{2},\frac{1}{2}\right)} = \mathrm{U}_n$, we obviously have

$$(\tilde{f}, \tilde{p}_n)_\mu = (f, p_n)_\nu, \qquad (\tilde{p}_m, \tilde{p}_n)_\mu = (p_m, p_n)_\nu = \delta_{mn}.$$

Moreover (see Theorem 7.32.2 in [33]),

$$|\tilde{p}_n(y)| = |p_n(y)| \left(\sqrt{1-y} + \frac{1}{n}\right) \left(\sqrt{1+y} + \frac{1}{n}\right) \cdot \\ \cdot \frac{\sqrt{1+y}}{\sqrt{1+y} + \frac{1}{n}} \cdot \frac{\sqrt{1-y}}{\sqrt{1-y} + \frac{1}{n}} \leq c \cdot 1 \cdot 1$$

with a positive constant c. Hence,

$$|(\tilde{f}, \tilde{p}_j)_\mu \tilde{p}_j(y)| \leq c |(\tilde{f}, \tilde{p}_j)_\mu| = c |(f, p_j)_\nu|$$

and, by the Cauchy-Schwarz inequality,

$$\sum_{j=0}^{\infty} |(\tilde{f}, \tilde{p}_j)_\mu| \leq \left(\sum_{j=0}^{\infty} (1+j)^{2s} |(f, p_j)_\nu|^2 \right)^{1/2} \cdot \\ \cdot \left(\sum_{j=0}^{\infty} (1+j)^{-2s} \right)^{1/2} \leq c' \|f\|_{\nu,s}$$

since $2s > 1$. Thus the series $\sum_{j=0}^{\infty} (\tilde{f}, \tilde{p}_j)_\mu \tilde{p}_j(y)$ is uniformly convergent with respect to y, $-1 \leq y \leq 1$, and, consequently, its sum is a continuous function F on $[-1, 1]$. Since this series coincides with the Fourier series of $\tilde{f} \in L^2_\mu$ with respect to the orthonormal system $\{\tilde{p}_n\}_{n=0}^{\infty}$ in L^2_μ, we have $F = \tilde{f}$ almost everywhere on $[-1, 1]$. Thus

$$\max_{-1 \leq y \leq 1} |\tilde{f}(y)| \leq c \cdot c' \|f\|_{\nu,s}$$

which coincides with Eq.35.

References

[1] PRANDTL, L., Tragflügeltheorie, Nachr. K. Ges. Wiss. Göttingen, Math. Phys. Klasse, 1918, pp.451-477

[2] PISTOLESI, E., Betrachtungen über die gegenseitige Beeinflussung von Tragflügelsystemen, in *Gesammelte Vorträge der Hauptversammlung 1937 der Lilienthal-Gesellschaft*, Berlin, 1937

[3] WEISSINGER, J., Über die Auftriebverteilung von Pfeilflügeln, Forsch. Ber. d. Zentr. f. wiss. Berichtswesen 1553, Berlin-Adlershof, 1942.

[4] PRÖßDORF, S., TORDELLA, D., On an extension of Prandtl's lifting line theory to curved wings, Impact Comp. Sc. Eng., Vol.3, 1991, pp.192-212.

[5] CICALA, P., Sul moto non stazionario di un'ala di allungamento finito, Rendiconti R. Acc. Lincei, Vol.25, 1937, pp.97-102

[6] CICALA, P., Le teorie approssimate dell'ala di allungamento finito, Atti R. Acc. Sc. Torino, Vol.76, 1941, pp.389-416

[7] BORBÉLY, V., Über einen Grenzfall der instationären räumlichen Tragflügelströmung, ZAMM, Vol.18, 1938, pp.319-342

[8] REISSNER, E., Effects of finite span on the airload distributions for oscillating wings, I: Aerodynamic theory of oscillating wings of finite span, NACA TN 1194, 1947

[9] REISSNER, E., STEVENS, J.E., Effects of finite span on the airload distributions for oscillating wings, II: Method of calculation and examples of application, NACA TN 1195, 1947

[10] POSSIO, C., Sulla determinazione dei coefficienti aerodinamici che interessano la stabilità del velivolo, Pont. Academ. Scient. Commentationes, Vol.III, n.6, 1938, pp.141-169

[11] POSSIO, C., L'azione aerodinamica su una superficie portante in moto oscillatorio, Atti R. Acc. Naz. Lincei, Classe Sc. Mat. Fis. Nat., Vol.28, 1938, pp.194-200

[12] POSSIO, C., Determinazione dell'azione aerodinamica corrispondente alle piccole oscillazioni del velivolo, L'Aerotecnica, Vol.XVII, n.12, 1938, pp.1323-1351

[13] POSSIO, C., Sul moto non stazionario di una superficie portante, Atti R. Acc. Scienze Torino, Parte Fisica, Vol.74, 1939, pp.285-299

[14] POSSIO, C., L'azione aerodinamica su di una superficie portante in moto vario, Atti R. Acc. Scienze Torino, Parte Fisica, Vol.74, 1939, pp.537-557

[15] KÜSSNER, H.G., Allgemeine Tragflügeltheorie, Luftfahrtforsch., Vol.17, 1940, pp.370-378

[16] LASCHKA, B., Zur Theorie der harmonisch schwingenden tragenden Fläche bei Unterschallanströmung, Zeitschr. f. Flugwissenschaften, Vol.11, n.7, 1963, pp.265-292

[17] SCHADE, T., Theorie der schwingenden kreisförmigen Tragfläche auf potentialtheoretischer Grundlage - I. Analytischer Teil, Luftfahrtforsch., Vol.17, 1940, pp.387-400

[18] KRIENES, K., SCHADE, T., Theorie der schwingenden kreisförmigen Tragfläche auf potentialtheoretischer Grundlage - II. Numerischer Teil, Luftfahrtforsch., Vol.19, 1942, pp.282-291

[19] ALBANO, E., RODDEN, W.P., A doublet lattice method for calculating lift distributions on oscillating surfaces in subsonic flows, AIAA J., vol.7, 1969, pp.279-285

[20] THURBER, J.K., An asymptotic method for determining the lift distribution of a swept-back wing of finite span , Comm. Pure Appl. Maths., Vol.18, 1965, pp.733-756

[21] JAMES, E.C., Lifting-line theory for an unsteady wing as a singular perturbation problem, J.Fluid Mech., Vol.70, n.4, 1975, pp.753-771

[22] VAN HOLTEN, TH., Some notes on unsteady lifting-line theory, J.Fluid Mech., Vol.77, n.3, 1976, pp.561-579

[23] CHENG, H.K., MURILLO, L.E., Lunate-tail swimming propulsion as a problem of curved lifting line in unsteady flow - Part 1. Asymptotic theory, J.Fluid Mech., Vol.143, 1984, pp.327-350

[24] AHMADI, A.R., WIDNALL, S.E., Unsteady lifting-line theory as a singular perturbation problem, J.Fluid Mech., Vol.153, 1985, pp.59-81

[25] GUERMOND, J.L., SELLIER, A., A unified unsteady lifting-line theory, J.Fluid Mech., Vol.229, 1991, pp.427-451

[26] CHIOCCHIA, G., The lift computation for an oscillating flat plate in incompressible potential flow, Meccanica, Vol.29, 1994, pp.43-60

[27] CHENG, H.K., On lifting-line theory in unsteady aerodynamics, in *Unsteady Aerodynamics, Proc. Symp. Univ. Arizona, 18-20 March*, ed.R.B.Kinney, 1975, pp.719-739

[28] PRANDTL, L., Beitrag zur Theorie der tragenden Fläche, ZAMM, Vol.16, n.6, 1936, pp.360-361

[29] MIKHLIN, S.G., PRÖßDORF, S., *Singular integral operators*, Springer Verlag, Berlin, N.Y., Tokyo, 1986

[30] TRICOMI, F.G., *Integral Equations*, Interscience, N.Y. 1957

[31] BERTHOLD, D., HOPPE, W., SILBERMANN, B., The numerical solution of the generalized airfoil equation, J. Integral Equations and Appl., Vol.4, no.3, 1992, pp.309-336

[32] ABRAMOWITZ, M., STEGUN, I.A., *Handbook of Mathematical Functions*, Dover Pub., 1970

[33] SZEGÖ, G., *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol.23, AMS, Providence, R.I., 1975

[34] BERTHOLD, D., HOPPE, W., SILBERMANN, B., A fast algorithm for solving the generalized airfoil equation, J. Comput. Appl. Mathematics, Vol.43, 1992, pp. 185-219.

[35] MONEGATO, G., PRÖßDORF, S., Uniform convergence estimates for a collocation and discrete collocation method for the generalized airfoil equation, in Contributions in Numer. Mathematics (ed. R.P. Agarwal), Vol.262, World Scientific Series in Applicable Analysis, 1993, pp. 285-299.

[36] GRADSHTEJN, I.S., RYZHIK, I.M., *Tables of Integrals, Series and Products*, Academic Press, N.Y. and London, 1965

List of Captions

Fig.1 - Reference scheme of the wing.

Fig.2 - Function T_u for small z. Solid line: by using Eq.22. Dashed line: approximation through $z = 0$ by means of expansion Eq.24.

Fig.3 - Motions of a generic wing section.

Fig.4 - Flapping motion of a straight rectangular wing (AR)=2: a) spanwise circulation (real part); b) spanwise circulation (imaginary part); c) complex lift coefficient versus reduced frequency $\omega_o^\star$. Solid lines: present method. Dots: Laschka's results.

Fig.5 - Pitching motion of a straight rectangular wing (AR=2): a) spanwise circulation (real part); b) spanwise circulation (imaginary part). Solid lines: present method. Dots: Laschka's results (not labelled; the sequence in $\omega_o^\star$ is the same as for the solid lines).

Fig.6 - Flapping motion of a delta wing (AR=4): a) spanwise circulation (real part); b) spanwise circulation (imaginary part). Solid lines: present method. Dots: Laschka's results.

Fig.7 - Flapping motion of a 45° swept wing (AR=2): spanwise circulation (modulus and phase). Solid lines: present method. Dots: Laschka's results.

Fig.8 - Flapping motion of a curved elliptic wing (AR=5.1): a) spanwise circulation (real part); b) spanwise circulation (imaginary part); c) complex lift coefficient versus reduced frequency $\omega_o^\star$.

Fig.9 - Function $\Lambda(z)$.