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A weak formulation for a rate-independent delamination evolution with inertial and viscosity effects subjected to unilateral constraint

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Abstract

We consider a system of two viscoelastic bodies attached on one edge by an adhesive where a delamination process occurs. We study the dynamic of the system subjected to external forces, suitable boundary conditions, and an unilateral constraint on the jump of the displacement at the interface between the bodies. The constraint arises in a graph inclusion, while the delamination coefficient evolves in a rate-independent way. We prove the existence of a weak solution to the corresponding system of PDEs.

1 Introduction

The mathematical problem. Within this paper we show the existence of solutions to the following evolution problem: we consider a system of two sufficiently smooth open and connected sets Ω_1 and Ω_2 in $\mathbb{R}^d$, with $d \leq 3$, which have Γ as common boundary. Let us denote by ν the normal versor on Γ oriented in such a way that it points outside Ω_2 (inside Ω_1), and by n the unit outer normal to $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$. Given an external force $f : [0,T] \times \Omega \to \mathbb{R}^d$, a boundary traction $g : [0,T] \times \partial_N\Omega \to \mathbb{R}^d$, and a boundary datum $w : [0,T] \times \partial_D\Omega \to \mathbb{R}^d$, we look for functions $u : [0,T] \times (\Omega = \Omega_1 \cup \Omega_2) \to \mathbb{R}^d$, $z : [0,T] \times \Gamma \to [0,1]$, and $\eta : [0,T] \times \Gamma \to \mathbb{R}$, satisfying

$$\rho\ddot{u} - \operatorname{div} \sigma = f \quad \text{on } \Omega, \tag{1.1a}$$

$$\sigma = \mathbb{C}^0 e(u) + \mu\mathbb{C}^1 e(\dot{u}), \tag{1.1b}$$

$$\sigma\nu = \mathbb{K}[u]z + \eta\nu \quad \text{on } \Gamma, \tag{1.1c}$$

$$\dot{z} \leq 0, \tag{1.1d}$$

$$\frac{1}{2}\mathbb{K}[u] \cdot [u] < \alpha \quad \Rightarrow \quad \dot{z} = 0, \tag{1.1e}$$

and

$$\dot{z}(\frac{1}{2}\mathbb{K}[u] \cdot [u] - \alpha) = 0 \quad \text{on } \{z > 0\} \subset \Gamma, \tag{1.1f}$$

$$\frac{1}{2}\mathbb{K}[u] \cdot [u] - \alpha \leq 0 \quad \text{on } \{z > 0\} \subset \Gamma, \tag{1.1g}$$

coupled with boundary conditions

$$u = w \quad \text{on } \partial_D\Omega, \qquad\qquad \frac{\partial u}{\partial n} = g \quad \text{on } \partial_N\Omega, \tag{1.1h}$$

and with the constraint

$$\eta \in \partial I_{[0,+\infty)}([u] \cdot \nu). \tag{1.1i}$$

Here $\partial I_{[0,+\infty)}$ denotes the subdifferential of the indicator function $I_{[0,+\infty)}$ of the interval $[0,+\infty)$, defined as the map that takes the value 0 on such interval, and $+\infty$ outside it. In the equations above $[u] := u_2 - u_1$ represents the jump of u at Γ, i.e., the difference between the two traces of u, respectively from Ω_2 and Ω_1. The real function $\alpha \geq 0$ on Γ is assumed constant in time, and $\mathbb{C}^0$, $\mathbb{C}^1$, and $\mathbb{K}$, are positive definite and symmetric tensors mapping $\mathbb{R}^{d\times d}$ into itself, $e(u) := \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized gradient of u, and ρ and μ are positive constants.

The system of equations above describes the evolution of a delamination process. Here Ω_1 and Ω_2 are the reference configurations of two visco-elastic bodies whose displacement is represented by u. The tensors $\mathbb{C}^0$ and $\mathbb{C}^1$ are the elasticity tensor and the visco-elasticity tensor, respectively. The variable σ represents the Cauchy stress tensor, so that the quantity $\sigma\nu$ is the force that the body Ω_2 acts on Ω_1. The two bodies are glued along the interface Γ, and the efficacy of the adhesive is represented by the variable z. An high value of z provides a great effect of the glue, while a small value means that deterioration of the adhesive, consequence of high stresses and movements of the bodies, has taken place and hence the glue is less effective. In particular $z = 1$ means that the adhesive is perfectly sane, and $z = 0$ corresponds to the status when all its macromolecular links have been broken and no resistance to bodies separation is observed. This dependence arises in the equation for the interaction force between the bodies (1.1c). The variable η in this equation represents a reaction which must avoid interpenetration of the bodies. Specifically, the constraint of interpenetration

$$[u] \cdot \nu \geq 0, \tag{1.2}$$

provides an instantaneous normal reaction at Γ as soon as $[u] \cdot \nu = 0$, forcing the bodies to separate. Equation (1.1i) is equivalent to the conditions

$$[u] \cdot \nu > 0 \quad \Rightarrow \quad \eta = 0, \tag{1.3}$$

$$[u] \cdot \nu = 0 \quad \Rightarrow \quad \eta \leq 0, \tag{1.4}$$

which might formally describe such phenomenon. Notice that such description is only formal, since the variable η, as we will see, is not defined in a pointwise sense (both in time and in space), but it will be well defined only in the dual of a suitable Sobolev space. This unilateral constraint is the classic *Signorini frictionless condition*, and a process satisfying it is also referred to as evolution in MODE I, in contrast with evolutions in MODE II, where the constraint is bilateral, i.e., $[u] \cdot \nu = 0$. The latter corresponds to processes where only shear displacements are allowed at the interface.

We study this process in the setting of a dynamic evolution, arising in the hyperbolic equation (1.1a) (that, to be precise, turns out to be parabolic due to the presence of the damping term $e(\dot{u})$ in (1.1b)). Here $\rho\ddot{u}$ is the inertial term, ρ being the mass density of the body, assumed constant, and the constant μ in (1.1b) is the viscosity of the material.

Delamination framework and main result. Delamination models are more and more studied in the recent years. As an introduction to evolution in delamination, see, e.g., [11] where the quasistatic model is considered, and [16] for a dynamic model where also thermal effects are considered (for evolution problems in delamination we also quote, among many, [14], [4], [15], [17], and references therein). The evolution of the internal variable z is based on the concept of Frèmond delamination (see [9]). The model we consider was previously introduced by T. Roubicek, who proves existence of solution of evolution in MODE II (i.e., with the bilateral constraint $[u] \cdot \nu = 0$) in [20]. Then the same model was considered with the addition of viscosity of the adhesive in the subsequent papers [21] and [22]. Notice that in our equations no spacial derivatives of z appear, even if some space regularity of z can be derived by the equation (1.1c), since the value of z at $x \in \Gamma$ depends implicitly on the values at the neighbor points, by such equation. However there exist other models of delamination where partial derivatives of z enter in the equations (see, e.g., [4] and references therein). In our model it is remarkable the presence of the viscosity term $\mu\mathbb{C}^1 e(\dot{u})$, that provides more regularity of the displacement. Without this, it seems not possible, at the present stage, to provide solutions to systems (1.1) with the unilateral constraint (1.2), not even in a very weak sense.

The main result of the paper states the existence of weak solutions to (1.1), thus extending the results of existence in [20] to evolutions in MODE I. In order to prove existence of a solution to problem (1.1) for every initial data for u, $\dot{u}$, and z in suitable spaces, we need to reformulate the equations in a weaker sense. In particular such weak formulation is needed to treat the unilateral constraint (1.2), which in turn represents the principal difficulty for the proof of existence. The main tool to face it is inspired by the pioneer paper [5], whose arguments we adapt to our situation. Different approaches to unilateral constraints for contact problems in the framework of dynamic evolutions (i.e., of hyperbolic systems of PDEs) exist and can be found in [2]. Here the obstacle is treated in an implicit way, by the use of variational inequalities.

The simplified model. To study problem (1.1) we first make some nonrestrictive simplifications. In what follows we assume that the constants ρ and μ are equal to 1. Moreover, since we treat homogeneous materials, the elasticity tensors $\mathbb{C}^0$, $\mathbb{C}^1$, and $\mathbb{K}$, are assumed constant, and without lose of generality we suppose they are all equal to Id, the identity matrix. Since we always fix a Dirichlet boundary datum, the Korn inequality ensures that we can replace the symmetrized gradient by the full gradient ∇u. The resulting model and the original one, apparently different, are instead mathematically equivalent, the technicalities involved in the simplified problem being exactly the same, and all the results can be trivially adapted to the original case. On the other hand we want to study a large class of unilateral constraints for the normal jump of the displacements, so that we are led to replace the function $I_{[0,+\infty)}$ by a general lower semicontinuous and convex function $j : \mathbb{R} \to [0, +\infty)$, with $j(0) = \min j = 0$. After simplifications, the resulting system of equations reads

$$\ddot{u} - \Delta u - \Delta \dot{u} = f \quad \text{on } \Omega, \tag{1.5a}$$
$$(\nabla u + \nabla \dot{u})\nu = [u]z + \eta\nu \quad \text{on } \Gamma, \tag{1.5b}$$
$$\dot{z} \leq 0, \tag{1.5c}$$
$$\dot{z}(\frac{1}{2}|[u]|^2 - \alpha) = 0, \tag{1.5d}$$

and, on the set $\{z > 0\} \subset \Gamma$,

$$\frac{1}{2}|[u]|^2 - \alpha \leq 0, \tag{1.5e}$$

with the constraint

$$\eta \in \partial j([u] \cdot \nu). \tag{1.5f}$$

Such system is coupled with the boundary conditions

$$u = w \quad \text{on } \partial_D\Omega, \qquad \nabla u \cdot n = g \quad \text{on } \partial_N\Omega, \tag{1.5g}$$

for some boundary datum $w : \partial_D\Omega \to \mathbb{R}^d$ and boundary force $g : \partial_N\Omega \to \mathbb{R}^d$. Equation (1.5d) implies the threshold condition

$$\frac{1}{2}|[u]|^2 < \alpha \quad \Rightarrow \quad \dot{z} = 0. \tag{1.6}$$

In Section 3 we reformulate Problem (1.5) in a weak sense. Such weak form is reminiscent of the energetic formulations for rate-independent systems (see [12], [13], and [19]; for the general theory of rate-independent systems, see [11]). The energetic formulation of a rate-independent system that evolves in a time interval $[0, T]$ usually arises in a equilibrium condition which holds at every time $t \in [0, T]$, and an energy equality, which provides that the energy stored and dissipated by the system balances the work done on the system by the external forces. Actually our formulation does not provide an energy balance, but only an energy inequality, since at this stage we are not able to prove that the additional dissipation due to the presence of the unilateral constraint (1.5f) exactly balances the external work. We can only show that the energy dissipated by the constraint, which provokes instantaneous reaction at Γ and then discontinuities in time of the velocity field $\dot{u}$, is less or equal to the external work. On the other hand, we prove that the flow rule for the variable z is still satisfied in a weak sense, condition expressed by property (b') of Definition 3.1 below. Let us emphasize that this equation is not needed in presence of an energy balance, since it can be readily deduced from it and the other weak equations of motion.

The approximate problem. In order to prove our existence result (Theorem 4.1), we proceed approximating the Problem (1.5) by a regularized one. Specifically, we fix $\epsilon \in (0, 1)$, and denote by j^ϵ the Moreau-Yosida regularization of j. Denoting the subdifferential of j^ϵ by $\beta^\epsilon := \partial j^\epsilon$, i.e., the Yosida approximation of ∂j, we study the approximate problem

$$\ddot{u}^\epsilon - \Delta u^\epsilon - \Delta \dot{u}^\epsilon = f \quad \text{on } \Omega, \tag{1.7a}$$
$$(\nabla u^\epsilon + \nabla \dot{u}^\epsilon)\nu = [u^\epsilon]z^\epsilon + \beta^\epsilon([u^\epsilon] \cdot \nu)\nu \quad \text{on } \Gamma, \tag{1.7b}$$
$$\dot{z}^\epsilon \leq 0, \tag{1.7c}$$
$$\dot{z}^\epsilon(\frac{1}{2}|[u^\epsilon]|^2 - \alpha) = 0, \tag{1.7d}$$

and, on the set $\{z^\epsilon > 0\} \subset \Gamma$,

$$\frac{1}{2}|[u^\epsilon]|^2 - \alpha \leq 0. \tag{1.7e}$$

The constraint is implicit in (1.7b), where, noting by η^ϵ the reaction term (thus replacing $\beta^\epsilon([u^\epsilon]\cdot\nu)$ by η^ϵ), it reads

$$\eta^\epsilon \in \partial j^\epsilon([u^\epsilon]\cdot\nu). \tag{1.7f}$$

Also in this approximate problem, the existence of a solution is provided in the framework of an energetic-type formulation. This consists of a weak equation of motion, a weak formula for the flow rule, and an energy balance. The former reads

$$\langle \ddot{u}^\epsilon, \varphi\rangle + (\nabla\dot{u}^\epsilon, \nabla\varphi) + (\nabla u^\epsilon, \nabla\varphi) + \langle \beta^\epsilon([u^\epsilon]\cdot\nu), [\varphi]\rangle = \langle f, \varphi\rangle - \langle z[u^\epsilon], [\varphi]\rangle, \tag{1.8}$$

for all test function φ in an appropriate space, and where the duality product $\langle\cdot,\cdot\rangle$ are intended in the respective topology. The first part of the flow rule is expressed by the condition that the function z^ϵ is nonincreasing in time, and that at every time $t \in [0,T]$ it holds

$$\text{either} \quad \frac{1}{2}[u^\epsilon(t,x)] \leq \alpha(x) \quad \text{or} \quad z^\epsilon(t,x) = 0 \quad \text{for a.e. } x \in \Gamma. \tag{1.9}$$

As already mentioned, equation (1.7d) can be deduced from the previous two conditions and the energy balance

$$\begin{aligned}
&\frac{1}{2}\|\dot{u}^\epsilon(t)\|^2 + \int_\Gamma j^\epsilon([u^\epsilon(t)]\cdot\nu) + \frac{1}{2}\int_\Gamma z^\epsilon(t)[u^\epsilon(t)]^2 dx + \frac{1}{2}\|\nabla u^\epsilon(t)\|^2 \\
&+ \int_0^t \|\nabla\dot{u}^\epsilon\|^2 dt - \int_\Gamma \alpha z^\epsilon(t) dx = \frac{1}{2}\|v_0\|^2 + \int_\Gamma j^\epsilon([u_0]\cdot\nu)dx + \frac{1}{2}\int_\Gamma z_0[u_0]^2 dx \\
&+ \frac{1}{2}\|\nabla u_0\|^2 - \int_\Gamma \alpha z_0 dx + \int_0^t \langle f, \dot{u}^\epsilon\rangle dt,
\end{aligned} \tag{1.10}$$

valid for every time $t \in [0,T]$.

Benefiting of the regularity of j^ϵ, the existence of an energetic solution to the approximate problem is readily obtained by adapting standard results in delamination theory. For this we mainly refer to [20] and references therein.

Then we pass to the limit as ϵ tends to 0. Thanks to standard a-priori estimates it is possible to show that the triple $(u^\epsilon, z^\epsilon, \beta^\epsilon([u^\epsilon]\cdot\nu))$ tends to a triple (u, z, η) with respect to suitable topologies, the latter being an energetic solution to Problem (1.5) as in Definition 3.1 below. In particular, it is seen that condition (1.9) passes to the limit, while in order to let (1.8) pass to the limit we have still to integrate it with respect to time, and thus getting rid of the second time derivative of u by parts integration. The resulting weak equation is

$$\begin{aligned}
&-(\!(\dot{u}, \dot{\varphi})\!) + (\dot{u}(T), \varphi(T)) + (\!(\nabla\dot{u}, \nabla\varphi)\!) + (\!(\nabla u, \nabla\varphi)\!) + \langle\!\langle \eta, [\varphi]\cdot\nu\rangle\!\rangle^\Gamma \\
&= (u_1, \varphi(0)) + \langle\!\langle \mathcal{L}, \varphi\rangle\!\rangle - (\!(z[u], [\varphi])\!)^\Gamma,
\end{aligned} \tag{1.11}$$

where the duality products are intended in appropriate spaces (see Section 2). As for the energy balance, as said, we prove that an energy inequality holds at the limit. In order to guarantee that (1.7d) is still valid at the limit, we prove an additional condition, obtained from (1.5d) integrating by parts in time, namely

$$\int_\Gamma z(t_2)(\frac{1}{2}|[u(t_2)]|^2 - \alpha)dx - \int_\Gamma z(t_1)(\frac{1}{2}|[u(t_1)]|^2 - \alpha)dx - \int_{t_1}^{t_2}\int_\Gamma z([u]\cdot[\dot{u}])dxdt = 0, \tag{1.12}$$

for every time interval $[t_1, t_2] \subset [0,T]$.

The main difficulty in the proof of Theorem 4.1 relies in the lack of compactness of the family of functions $\beta^\epsilon([u^\epsilon]\cdot\nu)$ in $L^2([0,T]\times\Gamma)$. Indeed it is possible to prove that these terms are only uniformly bounded in the larger space $H^{-1}([0,T], H^{-\frac{1}{2}}(\Gamma))$. Therefore, since the limit function η only belongs to $H^{-1}([0,T], H^{-\frac{1}{2}}(\Gamma))$, in order that (1.5f) makes sense, we have to relax the notion of subdifferential ∂j. This relaxation is described in Section 2.1 where we first extend the operator j to a new operator $\mathcal{J}$ defined on the space $L^2([0,T]\times\Gamma)$, then we restrict it on $H^1([0,T], H^{\frac{1}{2}}(\Gamma))$ and consider its subdifferential with respect to this new topology, noted by β_w. Within this weaker notion of constraint, it is no longer true that (1.5f) is satisfied in a pointwise sense. Nevertheless it is still possible to recover some regularity from the condition $\eta \in \beta_w([u]\cdot\nu)$., and a finer description of it also elucidates the link between the strong pointwise inclusion (1.5f) and that intended in the weak sense. This is a standard procedure which has been adapted from [23], [5], and is based upon convex analysis results contained in [6] and [10]. Similarly defining the correspondent operators $\mathcal{J}^\epsilon$ on $L^2([0,T]\times\Gamma)$, it is shown that their subdifferentials $\partial\mathcal{J}^\epsilon$, still noted by β^ϵ, tend in the sense of graphs to the weak operator β_w (see Lemma 2.3). Then, adapting standard results of the theory of maximal monotone operators allows us to prove that the limit constraint is satisfied, namely,

$$\eta \in \beta_w([u]\cdot\nu). \tag{1.13}$$

The previous argument, synthesized in Section 2.2 and Step 3 of the proof of Theorem 4.1 was previously used in [5], where the authors consider a general obstacle acting on the whole Ω. The argument turns out to be very general and can be easily adapted to the present situation.

Structure of the paper. The paper is organized as follows: in Section 2 we introduce the notation and all the preliminaries on the mechanical setting of the problem. Moreover, in Subsections 2.1 and 2.2 we describe the general procedure to relax and approximate the constraint. In Section 3 we introduce our notion of weak solution to Problem 1.5a and provide the existence of approximate solutions. The last Section 4 is devoted to the proof of the existence result, Theorem 4.1.

2 Preliminaries

Setting. The apparatus for the delamination process consists of two elastic bodies, whose reference configuration is represented by the disjoint bounded open sets Ω_1 and Ω_2. We assume that Ω_1 and Ω_2 are connected, and that their common boundary $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ has positive $(d-1)$-dimensional Hausdorff measure, i.e., $\mathcal{H}^{d-1}(\Gamma) > 0$. We denote by ν the unit normal vector to Γ, oriented in such a way that it points from Ω_1 into Ω_2. We set

$$\Omega := \Omega_1 \cup \Omega_2 \quad \text{while} \quad \tilde{\Omega} := \mathrm{int}(\overline{\Omega_1 \cup \Omega_2}),$$

the latter being the inner part of the closure of Ω. The external boundary of Ω, i.e. $\partial\tilde{\Omega}$, splits as $\partial\tilde{\Omega} = \partial_D\Omega \cup \partial_N\Omega$, representing the parts of the boundary where we will impose Dirichlet and Neumann conditions, respectively. We also denote by $\partial_D\Omega_1 := \partial_D\Omega \cap \partial\Omega_1$ and $\partial_D\Omega_2 := \partial_D\Omega \cap \partial\Omega_2$, and we will make the geometric assumptions that both $\partial_D\Omega_1$ and $\partial_D\Omega_2$ have positive $(d-1)$-dimensional Hausdorff measures. We denote by n the external unit normal to $\partial\tilde{\Omega}$. Crucial will be the hypothesis that

$$d(\partial_D\Omega, \Gamma) > 0. \tag{2.1}$$

The latter, where $d(\cdot,\cdot)$ is the Hausdorff distance between sets, ensures that there exists a smooth function ψ on $\mathbb{R}^d$ that takes the value 0 on $\partial_D\Omega$ and 1 on Γ.

Notation. We introduce the space

$$V := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial_D\Omega\}, \tag{2.2}$$

with dual V'. Note that in general $u \in V$ does not belong to $H^1(\tilde{\Omega})$, since it might have nonzero jump on the interface Γ. The jump of $u \in V$ on Γ, denoted by $[u]$, is defined by $[u] := u_2 - u_1$, the difference between the traces of u on Γ, from Ω_2 and Ω_1 respectively. With this convention the scalar product $[u]\cdot\nu$ represents the normal displacement between the two bodies, which in turn will be positive if they are a positive distance far, while a negative value means that interpenetration is occurring.

We also introduce the following space

$$\mathcal{V} := H^1([0,T],V). \tag{2.3}$$

Similarly, for all $t \in [0,T]$, we introduce the space $\mathcal{V}_t := H^1([0,t],V)$. Let

$$\mathcal{Z} := L^2(\Gamma,[0,1]). \tag{2.4}$$

The following space will play a crucial role in the following discussion.

$$\mathcal{H} := H^1([0,T],H^{\frac{1}{2}}(\Gamma)), \tag{2.5}$$

and its counterpart $\mathcal{H}_t := H^1([0,t],H^{\frac{1}{2}}(\Gamma))$ for all $t \in [0,T]$. Sometimes we will deal with

$$\tilde{H}^2(\Omega) := \{u \in H^2(\Omega,\mathbb{R}^d) : u = 0 \text{ on } \partial\tilde{\Omega}\}, \tag{2.6}$$

and with its dual space, denoted by $\tilde{H}^{-2}(\Omega)$.

The scalar products in $L^2(\Omega,\mathbb{R}^d)$ and $L^2(\Gamma)$ are noted by

$$(\cdot,\cdot) \qquad (\cdot,\cdot)^{\Gamma},$$

respectively, while the scalar products in $L^2([0,T]\times\Omega,\mathbb{R}^d)$ and $L^2([0,T]\times\Gamma)$ are

$$((\cdot,\cdot)) \qquad ((\cdot,\cdot))^{\Gamma}.$$

This convention reflects the idea that integration only in space is represented by only one bracket, while double brackets are used for integration both in time and space. When we integrate in a subinterval $[0,t]\subset[0,T]$ we will add a label t, namely,

$$((\cdot,\cdot))_t \qquad ((\cdot,\cdot))^{\Gamma}_t,$$

are the scalar products in $L^2([0,t]\times\Omega,\mathbb{R}^d)$ and $L^2([0,t]\times\Gamma)$, respectively. The symbol $\|\cdot\|$ usually denotes both the norms in $L^2(\Omega,\mathbb{R}^d)$ and $L^2(\Gamma)$. The norm in a general Banach space X is denoted by $\|\cdot\|_X$.

The duality pairing between a Banach space of functions on Ω and its dual (for instance the duality between V' and V) is denoted by $\langle\cdot,\cdot\rangle$, whereas if the functions are defined on Γ we will use the notation $\langle\cdot,\cdot\rangle^{\Gamma}$ (for instance the duality between $H^{\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)'$). We use the double brackets when we deal with a space of functions in the time-space. For instance, the duality pairing between $L^2([0,T],V')$ and $L^2([0,T],V)$ is denoted by $\langle\langle\cdot,\cdot\rangle\rangle$, and for any $t\in(0,T)$, the symbol $\langle\langle\cdot,\cdot\rangle\rangle_t$ denotes the duality pairing between $L^2([0,t],V')$ and $L^2([0,t],V)$. The duality pairing in $\mathcal{H}$ and $\mathcal{H}_t$ are denoted by $\langle\langle\cdot,\cdot\rangle\rangle^{\Gamma}$ and $\langle\langle\cdot,\cdot\rangle\rangle^{\Gamma}_t$, respectively.

We define, for all $u\in H^1(\Omega)$,

$$V(u) := \frac{1}{2}|[u]|^2. \tag{2.7}$$

It is also convenient to define the operator $T : \mathcal{Z}\times V \to V'$ as

$$\langle T(z,u),\varphi\rangle = \int_{\Gamma} z[u]\cdot[\varphi]dx \tag{2.8}$$

for all $\varphi\in V$. Since $0\le z\le 1$, by the continuity of the trace operator from V in $L^2(\Gamma)$ (whose norm is denoted by $C>0$), we have

$$\int_{\Gamma} z[u]\cdot[\varphi]dx \le \|[u]\|\|\varphi\| \le C\|u\|_V\|\varphi\|_V, \tag{2.9}$$

which implies $T(z,u) \in V'$ with $\|T(z,u)\|_{V'} \leq C\|u\|_V$, for all $(z,u) \in \mathcal{Z} \times V$.

Extension operators. We also need to introduce the linear operators $S_i : H^{\frac{1}{2}}(\Gamma, \mathbb{R}^d) \to V$, $i = 1, 2$, defined as follows. For all $\varphi \in H^{\frac{1}{2}}(\Gamma, \mathbb{R}^d)$ we define $u(\varphi)$ as the unique harmonic function in $H^1(\Omega_1 \cup \Omega_2, \mathbb{R}^d)$ with boundary condition $u = \varphi$ on Γ, $u = 0$ on $\partial\Omega_D$, and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega_N$. Then we define

$$S_i(\varphi) := u(\varphi)\chi_{\Omega_i} \tag{2.10}$$

for $i = 1, 2$, where χ_{Ω_i} is the characteristic function of Ω_i. It is easy to check that the operator S_i is linear and continuous, i.e., there exists a constant $c > 0$ such that

$$\|S_i(\varphi)\|_V \leq c\|\varphi\|_{H^{\frac{1}{2}}(\Gamma, \mathbb{R}^d)}, \tag{2.11}$$

for $i = 1, 2$, and that $[S_1(\varphi)] = -[S_2(\varphi)] = \varphi$.

External forces. If there are external forces $f \in L^2(\Omega, \mathbb{R}^d)$ and $g \in L^2(\partial_N\Omega, \mathbb{R}^d)$ the total external load is defined as

$$\langle \mathcal{L}, \varphi \rangle := (f, \varphi) + \int_{\partial_N\Omega} g \cdot \varphi dx,$$

for all $\varphi \in V$. It easily follows that $\mathcal{L} \in V'$. The weak equation for the stress field $\sigma \in L^2(\Omega, \mathbb{R}^{d\times d})$, that is

$$(\sigma, \nabla\varphi) = (f, \varphi) + \int_{\partial_N\Omega} g \cdot \varphi dx,$$

for all $\varphi \in V$, implies that

$$\text{div } \sigma = f \quad \text{a.e. on } \Omega,$$

and

$$\sigma \cdot n = g \quad \text{a.e. on } \partial_N\Omega.$$

In what follows, we will only assume that there exists an external load $\mathcal{L} \in V'$, so that, considering also the inertial term, the equation of motion becomes

$$\langle \ddot{u}, \varphi \rangle + (\sigma, \nabla\varphi) = \langle \mathcal{L}, \varphi \rangle, \tag{2.12}$$

for all $\varphi \in V$. Equations (1.5a) and (1.5b), when coupled with homogeneous Dirichlet and Neumann conditions, thanks to notations (2.12) and (2.8), can be expressed in weak form as

$$\langle \ddot{u}, \varphi \rangle + (\nabla\dot{u}, \nabla\varphi) + (\nabla u, \nabla\varphi) + \langle \beta([u] \cdot \nu), \varphi \rangle = \langle \mathcal{L}, \varphi \rangle - \langle T(z,u), \varphi \rangle. \tag{2.13}$$

for all $\varphi \in V$. Unfortunately we are not able to provide a solution of (2.13) for all times $t \in [0, T]$, but we will further need a weaker formulation (see Section 3).

As far as the evolution of the delamination variable z is concerned, we assume it satisfies equations (1.5c), (1.5d), and (1.5e). Here $\alpha \in L^\infty(\Gamma)$ is a positive function that represents the delamination threshold, defined as the potential that the elastic stored energy of the adhesive $V([u])$ must reach to start the delamination process (equation (1.6)). We assume that

$$\alpha > c \quad \text{a.e. on } \Gamma, \tag{2.14}$$

for a fixed positive constant $c > 0$.

2.1 The unilateral constraint

We assume that $j : \mathbb{R} \to [0, +\infty]$ is a convex and lower semicontinuous function such that $j(0) = \min j = 0$. We denote by $\beta := \partial j$ the subdifferential of j, which turns out to be a maximal monotone operator from $\mathbb{R}$ to $2^{\mathbb{R}}$.

We introduce the functional J on $L^2(\Gamma)$ as

$$J(v) := \int_\Gamma j(v)dx \quad v \in L^2(\Gamma), \tag{2.15}$$

where the value of the integral may well be $+\infty$ if $j(v) \notin L^1(\Gamma)$. The subdifferential of J in H is defined as the multivalued operator ∂J from $L^2(\Gamma)$ to $2^{L^2(\Gamma)}$ such that, given $u \in L^2(\Gamma)$,

$$L^2(\Gamma) \ni v \text{ belongs to } \partial J(u) \Leftrightarrow J(w) - J(u) \geq (v, w-u)^\Gamma \quad \forall w \in L^2(\Gamma). \tag{2.16}$$

It is well-known that ∂J coincides with the operator β in $L^2(\Gamma)$, in the sense that, $v \in \partial J(u)$ if and only if $v(x) \in \beta(u(x))$ for a.e. $x \in \Gamma$. In a similar way we introduce the functionals $\mathcal{J}$ and $\mathcal{J}_t$ on $L^2([0,T], L^2(\Gamma))$ and $L^2([0,t], L^2(\Gamma))$, respectively, by

$$\mathcal{J}(v) := \int_0^T \int_\Gamma j(v)dxds \qquad \mathcal{J}_t(v) := \int_0^t \int_\Gamma j(v)dxds. \tag{2.17}$$

The multivalued operator $\partial\mathcal{J}$ on $L^2([0,T]\times\Gamma)$ into $2^{L^2([0,T]\times\Gamma)}$, subdifferential of $\mathcal{J}$, is defined as follows:

$$L^2([0,T]\times\Gamma) \ni v \text{ belongs to } \partial\mathcal{J}(u) \Leftrightarrow \mathcal{J}(w) - \mathcal{J}(u) \geq ((v, w-u))^\Gamma, \tag{2.18}$$

for all $w \in L^2([0,T]\times\Gamma)$. As for J, the subdifferential of $\mathcal{J}$ (and the analogue $\mathcal{J}_t$) is still interpreted in the pointwise form β, and we will still adopt the notation $\beta = \partial\mathcal{J}$.

Relaxation of the constraint. We want now to introduce a relaxed notion for the operator β, seen as an operator on the space $\mathcal{H} \subset L^2([0,T], L^2(\Gamma))$. To this aim, we first set $\mathcal{J}_\mathcal{H} := \mathcal{J}\llcorner_\mathcal{H}$, the restriction of $\mathcal{J}$ to $\mathcal{H}$. Hence we can consider its subdifferential $\partial\mathcal{J}_\mathcal{H}$ with respect to the duality pairing between $\mathcal{H}$ and $\mathcal{H}'$. Namely, if $\xi \in \mathcal{H}'$ and $u \in \mathcal{H}$, we say that

$$\xi \in \partial\mathcal{J}_\mathcal{H}(u) \quad \Leftrightarrow \quad \mathcal{J}_\mathcal{H}(w) - \mathcal{J}_\mathcal{H}(u) \geq \langle\langle \xi, w-u \rangle\rangle^\Gamma \quad \forall w \in \mathcal{H}. \tag{2.19}$$

Consistently with the definition of β, we will denote the operator $\partial\mathcal{J}_\mathcal{H}$ by β_w (w standing for "weak"). Similarly proceeding for the functional $\mathcal{J}_t$, we are led to define the subdifferential $\partial\mathcal{J}_{t,\mathcal{H}}$ of the operator $\mathcal{J}_{t,\mathcal{H}} := \mathcal{J}_t\llcorner_\mathcal{H}$, and thus using equivalently the notation $\beta_{w,t}$.

In this general setting it is not true anymore that β_w coincides with the operator β in a pointwise sense. Indeed if $v \in \beta_w(u)$, the pointwise value of v is not anymore defined when $v \in \mathcal{H}' \setminus L^2(Q)$. However we can still recover some regularity of v from the condition $v \in \beta_w(u)$. Following the argument of [23, Prop. 2.1] (which, in turn, is based on the results of [7]), it is easily seen that if $\xi \in \beta_w(u)$ then there exists a bounded Borel measure $\mathcal{T}$ such that $\langle\langle \xi, \varphi \rangle\rangle = \int_0^T \int_\Gamma \varphi d\mathcal{T}$ for all $\varphi \in \mathcal{H} \cap C([0,T]\times\Gamma)$. We thus say that the measure $\mathcal{T}$ *represents* ξ on $C([0,T]\times\Gamma)$. Moreover, we obtain the following relation between the measure $\mathcal{T}$ and the original constraint β (cf. [7, Thm. 3] for further detail): noting as $\mathcal{T} = \mathcal{T}_a + \mathcal{T}_s$ the Radon-Nikodym decomposition of $\mathcal{T}$, where $\mathcal{T}_a$ ($\mathcal{T}_s$, respectively) is the absolutely continuous (singular, respectively) part with respect to the $\mathcal{L}^1 \times \mathcal{H}^{d-1}$ measure on $[0,T]\times\Gamma$, we then have

$$\mathcal{T}_a u \in L^1([0,T]\times\Gamma), \tag{2.20}$$

$$\mathcal{T}_a(t,x) \in \beta(u(t,x)) \text{ for a.e. } (t,x) \in [0,T]\times\Gamma, \tag{2.21}$$

$$\langle\langle \xi, u \rangle\rangle - \int_0^T \int_\Gamma \mathcal{T}_a u \, dxdt = \sup\Big\{ \int_0^T \int_\Gamma \eta \, d\mathcal{T}_s, \ \eta \in C([0,T]\times\Gamma), \ \eta \in [-1,1] \Big\}. \tag{2.22}$$

In other words, the absolutely continuous part $\mathcal{T}_a$ of $\mathcal{T}$ satisfies the constraint pointwise (in view of (2.21)), while the singular part $\mathcal{T}_s$ is characterized by (2.22).

Moreover, it could be said more about condition (2.22), in the case that $j = I_{[0,+\infty)}$. Namely, denoting by $\mathcal{T}_s = \rho|\mathcal{T}_s|$ the *polar decomposition* of $\mathcal{T}_s$, where $|\mathcal{T}_s|$ is the total variation of $\mathcal{T}_s$, following the lines of [10, Thm. 3] one may prove that

$$\rho \in \partial I_{[0,+\infty)}([u]\cdot n) \quad |\mathcal{T}_s| - \text{a.e. in } [0,T]\times\Gamma. \tag{2.23}$$

This means that the singular part of $\mathcal{T}$ is supported on the set where $[u] \cdot n = 0$ and that here it holds $\rho = -1$. In some sense, also the singular part of $\mathcal{T}$ is partially reminiscent of the expression of the operator β.

Actually, the characterization (2.23) is proved in [10] in the case when $\mathcal{H}$ is replaced by $H_0^1(\Omega)$, with Ω a bounded domain of $\mathbb{R}^N$, and may be likely extended to the present situation. We drop the proof since it would be much technical and of low interest.

2.2 Approximation of J

For all $\epsilon \in (0,1)$, we introduce the convex and lower semicontinuous map j^ϵ, the Moreau-Yosida regularization of j. As for j, we set

$$\beta^\epsilon := \partial j^\epsilon,$$

the Yosida approximation of β, and recall that β^ϵ is globally ϵ^{-1}-Lipschitz continuous

Similarly to J, we introduce the functional J^ϵ on $L^2(\Gamma)$ as

$$J^\epsilon(v) := \int_\Gamma j^\epsilon(v)dx \quad v \in L^2(\Gamma), \tag{2.24}$$

where again the value may well be $+\infty$ if $j^\epsilon(v) \notin L^1(\Gamma)$. Similarly the functionals $\mathcal{J}^\epsilon$ and $\mathcal{J}_t^\epsilon$ on $L^2([0,T], L^2(\Gamma))$ and $L^2([0,t], L^2(\Gamma))$ are defined by

$$\mathcal{J}^\epsilon(v) := \int_0^T \int_\Gamma j^\epsilon(v)dxds \qquad \mathcal{J}_t^\epsilon(v) := \int_0^t \int_\Gamma j^\epsilon(v)dxds, \tag{2.25}$$

respectively. The operator $\partial\mathcal{J}^\epsilon$, subdifferential of $\mathcal{J}^\epsilon$, is readily defined as

$$L^2([0,T]\times\Gamma) \ni v \text{ belongs to } \partial\mathcal{J}^\epsilon(u) \Leftrightarrow \mathcal{J}^\epsilon(w) - \mathcal{J}^\epsilon(u) \geq (\!(v, w-u)\!)^\Gamma, \tag{2.26}$$

for all $w \in L^2([0,T]\times\Gamma)$, and similarly $\partial\mathcal{J}_t^\epsilon$, the subdifferential of $\mathcal{J}_t^\epsilon$. Also in this situation the operators ∂J^ϵ, $\partial\mathcal{J}^\epsilon$, and $\partial\mathcal{J}_t^\epsilon$, coincide with the operator β^ϵ pointwise, that is, $v \in \partial\mathcal{J}^\epsilon(u)$ if and only if $v(t,x) \in \beta^\epsilon(u(t,x))$ for a.e. $(t,x) \in [0,T]\times\Gamma$.

Lemma 2.1. *The operators J^ϵ ($\mathcal{J}^\epsilon$, and $\mathcal{J}_t^\epsilon$) converge to J ($\mathcal{J}$ and $\mathcal{J}_t$, respectively) in the sense of Mosco-convergence in $L^2(\Gamma)$ ($L^2([0,T]\times\Gamma)$ and $L^2([0,t]\times\Gamma)$, respectively).*

The proof of this is a consequence of the fact that $j^\epsilon \nearrow j$ pointwise and of [1, Theorem 3.20].

We are now interested in restricting the operators $\mathcal{J}^\epsilon$ to the space $\mathcal{H}$ and looking at their subdifferential in this new topology. First, the following can be said.

Lemma 2.2. *There holds:*

(a) The function β^ϵ is a monotone operator from $\mathcal{H}$ into $\mathcal{H}'$.

(b) For all $u \in \mathcal{H}$, the function $\beta^\epsilon(u)$ belongs to the subdifferential of $\mathcal{J}^\epsilon$ at u (denoted by $\partial_\mathcal{H}\mathcal{J}^\epsilon$), seen as an operator from $\mathcal{H}$ into $\mathcal{H}'$ (actually, $\partial_\mathcal{H}\mathcal{J}^\epsilon$ is univalued and $\partial_\mathcal{H}\mathcal{J}^\epsilon = \beta^\epsilon$).

Proof. To prove (a), we see that if $v \in \mathcal{H}$ it results $\beta^\epsilon(v) \in L^2([0,T], L^2(\Gamma)) \subset \mathcal{H}'$ thanks to the Lipschitz continuity of β^ϵ. Moreover, β^ϵ is a monotone operator on $L^2([0,T], L^2(\Gamma))$, so that for all $u, v \in \mathcal{H}$

$$\langle\!\langle \beta^\epsilon(u) - \beta^\epsilon(v), u - v\rangle\!\rangle^\Gamma = (\!(\beta^\epsilon(u) - \beta^\epsilon(v), u - v)\!)^\Gamma \geq 0.$$

Let us prove (b). By definition, $\beta^\epsilon(u)$ belongs to the subdifferential of $\mathcal{J}^\epsilon$ at u as an operator on $L^2([0,T], L^2(\Gamma))$. Thus we have

$$\langle\!\langle \beta^\epsilon(u), v - u\rangle\!\rangle^\Gamma = (\!(\beta^\epsilon(u), v - u)\!)^\Gamma \leq \mathcal{J}^\epsilon(v) - \mathcal{J}^\epsilon(u),$$

for all $v \in \mathcal{H}$, and the thesis follows. $\square$

Now the desired approximation property of $\mathcal{J}^\epsilon$ is expressed by the following fact.

Lemma 2.3. *The monotone operators $\beta^\epsilon = \partial_{\mathcal{H}}\mathcal{J}^\epsilon$ converge to the maximal monotone operator $\partial_{\mathcal{H}}\mathcal{J} = \beta_w$ in the sense of graphs, i.e.,*

$$\forall [x,y] \in \beta_w \quad \exists [x^\epsilon, y^\epsilon] \in \beta_\epsilon \quad \text{such that} \quad [x^\epsilon, y^\epsilon] \to [x,y],$$

where the convergence is intended with respect to the strong topology of $\mathcal{H} \times \mathcal{H}'$.

The proof is obtained thanks to the monotonicity of the functionals $\mathcal{J}^\epsilon$, and then owing to [1, Theorem 3.20] and [1, Theorem 3.66].

It is straightforward that Lemmas 2.2 and 2.3 apply also the the operators β_t^ϵ and $\beta_{t,w}$, for every fixed $t \in [0,T]$.

The following Lemma will be crucial to prove our main result:

Lemma 2.4. *Let the monotone operators A_n tends to the maximal monotone operator A in the sense of graphs (operators from $\mathcal{H}$ into $2^{\mathcal{H}'}$). Let $v_n \rightharpoonup v$ weakly in $\mathcal{H}$, $\xi_n \rightharpoonup \xi$ weakly in $\mathcal{H}'$, and assume $\xi_n \in A_n(v_n)$. If*

$$\limsup \langle\langle \xi_n, v_n \rangle\rangle^\Gamma \le \langle\langle \xi, v \rangle\rangle^\Gamma,$$

then $\xi \in A(v)$.

Proof. The proof is an adaptation of [1, Proposition 3.59]. Since A_n tends to A in the graphs sense, for all $[x,y] \in A$ there exists a sequence $[x_n, y_n]$ tending to $[x,y]$ strongly in $\mathcal{H} \times \mathcal{H}'$. Then, by monotonicity of A_n, we have

$$\langle\langle \xi_n - y_n, v_n - x_n \rangle\rangle^\Gamma \ge 0. \tag{2.27}$$

Passing to the limit we get

$$\limsup \langle\langle \xi_n, v_n \rangle\rangle^\Gamma \ge \langle\langle y, x \rangle\rangle, \tag{2.28}$$

and so by hypothesis, $\langle\langle \xi, v \rangle\rangle^\Gamma \ge \langle\langle y, x \rangle\rangle^\Gamma$, which is equivalent to

$$\langle\langle \xi - y, v - x \rangle\rangle^\Gamma \ge 0.$$

Now the thesis follows by the arbitrariness of $[x,y] \in A$ and the maximality of A. $\square$

Remark 2.5. Let us remark that all the previous results do not appeal to the specific definition of the space $\mathcal{H}$. Indeed they hold true for a general Hilbert space $\mathcal{H}$, provided that $\mathcal{H} \subset L^2 \subset \mathcal{H}'$ is an Hilbert triple, i.e., the duality pairing between $\mathcal{H}'$ and $\mathcal{H}$ satisfies $\langle\langle v, u \rangle\rangle = ((v, u))$ whenever $v \in L^2$.

3 Weak formulation

We are now in position to define the notion of energetic solution to Problem (1.5).

Definition 3.1. Let $T > 0$, let $u_0, v_0 \in V$, $z_0 \in \mathcal{Z}$, and $\mathcal{L} \in L^2([0,T], V')$. Then we say that a triple (u, z, η) is an weak solution to (1.5) (of energetic type) on $[0,T]$ with initial conditions u_0, v_0, and z_0, if

$$u \in H^1([0,T], V) \cap W^{1,\infty}([0,T], L^2(\Omega)), \tag{3.1a}$$

$$\dot{u} \in H^1([0,T], H^{-1}(\tilde{\Omega})) \cap BV([0,T], \tilde{H}^{-2}(\Omega)), \tag{3.1b}$$

$$z \in L^\infty([0,T], \mathcal{Z}) \cap BV([0,T], L^1(\Gamma)), \tag{3.1c}$$

$$\eta \in \mathcal{H}', \tag{3.1d}$$

is such that $u(0) = u_0$, $\dot{u}(0) = v_0$, $z(0) = z_0$, and satisfies conditions (a), (a'), (a"), (b), (b'), and (c) below.

(a) The following weak equation of motion holds: for all $\varphi \in \mathcal{V}$ we have

$$
\begin{aligned}
&-(\!(\dot{u}, \dot{\varphi})\!) + (\dot{u}(T), \varphi(T)) + (\!(\nabla \dot{u}, \nabla \varphi)\!) + (\!(\nabla u, \nabla \varphi)\!) + \langle\!\langle \eta, [\varphi] \cdot \nu \rangle\!\rangle^{\Gamma} \\
&= (v_0, \varphi(0)) + \langle\!\langle \mathcal{L}, \varphi \rangle\!\rangle - (\!(z[u], [\varphi])\!)^{\Gamma}.
\end{aligned}
\tag{3.2}
$$

Moreover

$$
\langle\!\langle \ddot{u}, \varphi \rangle\!\rangle + (\!(\nabla \dot{u}, \nabla \varphi)\!) + (\!(\nabla u, \nabla \varphi)\!) = \langle\!\langle \mathcal{L}, \varphi \rangle\!\rangle, \tag{3.3}
$$

for all $\varphi \in H^1([0,T], H^1_0(\tilde{\Omega}))$, for all $t \in [0,T]$.

(a') The following restricted weak equations of motion holds: for all $t \in [0,T]$ there exists $\eta_t \in \mathcal{H}'_t \cap \mathcal{H}'$ such that

$$
\begin{aligned}
&-(\!(\dot{u}, \dot{\varphi})\!)_t + (\dot{u}(t), \varphi(t)) + (\!(\nabla \dot{u}, \nabla \varphi)\!)_t + (\!(\nabla u, \nabla \varphi)\!)_t + \langle\!\langle \eta_t, [\varphi] \cdot \nu \rangle\!\rangle^{\Gamma}_t \\
&= (v_0, \varphi(0)) + \langle\!\langle \mathcal{L}, \varphi \rangle\!\rangle_t - (\!(z[u], [\varphi])\!)^{\Gamma}_t,
\end{aligned}
\tag{3.4}
$$

for all $\varphi \in \mathcal{V}_t$. Moreover η_t satisfies the property that, for all $\varphi \in \mathcal{H}_t$ with $\varphi(t) = 0$, we have

$$
\langle\!\langle \eta_t, \varphi \rangle\!\rangle^{\Gamma}_t = \langle\!\langle \eta, \tilde{\varphi} \rangle\!\rangle^{\Gamma}, \tag{3.5}
$$

where $\tilde{\varphi}$ denotes the extension to $\mathcal{H}$ of $\varphi \in \mathcal{H}_t$ such that $\varphi(s) = 0$ for $s \in [t,T]$.

(a") We have

$$
\eta \in \beta_w([u] \cdot \nu), \tag{3.6}
$$

and for all $t \in [0,T]$ it also holds

$$
\eta_t \in \beta_{w,t}([u_{\llcorner [0,t]}] \cdot \nu). \tag{3.7}
$$

(b) for almost every $x \in \Gamma$ the function $t \mapsto z(t,x)$ is nonincreasing and

$$
\text{either} \quad \frac{1}{2}|[u(t,x)]|^2 \leq \alpha(x) \quad \text{or} \quad z(t,x) = 0 \quad \text{for a.e. } x \in \Gamma \tag{3.8}
$$

for all $t \in [0,T]$.

(b') for all times t_1 and t_2 with $0 \leq t_1 < t_2 \leq T$ it holds

$$
\int_{\Gamma} z(t_2)(\frac{1}{2}|[u(t_2)]|^2 - \alpha)dx - \int_{\Gamma} z(t_1)(\frac{1}{2}|[u(t_1)]|^2 - \alpha)dx - \int_{t_1}^{t_2} \int_{\Gamma} z[u] \cdot [\dot{u}] dx dt = 0. \tag{3.9}
$$

(c) the following energy inequality holds

$$
\begin{aligned}
&\frac{1}{2}\|\dot{u}(t_2)\|^2 + J([u(t_2)] \cdot \nu) + (V(u(t_2)), z(t_2))^{\Gamma} + \frac{1}{2}\|\nabla u(t_2)\|^2 \\
&+ \int_{t_1}^{t_2} \|\nabla \dot{u}\|^2 ds - (\alpha, z(t_2))_{\Gamma} + (\alpha, z(t_1))_{\Gamma} \leq \\
&\frac{1}{2}\|\dot{u}(t_1)\|^2 + J([u(t_1)] \cdot \nu) + (V(u(t_1)), z(t_1))^{\Gamma} + \frac{1}{2}\|\nabla u(t_1)\|^2 + \int_{t_1}^{t_2} \langle \mathcal{L}, \dot{u} \rangle ds,
\end{aligned}
\tag{3.10}
$$

for a.e. $t_1, t_2 \in [0,T]$, $t_1 < t_2$.

3.1 The approximate problem

In this section we introduce the energetic formulation of the approximate problem (1.7). Also for the approximate problem we restrict our attention to the homogeneous Dirichlet condition

$$u^\epsilon = 0 \text{ on } \partial_D\Omega \times [0,T]. \tag{3.11}$$

Definition 3.2. Let us fix $\epsilon \in (0,1)$, let $(u_0, v_0, z_0) \in V \times V \times \mathcal{Z}$, and $\mathcal{L} \in L^2([0,T], V')$. A couple (u^ϵ, z^ϵ) satisfying

$$u^\epsilon \in H^1([0,T], V) \cap W^{1,\infty}([0,T], L^2(\Omega)), \tag{3.12a}$$

$$\dot{u}^\epsilon \in H^1([0,T], V'), \tag{3.12b}$$

$$z^\epsilon \in L^\infty([0,T], \mathcal{Z}) \cap BV([0,T], L^1(\Gamma)), \tag{3.12c}$$

is called an weak (energetic) solution to Problem (1.7) if $u^\epsilon(0) = u_0$, $\dot{u}^\epsilon(0) = v_0$, $z^\epsilon(0) = z_0$, and the three following conditions hold:

(a^ϵ) for every time $t \in [0,T]$, it holds

$$\begin{aligned} &- (\!(\dot{u}^\epsilon, \varphi_t)\!)_t + (\dot{u}^\epsilon(t), \varphi(t)) + (\!(\nabla \dot{u}^\epsilon, \nabla\varphi)\!)_t + (\!(\nabla u^\epsilon, \nabla\varphi)\!)_t + (\!(\beta^\epsilon([u^\epsilon]\cdot\nu), \varphi)\!)_t^\Gamma \\ &= (u_1, \varphi(0)) + \langle\!\langle \mathcal{L}, \varphi\rangle\!\rangle_t - (\!(z^\epsilon[u^\epsilon], [\varphi])\!)_t^\Gamma, \end{aligned} \tag{3.13}$$

for all $\varphi \in \mathcal{V}_t$.

(b^ϵ) for almost every $x \in \Gamma$ the function $t \mapsto z^\epsilon(t,x)$ is nonincreasing and

$$\text{either} \quad V([u^\epsilon(t,x)]) \le \alpha(x) \quad \text{or} \quad z^\epsilon(t,x) = 0 \quad \text{for a.e. } x \in \Gamma \tag{3.14}$$

for all $t \in [0,T]$.

(c^ϵ) the following energy balance holds

$$\begin{aligned} &\frac{1}{2}\|\dot{u}^\epsilon(t)\|^2 + J^\epsilon([u^\epsilon(t)]\cdot\nu) + (V([u^\epsilon](t)), z^\epsilon(t))^\Gamma + \frac{1}{2}\|\nabla u^\epsilon(t)\|^2 + \int_0^t \|\nabla \dot{u}^\epsilon\|^2 ds - (\alpha, z^\epsilon(t))^\Gamma \\ &= \frac{1}{2}\|v_0\|^2 + J^\epsilon([u_0]\cdot\nu) + (V([u_0]), z_0)^\Gamma + \frac{1}{2}\|\nabla u_0\|^2 - (\alpha, z_0)^\Gamma + \langle\!\langle \mathcal{L}, \dot{u}^\epsilon\rangle\!\rangle_t \end{aligned} \tag{3.15}$$

for all $t \in [0,T]$.

Note that, thanks to (3.12a) and (3.12b), equation (3.13) can also be written in the standard form

$$\langle\!\langle \ddot{u}^\epsilon, \varphi\rangle\!\rangle + (\!(\nabla \dot{u}^\epsilon, \nabla\varphi)\!)_t + (\!(\nabla u^\epsilon, \nabla\varphi)\!)_t + (\!(\beta^\epsilon([u^\epsilon]\cdot\nu), [\varphi])\!)_t^\Gamma = \langle\!\langle \mathcal{L}, \varphi\rangle\!\rangle_t - (\!(z^\epsilon[u^\epsilon], [\varphi])\!)_t^\Gamma, \tag{3.16}$$

for all $\varphi \in \mathcal{V}$, for all $t \in [0,T]$.

Remark 3.3. Condition (b^ϵ) only ensures that (1.7c) and (1.7e) hold. Equation (1.7d) is not explicit, but the presence of both (b^ϵ) and (c^ϵ) ensures that it is satisfied in a weak sense. In fact (b^ϵ) and (c^ϵ) imply that for all times t_1 and t_2 with $0 \le t_1 < t_2 \le T$ it holds

$$\int_\Gamma z^\epsilon(t_2)(\frac{1}{2}|[u^\epsilon(t_2)]|^2 - \alpha)dx - \int_\Gamma z^\epsilon(t_1)(\frac{1}{2}|[u^\epsilon(t_1)]|^2 - \alpha)dx - \int_{t_1}^{t_2}\int_\Gamma z^\epsilon[u^\epsilon]\cdot[\dot{u}^\epsilon]dxdt = 0. \tag{3.17}$$

Equation (3.9) can be seen exactly as the integration by parts in time of (1.7d).

The existence of energetic solutions to problem (1.5) is standard. It can be carried out following the lines of the proof of existence of energetic solutions of the problem in [20, Definition 2.1]. We do not give a detailed proof, referring to [20, Appendix] and references therein for further detail. Here we just recover some fundamental steps in order to highlight the small differences between the cited case and the ours. The argument consists in a time discretization procedure and a variational implicit scheme as described below. To simplify notation in the rest of this section we drop the label ϵ.

For all integer $n > 0$ we divide the interval $[0, T]$ in n equal subintervals of length $\tau := T/n$. We set $t_i^n := i\tau$,

$$u_0^n = u_0, \qquad u_{-1}^n := u_0 - \tau v_0, \qquad z_0^n := z_0,$$

and define $\mathcal{L}_i^n := \frac{1}{\tau}\int_{t_i^n}^{t_{i+1}^n} \mathcal{L}(s)ds$ for all $n > 0$. Then for $1 \le i \le n$ we recursively define $u_i^n \in V$ as a minimizer of

$$\begin{aligned} U_i^n(u) :=& \frac{1}{2}\|\frac{u - u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2\tau}\|\nabla u - \nabla u_{i-1}^n\|^2 \\ &+ J^\epsilon([u]\cdot\nu) + (V([u]), z_{i-1}^n)^\Gamma - \langle g_i^n, u\rangle, \end{aligned} \tag{3.18}$$

and $z_i^n \in \mathcal{Z}$ as the minimizer of

$$W_i^n(z) := (V([u_i^n]), z)^\Gamma - (\alpha, z - z_{i-1}^n)^\Gamma, \tag{3.19}$$

among the class of all $z \in L^2(\Gamma, [0,1])$ such that $z \le z_{i-1}^n$. Computing variations at these minimizers we find out

$$\begin{aligned} &(\frac{u_i^n - u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}, \varphi) + (\nabla u_i^n, \nabla\varphi) + (\frac{\nabla u_i^n - \nabla u_{i-1}^n}{\tau}, \nabla\varphi) \\ &+ (\beta^\epsilon([u_i^n]\cdot\nu), [\varphi])^\Gamma + ([u_i^n]\cdot[\varphi], z_{i-1}^n)^\Gamma - \langle \mathcal{L}_i^n, \varphi\rangle = 0, \end{aligned} \tag{3.20}$$

for all $\varphi \in V$, while

$$\int_{\Gamma\cap\{z_i^n>0\}} V([u_i^n])\eta dx - \int_{\Gamma\cap\{z_i^n>0\}} \alpha\eta dx \ge 0, \tag{3.21}$$

for all $\eta \in L^2(\Gamma)$, $\eta \le 0$, and

$$(V([u_i^n]), \eta)^\Gamma - (\alpha, \eta)^\Gamma = 0, \tag{3.22}$$

if η is such that, for some $\epsilon > 0$, $z_i^n \pm \epsilon\eta \in [0, z_{i-1}]$ a.e. in Γ. The minimality of z_i^n implies also

$$(V([u_i^n]), z_i^n - z_{i-1}^n)^\Gamma - (\alpha, z_i^n - z_{i-1}^n)^\Gamma \le 0. \tag{3.23}$$

Now, standard a-priori bounds are provided for the functions u_τ, z_τ, and v_τ, defined as the unique piecewise affine (on $[t_{j-1}^n, t_j^n]$ for all $j = 1, \dots, n$) maps satisfying $u_\tau(t_j^n) = u_j^n$, $z_\tau(t_j^n) = z_j^n$, and $v_\tau(t_j^n) = v_j^n := \frac{1}{\tau}(u_j^n - u_{j-1}^n)$, for all $j = 1, \dots, n$. In particular we find

$$u_\tau \rightharpoonup u \quad \text{weakly in } H^1([0,T], V), \tag{3.24a}$$
$$u_\tau(t) \rightharpoonup u(t) \quad \text{weakly in } V, \text{ for every } t \in [0,T], \tag{3.24b}$$
$$z_\tau \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T], L^2(\Gamma)), \tag{3.24c}$$

as $\tau \to 0$. Moreover $\dot{u} \in H^1([0,T], V')$, $z \in BV([0,T], X')$ for any Banach space X such that $L^1(\Gamma) \subset X'$, $t \mapsto z(t,x)$ is nonincreasing, and

$$v_\tau \rightharpoonup \dot{u} \quad \text{weakly* in } L^\infty([0,T], H), \tag{3.24d}$$
$$\dot{v}_\tau \rightharpoonup \ddot{u} \quad \text{weakly in } L^2([0,T], V'), \tag{3.24e}$$
$$z_\tau \rightharpoonup z \quad \text{weakly* in } BV([0,T], X'), \tag{3.24f}$$
$$z_\tau(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for every } t \in [0,T],, \tag{3.24g}$$

To deduce (3.24e) we argued by comparison in (3.20) and used the fact that, for fixed ϵ, the function $|\beta^\epsilon(s)|$ has linear growth in s. Condition (3.20) is easily seen to pass to the limit in an integral form, thus providing condition (a^ϵ). Condition (b^ϵ) is proved in the following Lemma:

Lemma 3.4. *Condition (b^ϵ) holds for (u,z) in (3.24).*

Proof. Since $u_\tau \cdot u_\tau$ is bounded in $H^1([0,T],W^{1,q}(\Omega))$ for $q<\frac{d}{d-1}$, we can assume

$$[u_\tau]\cdot[u_\tau]\to[u]\cdot[u] \quad \text{strongly in } L^2([0,T],L^r(\Gamma)), \tag{3.25}$$

for $r<\frac{d-1}{d-2}=2$ if $d=3$, or $r<+\infty$ if $d\le 2$. Let $\hat{u}_\tau$ and $\hat{z}_\tau$ be the piecewise constant maps on $[0,T]$ such that $\hat{u}_\tau(t)=u^n_{i-1}$ and $\hat{z}_\tau(t)=z^n_{i-1}$ for all $t\in[t_{i-1},t_i)$ and $i=1,\dots,n-1$. Hence it is not difficult to see that, up to a further subsequence, it holds

$$[\hat{u}_\tau]\cdot[\hat{u}_\tau]\to[u]\cdot[u] \quad \text{strongly in } L^2([0,T],L^r(\Gamma)), \tag{3.26a}$$

$$\hat{z}_\tau \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T],L^2(\Gamma)), \tag{3.26b}$$

$$\hat{z}_\tau \rightharpoonup z \quad \text{weakly* in } BV([0,T],X'), \tag{3.26c}$$

$$\hat{z}_\tau(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for every } t\in[0,T]. \tag{3.26d}$$

Conditions (3.21) and (3.22) are equivalent to

$$\int_{\Gamma\cap\{\hat{z}_\tau(t)>0\}} V([\hat{u}_\tau(t)])\eta dx-\int_{\Gamma\cap\{\hat{z}_\tau(t)>0\}}\alpha\eta dx\ge 0, \tag{3.27}$$

for all $\eta\in L^2(\Gamma)$, $\eta\le 0$, and

$$(V([\hat{u}_\tau(t)]),\eta)^\Gamma-(\alpha,\eta)^\Gamma=0, \tag{3.28}$$

if η is such that, for some $\epsilon>0$, $z^n_i\pm\epsilon\eta\in[0,z^n_{i-1}]$, for $t\in[t_{i-1},t_i)$.

Moreover there exists $\zeta\in L^\infty([0,T]\times\Gamma)$ such that

$$\chi_{\{\hat{z}_\tau>0\}}\rightharpoonup\zeta \quad \text{weakly* in } L^\infty([0,T]\times\Gamma). \tag{3.29}$$

Thus from this, (3.26a), and (3.27), we infer

$$(\!(V([u]),\zeta\psi)\!)^\Gamma-(\!(\alpha,\zeta\psi)\!)^\Gamma,\ge 0 \tag{3.30}$$

for all $\psi\in L^2([0,T],L^2(\Gamma))$, $\psi\le 0$. Let us show that $\{\zeta>0\}\supseteq\{z>0\}$; from this and the arbitrariness of ψ we will obtain that

$$V([u(t,\cdot)])\le\alpha(\cdot) \qquad \text{a.e. on the set} \quad \{z(t)>0\}, \tag{3.31}$$

for a.e. $t\in[0,T]$. To this aim set $A:=\{(t,x)\in[0,T]\times\Gamma: 0=\zeta(t,x)<z(t,x)\}$. Using (3.26b), by Fubini and the Dominated Convergence Theorem, and then by the fact that $\hat{z}_\tau\le 1$, we find

$$0\le\int_A zdxdt=\lim_{\tau\to 0}\int_A\hat{z}_\tau dxdt\le\int_A\chi_{\{\hat{z}_\tau>0\}}dxdt=\int_A\zeta dxdt,$$

which proves that $|A|=0$ and the claim follows. □

To prove (c^ϵ) we first test (3.20) by $\varphi=u^n_i-u^n_{i-1}$, then sum the obtained expression with (3.23). Hence, summing over $i=1,\dots,n$, we obtain the approximate energy inequality

$$\begin{aligned}&\frac{1}{2}\|v_\tau(T)\|^2-\frac{1}{2}\|v_0\|^2+\frac{\tau}{2}\int_0^T\|\dot{v}_\tau\|^2dt+\frac{1}{2}\|\nabla u_\tau(T)\|^2\\&-\frac{1}{2}\|\nabla u_0\|^2+\frac{\tau}{2}\int_0^T\|\nabla\dot{u}_\tau\|^2ds+\int_0^T\|\nabla\dot{u}_\tau\|^2ds+J^\epsilon([u(T)]\cdot\nu)-J^\epsilon([u_0]\cdot\nu)\\&-(\alpha,z_\tau(T))^\Gamma+(\alpha,z_0)^\Gamma+(V([u_\tau(T)]),z_\tau(T))^\Gamma-(V([u_0]),z_0)^\Gamma\\&\le\langle\!\langle\mathcal{L}_\tau,\dot{u}_\tau\rangle\!\rangle+(\!([u_\tau]\cdot[\dot{u}_\tau]-[\hat{u}_\tau]\cdot[\dot{u}_\tau],\hat{z}_\tau)\!)^\Gamma.\end{aligned} \tag{3.32}$$

Passing to the limit in the last formula, where it is easily seen that the third and sixth terms in the left-hand side, and the last term in the right-hand side, tend to 0, we get (c^ϵ) with $\le$. To prove the opposite inequality the arguments are standard and we address to [20, Appendix] and references therein.

4 Existence result

In this section we state and prove our main result, which provides the existence of solutions as in Definition 3.1.

Theorem 4.1. *Let $T > 0$, u_0, $v_0 \in V$, $z_0 \in L^2(\Gamma, [0,1])$, $\mathcal{L} \in L^2([0,T], V')$, then there exists (u, z, η) an energetic solution of (1.5) in the sense of Definition 3.1.*

For all $\epsilon \in (0,1)$ let (u^ϵ, z^ϵ) be an approximate solution of Problem (1.5), as given in Definition 3.2. Now we divide the proof in several steps.

Step 1. The following apriori estimates for the approximate solutions (u^ϵ, z^ϵ) hold true. There exists a constant $M > 0$ such that

$$\|u^\epsilon\|_{H^1([0,T],V)} \leq M, \tag{4.1a}$$
$$\|u\|_{W^{1,\infty}([0,T],H)} \leq M, \tag{4.1b}$$
$$\|\dot{u}^\epsilon\|_{W^{1,1}([0,T],\tilde{H}^{-2}(\Omega))} \leq M, \tag{4.1c}$$
$$\|\ddot{u}^\epsilon\|_{L^2([0,T],H^{-1}(\tilde{\Omega}))} \leq M, \tag{4.1d}$$
$$\|z^\epsilon\|_{L^\infty([0,t],\mathcal{Z})} \leq M, \tag{4.1e}$$
$$\|z^\epsilon\|_{BV([0,T],L^1(\Gamma))} \leq M. \tag{4.1f}$$
$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\|_{L^1([0,T],L^1(\Gamma))} \leq M, \tag{4.1g}$$

for all $\epsilon \in (0,1)$. Moreover

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\|_{\mathcal{H}'} + \|\beta_\epsilon([u^\epsilon]\cdot\nu)\|_{\mathcal{V}'} \leq M, \tag{4.2}$$

and for all $t \in [0,T]$

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\llcorner_{(0,t)}\|_{\mathcal{H}'_t} + \|\beta_\epsilon([u^\epsilon]\cdot\nu)\llcorner_{(0,t)}\|_{\mathcal{V}'_t} \leq M, \tag{4.3}$$

for all $\epsilon \in (0,1)$.

Proof. For all $\epsilon \in (0,1)$ the energy balance (c^ϵ) of Definition 3.2 implies

$$\begin{aligned}
&\frac{1}{2}\|\dot{u}^\epsilon(t)\|^2 + J^\epsilon([u^\epsilon(t)]\cdot\nu) + (V([u^\epsilon](t)), z^\epsilon(t))^\Gamma + \frac{1}{2}\|\nabla u^\epsilon(t)\|^2 + \|\nabla\dot{u}^\epsilon\|^2_{L^2([0,t]\times\Omega)} \\
&+ (\alpha, z_0 - z^\epsilon(t))^\Gamma = C_0 + \langle\!\langle \mathcal{L}, \dot{u}^\epsilon \rangle\!\rangle_t \leq C_0 + \frac{1}{2}\|\mathcal{L}\|^2_{L^2([0,t],V')} + \frac{1}{2}\|\dot{u}^\epsilon\|^2_{L^2([0,t],V)} \\
&= C_1 + \frac{1}{2}\|\dot{u}^\epsilon\|^2_{L^2([0,t]\times\Omega)} + \frac{1}{2}\|\nabla\dot{u}^\epsilon\|^2_{L^2([0,t]\times\Omega)},
\end{aligned} \tag{4.4}$$

where $C_0 := \frac{1}{2}\|u_1\|^2 + (V([u_0^\epsilon]), z_0^\epsilon)^\Gamma + J^\epsilon([u_0]\cdot\nu)dx + \frac{1}{2}\|\nabla u_0\|^2$, and $C_1 := C_0 + \|\mathcal{L}\|^2_{L^2([0,t],V')}$. From (4.4) we obtain $\|\dot{u}^\epsilon(t)\|^2 \leq C(1 + \|\dot{u}^\epsilon\|^2_{L^2([0,t]\times\Omega)})$, and the Gronwall Lemma implies that there exists a constant $M > 0$ such that

$$\|\dot{u}^\epsilon(t)\|^2 \leq M \text{ for all } t \in [0,T], \tag{4.5a}$$

for all $\epsilon \in (0,1)$, and hence (4.1b) holds. Note that M is a positive constant depending on the problem data, but independent of ϵ. From (4.4) we also get

$$\|u^\epsilon\|_{H^1([0,T],V)} \leq M, \tag{4.5b}$$
$$J^\epsilon([u^\epsilon(t)]\cdot\nu) \leq M \text{ for all } t \in [0,T], \tag{4.5c}$$
$$(V([u^\epsilon](t)), z^\epsilon(t)) \leq M \text{ for all } t \in [0,T], \tag{4.5d}$$
$$\|z^\epsilon\|_{L^\infty([0,t],\mathcal{Z})} \leq M, \tag{4.5e}$$

for all $\epsilon \in (0,1)$. Thanks to the monotonicity of z^ϵ and (2.14), the boundedness of the term $(\alpha, z_0 - z(t)) = \int_0^t (\alpha, \dot{z}^\epsilon) ds$, implies (4.1f). Moreover we find

$$\|T(z^\epsilon(t), u^\epsilon(t))\|_{V'} \leq M \text{ for all } t \in [0,T], . \tag{4.5f}$$

Let now $\psi \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^d)$ be a test function such that $\psi \cdot \nu = 1$ on the whole Γ. Let φ be the extension of ψ on Ω_1 defined as $\varphi := S_1(\psi)$ (see (2.10)), so that $\varphi \in V$. Let us set $\Psi(t,x) := \varphi(x)$ for all $t \in [0,T]$ and $x \in \Omega$. Then we test (3.2) by $\varphi = u + \delta\Psi$, with $\delta \in (0,1)$. We obtain (recall $\varphi = 0$ on Ω_2)

$$\begin{aligned}
&(\dot{u}^\epsilon(T), u^\epsilon(T)) - (u_1^\epsilon, u_0^\epsilon) - \int_0^T \|\dot{u}^\epsilon\|^2 dt + \int_{\Omega_1} \delta u^\epsilon(t) \cdot \Psi dx - \int_{\Omega_1} \delta u_0^\epsilon \cdot \Psi dx + \int_0^T \|\nabla u^\epsilon\|^2 dt \\
&+ \frac{1}{2}\|\nabla u^\epsilon(T)\|^2 - \frac{1}{2}\|\nabla u_0^\epsilon\|^2 + \int_0^T \int_{\Omega_1} \delta \nabla u^\epsilon \cdot \nabla\Psi dx dt + \int_{\Omega_1} \delta \nabla u^\epsilon(T) \cdot \nabla\Psi dx dt - \int_{\Omega_1} \delta \nabla u_0^\epsilon \cdot \nabla\Psi dx dt \\
&+ \int_0^T \int_\Gamma \beta_\epsilon([u^\epsilon]\cdot\nu)([u^\epsilon]\cdot\nu - \delta) dx dt + ⟪z^\epsilon, |u^\epsilon|^2 - \delta u^\epsilon \cdot \nu⟫^\Gamma = ⟪\mathcal{L}, u^\epsilon⟫ + ⟪\mathcal{L}, \delta\Psi⟫.
\end{aligned}$$

Thus, since $|\beta_\epsilon(x)| \leq \delta^{-1}|\beta_\epsilon(x)(x-\delta)|$ for $\epsilon \in (0,1)$, it follows

$$\begin{aligned}
&\delta \int_0^T \int_\Gamma |\beta_\epsilon([u^\epsilon]\cdot\nu)| dx dt \leq \int_0^T \|z^\epsilon\|^2 dt + \frac{1}{2}\int_0^T \|[u^\epsilon]\|^4_{L^4(\Gamma)} dt + \frac{1}{2}\int_0^T \|[u^\epsilon]\cdot\nu\|^2 dt \\
&\frac{1}{2}\|\dot{u}^\epsilon(T)\|^2 + \frac{1}{2}\|u^\epsilon(T)\|^2 + \int_0^T \|\dot{u}^\epsilon\|^2 dt + \int_0^T \|\nabla u^\epsilon\|^2 dt + \frac{1}{2}\|\nabla u^\epsilon(T)\|^2 + \frac{1}{2}\int_0^T \|\nabla u^\epsilon\|_2 dt \\
&+ \frac{1}{2}\|\nabla u^\epsilon(T)\|^2 + \frac{1}{2}\int_0^T \|\mathcal{L}\|^2_{V'} dt + \frac{1}{2}\int_0^T \|u^\epsilon\|^2_V dt + \int_0^T \langle \mathcal{L}, \delta\Psi \rangle dt + C_1 \leq C_2,
\end{aligned} \tag{4.6}$$

for some constant $C_1, C_2 > 0$ independent of $\epsilon \in (0,1)$. Here we have used the Young inequality in the first estimate and the estimates obtained so far in the last one. This entails (4.1g). Thanks to the continuity of the embedding $L^1(\Gamma) \subset H^{-\frac{3}{2}}(\Gamma)$, valid for $d \leq 3$, the continuity of the trace, together with (3.13), implies that

$$\|\ddot{u}^\epsilon\|_{L^1([0,T],\tilde{H}^{-2}(\Omega))} \leq M, \tag{4.7}$$

so that (4.1c) follows. Moreover, arguing by comparison in (3.16) with $\varphi \in H^1([0,T], H_0^1(\tilde{\Omega}))$ (i.e., $[\varphi] = 0$), estimate (4.1a) implies (4.1d).

Let us now prove (4.2) and (4.3). For every $\varphi \in \mathcal{H}$ let $\Phi(t,\cdot) := S_1(\varphi(t)) \in V$, so that $\Phi \in H^1([0,T], V)$. Since $\Phi \in \mathcal{V}$, from (3.13) we write

$$\begin{aligned}
&|⟪\beta_\epsilon([u^\epsilon]\cdot\nu), \varphi⟫_t| \leq \|\dot{u}^\epsilon\|_{L^2([0,t]\times\Omega)}\|\dot{\Phi}\|_{L^2([0,t]\times\Omega)} + \|\dot{u}^\epsilon(t)\|\|\varphi(t)\|_H + \|u_1^\epsilon\|\|\Phi(0)\| \\
&+ \|\nabla\dot{u}^\epsilon\|_{L^2([0,t]\times\Omega)}\|\nabla\Phi\|_{L^2([0,t]\times\Omega)} + \|\nabla u^\epsilon\|_{L^2([0,t]\times\Omega)}\|\nabla\Phi\|_{L^2([0,t]\times\Omega)} \\
&+ \|T(z^\epsilon, u^\epsilon)\|_{L^2([0,t],V')}\|\Phi\|_{L^2([0,t],V)} + \|\mathcal{L}\|_{L^2([0,t],V')}\|\Phi\|_{L^2([0,t],V)} \\
&\leq C_1\|\Phi\|_{\mathcal{V}_t} \leq C\|\varphi\|_{\mathcal{H}_t},
\end{aligned} \tag{4.8}$$

for all $\varphi \in \mathcal{H}$, where we have used (4.1b), (4.1a), (4.5f), and the continuity of the map S_1. This shows that

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\llcorner_{(0,t)}\|_{\mathcal{H}'_t} \leq M \text{ for all } t \in [0,T], \tag{4.9}$$

and in particular

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\|_{\mathcal{H}'} \leq M, \tag{4.10}$$

for all $\epsilon \in (0,1)$. If we repeat the argument in (4.8) with an arbitrary extension $\Phi \in \mathcal{V}$ of φ, we see that estimates (4.9) and (4.10) hold also in the spaces $\mathcal{V}'_t$ and $\mathcal{V}'$, respectively, i.e.

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\llcorner_{(0,t)}\|_{\mathcal{V}'_t} \leq M \text{ for all } t \in [0,T], \tag{4.11}$$

$$\|\beta_\epsilon([u^\epsilon]\cdot\nu)\|_{\mathcal{V}'} \leq M, \tag{4.12}$$

for all $\epsilon \in (0,1)$. This concludes the proof of Step 1. □

Step 2. There exist (u, z, η) satisfying (3.1) such that, for a subsequence of $\epsilon \to 0$,

$$u^\epsilon \rightharpoonup u \quad \text{weakly in } H^1([0,T], V) \text{ and weakly* in } W^{1,\infty}([0,T], H), \tag{4.13a}$$
$$\dot{u}^\epsilon \to \dot{u} \quad \text{strongly in } L^2([0,T], H) \text{ and weakly in } H^1([0,T], H^{-1}(\tilde{\Omega})), \tag{4.13b}$$
$$\dot{u}^\epsilon(t) \rightharpoonup u(t) \quad \text{weakly in } \tilde{H}^{-2}(\Omega) \text{ for all } t \in [0,T], \tag{4.13c}$$
$$z^\epsilon(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for all } t \in [0,T], \tag{4.13d}$$
$$\beta^\epsilon([u^\epsilon] \cdot \nu) \rightharpoonup \eta \quad \text{weakly in } \mathcal{H}' \text{ and } \mathcal{V}', \tag{4.13e}$$

with

$$z \in BV([0,T], L^1(\Gamma)). \tag{4.13f}$$

Moreover (a) is satisfied, and for all $t \in [0,T)$ there exists $\eta_t \in \mathcal{H}_t$ such that, for the same subsequence,

$$\beta^\epsilon([u^\epsilon] \cdot \nu)_{\llcorner(0,t)} \rightharpoonup \eta_t \quad \text{weakly in } \mathcal{H}_t' \text{ and } \mathcal{V}_t', \tag{4.14}$$

with η_t satisfying (a').

Proof. From (4.1a), (4.1d), (4.1e), and (4.1f), we deduce that there exist $u \in H^1([0,T], V)$ and $z \in L^\infty([0,T], \mathcal{Z}) \cap BV([0,T]; L^1(\Gamma))$ such that, for a subsequence of ϵ tending to 0,

$$u^\epsilon \rightharpoonup u \quad \text{weakly in } H^1([0,T], V), \tag{4.15a}$$
$$u^\epsilon \rightharpoonup u \quad \text{weakly* in } W^{1,\infty}([0,T], H), \tag{4.15b}$$
$$\dot{u}^\epsilon \rightharpoonup \dot{u} \quad \text{weakly* in } H^1([0,T], H^{-1}(\tilde{\Omega})), \tag{4.15c}$$
$$z^\epsilon \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T], L^2(\Gamma)), \tag{4.15d}$$
$$z \in BV([0,T], L^1(\Gamma)), \tag{4.15e}$$

and in particular

$$u^\epsilon(t) \to u(t) \quad \text{strongly in } H \text{ for all } t \in [0,T], \tag{4.15f}$$
$$u^\epsilon(t) \rightharpoonup u(t) \quad \text{weakly in } V \text{ for all } t \in [0,T]. \tag{4.15g}$$

Moreover, the continuity of the trace from V to $H^{\frac{1}{2}}(\Gamma, \mathbb{R}^d)$ and the compactness of the embedding $H^{\frac{1}{2}}(\Gamma, \mathbb{R}^d) \subset L^r(\Gamma, \mathbb{R}^d)$, for all $r < \frac{2(d-1)}{d-2}$, imply that

$$[u^\epsilon] \to [u] \quad \text{strongly in } L^2([0,T], L^2(\Gamma, \mathbb{R}^d)), \tag{4.15h}$$
$$[u^\epsilon(t)] \to [u(t)] \quad \text{strongly in } L^r(\Gamma, \mathbb{R}^d) \text{ for all } t \in [0,T]. \tag{4.15i}$$

Similarly, by (4.15g), we find that

$$\begin{aligned} &T(z^\epsilon, u^\epsilon) \to T(z, u) \text{ weakly in } \mathcal{V}', \\ &T(z^\epsilon(t), u^\epsilon(t)) \to T(z(t), u(t)) \text{ weakly in } V' \text{ for all } t \in [0,T]. \end{aligned} \tag{4.15j}$$

Condition (4.1c) implies that $\dot{u}^\epsilon$ are functions uniformly bounded in $BV([0,T], \tilde{H}^{-2}(\Omega))$. We can then employ a generalization of Helly Theorem [8, Lemma 7.2], providing a function $v \in BV([0,T], \tilde{H}^{-2}(\Omega))$ such that

$$\dot{u}^\epsilon(t) \rightharpoonup v(t) \quad \text{weakly in } \tilde{H}^{-2}(\Omega) \text{ for all } t \in [0,T].$$

Since (4.15c) holds, we can identify v with $\dot{u}$, everywhere on $[0,T]$. Moreover condition (4.1b) entails that such convergence must hold in H, i.e.,

$$\dot{u}^\epsilon(t) \rightharpoonup \dot{u}(t) \quad \text{weakly in } H \text{ for all } t \in [0,T]. \tag{4.15k}$$

By (4.1f), again the Helly selection principle implies

$$z^\epsilon(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for all } t \in [0,T]. \tag{4.15l}$$

Since V is compactly embedded in H, thanks to condition (4.1a) and (4.1c), we can apply [24, Corollary 4] with $X = V$, $B = H$, $Y = \tilde{H}^{-2}(\Omega)$, and $p = 2$, in order to obtain that

$$\dot{u}^\epsilon \to \dot{u} \quad \text{strongly in } L^2([0,T], H). \tag{4.15m}$$

Besides, condition (4.10) and (4.12) imply that, up to a subsequence,

$$\beta_\epsilon([u^\epsilon]\cdot\nu) \rightharpoonup \eta \quad \text{weakly in } \mathcal{H}' \text{ and in } \mathcal{V}', \tag{4.15n}$$

for some $\eta \in \mathcal{H}'$. We have obtained (4.13).

Let us now define η_t as the element of $\mathcal{H}' \cap \mathcal{H}'_t$ such that

$$\begin{aligned}\langle\langle \eta_t, \varphi\rangle\rangle := (\!(\dot{u}, \dot{\Phi})\!)_t - (\dot{u}(t), \Phi(t)) + (u_1, \Phi(0)) - (\!(\nabla\dot{u}, \nabla\Phi)\!)_t \\ - (\!(\nabla u, \nabla\Phi)\!)_t + \langle\langle T(z,u), \Phi\rangle\rangle_t + \langle\langle \mathcal{L}, \Phi\rangle\rangle_t,\end{aligned} \tag{4.16}$$

where again $\Phi := S_1(\varphi)$ is the extension of φ to $\Omega \times [0,T]$ obtained by the map S_1 in (2.10), in such the way that $\Phi \in \mathcal{V}$ and $[\Phi(t)] = \varphi(t)$ for all $t \in [0,T]$. It is easy to check that, by the same estimates as in (4.8), the map η_t belongs to $\mathcal{H}' \cap \mathcal{H}'_t$ and it can be identified as an element of $\mathcal{V}' \cap \mathcal{V}'_t$. Now, convergences (4.15) imply that we can pass to the limit in (3.13), so that (with no need of extracting a further subsequence)

$$\beta^\epsilon([u^\epsilon]\cdot\nu)_{\llcorner(0,t)} \rightharpoonup \eta_t \quad \text{weakly in } \mathcal{H}' \text{ and } \mathcal{V}' \text{ for all } t \in [0,T]. \tag{4.17}$$

Moreover the same limit takes place in the weak topology of $\mathcal{H}'_t$ and $\mathcal{V}'_t$. In particular we have obtained equations (3.2) and (3.4). In the case that $\varphi \in H^1([0,T], H^1_0(\tilde{\Omega}))$ also equation (3.16) passes to the limit thanks to (4.15c), providing (3.3). □

Step 3. Conditions (b) and (b') hold.

Proof. Let us first see that condition (b^ϵ) of Definition 3.2 passes to the limit. Since $|u^\epsilon|^2$ is bounded in $H^1([0,T], W^{1,q})$ for $q < \frac{d}{d-1}$, we can assume, by (4.13a), that

$$|[u^\epsilon]|^2 \rightharpoonup |[u]|^2 \quad \text{weakly in } H^1([0,T], W^{1-1/q,q}(\Gamma)), \tag{4.18}$$

so by Sobolev embedding

$$|[u^\epsilon]|^2 \to |[u]|^2 \quad \text{strongly in } L^2([0,T], L^r(\Gamma)), \tag{4.19}$$

for $r < \frac{d-1}{d-2} = 2$ if $d = 3$, or $r < +\infty$ if $d \leq 2$. Moreover, since for all $t \in [0,T]$ it holds $|[u^\epsilon(t)]|^2 \rightharpoonup |[u(t)]|^2$ weakly in $W^{1-1/q,q}(\Gamma)$, we also get

$$|[u^\epsilon(t)]|^2 \to |[u(t)]|^2 \quad \text{strongly in } L^r(\Gamma). \tag{4.20}$$

Thus we argue as in the proof of Lemma 3.4, obtaining (b).

We prove that also condition (3.17) passes to the limit. Thanks to (4.13d) and (4.20) it is easily seen that for all $t \in [0,T]$ the convergence holds

$$\int_\Gamma z^\epsilon(t)(\frac{1}{2}|[u^\epsilon(t)]|^2 - \alpha)dx \to \int_\Gamma z(t)(\frac{1}{2}|[u(t)]|^2 - \alpha)dx.$$

In order to prove that for all $t_1 < t_2$ it holds

$$\int_{t_1}^{t_2}\int_\Gamma z^\epsilon[u^\epsilon]\cdot[\dot{u}^\epsilon]dxdt \to \int_{t_1}^{t_2}\int_\Gamma z[u]\cdot[\dot{u}]dxdt,$$

we first note that by (4.1f) we may apply the generalized Aubin-Lions Lemma [18, Corollary 7.9] obtaining

$$z^\epsilon \to z \quad \text{strongly in } L^2([0,T], W^{1-1/q,q}(\Gamma)'), \tag{4.21}$$

for some $q < \frac{d}{d-1}$, where we have used that the compact and dense embedding $W^{1-1/q,q}(\Gamma) \subset L^r$ implies $L^{r/(r-1)} \subset W^{1-1/q,q}(\Gamma)'$ compactly for all $1 < r < \frac{dq-q}{d-q}$. Thus the thesis follows from this and from (4.18). $\square$

Step 4. Condition (a") holds true. Moreover we have

$$u^\epsilon \to u \quad \text{strongly in } L^2([0,T], V), \tag{4.22}$$

$$u^\epsilon(t) \to u(t) \quad \text{strongly in } V \text{ for all } t \in [0,T]. \tag{4.23}$$

Proof. In order to prove (a") we apply Lemma 2.4 with $v_\epsilon = [u^\epsilon] \cdot \nu$, $\xi_\epsilon = \beta^\epsilon([u^\epsilon] \cdot \nu)$, $v = [u] \cdot \nu$, and $\xi = \eta$. Thanks to (4.13a) and (4.13e) it is sufficient to check that

$$\limsup_{\epsilon \to 0} \langle\langle \beta^\epsilon(u^\epsilon), u^\epsilon \rangle\rangle^\Gamma \le \langle\langle \eta, u \rangle\rangle^\Gamma. \tag{4.24}$$

Using (3.13), we write

$$\begin{aligned}\langle\langle \beta^\epsilon([u^\epsilon] \cdot \nu), [u^\epsilon] \cdot \nu \rangle\rangle^\Gamma &= \int_0^T \|\dot u^\epsilon\|^2 dt - (\dot u^\epsilon(T), u^\epsilon(T)) + (v_0, u_0) - \frac{1}{2}\|\nabla u^\epsilon(T)\|^2 + \frac{1}{2}\|\nabla u_0\|^2 \\ &- \int_0^T \|\nabla u^\epsilon\|^2 dt - \langle\langle T(z^\epsilon, u^\epsilon), u^\epsilon \rangle\rangle + \langle\langle \mathcal{L}, u^\epsilon \rangle\rangle.\end{aligned} \tag{4.25}$$

It is seen that

$$\begin{aligned}\lim_{\epsilon \to 0} \langle\langle T(z^\epsilon, u^\epsilon), u^\epsilon \rangle\rangle &= \int_0^T (z^\epsilon(t), |[u^\epsilon(t)]|^2)^\Gamma dt = \int_0^T (z(t), |[u(t)]|^2)^\Gamma dt \\ &= \langle\langle T(z(t), u(t)), [u(t)] \rangle\rangle^\Gamma,\end{aligned} \tag{4.26}$$

thanks to (4.13d) and (4.19). Therefore thanks to (4.13a), (4.15g), (4.15k), (4.13b), and (4.26), we see that the lim sup of (4.25) is less or equal to

$$\begin{aligned}&\int_0^T \|\dot u\|^2 dt - (\dot u(T), u(T)) + (v_0, u_0) - \frac{1}{2}\|\nabla u(T)\|^2 + \frac{1}{2}\|\nabla u_0\|^2 - \int_0^T \|\nabla u\|^2 dt \\ &+ \langle\langle T(z,u), [u] \rangle\rangle + \langle\langle \mathcal{L}, u \rangle\rangle = \langle\langle \eta, [u] \cdot \nu \rangle\rangle^\Gamma,\end{aligned} \tag{4.27}$$

by (3.2), and (4.24) is proved, i.e.,

$$\eta \in \beta_w([u] \cdot \nu). \tag{4.28}$$

If we fix any $t \in [0,T]$ and repeat the previous limit (4.25) with $T = t$, thanks to (4.14), the same argument shows that

$$\eta_t \in \beta_w^t([u_{\llcorner(0,t)}] \cdot \nu). \tag{4.29}$$

Now, thanks to the monotonicity of the operators β^ϵ we have

$$\langle\langle \beta^\epsilon([u^\epsilon] \cdot \nu) - \eta, [u^\epsilon] \cdot \nu - [u] \cdot \nu \rangle\rangle^\Gamma \ge 0,$$

hence passing to the limit we infer the opposite inequality in (4.24). In particular this implies that the limit of the expression (4.25) is exactly (4.27), and then we obtain

$$\lim_{\epsilon \to 0} \|\nabla u^\epsilon\|^2_{L^2([0,T]\times\Omega)} = \|\nabla u\|^2_{L^2([0,T]\times\Omega)}$$

$$\lim_{\epsilon \to 0} \|\nabla u^\epsilon(t)\|^2 = \|\nabla u(t)\|^2 \quad \text{for all } t \in [0,T],$$

getting (4.22). $\square$

Step 5. The energy inequality (c) holds.

Proof. In order to obtain this we first write the approximate energy balance (3.15) for a couple of times $t_1, t_2 \in [0,T]$, $t_1 < t_2$, and then let $\epsilon \to 0$. The convergences obtained so far show that all the terms pass to limit but $J^\epsilon(u^\epsilon(t))$ and $\int_0^T \|\nabla \dot{u}^\epsilon\| dt$. Convergence (4.13a) readily infer

$$\int_0^T \|\nabla \dot{u}\| dt \leq \liminf_{\epsilon \to 0} \int_0^T \|\nabla \dot{u}^\epsilon\| dt,$$

and then it remains to prove the convergence of the term $J^\epsilon(u^\epsilon(t))$ to $J(u(t))$ for a.e. $t \in [0,T]$. The inequality

$$J([u(t)] \cdot \nu) \leq \liminf_{\epsilon \to 0} J^\epsilon([u^\epsilon(t)] \cdot \nu). \tag{4.30}$$

is true thanks to (4.15f) and to Lemma 2.1. Moreover it can be proved that the liminf in the right hand side is actually a limit and that equality holds for a.e. $t \in [0,T]$. The proof of this fact is identical to the one in [5, Step 5], which we refer to. Therefore we can pass to the limit in (3.15) for a.e. $t_1, t_2 \in [0,T]$, $t_1 < t_2$.

□

4.1 Existence result: nonhomogeneous case

We describe here how to obtain existence of energetic dynamic solutions as in Theorem 4.1 satisfying a nonhomogeneous boundary condition. In order to impose a Dirichlet condition, we fix a map w satisfying the following hypotheses

$$w \in H^1([0,T], H^1(\tilde{\Omega})) \cap W^{1,\infty}([0,T], L^2(\Omega)), \tag{4.31a}$$

$$\dot{w} \in H^1([0,T], H^{-1}(\tilde{\Omega})) \cap BV([0,T], \tilde{H}^{-2}(\Omega)), \tag{4.31b}$$

$$w(0) = u_0 \quad \dot{w}(0) = v_0 \quad \text{ on } \partial_D \Omega. \tag{4.31c}$$

Note that the condition $w \in H^1(\tilde{\Omega})$ implies $[w] = 0$. Then the following Theorem holds true.

Theorem 4.2. *Let u_0, $v_0 \in H^1(\Omega)$, $z_0 \in L^2(\Gamma, [0,1])$, $\mathcal{L} \in L^2([0,T], V')$, then for any w satisfying hypotheses (4.31), there exists a triple (u, z, η) with*

$$u - w \in H^1([0,T], V) \cap W^{1,\infty}([0,T], L^2(\Omega)), \tag{4.32a}$$

$$\dot{u} \in H^1([0,T], H^{-1}(\tilde{\Omega})) \cap BV([0,T], \tilde{H}^{-2}(\Omega)), \tag{4.32b}$$

$$z \in L^\infty([0,T], \mathcal{Z}) \cap BV([0,T], L^1(\Gamma)), \tag{4.32c}$$

$$\eta \in \mathcal{H}', \tag{4.32d}$$

such that $u(0) = u_0$, $\dot{u}(0) = v_0$, $z(0) = z_0$, satisfying the conditions (a), (a'), (a"), (b), (b') of Theorem 4.1, and the following energy inequality

(c') for a.e. $t_1 < t_2 \in [0,T]$ it holds

$$\begin{aligned}
&\frac{1}{2}\|\dot{u}(t_2) - \dot{w}(t_2)\|_H^2 + J([u(t_2)] \cdot \nu) + (V(u(t_2)), z(t_2))^\Gamma + \frac{1}{2}\|\nabla u(t_2)\|^2 \\
&+ \int_{t_1}^{t_2} \|\nabla \dot{u}\|^2 ds - (\alpha, z(t_2))_\Gamma \leq \frac{1}{2}\|\dot{u}(t_1) - \dot{w}(t_1)\|^2 + J([u(t_1)] \cdot \nu) + (V(u(t_1)), z(t_1))^\Gamma \\
&+ \frac{1}{2}\|\nabla u(t_1)\|^2 - (\alpha, z_0)_\Gamma + \int_{t_1}^{t_2} (\sigma, \nabla \dot{w}) ds + \int_{t_1}^{t_2} \langle \mathcal{L} - \ddot{w}, \dot{u} - \dot{w} \rangle ds,
\end{aligned} \tag{4.33}$$

with $\sigma = \nabla u + \nabla \dot{u}$.

Let us remark that the boundary condition

$$u(t) = w(t) \quad \text{a.e. on } \partial_D\Omega, \quad \text{for all } t \in [0,T], \tag{4.34}$$

is implicit in condition (4.32a).

The technique of the proof is standard and we only sketch it. We apply Theorem 4.1 with external force $\mathcal{L}$ replaced by $\tilde{\mathcal{L}} := \mathcal{L} - \ddot{w} + \Delta w + \Delta \dot{w} \in L^2([0,T], V')$, hence providing a triple $(\tilde{u}, \tilde{z}, \eta)$ which is an energetic solution as in Definition 3.2, with homogeneous Dirichlet condition. Setting $u := \tilde{u} + w$ and observing that $[u] = [\tilde{u}]$ since $[w] = 0$, conditions (a), (a'), (a"), (b), (b') readily follow, as far as (4.32), and then (4.34). In order to obtain (c') we must argue in a different way, following the lines of the proof of (c) in Theorem 4.1. This relies in letting ϵ go to 0 in the energy balance of the approximate solution $\tilde{u}^\epsilon$, and dealing with some elementary algebra.

5 Concluding remarks

Within this paper we have proved the existence of a solution to Problem 1.5 in a weak form. In particular, as we have seen, this weak form involves an equation of motion written in duality with test functions in the space $\mathcal{V} := L^2(0,T;V) \cap H^1(0,T;H)$. Notice that this space is a space of functions in both the time and space variables. To the present stage, it seems very difficult to find a stronger formulation involving the duality with a space of test functions independent of time. This is due to the fact that the reaction term η is found only as an element of $\mathcal{V}'$, and in particular might be a measure concentrated in some discrete time set. The presence of such concentration phenomenon is quite intuitive in the one dimensional case, where we might imagine that the two bodies (strings) are separate and collide in a precise instant, after which they separate again. The instant of collision is the only one where the reaction is nonzero, and then concentrated. On the other hand, the presence of such concentration points being the only responsible for the discontinuities of the velocity field $\dot{u}$, cannot be apriori ruled out, as shown in the example of [5, Remark 2.4]. In this paper the authors treat a general evolution driven by a damped wave equation with unilateral constraint, which, neglecting the internal variable z, overlap also the situation considered in the present paper.

Another consequence of this concentration phenomenon, and then of the discontinuities of the velocity field, is the difficulty to establish an energy balance. This is somehow due to the fact that we cannot test equation (3.2) by $\varphi = \dot{u}$, since this does not belong to $\mathcal{V}$. On the other hand it is reasonable to claim the existence of solutions satisfying the energy balance, and to consider them as the "physically admissible" ones (in the specific example in [5, Remark 2.4] it is shown as there exist more then one solutions, some of whose satisfying the energy balance). It seems to us that the method provided here of approximating the solution by regularized ones fails if we wish to prove the energy balance, since it does not give sufficiently strong compactness criterion for the approximating evolutions. The proof of the energy balance is, at the present stage, the most challenging open question left by the argument proposed so far.

Let us finally remark that the method of approximation has been firstly proposed in [5] and then adapted to a problem of delamination in [22]. In this last paper the author consider a problem similar to (1.5), but with the addition of viscosity in the adhesive which provide different difficulties in order to argue by approximation. Some other techniques to treat second order evolutionary problems with unilateral constraints, based on the use of variational inequality, exist and can be found, for instance, in [2].

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