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Shift spaces and attractors in non invertible horse shoes

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Abstract

As well known, a horse shoe map, i.e. a special injective reimbedding of the unit square I^2 in $\mathbb{R}^2$ (or more general, of the cube I^m in $\mathbb{R}^m$) as considered first by S. Smale [4], defines a shift dynamics on the maximal invariant subset of I^2 (or I^m). It is shown that this remains true almost surely for non injective maps provided the contraction of the mapping in the stable direction in sufficiently strong.

Contents

1 Definitions and results

For an integer $\theta \geq 2$ the set Σ_θ of all doubly infinite sequences $\underline{i} = (\ldots, i_{-1}, i_0, i_1, \ldots)$, where $i_l \in \{1, \ldots, \theta\}$, equipped with the metric

$$d\left((\ldots, i_{-1}, i_0, i_1, \ldots), (\ldots, j_{-1}, j_0, j_1, \ldots)\right) = \sum_{l=-\infty}^{\infty} 2^{-|l|} \left(|i_l - j_l|\right)$$

is a Cantor set. the shift mapping $\sigma : \Sigma_\theta \to \Sigma_\theta$ given by

$$\sigma(\ldots, i_{-1}, i_0, i_1, \ldots) = (\ldots, j_{-1}, j_0, j_1, \ldots) \qquad \text{with } j_l = i_{l+1}$$

is a homeomorphism which defines a simple but nevertheless non trivial dynamics on Σ_θ : the periodic points are dense, and there are dense orbits e.g. Therefore, to ask whether or not a given diskrete dynamical system contains a subsystem conjugate to a shift space of this kind is a natural question.

Let R be a topological space with metric d, R^* a compact subset of R and $f : R^* \to R$ continuous. For $k \geq 1$ we define the compact sets

$$\begin{aligned} R_k^* &= \{p \in R |\ f^k(p) \ \text{ is defined}\} \\ A_k &= f^k(R_k^*). \end{aligned}$$

Then $R_1^* = R^* \supset R_2^* \supset R_3^* \ldots$, $A_1 \supset A_2 \supset \ldots$, and we consider the compact sets

$$\begin{aligned} R_\infty^* &= \bigcap_{k=1}^{\infty} R_k^*, \\ A &= \bigcap_{k=1}^{\infty} A_k, \\ Z &= R_\infty^* \cap A. \end{aligned}$$

The set A can be regarded as the global attractor of k. Indeed, $f(A \cap R^*) = A$, and there is a sequence $\varepsilon_1 > \varepsilon_2 > \ldots$ of real numbers tending to 0 such that for any $k > 1$ and any $p \in R_k^*$ we have $d(f^k(q), A) \leq \varepsilon_k$. The set Z is the maximal invariant subset of R, i.e. the maximal set on which f is defined, and $f(Z) = Z$.

A subset S of R^* will be called a *shift space in* R, if for some $\theta \geq 2$ there is a homeomorphisms $h : \Sigma_\theta \to S$ such that $h\sigma = fh$. Obviously, if S is a shift space in R then $S \subset Z$. If Z itself is a shift space in R then we say that f *concentrates to a shift space.*

Among the best known examples of mappings which concentrate to a shift are the so called horse shoe mappings which can be defined as follows. Let $R_0 = \mathbb{R}^{m+1}$ $(m \geq 1)$ and $R_0^* = I^{m+1} = I \times I^m$ the $(m+1)$-dimensional unit cube in $\mathbb{R}^{m+1}$ which is regarded as the cartesian product of the unit interval $I = [0,1]$ with the m-dimensional unit cube. To define a horse shoe mapping we fix disjoint subintervals $I_1, \ldots, I_\theta$ in I $(\theta \geq 2)$ and choose $f : R_0^* \to R_0$ so that the following conditions are satisfied, where $I^* = I_1 \cup \cdots \cup I_\theta$.

(i) $f(R_0^*) \cap R_0^* = f(I^* \times I^m)$.

(ii) For some $\lambda \in (0,1)$ there are C^1 a mapping $\varphi : I^* \to I$ whose restriction to each component I_i of I^* is an expanding C^1 mapping onto I and a C^1 mapping $\psi : I^* \to [0, 1-\lambda]^m$ such that

$$f(t,x) = (\varphi(t), \psi(t) + \lambda \cdot x) \quad ((t,x) \in I^* \times I^m).$$

(iii) f is injective on $I^* \times I^m$.

(see Fig. 1, where $m = 2, \theta = 3$.)

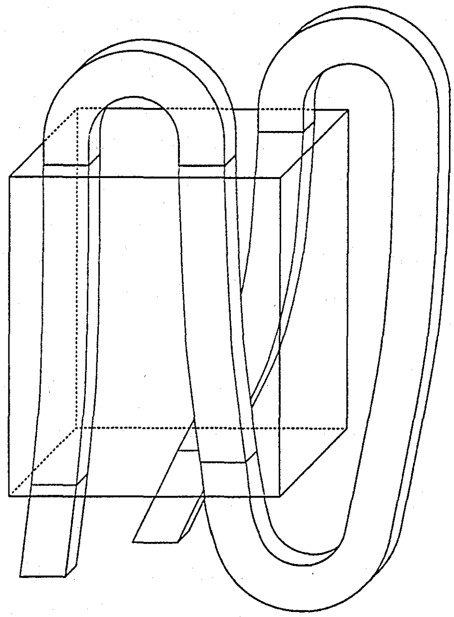

Figure 1

It is well known (and not hard to prove) that f concentrates to a shift space Z. Moreover, the global attractor A of f is homeomorphic to the cartesian product $I \times C^0$ of I with a Cantor set C^0, and each component of A is a C^1 arc running upwards from the bottom $\{0\} \times I^m$ of R_0^* to the top $\{1\} \times I^m$. These facts remain true for more general mappings f (see [3] Ch. III, e.g.), but they may fail to hold if (iii) is dropped from our assumptions (see Fig. 2, where $m = 2, \theta = 2$).

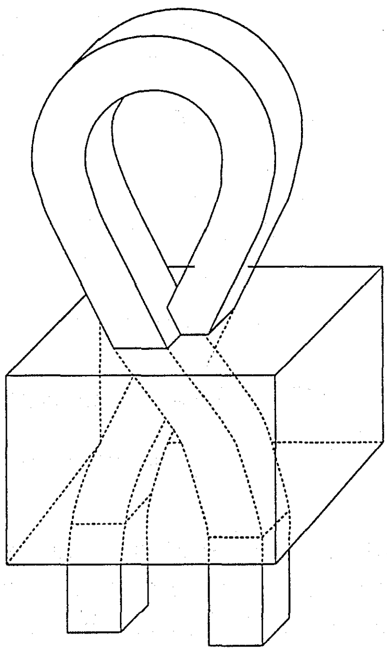

Figure 2

This paper is concerned with mappings f satisfying (i), (ii). If θ, φ are fixed we shall show that for "almost all" ψ the mapping f concentrates to a shift space and A has the structure mentioned above even if f is not injective on $I^* \times I^m$, provided λ is sufficiently small.

A natural technical simplification in the definition is obtained by neglecting the part of $R_0 = \mathbb{R}^{m+1}$ outside $R_0^* = I \times I^m$, i.e., we shall start with $R = I \times I^m, R^* = I^* \times I^m$ and the restriction of the original f to $f : R^* \to R$. Moreover, to avoid considerably technical difficulties as piecewise linear approximations of φ and ψ e.g. we assume that the restrictions of φ and ψ to the components I_i of I^* are affine mappings onto I or into $[0, 1-\lambda]^m$, respectively. (See [1], where for nonlinear mapppings in a similar situation the attractor A is considered. Indeed, using the techniques applied there, facts analogous to those stated an Corollary 1 and Corollary 2 can be proved in the nonlinear case provided "full measure in $J^{2\theta m}$" is replaced by "open and dense in the space of all C^1 mappings $\psi : I^* \to J^m$".)

So we define R, R^*, f as follows. $R = I \times I^m, R^* = I^* \times I^m$, where $I^* = I_1 \cup \cdots \cup I_\theta$ is the union of $\theta \geq 2$ disjoint closed subintervals of I and $f : r^* \to R$ is given by

$$f(t, x) = (\varphi(t), \psi(t) + \lambda \cdot x) \qquad ((t, x) \in I \times I^m), \tag{1}$$

where $\lambda \in (0, 1), \varphi : I^* \to I$ is a mapping whose restrictions to the intervals I_i are affine mappings onto I, and $\psi : I^* \to [0, 1-\lambda]^m$ is a mapping whose restrictions to the intervals I_i are affine. The interval $[0, 1 - \lambda]$ will be denoted by J.

The maximal subset I_k^* of I on which φ^k is defined ($k = 0, 1, 2, \ldots$) consists of θ^k disjoint intervals, where $I_0^* = I \supset I_1^* = I^* \supset I_2^* \supset I_3^* \supset \cdots$, and

$$I_\infty^* = \bigcap_{k=0}^{\infty} I_k^*$$

is a Cantor set in I. The Hausdorff dimension $\dim_H I_\infty^*$ of I_∞^* coincides with the box counting dimension $\dim_B I_\infty^*$ (see [2]) and will be denoted by d^*. It is determined by $|I_1|^{d^*} + \cdots + |I_\theta|^{d^*} = 1$, where $|I_i|$ denotes the length of I_i.

We assume that $\theta, I^*, \varphi : I^* \to I$ and $\lambda \in (0, 1)$ are fixed while ψ is variable. Then the mapping f in (1) is determined by ψ and will sometimes be denoted by f_ψ.

Let s_i, t_i be the end points of I_i which are chosen so that $\varphi(s_i) = 0, \varphi(t_i) = 1$, and let $a_i = \psi(s_i), b_i = \psi(t_i)$. Then, since ψ is piecewise affine, it is determined by these points $a_i, b_i \in J^m$ or, equivalently, by the point $(a_1, b_1, a_2, b_2, \ldots, a_\theta, b_\theta)$ in $J^{2\theta m}$. So all possible mappings ψ are in 1-to-1 correspondence with the points in $J^{2\theta m}$, and we shall not distinguish between ψ and the corresponding point.

The following sets will play an important role. (A denotes the global attractor of f_ψ.)

$$\Psi = \left\{\psi \in J^{2\theta m} \middle| f_\psi \text{ does not concentrate to a shift space}\right\},$$
$$\Psi_A = \left\{\psi \in J^{2\theta m} \middle| f_\psi\big|_{A \cap R^*} \text{ is not injective}\right\}.$$

In Section 2 (Proposition 3) we shall see that Ψ, Ψ_A are compact, $\Psi \subset \Psi_A$ and that for $\psi \in J^{2\theta m} \backslash \Psi_A$ the global attractor A of f_ψ is homoeomorphic to the cartesian product of an interval with a Cantor set. Moreover, since $A \cap R^*$ is compact and $f_\psi(A \cap R^*) = A$, for each $\psi \in J^{2\theta m} \Psi_A$ the restriction $f|_{A \cap R^*} : A \cap R^* \to A$ is a homeomorphism. The main results of this paper as stated in the following two theorems concern the Hausdorff dimensions of Ψ and Ψ_A.

Theorem 1 *If $\lambda < \frac{1}{2}$ then*

$$\dim_H \Psi \leq 2\theta m - m + d^* + \frac{2\log\theta}{\log 1/\lambda}.$$

Theorem 2 *If $\lambda < \frac{1}{2}$ then*

$$\dim_H \Psi_A \leq 2\theta m - m + 1 + \frac{2\log\theta}{\log 1/\lambda}.$$

Corollary 1 *If $\lambda < \theta^{-2/(m-d^*)}, \lambda < \frac{1}{2}$, then the set of all those $\varphi \in J^{2\theta m}$ for which f_ψ concentrates to a shift space is open in $J^{2\theta m}$ and has full measure $(1-\lambda)^{2\theta m}$.*

Corollary 2 *If $m > 1, \lambda < \theta^{-2/(m-1)}$ and $\lambda < \frac{1}{2}$, then for all ψ in an open subset of $J^{2\theta m}$ with full measure $(1-\lambda)^{2\theta m}$ the global attractor A of f_ψ is the cartesian product of an interval with a Cantor set, and $f_\psi|_{A\cap R^*} : A \cap R^* \to A$ is a homeomorphisms.*

Proof of the corollaries. In these cases $\dim_H \Psi < 2\theta m$ or $\dim_H \Psi_A < 2\theta m$, respectively, and, by Proposition 3, Ψ, Ψ_A are compact. □

Propositions 1 and 2 in Section 2 will yield some further details.

Remark 1 Our condition $\lambda < \frac{1}{2}$ is void unless

$$-m + d^* + \frac{2\log\theta}{\log 1/\lambda} < 0 \text{ or } -m + 1 + \frac{2\log\theta}{\log 1/\lambda} < 0,$$

respectively, i.e.

$$m > 2\frac{\log\theta}{\log 2} - d^* \text{ or } m > 2\frac{\log\theta}{\log 2} - 1.$$

This condition reflects the fact that two m-dimensional cubes in I^m of edge length at least $1/2$ and with edges parallel to those of I^m must intersect. We do not know whether it is necessary. (Here it is essentially used only in the proof of Lemma 1.)

Remark 2 We do not know whether the bounds for $\dim_B \Psi, \dim_B \Psi_A$ in the theorems are sharp. As easily seen all points

$$\psi = (a_1, a_1, a_2, \tfrac{1}{t}a_1, +(1-\tfrac{1}{t})a_2, a_3, b_3, \ldots, a_\theta, b_\theta)$$

belong to Ψ if $t \in I^*_\infty \backslash \{0\}$ and to Ψ_A if $t \in (0,1]$. Therefore

$$\dim_H \Psi \geq 2\theta m - 2m + d^*,$$
$$\dim_H \Psi_A \geq 2\theta m - 2m + 1,$$

but these lower bounds are rather weak, and they don't depend on λ.

The following fact concerning Corollary 2 seems to be more interseting. If $m \geq 3$ is odd and $\lambda > 12\theta^{-2/(m-1)}$, then the set Ψ_A contains interior points, i.e. the exponent $-2/(m-1)$ in Corollary 2 is sharp at least for m odd. This can be proved by modifying the proof of a similar fact (Theorem 2) in [1].

2 Preliminaries

For integers $\theta \geq 2, k' \leq k''$ let $\theta^{[k',k'']}$ be the set of all sequences $(i_{k'}, i_{k'+1}, \ldots, i_{k''})$ where $i_l \in \{1, \ldots, \theta\}$, and let $\theta^{[-\infty,k'']}, \theta^{[k',\infty]}, \theta^{[-\infty,\infty]}$ consists of the sequences which are infinite to the left, the right or in both directions, respectively. So $\theta^{[-\infty,\infty]}$ coincides with the Cantor set Σ_θ of Section 1, and $\theta^{[-\infty,k'']}, \theta^{[k',\infty]}$ have a natural Cantor set structure too. The shift map $\sigma : \theta^{[k',k'']} \to \theta^{[k'-1,k''-1]}$ is defined in the obvious way.

As in Section 1 we assume that $I^* = I_1 \cup \cdots \cup I_\theta$ $(\theta \geq 2)$ is the union of θ disjoint closed subintervals of I and that $\varphi : I^* \to I$ is a mapping whose restrictions to the intervals I_i are affine mappings onto I. Moreover, for some $\psi \in J^{2\theta m}$ let $f : R^* = I^* \times I^m \to R = I \times I^m$ be defined by (1).

The θ^k components of the domain I_k^* of φ^k $(k \geq 1)$ will be denoted by $I_{\underline{i}}$ $(\underline{i} \in \theta^{[1,k]})$ where the indices are chosen so that for $k > 1$

$$\begin{aligned} I_{(i_1,\ldots,i_k)} &\subset I_{(i_1,\ldots,i_{k-1})} \\ \varphi\left(I_{(i_1,\ldots,i_k)}\right) &= I_{(j_i,\ldots,j_{k-1})}, \text{ where } j_l = i_{l+1}. \end{aligned}$$

For $\underline{i} = (i_1, i_2, \ldots) \in \theta^{[0,\infty]}$ the intersection $\bigcap_{k=1}^{\infty} I_{(i_1,\ldots,i_k)}$ contains exactly one point which will be denoted by $t_{\underline{i}}$. The sets $R_{\underline{i}} = I_{\underline{i}} \times I^m$ $(\underline{i} \in \theta^{[1,k]}, 1 \leq k < \infty)$ are slices of $R = I \times I^m$ while for $\underline{i} = (i_1, i_1, \ldots) \in \theta^{[1,\infty]}$

$$R_{\underline{i}} = \bigcap_{k=1}^{\infty} R_{(i_1,\ldots,i_k)}$$

is the m-dimensional cube $\{t_{\underline{i}}\} \times I^m$.

For $\underline{i} \in \theta^{[1,k'']}$ $(1 \leq k'' \leq \infty)$ and $1 \leq k' \leq k'', k' < \infty$ the image $f^{k'}(R_{\underline{i}})$ is well defined and will be denoted by $R_{\sigma^{k'}(\underline{i})}$. So $R_{\underline{i}}$ is now defined for all $\underline{i} \in \theta^{[k',k'']}$ provided $k' \leq k'', -\infty < k' \leq 1, 0 \leq k'' \leq \infty$. By

$$R_{\underline{i}} = \bigcap_{k=0}^{-\infty} R_{(i_k,\ldots,i_0,\ldots)}$$

for $\underline{i} = (\ldots, i_{-1}, i_0, \ldots) \in \theta^{[-\infty,k'']}$ $(0 \leq k'' \leq \infty)$ we include the case $k' = -\infty$ into our definition.

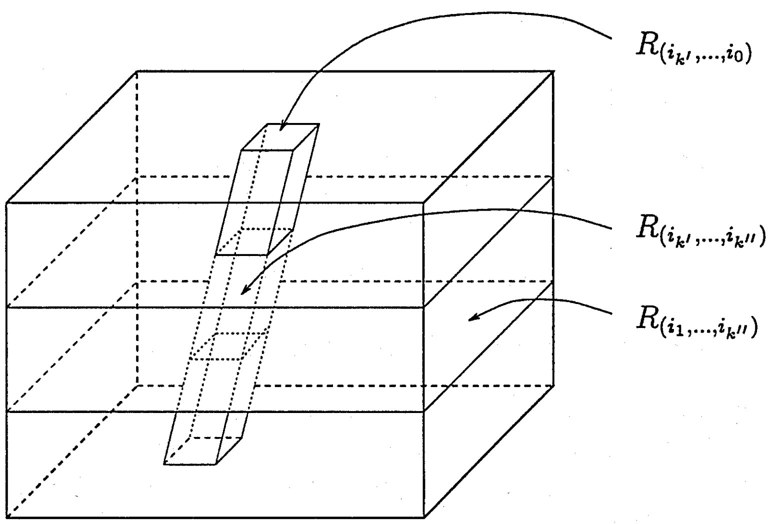

Figure 3

For k', k'' finite, $k' \le 0$ the set $R_{\underline{i}}$ is an $(m+1)$-dimensional prism over an m-dimensional cube with edge length $\lambda^{-k'+1}$ which for $k'' = 0$ has its bottom in $\{0\} \times I^m$ and its top in $\{1\} \times I^m$, while for $k' \le 0, k'' \ge 1$

$$R_{(i_{k'},\dots,i_{k''})} = R_{(i_{k'},\dots,i_0)} \cap R_{(i_1,\dots,i_{k''})}$$

(see Fig. 3). For $\underline{i} \in \theta^{[-\infty,0]}$ the set $R_{\underline{i}}$ is a straight segment running from a point in $\{0\} \times I^m$ to a point on $\{1\} \times I^m$, and if $\underline{i} \in \theta^{[-\infty,\infty]}$ then $R_{\underline{i}}$ contains exactly one point which will be denoted by $p_{\underline{i}}$. As easily seen

$$f(R_{\underline{i}}) = R_{\sigma(i)} \tag{2}$$

holds wherever $R_{\underline{i}}$ and $R_{\sigma(i)}$ are defined. Moreover, $R_{\underline{j}} \subset R_{\underline{i}}$ holds if and only if $\underline{i}$ is a part of $\underline{j}$, i.e., if $\underline{i}$ can be obtained from $\underline{j}$ by cancelling digits on one or both ends. The domain of f^k $(k \ge 1)$ is

$$R_k^* = I_k^* \times I^m = \bigcup_{\underline{i} \in \theta^{[1,k]}} R_{\underline{i}},$$

and

$$R_\infty^* = I_\infty^* \times I^m = \bigcap_{k=1}^{\infty} R_k^*$$

is the maximal set on which all iterations f^k $(k \ge 1)$ are defined.

The global attractor of f is given by

$$A = \bigcup_{\underline{i} \in \theta^{[-\infty,\infty]}} R_{\underline{i}}.$$

The maximal invariant set of f is

$$Z = \bigcup_{\underline{i} \in \theta^{[-\infty,\infty]}} R_{\underline{i}},$$

i.e. Z consists of the points $p_{\underline{i}}$ $(\underline{i} \in \theta^{[-\infty,\infty]})$, and by $h(\underline{i}) = p_{\underline{i}}$ we get a surjective mapping $h : \Sigma_\theta = \theta^{[-\infty,\infty]} \to Z$. As easily seen h is continuous, and (2) implies $h\sigma = fh$. For $t \in I, \underline{i} \in \theta^{[-\infty,0]}$ we define $g(t, \underline{i})$ to be the intersection point of $\{t\} \times I^m$ and $R_{\underline{i}}$. So we get a surjective continuous mapping $g : I \times \theta^{[-\infty,0]} \to A$.

Proposition 1 *The following conditions are equivalent.*

(i) f concentrates to a shift space.

(ii) $h : \Sigma_\theta \to Z$ is a homeomorphism.

(iii) If $\underline{i}, \underline{j} \in \theta^{[-\infty,0]}, \underline{i} \ne \underline{j}$ then

$$R_{\underline{i}} \cap R_{\underline{j}} \cap R_\infty^* = \emptyset.$$

Proof. The equivalence between *(ii)* and *(iii)* is an immediate consequence of the following fact. If $\underline{i}^- = (\dots, i_{-1}, i_0) \in \theta^{[-\infty,0]}$ then the mapping $h_{\underline{i}^-} : \theta^{[1,\infty]} \to R_{\underline{i}^-} \cap R_\infty^*$ given by $h_{\underline{i}^-}(i_1, i_2, \dots) = h(\dots, i_{-1}, i_0, i_1, \dots)$ is a homeomorphism.

The implication *(ii)* $\Rightarrow$ *(i)* follows from (2).

To complete the proof we assume *(i)* and prove *(ii)*. Since Σ_θ is compact and h is surjective it is sufficient to show that h is injective.

If for $\underline{i} = (\ldots, i_{-1}, i_0, i_1, \ldots), \underline{j} = (\ldots, j_{-1}, j_0, j_1, \ldots) \in \theta^{[-\infty,\infty]}$ the positive halves $\underline{i}^+ = (i_1, i_2, \ldots), \underline{j}^+ = (j_1, j_2, \ldots)$ are different, then $h(\underline{i}) \in R_{\underline{i}^+}, h(\underline{j}) \in R_{\underline{j}^+}, R_{\underline{i}^+} \cap R_{\underline{j}^+} = \emptyset$ implies $h(\underline{i}) \neq h(\underline{j})$. If $\underline{i}^+ = \underline{j}^+$ but $\underline{i} \neq \underline{j}$ then for some $k < 0$ the positive halves $\sigma^k(\underline{i})^+, \sigma^k(\underline{j})^+$ of $\sigma^k(\underline{i}), \sigma^k(\underline{j})$ will differ, and we get

$$h\left(\sigma^k(\underline{i})\right) \neq h\left(\sigma^k(\underline{j})\right).$$

By *(i)* $f|_Z : Z \to Z$ is a homeomorphism, and $(f|_Z)^k h = h\sigma^k$ holds for our negative exponent k. So we get $(f|_Z)^k h(\underline{i}) \neq (f|_Z)^k h(\underline{j})$ and therefore $h(\underline{i}) \neq h(\underline{j})$. □

Proposition 2 *The following conditions are equivalent.*

(i) $f|_{A\cap R^*} : A \cap R^* \to A$ *is a homeomorphism.*

(ii) $g : I \times \theta^{[-\infty,0]} \to A$ *is a homeomorphism.*

(iii) If $\underline{i}, \underline{j} \in \theta^{[-\infty,0]}, \underline{i} \neq \underline{j}$ *then* $R_{\underline{i}} \cap R_{\underline{j}} = \emptyset$.

Proof. Since g maps each interval $I \times \{\underline{i}\}$ injectively onto $R_{\underline{i}}$, the equivalence of *(ii)* and *(iii)* is obvious.

Now we prove *(i)* $\Rightarrow$ *(iii)*. By *(i)* for $k \geq 1$ the mapping $f^k : A \cap R_k^* \to A$ is a homeomorphism. To prove *(iii)* we show that for $\underline{i} = (\ldots, i_{-1}, i_0), \underline{j} = (\ldots, j_{-1}, j_0) \in \theta^{[-\infty,0]}$ the existence of a common point $p = (t, x)$ of $R_{\underline{i}}$ and $R_{\underline{j}}$ $(t \in I, x \in I^m)$ implies $\underline{i} = \underline{j}$.

For $k \geq 1$ there is a unique $p^* = (t^*, x) \in A \cap R_k^*$ such that $f^k(p^*) = p$. Here $t^* \in I_{\underline{i}^*}$, where $\underline{i}^* = (i_1^*, \ldots, i_k^*) \in \theta^{[1,k]}$ with $i_l^* = i_{l-k} = j_{l-k}$ $(1 \leq l \leq k)$. Since $k \geq 1$ is arbitrary this shows $i_n = j_n$ for all $n \leq 0$.

To prove *(iii)* $\Rightarrow$ *(i)* we assume that all segments $R_{\underline{i}}$ $(\underline{i} \in \theta^{[-\infty,0]})$ are disjoint. Then each component of $A \cap R^*$ is a segment $R_{\underline{i}} \cap R_i$ $(\underline{i} = (\ldots, i_{-1}, i_0) \in \theta^{[-\infty,0]}, 1 \leq i \leq \theta)$, and f maps this segment injectively onto $R_{\underline{j}}$, where $\underline{j} = (\ldots, j_{-1}, j_0) \in \theta^{[-\infty,0]}$ is given by $j_l = j_{l+1}$ if $l < 0, j_0 = i$. So f is injective on each component of $A \cap R^*$, and by *(iii)* different components have disjoint images. Since $A \cap R^*$ is compact injectivity together with $f(A \cap R^*) = A$ of $f|_{A\cap R^*}$ implies *(i)*. □

Proposition 3 Ψ *and* Ψ_A *are compact.*

Proof. Since the proofs in both cases are similar we consider Ψ only. For $\psi \in J^{2\theta m}, f = f_\psi : R^* \to R$ the corresponding mapping and $1 \leq i \leq \theta$ let $Z_i(\psi)$ denote the union of all $R_{\underline{i}} \cap R_\infty^*$, where $\underline{i} = (\ldots, i_{-1}, i_0) \in \theta^{[-\infty,0]}, i_0 = i$. Obviously $Z_1(\psi), \ldots, Z_\theta(\psi)$ are compact and their union is the set Z belonging to f_ψ.

We show that f_ψ concentrates to a shift space provided the θ sets $Z_i(\psi)$ are disjoint. Let $\underline{i} = (\ldots, i_{-1}, i_0, i_1, \ldots), \underline{j} = (\ldots, j_{-1}, j_0, j_1, \ldots) \in \theta^{[-\infty,\infty]}, \underline{i} \neq \underline{j}$ be given. We have to show $h(\underline{i}) \neq h(\underline{j})$. If $i_l \neq j_l$ for some $l \geq 1$, then $h(\underline{i})$ and $h(\underline{j})$ lie in different components of R_∞^*, and $h(\underline{i}) \neq h(\underline{j})$ is obvious. Now we assume that $l_0 \leq 0$ is the maximal index with $i_{l_0} \neq j_{l_0}$. Then for $\underline{i}' = (\ldots, i'_{-1}, i'_0, i'_1, \ldots) = \sigma^{l_0}(\underline{i}), \underline{j}' = (\ldots, j'_{-1}, j'_0, j'_1, \ldots) = \sigma^{l_0}(\underline{j})$ we have $i'_0 \neq j'_0$ but $i'_l = j'_l$ if $l \geq 1$. The points $h(\underline{i}'), h(\underline{j}')$ lie in the same component $\{t\} \times I^m$ of R_∞^* but in different and therefore disjoint sets $Z_{i'_0}(\psi), Z_{j'_0}(\psi)$. So $h(\underline{i}') \neq h(\underline{j}')$, and since f^{-l_0} is injective on $\{t\} \times I^m$ this gives

$$h(\underline{i}) = h\sigma^{-l_0}(\underline{i}') = f^{-l_0}h(\underline{i}') \neq f^{-l}h(\underline{j}') = h\sigma^{-l_0}(\underline{j}') = h(\underline{j}).$$

To prove that Ψ is compact we show that each point $\psi \in J^{2\theta m} \backslash \Psi$ has a neighbourhood which does not intersect Ψ. If $\psi \notin \Psi$, by Proposition 1 the corresponding sets $Z_1(\psi), \ldots, Z_\theta(\psi)$ are disjoint and since they are compact there is a positive ε such that the distance between each two of them is at least ε. As easily seen the end points of the segments $R_{\underline{i}}$ ($\underline{i} \in \theta^{[-\infty,0]}$) depend continuously on ψ, and this continuity is uniform with respect to $\underline{i}$. Therefore, if $\psi' \in J^{2\theta m}$ is sufficiently close to ψ the sets $Z_i(\psi')$ belonging to ψ' will still be mutually disjoint and $\psi' \notin \Psi$. □

3 Proof of the two theorems

We assume that $\varphi : I^* \to I$ and $\lambda \in (0, \frac{1}{2})$ and therefore θ, I_k^* $(1 \le k \le \infty), I_{\underline{i}}, R_{\underline{i}}$ ($\underline{i} \in \theta^{[1,k]}, 1 \le k \le \infty$), $t_{\underline{i}}$ ($\underline{i} \in \theta^{[1,\infty]}$) are fixed. Let H denote one of the sets I_∞^* or I, and let $q^* = \dim_H H = \dim_B H$. We define

$$\Psi^* = \left\{\psi \in J^{2\theta m} \middle| R_{\underline{i}}(\psi) \cap R_{\underline{j}}(\psi) \cap (H \times I^m) \neq 0 \text{ for at least one pair } \underline{i} \neq \underline{j} \in \theta^{[-\infty,0]}\right\},$$

where $R_{\underline{i}}(\psi)$ denotes the set $R_{\underline{i}}$ which is constructed with the mapping ψ. Looking at the equivalences between *(i)* and *(iii)* of the propositions in section 2 we see that both theorems of section 1 are combined in

$$\boxed{\dim_H \Psi^* \le 2\theta m - m + q^* + \frac{2\log\theta}{\log 1/\lambda}.} \tag{3}$$

We shall prove (3) at the end of this section after some lemmas are stated and proved.

Besides Ψ^* for $1 \le k < \infty, \underline{i} = (i_1, \ldots, i_k), \underline{j} = (j_1, \ldots, j_k) \in \theta^{[1,k]}, \underline{i} \neq \underline{j}$ we shall consider the sets

$$\begin{aligned} \Psi_{\underline{i},\underline{j}}^* &= \left\{\psi \in J^{2\theta m} \middle| R_{\sigma^k(\underline{i})}(\psi) \cap R_{\sigma^k(\underline{j})}(\psi) \cap (H \times I^m) \neq \emptyset\right\} \\ \Psi_k^* &= \bigcup_{\substack{\underline{i},\underline{j} \in \theta^{[1,k]} \\ i_k \neq j_k}} \Psi_{\underline{i},\underline{j}}^*. \end{aligned} \tag{4}$$

Since $R_{(l_{-k},\ldots,l_0)} \subset R_{(l_{-k+1},\ldots,l_0)}$, we have $\Psi_1^* \supset \Psi_2^* \supset \cdots$, and

$$\Psi^* = \bigcap_{k=1}^{\infty} \bigcup_{\substack{\underline{i},\underline{j} \in \theta^{[1,k]} \\ \underline{i} \neq \underline{j}}} \Psi_{\underline{i},\underline{j}}^*$$

together with the proof of Proposition 3 implies

$$\Psi^* = \bigcap_{k=1}^{\infty} \Psi_k^*. \tag{5}$$

For $k \ge 1, \underline{i}, \underline{j} \in \theta^{[1,k]}, \underline{i} \neq \underline{j}$ we define the mapping

$$\pi_{\underline{i},\underline{j}} : J^{2\theta m} \to I^{4m} = (I^m)^4$$

by

$$\pi_{\underline{i},\underline{j}}(\psi) = (a, b, c, d),$$

where the points $a, b, c, d \in I^m$ are determined by

$$\begin{aligned} f_\psi^k(s_{\underline{i}}, o) &= (0, a), & f_\psi^k(t_{\underline{i}}, o) &= (1, b), \\ f_\psi^k(s_{\underline{j}}, o) &= (0, c), & f_\psi^k(t_{\underline{j}}, o) &= (1, d), \end{aligned}$$

with $s_{\underline{i}}, t_{\underline{i}}$ the end points of $I_{\underline{i}}$ such that $\varphi^k(s_{\underline{i}}) = 0, \varphi^k(t_{\underline{i}}) = 1$ and $o = (0, \ldots, 0) \in I^m$. Therefore $(0,a),(1,b)$ are the end points of the segment $f^k_\psi(I_{\underline{i}} \times \{o\})$ and $(0,c),(1,d)$ those of $f^k_\psi(I_{\underline{j}} \times \{o\})$. Moreover the segments $[(0,a),(1,b)], [(0,c),(1,d)]$ are edges of the prisms $f^k(R_{\underline{i}}) = R_{\sigma^k(\underline{i})}, f^k(R_{\underline{j}}) = R_{\sigma^k(\underline{j})}$, respectively, such that for $(t,y) \in [(0,a),(1,b)]$, $(t,z) \in [(0,c),(1,d)]$ we have the cubes

$$\begin{aligned} R_{\sigma^k(\underline{i})} \cap (\{t\} \times I^m) &= \{t\} \times (y + [0, \lambda^k]^m), \\ R_{\sigma^k(\underline{j})} \cap (\{t\} \times I^m) &= \{t\} \times (z + [0, \lambda^k]^m). \end{aligned} \tag{6}$$

For $(a,b,c,d) \in (I^m)^4 = I^{4m}$ we define

$$\pi(a,b,c,d) = (c-a, d-b)$$

and get a mapping

$$\pi : I^{4m} \to [-1,1]^{2m}.$$

Finally we consider the composition

$$\rho_{\underline{i},\underline{j}} = \pi \pi_{\underline{i},\underline{j}} : J^{2\theta m} \to I^{2m}.$$

Lemma 1 *There is a real $\alpha_1 > 0$ not depending on $k, \underline{i} = (i_1, \ldots, i_k), \underline{j} = (j_1, \ldots, j_k) \in \theta^{[1,k]}$ such that for any measurable set X in I^{4m}*

$$\mathrm{vol}^{2\theta m}\left(\pi^{-1}_{\underline{i},\underline{j}}(X)\right) \leq \alpha_1 \,\mathrm{vol}^{4m}(X),$$

provided $i_k \neq j_k$. (By vol^p we denote the p-dimensional Lebesgue measure in $\mathbb{R}^p$.)

Lemma 2 *There is a real $\alpha_2 > 0$ such that for any measurable set in $[-1,1]^{2m}$*

$$\mathrm{vol}^{4m}\left(\pi^{-1}(X)\right) \leq \alpha_2 \,\mathrm{vol}^{2m}(X).$$

Corollary *There is a real $\alpha > 0$ not depending on $k, \underline{i} = (i_1, \ldots, i_k), \underline{j} = (j_1, \ldots, j_k) \in \theta^{[1,k]}$ such that for any measurable set X in $[-1,1]^{2m}$*

$$\mathrm{vol}^{2\theta m}\left(\rho^{-1}_{\underline{i},\underline{j}}(X)\right) \leq \alpha \,\mathrm{vol}^{2m}(X),$$

provided $i_k \neq j_k$.

Since the proof of Lemma 2 is trivial it is sufficient to prove Lemma 1.

Proof of Lemma 1. We start with the remark that $\pi_{\underline{i},\underline{j}}$ can be extended to a linear mapping

$$\overline{\pi}_{\underline{i},\underline{j}} : \mathbb{R}^{2\theta m} \to \mathbb{R}^{4m}.$$

The proof will proceed as follows. We define a $4m$-dimensional linear subspace L of $\mathbb{R}^{2\theta m}$ (depending on $\underline{i}, \underline{j}$) such that $\overline{\pi}_{\underline{i},\underline{j}}|_L : L \to \mathbb{R}^{4m}$ is a linear isomorphism and for any measurable set X in $\mathbb{R}^{4m}$ we have

$$\mathrm{vol}^{4m}\left(\left(\overline{\pi}_{\underline{i},\underline{j}}|_L\right)^{-1}(X)\right) \leq \alpha^* \,\mathrm{vol}^{4m}(X), \tag{7}$$

where $\alpha^* = \left(\frac{1-\lambda}{1-2\lambda}\right)^{4m}$. (This is the point where we need $\lambda < \frac{1}{2}$!) Obviously $\overline{\pi}_{\underline{i},\underline{j}} = \overline{\pi}_{\underline{i},\underline{j}}|_L \pi^*$ with a linear projection $\pi^* : \mathbb{R}^{2\theta m} \to L$, and therefore, if $X \subset I^{4m}$

$$\begin{aligned}
\operatorname{vol}^{2\theta m}\left(\pi_{\underline{i},\underline{j}}^{-1}(X)\right) &= \operatorname{vol}^{2\theta m}\left(\overline{\pi}_{\underline{i},\underline{j}}^{-1}(X) \cap J^{2\theta m}\right) \\
&= \operatorname{vol}^{2\theta m}\left(\pi^{*-1}\left(\overline{\pi}_{\underline{i},\underline{j}}|_L\right)^{-1}(X) \cap J^{2\theta m}\right) \\
&\le \left(\operatorname{diam} J^{2\theta m}\right)^{2\theta m - 4m} \operatorname{vol}^{4m}\left(\left(\overline{\pi}_{\underline{i},\underline{j}}|_L\right)^{-1}(X)\right) \\
&\le \left(\operatorname{diam} J^{2\theta m}\right)^{2\theta m - 4m} \alpha^* \operatorname{vol}^{4m}(X),
\end{aligned}$$

such that the lemma will be proved with $\alpha_2 = (\operatorname{diam} J^{2\theta m})^{2\theta m - 4m}(\frac{1-\lambda}{1-2\lambda})^{4m}$, provided (7) is proved.

Thinking at our identification of the mappings $\psi : I^* \to J^m$ with the points in $J^{2\theta m}$ we regard $J^{2\theta m}$ as $(J^m)^{2\theta}$ and its points as sequences $(a_1, b_1, \ldots, a_\theta, b_\theta)$, where $a_i, b_i \in J^m$. Let $J^{4m}_{\underline{i},\underline{j}}$ denote the $4m$-dimensional face of $J^{2\theta m}$ consisting of all $(a_1, b_1, \ldots, a_\theta, b_\theta)$ with $a_i = b_i = o$ for $i_k \ne i \ne j_k$. (Here i_k, j_k are the last digits of $\underline{i}, \underline{j}$, respectively, and o denotes the point $(0, \ldots, 0)$ in $\mathbb{R}^m$.) Then L is defined to be the $4m$-dimensional linear subspace of $\mathbb{R}^{2\theta m}$ which contains $J^{4m}_{\underline{i},\underline{j}}$.

Since $\overline{\pi}_{\underline{i},\underline{j}}$ is linear there is a real δ such that for any measurable Y in L we have

$$\operatorname{vol}^{4m}\left(\overline{\pi}_{\underline{i},\underline{j}}(Y)\right) = \delta \operatorname{vol}^{4m}(Y),$$

and, since $\operatorname{vol}^{4m} J^{4m}_{\underline{i},\underline{j}} = (1-\lambda)^{4m}$, to prove (7) it is sufficient to show that

$$\begin{aligned}
\operatorname{vol}^{4m}\left(\overline{\pi}_{\underline{i},\underline{j}}\left(J^{4m}_{\underline{i},\underline{j}}\right)\right) &\ge \left(\frac{1-2\lambda}{1-\lambda}\right)^{4m}(1-\lambda)^{4m} \\
&= (1-2\lambda)^{4m}
\end{aligned}$$

or that $\overline{\pi}_{\underline{i},\underline{j}}(J^{4m}_{\underline{i},\underline{j}})$ contains the cube

$$Q = [\lambda, 1-\lambda]^{4m}.$$

It will be convinient to identify L with $\mathbb{R}^{4m}$ via the mapping $L \to \mathbb{R}^{4m}$ which is obtained by neglecting in points

$$(x_1, \ldots, x_{2\theta m}) = (a_1, b_1, \ldots, a_\theta, b_\theta) \in L$$

($a_i, b_i \in \mathbb{R}^m$) all coordinates not belonging to $a_{i_k}, b_{i_k}, a_{j_k}, b_{j_k}$. Then $J^{4m}_{\underline{i},\underline{j}} = J^{4m}$ and we have to show

$$\overline{\pi}_{\underline{i},\underline{j}}(J^{4m}) \supset Q. \tag{8}$$

Starting with the cube

$$Q^* = [0, \lambda]^{4m}$$

for each vertex ψ of J^{4m} we define the cube

$$Q^*_\psi = \psi + Q^*.$$

By a simple geometric argument illustrated in Figure 4 it can be proved that any convex set which intersects all 2^{4m} cubes Q^*_ψ must contain Q. Therefore to prove (8) it is sufficient to show that for any vertex ψ of J^{4m}

$$\overline{\pi}_{\underline{i},\underline{j}}(\psi) \in Q^*_\psi$$

or, equivalently

$$\overline{\pi}_{\underline{i},\underline{j}}(\psi) - \psi \in [0, \lambda]^{4m}. \tag{9}$$

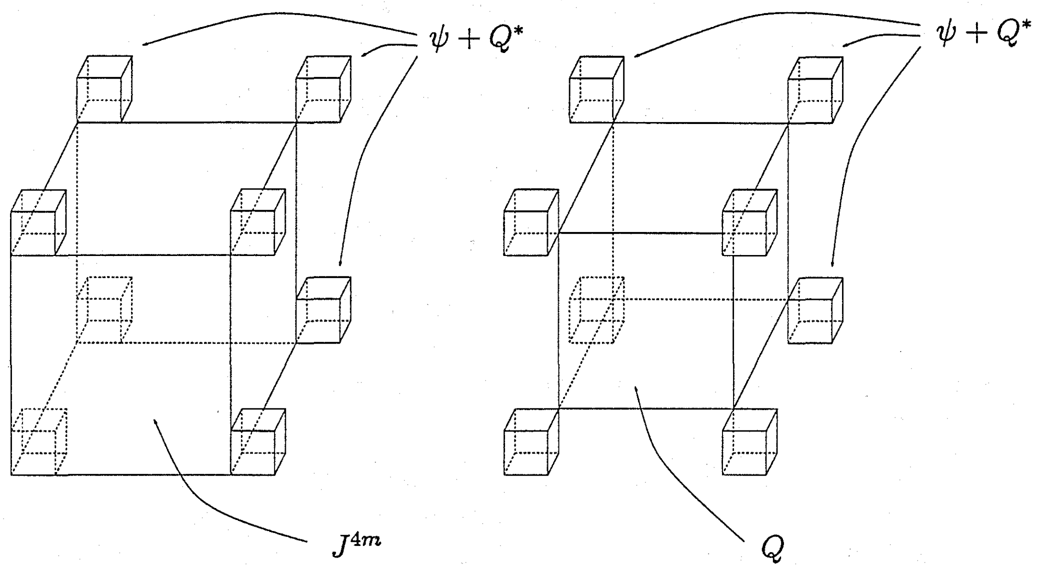

Figure 4

Let us assume $i_k < j_k$. For a vertex $\psi = (a_{i_k}, b_{i_k}, a_{j_k}, b_{j_k})$ of J^{4m} we shall write $\overline{\pi}_{\underline{i},\underline{j}}(\psi) = \pi_{\underline{i},\underline{j}}(\psi) = (a,b,c,d)$. To prove (9) it is sufficient to prove

$$a - a_{i_k}, \quad b - b_{i_k}, \quad c - a_{j_k}, \quad d - b_{j_k} \in [0,\lambda]^m. \tag{10}$$

We consider $a - a_{i_k}$; the remaining cases are analoguous. Our identification $\psi = (a_1, b_1, \ldots, a_\theta, b_\theta)$ made in Section 1 implies for $1 \le i \le \theta$

$$f_\psi(R_i) \cap (\{0\} \times I^m) = f_\psi(\{s_i\} \times I^m) = \{0\} \times (a_i + [0,\lambda]^m).$$

Therefore we have by the definition of $\pi_{\underline{i},\underline{j}}$

$$(0,a) = f_\psi^k(s_{\underline{i}}, o) = f_\psi f_\psi^{k-1}(s_{\underline{i}}, o)$$

and, since $\varphi(s_{(i_1,\ldots,i_l)}) = s_{(i_2,\ldots,i_l)}$ $((i_2,\ldots,i_l)$ regarded as element of $\theta^{[1,l-1]})$

$$\begin{aligned} f_\psi^{k-1}(s_{\underline{i}}, o) &\in \left\{\varphi^{k-1}(s_{\underline{i}})\right\} \times I^m \\ &= \{s_{i_k}\} \times I^m \\ &\subset R_{i_k}. \end{aligned}$$

Therefore

$$\begin{aligned} (0,a) &\in f_\psi(R_{i_k}) \cap (\{0\} \times I^m) \\ &= \{0\} \times (a_{i_k} + [0,\lambda]^m) \end{aligned}$$

which proves (10) for $a - a_{i_k}$ and the lemma. □

We consider the compact subset

$$K = \left\{(a,b) \in ([-1,1]^m)^2 = [-1,1]^{2m} \middle| (1-t)a + tb = o \text{ for some } t \in H\right\}$$

of $[-1,1]^{2m}$.

Lemma 3 *Let $(a,b,c,d) \in I^{4m}$. Then the segments $[(0,a),(1,b)], [(0,c),(1,d)]$ intersect in a point (t,x) with $t \in H, x \in I^m$ if and only if $\pi(a,b,c,d) \in K$.*

This lemma is an immediate consequence of the definitions of π and of K. $\square$

Lemma 4 *There is a real $\beta > 0$ such that for any $k \geq 1, \underline{i}, \underline{j} \in \theta^{[1,k]}, \underline{i} \neq \underline{j}$ we have*

$$N_{\lambda^k}\left(\Psi^*_{\underline{i},\underline{j}}\right) \subset \rho^{-1}_{\underline{i},\underline{j}}\left(N_{\beta\lambda^k}(K)\right)$$

where $N_{\lambda^k}(\Psi^*_{\underline{i},\underline{j}})$ denotes the λ^k-neighbourhood of $\Psi^*_{\underline{i},\underline{j}}$ in $J^{2\theta m}$ while $N_{\beta\lambda^k}(K)$ is the $\beta\lambda^k$-neighbourhood of K in $[-1,1]^{2m}$.

Proof. For an arbitrarily given $\psi = (a_1, b_1, \ldots, a_\theta, b_\theta) \in N_{\lambda^k}(\Psi^*_{\underline{i},\underline{j}})$ we choose $\psi' = (a'_1, b'_1, \ldots, a'_\theta, b'_\theta) \in \Psi^*_{\underline{i},\underline{j}}$ so that

$$|a'_i - a_i| \leq \lambda^k, \quad |b'_i - b_i| \leq \lambda^k \qquad (1 \leq i \leq \theta).$$

A simple geometric argument (by induction with respect to k) shows that for

$$(a,b,c,d) = \pi_{\underline{i},\underline{j}}(\psi), \qquad (a',b',c',d') = \pi_{\underline{i},\underline{j}}(\psi')$$

each of the distances $|a'-a|, |b'-b|, |c'-c|, |d'-d|$ is at most

$$\lambda^k \sum_{i=0}^{k-1} \lambda^i < \frac{\lambda^k}{1-\lambda} < 2\lambda^k. \tag{11}$$

(The last inequality is a consequence of our assupmtion $\lambda < \frac{1}{2}$. Instead of applying this assumption we could proceed with $\frac{1}{1-\lambda}$ instead of 2 and choose $\beta = 4/(1-\lambda) + 4\sqrt{m}$. Therefore in this proof $\lambda < \frac{1}{2}$ is inessential.) As an immediate consequence of (11) we have

$$\left|\pi_{\underline{i},\underline{j}}(\psi') - \pi_{\underline{i},\underline{j}}(\psi)\right| < 4\lambda^k$$

and by $|\pi(p) - \pi(q)| < 2|p-q|$ we get

$$\left|\rho_{\underline{i},\underline{j}}(\psi') - \rho_{\underline{i},\underline{j}}(\psi)\right| < 8\lambda^k. \tag{12}$$

Since $\psi' \in \Psi^*_{\underline{i},\underline{j}}$, we can find points $t \in H, x \in I^m$ such that

$$(t,x) \in R_{\sigma^k(\underline{i})}(\psi') \cap R_{\sigma^k(\underline{j})}(\psi'). \tag{13}$$

Let $(t,y), (t,z)$ be the points at which $\{t\} \times I^m$ intersects the segments $f^k_{\psi'}(I_{\underline{i}} \times \{o\}), f^k_{\psi'}(I_{\underline{j}} \times \{o\})$, respectively. The end points of these segments are $(0,a'), (1,b'); (0,c'), (1,d')$ respectively.

Moreover, (6) together with $f^k_{\psi'}(I_{\underline{i}} \times \{o\}) \subset R_{\sigma^k(\underline{i})}(\psi'), f^k_{\psi'}(I_{\underline{j}} \times \{o\}) \subset R_{\sigma^k(\underline{j})}(\psi')$ and (13) implies

$$|x-y| \leq \sqrt{m}\,\lambda^k, \qquad |x-z| \leq \sqrt{m}\,\lambda^k. \tag{14}$$

Let $a^* = a'+x-y,\ b^* = b'+x-y,\ c^* = c'+x-z,\ d^* = d'+x-z$. Then $(a^*, b^*, c^*, d^*) \in I^{4m}$, and since $(t,x) \in [a^*, b^*] \cap [c^*, d^*], t \in H$ by Lemma 3 we have $\pi(a^*, b^*, c^*, d^*) \in K$.

Applying (14) we get

$$|(a',b',c',d') - (a^*,b^*,c^*,d^*)| \leq 2\sqrt{m}\,\lambda^k$$

and therefore

$$\begin{aligned}\operatorname{dist}(\rho_{\underline{i},j}(\psi'), K) &\leq |\pi(a', b', c', d') - \pi(a^*, b^*, c^*, d^*)| \\ &\leq 4\sqrt{m}\,\lambda^k.\end{aligned}$$

This together with (12) shows

$$\rho_{\underline{i},j}(\psi) \in N_{\beta\cdot\lambda^k}(K),$$

where $\beta = 8 + 4\sqrt{m}$. $\square$

Lemma 5 $$\dim_B K = m + q^*.$$

Proof. K is the intersection of a cone with $[-1,1]^{2m}$, i.e., if $v \in K, \gamma \in \mathbb{R}$ and $\gamma v \in [-1,1]^{2m}$, then $\gamma v \in K$. The full cone is

$$\begin{aligned}\overline{K} &= \left\{\gamma v \middle| v \in K, \gamma \in \mathbb{R}\right\} \\ &= \left\{(a,b) \in (\mathbb{R}^m)^2 = \mathbb{R}^{2m} \middle| (1-t)a + tb = 0 \text{ for some } t \in H\right\},\end{aligned}$$

and $K = \overline{K} \cap [-1,1]^{2m}$. So it is sufficient to prove

$$\dim_B \overline{K} = m + q^*.$$

To describe $\overline{K}$ we consider the boundary $\partial(\mathbb{D}^m \times \mathbb{D}^m) = (\mathbb{S}^{m-1} \times \mathbb{D}^m) \cup (\mathbb{D}^m \times \mathbb{S}^{m-1})$ of the ball $\mathbb{D}^m \times \mathbb{D}^m$ in $\mathbb{R}^{2m}$, where $\mathbb{D}^m = \{a \in \mathbb{R}^m |\ |a| \leq 1\}, S^{m-1} = \{a \in \mathbb{R}^m |\ |a| = 1\}$. Then, since

$$\dim_B \overline{K} = 1 + \dim_B \left(\partial(\mathbb{D}^m \times \mathbb{D}^m) \cap \overline{K}\right)$$

it is sufficient to show

$$\max\left[\dim_B \left((\mathbb{S}^{m-1} \times \mathbb{D}^m) \cap \overline{K}\right), \dim_B \left((\mathbb{D}^m \times \mathbb{S}^{m-1}) \cap \overline{K}\right)\right] = m - 1 + q^*. \tag{15}$$

We consider the first term

$$(\mathbb{S}^{m-1} \times \mathbb{D}^m) \cap \overline{K} = \left\{\left(a, \tfrac{t-1}{t}a\right) \middle| a \in \mathbb{S}^{m-1}, t \in H \cap \left[\tfrac{1}{2}, 1\right]\right\}.$$

Let $F = \mathbb{S}^{m-1} \times [\frac{1}{2}, 1]$, and let

$$\chi : F \to \mathbb{S}^{m-1} \times \mathbb{D}^m$$

be the mapping given by

$$\chi(a,t) = \left(a, \tfrac{t-1}{t}a\right).$$

Obviously, χ is a C^∞ embedding which is injective on $\mathbb{S}^{m-1} \times (\frac{1}{2}, 1]$, and since for $H \cap [\frac{1}{2}, 1] \neq \emptyset$

$$\dim_B \left(\mathbb{S}^{m-1} \times \left(H \cap \left[\tfrac{1}{2}, 1\right]\right)\right) = m - 1 + \dim_B \left(H \cap \left(\tfrac{1}{2}, 1\right]\right),$$

we have

$$\dim_B \left(\left(\mathbb{S}^{m-1} \times \mathbb{D}^m\right) \cap \overline{K}\right) = m - 1 + \dim_B \left(H \cap \left[\tfrac{1}{2}, 1\right]\right), \text{ if } H \cap \left[\tfrac{1}{2}, 1\right] \neq \emptyset.$$

In the same way we get

$$\left(\mathbb{D}^m \times \mathbb{S}^{m-1}\right) \cap \overline{K} = \left\{\left(\tfrac{t}{t-1}b, b\right) \middle| b \in \mathbb{S}^{m-1},\ t \in H \cap \left[0, \tfrac{1}{2}\right]\right\},$$

$$\dim_B \left(\left(\mathbb{D}^m \times \mathbb{S}^{m-1}\right) \cap \overline{K}\right) = m - 1 + \dim_B \left(H \cap \left[0, \tfrac{1}{2}\right]\right), \text{ if } H \cap \left[0, \tfrac{1}{2}\right] \neq \emptyset.$$

Since

$$q^* = \max\left(\dim_B\left(H \cap \left[0, \tfrac{1}{2}\right]\right), \dim_B\left(H \cap \left[\tfrac{1}{2}, 1\right]\right)\right)$$

this implies (15). □

To prove (3) we apply the following result of C. Tricot Jr. [5], in which $\overline{\dim}_B, \underline{\dim}_B$ denote the upper and the lower box counting dimension, respectively, (see [2] e.g.).

Lemma 6 *If X is a bounded subset of $\mathbb{R}^p$ then*

$$\overline{\dim}_B X = p - \liminf_{\varepsilon \to 0} \frac{\log \operatorname{vol}^p N_\varepsilon(X)}{\log \varepsilon}, \tag{16}$$

$$\underline{\dim}_B X = p - \limsup_{\varepsilon \to 0} \frac{\log \operatorname{vol}^p N_\varepsilon(X)}{\log \varepsilon}, \tag{17}$$

where $N_\varepsilon(X)$ denotes the ε-neighbourhood of X in $\mathbb{R}^p$. □

Proof of (3). Lemma 6 for $X = K$ together with lemma 5 implies

$$2m - \lim_{\varepsilon \to 0} \frac{\log \operatorname{vol}^{2m} N_\varepsilon(K)}{\log \varepsilon} = m + q^*,$$

$$\lim_{\varepsilon \to 0} \frac{\log \operatorname{vol}^{2m} N_\varepsilon(K)}{\log \varepsilon} = m - q^*. \tag{18}$$

Applying Lemma 4 and the corollary to Lemma 1 and Lemma 2 we get for $k \geq 1, \underline{i} = (i_1, \ldots, i_k), \underline{j} = (j_1, \ldots, j_k) \in \theta^{[1,k]}, i_k = j_k$

$$\operatorname{vol}^{2\theta m} N_{\lambda^k}\left(\Psi^*_{\underline{i},\underline{j}}\right) \leq \alpha \operatorname{vol}^{2m} N_{\beta\lambda^k}(K),$$

where α, β do not depend on $k, \underline{i}, \underline{j}$. By (4), (5) we have for $k \geq 1$

$$N_\varepsilon(\Psi^*) \subset N_\varepsilon(\Psi^*_k) = \bigcup_{\substack{\underline{i},\underline{j} \in \theta^{[1,k]} \\ i_k \neq j_k}} N_\varepsilon\left(\Psi^*_{\underline{i},\underline{j}}\right)$$

and therefore, since there are less than θ^{2k} summands on the right hand side,

$$\operatorname{vol}^{2\theta m} N_{\lambda^k}(\Psi^*) \leq \theta^{2k} \alpha \operatorname{vol}^{2m}\left(N_{\beta\lambda^k}(K)\right).$$

Since $\lambda < 1$, i.e. $\log \lambda < 0$, this together with (18) implies

$$\begin{aligned}
\limsup_{k \to \infty} \frac{\log \operatorname{vol}^{2\theta m} N_{\lambda^k}(\Psi^*)}{\log \lambda^k} &\geq \frac{2 \log \theta}{\log \lambda} + \lim_{k \to \infty} \frac{\log \alpha}{\log \lambda^k} + \limsup_{k \to \infty} \frac{\log \operatorname{vol}^{2m} N_{\beta\lambda^k}(K)}{\log \lambda^k} \\
&= \frac{2 \log \theta}{\log \lambda} + \lim_{k \to \infty} \frac{\log \operatorname{vol}^{2m} N_{\beta\lambda^k}(K)}{\log \lambda^k - \log \beta} \\
&= \frac{2 \log \theta}{\log \lambda} + m - q^*,
\end{aligned}$$

and a fortiori

$$\limsup_{\varepsilon \to 0} \frac{\log \operatorname{vol}^{2\theta m} N_\varepsilon(\Psi^*)}{\log \varepsilon} \geq \frac{2 \log \theta}{\log \lambda} + m - q^*.$$

Then

$$2\theta m - \limsup_{\varepsilon \to 0} \frac{\log \operatorname{vol}^{2\theta m} N_\varepsilon(\Psi^*)}{\log \varepsilon} \leq 2\theta m - m + q^* - \frac{2 \log \theta}{\log \lambda},$$

and, since Ψ^* lies in $\mathbb{R}^{2\theta m}$, (17) implies

$$\underline{\dim}_B \Psi^* \leq 2\theta m - m + q^* + \frac{2\log\theta}{\log 1/\lambda}.$$

Now (3) is a consequence of the well known inequality $\dim_H \leq \underline{\dim}_B$, and the theorems are proved. ∎

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