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On some problems of hypothesis testing leading to infinitely divisible distributions

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ABSTRACT. We observe an n-dimensional Gaussian random vector $x = \xi + v$ where ξ is a standard n-dimensional Gaussian vector and $v \in R^n$ is an unknown mean and we consider the hypothesis testing problem $H_0 : v = 0$ against two related types of alternatives:

Bayesian: the coordinates of v may be equal to $-b$, 0 or $+b$ only and the number of nonzero coordinates is random with binomial distribution $Bi(h_n, n)$;

Minimax: the coordinates of v may be equal to $-b$, 0 or $+b$ only and the number, k, of nonzero coordinates is nonrandom.

The values $b = b_n > 0$, $h = h_n \in (0, 1]$ or an integer $k = k_n \in [1, n]$ are given.

These problems are of importance for many applications, for example for multi-channel detection and communication systems.

We study the asymptotics of the log-likelihood distribution for Bayesian alternatives and show that they are either Gaussian or degenerate or belong to a special two-parametric class of infinitely divisible distributions. The latter corresponds to the case $b_n \asymp \sqrt{\log n}$ and h_n is small enough.

We also show that randomization in the Bayesian alternative corresponds to asymptotically least favorable priors for minimax alrernative if $nh_n = k_n \to \infty$.

1. INTRODUCTION

Let an n-dimensional Gaussian random vector $x = \xi + v$ be observed where ξ is a standard Gaussian random vector with zero mean and unit covariance matrix and $v \in R^n$ is an unknown mean. We test the null hypothesis $H_0 : v = 0$ and consider two variants of alternatives.

1.1. **Bayesian alternative** H_{n,π^n}. Let values $b = b_n$ and $h = h_n$ be given. Let $v \in R^n$ be a random vector of the form $v = b(t_1, ..., t_n)$ where $t_1, \dots, t_n$ are i.i.d. random variables taking values in the set $\{-1, 0, +1\}$ and $Pr(t = -1) = Pr(t = +1) = h/2$, $Pr(t = 0) = 1 - h$. In other words we deal with the following product prior

$$\pi^n = \pi^n(b, h) = \pi \times \ldots \times \pi \tag{1.1}$$

where $\pi = \pi_n$ is the three-point measure on the real line

$$\pi = \pi(b, h) = (1 - h)\delta_0 + \frac{h}{2}(\delta_{-b} + \delta_b) \tag{1.2}$$

(or a two-point measure if $h = 1$); here δ_t is the Dirac mass at point $t \in R^1$.

It is clear that each coordinate of v takes values in the three-point set $\{-b, 0, +b\}$ (or the two-point set $\{-b, b\}$ if $h = 1$) and the number, k, of nonzero coordinates of v is random with the binomial distribution $Bi(h, n)$ (or equals n if $h = 1$) and if $nh \to \infty$ then $k = nh(1 + o(1))$ with probability close to 1.

Let $P_{n,v}$ be the Gaussian measure on $(R^n, \mathcal{B}^n)$ with the mean $v \in R^n$ and unit covariance matrix. The null hypothesis H_0 corresponds to the measure $P_{n,0}$ and the alternative H_{n,π^n} corresponds to the mixture

$$P_{n,\pi^n} = \int P_{n,v}\pi^n(dv) = \prod_{i=1}^n \int P_{1,v}\pi(dv).$$

and

$$\int P_{1,v}\pi(dv) = (1-h)P_{1,0} + \frac{h}{2}(P_{1,-b} + P_{1,b})$$

For given $\alpha \in (0,1)$ the optimal test $\psi_{n,\alpha}$ of level α is based on the log-likelihood ratio

$$l_n = \log \frac{dP_{n,\pi^n}}{dP_{n,0}} = \sum_{i=1}^n \log(1 + h_n \xi(x_i, b)) \tag{1.3}$$

where

$$\begin{aligned} \xi(x,b) &= \frac{1}{2}(\frac{dP_{1,-b}}{dP_{1,0}}(x) + \frac{dP_{1,b}}{dP_{1,0}}(x)) - 1 \\ &= \exp\left(-\frac{b^2}{2}\right)\cosh(bx) - 1. \end{aligned} \tag{1.4}$$

It means that

$$\psi_{n,\alpha} = \mathbf{1}_{\{l_n > t_{n,\alpha}\}}$$

where $t_{n,\alpha}$ is the $(1-\alpha)$ - quantile of $P_{n,0}$ - distribution of l_n:

$$P_{n,0}(l_n > t_{n,\alpha}) = \alpha$$

and its second kind error probability is

$$\beta_n(\alpha) = \beta_n(\alpha, \pi^n(h,b)) = P_{n,\pi^n}(l_n \le T_{n,\alpha}). \tag{1.5}$$

It is clear that

$$0 \le \beta_n(\alpha) \le 1 - \alpha. \tag{1.6}$$

Our goal is to investigate the asymptotics of the probabilities (1.5) as $n \to \infty$ for any $\alpha \in (0,1)$ which are determined by the asymptotic distributions of the statistics (1.3) for the measures $P_{n,0}$ and P_{n,π^n}.

1.2. **Minimax alternative** V_n. Let the values $b = b_n > 0$ and integers $k = k_n \in [1, n]$ be given. Put

$$\begin{aligned} V_n &= V_n(b,k) \\ &= \left\{ v = b(t_1, \dots, t_n), t_i \in \{-1,0,1\}, \quad \sum_{i=1}^n t_i = k \right\}. \end{aligned}$$

We consider the composite alternative $H_1 : v \in V_n$. Let $\Psi_{n,\alpha}$ be the set of tests of level α, $\alpha \in (0,1)$, i.e. the set of measurable functions $\psi : R^n \to [0,1]$ such that $\alpha(\psi) \le \alpha$ where $\alpha(\psi) = E_{n,0}\psi$ is the first kind error probability. Here and below $E_{n,v}$ means the expectation with respect to measure $P_{n,v}$.

Let $\beta_n(\psi, v) = E_{n,v}(1-\psi)$ be the second kind error probability and let

$$\beta_n(\psi, V_n) = \sup_{v \in V_n} \beta_n(\psi, v)$$

be the maximum value of the second kind error probability for test ψ. Let

$$\beta_n(\alpha) = \beta_n(\alpha, V_n) = \inf_{\psi\in\Psi_{(n,\alpha)}} \beta_n(\psi, V_n)$$

be the minimax second kind error probability. It is clear that inequalities (1.6) hold.

We are interested in the dependence of the asymptotics of $\beta_n(\alpha)$ on the behavior of b_n and h_n as $n \to \infty$ for any $\alpha \in (0,1)$ and in the structure of asymptotically minimax tests $\psi_{n,\alpha}$ such that

$$\alpha_n(\psi_{n,\alpha}) \leq \alpha + o(1), \quad \beta_n(\psi_{n,\alpha}) \leq \beta_n(\alpha, V_n) + o(1).$$

Here and below the asymptotic relations are undersdood as $n \to \infty$.

One can easily see that the least favorable prior for the minimax alternative H_1 is the prior $\pi^n_{b,k}$ which is uniform discrete measure on the set $V_n(b,k)$. But it is difficult to study the likelihood ratio $L_n = dP_{n,\pi^n_{b,k}}/dP_{n,0}$ directly.

Note that for $k = n$ the least favorable prior $\pi^n_{b,n}$ is the same as the prior π^n for the Bayesian alternative with $h = 1$.

1.3. **Discussion.** The type of problems we consider are of importance for many applications in multi-channel detection and communication systems (see for example Dobrushin (1958)), and there are a lot of publications on this topic (see Urkowitz (1967), Bakut (1984), Krasner (1986)). Such problems also arise in the constructions of lower bounds for various statistical problems (see Burnashev (1979) for $k = 1$; Assouad (1983), Birge (1985), Birge and Massart (1995) for $k = n$ or $h = 1$; Ingster (1985, 1986, 1990, 1993), Suslina (1993, 1995), Lepski and Spokoiny (1995) and others). Product measures of the form (1.1), (1.2) arised in minimax nonparametric estimation problems also (Donoho and Jonhstone, 1994, Donoho, Jonhstone, Kerkyacharian and Picard, 1995).

For the case $k = n$ or $h = 1$ it was shown in Ingster (1990, 1993) that log-likelihood statistics (1.3) are asymptotically Gaussian $N(-u_n^2/2, u_n^2/2)$ under the null hypothesis and $N(u_n^2/2, u_n^2/2)$ under the alternative where $u_n^2 = nb^4/2$. This implies that

$$\beta_n(\alpha, V_n(b,n)) = \beta_n(\alpha, \pi^n(b,1)) = \Phi(t_\alpha - u_n) + o(1). \tag{1.7}$$

Asymptotically minimax tests $\psi_{n,\alpha} = \mathbf{1}_{\{R_n > t_\alpha\}}$ in this case may be based on the chi-square statistics

$$R_n = \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} (x_i^2 - 1).$$

Here and onwards $\Phi(t)$ stands for the standard normal distribution function and t_α for its $(1-\alpha)$ - quantile: $\Phi(t_\alpha) = 1 - \alpha$.

For $k = n$ or $h = 1$ it follows from (1.7) that $\beta_n(\alpha) \to 1 - \alpha$ iff $b_n n^{1/4} \to 0$ and $\beta_n(\alpha) \to 1 - \alpha$ iff $b_n n^{1/4} \to \infty$. Also $\beta_n(\alpha)$ is separated away from 0 and $1 - \alpha$ iff $b_n \asymp n^{-1/4}$.

For Bayesian alternative with $b_n = O(1)$ it follows from Ingster (1990, 1993) that log-likelihood statistics (1.3) are also asymptotically Gaussian with

$$u_n^2 = 2nh_n^2 \left(sinh \frac{b_n^2}{2} \right)^2 \tag{1.8}$$

and

$$\beta_n(\alpha) = \Phi(t_\alpha - u_n). \tag{1.9}$$

In this case the asymptotically minimax tests

$$\psi_{n,\alpha} = \mathbf{1}_{\{R_n > t_\alpha\}}$$

may be based on the statistics

$$R_n = \frac{1}{\sqrt{2n}\sinh(b_n^2/2)} \sum_{i=1}^{n} \xi(x_i, b_n)$$

where $\xi(x,b)$ is defined in (1.4). For $b_n \asymp 1$ it follows from (1.8), (1.9) that $\beta_n(\alpha) \to 0$ iff $h_n n^{1/2} \to \infty$ and $\beta_n(\alpha) \to 1-\alpha$ iff $h_n n^{1/2} \to 0$ and also $\beta_n(\alpha)$ is separated from 0 and $1-\alpha$ iff $h_n \asymp n^{-1/2}$. Similar distinguishability conditions were obtained in Ingster (1985, 1986) for minimax alternatives.

On the other hand it was shown in Burnashev and Begmatov (1990), Ingster (1990) that for the case of $k=1$ we have an essentially different type of the asymptotics for minimax alternative. The statistics $L_n = dP_{\pi^n_{b,1}}/dP_{n,0}$ are asymptotically constant under $P_{n,0}$ - probability:

$$L_n = \Phi(-H_n) + \eta_n,$$

where $\eta_n \to 0$ under $P_{n,0}$ - probability and

$$\beta_n(\alpha, V_n(b_n, 1)) = (1-\alpha)\Phi(-H_n) + o(1). \tag{1.10}$$

Here

$$H_n = b_n - \sqrt{2\log n}. \tag{1.11}$$

This implies that the asymptotic distributions of the log-likelihood ratio statistics l_n and $\log L_n$ can be non Gaussian if $b_n \to \infty$.

In the next section we formulate the results for the Bayesian alternative. There are three types of the asymptotic distributions of the log-likelihood statistics (1.3): Gaussian, degenerate and special two-parameter infinitely divisible of Poisson type.

Asymptotics of Gaussian and degenerate types have been considered by Ingster (1990, 1993) also for minimax nonparametric hypotheses testing problems. But to author's knowledge, asymptotics of infinitely divisible type seem to appear at the first time.

We describe these families of distributions and consider their properties. Such distribution have not been before.

Also we show that the considered Bayesian alternatives are asymptotically least favorable for minimax alternatives.

In Sections 3 - 6 we give proofs.

In another paper we will consider the minimax alternatives $V_n = V_n(p, q, R_1, R_2)$ corresponding to l_q^n - balls of radius $R_2 = R_{n,2}$ with l_p^n - balls of radius $R_1 = R_{n,1}$ removed. We will show that the product measures of type (1.1), (1.2) for some $b_n = b_n(p, q, R_1, R_2)$ and $h_n = h_n(p, q, R_1, R_2)$ are asymptotically least favorable priors which implies similar effects in this problem if $p > q$.

Note that close minimax estimation problem have been considered by Donoho and Jonhstone (1994).

2. Results

2.1. Required values. To formulate the results for the cases $b_n \to \infty$ and $h_n \to 0$ we need to define two sequences T_n and τ_n. Let T_n be defined by

$$h_n \xi(T_n, b_n) = 1 + o(1).$$

One can easily see that

$$T_n = \frac{b_n}{2} + \frac{\log 2h_n^{-1}}{b_n} + o(b_n^{-1}). \tag{2.1}$$

Let us put

$$\tau_n = \frac{T_n}{b_n} = \frac{1}{2} + \frac{\log h_n^{-1}}{b_n^2} + o(1) \tag{2.2}$$

and assume without loss of generality that

$$\tau_n \to \tau \in [1/2, \infty].$$

Also put $\tau = \infty$ if $b_n = O(1)$ or $h_n \asymp 1$.

Three different types of the limit distributions of the log-likelihood statistics (1.3) and three types of the asymptotics of the second kind error probabilities correspond to three intervals of τ: $\tau \in [2, \infty]$, $\tau \in (1, 2)$ and $\tau \in (1/2, 1]$.

2.2. Gaussian case: $\tau \in [2, \infty]$. Put $u_n = +\sqrt{u_n^2}$ where

$$u_n^2 = \begin{cases} 2nh_n^2(\sinh(b_n^2/2))^2, & \text{if } \tau \in (2, \infty], \\ \\ \frac{1}{2}nh_n^2 e^{b_n^2}\Phi(T_n - 2b_n), & \text{if } \tau = 2. \end{cases} \tag{2.3}$$

Theorem 1. *If $\tau \geq 2$, then*

$$\beta_n(\alpha) = \Phi(t_\alpha - u_n) + o(1)$$

and if $u_n \asymp 1$, then the log-likelihood statistics l_n in (1.3) are asymptotically Gaussian $N(-u_n^2/2, u_n^2/2)$ under the null hypothesis H_0 and $N(u_n^2/2, u_n^2/2)$ under the Bayesian alternative H_{n,π^n}.

Let us put

$$x_n = \frac{b^2}{2\log n}, \quad y_n = \frac{\log nh_n}{\log n}. \tag{2.4}$$

If $u_n \asymp 1$ and $\tau \in [2, \infty)$ then one has using (2.1) - (2.3) that

$$x_n \sim x = 1/4(\tau - 1), \quad y_n \sim y = (2\tau - 3)/4(\tau - 1) \tag{2.5}$$

where $x + y = 1/2$; $\quad 0 < x \leq 1/4, \quad 1/4 \leq y < 1/2$.

For $\tau > 2$ Theorem 1 extends the relations (1.7), (1.9) to the case when $b_n \to \infty$ but $\limsup x_n < 1/2$ and $\beta_n(\alpha)$ is separated away from 0 and $1 - \alpha$. For $\tau = 2$ expression (2.3) for the value u_n is different from (1.8).

2.3. **Infinitely divisible case:** $\tau \in (1,2)$. Put $c_n = 2n\Phi(-T_n)$ and assume without loss of generality that $c_n \to c \in [0,\infty]$.

For $\tau \in (1,2)$ and $c \in (0,\infty)$ let us define two independent infinitely divisible random variables $\zeta^0 = \zeta^0_{c,\tau}$ and $\zeta^\Delta = \zeta^\Delta_{c,\tau}$ with the characteristic functions

$$\log\varphi^0(z) = iz\gamma^0 + \int_{+0}^{\infty}(e^{izt} - 1 - \frac{izt}{1+t^2})dL^0(t), \tag{2.6}$$

$$\log\varphi^\Delta(z) = \int_{+0}^{\infty}(e^{izt} - 1)dL^\Delta(t). \tag{2.7}$$

Here $L^0 = L^0_{c,\tau}$ and $L^\Delta = L^\Delta_{c,\tau}$ are the Levi spectral functions (see Petrov, 1975) which are zero for $t < 0$ and for $t > 0$

$$L^0(t) = -c(e^t - 1)^{-\tau}, \tag{2.8}$$

$$L^\Delta(t) = -\frac{d}{dt}L^0(t) = -\frac{c}{\tau - 1}(e^t - 1)^{1-\tau}. \tag{2.9}$$

The constant γ^0 in (2.6) is defined by the relation

$$E\zeta^0 = \gamma^0 + \int_0^\infty \frac{t^3}{1+t^2}dL^0(t) = cI^0(\tau) \tag{2.10}$$

where

$$I^0(\tau) = \int_0^\infty (\log(1 + u^{-1/\tau}) - u^{-1/\tau})du \tag{2.11}$$

and the first equality in (2.10) follows from the relation

$$\frac{d}{dz}\log\varphi^0(z)\mid_{z=0} = iE\zeta^0.$$

The representations (2.6), (2.7) mean that the random variables ζ^0 and ζ^Δ do not have Gaussian components. Using the relation

$$\log\varphi^0(-i) = \log E\exp\zeta^0 = \gamma^0 + \int_{+0}^{\infty}(e^t - 1 - \frac{t}{1+t^2})dL^0(t)$$

and (2.10), (2.11) one has

$$\begin{aligned}\log E\exp\zeta^0 &= c(\int_0^\infty (\log(1+u^{-1/\tau}) - u^{-1/\tau}))du \\ &- \int_0^\infty (t + 1 - e^t)(e^t - 1)^{-\tau}dt).\end{aligned} \tag{2.12}$$

Using the change of variables $(e^t - 1)^{-\tau} = u$ one has the equality of the integrals on the right side of (2.12). This implies the equivalence of the relations (2.10), (2.11) to the following equality

$$E\exp\zeta^0 = 1. \tag{2.13}$$

Note that for the Levi spectral functions $L = L^0$ and $L = L^\Delta$ one has from (2.8) and (2.9) that

$$\int_{|x|>1}|x|^p dL(x) < \infty$$

for any $p > 0$ which implies that the random variables ζ^0 and ζ^Δ have finite moments of any order. The distributions of ζ^0 and ζ^Δ are absolutely continuous.

The support of ζ^0 is R^1 but the support of ζ^Δ is the positive halfline $R^1_+ = \{t \geq 0\}$ (see Petrov (1975) for general theorems which imply these properties).

Let $F^0 = F^0_{c,\tau}$ and $F^1 = F^1_{c,\tau}$ be the distribution functions of ζ^0 and of $\zeta^1 = \zeta^1_{c,\tau} = \zeta^0 + \zeta^\Delta$ and let $t^0 = t^0_{c,\tau}$ be the $(1-\alpha)$-quantile of ζ^0: $F^0(t^0) = 1 - \alpha$.

Theorem 2. *Let $\tau \in (1,2)$.*

1. If $c = 0$, then $\beta_n(\alpha) \to 1 - \alpha$.

2. If $c = \infty$, then $\beta_n(\alpha) \to 0$.

3. Let $c \in (0, \infty)$. Then $l_n \to \zeta^0$ under $P_{n,0}$ - probability, $l_n \to \zeta^1$ under P_{n,π^n} - probability and

$$\beta_n(\alpha) \to F^1(t^0).$$

Note that equality (2.13) and Theorem 2 imply the contiguity of the sequences of the measures $P_{n,0}$ and P_{n,π^n} according to first Le Cam's lemma.

For $\tau \in (1,2)$ and $c \asymp 1$ using the inequalities (2.1) and (2.2) one has the asymptotic relations for x_n and y_n defined in (2.4):

$$x_n \sim x = \tau^{-2}, \quad y_n \sim y = (1 - \tau^{-1})^2; \quad x^{1/2} + y^{1/2} = 1 \tag{2.14}$$

where $1/4 < x < 1, \quad 0 < y < 1/4$.

2.4. **Degenerate case:** $\tau \in [1/2, 1]$. Let us put

$$\lambda_n = nh_n\Phi(b_n - T_n)$$

and assume without loss of generality that $\lambda_n \to \lambda \in [0, \infty]$.

Theorem 3. *Let $\tau \in [1/2, 1]$.*

1. If $\lambda = 0$, then $\beta_n(\alpha) \to 1 - \alpha$.

2. If $\lambda = \infty$, then $\beta_n(\alpha \to 0$.

3. Let $\lambda \in (0, \infty)$. Then $l_n \to -\lambda$ under null hypothesis H_0 and

$$\beta_n(\alpha) \to (1 - \alpha)\exp(-\lambda).$$

Note that if $\lambda > 0$, then the sequences of the measures $P_{n,0}$ and P_{n,π^n} are not contiguous.

In the case $\tau \in [1/2, 1], \lambda \in (0, \infty)$ one has the asymptotic relations for x_n and y_n defined in (2.4):

$$x_n \sim x = \frac{1}{2\tau - 1}, \quad y_n \sim y \equiv 0; \quad x > 0. \tag{2.15}$$

2.5. **Minimax alternative.** It was noted above that if $k_n = nh_n \to \infty$, then Bayesian alternative is close to the minimax alternative. More exactly

Theorem 4. *Let $k_n = nh_n \to \infty$. Then*

$$\beta_n(\alpha, V_n(b_n, k_n)) = \beta_n(\alpha, \pi^n(b_n, h_n)) + o(1).$$

Theorem 4 implies that the statements of Theorems 1 - 3 dealing with Bayesian alternatives carry over to the case of minimax alternatives with the same b_n and with $k_n = nh_n \to \infty$. The relations (2.4), (2.14) and (2.15) hold with the change of nh_n to k_n, of the interval $(-\infty, +\infty)$ of y_n to $(0, \infty)$ because $k_n \geq 1$, and of the interval $(0, \infty)$ of x_n to $(0, 1]$ because $\beta_n(\alpha, V_n) \to 0$ for $x_n > 1$.

Remark. If $k_n = nh_n \asymp 1$, then Bayesian and minimax alternatives are essentially different. In fact, the function $\beta_n(\alpha, V_n(b_n, k_n))$ is decreasing in both b_n and k_n which follows from Anderson's lemma (Ibragimov and Khasminskii, 1981). Using this fact and the relations (1.10), (1.11), one has $\beta_n(\alpha, V_n(b_n, k_n)) \to 0$ as $b_n - \sqrt{2\log n} \to \infty$ for any $k_n \geq 1$.

On the other hand, it follows from Theorem 3 for Bayesian alternatives that if $\tau < 1$ and $nh_n \sim \lambda \asymp 1$, then $\lambda_n \sim \lambda$ and $\beta_n(\alpha)$ is separated away from 0 for arbitrary large b_n.

The reason for this is that if $b_n - \sqrt{\log 2n} \to \infty$, then the second kind error probabilities for Bayesian alternatives are specified by the π^n-probabilities of the events

$$\Im_{n,0} = \{\sum_{i=1}^{n} t_i = 0\}$$

and if $nh_n \to \lambda$, then the random variables $\sum_{i=1}^{n} t_i$ tend to the Poisson random variable with parameter λ under π^n-probability which implies the relation

$$\pi^n(\Im_{n,0}) \to \exp(-\lambda).$$

We do not have results on the minimax alternative with $k_n \asymp 1$ except for the case $k_n = 1$.

2.6. **Graphical representation.** Let us consider the three-piece curve $\Gamma^0 = \{y = f(x)\}$ in the coordinates

$$x \sim x_n = \frac{b_n^2}{2\log n}, \quad y \sim y_n = \frac{\log nh_n}{\log n} = \frac{\log k_n}{\log n}.$$

The pieces of this curve correspond to the equations in (2.5), (2.14) and (2.15):

$$x + y = \frac{1}{2}, \quad 0 < x \leq \frac{1}{4}; \quad x^{1/2} + y^{1/2} = 1, \quad \frac{1}{4} < x < 1; \quad y = 0, \quad x \geq 1;$$

for minimax alternatives we consider only the first two pieces.

This curve roughly characterizes the dependence of distinguishability conditions on b_n and h_n or k_n in the considered problems. This divides the halfplane $\{x > 0, \quad y\}$ for Bayesian alternative or the quarter plane $\{x > 0, \quad y > 0\}$ for minimax alternative into two regions $\Gamma^+ = \{y > f(x)\}$ and $\Gamma^- = \{y < f(x)\}$. The region Γ^+ corresponds to distinguishability: $\beta_n(\alpha) \to 0$, the region Γ^- corresponds to non distinguishability: $\beta_n(\alpha) \to 1 - \alpha$. The curve Γ^0 corresponds to the case when $\beta_n(\alpha)$ is separated away from 0 and from $1 - \alpha$. Each point of this curve corresponds to the family of Gaussian distributions (the first piece), of infinitely divisible distributions with parameter $\tau = x^{-1/2}$ (the second piece) and degenerate distributions (the third piece).

3. Proof of Theorem 1

First, note that it is enough to consider the case $b_n \to \infty$ because for $b_n \asymp 1$ the results follow from Ingster (1990, 1993).

Next, it is enough to consider the case $u_n = O(1)$. In fact, $\beta_n(\alpha, \pi^n(b, h))$ is a decreasing function of b_n which is an easy consequence of Anderson's lemma (see Ibragimov and Khasminskii, 1981). Also τ_n in (2.2) decrease and u_n increase in b_n. This implies that if $u_n \to \infty$, then by making b_n smaller we can reduce the consideration to the case of arbitrarily large $u_n \asymp 1$ and $\tau \geq 2$. Theorem 1 implies that $\beta_n(\alpha)$ will then be arbitrarily small.

Also note that in the cases $b_n \to \infty$ and $u_n = O(1)$ or $c_n = O(1)$ or $\lambda_n = O(1)$ one can easily see from (2.1) that

$$T_n \to \infty, \quad h_n \to 0, \quad h_n \sim 2\exp(\frac{b_n^2}{2} - T_n b_n). \tag{3.1}$$

Let us put

$$z_n(x) = h_n \xi(x, b_n), \quad w_n(x) = \log(1 + z_n(x))$$

and note that for any $\delta > 0$ and $y = |x| - T_n > \delta$ the following representations hold

$$z_n(x) = \exp b_n y(1 + o(1)), \quad w_n(x) = \log(1 + \exp b_n y(1 + o(1))) \tag{3.2}$$

and

$$\sup_{|x| \leq T_n - \delta} |w_n(x)| \to 0. \tag{3.3}$$

Define the function $T_n(t)$ by

$$w_n(T_n(t)) = t, \quad T_n(t) \geq 0.$$

It follows from (3.2), (2.1), since $w_n(x)$ is increasing in x, that for any $t > 0$

$$T_n(t) = T_n + \frac{\log(e^t - 1)}{b_n} + o(\frac{1}{b_n}), \tag{3.4}$$

$$\{x : w_n(x) < t\} = (-T_n(t), T_n(t)). \tag{3.5}$$

Under the assumptions $u_n = O(1)$ and $\tau \leq 2$ the following holds

$$n\Phi(-T_n) \to 0. \tag{3.6}$$

In fact, using the well known relation

$$\Phi(-x) \sim \frac{1}{x\sqrt{2\pi}} \exp(-\frac{x^2}{2}) \tag{3.7}$$

and (3.1) one has for the case $\liminf(T_n - 2b_n) > -\infty$ that

$$u_n^2 \asymp n h_n^2 \exp b_n^2 \asymp n \exp(\frac{(T_n - 2b_n)^2}{2} - \frac{T_n^2}{2})$$

which implies

$$n\Phi(-T_n) = o(n \exp(-\frac{T_n^2}{2})) = o(u_n^2) \to 0.$$

If $d_n = T_n - 2b_n \to -\infty$, then $d_n = o(T_n)$ because $\tau_n \geq 2$ and similarly

$$n\Phi(-T_n) \sim u_n^2 \frac{d_n}{T_n} \to 0.$$

Using the relation (3.6) one can choose values H_n, t_n such that

$$H_n = T_n - \frac{t_n}{b_n}, \quad t_n \to \infty, \quad t_n = o(b_n), \quad n\Phi(-H_n) \to 0$$

which implies by (3.2) - (3.6) that

$$P_{n,0}(\max_{1 \leq i \leq n} |x_i| > H_n) \to 0, \ \sup_{|x| \leq H_n} |z_n(x)| \to 0 \tag{3.8}$$

and for $|x| \le H_n$ one has

$$w_n(x) = z_n(x) - \frac{1}{2} z_n^2(x)(1 + o(1)). \tag{3.9}$$

The relations (3.8), (3.9) imply the asymptotic normality of the statistics

$$l_n = \sum_{i=1}^{n} w_n(x_i)$$

under $P_{n,0}$ - distributions (see Petrov, 1975) and it is enough to show that

$$E_{n,0} l_n^* \sim -u_n^2/2, \quad D_{n,0} l_n^* \sim u_n^2/2 \tag{3.10}$$

where $l_n^* = \sum_{i=1}^n w_n^*(x_i)$ is the sum of the H_n - truncated random variables $w_n^*(x_i)$ and $D_{n,v}$ stands for the variance under $P_{n,v}$. In view of third Le Cam's lemma this implies the statement of Theorem 1.

The equivalencies (3.10) follow from (3.9) and from the relations

$$nE_{n,0} z_n^* \to 0, \quad nE_{n,0}(z_n^*)^2 \sim u_n^2, \tag{3.11}$$

where z_n^* denotes z_n truncated at level H_n.

To obtain (3.11) note that for $b \to \infty, T \to \infty$ the following equalities hold:

$$\int_{|x|<T} \zeta(x,b) d\Phi(x) = -\Phi(b-T)(1+o(1)), \tag{3.12}$$

$$\int_{|x|<T} \zeta^2(x,b) d\Phi(x) = \frac{1}{2}(\exp b^2)\Phi(T-2b) + 2\Phi(T-b) - 1 + o(1). \tag{3.13}$$

In fact, a direct calculation gives

$$\int_{|x|<T} \zeta(x,b) d\Phi(x) = 2\Phi(-T) - \Phi(-T+b) - \Phi(-T-b),$$

and using (3.7) one has (3.12). Also

$$\begin{aligned}\int_{|x|<T} \zeta^2(x,b) d\Phi(x) &= \{e^{b^2}[\Phi(T-2b) - \Phi(-T-2b)] + e^{-b^2}[\Phi(T) - \Phi(-T)]\}/2 \\ &+ \Phi(T) - \Phi(-T) + 2\Phi(-T+b) - 2\Phi(T+b),\end{aligned}$$

which implies (3.13).

Using (3.12), (3.13) and the relations

$$nh_n \Phi(b_n - T_n) \sim 2n\Phi(-T_n)\frac{\tau_n}{\tau_n - 1} \to 0, \quad nh_n^2 \to 0,$$

which follow from (3.6), (3.7) and from the assumptions $u_n = O(1)$ and $b_n \to \infty$ one has (3.11). Theorem 1 is proved.

4. Proof of Theorem 2

Analogously to the proof of Theorem 1 it is enough to consider the case $c_n = O(1)$. In fact, for $c_n = 2n\Phi(-T_n) \to \infty$, since $b_n/2 + \log h_n^{-1}/b_n^2$ is decreasing in b_n for $\tau_n > 1$, by making b_n decrease one can pass to new $c_n^* = O(1)$, which can be chosen arbitrarily large, and to new $\tau^* > 1$. If $\tau^* \geq 2$, then the new $u_n^* \to \infty$ by (3.6) and the statement of Theorem 1 implies that $\beta_n(\alpha)$ will be arbitrarily small; if $\tau^* < 2$, then the above follows from the statement of Theorem 2.

To prove Theorem 2 it is enough, for $c > 0$, to check the conditions of (Petrov, 1975, ch.4, §4) for the convergence of $P_{n,0}$ - and P_{n,π^n} - distributions of the statistics l_n to the infinitely divisible distributions which are described above. For $c = 0$ it is sufficient to show that $l_n \to 0$ under these distributions .

First, let us consider the case of null-hypothesis H_0.

To check that the Levi spectral function is of requried form for $c > 0$ it is necessary to show that $L_n^0(t) \to L^0(t)$ for any $t \neq 0$ where

$$L_n^0(t) = nP_{1,0}(w_n(x) < t)$$

for $t < 0$ and

$$L_n^0(t) = -nP_{1,0}(w_n(x) > t)$$

for $t > 0$. The inequality

$$\xi(x,b) \geq e^{-b^2/2} - 1 \tag{4.1}$$

implies this relation for $t < 0$. For $t > 0$ using (3.7) and (3.4), (3.5) one has

$$-L_n^0(t) = 2n\Phi(-T_n(t)) \sim 2n\Phi(-T_n)(e^t - 1)^{-\tau_n} \to -L^0(t). \tag{4.2}$$

To check the condition on the constant γ^0 for $c > 0$ let us show that

$$E_{n,0}l_n = n\int w_n(x)d\Phi(x) \to E\zeta^0. \tag{4.3}$$

We show below that $D_{n,0}l_n = O(1)$ which together with (4.3) implies the required representation γ^0.

In fact, because $L^0(t)$ is continuous and

$$\int_1^\infty xdL^0(x) < \infty$$

one easily has from (Petrov, 1975) that for any $t > 0$

$$\begin{aligned}
\gamma^0 &= \lim_{n\to\infty} n\int_{|w_n|<t} w_n(x)d\Phi(x) - \int_0^t \frac{x^3}{1+x^2}dL^0(x) \\
&+ \int_t^\infty \frac{x}{1+x^2}dL^0(x) = \lim_{n\to\infty} n\int_{|w_n|<t} w_n(x)d\Phi(x) \\
&- \int_0^\infty \frac{x^3}{1+x^2}dL(x) + \int_t^\infty xdL(x)
\end{aligned} \tag{4.4}$$

and because $nD_{1,0}w_n = D_{n,0}l_n = O(1)$ we can assume that $t \to \infty$ in (4.4) which implies equality (2.10).

To show (4.3), by the equality

$$\int \xi(x,b)d\Phi(x) = 0,$$

one has

$$E_{n,0}l_n = n\int(\log(1+z_n(x)) - z_n(x))d\Phi(x) = A_n + B_n,$$

where A_n and B_n are integrals over the sets $\{|x| < T_n - \delta_n\}$ and $\{|x| > T_n - \delta_n\}$. Here and below in this section sequence δ_n is such that $\delta_n \to 0$ and $b_n\delta_n \to \infty$. Using the formulas (3.1), (3.2), (3.7), (3.13) and the relations

$$\log(1+z) - z \sim -z^2/2$$

as $z \to 0$ and $nh_n^2 \to 0$ (the latter follows from (2.5), (2.14) and (2.15) for any $\tau < 2$ since $y < 1/2$ in this relations), one has

$$\begin{aligned} -A_n &\sim \frac{n}{2}\int_{|x|<T_n-\delta_n} z_n^2(x)d\Phi(x) \\ &\sim \frac{n}{2}h_n^2\exp(b_n^2)\Phi(T_n - 2b_n - \delta_n) \\ &\sim \frac{2\tau_n}{2-\tau_n}n\Phi(-T_n)\exp((\tau_n - 2)b_n\delta_n) \to 0. \end{aligned} \tag{4.5}$$

Also using (3.2), (3.7) and (2.11) one has for $y = |x| - T_n$, $z = b_n y$, $u = \exp(-\tau_n z)$:

$$\begin{aligned} B_n &\sim 2\int_{-\delta_n}^{\infty}(\log(1+e^{b_n y}) - e^{b_n y})d\Phi(y + T_n) \\ &\sim 2n\tau_n\Phi(-T_n)\int_{-b_n\delta_n}^{\infty}(\log(1+e^z) - e^z)\exp\left(-\tau_n z - \frac{z^2}{2b_n^2}\right)dz \\ &\sim c_n\int_0^{\infty}(\log(1+e^z) - e^z)e^{-\tau_n z}dz \\ &= c_n\int_0^{\infty}(\log(1+u^{-1/\tau_n}) - u^{-1/\tau_n})du \to cI^0(\tau). \end{aligned} \tag{4.6}$$

Relations (4.5), (4.6) and (2.10) imply (4.3).

To verify that the limit $P_{n,0}$ - distribution of l_n has no Gaussian component it is sufficient to show that

$$\lim_{t\to 0}\lim_{n\to\infty} n\int_{|w_n(x)|<t} w_n^2(x)d\Phi(x) = 0. \tag{4.7}$$

Relation (4.7) follows from the formulas

$$A_n = n\int_{|x|<T_n-\delta_n} z_n^2(x)d\Phi(x) \to 0, \quad \lim_{t\to 0}\lim_{n\to\infty} B_n(t) = 0 \tag{4.8}$$

where

$$B_n(t) = n\int_{T_n-\delta_n<|x|<T_n(t)} w_n^2(x)d\Phi(x).$$

But the first formula in (4.8) was checked in (4.5) and the second in analogy with (4.6) follows from (3.2), (3.7): for any $t > 0$, $y = |x| - T_n$, $z = b_n y$, $u = e^z$ one has

$$\begin{aligned} B_n(t) &\sim 2n\int_{-\delta_n}^{b_n^{-1}\log(e^t-1)}\log^2(1+e^{b_n y})d\Phi(T_n + y) \\ &\sim c_n\tau_n\int_{-\infty}^{\log(e^t-1)}\log^2(1+e^z)e^{-\tau_n z}dz \\ &\sim c\tau\int_0^{e^t-1}\log^2(1+u)u^{-(\tau+1)}du \end{aligned} \tag{4.9}$$

because as $t \to 0$ the last integral in (4.9) is of order

$$\int_0^t u^{1-\tau} du \asymp t^{2-\tau} \to 0$$

for any $\tau < 2$.

Note that calculations similar to (4.5) and (4.9) give the asymptotics of $P_{n.0}$ - variances $D_{n,0} l_n$: for any $\tau \in [1/2, 2)$

$$D_{n,0} l_n \sim c_n D^0(\tau), \quad D^0(\tau) = \tau \int_0^\infty \log^2(1+u) u^{-(\tau+1)} du < \infty. \tag{4.10}$$

Formulas (4.3), (2.10), (4.10) and Chebyshev's inequality imply that $l_n \to 0$ under $P_{n,0}$ - probability if $c = 0$ which completes the proof for the null-hypothesis.

Let us assume that the Bayesian alternative H_{n,π^n} holds. In this case the random variables x_i, $i = 1, \dots, n$ are i.i.d. with the distribution function $\Phi_{\pi_n}(t) = \Phi(t) + \Delta\Phi_{\pi_n}(t)$ where

$$\Delta\Phi_{\pi_n}(t) = \frac{h_n}{2}[\Phi(t+b_n) + \Phi(t-b_n) - 2\Phi(t)]$$

and

$$\frac{d\Delta\Phi_{\pi_n}}{d\Phi}(x) = z_n(x).$$

To prove the statement of Theorem 2 for the limit P_{n,π^n} - distribution of the statistics l_n for the case $c > 0$ it is enough to show that for any $t > 0$

$$\begin{aligned} L^\Delta_{\pi_n}(t) &= n[P_{1,0}(w_n(x) > t) - P_{1,\pi_n}(w_n(x) > t)] \\ &= -2n\Delta\Phi_{\pi_n}(-T_n(t)) \to L^\Delta(t); \end{aligned} \tag{4.11}$$

$$\lim_{t\to 0} \lim_{n\to\infty} n \int_{|w_n|} w_n(x) d\Delta\Phi_{\pi_n}(x) = 0; \tag{4.12}$$

$$\lim_{t\to 0} \lim_{n\to\infty} n \int_{|w_n|} w_n^2(x) d\Delta\Phi_{\pi_n(x)} = 0. \tag{4.13}$$

In fact, relations (4.1), (4.2) and (4.11) imply the required form of the Levi spectral function. The relation (4.12) and the equalities $\gamma^1 = \gamma^0 + \gamma^\Delta$,

$$\begin{aligned} \gamma^\Delta &= \lim_{n\to\infty} n \int_{|w_n|} w_n(x) d\Delta\Phi_{\pi_n}(x) \\ &- \int_{+0}^t \frac{x^3}{1+x^2} dL^\Delta(x) + \int_t^\infty \frac{x}{1+x^2} dL^\Delta(x) \end{aligned}$$

imply as $t \to 0$ that

$$\gamma^\Delta = \int_{+0}^\infty \frac{x}{1+x^2} dL^\Delta(x)$$

which is equivalent to representation (2.7) up to a quadratic term corresponding to a Gaussian component. The absence of this component follows from (4.7) and (4.13) (see Petrov, 1975).

because as $t \to 0$ the last integral in (4.9) is of order

$$\int_0^t u^{1-\tau} du \asymp t^{2-\tau} \to 0$$

for any $\tau < 2$.

Note that calculations similar to (4.5) and (4.9) give the asymptotics of $P_{n.0}$ - variances $D_{n,0} l_n$: for any $\tau \in [1/2, 2)$

$$D_{n,0} l_n \sim c_n D^0(\tau), \quad D^0(\tau) = \tau \int_0^\infty \log^2(1+u) u^{-(\tau+1)} du < \infty. \tag{4.10}$$

Formulas (4.3), (2.10), (4.10) and Chebyshev's inequality imply that $l_n \to 0$ under $P_{n,0}$ - probability if $c = 0$ which completes the proof for the null-hypothesis.

Let us assume that the Bayesian alternative H_{n,π^n} holds. In this case the random variables x_i, $i = 1, \ldots, n$ are i.i.d. with the distribution function $\Phi_{\pi_n}(t) = \Phi(t) + \Delta\Phi_{\pi_n}(t)$ where

$$\Delta\Phi_{\pi_n}(t) = \frac{h_n}{2}[\Phi(t+b_n) + \Phi(t-b_n) - 2\Phi(t)]$$

and

$$\frac{d\Delta\Phi_{\pi_n}}{d\Phi}(x) = z_n(x).$$

To prove the statement of Theorem 2 for the limit P_{n,π^n} - distribution of the statistics l_n for the case $c > 0$ it is enough to show that for any $t > 0$

$$\begin{aligned} L^\Delta_{\pi_n}(t) &= n[P_{1,0}(w_n(x) > t) - P_{1,\pi_n}(w_n(x) > t)] \\ &= -2n\Delta\Phi_{\pi_n}(-T_n(t)) \to L^\Delta(t); \end{aligned} \tag{4.11}$$

$$\lim_{t\to 0}\lim_{n\to\infty} n \int_{|w_n|} w_n(x) d\Delta\Phi_{\pi_n}(x) = 0; \tag{4.12}$$

$$\lim_{t\to 0}\lim_{n\to\infty} n \int_{|w_n|} w_n^2(x) d\Delta\Phi_{\pi_n(x)} = 0. \tag{4.13}$$

In fact, relations (4.1), (4.2) and (4.11) imply the required form of the Levi spectral function. The relation (4.12) and the equalities $\gamma^1 = \gamma^0 + \gamma^\Delta$,

$$\begin{aligned} \gamma^\Delta &= \lim_{n\to\infty} n \int_{|w_n|} w_n(x) d\Delta\Phi_{\pi_n}(x) \\ &- \int_{+0}^t \frac{x^3}{1+x^2} dL^\Delta(x) + \int_t^\infty \frac{x}{1+x^2} dL^\Delta(x) \end{aligned}$$

imply as $t \to 0$ that

$$\gamma^\Delta = \int_{+0}^\infty \frac{x}{1+x^2} dL^\Delta(x)$$

which is equivalent to representation (2.7) up to a quadratic term corresponding to a Gaussian component. The absence of this component follows from (4.7) and (4.13) (see Petrov, 1975).

Using (3.2) - (3.5) and (3.7) one has relation (4.11) by the equivalencies

$$\begin{aligned} -L^{\Delta}_{\pi_n}(t) &\sim nh_n\Phi(b_n - T_n(t)) \\ &\sim nh_n\Phi(b_n - T_n)(e^t-1)^{1-\tau_n} \\ &\sim c_n\frac{\tau_n}{\tau_n-1}(e^t-1)^{1-\tau} \to -L^{\Delta}(t) \end{aligned} \tag{4.14}$$

where the third equivalence in (4.14) follows from relations (3.1), (3.7) and

$$nh_n\Phi(b_n - T_n) \sim \frac{nT_n e^{-T_n^2/2}}{T_n(T_n-b_n)\sqrt{2\pi}}h_n\exp(\frac{b_n^2}{2} - T_nb_n) \sim c_n\frac{\tau_n}{\tau_n-1}.$$

To obtain (4.12) let us note that using equivalencies (4.5) one has

$$\begin{aligned} &n\int_{|x|<T_n-\delta_n} w_n(x)d\Delta\Phi_{\pi_n}(x) \\ &= n\int_{|x|<T_n-\delta_n}\log(1+z_n(x))z_n(x)d\Phi(x) \\ &\sim n\int_{|x|<T_n-\delta_n} z_n^2(x)d\Phi(x) \to 0 \end{aligned}$$

and using relations (3.2) - (3.5), (3.7) and in a manner to (4.6) one has for any $t>0$

$$\begin{aligned} &n\int_{T_n-\delta_n<|x|<T_n(t)} w_n(x)d\Delta\Phi_{\pi_n}(x) \\ &\sim c_n\tau_n\int_{-\delta_n b_n}^{\log(e^t-1)}\log(1+e^z)e^{z(1-\tau)}dz \\ &\sim c\tau\int_0^{e^t-1}\log(1+u)u^{-\tau}du. \end{aligned} \tag{4.15}$$

Relation (4.12) follows from the relations above if one notes that the integral in (4.15) is of order

$$\int_0^t u^{1-\tau}du \asymp t^{2-\tau} \to 0$$

as $t \to 0$ for any $\tau < 2$.

Finaly, using formulas (3.2) - (3.5), (3.7), (4.5) as above one has that

$$\begin{aligned} &n\int_{|x|<T_n-\delta_n}\log^2(1+z_n(x))z_n(x)d\Phi(x) \\ &= o(n\int_{|x|<T_n-\delta_n} z_n^2(x)d\Phi(x)) \to 0, \end{aligned}$$

$$\begin{aligned} &n\int_{T_n-\delta_n<|x|<T_n(t)}\log^2(1+z_n(x))z_n(x)d\Phi(x) \\ &\sim c\tau\int_0^{e^t-1}\log^2(1+u)u^{-\tau}du, \end{aligned} \tag{4.16}$$

which implies relation (4.13) because the integral in (4.16) tends to 0 as $t \to 0$.

If $c = 0$ then the above estimates and results for the null hypothesis show that $l_n \to 0$ under P_{n,π^n} - probability. Theorem 2 is proved.

5. Proof of Theorem 3

Analogously to the proofs of Theorems 1 and 2 it is enough to consider the case $\lambda_n = O(1)$. In fact,

$$\Phi(\frac{b_n}{2} - \frac{\log 2h_n^{-1}}{b_n^2}) \sim \Phi(b_n - T_n)$$

increases in b_n and if $\lambda_n = nh_n\Phi(b_n - T_n) \to \infty$ one can pass by decreasing b_n to new $\lambda_n^* = O(1)$, which can be chosen arbitrarily large, and to new τ^* with $\tau^* \geq \tau$ and $\tau^* < 2$, or to $\lambda_n^* \to \infty$ and $\tau^* \in (1,2)$. If $\tau^* < 1$, then $\beta_n(\alpha)$ is arbitrarily small by Theorem 3. If $\tau^* > 1$, then for the new values T_n^*, b_n^* and $c_n^* = 2n\Phi(-T_n^*)$ one has $T_n^* - b_n^* \to \infty$ and using (3.1), (3.7) one can easily see that

$$\lambda_n^* \sim c_n^*\tau_n^*/(\tau_n^* - 1) \tag{5.1}$$

which also implies that $\beta_n(\alpha)$ will be arbitrarily small by Theorem 2.

Note that for $\tau \leq 1$ one has

$$c_n = o(\lambda_n) \to 0. \tag{5.2}$$

This follows from (3.1), (3.7), in analogy with (5.1), if $\tau = 1$ and $T_n - b_n \to \infty$. If $\limsup T_n - b_n < \infty$, then the assumption $\lambda_n = O(1)$ implies that $\lambda_n \asymp nh_n = O(1)$ and using (3.1) one has

$$c_n \sim nh_n \exp(-(T_n - b_n)^2/2)/T_n\sqrt{2\pi} = o(nh_n).$$

Let us show that

$$E_{n,0}l_n = -\lambda_n + o(1). \tag{5.3}$$

For any $t > 0$ let us consider the representation

$$E_{n,0}l_n = A_n(t) + B_n(t)$$

where

$$\begin{aligned} A_n(t) &= n\int_{|x|<T_n(t)} w_n(x)d\Phi(x) = n\int_{|x|<T_n(t)} z_n(x)d\Phi(x) \\ &+ O(\int_{|x|<T_n(t)} z_n^2(x)d\Phi(x)) \end{aligned} \tag{5.4}$$

and

$$B_n(t) = n\int_{|x|>T_n(t)} w_n(x)d\Phi(x).$$

Analogously to estimates (4.8), (4.9) and using (5.2) one can see that the remainder term in (5.4) is $O(c_n) \to 0$ and according to (3.12) one has

$$A_n(t) = -h_n\Phi(b_n - T_n(t)) + o(1). \tag{5.5}$$

Let us observe that

$$\Phi(b_n - T_n(t)) \sim \Phi(b_n - T_n). \tag{5.6}$$

In fact, using (3.4) one has $T_n(t) = T_n + O(b_n^{-1})$ which implies (5.6) if $\limsup(T_n - b_n) < \infty$. If $d_n = T_n - b_n \to \infty$, then $d_n = o(b_n)$ because $\tau \leq 1$ and using (3.7) one has

$$\begin{aligned} \Phi(b_n - T_n(t)) &= \Phi(-d_n + O(b_n^{-1})) \\ &\sim \Phi(-d_n)\exp(O(d_n/b_n)) \sim \Phi(-d_n) \end{aligned}$$

which is the same as (5.6).

Relations (5.5) and (5.6) imply that

$$A_n(t) = -\lambda_n + o(1).$$

In the end, according to relations (3.2) - (3.5), (3.7) analogously to estimates (4.6), (4.9) one has

$$\begin{aligned} B_n(t) &\sim 2n \int_{b_n^{-1}\log(e^t-1)}^{\infty} \log(1+e^{b_n y}) d\Phi(T_n + y) \\ &\sim c_n \tau_n \int_{\log(e^t-1)}^{\infty} \log(1+e^z) e^{-\tau_n z} dz \\ &\sim c_n \tau_n \int_{e^t-1}^{\infty} \log(1+u) u^{-(\tau_n+1)} du = O(c_n) \to 0. \end{aligned}$$

Relation (5.3) is proved.

According to formulas (4.10), (5.3) and to Chebyshev's inequality one has that for any $\epsilon > 0$

$$P_{n,0}(|l_n + \lambda_n| > \epsilon) \to 0$$

which implies the statement of Theorem 3.

6. Proof of Theorem 4

First, let us observe the following asymptotical continuity property for Bayesian alternatives which follows from Theorems 1 - 3 : if $h_n^* \sim h_n$, then for any $\alpha \in (0,1)$ and $b_n > 0$

$$\beta_n(\alpha, \pi^n(b_n, h_n^*)) = \beta_n(\alpha, \pi^n(b_n, h_n)) + o(1).$$

Let $k_n \to \infty$, $h_n = k_n/n$ and δ_n be a sequence such that $\delta_n \to 0$, $k_n\delta_n^2 \to \infty$. Put for $r = 1, 2$

$$k_{n,r}^- = \max\{0, [k_n(1 - r\delta_n)]\}, \quad k_{n,r}^+ = \min\{n, [k_n(1 + r\delta_n)]\}$$

where $[t]$ denotes the integer part of t. Let us consider the sets

$$V_n^+ = \{v \in V_{b_n,n} : k_n \le k_n(v) \le k_{n,2}^+\},$$

$$V_n^- = \{v \in V_{b_n,n} : k_{n,2}^- \le k_n(v) \le k_n\}.$$

Here $k_n(v) = b_n^{-1} \sum_{i=1}^n |v_i|$ and

$$V_{b,n} = \{v = (v_1, \cdots, v_n) \in R^n : v_i = bt_i, t_i \in \{-1, 0, +1\}\}.$$

Put also $h_n^+ = k_{n,1}^+/n$, $h_n^- = k_{n,1}^-/n$ and consider the product measures $\pi_+^n = \pi^n(b_n, h_n^+)$, $\pi_-^n = \pi^n(b_n, h_n^-)$ of type (1.1), (1.2) and the conditional measures $\pi_{+,*}^n = \pi^+$ and $\pi_{-,*}^n = \pi^-$ with respect to the sets V_n^+ and V_n^-:

$$\pi_{+,*}^n(A) = \pi_+^n(A \bigcap V_n^+)/\pi_+^n(V_n^+),$$

$$\pi_{-,*}^n(A) = \pi_-^n(A \bigcap V_n^-)/\pi_-^n(V_n^-).$$

The asymptotical continuity above implies that

$$\beta_n(\alpha, \pi^n(b_n, h_n)) = \beta_n(\alpha, \pi_+^n) + o(1) = \beta_n(\alpha, \pi_-^n) + o(1). \tag{6.1}$$

Note that the random variables $k_n = k_n(v)$ are binomial $Bi(h_n^+, n)$ and $Bi(h_n^-, n)$ with respect to the measures π_+^n and π_-^n which are supported by $V_{b_n,n}$ and using Chebyshev's inequality one can easily check that

$$\pi_+^n(V_n^+) \to 1, \quad \pi_-^n(V_n^-) \to 1$$

which imply the relations

$$\begin{aligned} \beta_n(\alpha, \pi_+^n) &= \beta_n(\alpha, \pi^+) + o(1), \\ \beta_n(\alpha, \pi_-^n) &= \beta_n(\alpha, \pi^-) + o(1) \end{aligned} \tag{6.2}$$

(see Ingster, 1993, sec.4.1).

Let $\pi^0 = \pi_{b_n,k_n}^n$ be the uniform discrete measure on the set $V_n(b_n, k_n)$ which, by the above, is the least favorable prior for the minimax alternative . This means that

$$\beta_n(\alpha, V_n(b_n, k_n)) = \beta_n(\alpha, \pi^0). \tag{6.3}$$

Let us show that the following inequalities hold:

$$\beta_n(\alpha, \pi^+) \leq \beta_n(\alpha, \pi^0) \leq \beta_n(\alpha, \pi^-). \tag{6.4}$$

Relations (6.1) - (6.3) and inequalities (6.4) evidently imply the statement of Theorem 4.

To prove inequalities (6.4) let us consider the critical sets $X^0 = X_{n,\alpha}^0$ and $X^- = X_{n.\alpha}^-$ of the optimal tests of level α for Bayesian alternatives H_{n,π^0} and H_{n,π^-}:

$$X^0 = \{x \in R^n : \frac{dP_{n,\pi^0}}{dP_{n,0}}(x) < T^0\},$$

$$X^- = \{x \in R^n : \frac{dP_{n,\pi^-}}{dP_{n,0}}(x) < T^-\}$$

where $T^0 = T_{n,\alpha}^0$ and $T^- = T_{n,\alpha}^-$ are the thresholds such that

$$P_{n,0}(X^0) = P_{n,0}(X^-) = 1 - \alpha.$$

According to the optimality of the Bayesian tests one has the inequalities

$$\beta_n(\alpha, \pi^+) \leq \int_{V_n^+} P_{n,v}(X^0)\pi^+(dv), \tag{6.5}$$

$$\beta_n(\alpha, \pi^0) \leq \int_{V_n^0} P_{n,v}(X^-)\pi^0(dv). \tag{6.6}$$

Let us observe that the sets X^0 and X^- are symmetric with respect to all permutations and changes of signs of coordinates which implies that the probabilities $P_{n,v}(X^0)$ and $P_{n,v}(X^-)$ depend only on $k_n(v)$ for $v \in V_{n,b_n}$. In particular one has

$$\beta_n(\alpha, \pi^0) = P_{n,v^0}(X^0), \quad P_{n,v_0}(X^-) = P_{n,v^0}(X^-) \tag{6.7}$$

for any $v_0 \in V_n^0$ where $v^0 = (\underbrace{b, \cdots, b}_{k}, 0, \cdots, 0)$ and $k = k_n$, $b = b_n$.

Let us show that for any $v_+ \in V_n^+$ and for any $v_- \in V_n^-$ the following inequalities hold:

$$P_{n,v_+}(X^0) \leq P_{n,v^0}(X^0), \quad P_{n.v^0}(X^-) \leq P_{n,v_-}(X^-). \tag{6.8}$$

It is enough to consider the case

$$v_+ = (\underbrace{b, \cdots, b}_{k_+}, 0, \cdots, 0), \quad v_- = (\underbrace{b, \cdots, b}_{k_-}, 0, \cdots, 0)$$

where $k_+ > k > k_-$. Note that the sets X^0 and X^- are convex as the critical sets of Bayesian tests (see Burnashev, 1979 for example) and all the coordinate cross-sections of these sets are convex and symmetric. Applying Anderson's lemma (see Ibragimov and Khasminskii, 1981) to the $(k_+ - k)$ - dimensional and to the $(k - k_-)$ - dimensional cross-sections of X^0 and X^- which correspond to the distinction between v_+ and v^0, and between v^0 and v_-, one easily obtains inequalities (6.8).

Using inequalities (6.8) and equalities (6.7) one has

$$\begin{aligned} \int_{V_n^+} P_{n,v}(X^0)\pi^+(dv) &\leq \max_{v \in V_n^+} P_{n,v}(X^0) \\ &\leq P_{n,v^0}(X^0) = \beta_n(\alpha, \pi^0), \end{aligned} \tag{6.9}$$

$$\begin{aligned} \int_{V_n^0} P_{n,v_0}(X^-)\pi^0(dv_0) &= P_{n,v^0}(X^-) \\ &\leq \min_{v \in V_n^-} P_{n,v}(X^-) \leq \beta_n(\alpha, \pi^-). \end{aligned} \tag{6.10}$$

Relations (6.5), (6.6), (6.9), (6.10) imply inequalities (6.4). Theorem 4 is proved.

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