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**Attractor properties for irreversible and reversible
interacting particle systems**

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1. INTRODUCTION

The last years have seen an interest in the analysis of lattice measures under stochastic time-evolutions, with a particular view on the possible production of singularities of such measures [4–7, 19–21]. These singularities are related to the emergence of long spatial memory in the conditional probabilities of the time-evolved measures at given transition times. When the initial measure is a Gibbs measure in a low-temperature phase for some absolutely summable potential it may happen that a time-evolved potential ceases to exist, and one speaks of a Gibbs-non Gibbs transition. Such phenomena are proved to occur on the lattice for weakly interacting Glauber dynamics, based on the detection of 'hidden phase transitions'. Suggested by mean-field analogues, singularities are expected to appear (and even more easily so) for strongly interacting reversible dynamics. While the focus of this research has been much on reversible dynamics, one expects similar singularities during time-evolution in the huge field of irreversible dynamics, see for example [23, 24], which are even harder to analyze.

This possible occurrence of non-localities in turn poses difficulties to control the large time behavior of trajectories of time-evolved measures [14]. It is the purpose of this paper to exploit the key concept of relative entropy change along trajectories in this context, including situations of multiple phases, and including situations of irreversible dynamics. Relative entropy has a huge importance in the probability theory of statistical mechanics in infinite volume, via its relevance in large deviations, via the Gibbs variational principle, see for example [2, 8, 20], and also via a new formulation to analyze Gibbs-non Gibbs transitions in terms of a variational principle in path space, see [5, 7]. Its successful use as a Lyapunov function in the context of stochastic time evolutions goes back to very early work of Holley [12, 13]. In [22] zero entropy loss is used to classify invariant states, but the more difficult issue of the behavior of trajectories for starting measures off the invariant states is not studied.

In this paper we build up on these initial steps, go beyond the reversible case, and provide also a treatment of general Glauber dynamics (which to our knowledge has not appeared in the literature).

We work in the setting of stochastic dynamics for lattice systems in the infinite volume and in continuous time. Our local state spaces are finite and the dynamics is specified by giving the rates to jump between different symbols in this alphabet. These rates depend on the initial configuration around the site at which the jump occurs, but usually they depend also on the configurations *around* the site at which the jump occurs. This makes the jump processes non-independent over the sites and creates the possibility for macroscopically non-trivial collective behavior. In all what follows we assume lattice-translation invariance for the rules specifying the dynamics, namely the possible sets of sites on which the spins are jointly updated and their rates. We will not assume that the dynamics is reversible for a particular measure.

We look at initial configurations which are chosen from lattice-translation invariant starting measures, and will then be interested in the corresponding trajectories of lattice-translation invariant infinite-volume measures. We ask for possible large time limiting behavior. In the language of dynamical systems, we want to know the omega-limit sets of the dynamics, that is the set of possible weak limit points of ν_{t_n} where t_n tends to infinity. Here the usual weak convergence is chosen in which convergence of measures is checked in terms of local observables. In particular, by compactness, there are always weak limit points. The dynamics has at least one time-stationary measure μ , which might be ergodic w.r.t. lattice translations or not. In fact we have examples for both situations. For this measure we will assume that it is even a Gibbs measure w.r.t. a quasilocal specification γ , in other words $\mu \in \mathcal{G}(\gamma)$. This is the case for Glauber dynamics, and for a class of irreversible dynamics [15]. However, there are examples of irreversible dynamics with non-Gibbsian invariant measures [3]. If there is one non-Gibbsian invariant measure, the other invariant measures must be non-Gibbsian too [22].

We want to use the relative entropy $h(\nu|\mu)$ w.r.t. to a time-stationary measure $\mu \in \mathcal{G}(\gamma)$ as a Lyapunov function to investigate trajectories and limit points. Let us note that in the uniqueness regime, $|\mathcal{G}(\gamma)| = 1$, the subject of entropy decay under time-evolutions is intimately linked to Log-Sobolev inequalities for infinite-volume measures, see [1, Chapter 5] or [10]. Proving a Log-Sobolev inequality for a non-equilibrium model implies the exponential decay of the relative entropy distance and thus gives not only the attractor property but also the rate of convergence to the unique equilibrium. We cannot use these methods here since our interest goes beyond situations of uniqueness to situations where multiple invariant measures may occur.

The difficulties using the relative entropy as a Lyapunov function in the infinite volume are caused by the potential lack of continuity. Recall that the relative entropy density $\nu \mapsto h(\nu|\mu)$ is a lower semicontinuous (l.s.c.) function in the weak topology, but in general it is not upper semicontinuous (u.s.c.). Looking at the time-derivative of the relative entropy $g(\nu_{t_n}|\mu)$, as defined in (3), along trajectories which are sampled at time instances t_n tending to infinity, we have $\lim_{n \uparrow \infty} g(\nu_{t_n}|\mu) = 0$. We would like to conclude that $g(\nu^*|\mu) = 0$ where ν^* denotes a weak limit point of the trajectory. This equation expresses zero entropy loss of the limiting measure and is in itself very useful to characterize possible limits. In interesting cases it may have multiple solutions ν^* . In many cases, and even irreversible situations as in [15], it allows to characterize ν^* by concluding that these solutions must be elements of $\mathcal{G}(\gamma)$.

Now, in order to prove $g(\nu^*|\mu) = 0$, we would have to know that $\nu \mapsto g(\nu|\mu)$ is u.s.c.. A proof that g is u.s.c. has been given in a reversible situation for the particular case of the stochastic Ising model for which the corresponding Ising Gibbs measures are reversible measures in [11, 12]. It is the prime aim of the present paper to move into the realm of non-reversibility. As our main result, we prove that g is u.s.c. also for general types of non-reversible dynamics and also in the situation of general finite state spaces. As a byproduct we also prove the attractor property for reversible dynamics w.r.t. Gibbs measures for irreducible finite state space interacting particle systems (IPS) on the lattice.

Here is a rough outline of the proof. We are looking for monotonicity in certain finite-volume approximations of g to conclude that g is u.s.c.. This is what is done in [11] successfully in the reversible two-spin situation. In our non-reversible situation there is more, and we give a useful decomposition of g into a weakly continuous term and a potentially dangerous term, the latter of which can be realized as a monotone limit in the volume. To show monotonicity in the volume of the approximating sequence of the latter term, we use Jensen's inequality for certain conditional measures on outer annuli appearing as volume differences, after things have been rewritten such that convex functions appear as integrands. In order to show that boundary terms do not spoil this picture we only need to impose as some minimal regularity the non-nullness of the limiting measure. We stress that we do not need any assumption on quasilocality along the trajectory which in many cases indeed would not hold. Even the non-nullness requirement can be dropped if an alternate zero entropy loss condition, without the boundary terms, is satisfied. In the reversible situation, where the dynamics is also assumed to be irreducible, we show that the alternate conditions are indeed satisfied, leading to the attractor property for such IPS.

Let us compare the present result with the results of an earlier paper of us [14], where we treated the same problem with a different approach, and explain why the present result is much stronger. Rather than looking at semicontinuity of g along any trajectory, as we do now, we were in that paper proving continuity of g but only along nice trajectories. In particular we proved: If a trajectory ν_{t_n} is uniformly Gibbs, meaning that for each time instant there is a Gibbsian potential Φ_n and those potentials all stay inside of a ball in the Banach space of Gibbsian potentials, we also have that a weak limit ν^* satisfies zero entropy loss $g(\nu^*|\mu) = 0$. A version of this theorem with weaker assumptions was also given, allowing for some degree of non-Gibbsianness in the measures appearing along the trajectory. The

proofs of the latter statements are simpler than the proof of the general u.s.c. result of the present paper. This explains that, even though we do not need to mention non-Gibbsian measures explicitly in the present paper, their appearance poses the main difficulty to exclude discontinuities in entropy gain in a simple way.

One particular motivation for considering the relative entropy decay under irreversible dynamics comes from a class of models we consider in [15, 17]. These models exhibit dynamical non-ergodicity, in the sense of IPS, in the presence of a unique time-stationary Gibbs measure, making rigorous a heuristics of [25]. In the analysis of a mean-field version of these rotation dynamics in [16] we were able to show the attractor property of the limiting cycle using relative entropy techniques on finite-dimensional simplexes. This proves synchronization in the sense of attractivity of macroscopically coherent rotating states. Let us mention that a similar type of synchronization is also frequently studied for other, however mostly mean-field models, for example the Kuramoto model for coupled noisy phase oscillators [9].

2. ENTROPY DECAY FOR INTERACTING PARTICLE SYSTEMS

2.1. Gibbs measures and relative entropy. Consider translation-invariant probability measures μ and ν on the configuration space $\Omega = \{1, \dots, q\}^{\mathbb{Z}^d}$ equipped with the usual product topology and the Borel sigma-algebra \mathcal{S} . For a finite set of sites $\Lambda \Subset \mathbb{Z}^d$ define the *local relative entropy* via

$$h_\Lambda(\nu|\mu) := \begin{cases} \sum_{\omega_\Lambda \in \{1, \dots, q\}^\Lambda} \nu(\omega_\Lambda) \log \frac{\nu(\omega_\Lambda)}{\mu(\omega_\Lambda)}, & \text{if } \nu \ll \mu, \\ \infty, & \text{else.} \end{cases}$$

and the *relative entropy density* via

$$h(\nu|\mu) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} h_\Lambda(\nu|\mu)$$

where Λ runs over hypercubes centered at the origin, whenever the limit exists. We use notations like $\omega_\Lambda := \{\sigma \in \Omega : \sigma_i = \omega_i \text{ for all } i \in \Lambda\}$, $\omega_\Delta \omega_{\Lambda \setminus \Delta} := \omega_\Delta \cap \omega_{\Lambda \setminus \Delta}$, $\Delta^c := \mathbb{Z}^d \setminus \Delta$ etc.

We will be interested in situations where μ is a Gibbs measures for a translation-invariant non-null quasilocal specification on Ω . A *specification* is a family $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$ of proper probability kernels $\gamma_\Lambda(\eta_\Lambda | \eta_{\Lambda^c})$ satisfying the consistency condition $\gamma_\Lambda(\gamma_\Delta(\eta_\Delta | \cdot) | \eta_{\Lambda^c}) = \gamma_\Lambda(\eta_\Delta | \eta_{\Lambda^c})$ when $\Delta \subset \Lambda$.

Definition 2.1. *The specification γ is called*

- 1 *translation invariant, if for all $\Lambda \Subset \mathbb{Z}^d$ and $i \in \mathbb{Z}^d$ we have $\gamma_{\Lambda+i}(\eta_{\Lambda+i} | \eta_{(\Lambda+i)^c}) = \gamma_\Lambda(\eta_\Lambda | \eta_{\Lambda^c})$ where $\Lambda + i$ denotes the lattice translate of Λ by i ;*
- 2 *non-null, if $\inf_\eta \gamma_0(\eta_0 | \eta_{0^c}) \geq \delta$ for some $\delta > 0$;*
- 3 *quasilocal, if for all $\Lambda \Subset \mathbb{Z}^d$, $\lim_{\Delta \uparrow \mathbb{Z}^d} \sup_{\eta, \xi} |\gamma_\Lambda(\eta_\Lambda | \eta_{\Delta \setminus \Lambda} \xi_{\Delta^c}) - \gamma_\Lambda(\eta_\Lambda | \eta_{\Lambda^c})| = 0$.*

The infinite-volume probability measure μ is called a *Gibbs measure* for γ , i.e. $\mu \in \mathcal{G}(\gamma)$, if μ satisfies the DLR equation, namely for all $\Lambda \Subset \mathbb{Z}^d$ and η_Λ we have $\mu(\gamma_\Lambda(\eta_\Lambda | \cdot)) = \mu(\eta_\Lambda)$. For details on Gibbs measures and specifications see [3, 8].

In order to guarantee existence of the relative entropy density, μ has to be *asymptotically decoupled* as defined in [20, 26]. Denote Λ_n the centered box with side-length $2n + 1$.

Definition 2.2. *A positive measure μ on (Ω, \mathcal{S}) is called asymptotically decoupled if*

- 1 *there exist $d : \mathbb{N} \mapsto \mathbb{N}$ and $c : \mathbb{N} \mapsto [0, \infty)$, such that*

$$\lim_{n \uparrow \infty} d(n)/n = 0 \text{ and } \lim_{n \uparrow \infty} c(n)/|\Lambda_n| = 0.$$

2 for all $i \in \mathbb{Z}^d$, $n \in \mathbb{N}$, $A \in \mathcal{S}$ measurable w.r.t. $\Lambda_n + i$ and $B \in \mathcal{S}$ measurable w.r.t. $(\Lambda_{n+d(n)} + i)^c$, we have

$$e^{-c(n)}\mu(A)\mu(B) \leq \mu(A \cap B) \leq e^{c(n)}\mu(A)\mu(B).$$

The following result, proved in [26, Proposition 3.2], guarantees existence of the relative entropy density w.r.t. asymptotically decoupled measures.

Lemma 2.3. *Let ν and μ be translation-invariant probability measures on (Ω, \mathcal{S}) and μ asymptotically decoupled. Then the relative entropy density $h(\nu|\mu)$ exists.*

For example specifications defined via translation-invariant absolutely summable potentials $\Phi = (\Phi_A)_{A \in \mathbb{Z}^d}$ are translation invariant, non-null and quasilocal. Gibbs measures for such *Gibbsian specifications* are moreover asymptotically decoupled and hence the relative entropy density exists.

Note that for general translation-invariant specifications without any further assumptions on locality properties, existence of an absolutely summable *translation-invariant* potential is not guaranteed, see [3, 18, 27]. This is why we are imposing asymptotic decoupledness as an additional requirement.

The equilibrium model considered in [15] provides an example of such asymptotically decoupled Gibbs measures, where the specification is a priori not given in terms of an absolutely summable translation-invariant potential. More precisely, the translation-invariant specification γ' on (Ω, \mathcal{S}) is given via

$$\gamma'_\Lambda(\omega'_\Lambda | \omega'_{\Lambda^c}) = \frac{\mu_{\Lambda^c}[\omega'_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda} \mathbf{1}_{\omega'_\Lambda}))}{\mu_{\Lambda^c}[\omega'_{\Lambda^c}](\lambda^\Lambda(e^{-H_\Lambda}))} \quad (1)$$

where $\mu_{\Lambda^c}[\omega'_{\Lambda^c}]$ is the unique continuous-spin Gibbs measure for the continuous spin XY -model with Hamiltonian $H_\Lambda = \sum_{A \cap \Lambda \neq \emptyset} \Phi_A$ on the volume Λ^c , not interacting with Λ and conditioned to a discrete configuration $\omega'_{\Lambda^c} \in \{1, \dots, q\}^{\Lambda^c}$. λ here denotes the Lebesgue measure on the one-dimensional unit sphere. Under further suitable conditions on parameter values appearing in the discussion in [15], one could construct a translation-invariant potential for γ' . Nevertheless, it is much easier to verify the condition of asymptotic decoupledness for elements of $\mathcal{G}(\gamma')$ directly from (1), more specifically with parameters $d \equiv 0$ and $c(n) = 4 \sum_{A \cap \Lambda_n \neq \emptyset, A \not\subset \Lambda_n} \|\Phi_A\|$. Moreover, γ' is again translation invariant, non-null and quasilocal.

2.2. IPS dynamics and relative entropy. Consider time-continuous, translation-invariant Markovian dynamics on Ω , namely IPS characterized by time-homogeneous generators L with domain $D(L)$ and its associated Markovian semigroup $(P_t^L)_{t \geq 0}$. For the IPS we adopt the exposition given in [23, Chapter I]. In all generality the generator L is given via jump-rates $c_\Delta(\eta, \xi_\Delta)$ in finite volumes $\Delta \in \mathbb{Z}^d$, continuous in the starting configurations $\eta \in \Omega$

$$Lf(\eta) = \sum_{\Delta \in \mathbb{Z}^d} \sum_{\xi_\Delta} c_\Delta(\eta, \xi_\Delta) [f(\xi_\Delta \eta_{\Delta^c}) - f(\eta)]. \quad (2)$$

To ensure well-definedness, the jump-measures must satisfy a number of conditions, most importantly the single-site jump-intensities have to be bounded, i.e. for $c_\Delta(\eta) := \sum_{\xi_\Delta} c_\Delta(\eta, \xi_\Delta)$ and $c_\Delta := \sup_\eta c_\Delta(\eta)$ we assume $\sum_{\Delta \ni 0} c_\Delta < \infty$. In fact the definition of L in (2) should be read in such a way that the two summations are only over those Δ and ξ_Δ with $c_\Delta(\eta, \xi_\Delta) > 0$. We will call an IPS well-defined if it is well-defined in the sense of [23, Chapter I].

The following additional conditions on IPS will be used in the sequel.

Definition 2.4. *Let $Lf(\eta) = \sum_{\Delta \in \mathbb{Z}^d} \sum_{\xi_\Delta} c_\Delta(\eta, \xi_\Delta) [f(\xi_\Delta \eta_{\Delta^c}) - f(\eta)]$ be a well-defined translation-invariant IPS. We say that*

- 1 for L there are only finitely many types of transitions, if there exists a finite set $0 \in \Gamma \subseteq \mathbb{Z}^d$ such that $c_\Delta = 0$ if $0 \in \Delta \not\subseteq \Gamma$;
- 2 for L the rates are uniformly continuous, if $\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{\Delta \ni 0} \sup_{\eta, \xi, \sigma} |c_\Delta(\eta_\Lambda \xi_{\Lambda^c}, \sigma_\Delta) - c_\Delta(\eta, \sigma_\Delta)| = 0$;
- 3 L can not enter trap states, if for all $\eta \in \{1, \dots, q\}^{\mathbb{Z}^d}$ and $\sigma \in \{1, \dots, q\}^\Delta$ we have that $c_\Delta(\eta, \sigma_\Delta) > 0$ implies that $c_\Delta(\sigma_\Delta \eta_{\Delta^c}) > 0$;
- 4 L has a minimal transition rate, if for all $\Delta \subseteq \mathbb{Z}^d$ with $c_\Delta > 0$ we have $\inf_{\eta, \sigma_\Delta: c_\Delta(\eta, \sigma_\Delta) > 0} c_\Delta(\eta, \sigma_\Delta) > 0$.

Examples of IPS satisfying the above conditions are the stochastic Ising model or more general Glauber dynamics. As another example consider the exclusion process on $\{0, 1\}^{\mathbb{Z}^d}$ with rates

$$c_{\{x,y\}}(\eta; (1 - \eta_x), (1 - \eta_y)) = p(x, y)\eta_x(1 - \eta_y) + p(y, x)\eta_y(1 - \eta_x).$$

Here $p(x, y)$ describes the possibly non-symmetric rate of moving a particle from x to site y . Such processes are contained in the class of IPS satisfying above conditions as long as $p(x, y) > 0$ implies $p(y, x) > 0$. Contact processes or voter models have trap states and thus our approach can not be applied.

In this paper we want to consider models given in terms of translation-invariant non-null quasilocal specifications γ and transformations given by translation-invariant IPS dynamics. Recall the notion of non-nullness for probability measures.

Definition 2.5. *A random field ν is non-null if there exists $\delta > 0$ and a version of the single-site conditional probabilities such that $\nu(\eta_0 | \eta_{0^c}) \geq \delta$ for ν -a.a. η .*

For any translation invariant probability measure ν , let us define the *relative entropy loss density* via

$$g_L(\nu | \mu) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \frac{d}{dt} \Big|_{t=0} h_\Lambda(P_t^L \nu | \mu) \quad (3)$$

where Λ runs over hypercubes centered at the origin, whenever the limit exists. As we will explain in more detail in the next subsection, the limit exists if ν is non-null. In particular in that case $g_L(\nu | \mu) \leq 0$ as proved in the following lemma.

Lemma 2.6. *Let ν be a translation-invariant and non-null probability measure and L a well-defined translation-invariant IPS generator with finitely many types of transitions and uniformly continuous rates. Assume that for L there exists a translation-invariant asymptotically-decoupled time-stationary Gibbs measure $\mu \in \mathcal{G}(\gamma)$ where γ is translation-invariant non-null and quasilocal. Then $g_L(\nu | \mu) \leq 0$.*

Let us note, that the relative entropy density w.r.t. translation-invariant probability measures is a non-increasing function under rather general transformations see for example [3, Lemma 3.3]. We will assume that under the dynamics the following *zero entropy loss condition* holds:

Condition 2.7. *Let the well-defined IPS dynamics L be such that there exists a translation-invariant asymptotically-decoupled time-stationary Gibbs measure $\mu \in \mathcal{G}(\gamma)$ where γ is translation-invariant non-null and quasilocal. Further, for any translation-invariant measure ν with $g_L(\nu | \mu) = 0$ it follows that $\nu \in \mathcal{G}(\gamma)$.*

Without the time derivative this condition is one direction of the Gibbs variational principle, see for example [8, Theorem 15.37]. All conditions given in Definition 2.4 plus the above Condition 2.7 involving the time-derivative is proved to hold for example for the stochastic Ising model in [12, 13, 23] or more general Glauber dynamics and even non-reversible rotation dynamics see [15]. It is also satisfied for the symmetric exclusion process if we assume that ν and μ have the same particle density see [14].

In the following subsection we give a representation of $g_L(\nu|\mu)$ and state our main result about the attractor property for IPS.

2.3. A representation of the relative entropy loss density. If μ is a translation-invariant Gibbs measure for the Gibbsian specification γ^Φ , then the relative entropy density $h(\nu|\mu)$ is just the *free energy per site* of ν with respect to the absolutely summable potential Φ , i.e.

$$h(\nu|\mu) = \int \nu(d\omega) \sum_{A \ni 0} \frac{1}{|A|} \Phi_A(\omega) - \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} h_\Lambda(\nu) + P(\Phi). \quad (4)$$

Here the first summand is the *specific energy*, the second summand is the *specific entropy* with

$$h_\Lambda(\nu) := - \sum_{\omega_\Lambda \in \{1, \dots, q\}^\Lambda} \nu(\omega_\Lambda) \log \nu(\omega_\Lambda)$$

and $P(\Phi)$ is a constant often referred to as the *pressure*. For details see for example [8, Theorem 15.30].

A similar decomposition can be given also for the relative entropy loss density $g_L(\nu|\mu)$ if μ is a translation-invariant asymptotically-decoupled time-stationary Gibbs measure for the translation-invariant non-null quasilocal specifications γ . Let us define the *specific entropy loss* by

$$g_L(\nu) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{\omega_\Lambda \in \{1, \dots, q\}^\Lambda} \nu(L1_{\omega_\Lambda}) \log \nu(\omega_\Lambda)$$

whenever the limit exists. In [14, Proposition 2.7] we give a representation of $g_L(\nu)$ for general well-defined dynamics L and ν assumed to be non-null of the following form

$$g_L(\nu) = \sum_{\Delta \ni 0} \sum_{\xi_\Delta} \int \nu(d\eta) c_\Delta(\eta, \xi_\Delta) \frac{1}{|\Delta|} \log \frac{\nu(\xi_\Delta | \eta_{\Delta^c})}{\nu(\eta_\Delta | \eta_{\Delta^c})}$$

and derive condition under which $\nu \mapsto g_L(\nu)$ is weakly continuous. Let us further define the *specific energy loss* by

$$\rho_L(\nu, \mu) := - \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{\omega_\Lambda \in \{1, \dots, q\}^\Lambda} \nu(L1_{\omega_\Lambda}) \log \mu(\omega_\Lambda)$$

whenever it exists and note that

$$g_L(\nu|\mu) = \rho_L(\nu, \mu) + g_L(\nu)$$

if the right hand side is well-defined. Observe the analogy to the first two terms on the right hand side of (4). In [14, Equation 4] we give a representation of $\rho_L(\nu, \mu)$ for general well-defined dynamics L and μ being a Gibbs measure for a Gibbsian specification with potential Φ of the form

$$\rho_L(\nu, \mu) = \sum_{\Delta \ni 0} \sum_{\zeta_\Delta} \int \nu(d\eta) c_\Delta(\eta, \zeta_\Delta) \frac{1}{|\Delta|} \sum_{A \cap \Delta \neq \emptyset} [\Phi_A(\zeta_\Delta \eta_{\Delta^c}) - \Phi_A(\eta)]$$

and show continuity of $\nu \mapsto \rho_L(\nu, \mu)$ w.r.t. the weak topology.

Let us present here a generalization of the representation of $\rho_L(\nu, \mu)$ for cases where μ is a Gibbs measure for a quasilocal specification which is not necessarily coming from an absolutely summable translation-invariant potential.

Proposition 2.8. *Let L be a well-defined IPS and μ a translation-invariant Gibbs measure for the translation-invariant non-null quasilocal specifications γ . Then*

$$\rho_L(\nu, \mu) = \sum_{\Delta \ni 0} \sum_{\xi_\Delta} \int \nu(d\eta) c_\Delta(\eta, \xi_\Delta) \frac{1}{|\Delta|} \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})} \quad (5)$$

and $\nu \mapsto \rho_L(\nu, \mu)$ is continuous w.r.t. the weak topology.

We come to our main result which states the existence and upper semicontinuity of $\nu \mapsto g_L(\nu)$ in ν if ν is non-null. The approach is inspired by the works [11, 12] which only deal with the case of the stochastic Ising model.

Theorem 2.9. *Let L be a well-defined translation-invariant IPS with finitely many types of transitions, where the rates are uniformly continuous and have a minimal transition rate and L can not enter trap states. Then $\nu \mapsto g_L(\nu)$ exists and is upper semicontinuous in ν if ν is non-null and translation-invariant.*

Under the zero entropy loss Condition 2.7 this implies the attractor property of the set of translation-invariant Gibbs measures.

Corollary 2.10. *Let L be a well-defined translation-invariant IPS with finitely many types of transitions, where the rates are uniformly continuous and have a minimal transition rate. Also assume that L can not enter trap states and satisfies Condition 2.7 with time-stationary $\mu \in \mathcal{G}(\gamma)$. Then, for any translation-invariant starting measure ν where the sequence $(P_{t_n}^L \nu)_{n \in \mathbb{N}}$ consists of non-null probability measures and converges weakly to the non-null probability measure ν_* as $t_n \uparrow \infty$, $\nu_* \in \mathcal{G}(\gamma)$.*

Let us remark that the non-nullness condition is stronger than necessary for the proof of the above results. What is really needed is that

$$\sup_{\Lambda \ni 0} \sum_{\Delta \ni 0} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta(\eta, \sigma_\Delta) \log \frac{\nu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\nu(\sigma_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} < \infty$$

which is implied if ν is non-null. In the next subsection we show that the non-nullness requirement can be dropped if the zero entropy loss Condition 2.7 is replaced by an approximating zero entropy loss condition. Moreover we prove that this approximating condition is satisfied if the time-stationary measure μ is even reversible for L . This in particular implies the attractor property for reversible dynamics.

2.4. Avoiding the non-nullness condition and the attractor property for reversible dynamics.

Imposing non-nullness for the measure ν in $g_L(\nu)$ as well as for μ in $\rho_L(\nu, \mu)$ is a way to avoid degeneracies which could lead for example to $g_L(\nu)$ being minus infinity or $\rho_L(\nu, \mu)$ being infinity. Requiring the specification γ for the Gibbs measure μ to be non-null is not a strong condition, since it is satisfied for example for every Gibbsian specification. It is natural to believe that under dynamics which have a non-null Gibbs measure as a time-stationary measure and satisfy the additional conditions given in Definition 2.4, also measures propagated by the dynamics should be non-null for positive times. But we could not prove it.

Consider cubes of the form $\Lambda_n := [-2^n + 1, 2^n - 1]^d$ and $\tilde{\Lambda}_n := [-2^n + n + 1, 2^n - n - 1]^d$ and the approximated specific entropy loss given by $\tilde{g}_L(\nu) := \lim_{n \uparrow \infty} \tilde{g}_L^n(\nu)$ whenever the limit exists, where

$$\tilde{g}_L^n(\nu) := \frac{1}{|\Lambda_n|} \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta(\eta, \sigma_\Delta) \frac{1}{|\Delta|} \log \frac{\nu(\sigma_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\nu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}$$

and L is assumed to be well-defined. Further define the *approximated relative entropy loss* as $\tilde{g}_L(\nu|\mu) := \tilde{g}_L(\nu) + \rho_L(\nu, \mu)$. Working around the argument where we used non-nullness in the proof of Theorem 2.9 we can show the following.

Theorem 2.11. *Let L be a well-defined translation-invariant IPS with finitely many types of transitions, where the rates are uniformly continuous and have a minimal transition rate and L can not enter trap states. Then $\tilde{g}_L(\nu)$ exists, $\tilde{g}_L(\nu) \geq g_L(\nu)$ and $\nu \mapsto \tilde{g}_L(\nu)$ is upper semicontinuous.*

Let us assume that under the dynamics the following *approximated zero entropy loss condition* holds:

Condition 2.12. *Let the well-defined IPS dynamics L be such that there exists a translation-invariant asymptotically-decoupled time-stationary Gibbs measure $\mu \in \mathcal{G}(\gamma)$ where γ is translation-invariant non-null and quasilocal. Further, for any translation-invariant measure ν we have existence of $\tilde{g}_L(\nu|\mu) \leq 0$ and the following property: $\tilde{g}_L(\nu|\mu) = 0$ implies that $\nu \in \mathcal{G}(\gamma)$.*

Under the approximated zero entropy loss condition we can prove the attractor property avoiding a non-nullness requirement.

Corollary 2.13. *Let L be a well-defined translation-invariant IPS with finitely many types of transitions, where the rates are uniformly continuous and have a minimal transition rate. Further assume that L can not enter trap states and satisfies Condition 2.12 with time-stationary measure $\mu \in \mathcal{G}(\gamma)$. Then, for any translation-invariant starting measure ν where the sequence $(P_{t_n}^L \nu)_{n \in \mathbb{N}}$ converges weakly to ν_* as $t_n \uparrow \infty$, $\nu_* \in \mathcal{G}(\gamma)$.*

Finally we show that Condition 2.12 can be verified if μ is a reversible measure for L and the requirement that L has no trap states is replaced by the following stronger assumption of irreducibility.

Definition 2.14. *Let $Lf(\eta) = \sum_{\Delta \in \mathbb{Z}^d} \sum_{\xi_\Delta} c_\Delta(\eta, \xi_\Delta) [f(\xi_\Delta \eta_{\Delta^c}) - f(\eta)]$ be a well-defined translation-invariant IPS. We say that L is irreducible, if for all $\eta^{(0)} \in \Omega$ and $\sigma \in \{1, \dots, q\}^\Delta$ with $\Delta \in \mathbb{Z}^d$ there exists a finite sequence of configurations $\{\eta^{(1)}, \dots, \eta^{(n)}\}$ with $\eta^{(i)} \in \Omega$ and $\eta^{(n)} = \eta_{\Delta^c}^{(0)} \sigma_\Delta$ such that the transition rates to jump from $\eta^{(i-1)}$ to $\eta^{(i)}$ are positive for all $i \in \{1, \dots, n\}$.*

The following proposition together with Corollary 2.13 implies the attractor property for reversible dynamics.

Proposition 2.15. *Let L be a well-defined translation-invariant irreducible IPS with finitely many types of transitions, where the rates are uniformly continuous and have a minimal transition rate. Further let $\mu \in \mathcal{G}(\gamma)$ be a translation-invariant asymptotically-decoupled Gibbs measure and γ translation-invariant non-null and quasilocal. If μ is reversible w.r.t. L then $\tilde{g}_L(\nu|\mu)$ exists and $\tilde{g}_L(\nu) \leq 0$. Further, the assumption $\tilde{g}_L(\nu|\mu) = 0$ implies that $\nu \in \mathcal{G}(\gamma)$.*

3. PROOFS

3.1. Proof of Lemma 2.6. The proof is based on a finite-volume argument for an approximating dynamics, using Jensen's inequality. Consider the approximating finite-volume process L_Δ

$$L_\Delta 1_{\omega_\Lambda}(\eta_\Lambda) = \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\xi_\Delta} c_\Delta(\eta_\Lambda, \xi_\Delta) [1_{\omega_\Lambda}(\xi_\Delta \eta_{\Lambda \setminus \Delta}) - 1_{\omega_\Lambda}(\eta_\Lambda)]$$

where the approximating rates are defined by $c_\Delta(\eta_\Lambda, \xi_\Delta) = \int \mu(d\sigma | \eta_\Lambda) c_\Delta(\eta_\Lambda \sigma_{\Lambda^c}, \xi_\Delta)$ with μ the time-stationary Gibbs measure for L . Note that also L_Δ is well-defined. This construction in particular

implies, that μ as a measure on $\{1, \dots, q\}^\Lambda$ is invariant w.r.t. L_Λ . Indeed, for every ω_Λ we have

$$\begin{aligned} \mu(L_\Lambda 1_{\omega_\Lambda}) &= \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\xi_\Delta} \sum_{\eta_\Lambda} \mu(\eta_\Lambda) c_\Delta(\eta_\Lambda, \xi_\Delta) [1_{\omega_\Lambda}(\xi_\Delta \eta_{\Lambda \setminus \Delta}) - 1_{\omega_\Lambda}(\eta_\Lambda)] \\ &= \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\xi_\Delta} \int \mu(d\eta) c_\Delta(\eta, \xi_\Delta) [1_{\omega_\Lambda}(\xi_\Delta \eta_{\Lambda \setminus \Delta}) - 1_{\omega_\Lambda}(\eta_\Lambda)] = 0. \end{aligned}$$

Let $(P_t^{L_\Lambda})_{t \geq 0}$ denote the semigroup associated to L_Λ , then by Jensen's inequality applied to the non-positive concave function $\Psi(u) := -u \log u + u - 1$ we have

$$h_\Lambda(P_t^{L_\Lambda} \nu | \mu) = - \sum_{\eta_\Lambda} \mu(\eta_\Lambda) \Psi\left(\frac{P_t^{L_\Lambda} \nu(\eta_\Lambda)}{\mu(\eta_\Lambda)}\right) \leq - \sum_{\eta_\Lambda} \mu(\eta_\Lambda) \Psi\left(\frac{\nu(\eta_\Lambda)}{\mu(\eta_\Lambda)}\right) = h_\Lambda(\nu | \mu).$$

This is a standard argument for finite Markov processes, see for example [8, Theorem 3.A3]. Consequently the derivative

$$\frac{d}{dt} \Big|_{t=0} h_\Lambda(P_t^{L_\Lambda} \nu | \mu) = \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\xi_\Delta} \sum_{\eta_\Lambda} \nu(\eta_\Lambda) c_\Delta(\eta_\Lambda, \xi_\Delta) \log \frac{\nu(\eta_\Lambda) \mu(\xi_{\Delta \cap \Lambda} \eta_{\Lambda \setminus \Delta})}{\nu(\xi_{\Delta \cap \Lambda} \eta_{\Lambda \setminus \Delta}) \mu(\eta_\Lambda)}$$

must be non-positive. What remains to show, is that the approximation of the dynamics is of boundary order. But this is the case, we have

$$\begin{aligned} & \left| \frac{d}{dt} \Big|_{t=0} h_\Lambda(P_t^L \nu | \mu) - \frac{d}{dt} \Big|_{t=0} h_\Lambda(P_t^{L_\Lambda} \nu | \mu) \right| \\ & \leq \sup_{\Delta \ni 0} \sup_{\eta, \xi, \sigma} |c_\Delta(\eta_\Lambda \sigma_{\Lambda^c}, \xi_\Delta) - c_\Delta(\eta, \xi_\Delta)| \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\xi_\Delta} \int \nu(d\eta) \left| \log \frac{\nu(\eta_\Lambda) \mu(\xi_{\Delta \cap \Lambda} \eta_{\Lambda \setminus \Delta})}{\nu(\xi_{\Delta \cap \Lambda} \eta_{\Lambda \setminus \Delta}) \mu(\eta_\Lambda)} \right| \end{aligned}$$

where $\sup_{\Delta \ni 0} \sup_{\eta, \xi, \sigma} |c_\Delta(\eta_\Lambda \sigma_{\Lambda^c}, \xi_\Delta) - c_\Delta(\eta, \xi_\Delta)|$ tends to zero as Λ tends to \mathbb{Z}^d by the uniform continuity condition on the rates. Note that by the chain rule of conditional expectations as well as the non-nullness condition on ν and μ we have

$$\frac{\nu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta}) \mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\nu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \leq \frac{1}{\nu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \leq \frac{1}{\delta^{2|\Delta|}}$$

for some $\delta > 0$. This implies

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\xi_\Delta} \int \nu(d\eta) \log \frac{\nu(\eta_\Lambda) \mu(\xi_{\Delta \cap \Lambda} \eta_{\Lambda \setminus \Delta})}{\nu(\xi_{\Delta \cap \Lambda} \eta_{\Lambda \setminus \Delta}) \mu(\eta_\Lambda)} \leq 2 \log \frac{1}{\delta} \sum_{\Delta \ni 0} q^{|\Delta|} < \infty.$$

Using the exact same argument, we also have

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\xi_\Delta} \int \nu(d\eta) \log \frac{\nu(\xi_{\Delta \cap \Lambda} \eta_{\Lambda \setminus \Delta}) \mu(\eta_\Lambda)}{\nu(\eta_\Lambda) \mu(\xi_{\Delta \cap \Lambda} \eta_{\Lambda \setminus \Delta})} \leq 2 \log \frac{1}{\delta} \sum_{\Delta \ni 0} q^{|\Delta|} < \infty.$$

This finishes the proof. \square

Let us remark, that instead of imposing the condition that L has only finitely many types of transitions, a similar proof can be given if L has the minimal transition rate property.

3.2. Proof of Proposition 2.8. The finite-volume specific energy loss is given by

$$\begin{aligned} \rho_L^\Lambda(\nu, \mu) &:= - \frac{1}{|\Lambda|} \sum_{\sigma_\Lambda} \nu(L 1_{\sigma_\Lambda}) \log \mu(\sigma_\Lambda) \\ &= \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \int \nu(d\eta) \sum_{\Delta \ni i} \int \frac{1}{|\Delta|} c(\eta, d\xi_\Delta) \log \frac{\mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \end{aligned}$$

On the other hand by translation-invariance the r.h.s. of (5) can be written as

$$\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \int \nu(d\eta) \sum_{\Delta \ni i} \int \frac{1}{|\Delta|} c(\eta, d\xi_\Delta) \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})} =: R_L(\nu, \mu).$$

Thus the finite-volume difference can be expressed as

$$\begin{aligned} R_L(\nu, \mu) - \frac{1}{|\Lambda|} \sum_{\omega_\Lambda} \nu(L1_{\omega_\Lambda}) \log \mu(1_{\omega_\Lambda}) \\ = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \int \nu(d\eta) \int c(\eta, d\xi_\Delta) \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \end{aligned} \quad (6)$$

and it suffices to show that this difference tends to zero as $\Lambda \uparrow \mathbb{Z}^d$. For any fixed $\Delta \in \mathbb{Z}^d$ with $\Delta \subset \Lambda$ we can estimate

$$\begin{aligned} \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_\Delta | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_\Delta | \eta_{\Lambda \setminus \Delta})} &= \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\gamma_\Lambda(\xi_\Delta \eta_{\Lambda \setminus \Delta} | \sigma_{\Lambda^c}))}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\gamma_\Lambda(\eta_\Delta \eta_{\Lambda \setminus \Delta} | \tilde{\sigma}_{\Lambda^c}))} \\ &= \frac{\mu\left(\frac{\gamma_\Delta(\xi_\Delta | \eta_{\Lambda \setminus \Delta} \sigma_{\Lambda^c})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})} \gamma_\Lambda(\eta_{\Lambda \setminus \Delta} | \sigma_{\Lambda^c})\right)}{\mu\left(\frac{\gamma_\Delta(\eta_\Delta | \eta_{\Lambda \setminus \Delta} \tilde{\sigma}_{\Lambda^c})}{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c})} \gamma_\Lambda(\eta_{\Lambda \setminus \Delta} | \tilde{\sigma}_{\Lambda^c})\right)} \leq \frac{\sup_{\xi, \eta, \sigma} \frac{\gamma_\Delta(\xi_\Delta | \eta_{\Lambda \setminus \Delta} \sigma_{\Lambda^c})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})}}{\inf_{\eta, \tilde{\sigma}} \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Lambda \setminus \Delta} \tilde{\sigma}_{\Lambda^c})}{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c})}}. \end{aligned}$$

By the chain rule for conditional probabilities and the non-nullness assumption we have $\inf_\eta \gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \geq \delta^{|\Delta|}$ and hence

$$\begin{aligned} 1 - \delta^{-|\Delta|} |\gamma_\Delta(\xi_\Delta | \eta_{\Lambda \setminus \Delta} \sigma_{\Lambda^c}) - \gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})| &\leq \frac{\gamma_\Delta(\xi_\Delta | \eta_{\Lambda \setminus \Delta} \sigma_{\Lambda^c})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})} \\ &\leq 1 + \delta^{-|\Delta|} |\gamma_\Delta(\xi_\Delta | \eta_{\Lambda \setminus \Delta} \sigma_{\Lambda^c}) - \gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})| \end{aligned} \quad (7)$$

where left and right hand side tend to one by the quasilocality assumption on the specification uniformly in the configurations.

For any $\Gamma, \Theta \in \mathbb{Z}^d$ we can split the sum in (6) and write

$$\begin{aligned} &\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \int \nu(d\eta) \int c(\eta, d\xi_\Delta) \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \\ &= \frac{1}{|\Lambda|} \sum_{i \in \Lambda: \Gamma + i \subset \Lambda} \sum_{\Delta \ni i, \Delta \subset \Theta + i} \frac{1}{|\Delta|} \int \nu(d\eta) \int c(\eta, d\xi_\Delta) \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \\ &\quad + \frac{1}{|\Lambda|} \sum_{i \in \Lambda: \Gamma + i \subset \Lambda} \sum_{\Delta \ni i, \Delta \not\subset \Theta + i} \frac{1}{|\Delta|} \int \nu(d\eta) \int c(\eta, d\xi_\Delta) \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \\ &\quad + \frac{1}{|\Lambda|} \sum_{i \in \Lambda: \Gamma + i \not\subset \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \int \nu(d\eta) \int c(\eta, d\xi_\Delta) \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \\ &=: I + II + III. \end{aligned}$$

For the boundary term *III* we can use

$$\left| \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \right| \leq (|\Delta| + |\Delta \cap \Lambda|) \log \frac{1}{\delta} \leq 2|\Delta| \log \frac{1}{\delta}$$

and estimate

$$|III| \leq \frac{1}{|\Lambda|} \sum_{i \in \Lambda: \Gamma + i \not\subset \Lambda} 2 \log \frac{1}{\delta} \sum_{\Delta \ni 0} c_\Delta \leq \frac{\#\{i \in \Lambda : \Gamma + i \not\subset \Lambda\}}{|\Lambda|} 2 \log \frac{1}{\delta} \sum_{\Delta \ni 0} c_\Delta$$

which tends to zero for $\Lambda \uparrow \mathbb{Z}^d$. For the error term arising from the truncation of the rates represented by II , pick Θ such that $\sum_{\Delta \ni 0, \Delta \not\subset \Theta} c_\Delta < \varepsilon$. As a consequence we have

$$|II| \leq \log \frac{1}{\delta^2} \sum_{\Delta \ni 0, \Delta \not\subset \Theta} c_\Delta < 2\varepsilon \log \frac{1}{\delta}$$

by the same estimate as for III . Finally for the bulk term I we can pick $\Gamma(\Theta)$ such that, using the estimate (7) for finitely many finite sets $\Delta \subset \Theta$,

$$\max_{\Delta \ni 0, \Delta \subset \Theta} \left| \int \nu(d\eta) \int c(\eta, d\xi_\Delta) \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})} \right| < \varepsilon$$

for all $\Gamma(\Theta) \subset \Lambda - i$. Hence

$$\begin{aligned} |I| &= \frac{1}{|\Lambda|} \sum_{i \in \Lambda: \Gamma(\Theta) \subset \Lambda - i} \sum_{\Delta \ni 0, \Delta \subset \Theta} \frac{1}{|\Delta|} \left| \int \nu(d\eta) \int c(\eta, d\xi_\Delta) \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c}) \mu(\xi_\Delta | \eta_{\Lambda \setminus \Delta})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c}) \mu(\eta_\Delta | \eta_{\Lambda \setminus \Delta})} \right| \\ &\leq \varepsilon \sum_{\Delta \ni 0, \Delta \subset \Theta} \frac{1}{|\Delta|} = \text{Const } \varepsilon. \end{aligned}$$

This finishes the representation part of the proof.

For the continuity let $\Gamma \Subset \mathbb{Z}^d$ then

$$\begin{aligned} \rho_L(\nu, \mu) &= \int \nu(d\eta) \sum_{\Delta \ni 0, \Delta \subset \Gamma} \int c(\eta, d\xi_\Delta) \frac{1}{|\Delta|} \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})} \\ &\quad + \int \nu(d\eta) \sum_{\Delta \ni 0, \Delta \not\subset \Gamma} \int c(\eta, d\xi_\Delta) \frac{1}{|\Delta|} \log \frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c})}{\gamma_\Delta(\xi_\Delta | \eta_{\Delta^c})} =: \rho_L^\Gamma(\nu) + \rho_L^{\Gamma^c}(\nu) \end{aligned}$$

and the maps $\nu \mapsto \rho_L^\Gamma(\nu)$ is weakly continuous as a finite sum of weakly continuous functions by the continuity of the rates and the quasilocality of the specification. The second summand can be bounded from above and below by

$$-\log \frac{1}{\delta} \sum_{\Delta \ni 0, \Delta \not\subset \Gamma} c_\Delta \leq \rho_L^{\Gamma^c}(\nu) \leq \log \frac{1}{\delta} \sum_{\Delta \ni 0, \Delta \not\subset \Gamma} c_\Delta$$

which can be made arbitrarily small since we assumed $\sum_{\Delta \ni 0} c_\Delta < \infty$. Thus $\rho_L(\nu, \mu)$ is continuous as a uniform limit of continuous functions. \square

3.3. Proof of Theorem 2.9. The strategy of the proof is the following: We consider the entropy loss before the volume limit $\Lambda \uparrow \mathbb{Z}^d$ and eliminate boundary terms in the summation over sites in Λ . The bulk summation can be written as a sum of two terms, where additional rates are included in the logarithm in such a way, that the entropy loss appears like a new relative entropy. The compensation term is continuous and can be ignored. For a sequence of finite boxes with exponentially growing size, the new relative entropy can be approximated by a non-increasing sequence of continuous functions which in the volume limit gives the upper semicontinuity. The crucial ingredient for the monotonicity is to subdivide given boxes into congruent subboxes, apply Jensen's inequality and use translation invariance.

For convenience let us write $c_\Delta^{\sigma_\Delta}(\eta) := c_\Delta(\eta, \sigma_\Delta)$ and recall $c_\Delta(\eta) := \sum_{\sigma_\Delta} c_\Delta^{\sigma_\Delta}(\eta)$, $c_\Delta := \sup_\eta c_\Delta(\eta)$. The finite-volume unnormalized entropy loss is given by

$$g_L^\Lambda(\nu) := \sum_{\sigma_\Lambda} \nu(L1_{\sigma_\Lambda}) \log \nu(\sigma_\Lambda) = \sum_{i \in \Lambda} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{\nu(\sigma_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}{\nu(\eta_{\Delta \cap \Lambda} | \eta_{\Lambda \setminus \Delta})}$$

and note that by non-nullness of ν and the chain rule for conditional measures,

$$-|\Lambda| \log \frac{1}{\delta} \sum_{\Delta \ni 0} c_\Delta \leq g_L^\Lambda(\nu) \leq |\Lambda| \log \frac{1}{\delta} \sum_{\Delta \ni 0} c_\Delta. \quad (8)$$

Consider cubes of the form $\Lambda_n := [-2^n + 1, 2^n - 1]^d$ and $\tilde{\Lambda}_n := [-2^n + n + 1, 2^n - n - 1]^d$. Further consider 2^d disjoint and congruent subcubes $\Delta_{n,k}$ of Λ_n with total side length $2^n - 1$ as well as 2^d disjoint congruent subcubes $\tilde{\Delta}_{n,k}$ of $\tilde{\Lambda}_n$ with total side length $2^n - n - 1$. Let the subcubes be centered such that $\tilde{\Delta}_{n,k} \subset \Delta_{n,k}$. Moreover we will consider balls w.r.t. the Euclidian norm $B_n(i) := \{j \in \mathbb{Z}^d : |i - j| \leq n\}$. Let us take away boundary terms of the i -summation in $g_L^\Lambda(\nu)$ and define

$$\tilde{g}_L^{\Lambda_n}(\nu) = \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{\nu(\sigma_\Delta \cap \Lambda_n | \eta_{\Lambda_n \setminus \Delta})}{\nu(\eta_{\Delta \cap \Lambda_n} | \eta_{\Lambda_n \setminus \Delta})}.$$

Note that the error $|g_L^{\Lambda_n}(\nu) - \tilde{g}_L^{\Lambda_n}(\nu)|$ is of boundary order $o(|\Lambda_n|)$ which is immediate from equation (8). Let us rewrite $\tilde{g}_L^{\Lambda_n}(\nu)$ as a sum of two terms

$$\begin{aligned} \tilde{g}_L^{\Lambda_n}(\nu) &= - \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{\nu(\eta_{\Lambda_n}) q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\eta)}{\nu(\sigma_\Delta \cap \Lambda_n | \eta_{\Lambda_n \setminus \Delta}) c_\Delta(\eta_{\Delta^c \sigma_\Delta})} \\ &\quad + \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c \sigma_\Delta})} =: s_n(L, \nu) + r_n(L, \nu) \end{aligned} \quad (9)$$

where the well-definedness of $r_n(L, \nu)$ is guaranteed by the no-trap Condition 3 in Definition 2.4. By translation invariance the density of the second summand is given by

$$\lim_{n \uparrow \infty} \frac{1}{|\Lambda_n|} r_n(L, \nu) = \sum_{\Delta \ni 0} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c \sigma_\Delta})} =: r(L, \nu)$$

and $\nu \mapsto \langle \nu, L \rangle$ is continuous by the continuity of the rates and the finite-range property of L . Thus it suffices to show upper semicontinuity for $s_n(L, \nu)$.

Instead of $s_n(L, \nu)$ we want to consider an approximation by using suitable rate truncations and expressing the integral as a sum over suitable concave functions

$$\begin{aligned} f_n(\nu) &:= \frac{1}{q^{|\Delta|}} \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \sum_{\eta_{\Lambda_n}} \nu(\sigma_\Delta \cap \Lambda_n | \eta_{\Lambda_n \setminus \Delta}) \tilde{c}_\Delta(\eta_{B_{n-1}(i)} \setminus \Delta | \sigma_{B_{n-1}(i)} \cap \Delta) \\ &\quad \times \Psi \left[\frac{1}{\nu(\sigma_\Delta \cap \Lambda_n | \eta_{\Lambda_n \setminus \Delta})} \int \nu(d\xi) 1_{\eta_{\Lambda_n}}(\xi) \frac{q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c \sigma_\Delta})} \right]. \end{aligned}$$

Here the truncated rates $\tilde{c}_\Delta^{\sigma_\Delta}(\eta_{B_{n-1}(i)}) := \inf_\xi c_\Delta^{\sigma_\Delta}(\eta_{B_{n-1}(i)} \xi_{B_{n-1}(i)^c})$ depend only on the sites in $B_{n-1}(i) \subset \Lambda_n$, $\tilde{c}_\Delta(\eta_{B_{n-1}(i)}) := \sum_{\sigma_\Delta} \tilde{c}_\Delta^{\sigma_\Delta}(\eta_{B_{n-1}(i)})$ and $\Psi(u) := -u \log u + u - 1$ is a non-positive concave function. We will show that

- 1 $\lim_{n \uparrow \infty} |\Lambda_n|^{-1} f_n(\nu) = f(\nu)$ exists and is upper semicontinuous by an application of Jensen's inequality w.r.t. a partial summation and
- 2 the error $s_n(L, \nu) - f_n(\nu)$ is of boundary order.

This gives the upper semicontinuity of $\nu \mapsto g_L(\nu)$.

Step 1: First note since Ψ is non-positive and $\bigcup_k \tilde{\Delta}_{n,k} \subsetneq \tilde{\Lambda}_n$, by dropping some terms in the sum we have the inequality

$$\begin{aligned}
f_n(\nu) &\leq \frac{1}{q^{|\Delta|}} \sum_{j=1}^{2^d} \sum_{i \in \tilde{\Delta}_{n,j}} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \sum_{\eta_{\Lambda_n}} \nu(\sigma_{\Delta \cap \Lambda_n} \eta_{\Lambda_n \setminus \Delta}) \tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_{B_{n-1}(i) \cap \Delta}) \\
&\quad \times \Psi \left[\frac{1}{\nu(\sigma_{\Delta \cap \Lambda_n} \eta_{\Lambda_n \setminus \Delta})} \int \nu(d\xi) 1_{\eta_{\Lambda_n}}(\xi) \frac{q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c} \sigma_\Delta)} \right] \\
&= \frac{1}{q^{|\Delta|}} \sum_{j=1}^{2^d} \sum_{i \in \tilde{\Delta}_{n,j}} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \sum_{\eta_{\Delta_{n,j}}} \tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_{B_{n-1}(i) \cap \Delta}) \sum_{\eta_{\Lambda_n \setminus \Delta_{n,j}}} \nu(\sigma_{\Delta \cap \Lambda_n} \eta_{\Lambda_n \setminus \Delta}) \\
&\quad \times \Psi \left[\frac{1}{\nu(\sigma_{\Delta \cap \Lambda_n} \eta_{\Lambda_n \setminus \Delta})} \int \nu(d\xi) 1_{\eta_{\Lambda_n}}(\xi) \frac{q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c} \sigma_\Delta)} \right]
\end{aligned}$$

where we also used $i \in \tilde{\Delta}_{n,j}$, $B_{n-1}(i) \subset \Delta_{n,j}$ to move the truncated rates in front of the sum over configurations in $\Lambda_n \setminus \Delta_{n,j}$. There exists $m \in \mathbb{N}$ such that $c_\Delta = 0$ if $0 \in \Delta \not\subset B_{m-1}(0)$ by the finite range condition on L . For $n \geq m$ from $\Delta \ni i$ and $i \in \tilde{\Delta}_{n,j}$ follows $\Delta \subset \Delta_{n,j}$. Thus for $n \geq m$, by translation invariance of ν and the rates and an application of Jensen's inequality w.r.t. the partial sum over configurations in $\Lambda_n \setminus \Delta_{n,j}$ to the concave function Ψ we have

$$\begin{aligned}
f_n(\nu) &\leq \frac{1}{q^{|\Delta|}} \sum_{j=1}^{2^d} \sum_{i \in \tilde{\Delta}_{n,j}} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \sum_{\eta_{\Delta_{n,j}}} \tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_{B_{n-1}(i) \cap \Delta}) \nu(\sigma_{\Delta \cap \Delta_{n,j}} \eta_{\Delta_{n,j} \setminus \Delta}) \\
&\quad \times \Psi \left[\frac{1}{\nu(\sigma_{\Delta \cap \Delta_{n,j}} \eta_{\Delta_{n,j} \setminus \Delta})} \int \nu(d\xi) 1_{\eta_{\Delta_{n,j}}}(\xi) \frac{q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c} \sigma_\Delta)} \right] \leq 2^d f_{n-1}(\nu).
\end{aligned}$$

Notice that in the last inequality we used that truncating the rates over smaller volumes only decreases the rates which gives the upper bound by non-positivity of Ψ . To compensate for the different volumes define $G(n) := \prod_{l=n}^{\infty} \frac{(2^{l+2}-2)^d}{(2^{l+2}-1)^d}$ which goes to one for $n \uparrow \infty$, then

$$\frac{G(n)}{(2^{n+1}-1)^d} f_n(\nu)$$

is non-increasing in n since $f_n(\nu) \leq 2^d f_{n-1}(\nu)$ and thus

$$\lim_{n \uparrow \infty} \frac{G(n)}{(2^{n+1}-1)^d} f_n(\nu) = f(\nu) \geq -\infty$$

exists which implies $\lim_{n \uparrow \infty} \frac{1}{|\Lambda_n|} f_n(\nu) = f(\nu)$. Since $\nu \mapsto f_n(\nu)$ is continuous $\nu \mapsto f(\nu)$ is upper semicontinuous.

Step 2: We show $f_n(\nu) = s_n(L, \nu) + o(|\Lambda_n|)$. For $n \geq m$ let us start by decomposing Ψ in $f_n(\nu)$, we have

$$\begin{aligned}
f_n(\nu) &= \frac{1}{q^{|\Delta|}} \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \sum_{\eta_{\Lambda_n}} \nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) \tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) \\
&\quad \times \Psi \left[\frac{1}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta})} \int \nu(d\xi) 1_{\eta_{\Lambda_n}}(\xi) \frac{q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c} \sigma_\Delta)} \right] \\
&= - \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) \frac{\tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} \\
&\quad \times \log \left[\frac{1}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta})} \int \nu(d\xi) 1_{\eta_{\Lambda_n}}(\xi) \frac{q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c} \sigma_\Delta)} \right] \\
&\quad + \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\xi) \frac{\tilde{c}_\Delta(\xi_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c} \sigma_\Delta)} \\
&\quad - \frac{1}{q^{|\Delta|}} \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \sum_{\eta_{\Lambda_n}} \nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) \tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta).
\end{aligned} \tag{10}$$

The sum given by the last two lines on the right hand side of (10) is of boundary order. This can be seen by rewriting this sum as

$$\begin{aligned}
&\sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \left[\sum_{\sigma_\Delta} \int \nu(d\xi) \frac{\tilde{c}_\Delta(\xi_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c} \sigma_\Delta)} - \sum_{\eta_{\Lambda_n}} \nu(\eta_{\Lambda_n}) \tilde{c}_\Delta(\eta_{B_{n-1}(i)}) \right] \\
&= \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \int \nu(d\eta) \left[\sum_{\sigma_\Delta} \frac{\tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} - \tilde{c}_\Delta(\eta_{B_{n-1}(i)}) \right]
\end{aligned} \tag{11}$$

and showing that the term in square brackets goes to zero as $n \uparrow \infty$ uniformly in η and i . But this is the case, indeed if $c_\Delta = 0$ by the definition L , Δ is not included in the summation and hence there is nothing to show. If $c_\Delta > 0$ with $\Delta \ni i$ we have for all η with $c_\Delta(\eta) > 0$,

$$\begin{aligned}
&\left| \sum_{\sigma_\Delta} c_\Delta^{\sigma_\Delta}(\eta) \frac{\tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta)}{c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} - \tilde{c}_\Delta(\eta_{B_{n-1}(i)}) \right| \\
&\leq \left| \sum_{\sigma_\Delta} c_\Delta^{\sigma_\Delta}(\eta) \frac{\tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) - c_\Delta(\eta_{\Delta^c} \sigma_\Delta)}{c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} \right| + \sup_\eta |c_\Delta(\eta) - \tilde{c}_\Delta(\eta_{B_{n-1}(i)})| \\
&\leq \sup_\eta |c_\Delta(\eta) - \tilde{c}_\Delta(\eta_{B_{n-1}(i)})| \sum_{\sigma_\Delta} \frac{c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} + \sup_\eta |c_\Delta(\eta) - \tilde{c}_\Delta(\eta_{B_{n-1}(i)})|
\end{aligned}$$

where

$$\begin{aligned}
\sum_{\sigma_\Delta} \frac{c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} &= \sum_{\sigma_\Delta: c_\Delta^{\sigma_\Delta}(\eta) > 0} \frac{c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} \leq \frac{c_\Delta(\eta)}{\min_{\sigma_\Delta: c_\Delta^{\sigma_\Delta}(\eta) > 0} c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} \\
&\leq \sup_{\eta: c_\Delta(\eta) > 0} \frac{c_\Delta(\eta)}{\min_{\sigma_\Delta: c_\Delta^{\sigma_\Delta}(\eta) > 0} c_\Delta(\eta_{\Delta^c} \sigma_\Delta)} \leq \frac{c_\Delta}{\inf_{\eta, \sigma_\Delta: c_\Delta^{\sigma_\Delta}(\eta) > 0} c_\Delta(\eta_{\Delta^c} \sigma_\Delta)}
\end{aligned}$$

which is finite by Condition 4 in Definition 2.4. Hence by the uniform continuity of the rates (11) is of boundary order.

It remains to compare the first line on the right hand side of (10) with $s_n(L, \nu)$. This amounts to showing that for all $i \in \tilde{\Lambda}_n$

$$\begin{aligned} & \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{\nu(\eta_{\Lambda_n}) q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\eta)}{\nu(\sigma_\Delta \cap \Lambda_n \eta_{\Lambda \setminus \Delta}) c_\Delta(\eta_{\Delta^c \sigma_\Delta})} \\ & - \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) \frac{\tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c \sigma_\Delta})} \log \left[\int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) c_\Delta(\xi_{\Delta^c \sigma_\Delta})} \right] \end{aligned}$$

tends to zero as $n \uparrow \infty$. Adding and subtracting the mixed term we first show boundary order of

$$- \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \left[\int \nu(d\xi | \eta_{\Lambda_n}) \frac{c_\Delta^{\sigma_\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n}) c_\Delta(\eta_{\Delta^c \sigma_\Delta})}{c_\Delta^{\sigma_\Delta}(\eta) c_\Delta(\xi_{\Lambda_n^c} \eta_{\Lambda_n \setminus \Delta} \sigma_\Delta)} \right] \dots$$

Define the minimal transition rate guaranteed by Condition 4 in Definition 2.4 as c_Δ^{\min} then for the lower bound

$$\begin{aligned} \frac{c_\Delta^{\sigma_\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n})}{c_\Delta^{\sigma_\Delta}(\eta)} & \leq 1 + \frac{1}{c_\Delta^{\min}} \sup_{\eta, \xi, \sigma_\Delta} |c_\Delta^{\sigma_\Delta}(\eta_{\Lambda_n} \xi_{\Lambda_n^c}) - c_\Delta^{\sigma_\Delta}(\eta)| \quad \text{and} \\ \frac{c_\Delta(\eta_{\Delta^c \sigma_\Delta})}{c_\Delta(\xi_{\Lambda_n^c} \eta_{\Lambda_n \setminus \Delta} \sigma_\Delta)} & \leq 1 + \frac{1}{c_\Delta^{\min}} \sup_{\eta, \xi} |c_\Delta(\eta_{\Lambda_n} \xi_{\Lambda_n^c}) - c_\Delta(\eta)| \end{aligned} \quad (12)$$

and similar from below for the upper bound. This yields the boundary order.

Secondly we show boundary order of

$$\begin{aligned} & \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \left[\int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) c_\Delta(\xi_{\Delta^c \sigma_\Delta})} \right] \\ & - \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) \frac{\tilde{c}_\Delta(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta(\eta_{\Delta^c \sigma_\Delta})} \log \left[\int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) c_\Delta(\xi_{\Delta^c \sigma_\Delta})} \right]. \end{aligned}$$

Note that by equation (11) the second summand equals $f_n(\nu) + o(|\Lambda_n|)$ where $f_n(\nu)$ is written in terms of the function Ψ . We want to write also the first summand in terms of Ψ . We have for all $\Delta \ni i$

$$\begin{aligned} & \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \left[\int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) c_\Delta(\xi_{\Delta^c \sigma_\Delta})} \right] \\ & = - \frac{1}{q^{|\Delta|}} \sum_{\sigma_\Delta} \int \nu(d\eta) \frac{c_\Delta^{\sigma_\Delta}(\eta) \nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta})}{\int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c \sigma_\Delta})}} \Psi \left[\int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) q^{|\Delta|} c_\Delta^{\sigma_\Delta}(\xi)}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) c_\Delta(\xi_{\Delta^c \sigma_\Delta})} \right] \\ & \quad + \int \nu(d\eta) c_\Delta(\eta) - \frac{1}{q^{|\Delta|}} \sum_{\sigma_\Delta} \int \nu(d\eta) \frac{c_\Delta^{\sigma_\Delta}(\eta) \nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta})}{\int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta(\xi_{\Delta^c \sigma_\Delta})}}. \end{aligned} \quad (13)$$

The last line is of boundary order. Indeed it can be reexpressed as

$$\begin{aligned} & \sum_{\eta_{\Lambda_n}} \nu(\eta_{\Lambda_n}) \tilde{c}_\Delta(\eta_{\Lambda_n}) \int \nu(d\eta | \eta_{\Lambda_n}) \frac{c_\Delta(\eta)}{\tilde{c}_\Delta(\eta_{\Lambda_n})} \\ & - \frac{1}{q^{|\Delta|}} \sum_{\sigma_\Delta} \sum_{\eta_{\Lambda_n}} \nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) \tilde{c}_\Delta(\eta_{\Lambda_n \setminus \Delta} \sigma_\Delta) \frac{\int \nu(d\eta | \eta_{\Lambda_n}) c_\Delta^{\sigma_\Delta}(\eta)}{\int \nu(d\xi | \eta_{\Lambda_n}) \frac{c_\Delta^{\sigma_\Delta}(\eta_{\Lambda_n} \xi_{\Lambda_n^c}) \tilde{c}_\Delta(\eta_{\Lambda_n \setminus \Delta} \sigma_\Delta)}{c_\Delta(\xi_{\Lambda_n^c} \eta_{\Lambda_n \setminus \Delta} \sigma_\Delta)}} \end{aligned}$$

where $\sum_{\eta_{\Lambda_n}} \nu(\eta_{\Lambda_n}) \tilde{c}_{\Delta}(\eta_{\Lambda_n}) - \frac{1}{q^{|\Delta|}} \sum_{\sigma_{\Delta}} \sum_{\eta_{\Lambda_n}} \nu(\sigma_{\Delta} \eta_{\Lambda_n} \setminus \Delta) \tilde{c}_{\Delta}(\eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta}) = 0$. Hence it suffices to note that

$$\begin{aligned} \left| \int \nu(d\eta | \eta_{\Lambda_n}) \frac{c_{\Delta}(\eta)}{\tilde{c}_{\Delta}(\eta_{\Lambda_n})} - 1 \right| &\leq \int \nu(d\eta | \eta_{\Lambda_n}) \left| \frac{c_{\Delta}(\eta) - \tilde{c}_{\Delta}(\eta_{\Lambda_n})}{\tilde{c}_{\Delta}(\eta_{\Lambda_n})} \right| \\ &\leq \frac{1}{c_{\Delta}^{\min}} \sup_{\eta} |\tilde{c}_{\Delta}(\eta_{\Lambda_n}) - c_{\Delta}(\eta)| \end{aligned} \quad (14)$$

tends to zero as $n \uparrow \infty$ and also

$$\begin{aligned} &\left| \frac{\int \nu(d\eta | \eta_{\Lambda_n}) c_{\Delta}^{\sigma_{\Delta}}(\eta)}{\int \nu(d\xi | \eta_{\Lambda_n}) \frac{c_{\Delta}^{\sigma_{\Delta}}(\eta_{\Lambda_n} \xi_{\Lambda_n^c}) \tilde{c}_{\Delta}(\eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta})}{c_{\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta})}} - 1 \right| \\ &\leq \int \nu(d\eta | \eta_{\Lambda_n}) \frac{\int \nu(d\xi | \eta_{\Lambda_n}) \left| 1 - \frac{c_{\Delta}^{\sigma_{\Delta}}(\eta_{\Lambda_n} \xi_{\Lambda_n^c}) \tilde{c}_{\Delta}(\eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta})}{c_{\Delta}^{\sigma_{\Delta}}(\eta) c_{\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta})} \right|}{\int \nu(d\xi | \eta_{\Lambda_n}) \frac{c_{\Delta}^{\sigma_{\Delta}}(\eta_{\Lambda_n} \xi_{\Lambda_n^c}) \tilde{c}_{\Delta}(\eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta})}{c_{\Delta}^{\sigma_{\Delta}}(\eta) c_{\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta})}} \end{aligned}$$

which also tends to zero as $n \uparrow \infty$ as can be seen by the estimates given in (12). Hence all that remains is to compare the first line on the right hand side of (13) with $f_n(\nu)$. This can be written as

$$\begin{aligned} &\frac{1}{q^{|\Delta|}} \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_{\Delta}} \sum_{\eta_{\Lambda_n}} \nu(\sigma_{\Delta} \eta_{\Lambda_n} \setminus \Delta) \tilde{c}_{\Delta}(\eta_{B_{n-1}(i)} \setminus \Delta \sigma_{\Delta}) \\ &\times \Psi \left[\int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) q^{|\Delta|} c_{\Delta}^{\sigma_{\Delta}}(\xi)}{\nu(\sigma_{\Delta} \eta_{\Lambda_n} \setminus \Delta) c_{\Delta}(\xi_{\Delta^c} \sigma_{\Delta})} \right] \left[1 - \frac{\int \nu(d\eta | \eta_{\Lambda_n}) c_{\Delta}^{\sigma_{\Delta}}(\eta)}{\int \nu(d\xi | \eta_{\Lambda_n}) \frac{c_{\Delta}^{\sigma_{\Delta}}(\xi_{\Lambda_n^c} \eta_{\Lambda_n}) \tilde{c}_{\Delta}(\eta_{B_{n-1}(i)} \setminus \Delta \sigma_{\Delta})}{c_{\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta})}} \right]. \end{aligned} \quad (15)$$

Notice, that by the estimates given in (12)

$$a_1(n) \leq 1 - \frac{\int \nu(d\eta | \eta_{\Lambda_n}) c_{\Delta}^{\sigma_{\Delta}}(\eta)}{\int \nu(d\xi | \eta_{\Lambda_n}) \frac{c_{\Delta}^{\sigma_{\Delta}}(\xi_{\Lambda_n^c} \eta_{\Lambda_n}) \tilde{c}_{\Delta}(\eta_{B_{n-1}(i)} \setminus \Delta \sigma_{\Delta})}{c_{\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n} \setminus \Delta \sigma_{\Delta})}} \leq a_2(n)$$

where $a_1(n)$ and $a_2(n)$ tend to zero as n tends to infinity. Thus the term in (15) is bounded from above by $a_1(n) f_n(\nu)$ and from below by $a_2(n) f_n(\nu)$. From step one we know that the limit $\lim_{n \uparrow \infty} |\Lambda_n|^{-1} f_n(\nu) \geq -\infty$ exists. If $\lim_{n \uparrow \infty} |\Lambda_n|^{-1} f_n(\nu) > -\infty$ then the term in (15) tends to zero as n tends to infinity. The case $\lim_{n \uparrow \infty} |\Lambda_n|^{-1} f_n(\nu) = -\infty$ implies $\lim_{n \uparrow \infty} |\Lambda_n|^{-1} s_n(L, \nu) = -\infty$. This completes the proof. \square

3.4. Proof of Corollary 2.10. Note that by Lemma 2.6, Proposition 2.8 and Theorem 2.9 we have

$$0 = \lim_{k \uparrow \infty} g_L(\nu_{t_k} | \mu) = \lim_{k \uparrow \infty} g_L(\nu_{t_k}) + \rho_L(\nu_*, \mu) \leq g_L(\nu_*) + \rho_L(\nu_*, \mu) = g_L(\nu_* | \mu) \leq 0.$$

Thus $g_L(\nu_* | \mu) = 0$ and by Condition 2.7 we have $\nu_* \in \mathcal{G}(\gamma)$. \square

3.5. Proof of Theorem 2.11. Inspecting the proof of Theorem 2.9 we see that the non-nullness assumption on ν appears only in one place, namely in the boundedness of $g_L^{\Lambda}(\nu)$ given in (8). Note that the upper and lower bound is only used in order to establish boundary order of $|g_L^{\Lambda_n}(\nu) - \tilde{g}_L^{\Lambda_n}(\nu)|$. For this theorem we only need to prove $g_L^{\Lambda_n}(\nu) \leq \tilde{g}_L^{\Lambda_n}(\nu)$ which can be seen using $\log x \leq x$. More

precisely we have

$$\begin{aligned}
g_L^{\Lambda_n}(\nu) - \tilde{g}_L^{\Lambda_n}(\nu) &= \sum_{i \in \Lambda_n \setminus \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{\nu(\sigma_\Delta \cap \Lambda_n | \eta_{\Lambda_n \setminus \Delta})}{\nu(\eta_\Delta \cap \Lambda_n | \eta_{\Lambda_n \setminus \Delta})} \\
&\leq \sum_{i \in \Lambda_n \setminus \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \sum_{\eta_{\Lambda_n}} \int \nu(d\eta | \eta_{\Lambda_n}) c_\Delta^{\sigma_\Delta}(\eta) \nu(\sigma_\Delta \cap \Lambda_n | \eta_{\Lambda_n \setminus \Delta}) \\
&\leq \sum_{\Delta \ni 0} \frac{1}{|\Delta|} c_\Delta q^{|\Delta|} |\Lambda_n \setminus \tilde{\Lambda}_n| = o(|\Lambda_n|).
\end{aligned}$$

The existence and upper semicontinuity of $\tilde{g}_L(\nu)$ is what is in fact proven in Theorem 2.9. \square

3.6. Proof of Corollary 2.13. Note that by Lemma 2.6, Proposition 2.8 and Theorem 2.11 we have

$$\begin{aligned}
0 = \lim_{k \uparrow \infty} g_L(\nu_{t_k} | \mu) &= \lim_{k \uparrow \infty} g_L(\nu_{t_k}) + \rho_L(\nu_*, \mu) \leq \lim_{k \uparrow \infty} \tilde{g}_L(\nu_{t_k}) + \rho_L(\nu_*, \mu) \\
&\leq \tilde{g}_L(\nu_*) + \rho_L(\nu_*, \mu) = \tilde{g}_L(\nu_* | \mu).
\end{aligned}$$

Since by Condition 2.12 also $\tilde{g}_L(\nu_* | \mu) \leq 0$ we have $\tilde{g}_L(\nu_* | \mu) = 0$ and thus again by Condition 2.12, ν_* is a Gibbs measure for γ . \square

3.7. Proof of Proposition 2.15. The first part of the proof is similar to the proof of Theorem 2.9. Recall that in (9) we write $\tilde{g}_L^{\Lambda_n}(\nu)$ as a sum of two terms. To simplify notation let us assume n to be large enough such that $\sum_{\Delta \ni 0: \Delta \not\subset B_{n-1}(0)} c_\Delta = 0$. This can be done without loss of generality since we are interested in the large n limit and L is assumed to have the property that there are only finitely many types of transitions.

Since we are now in a reversible setting it is more convenient to extend $\tilde{g}_L^{\Lambda_n}(\nu)$ in the following way

$$\begin{aligned}
\tilde{g}_L^{\Lambda_n}(\nu) &= - \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{\nu(\eta_{\Lambda_n}) c_\Delta^{\sigma_\Delta}(\eta)}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) c_\Delta^{\eta_\Delta}(\eta_{\Delta^c} \sigma_\Delta)} \\
&\quad + \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta^{\eta_\Delta}(\eta_{\Delta^c} \sigma_\Delta)} =: s_n(L, \nu) + r_n(L, \nu)
\end{aligned}$$

where, by the continuity of the rates

$$\lim_{n \uparrow \infty} \frac{1}{|\Lambda_n|} r_n(L, \nu) = \sum_{\Delta \ni 0} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{c_\Delta^{\sigma_\Delta}(\eta)}{c_\Delta^{\eta_\Delta}(\eta_{\Delta^c} \sigma_\Delta)} =: r(L, \nu).$$

Note that $s_n(L, \nu)$ is still well-defined since by the reversibility assumption $c_\Delta^{\sigma_\Delta}(\eta) > 0$ implies that $c_\Delta^{\eta_\Delta}(\eta_{\Delta^c} \sigma_\Delta) > 0$. Indeed, the reversibility implies that for all η_Λ and σ_Δ with $\Delta \subset \Lambda$

$$\int \mu(d\xi) \gamma_\Lambda(\eta_\Lambda | \xi_{\Lambda^c}) c_\Delta^{\sigma_\Delta}(\xi_{\Lambda^c} \eta_\Lambda) = \int \mu(d\xi) \gamma_\Lambda(\eta_\Lambda \setminus \Delta \sigma_\Delta | \xi_{\Lambda^c}) c_\Delta^{\eta_\Delta}(\xi_{\Lambda^c} \eta_\Lambda \setminus \Delta \sigma_\Delta). \quad (16)$$

Hence, if $c_\Delta^{\sigma_\Delta}(\eta) > 0$ by the continuity also $c_\Delta^{\sigma_\Delta}(\xi_{\Lambda^c} \eta_\Lambda) > 0$ for any ξ , for a large enough volume Λ . Further since the specification is assumed to be non-null, also $c_\Delta^{\eta_\Delta}(\xi_{\Lambda^c} \eta_\Lambda \setminus \Delta \sigma_\Delta) > 0$ and $c_\Delta^{\eta_\Delta}(\eta_{\Delta^c} \sigma_\Delta) > 0$ for any ξ , for the same large volume Λ .

The reversibility in particular implies that $r(L, \nu) + \rho(\nu, \mu) = 0$, i.e.

$$0 = \sum_{\Delta \ni 0} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \int \nu(d\eta) c_\Delta^{\sigma_\Delta}(\eta) \log \frac{c_\Delta^{\sigma_\Delta}(\eta) \gamma_\Delta(\eta_\Delta | \eta_{\Delta^c})}{c_\Delta^{\eta_\Delta}(\eta_{\Delta^c} \sigma_\Delta) \gamma_\Delta(\sigma_\Delta | \eta_{\Delta^c})}.$$

This can be seen in the following way. As a consequence of (16) we have

$$\frac{\gamma_\Delta(\eta_\Delta|\eta_{\Delta^c})c_\Delta^{\sigma_\Delta}(\eta)}{\gamma_\Delta(\sigma_\Delta|\eta_{\Delta^c})c_\Delta^{\eta_\Delta}(\eta_{\Delta^c}\sigma_\Delta)} = \frac{\int \mu(d\xi) \frac{\gamma_\Delta(\sigma_\Delta|\eta_{\Lambda\setminus\Delta}\xi_{\Lambda^c})c_\Delta^{\eta_\Delta}(\xi_{\Lambda^c}\eta_{\Lambda\setminus\Delta}\sigma_\Delta)}{\gamma_\Delta(\sigma_\Delta|\eta_{\Delta^c})c_\Delta^{\eta_\Delta}(\eta_{\Delta^c}\sigma_\Delta)}}{\int \mu(d\xi) \frac{\gamma_\Delta(\eta_\Delta|\eta_{\Lambda\setminus\Delta}\xi_{\Lambda^c})c_\Delta^{\sigma_\Delta}(\xi_{\Lambda^c}\eta_\Delta)}{\gamma_\Delta(\eta_\Delta|\eta_{\Delta^c})c_\Delta^{\sigma_\Delta}(\eta)}}.$$

where the right hand side tends to one as Λ tends to infinity by the continuity and non-nullness assumptions on the rates as well as on the specification.

In other words, in a reversible setting, $\tilde{g}_L(\nu|\mu) := \lim_{n \uparrow \infty} \frac{1}{|\Lambda_n|} s_n(L, \nu)$. Very similar to the proof of Theorem 2.9, one can show, using Jensen's inequality, that in the limit as n tends to infinity, $s_n(L, \nu)$ can be replaced by

$$f_n(\nu) := \sum_{i \in \tilde{\Lambda}_n} \sum_{\Delta \ni i} \frac{1}{|\Delta|} \sum_{\sigma_\Delta} \sum_{\eta_{\Lambda_n}} \nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) \tilde{c}_\Delta^{\eta_\Delta}(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) \\ \times \Psi \left[\frac{1}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta})} \int \nu(d\xi) 1_{\eta_{\Lambda_n}}(\xi) \frac{c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta^{\eta_\Delta}(\xi_{\Delta^c} \sigma_\Delta)} \right].$$

where $a_n f_n(\nu)$, with $a_n > 0$ some volume-factor, is a non-increasing sequence of non-positive functions. Since $\Psi \leq 0$ this in particular implies that $\tilde{g}_L(\nu|\mu)$ exists and $\tilde{g}_L(\nu|\mu) \leq 0$, which is the first statement of the proposition.

As for the second statement, assume that $\tilde{g}_L(\nu|\mu) = 0$ which then implies that $f_n(\nu) = 0$ for every large n . Consequently, for all $i \in \tilde{\Lambda}_n$, $\Delta \ni i$, σ_Δ and η_{Λ_n} we have

$$\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) \tilde{c}_\Delta^{\eta_\Delta}(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) \Psi \left[\frac{1}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta})} \int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta^{\eta_\Delta}(\xi_{\Delta^c} \sigma_\Delta)} \right] = 0. \quad (17)$$

Let us assume $\tilde{c}_\Delta^{\eta_\Delta}(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) > 0$ and note, as above, that this implies $c_\Delta^{\eta_\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n \setminus \Delta} \sigma_\Delta) > 0$ and $c_\Delta^{\sigma_\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n}) > 0$ for all ξ , by continuity and reversibility. Under this assumption from $\nu(\eta_{\Lambda_n}) = 0$ it must follow $\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) = 0$ since otherwise

$$\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) \tilde{c}_\Delta^{\eta_\Delta}(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) \Psi \left[\frac{1}{\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta})} \int \nu(d\xi) \frac{1_{\eta_{\Lambda_n}}(\xi) c_\Delta^{\sigma_\Delta}(\xi)}{c_\Delta^{\eta_\Delta}(\xi_{\Delta^c} \sigma_\Delta)} \right] < 0.$$

In other words, whenever a jump is possible from a configuration η_{Λ_n} to a configuration $\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}$ then $\nu(\eta_{\Lambda_n}) = 0$ implies $\nu(\sigma_\Delta \eta_{\Lambda_n \setminus \Delta}) = 0$. By the condition that L is irreducible this implies that from $\nu(\eta_{\Lambda_n}) = 0$ it follows that $\nu(\xi_{\tilde{\Lambda}_n} \eta_{\Lambda_n \setminus \tilde{\Lambda}_n}) = 0$ for all $\xi_{\tilde{\Lambda}_n}$. Further assume $\nu(\eta_{\Lambda_n}) = 0$ for some η_{Λ_n} . Let $m \geq n$ be such that $\tilde{\Lambda}_m \supset \Lambda_n$, then it follows $\nu(\xi_{\Lambda_m \setminus \Lambda_n} \eta_{\Lambda_n}) = 0$ for all $\xi_{\Lambda_m \setminus \Lambda_n}$. Consequently $\nu(\xi_{\Lambda_m \setminus \Lambda_n} \xi_{\Lambda_n}) = 0$ for all ξ_{Λ_n} and thus $\nu(\xi_{\Lambda_n}) = 0$ for all ξ_{Λ_n} which is a contradiction. Hence $\nu(\eta_{\Lambda_n}) > 0$ for all η_{Λ_n} .

Finally, let η be given with $\tilde{c}_\Delta^{\eta_\Delta}(\eta_{B_{n-1}(i) \setminus \Delta} \sigma_\Delta) > 0$, then using (17) and the reversibility (16), we have

$$1 = \int \nu(d\xi | \eta_{\Lambda_n}) \frac{c_\Delta^{\sigma_\Delta}(\eta_{\Lambda_n} \xi_{\Lambda_n^c}) \nu(\eta_\Delta | \eta_{\Lambda_n \setminus \Delta})}{c_\Delta^{\eta_\Delta}(\xi_{\Lambda_n^c} \eta_{\Lambda_n \setminus \Delta} \sigma_\Delta) \nu(\sigma_\Delta | \eta_{\Lambda_n \setminus \Delta})} \\ = \int \nu(d\xi | \eta_{\Lambda_n}) \frac{\gamma_\Delta(\sigma_\Delta | \eta_{\Lambda_n \setminus \Delta} \xi_{\Lambda_n^c}) \nu(\eta_\Delta | \eta_{\Lambda_n \setminus \Delta})}{\gamma_\Delta(\eta_\Delta | \eta_{\Lambda_n \setminus \Delta} \xi_{\Lambda_n^c}) \nu(\sigma_\Delta | \eta_{\Lambda_n \setminus \Delta})}.$$

By martingale convergence, this implies that ν almost surely

$$\frac{\gamma_\Delta(\eta_\Delta | \eta_{\Delta^c})}{\gamma_\Delta(\sigma_\Delta | \eta_{\Delta^c})} = \frac{\nu(\eta_\Delta | \eta_{\Delta^c})}{\nu(\sigma_\Delta | \eta_{\Delta^c})}. \quad (18)$$

Again by the assumption that L is irreducible the above equation is true for ν almost all η and $\sigma_\Delta \in \{1, \dots, q\}^\Delta$. Recall the following general fact: Let (a_1, \dots, a_q) and (b_1, \dots, b_q) be probability vectors with $\frac{a_l}{a_k} = \frac{b_l}{b_k}$ for all $k, l \in \{1, \dots, q\}$ then

$$a_l = \frac{a_l}{\sum_{k=1}^q a_k} = \frac{1}{1 + \sum_{k \neq l} \frac{a_k}{a_l}} = \frac{1}{1 + \sum_{k \neq l} \frac{b_k}{b_l}} = b_l.$$

Hence (18) implies $\gamma_\Delta(\sigma_\Delta | \eta_{\Delta^c}) = \nu(\sigma_\Delta | \eta_{\Delta^c})$ for ν almost all η and $\sigma_\Delta \in \{1, \dots, q\}^\Delta$. But this implies that ν is a Gibbs measure for the specification γ . \square

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