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Generalized gradient flow structure of internal energy driven phase field systems

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Abstract

In this paper we introduce a general abstract formulation of a variational thermomechanical model, by means of a unified derivation via a generalization of the principle of virtual powers for all the variables of the system, including the thermal one. In particular, choosing as thermal variable the entropy of the system, and as driving functional the internal energy, we get a gradient flow structure (in a suitable abstract setting) for the whole nonlinear PDE system. We prove a global in time existence of (weak) solutions result for the Cauchy problem associated to the abstract PDE system as well as uniqueness in case of suitable smoothness assumptions on the functionals.

1 Introduction

In this paper we introduce a general derivation of thermo-mechanical phase transition models by use of a generalization of the principle of virtual powers, in which micro-forces and thermal forces are included. It is known that a recent field of research, in the framework of phase transitions, has concerned models with some micro-forces (see, e.g., the approaches by Frémond [11] and by Gurtin [14]). The main idea is that the equations governing the evolution of phase transition phenomena may be derived by a variational principle, i.e. the principle of virtual powers, in which micro-forces, responsible for phase transitions (i.e. for changes in the microstructure level of the materials), are included. As a consequence, the resulting PDE system provides an intrinsic variational structure, at least concerning equations for displacements and internal quantities, as phase or order parameters. Many authors have dealt with this kind of approach. We mention, among the others we quote some contributions as [15], [10], and [18].

On the other hand, as far as thermal properties are concerned, in the recent years several efforts have been spent to investigate models in which an entropy balance (or imbalance) equation was introduced in place of the more classical “heat equation”. We recall, e.g., the contribution by [3], [4], and [5]. In particular, let us mention that the last paper shows a derivation of the equation on the entropy by convex analysis tools and the application of a Legendre transformation for the free energy. It is interesting to observe that in this framework, also thermal memory is formally justified from the point of view of the derivation of the model. In a different direction Podio-Guidugli, in relation to a theory proposed by Green and Naghdi, introduced the possibility of including thermal displacements and forces in the whole balance of the principle of virtual powers, so that the entropy equation may be recovered, as well as the momentum equation, as a “balance of forces”, forcing the system on the base of some “reluctance to order”. Indeed, starting from the consideration that some virtual power principle may be used to deduce all balance and imbalance laws of thermomechanics, he suggested to use it also for the derivation of thermal evolution, through the notion of thermal displacement. As a consequence, he derives an equation for the entropy of the system, which is combined with momentum balance. This approach turns out to be consistent with thermodynamical principles. See, among the others, [20] and the papers by Green and Naghdi [12], [13], and references therein. Finally, we can quote the recent contribution [16], where a gradient structure of systems in thermoplasticity is introduced by means of

a free entropy functional instead of the internal energy, which is the driving functional in the present contribution.

Indeed, in this paper, we aim to combine the previous approaches and provide a general abstract formulation of a variational thermomechanical model which can be applied to recover different phase transitions and phase separation phenomena, also accounting for mechanical or thermal effects. Hence, we introduce an unified approach which formally justifies the evolution of the thermal variable (represented here by the entropy of the system), the phase parameters, and (possibly) the displacements. Actually, in the following, we are dealing with two state variables: s, which mainly plays the role of the entropy, and a phase parameter χ, representing the internal mechanical variable. The main advantage of the gradient structure is the possibility of deriving a time-incremental minimization procedure, where the internal energy functional is minimized with respect to the entropy and the internal variables and so the existence of weak solutions for the associated Cauchy problem can be deduced under quite general assumptions on the involved nonlinearities.

Indeed, the choice of the energy functional and the dissipation potential are fairly general. In particular, in the internal energy functional we can include multivalued operators to ensure some internal constraints. Since the resulting gradient flow structure is nonlinear and non smooth, we have to introduce a suitable notion of (weak) solution in order to get a global in time existence result. However, the weak notion of solution we are introducing is naturally in accordance with the physical meaning of the problem under consideration as well as with the classical principles of thermodynamics. The proof is performed by means of a combined regularization and time discretization procedure. Moreover, uniqueness of solutions is proved under some further smoothness assumption on the internal energy functional.

The paper is organized as follows. In the next Section 2 we derive the model and state the main assumptions on the involved physical quantities and functionals. The main existence result is stated and proved in Section 3, as well as the uniqueness of solutions.

2 The model and the main assumptions

Let $\Omega \subset \mathbb{R}^3$ be a bounded and sufficiently regular domain with boundary $\Gamma := \partial\Omega$. We introduce an Hilbert triplet $V \subset H \subset V'$ (with dense and compact injections), where $V = H^1(\Omega)$, $H = L^2(\Omega)$, and H is identified as usual with its dual. We introduce the notations $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V' and $(\cdot, \cdot)$ for the usual scalar product both in H and in $L^2(\Omega)^3$. To simplify the notation, we write H in place of $L^2(\Omega)^3$, or V in place of $H^1(\Omega)^3$, when vector-valued functions are considered. For every $f \in V'$ we indicate by $\overline{f}$ the *spatial mean* of f over Ω, i.e.

$$\overline{f} := \frac{1}{|\Omega|}\langle f, 1 \rangle,$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. We note as H_0, V_0 and V_0' the closed subspaces of functions (or functionals) having zero mean value in H, V, and in V', respectively. Then, by the Poincaré-Wirtinger inequality,

$$\|v\|_{V_0} := \left(\int_\Omega |\nabla v|^2 \, \mathrm{d}x \right)^{1/2}$$

represents a norm on V_0 which is equivalent to the norm naturally inherited from V. In particular $\|\cdot\|_{V_0}$ is a Hilbert norm associated to a scalar product $((\cdot, \cdot))_{V_0}$ (defined in (2.1)), and thus we can introduce

the associated Riesz isomorphism mapping $A : V_0 \to V_0'$ by setting, for $u, v \in V_0$,

$$\langle Au, v\rangle := ((u, v))_{V_0} := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \tag{2.1}$$

so that $\langle Au, u\rangle = \|u\|^2_{V_0}$ for every $u \in V_0$ and $\langle v, A^{-1}(v)\rangle = \|v\|^2_{V_0'}$ for every $v \in V_0'$. Finally, we can identify H_0 with H_0' by means of the scalar product of H so to obtain the Hilbert triplet $V_0 \subset H_0 \subset V_0'$, where inclusions are continuous and dense. In particular, if $z \in V$ and $v \in V_0$, it is easy to see that

$$\int_\Omega \nabla z \cdot \nabla(A^{-1}v) \, \mathrm{d}x = \int_\Omega (z - \bar{z})v \, \mathrm{d}x = \int_\Omega zv \, \mathrm{d}x. \tag{2.2}$$

In what follows in this section we introduce our modelling approach and the set of PDEs and initial and boundary conditions which we are going to analyze in the next sections.

2.1 The Principle of Virtual Powers

The model is derived by using a variational principle in mechanics which is known as (generalized) principle of virtual power. Indeed, we refer to some generalization of the well known mechanical principle as we are including in the involved forces the microscopic forces, acting on some “micro-scale”, and also possible “thermal forces". Without entering the details of this argumentation, let us point out that this principle is formally based on the fact that velocities are considered in a suitable linear space and thus forces are defined as elements acting on velocities with respect to some duality relation between the two spaces. This is done for any (sufficiently smooth) subdomain $\mathcal{D} \subseteq \Omega$. Hence, before proceeding we make precise the virtual velocities we are considering. More precisely, let us take the couple of virtual velocities (δ_t, v_t) (whose physical meaning may change time to time). In the case when no accelerations are included, the principle of virtual powers can be written considering the power of internal forces $\mathcal{P}_{int}$ and of external forces $\mathcal{P}_{ext}$ (depending on $\mathcal{D}$, δ_t, v_t) as follows:

$$\mathcal{P}_{int}(\mathcal{D}, \delta_t, v_t) + \mathcal{P}_{ext}(\mathcal{D}, \delta_t, v_t) = 0.$$

We assume that the power of internal forces is introduced as follows (in Ω and for any virtual velocities $\delta_t \in V_0$ and $v_t \in V$)

$$\mathcal{P}_{int} = \langle\langle F, \delta_t\rangle\rangle + \langle\langle \mathcal{G}, v_t\rangle\rangle = \langle F, \delta_t\rangle + \int_\Omega B \, v_t \, \mathrm{d}x + \int_\Omega E \, \nabla v_t \, \mathrm{d}x, \tag{2.3}$$

where F, B and E denote interior thermal and mechanical (micro) forces and stresses, respectively and the duality relation $\langle\langle \cdot, \cdot \rangle\rangle$ is suitably defined between forces and velocities spaces.

Analogously, the power of external forces is

$$\mathcal{P}_{ext} = \langle\langle \mathcal{R}, \delta_t\rangle\rangle + \langle\langle \mathcal{Z}, v_t\rangle\rangle.$$

We let $v_t \in V$ and $\delta_t \in V_0$ and we assume there exists Z, z such that

$$\langle\langle \mathcal{R}, \delta_t\rangle\rangle = \langle \mathcal{R}, \delta_t\rangle \quad \text{and} \quad \langle \mathcal{Z}, v_t\rangle = \int_\Omega Z \, v_t + \int_\Gamma z \, v_t, \tag{2.4}$$

where Z and z stand for the external forces acting in the bulk Ω and at the boundary Γ, respectively.

It is clear that that we are considering a different behavior of the forces on the two types of virtual velocities. Indeed, we note that the elements $\mathcal{G}$ and $\mathcal{Z}$ are defined as a.e. forces living

in the bulk and on the boundary (with suitable summability), while we take F and $\mathcal{R}$ as general as possible to include all the different (and less regular) situations we will face. In particular, as it will be clear once we will make a precise choice of the actual velocities (cf. (2.5) and (2.6)), of the energy functional (2.11), and of the dissipation potential (2.14), we aim to write down an equation for the thermal variable of *conservative* type: it will result indeed as a *conservation of energy*, while the equation for the mechanical variable will be on *non conservative type*. This mainly motivates the choice we have made for the power of internal and external forces. Other choices are possible (cf., e.g., Remark 2.8), but we prefer not to move in this direction in the present contribution.

2.2 The constitutive relations and the PDEs

The state variables. We are dealing with a physical system governed by the state variables $(s, \chi, \nabla\chi)$ whose evolution is ruled by different thermomechanical relations. Note that we are distinguishing between different *dependence* of the energy with respect to the two variables s and χ: we consider, in particular, the gradient $\nabla\chi$ but not ∇s as state variable (cf. (2.11)). This corresponds to the specific choice we have done for the forces F, $\mathcal{G}$ and $\mathcal{R}$, $\mathcal{Z}$ we have made in (2.3) and (2.4).

In order to get the evolution of s, we take the actual velocities as $\delta_t = A^{-1}(\xi_t)$, where $\xi_t \in V_0'$ and $v_t = 0$, in order to get (for $\mathcal{R} = 0$) $\langle F, A^{-1}(\xi_t)\rangle = 0$ for all $\xi_t \in V_0'$ and so we obtain

$$A(F) = 0 \quad \text{in } V_0'. \tag{2.5}$$

The evolution of χ is obtained by integrating by parts in $\mathcal{P}_{int}$ and choosing $Z = z = 0$ and $\delta_t = 0$ as well:

$$B - \operatorname{div} E = 0 \quad \text{in } V', \tag{2.6}$$

with the no-mass flux through the boundary of Ω:

$$E \cdot \mathbf{n} = 0 \quad \text{on } \Gamma\,, \tag{2.7}$$

where we have denoted by $\mathbf{n}$ the outward unit normal vector to Γ. Notice that we have chosen here to have 0 ezternal forces $\mathcal{R}$ and $\mathcal{Z}$ only for simplicity of notation.

Remark 2.1. *In the following, we mainly refer to the variable χ as a phase or order parameter, i.e. related to the micro-structure of the physical system. However, let us point out that we could formally include in our procedure the derivation of the (more) classical momentum balance equation (letting, e.g., χ stand for displacements). In this case, the force B has to be equal to 0, due to the principle of rigid motions.*

The functionals and the main assumptions. We introduce two functionals governing the evolution and the equilibrium of our (thermo)mechanical system. These functionals depend on the state variables and on the dissipative variables, respectively. As far as the equilibrium, it is governed by an energy functional, and we choose to make use of an internal energy functional (in place of the free energy functional). This choice is motivated by the fact that we may interprete s as the entropy of the system (see [16] for a physical justification). However, it is well known that, under suitable assumptions, the internal energy may be introduced as the Legendre transformed of the free energy.

Before we make precise the choices of the internal energy functional and of the dissipation potential, let us introduce a function W, depending on χ, as the sum of a convex possibly non-smooth part and non-convex but regular function and it satisfies some smoothness and growth assumptions, in particular, we need:

Hypothesis 2.2. *Assume* $W(\chi) = \widehat{\beta}(\chi) + \widehat{\gamma}(\chi)$*, where*

(w1) $\widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to [0, +\infty]$ *is convex, proper, and lower semicontinuous,*

(w2) $\widehat{\gamma} : \mathbb{R} \to \mathbb{R}$ *is a* $C^{1,1}$ *function on* $\mathbb{R}$*.*

Remark 2.3. *Notice that particularly meanigful choices of* W *used in the literature of phase transitions (when* χ *assumes the meaning of phase variable) are, for example,*

1. *the double well potential* $W(\chi) = (\chi^2 - 1)^2$
2. *the logarithmic potential* $W(\chi) = \chi \log(\chi) + (1 - \chi) \log(1 - \chi) - \chi^2$
3. *the double obstacle potential* $W(\chi) = I_{[0,1]}(\chi) - \chi^2$*, where* $I_{[0,1]}$ *denotes the indicator function of the interval* $[0, 1]$ *and it is defined as* $I_{[0,1]}(x) = 0$ *if* $x \in [0, 1]$ *and* $I_{[0,1]}(x) = +\infty$ *otherwise.*

Moreover, we introduce a function $j(\theta, \chi) : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ such that

$\theta \mapsto j(\theta, \chi)$ is a convex, proper, and lower semicontinuous for every $\chi \in \mathbb{R}$ and
$\chi \mapsto j(\theta, \chi)$ is a C^1 function for every $\theta \in \mathbb{R}$,

and let

$$J_H(\theta, \chi) = \begin{cases} \displaystyle\int_\Omega j(\theta, \chi) & \text{if } (\theta, \chi) \in H \times H \quad \text{and } j(\theta, \chi) \in L^1(\Omega) \\ +\infty & \text{if } (\theta, \chi) \in H \times H \quad \text{and } j(\theta, \chi) \notin L^1(\Omega) \end{cases} \tag{2.8}$$

$$J_V(\theta, \chi) = J_H(\theta, \chi) \quad \text{on } V \times H. \tag{2.9}$$

Hence, we can introduce the convex conjugate of J_V as follows $J_V^*(s, \chi) : V' \times H \to [0, +\infty]$ is defined as

$$J_V^*(s, \chi) = \sup_{\theta \in V} \left(\langle s, \theta \rangle - J_V(\theta, \chi) \right), \quad (s, \chi) \in V' \times H. \tag{2.10}$$

Now, we are in the position of introducing the energy functional $e : V' \times H \times H \to (-\infty, +\infty]$:

$$e(s, \chi, \nabla\chi) = J_V^*(s, \chi) + \int_\Omega \left(\frac{1}{2} |\nabla\chi|^2 + W(\chi) \right) \mathrm{d}x. \tag{2.11}$$

Let us note here that the first term in (2.11) contains both the purely caloric part of the energy functional (i.e. the one depending only on s as well as the coupling terms depending on both s and χ) (cf. Subsection 2.3 for possible choices of J_V^*), while inside the integral over Ω we have the parts accounting for the *nonlocal* interfacial energy effects (the $|\nabla\chi|^2$) and the mixing potential W (cf. Remark 2.3 for examples of possible choices of functions W). We intentionally choose not to consider interfacial (*nonlocal*) energy effects in the variable s in order to differentiate the roles of the caloric and the mechanical parts (s and χ, respectively) in our approach.

Then, we can define the subdifferential (with respect to the variable s) $\partial_{V',V} J_V(s, \chi)$ which maps $V' \times H$ into 2^V as (cf., e.g., [6]):

$$\begin{aligned} v \in \partial_{V',V} J_V^*(s, \chi) \quad & \text{if and only if } v \in V, \\ & (s, \chi) \in D(J_V^*), \text{ and } J_V^*(s, \chi) \le \langle s - w, v \rangle + J_V^*(w, \chi) \quad \forall (w, \chi) \in V' \times H\,. \end{aligned} \tag{2.12}$$

Actually, in what follows we will always work in the space $V' \times H$ and so we will state directly the assumptions we need on the functional J_V^* defined in (2.10). In particular, we need the following assumptions:

Hypothesis 2.4. *We assume that $J_V^* : V' \times H \to [0, +\infty]$ is such that: there exist two positive constants $c_1, c_2 \in \mathbb{R}^+$ such that the functional J_V^* defined in* (2.10) *satisfies:*

(J1) *$\chi \mapsto J_V^*(s, \chi)$ is Fréchet differentiable in H for every $s \in V'$,*

(J2) $\left\| \dfrac{\partial J_V^*(s,\chi)}{\partial \chi} \right\|_H \leq c_1 \|\eta\|_H + c_2$, *for every $\eta \in \partial_{V',V} J_V^*(s, \chi)$ and $(s, \chi) \in D(J_V^*)$,*

where $\frac{\partial(\cdot)}{\partial \chi}$ denotes the partial derivative with respect to χ (which will be denoted also by $\partial_\chi(\cdot)$ and by $(\cdot)_\chi$ in the paper).

Moreover, we assume that

(J3) *$s \mapsto J_V^*(s, \chi)$ is proper, convex and lower semicontinuous from V' to $[0, +\infty]$, for every $\chi \in H$,*

so that the subdifferential $\partial_{V',V} J_V(s, \chi)$ which maps $V' \times H$ into 2^V according to the definition (2.12) *turns out to be a maximal monotone operator acting from V' to 2^V, for every $\chi \in H$ (cf. [1]).*

Note that, the assumption **(J3)** follows from assumptions on j and (2.10) and that possible examples of functions j complying with our assumptions will be listed in the next Subsection 2.3.

Remark 2.5. *Observe that the assumptions on the positivity of the maps j and J_V^* could be weakened: we need indeed to have only a lower bound (possibly with a negative constant) for them in order to perform the first a-priori estimate* (3.16)*. Moreover, let us nothe that the assumption* **(J2)** *could be relaxed: we could indeed assume c_1, c_2 to be two continuous functions of χ bounded on $\mathrm{dom}(\widehat{\beta})$. However, we put ourselves in this setting to avoid further technicalities for the reader's convenience.*

Remark 2.6. *Note that, under particular assumptions on the function j (for example in case $\mathrm{dom}(j) = \mathbb{R}$), we could also rewrite the functional e in* (2.11) *as (cf. Remark 2.7 for more details and [1])*

$$e(s, \chi, \nabla\chi) = \int_\Omega \left(j^*(s, \chi) + \frac{1}{2}|\nabla\chi|^2 + W(\chi) \right) \mathrm{d}x \,, \tag{2.13}$$

where j^* is the conjugate function of j with respect to the variable s, i.e.

$$j^*(s, \chi) = \sup_{\theta \in \mathbb{R}} \left(s\theta - j(\theta, \chi) \right), \quad \forall (s, \chi) \in \mathbb{R} \times \mathbb{R} \,.$$

We introduce as dissipative variables the time derivatives s_t and χ_t (see, e.g. [11], for a definition of the pseudo-potential of dissipation *à la Moreau*) and we include dissipation in the model by choosing the following form for the pseudopotential of dissipation depending on the dissipative variables s_t and χ_t. Note that we suppose the evolution to be rate dependent. The first possibility we consider for the dissipation functional is

$$\varphi : V_0' \times H \to \mathbb{R}, \quad \varphi(s_t, \chi_t) = \frac{1}{2}\langle s_t, A^{-1} s_t \rangle + \frac{1}{2} \int_\Omega |\chi_t|^2 \, \mathrm{d}x \,. \tag{2.14}$$

Note that we have a natural scalar product in V_0' $((\cdot,\cdot))_{V_0'}$ defined as $((s_t,s_t))_{V_0'} = \langle s_t, A^{-1}s_t\rangle = \|s_t\|^2_{V_0'}$, $\int_\Omega |\chi_t|^2\,\mathrm{d}x = (\chi_t,\chi_t) = \|\chi_t\|^2_H$. In the defintions of e and φ we have normalized all the physical constants to 1 for simplicity and without any loss of generality. Another possibility consists in letting

$$\varphi(s_t,\chi_t) = \frac{1}{2}\langle s_t, A^{-1}s_t\rangle + \frac{1}{2}\langle \chi_t, A^{-1}\chi_t\rangle, \tag{2.15}$$

but we prefer not to exploit this case in the present contribution in order to distinguish between the roles of the two variables: the thermal variable s (*conserved*) and the mechanical variable χ (*non conserved*). Moreover, a rate-independent model could be introduced in place of the rate-dependent one we analyze here by suitably modifying the choice of the dissipation functional (2.14) (cf., e.g., [17]). However, the analysis we are performing does not apply to this case, which would require ad hoc techniques and some suitable notion of weak solution.

The constitutive relations and the PDEs. Now, according to the definition of $\mathcal{P}_{int}$ and of e and φ (cf. (2.11) and (2.14)), we let the thermal force F be

$$F = \partial_s e + \partial_{s_t}\varphi = \partial_{V',V} J_V^*(s,\chi) + A^{-1}(s_t)\,. \tag{2.16}$$

Hence, for the evolution of χ we prescribe the following mechanical (micro) forces and stresses B and E:

$$B = \partial_\chi e + \partial_{\chi_t}\varphi = \partial_\chi J_V^*(s,\chi) + \partial\widehat{\beta}(\chi) + \widehat{\gamma}'(\chi) + \chi_t, \quad E = \partial_{\nabla\chi} e = \nabla\chi\,. \tag{2.17}$$

From (2.5–2.7) and the above constitutive relations we deduce the following PDE system for the evolution of s and χ:

$$s_t + A\eta = 0 \text{ in } V_0',\ \eta \in \partial_{V',V} J_V^*(s,\chi), \quad \text{a.e. in } (0,T)\,, \tag{2.18}$$

$$\chi_t - \Delta\chi + \xi + \gamma(\chi) + \partial_\chi J_V^*(s,\chi) = 0,\ \xi \in \beta(\chi) \quad \text{a.e. in } \Omega\times(0,T) \tag{2.19}$$

$$\nabla\chi\cdot\mathbf{n} = 0 \quad \text{a.e. on } \Gamma\times(0,T)\,, \tag{2.20}$$

where we denote by β the subdiferential of $\widehat{\beta}$ ($\beta = \partial\widehat{\beta}$) and by $\gamma = \widehat{\gamma}'$.

Notice that system (2.18)–(2.19) can be rewritten in terms of the vector $u := (s,\chi)$ in a more general framework, as the gradient-flow associated to the functional

$$\Phi(u) = \Phi(s,\chi) = J_V^*(s,\chi) + \int_\Omega \left(\frac{1}{2}|\nabla\chi|^2 + W(\chi)\right)\,\mathrm{d}x\,, \tag{2.21}$$

as

$$\mathcal{N}(u_t) + \frac{\delta\Phi}{\delta u} \ni 0 \quad \text{in } (0,T)\,, \tag{2.22}$$

where $\mathcal{N}$ is the duality map between $\mathcal{H} := V_0'\times H$ and $V_0\times H$ induced by the norm

$$\|u\|_{\mathcal{H}} := \langle A^{-1}(s), s\rangle + \int_\Omega |\chi|^2\,\mathrm{d}x, \quad \text{so that } \mathcal{N}(s,\chi) := (A^{-1}(s),\chi)\,. \tag{2.23}$$

Remark 2.7. *Let us notice that in case* $\mathrm{dom}(j) = \mathbb{R}$*, which is also equivalent to assume*

$$\lim_{|r|\to+\infty} \frac{j^*(r)}{|r|} = +\infty\,,$$

then we can prove that (cf. [2]) the functional e *defined in* (2.11) *can be rewritten as*

$$e(s,\chi,\nabla\chi)=\int_\Omega\left(j^*(s,\chi)+\frac{1}{2}|\nabla\chi|^2+W(\chi)\right)\,\mathrm{d}x\,,$$

where j^* *is the conjugate function of* j *with respect to the variable* s*, i.e.*

$$j^*(s,\chi)=\sup_{\theta\in\mathbb{R}}\left(s\theta-j(\theta,\chi)\right),\quad\forall(s,\chi)\in\mathbb{R}\times\mathbb{R}\,.$$

Moreover, the inclusion (2.18) *can be rewritten as the following gradient flow in* V_0'*:*

$$s_t+\partial_{V'}J_V^*(s,\chi)\ni 0\,,$$

where $\partial_{V'}J_V^*$ *is defined as the subdifferential of* J_V^* *in* V' *mapping* $V'\times H$ *into* $2^{V'}$ *as follows:*

$$\begin{aligned}&\xi\in\partial_{V'}J_V^*(s,\chi)\quad\text{ if and only if }\xi\in V',\\&\quad(s,\chi)\in D(J_V^*),\ \text{and } J_V^*(s,\chi)\le((s-w,\xi))_*+J_V^*(w,\chi)\quad\forall(w,\chi)\in V'\times H\,,\end{aligned}\tag{2.24}$$

where $((\cdot,\cdot))_*$ *denotes the scalar product in* V'*. The reader can refer to [1] and to [2, Section 2] for the proofs of these results. Finally in this case we have* $u\in\partial_{V',V}J_V(s,\chi)$ *in* V *iff* $u\in\partial_s j^*(s,\chi)$ *a.e. in* Ω*, where* ∂_s *denotes here the subdifferential of convex analysis with respect to the variable* s *(cf., e.g., [6]).*

2.3 Possible choices of j^*

In this section we show how to derive different types of phase-field models by our general system.

The Caginalp model of phase transitions. Choose $j^*(s,\chi)=\frac{s^2}{2}-s\chi+\frac{\chi^2}{2}$. Denote by $\theta:=\partial_s j^*=s-\chi$. Then, the Hyp. 2.4 is obviously satisfied and the PDEs (2.18–2.19) can be rewritten as

$$\begin{aligned}&\theta_t+\chi_t-\Delta\theta=0\,,\\&\chi_t-\Delta\chi+\beta(\chi)+\gamma(\chi)-\theta\ni 0\,,\end{aligned}$$

coupled with Neumann homogeneous boundary conditions on θ and χ. This PDE system can be easily identified with the "standard" phase field model of Caginalp type (cf. [7]), letting θ be the relative temperature of the system and χ the local proportion of one of the two phases of the substance undergoing phase transitions.

The entropy model for phase transitions. Choosing $j^*(s,\chi)=j^*(s-\lambda(\chi))=\exp(s-\lambda(\chi))$, we have that Hyp. 2.4 is satisfied in case λ is a Lipschitz continuous function on the domain of β. Then, defining $\theta:=\partial_s j^*=\exp(s-\lambda(\chi))$, we get $s=\log\theta+\lambda(\chi)$ and the PDEs (2.18–2.19) can be rewritten as

$$\begin{aligned}&(\log\theta+\lambda(\chi))_t-\Delta\theta=0\,,\\&\chi_t-\Delta\chi+\beta(\chi)+\gamma(\chi)-\lambda'(\chi)\theta\ni 0\,,\end{aligned}$$

again with Neumann homogeneous boundary conditions for both θ and χ. This system can be easily identified with the "entropy" phase field model introduced in [3] and [4]. Here θ respresents the absolute temperature of the system which is forced to be positive, by the presence of the logarithmic nonlinearity in the θ-equation. Let us notice that in this case the assumption $D(j^*)=\mathbb{R}$ is not verified, hence we are not entitled to use the function j^* instead of the operator J_V^* in e (cf. Remark 2.6), so, the choice we made here is only formal. For a rigorous analysis of this case the reader can refer to [5].

The Penrose-Fife model for phase transitions. We choose $j^*(s,\chi) = -\log(s-\chi)$, for $s > \chi$ and define $\theta := \partial_s j^*(s,\chi) = -\frac{1}{s-\chi}$. Then, we observe that we can formally get the Penrose-Fife mode. Indeed, it results that $\partial_\chi j^*(s,\chi) = \frac{1}{s-\chi} = \frac{1}{\theta}$. Thus, we can rewrite (2.18–2.19) as

$$(\theta \pm \chi)_t \mp \Delta\left(-\frac{1}{\theta}\right) = 0\,,$$
$$\chi_t - \Delta\chi + \beta(\chi) + \gamma(\chi) + \frac{1}{\theta} \ni 0\,,$$

coupled with Neumann homogeneous boundary conditions on $-\frac{1}{\theta}$ and χ. This system can be easily identified with the Penrose-Fife model of phase transitions introduced in [19].

Remark 2.8. *Let us notice that in case we choose as pseudopotential of dissipation the functional* (2.15) *in* (2.16) *and* (2.17)*, the first equation is the same as* (2.18)*, while the equation for χ results as:*

$$A^{-1}\chi_t - \Delta\chi + \beta(\chi) + \gamma(\chi) + \partial_\chi j^*(s,\chi) \ni 0\,, \tag{2.25}$$

and thus

$$s_t + A\left(\partial_s j^*(s,\chi)\right) \ni 0 \quad \text{in } V_0'\,, \tag{2.26}$$
$$\chi_t + Aw = 0, \quad w \in -\Delta\chi + \beta(\chi) + \gamma(\chi) + \partial_\chi j^*(s,\chi)\,, \tag{2.27}$$

coupled with Neumann homogeneous boundary conditions for $\partial_s j^*(s,\chi)$, χ and w. In this case the evolution of χ is ruled by the well-known fouth order Cahn-Hilliard equation modelling phase separation phenomena (cf., e.g., [8]). However, as we already mentioned, we prefer not to deal with this case here.

3 Main results

In this section we state the main results of this paper, the first one (Thm. 3.1) concerns the existence of global in time solutions for system (2.18)–(2.19) coupled with the boundary condition (2.7) and the initial conditions

$$s(0) = s_0 \quad \text{in } D(J_V^*)\,, \tag{3.1}$$
$$\chi(0) = \chi_0 \quad \text{a.e. in } \Omega\,, \tag{3.2}$$

while the second one (Thm. 3.4) regards uniqueness of solutions under more restrictive assumptions on the nonlinearities involved. Let us start with the first result.

Theorem 3.1. *Assume Hypotheses 2.4 and 2.2 and take $s_0 \in D(J_V^*)$, $\chi_0 \in V \cap \mathrm{dom}(\widehat{\beta})$. Then, for every $T > 0$ there exists at least one solution (s,χ) to* (2.18)–(2.20) *and* (3.1)–(3.2) *satisfying the regularity properties:*

$$s \in H^1(0,T;V_0')\,, \tag{3.3}$$
$$\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))\,. \tag{3.4}$$

Proof. In order to prove Theorem 3.1, we first approximate system (2.18–2.19) with a regularized problem depending on a positive small parameter ε and then we pass to the limit by (weak-strong) compactness arguments and semicontinuity results based on sufficient a-priori estimates – independent of ε – we are going to prove on the approximating solutions.

The approximated problem. Let us fix $\varepsilon > 0$. Then, for every $T > 0$ and $(s_{0,\varepsilon}, \chi_0) \in (D(J_V^*) \cap H) \times (V \cap \mathrm{dom}(\widehat{\beta}))$, we aim to find a solution $(s_\varepsilon, \chi_\varepsilon) \in H^1(0,T; V_0' \times H)$ to the following differential inclusions:

$$\partial_t s_\varepsilon + A\left(\eta_\varepsilon + \varepsilon s_\varepsilon\right) = 0 \text{ in } V_0',\ \eta_\varepsilon \in \partial_{V',V} J_V^*(s_\varepsilon, \chi_\varepsilon), \quad \text{for a.e. } t \in (0,T)\,, \tag{3.5}$$

$$\partial_t \chi_\varepsilon - \Delta \chi_\varepsilon + \xi_\varepsilon + \gamma(\chi_\varepsilon) + \partial_\chi J_V^*(s_\varepsilon, \chi_\varepsilon) = 0 \quad \xi_\varepsilon \in \beta(\chi_\varepsilon), \quad \text{a.e. in } \Omega \times (0,T)\,, \tag{3.6}$$

coupled with the boundary and initial conditions (2.20) and (3.1–3.2), with $s_{0,\varepsilon}$ in place of s_0. In particular, we assume that

$$s_{0,\varepsilon} \in D(J_V^*) \cap H, \quad s_{0,\varepsilon} \to s_0 \quad \text{in } V_0' \quad \text{as } \varepsilon \searrow 0. \tag{3.7}$$

We first observe that we can recover (3.5) and and (3.6), by approximating the energy functional (2.21) as follows:

$$\Phi_\varepsilon(s,\chi) := \Psi_\varepsilon(s,\chi) + \int_\Omega \left(\frac{1}{2}|\nabla \chi|^2 + W(\chi)\right) \mathrm{d}x\,,\ \Psi_\varepsilon(s,\chi) := J_V^*(s,\chi) + \frac{\varepsilon}{2}\int_\Omega |s|^2 \,\mathrm{d}x.$$

Actually, note that now Φ_ε is defined in $(V_0' \cap H) \times H$. Hence, we can construct its subdifferential in the duality between V_0' and V_0, and rewrite the equation (3.5) as

$$\partial_t s_\varepsilon + A\zeta_\varepsilon = 0 \text{ in } V_0',\ \zeta_\varepsilon \in \partial_{V',V}\Phi_\varepsilon(s_\varepsilon, \chi_\varepsilon) \qquad \text{for a.e. } t \in (0,T)\,. \tag{3.8}$$

Now, our aim is to prove the existence of solutions of (3.5)–(3.6), (3.1)–(3.2) with $s_{0,\varepsilon}$ istead of s_0, and (2.20) by a time-discrete approximation, as follows (cf. also [21] for a similar procedure). Here we drop the index ε in order to simplify the notation. Let us fix a time step $\tau = T/N$, $N \in \mathbb{N}$ and introduce a uniform partition

$$P_\tau := \{t_0 = 0,\, t_1 = \tau,\, \ldots,\, t_n = n\tau,\, \ldots,\, t_N = T\}$$

of the interval $(0,T)$. Then, we need to find a discrete approximation $s^n \sim s(t_n)$, $\chi^n \sim \chi(t_n)$ by solving the implicit Euler scheme (cf. also (2.22)):

$$\mathcal{N}\left(\frac{U^n - U^{n-1}}{\tau}\right) + \zeta^n = 0, \quad n = 1,\, \ldots,\, N\,; U^0 := u_0\,, \tag{3.9}$$

where $\zeta^n \in \frac{\delta \Phi_\varepsilon}{\delta u}(U^n)$ and we have defined $U^n = (s^n, \chi^n)$, $u_0 = (s_{0,\varepsilon}, \chi_0)$. Using the functional space, we have already introduced to define the operator $\mathcal{N}$, $\mathcal{H} = (V_0' \cap H) \times H$, we notice that (3.9) is the Euler equation for the variational problem

$$\begin{cases} \text{find } U^n \in \mathcal{H} \quad \text{minimizing} \\ F_\varepsilon(\tau, U^{n-1}; U) := \frac{1}{2\tau}\left\|U - U^{n-1}\right\|_{\mathcal{H}}^2 + \Phi_\varepsilon(U), \quad U \in \mathcal{H}\,. \end{cases} \tag{3.10}$$

It is not difficult to see that this minimization problem is solvable due to the lower-semicontinuity and coercivity properties of Φ_ε (cf., e.g., [21, 22] and references therein for a similar variational approach to find a discrete solution).

Then, we can construct the piecewise constant interpolants $\bar{U}_\tau(t) := U^n$ if $t \in ((n-1)\tau,\, n\tau]$. In particular, we get that and we recover the solution $U := (s, \chi)$ of (3.5–3.6) as the limit of $\bar{U}_\tau$ as $\tau \searrow 0$.This can be done, by using suitable a priori estimates (independent of τ and then passing to the limit by compactness and semicontinuity arguments. We do not enter the details of the proof, as it is very similar to the estimates and passage to the limit procedure we are going to detail in the next sections to pass to the limit as $\varepsilon \searrow 0$. Note that in this case some technicalities are avoided due to the more regular setting for the variable s_ε (recall the strict positivity of ε). Thus, we can easily prove the following existence result.

Theorem 3.2. *Under the same assumptions of Theorem 3.1, letting* $\varepsilon > 0$ *be fixed and* (3.7) *holds, then there exists a solution to* (3.5)-(3.6) *with* $s_\varepsilon(0) = s_{0,\varepsilon}$ *and* $\chi_\varepsilon(0) = \chi_0$*, with the following regularity*

$$s_\varepsilon \in H^1(0,T;V_0') \cap L^\infty(0,T;V_0), \tag{3.11}$$
$$\chi_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)). \tag{3.12}$$

A priori estimates (uniform in ε). Let us consider the system (3.5–3.6), where for the sake of simplifying notation we neglect the index ε for solutions and involved functions.We now perform the a priori estimates independent of ε, so, we use here the same symbol c for positive constants, possibily different from line to line, depending on the problem data, but independent of ε.

In order to perform the first estimate we need to prove here a small variant of the chain rule formula stated, e.g., in [9, Prop. 4.2].

Proposition 3.3. *Let* $G : V' \times H \to [0,+\infty]$ *be a map such that*

$v \mapsto G(u,v)$ *is Fréchet differentiable for every* $u \in V'$*,*

$u \mapsto G(u,v)$ *is a proper convex lower semicontinuous mapping for every* $v \in H$*,*

and let $u \in H^1(0,T;V') \cap L^2(0,T;V_0)$*,* $v \in H^1(0,T;H) \cap L^2(0,T;V)$*, and* $\delta(t) \in \partial_{V',V} G(u(t),v(t))$ *for a.e.* $t \in (0,T)$*, where the subdifferential* $\partial_{V',V}$ *is defined as in* (2.12)*. Then the function* $g = G(u(\cdot),v(\cdot))$ *is absolutely continuous in* $[0,T]$ *and* $g'(t) = \langle u'(t), \delta(t)\rangle + (v'(t), \partial_v G(u(t),v(t)))$ *for a.e.* $t \in (0,T)$*.*

Proof. Here we follow the lines of the proof of [9, Prop. 4.2]. Let $w \in W^{1,\infty}(0,T)$ be a non-negative function with compact support in $(0,T)$. We choose $h > 0$ such that supp$(w) \subset [h, T-h]$. For a.e. $t \in (0,T)$, by definition of sub-differentials we can infer that

$$\begin{aligned}&\langle u(t) - u(t-h), \delta(t-h)\rangle + (v(t) - v(t-h), \partial_v G(u(t-h), v(t-h))\\ &\le g(t) - g(t-h) \le \langle u(t) - u(t-h), \delta(t)\rangle + (v(t) - v(t-h), \partial_v G(u(t),v(t)).\end{aligned}$$

Indeed, observe that $(\delta, \partial_v G)$ belongs to ∂G, i.e. to the sub-differential of the function $G : V' \times H \to [0,+\infty]$ defined w.r.t. the variable (u,v). Observe that we can extend w outside of $(0,T)$ with the 0 value. Hence, multiplying by $w(t)$, integrating with respect to t, and letting $h \searrow 0$, we obtain

$$\begin{aligned}\frac{1}{h}\int_h^T \langle u(t) - u(t-h), \delta(t-h)\rangle w(t)dt &= \frac{1}{h}\int_0^{T-h} \langle u(t+h) - u(t), \delta(t)\rangle w(t+h)dt\\ &\to \int_0^T \langle u'(t), \delta(t)\rangle w(t)dt,\\ \frac{1}{h}\int_h^T (g(t) - g(t-h))w(t)dt &= \frac{1}{h}\int_0^T \delta(t)(w(t) - w(t+h))dt\\ &\to -\int_0^T \delta(t)w'(t)dt,\\ \frac{1}{h}\int_h^T \langle u(t) - u(t-h), \delta(t)\rangle w(t)dt &\to \int_0^T \langle u'(t), \delta(t)\rangle w(t)dt.\end{aligned}$$

Moreover,

$$\frac{1}{h}\int_h^T (v(t)-v(t-h),\partial_v G(u(t-h),v(t-h))w(t)dt$$
$$=\frac{1}{h}\int_0^{T-h}(v(t+h)-v(t),\partial_v G(u(t),v(t))w(t+h)dt \to \int_0^T (v'(t),\partial_v G(u(t),v(t))w(t)dt,$$
$$\frac{1}{h}\int_h^T (v(t)-v(t-h),\partial_v G(u(t),v(t))w(t)dt \to \int_0^T (v'(t),\partial_v G(u(t),v(t))w(t)dt.$$

Therefore, we conclude that

$$-\int_0^T g(t)w'(t)dt \;=\; \int_0^T \left(\langle u'(t),\delta(t)\rangle + (v'(t),\partial_v G(u(t),v(t)))\right) w(t)dt$$

for all non-negative Lipschitz continuous test functions w with compact support. Since both the positive and the negative part of a Lipschitz continuous function are Lipschitz continuous, we obtain the assertion. ∎

First a priori estimate. We test (3.5) by $A^{-1}s_t$ getting

$$\langle s_t, A^{-1}s_t\rangle = \|s_t\|^2_{V_0'}, \tag{3.13}$$

and in addition, using the definition of J_V^* and a variant of the chain rule formula stated in Proposition 3.3 with the choices $G = J_V^*$, $u = s$, $v = \chi$, we get

$$\langle A\eta, A^{-1}s_t\rangle = \langle s_t,\eta\rangle = \frac{d}{dt}J_V^*(s(t),\chi(t)) - (\chi_t,\partial_\chi J_V^*(s,\chi))\,. \tag{3.14}$$

Testing (3.6) by χ_t, we get

$$\|\chi_t\|_H^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\chi\|_H^2 + \frac{d}{dt}\int_\Omega W(\chi) + (\chi_t,\partial_\chi J_V^*(s,\chi)) = 0\,. \tag{3.15}$$

Adding the resulting equations and integrating over $(0,t)$, $t\in(0,T)$, and using the definition of Ψ_ε, we obtain

$$\int_0^t \left(\|s_t\|^2_{V_0'} + \|\chi_t\|_H^2\right)\,\mathrm{d}\tau + \Psi_\varepsilon(s(t),\chi(t)) + \int_\Omega W(\chi(t)) + \|\nabla\chi(t)\|^2 \le c, \tag{3.16}$$

where here c depends in particular on the initial data. Adding to both sides in (3.16)

$$\|\chi(t)\|_H^2 = \|\chi_0\|_H^2 + 2\int_0^t (\chi(\tau),\chi_t(\tau))\,\mathrm{d}\tau\,,$$

and using Hölder and Young inequalities together with Hyp. 2.2 and 2.4 and a standard Gronwall lemma, we obtain

$$\|s_t\|^2_{L^2(0,T;V_0')} + \varepsilon\|s\|^2_{L^\infty(0,T;H)} \le c, \tag{3.17}$$
$$\|\chi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c, \tag{3.18}$$
$$\|\widehat{\beta}(\chi)\|_{L^\infty(0,T;L^1(\Omega))} \le c\,. \tag{3.19}$$

Second a priori estimate. We proceed by a comparison in (3.8). Due to $(3.17)_1$, we have that $A\zeta_\varepsilon$ is bounded in $L^2(0,T;V_0')$, and thus

$$\|\zeta_\varepsilon\|_{L^2(0,T;V_0)} \le c\,. \tag{3.20}$$

Hence, using $(3.17)_2$, we get

$$\|\eta_\varepsilon\|_{L^2(0,T;H)} \le c\,. \tag{3.21}$$

Indeed, due to the definition of Ψ_ε we can infer that $\zeta_\varepsilon = \eta_\varepsilon + \partial\psi_\varepsilon(s)$, where we have used the notation $\psi_\varepsilon(s) = \frac{\varepsilon}{2}\int_\Omega s^2 \mathrm{d}x$ and the fact that ψ_ε has sa domain the whole real line and thus its subdifferential in the duality V', V corresponds to the standard subdifferential of the convex analysis (cf. [6]).

Third a priori estimate. Using now Hyp. 2.4 together with (3.21), we get

$$\|\partial_\chi J_V^*(s,\chi)\|_{L^2(0,T;H)} \le c_1\|\eta_\varepsilon\|_{L^2(0,T;H)} + c_2 \le c\,. \tag{3.22}$$

Moreover, by comparison in (3.6) and by standard monotonicity and regularity results, we get

$$\|\xi\|_{L^2(0,T;H)} + \|\chi\|_{L^2(0,T;H^2(\Omega))} \le c\,. \tag{3.23}$$

Passage to the limit as $\varepsilon \searrow 0$. Now, we aim to pass to the limit in (3.5)–(3.6) as $\varepsilon \searrow 0$, revovering finally a solution to (2.18–2.19). By virtue of (3.17–3.23) and by compactness results we get (at least for some subsequences of $\varepsilon \searrow 0$):

$$s_\varepsilon \overset{*}{\rightharpoonup} s \quad \text{in } H^1(0,T;V_0'), \tag{3.24}$$

$$\chi_\varepsilon \overset{*}{\rightharpoonup} \chi \quad \text{in } H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))\,, \tag{3.25}$$

$$\zeta_\varepsilon \rightharpoonup \zeta \quad \text{in } L^2(0,T;V_0)\,, \tag{3.26}$$

$$\eta_\varepsilon \rightharpoonup \eta \quad \text{in } L^2(0,T;H)\,, \tag{3.27}$$

$$\partial_\chi J_V^*(s_\varepsilon,\chi_\varepsilon) \rightharpoonup z \quad \text{in } L^2(0,T;H)\,, \tag{3.28}$$

$$\varepsilon^{1/2} s_\varepsilon \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;H)\,. \tag{3.29}$$

Notice that, by the definition of Ψ_ε and by (3.26), (3.27), and (3.28), we immediately deduce that $\zeta = \eta$ a.e.. Moreover, by strong compactness, from (3.25) we can also deduce (at least) (cf. [23])

$$\chi_\varepsilon \to \chi \quad \text{in } C^0([0,T];H)\cap L^2(0,T;V)\,. \tag{3.30}$$

We aim to identify $\zeta \in \partial_{V',V} J_V^*(s,\chi)$ (see (3.26)). By definition of sub differential this corresponds to prove that

$$\int_0^T \langle v-s,\zeta\rangle\,\mathrm{d}\tau \le \int_0^T \left(J_V^*(v,\chi) - J_V^*(s,\chi)\right)\mathrm{d}\tau \quad \forall v\in V_0' \quad \text{and } \chi\in H\,. \tag{3.31}$$

Note that, if we test the equation (3.5) by $A^{-1}s_\varepsilon$ and integrate in time, by weak lower semicontinuity of norm we have (for $\zeta_\varepsilon \in \partial_{V',V}\Psi_\varepsilon(s_\varepsilon,\chi_\varepsilon)$)

$$\int_0^T \limsup_{\varepsilon\searrow 0} \langle s_\varepsilon,\zeta_\varepsilon\rangle\,\mathrm{d}\tau \le \int_0^T \langle s,\zeta\rangle\,\mathrm{d}\tau\,. \tag{3.32}$$

Hence, by employing (3.26), (3.27), and (3.29), and using the fact that ζ_ε belongs to $\partial_{V',V}\Psi_\varepsilon(s_\varepsilon,\chi_\varepsilon)$,we get, for all $v \in V_0'$ and $\chi_\varepsilon \in H$,

$$\begin{aligned}
&\int_0^T \langle v-s,\zeta\rangle \,\mathrm{d}\tau \le \int_0^T \langle v,\zeta\rangle \,\mathrm{d}\tau - \limsup_{\varepsilon\searrow 0}\int_0^T \langle s_\varepsilon,\zeta_\varepsilon\rangle \,\mathrm{d}\tau \qquad (3.33)\\
&= \int_0^T \langle v,\zeta\rangle \,\mathrm{d}\tau + \liminf_{\varepsilon\searrow 0}\int_0^T \left(-\langle s_\varepsilon,\zeta_\varepsilon\rangle\right)\,\mathrm{d}\tau\\
&\le \liminf_{\varepsilon\searrow 0}\int_0^T \left(\langle v,\zeta_\varepsilon\rangle - \langle s_\varepsilon,\zeta_\varepsilon\rangle\right)\,\mathrm{d}\tau\\
&\le \limsup_{\varepsilon\searrow 0}\int_0^T \left(\Psi_\varepsilon(v,\chi_\varepsilon) - \Psi_\varepsilon(s_\varepsilon,\chi_\varepsilon)\right)\,\mathrm{d}\tau\\
&= -\liminf_{\varepsilon\searrow 0}\int_0^T \left(-J_V^*(v,\chi_\varepsilon) + J_V^*(s_\varepsilon,\chi_\varepsilon) - \frac{\varepsilon}{2}\|v\|_H^2 + \frac{\varepsilon}{2}\|s_\varepsilon\|_H^2\right)\,\mathrm{d}\tau\\
&\le -\liminf_{\varepsilon\searrow 0}\int_0^T \left(-J_V^*(v,\chi_\varepsilon) + J_V^*(s_\varepsilon,\chi_\varepsilon)\right)\,\mathrm{d}\tau\,.
\end{aligned}$$

Using Hyp. 2.4 and (3.30), we can deduce

$$\lim_{\varepsilon\searrow 0} J_V^*(v,\chi_\varepsilon) = J_V^*(v,\chi)\,. \qquad (3.34)$$

Hence, let us observe that

$$J_V^*(s_\varepsilon,\chi_\varepsilon) - J_V^*(s,\chi) = J_V^*(s_\varepsilon,\chi_\varepsilon) - J_V^*(s_\varepsilon,\chi) + J_V^*(s_\varepsilon,\chi) - J_V^*(s,\chi), \qquad (3.35)$$

where

$$I_1 := J_V^*(s_\varepsilon,\chi_\varepsilon) - J_V^*(s_\varepsilon,\chi),$$

and

$$I_2 := J_V^*(s_\varepsilon,\chi) - J_V^*(s,\chi).$$

Hence, by (3.22) and (3.30), we have

$$\int_0^T |I_1|\,\mathrm{d}\tau \le \int_0^T \|\partial_\chi J_V^*(s_\varepsilon,\chi_\varepsilon)\|_H \|\chi_\varepsilon-\chi\|_H \,\mathrm{d}\tau \to 0 \quad \text{as } \varepsilon\searrow 0\,. \qquad (3.36)$$

The second integral

$$\int_0^T I_2 \,\mathrm{d}\tau = \int_0^T \left(J_V^*(s_\varepsilon,\chi) - J_V^*(s,\chi)\right)\,\mathrm{d}\tau$$

is treated by using the fact that J_V^* is lower semicontinuous in s_ε with respect to weak convergence as it is convex (for χ fixed) and so

$$\liminf_{\varepsilon\searrow 0}\int_0^T I_2\,\mathrm{d}\tau \ge 0$$

Hence, coming back to (3.33), and using (2.4), we have, for all $v \in V_0'$ and $\chi_\varepsilon \in H$,

$$\begin{aligned}
&-\int_0^T \liminf_{\varepsilon\searrow 0}\left(-J_V^*(v,\chi_\varepsilon) + J_V^*(s_\varepsilon,\chi_\varepsilon)\right)\,\mathrm{d}\tau\\
&\le -\int_0^T \liminf_{\varepsilon\searrow 0}(-J_V^*(v,\chi_\varepsilon))\,\mathrm{d}\tau - \int_0^T \liminf_{\varepsilon\searrow 0}\left(J_V^*(s_\varepsilon,\chi_\varepsilon) - J_V^*(s,\chi) + J_V^*(s,\chi)\right)\,\mathrm{d}\tau\\
&\le \int_0^T J_V^*(v,\chi) - \int_0^T \liminf_{\varepsilon\searrow 0} I_1 - \int_0^T \liminf_{\varepsilon\searrow 0} I_2 - \int_0^T J_V^*(s,\chi)\,\mathrm{d}\tau\\
&\le \int_0^T \left(J_V^*(v,\chi) - J_V^*(s,\chi)\right)\,\mathrm{d}\tau\,,
\end{aligned}$$

and this concludes the proof of (3.31). Finally, using the previous convergences, where we have now identified $\zeta = \eta \in \partial_{V',V} J_V^*(s,\chi)$ in $L^2(0,T;H)$, we can pass to the limit in the approximated system (3.5–3.6) as well as in the corresponding boundary and initial conditions for $\varepsilon \searrow 0$. This concludes the proof of Theorem 3.1. ∎

We conclude now with the last result of our paper concerning uniqueness of solutions for problem (2.18–2.20), (3.1–3.2).

Theorem 3.4. *Assume Hypotheses 2.4 and 2.2 and take $s_0 \in D(J_V^*)$, $\chi_0 \in V \cap \mathrm{dom}(\widehat{\beta})$ and suppose moreover that*

$$\text{the maps } \chi \mapsto \partial_{V,V'} J_V^*(s,\chi) \quad \text{and } \chi \mapsto \partial_\chi J_V^*(s,\chi) \text{ are Lipschitz continuous} \tag{3.37}$$

from H to V_0 and from H to H, respectively, for every $s \in V'$. Then the solution (s,χ) of (2.18–2.20), (3.1–3.2) is uniquely determined and the following continuous dependence estimate holds true:

$$\begin{aligned}\|(s_1-s_2)(t)\|_{V'}^2 + \|(\chi_1-\chi_2)(t)\|_H^2 + \int_0^t \|\nabla(\chi_1-\chi_2)\|_H^2 \,\mathrm{d}\tau \le & C\left(\|(s_1-s_2)(0)\|_{V'}^2\right. \\ & \left. + \|(\chi_1-\chi_2)(0)\|_H^2\right).\end{aligned} \tag{3.38}$$

Proof. Let us take the difference of the two equations (2.18) and the two relations (2.19) corresponding to two different solutions (s_i,χ_i), $i=1,\,2$ and test them by $A^{-1}(s_1-s_2)$ and $(\chi_1-\chi_2)$, respectively. Integrating over $(0,t)$, for $t \in [0,T]$, using Hyp. 2.4 and Hyp. 2.2, we get

$$\begin{aligned}\|(s_1-s_2)(t)\|_{V'}^2 + \|(\chi_1-\chi_2)(t)\|_H^2 + 2\int_0^t \|\nabla(\chi_1-\chi_2)\|_H^2 \,\mathrm{d}\tau \le \|(s_1-s_2)(0)\|_{V'}^2 \\ + \|(\chi_1-\chi_2)(0)\|_H^2 - 2\int_0^t (\gamma(\chi_1)-\gamma(\chi_2),\chi_1-\chi_2)\,\mathrm{d}\tau \\ - 2\int_0^t \langle s_1-s_2, \partial_{V,V'} J_V^*(s_2,\chi_1) - \partial_{V,V'} J_V^*(s_2,\chi_2)\rangle \,\mathrm{d}\tau \\ - 2\int_0^t (\chi_1-\chi_2, \partial_\chi J_V^*(s_2,\chi_1) - \partial_\chi J_V^*(s_2,\chi_2))\,\mathrm{d}\tau\,.\end{aligned}$$

Using then the Lipschitz continuity of γ and assumption (3.37), together with Gronwall lemma, we obtain exactly (3.38). ∎

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