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Analysis of $p(x)$-Laplace thermistor models describing the electrothermal behavior of organic semiconductor devices

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Abstract

We study a stationary thermistor model describing the electrothermal behavior of organic semiconductor devices featuring non-Ohmic current-voltage laws and self-heating effects. The coupled system consists of the current-flow equation for the electrostatic potential and the heat equation with Joule heating term as source. The self-heating in the device is modeled by an Arrhenius-like temperature dependency of the electrical conductivity. Moreover, the non-Ohmic electrical behavior is modeled by a power law such that the electrical conductivity depends nonlinearly on the electric field. Notably, we allow for functional substructures with different power laws, which gives rise to a $p(x)$-Laplace-type problem with piecewise constant exponent.

We prove the existence and boundedness of solutions in the two-dimensional case. The crucial point is to establish the higher integrability of the gradient of the electrostatic potential to tackle the Joule heating term. The proof of the improved regularity is based on Caccioppoli-type estimates, Poincaré inequalities, and a Gehring-type Lemma for the $p(x)$-Laplacian. Finally, Schauder's fixed-point theorem is used to show the existence of solutions.

1 Introduction

This paper is devoted to the analysis of a stationary thermistor model that was recently introduced in [18] to describe electrothermal effects, such as self-heating and inhomogeneous current distributions, in large-area Organic Light-Emitting Diodes (OLEDs). The model consists of the current-flow equation for the electrostatic potential φ and the heat equation with Joule heat source term for the temperature T in a domain Ω and reads as

$$\begin{aligned} -\nabla\cdot(\sigma(x,T,|\nabla\varphi|)\nabla\varphi) &= 0, \\ -\nabla\cdot(\lambda(x)\nabla T) &= (1-\eta)\sigma(x,T,|\nabla\varphi|)|\nabla\varphi|^2. \end{aligned} \tag{1.1}$$

Here, σ and λ are the electrical and thermal conductivities, respectively, and $\eta \in [0,1]$ is the efficiency of the light outcoupling, which describes how much of the electric power is emitted as light and not converted into heat. The key feature of the model is that the electrical conductivity σ depends on the temperature and on the electric field $E = -\nabla\varphi$. In [18], for organic semiconductor devices σ is proposed to be of the general form

$$\sigma(x,T,|\nabla\varphi|) = \sigma_0(x)F(x,T)\Big[\frac{|\nabla\varphi|}{V_{\mathrm{ref}}/d}\Big]^{p(x)-2} \tag{1.2}$$

with σ_0 being an effective conductivity, $F(x,T)$ is an Arrhenius-type temperature factor, and V_{ref} and d are a reference voltage and thickness, respectively. Notably, $p(x) \geq 2$ is a power-law exponent that depends on the spatial coordinate. In particular, OLEDs are thin-film heterostructure devices based on organic molecules or polymers, where each functional layer (electrode layer, electron and hole transport layers, emitting layer, see Fig. 1) has, in general, its own current-voltage characteristics and material parameters. In the electrode (made of Indium-Tin-Oxide), for example, an Ohmic behavior can be observed, which means that $p(x) = p_{\mathrm{ITO}} = 2$. For organic semiconductor materials we have non-Ohmic behavior corresponding to exponents $p_{\mathrm{org}} > 2$. This has been experimentally verified in [9] and [11] (in [9] a value of $p_{\mathrm{org}} = 9.7$ was obtained for OLED materials from fitting to experimental data).

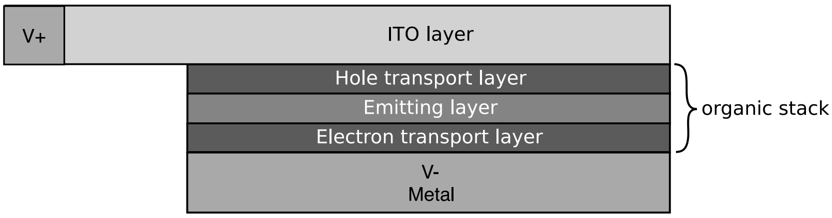

Figure 1: Schematic cross section of an OLED with crossbar contacts.

To take this behavior into account, we allow $p(x)$ to be piecewise constant with different values in each substructure of the device. In particular, this means that the current-flow equation in (1.1) is of $p(x)$-Laplace type, which makes the mathematical analysis challenging.

The temperature dependence of the conductivity is given by an Arrhenius-type factor

$$F(x,T) = \exp\Big[-\frac{E_{\mathrm{act}}(x)}{k_{\mathrm{B}}}\Big(\frac{1}{T}-\frac{1}{T_a}\Big)\Big], \tag{1.3}$$

where $E_{\mathrm{act}}(x) \geq 0$ represents the activation energy in the materials, T_a is the ambient temperature and k_{B} is Boltzmann's constant. Since the coefficient in front of the inverse temperature is negative, a rising temperature leads to an increase of the electric current for a constant applied voltage. By the Joule heat term in the second equation in (1.1), this leads to even higher temperatures in the device. Thus, a positive feedback loop is obtained, which continuously heats up the structure. Often physical experiments of this kind lead to the destruction of the device by thermal runaway, see [10].

In the zero-dimensional (i.e. spatially homogeneous) setting discussed in [11], the current-voltage characteristics for such devices show an S-shaped behavior for sufficiently high activation energies ($E_{\mathrm{act}} > 4k_{\mathrm{B}}T_a$). In particular, a region of negative differential resistance appears, which was also experimentally verified. This mechanism, together with the high resistivity of the optically transparent ITO anode, is considered as explanation for inhomogeneities in current distributions and unwanted pattern formation in the luminance of large-area OLEDs, see [3, 9].

The system in (1.1) is complemented by boundary conditions that model the electrical contacts and the thermal coupling to the environment. They read as

$$\begin{aligned} &\varphi = \varphi^D \quad \text{on} \quad \Gamma_D, \quad \sigma(x,T,|\nabla\varphi|)\nabla\varphi\cdot\nu = 0 \quad \text{on} \quad \Gamma_N, \\ &-\lambda(x)\nabla T\cdot\nu = \kappa(x)(T-T_a) \quad \text{on} \quad \Gamma := \partial\Omega. \end{aligned} \tag{1.4}$$

Here, φ^D is the Dirichlet data corresponding to the applied voltage at the contacts located at Γ_D and Γ_N is formed by that part of $\partial\Omega$ which does not belong to Γ_D. The Robin boundary condition for the heat flow equation expresses the heat transfer to the environment. The spatially dependent heat transfer coefficient κ takes care of the varying heat conduction of the surrounding materials. For a detailed discussion of the physical background of the thermistor model, we refer to [18].

For the mathematical analysis of the thermistor system in (1.1) – (1.4) several features of OLEDs have to be taken into account: We work in nonsmooth domains with mixed boundary conditions. The parameters $p, E_{\text{act}}, \sigma_0, \lambda, \eta$, and κ jump at interfaces between different materials. In particular, the exponent p is spatially varying and piecewise constant and takes values in the range of 2 (Ohmic material, e.g. ITO contacts) to 10 (organic semiconductor material). Thus, subdomains $\Omega_i \subset \Omega$ with different exponents $p_i > 2$ have to be considered in the problem.

To treat the spatially varying exponent p we work in the generalized Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ (see Subsection 2.2 or [5]). While the spaces $W^{1,p(\cdot)}(\Omega)$ share several characteristics with their classical counterparts $W^{1,\overline{p}}(\Omega)$ for constant $\overline{p}$, there are a number of properties (e.g. Poincaré and Sobolev inequalities), which do not follow naturally without additional assumptions on p. In particular, many results for $W^{1,p(\cdot)}(\Omega)$ rely on the assumption that $x \mapsto p(x)$ is log-Hölder continuous, which is not satisfied in our setting.

To prove the existence of solutions to (1.1) – (1.4) we apply Schauder's fixed-point theorem for the temperature distribution T. First, for a given $\widetilde{T}$ we obtain a unique solution $\varphi(\widetilde{T})$ of the current flow equation and prove L^∞-bounds and regularity results for the potential $\varphi(\widetilde{T})$. Next, exploiting these regularity results we give a weak formulation for the coupled problem and establish a priori estimates for the solution. Finally, we show that this solution can be obtained via a fixed-point map $Q : \widetilde{T} \mapsto T$, where T solves the heat equation (2.1b) for the Joule heat given by the electrostatic potential $\varphi(\widetilde{T})$ and $F(\widetilde{T})$. In particular, the proof follows the ideas in [18], where the case of a constant exponent p was considered.

The crucial point in this procedure now is the regularity result for φ, which allows us to exploit the elliptic theory for the heat equation. In particular, we show that $|\nabla\varphi| \in L^{s^*p(\cdot)}(\Omega)$, where $s^* > 1$ is a uniform exponent that does not depend on the temperature $\widetilde{T}$. Following the ideas in [8] and adapting some steps to the case of two different values of p in the localized situation depicted in Fig. 2, the higher regularity is obtained from Caccioppoli-type estimates, Poincaré inequalities, and a Gehring-type Lemma in the version of Giaquinta-Modica (see [14, Theorem 6.6]) for the $p(x)$-Laplacian. Let us note that higher regularity results for the $p(x)$-Laplacian are available in the log-Hölder continuous case [1, 6, 22, 23]. In particular, the regularity result presented here is, as far as we know, the first result for nonsmooth exponents $p(x)$.

Plan of the paper. We start in Section 2 with a non-dimensionalization of the thermistor system (1.1) – (1.4), which allows us to rewrite the system in a simpler, dimensionless form. Moreover, we state the main assumptions on the data and the underlying domain and introduce the function spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ for variable exponents. In Section 3 we proof the main result of this paper – the existence of solutions to the coupled thermistor system (1.1) – (1.4). Here, we apply Schauder's fixed-point theorem and exploit that solutions of the current-flow equation have a higher regularity in our setting. The proof of the higher integrability of the gradient of φ is postponed to Section 4. In particular, the result is obtained by localization of the problem to squares and careful estimates that are uniform with respect to the diameter of subsquares. Finally, inequalities and auxiliary results that are used throughout the proofs are collected in the Appendix.

2 Preliminaries

2.1 Non-dimensionalization of the system

For notational simplicity we work with dimensionless quantities. To this end, we introduce reference values V_0, I_0, T_0, and L_0 for voltage, current, temperature, and spatial coordinate, respectively. We set

$$\widehat{\varphi} = \frac{\varphi}{V_0}, \qquad \widehat{T} = \frac{T}{T_0}, \quad \text{and} \quad \widehat{x} = \frac{x}{L_0}$$

and define the non-dimensionalized coefficients

$$\begin{aligned} \widehat{\sigma}_0(\widehat{x}) &= \frac{V_0 L_0}{I_0}\sigma_0(L_0\widehat{x}), \qquad & \widehat{\beta}(\widehat{x}) &= \frac{E_{\text{act}}(L_0\widehat{x})}{k_{\text{B}}T_0}, \\ \widehat{\lambda}(\widehat{x}) &= \frac{T_0 L_0}{I_0 V_0}\lambda(L_0\widehat{x}), \qquad & \widehat{\kappa}(\widehat{x}) &= \frac{T_0 L_0^2}{I_0 V_0}\kappa(L_0\widehat{x}). \end{aligned}$$

We write the exponent p and the Arrhenius factor F as

$$\widehat{p}(\widehat{x}) = p(L_0\widehat{x}), \qquad \widehat{F}(\widehat{x},\widehat{T}) = \exp\Big[-\widehat{\beta}(\widehat{x})\Big(\frac{1}{\widehat{T}} - \frac{1}{\widehat{T}_{\text{a}}}\Big)\Big] \quad \text{with} \quad \widehat{T}_{\text{a}} = \frac{T_a}{T_0}$$

and choose V_0 and L_0 such that $(V_0 d)/(V_{\text{ref}}L_0) = 1$ is satisfied. Having in mind that $\nabla_x = \frac{1}{L_0}\nabla_{\widehat{x}}$ we can rewrite the system in (1.1) – (1.4) in a dimensionless form for $\widehat{\varphi}$ and $\widehat{T}$. Finally, in the resulting system of equations we drop the hats above the symbols and arrive at

$$-\nabla\cdot(\sigma(x,T,|\nabla\varphi|)\nabla\varphi) = 0 \qquad \text{on } \Omega, \tag{2.1a}$$

$$-\nabla\cdot(\lambda(x)\nabla T) = (1-\eta(j,T))\sigma(x,T,|\nabla\varphi|)|\nabla\varphi|^2 \quad \text{on } \Omega, \tag{2.1b}$$

where

$$\sigma(x,T,|\nabla\varphi|) = \sigma_0(x)F(x,T)|\nabla\varphi|^{p(x)-2}. \tag{2.1c}$$

The system is complemented with the mixed boundary conditions

$$\varphi = \varphi^D \quad \text{on} \quad \Gamma_D, \quad \sigma(x,T,|\nabla\varphi|)\nabla\varphi\cdot\nu = 0 \quad \text{on} \quad \Gamma_N, \tag{2.1d}$$

$$-\lambda(x)\nabla T\cdot\nu = \kappa(x)(T-T_a) \quad \text{on} \quad \Gamma := \partial\Omega, \tag{2.1e}$$

where φ^D is a function representing the non-dimensionalized Dirichlet values at all electrical contacts.

2.2 Assumptions and notation

Here, we collect the essential assumptions on the domain Ω as well as on the given data and fix the notation for the subsequent sections. In the following, we denote by $C_r(y) \subset \mathbb{R}^2$ the square with center $y \in \mathbb{R}^2$ and side length $2r$ and $|\cdot|_\infty$ is the supremum norm on $\mathbb{R}^2$. We start with the definition of regular domains due to Gröger.

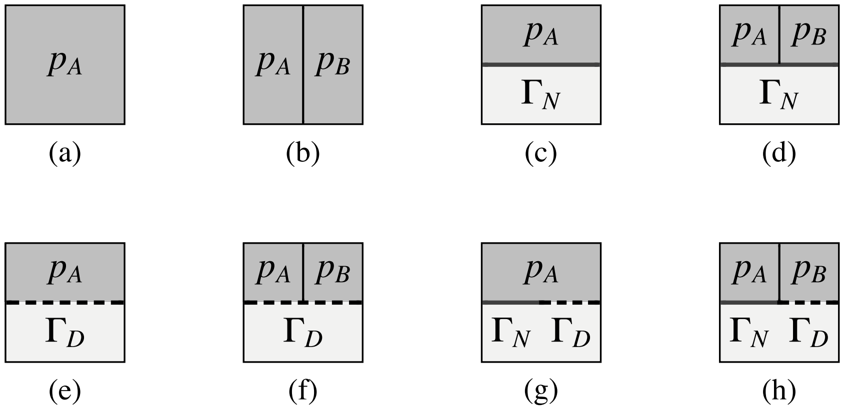

Figure 2: Model sets $C_1(0)$ with different constant exponents p with $p_A < p_B$ and different types of boundary conditions.

Definition 2.1 (Regular domain [15]) *We call $G \subset \mathbb{R}^2$ regular, if G is bounded and if for every $x^0 \in \partial G$ there exist subsets $U_{x^0} \subset \mathbb{R}^2$ and a bi-Lipschitz transformation $\Phi_{x^0} : U_{x^0} \to C_1(0)$ such that U_{x^0} is an open neighborhood of $x^0 \in \mathbb{R}^2$, $\Phi_{x^0}(U_{x^0}) = C_1(0)$, and $\Phi_{x^0}(x^0) = 0$. Furthermore, the image $\Phi_{x^0}(U_{x^0} \cap G)$ is one of the following sets:*

$$
\begin{aligned}
E_1 &:= \{y \in \mathbb{R}^2 : |y|_\infty < 1,\ y_2 > 0\},\\
E_2 &:= \{y \in \mathbb{R}^2 : |y|_\infty < 1,\ y_2 \geq 0\},\\
E_3 &:= \{y \in E_2 : y_2 > 0 \ \mathit{or}\ ,\ y_1 > 0\}.
\end{aligned}
$$

Note that Poincaré-type inequalities and Sobolev's embedding theorems are available on regular domains, see e.g. [14, Theorems 3.11-3.13]. To treat the mixed boundary conditions in (2.1d), we consider $G = \Omega \cup \Gamma_N$ and make the following assumptions for the analytical investigations:

Assumption (A1)

(i) $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitzian domain and Γ_D, Γ_N are disjoint open subsets of $\Gamma := \partial\Omega$ satisfying $\mathrm{mes}(\Gamma_D) > 0$, $\Gamma = \Gamma_D \cup \Gamma_N \cup (\overline{\Gamma_D} \cap \overline{\Gamma_N})$, and $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ consists of finitely many points. In particular, $\Omega \cup \Gamma_N$ is regular in the sense of Gröger [15], see Definition 2.1.

(ii) Ω satisfies $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega}_i$, where Ω_i are disjoint subdomains, and $x \mapsto p(x)$ is such that $p(x) = p_i \in [2, \infty)$ for $x \in \Omega_i$.

(iii) There exists a finite number of points $x_j^0 \in \partial\Omega$, for $j = 1, \ldots, N$, and $x_j^0 \in \Omega$, for $j = N{+}1, \ldots, M$, with corresponding neighborhoods U_j and one-to-one bi-Lipschitzian maps $\Phi_{x_j^0} : U_j \to C_1(0)$ such that $\mathrm{mes}(U_j \cap \Omega_i) \neq 0$ for at most two subdomains Ω_{j_A} and Ω_{j_B}. Additionally, we assume that $\Phi_j(U_j)$ is one of the model sets $C_1(0)$ given in Fig. 2 and $\bigcup_{j=1}^M (\Phi_{x_j^0})^{-1}(C_{\frac{1}{24}}(0)) \supset \overline{\Omega}$ is a finite covering of Ω.

Assumption (A2)

(i) The Dirichlet datum satisfies $\varphi^D \in W^{1,\infty}(\Omega)$.

(ii) The electrical conductivity $\sigma : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is of the form $\sigma(x,T,z) = \sigma_0(x)F(x,T)z^{p(x)-2}$, where $\sigma_0 \in L^\infty(\Omega)$ satisfies $\underline{\sigma_0} \le \sigma_0 \le \overline{\sigma_0}$ a.e. on Ω. The Arrhenius factor is of the form $F(x,T) = \exp\big[-\beta(x)\big(\frac{1}{T} - \frac{1}{T_a}\big)\big]$ with $\beta \in L^\infty_+(\Omega)$ and $T_a \in \mathbb{R}$, $T_a > 0$.

(iii) The heat conductivity λ satisfies $\lambda \in L^\infty(\Omega)$ and $\lambda \ge c > 0$ a.e. on Ω. The heat transfer coefficient κ is such that $\kappa \in L^\infty_+(\Gamma)$ and $\|\kappa\|_{L^1(\Gamma)} > 0$.

(iv) The light-outcoupling factor $\eta = \eta(x,T,j)$ is such that $\eta : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a Caratheodory function and $\eta(x,T,j) \in [0,1]$ holds f.a.a. $x \in \Omega$ and $\forall (T,j) \in \mathbb{R} \times \mathbb{R}^2$.

For constant $p \in (1,\infty)$, we work with the Sobolev spaces

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \le 1\}$$

equipped with the norm

$$\|u\|^p_{W^{1,p}} = \sum_{|\alpha|\le 1} \|D^\alpha u\|^p_{L^p}$$

and

$$W^{1,p}_D(\Omega \cup \Gamma_N) = \{u \in W^{1,p}(\Omega) :\ u|_{\Gamma_D} = 0\}.$$

For $p = 2$, we also write $H^1(\Omega)$ instead of $W^{1,2}(\Omega)$. Moreover, the dual space of a Banach space X is denoted by X^*.

Following [17, 7, 5], we introduce the generalized function spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, where $x \mapsto p(x)$ is a measurable function satisfying $p : \Omega \to (1,\infty)$. In particular, we write $p \in \mathcal{P}(\Omega)$ if $p : \Omega \to (1,\infty)$ is measurable and define

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$$

and consider bounded variable exponents $p \in \mathcal{P}(\Omega)$ with $p_+ < \infty$. The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions f for which the modular

$$\rho_{p(\cdot)}(f) := \int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x$$

is finite, see [17, 5]. With the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}} := \inf\Big\{\tau > 0 :\ \rho_{p(\cdot)}\Big(\frac{f}{\tau}\Big) \le 1\Big\}$$

$L^{p(\cdot)}(\Omega)$ becomes a Banach space and it holds that $\rho_{p(\cdot)}(f) \le 1$ if and only if $\|f\|_{L^{p(\cdot)}(\Omega)} \le 1$.

We collect some properties for $L^{p(\cdot)}(\Omega)$ spaces for the case that $1 < p_- \le p_+ < \infty$, for the more general situation see [17]: For all f with $0 < \|f\|_{L^{p(\cdot)}} < \infty$ it holds true that

$$\rho_{p(\cdot)}(f/\|f\|_{L^{p(\cdot)}}) = 1, \tag{2.2}$$

and [5, Lemma 3.2.5] ensures for all $f \in L^{p(\cdot)}(\Omega)$ the inequality

$$\min\left\{\rho_{p(\cdot)}(f)^{\frac{1}{p_-}}, \rho_{p(\cdot)}(f)^{\frac{1}{p_+}}\right\} \le \|f\|_{L^{p(\cdot)}} \le \max\left\{\rho_{p(\cdot)}(f)^{\frac{1}{p_-}}, \rho_{p(\cdot)}(f)^{\frac{1}{p_+}}\right\}. \tag{2.3}$$

Moreover, according to [17, Formula (2.28)]

$$\text{If } p_+ < \infty, \quad \text{then} \quad \rho_{p(\cdot)}(f_n) \to 0 \quad \text{if and only if} \quad \|f_n\|_{L^{p(\cdot)}} \to 0. \tag{2.4}$$

The generalized Sobolev space $W^{1,p(\cdot)}(\Omega)$ is the class of functions on Ω such that $D^\alpha f \in L^{p(\cdot)}(\Omega)$ for every multi-index α with $|\alpha| \le 1$. It is equipped with the norm

$$\|f\|_{W^{1,p(\cdot)}} := \sum_{|\alpha|\le 1} \|D^\alpha f\|_{L^{p(\cdot)}}, \tag{2.5}$$

see [17] and [5, Def. 8.1.4., Rem. 8.1.5]. By the mapping $u \mapsto (u, \nabla u)$, the space $W^{1,p(\cdot)}(\Omega)$ is a closed subspace of $L^{p(\cdot)}(\Omega) \times (L^{p(\cdot)}(\Omega))^n$. Under assumption (A1) we introduce the (closed) subspace $W^{1,p(\cdot)}_D(\Omega) \subset W^{1,p(\cdot)}(\Omega)$ of functions with homogeneous Dirichlet values at Γ_D,

$$W^{1,p(\cdot)}_D(\Omega) = \{\varphi \in W^{1,p(\cdot)}(\Omega) : \varphi|_{\Gamma_D} = 0\}$$

equipped with the norm (2.5). The spaces $W^{1,p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}_D(\Omega)$ are separable, reflexive Banach spaces. Note, that for $p_- \ge 2$ we always have the continuous embedding $W^{1,p(\cdot)}(\Omega) \subset H^1(\Omega)$.

In our estimates, positive constants, which may depend at most on the data of our problem, are denoted by c. In particular, we allow them to change from line to line.

For the local treatment of the $p(x)$-Laplace expressions we make use of the following inequalities: For an arbitrary, constant exponent $p \ge 1$ we consider the function $z \mapsto |z|^p$, which due to its convexity satisfies the inequality

$$\left|\frac{z_1 + z_2}{2}\right|^p \le \frac{|z_1|^p + |z_2|^p}{2} \qquad \text{for } z_1,\, z_2 \in \mathbb{R}^n. \tag{2.6}$$

Exploiting the subdifferential estimate gives the inequality

$$|z_1|^p \ge |z_2|^p + p|z_2|^{p-2} z_2 \cdot (z_1 - z_2) \qquad \text{if } p \ge 1, \quad z_1,\, z_2 \in \mathbb{R}^n. \tag{2.7}$$

Additionally, we apply the inequality

$$\begin{aligned}\big(|z_1|^{p-2} z_1 - |z_2|^{p-2} z_2\big) \cdot (z_1 - z_2) &\ge 2^{-1}\big(|z_1|^{p-2} + |z_2|^{p-2}\big)|z_1 - z_2|^2 \\ &\ge 2^{2-p}|z_1 - z_2|^p \qquad \text{if } p \ge 2, \quad z_1,\, z_2 \in \mathbb{R}^n,\end{aligned} \tag{2.8}$$

which can be found in [19, Chapter 10]. Moreover, we use the lower estimate

$$(|z_1|^{p-2} z_1 - |z_2|^{p-2} z_2) \cdot (z_1 - z_2) \ge c\big(|z_1| + |z_2|\big)^{p-2} |z_1 - z_2|^2 \quad \text{for } z_1,\, z_2 \in \mathbb{R}^n, \tag{2.9}$$

see [16, Lemma A.1] (with $F(A) = \frac{1}{p}|A|^p$). Finally, for $p \ge 2$ we note the estimate

$$\big||z_1|^{p-2} z_1 - |z_2|^{p-2} z_2\big| \le c\big(|z_1| + |z_2|\big)^{p-2} |z_1 - z_2| \quad \text{for } z_1,\, z_2 \in \mathbb{R}^n. \tag{2.10}$$

For a spatially dependent $p \in \mathcal{P}(\Omega)$ we have to distinguish in some integral estimates the subsets of Ω for which we have $p(x) = 2$ and $p(x) > 2$, respectively. Therefore, we introduce the notation

$$\Omega_1 := \{x \in \Omega : p(x) = 2\}, \quad \Omega_0 := \Omega \setminus \Omega_1, \quad \text{and} \quad p_{0-} := \operatorname*{ess\,inf}_{x \in \Omega_0} p(x). \tag{2.11}$$

3 Analysis for the $p(x)$-Laplace thermistor model

3.1 Results for the current flow equation

In the first step, we turn our attention to the current-flow equation (2.1a) for the potential φ. In particular, we consider an arbitrary but fixed T, which is assumed to lie in the set of relevant temperature distributions given by

$$\mathcal{T} := \{T \in H^1(\Omega) \cap L^\infty(\Omega) \,:\, T \geq T_a \text{ a.e. on } \Omega\}. \tag{3.1}$$

According to (A2), we find for $T \in \mathcal{T}$ that

$$\sigma_0 F(\cdot, T) \in L^\infty(\Omega) \quad \text{and} \quad \sigma_0(x) F(x, T(x)) \in \big[\underline{\sigma_0}, \overline{\sigma_0}\mathrm{e}^{\|\beta\|_{L^\infty}/T_a}\big) =: [\sigma_1, \sigma_2). \tag{3.2}$$

For fixed $T \in \mathcal{T}$, we introduce the operator $A_T : \varphi^D + W_D^{1,p(\cdot)}(\Omega) \to (W_D^{1,p(\cdot)}(\Omega))^*$

$$\langle A_T(\varphi), v\rangle_{W_D^{1,p(\cdot)}} := \int_\Omega \sigma(x, T, \nabla\varphi)\nabla\varphi \cdot \nabla v \,\mathrm{d}x, \quad v \in W_D^{1,p(\cdot)}(\Omega),$$

and consider the following problem: Find $\varphi \in \varphi^D + W_D^{1,p(\cdot)}(\Omega)$ such that

$$\langle A_T(\varphi), v\rangle_{W_D^{1,p(\cdot)}} = 0 \quad \text{for all } v \in W_D^{1,p(\cdot)}(\Omega), \tag{3.3}$$

which corresponds to finding a weak solution $\varphi \in \varphi^D + W_D^{1,p(\cdot)}(\Omega)$ of the current-flow equation (2.1a) with boundary conditions (2.1d) and fixed temperature distribution $T \in \mathcal{T}$.

Lemma 3.1 *We assume (A1) and (A2). Let $T \in \mathcal{T}$ be a fixed given function. Then* (3.3) *has exactly one solution φ, and for almost all $x \in \Omega$ this solution is bounded by*

$$\operatorname*{ess\,inf}_{x\in\Omega} \varphi^D \leq \varphi(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} \varphi^D. \tag{3.4}$$

Moreover, there are constants $c_\varphi > 0$ and $c_{int} > 0$, depending only on the data (Ω, φ^D, $\underline{\sigma_0}$, $\overline{\sigma_0}$, T_a, and β) but not on T, such that

$$\|\varphi\|_{W^{1,p(\cdot)}} \leq c_\varphi, \quad \rho_{p(\cdot)}(|\nabla\varphi|) = \int_\Omega |\nabla\varphi|^{p(x)} \,\mathrm{d}x \leq c_{int}. \tag{3.5}$$

Proof. In the following, we denote with $h^+ = \max(0, h)$ the positive and with $h^- = \max(-h, 0)$ the negative part of a function h, respectively.

1. Uniform bounds. First, we show the bounds of solutions to (3.3). Let $\overline{\varphi^D} := \|\varphi^D\|_{L^\infty}$, $\underline{\varphi^D} :=$ ess $\inf_{x\in\Omega}\varphi^D$. Then, the test of (3.3) with $(\varphi - \overline{\varphi^D})^+ \in W_D^{1,p(\cdot)}(\Omega)$ gives

$$0 = \int_\Omega \sigma(x, T, |\nabla\varphi|)\big|\nabla(\varphi - \overline{\varphi}^D)^+\big|^2 \,\mathrm{d}x \geq \int_\Omega \sigma_1\big|\nabla(\varphi - \overline{\varphi^D})^+\big|^{p(x)} \,\mathrm{d}x$$

leading to $\varphi \leq \overline{\varphi^D}$ a.e. in Ω. On the other hand, the test of (3.3) with $-(\varphi - \underline{\varphi^D})^-$ ensures

$$0 = \int_\Omega \sigma(x, T, |\nabla\varphi|)\big|\nabla(\varphi - \underline{\varphi}^D)^-\big|^2 \,\mathrm{d}x \geq \int_\Omega \sigma_1\big|\nabla(\varphi - \underline{\varphi^D})^-\big|^{p(x)} \,\mathrm{d}x$$

and therefore $\varphi \geq \underline{\varphi^D}$ a.e. in Ω. Therefore, (3.4) is verified.

To obtain the integral estimate (3.5) for the powers of the gradient, we use the test function $\varphi - \varphi^D$ for (3.3) to obtain

$$\begin{aligned}\int_\Omega \sigma_1 |\nabla\varphi|^{p(x)} \mathrm{d}x &\leq \int_\Omega \sigma(x,T,|\nabla\varphi|)|\nabla\varphi|^2 \mathrm{d}x \\ &= \int_\Omega \sigma(x,T,|\nabla\varphi|)\nabla\varphi\cdot\nabla(\varphi-\varphi^D)\mathrm{d}x + \int_\Omega \sigma(x,T,|\nabla\varphi|)\nabla\varphi\cdot\nabla\varphi^D \mathrm{d}x \\ &\leq \int_\Omega \sigma_2 |\nabla\varphi|^{p(x)-1}|\nabla\varphi^D|\mathrm{d}x \leq \int_\Omega \frac{\sigma_1}{2}|\nabla\varphi|^{p(x)}\mathrm{d}x + c\int_\Omega |\nabla\varphi^D|^{p(x)}\mathrm{d}x,\end{aligned}$$

where we have used that the first term in the second line vanishes since φ is a solution to (3.3). Together with the assumed L^∞-bounds for $\nabla\varphi^D$ (see (A2)) and φ (see (3.4)), this estimate leads to the desired estimate for $\int_\Omega |\nabla\varphi|^{p(x)}\,\mathrm{d}x$ and $\int_\Omega |\varphi|^{p(x)}\,\mathrm{d}x$. Thus, by (2.3) we have proven (3.5).

2. Existence of a solution to (3.3). Since we assume that $p \geq 2$ a.e. on Ω we obtain from a pointwise application of (2.8) that

$$\begin{aligned}&\langle A_T\varphi_1 - A_T\varphi_2, \varphi_1 - \varphi_2\rangle_{W_D^{1,p(\cdot)}} \\ &\qquad = \int_\Omega \sigma_0 F(\cdot,T)\big(|\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1 - |\nabla\varphi_2|^{p(x)-2}\nabla\varphi_2\big)\cdot\nabla(\varphi_1-\varphi_2)\,\mathrm{d}x \\ &\qquad \geq \int_\Omega \sigma_1 2^{2-p(x)}|\nabla(\varphi_1-\varphi_2)|^{p(x)}\,\mathrm{d}x \geq 0,\end{aligned}$$

which verifies that the operator A_T is monotone. To prove the continuity of A_T, we take an arbitrary sequence $\varphi_n - \varphi \to 0$ in $W_D^{1,p(\cdot)}(\Omega)$ and show that $A_T\varphi_n - A_T\varphi \to 0$ in $W_D^{1,p(\cdot)}(\Omega)^*$. Having in mind (2.10) we have to estimate

$$\begin{aligned}&\|A_T\varphi_n - A_T\varphi\|_{W_D^{1,p(\cdot)}(\Omega)^*} \\ &= \sup_{v\in W_D^{1,p(\cdot)}(\Omega),\, \|v\|_{W^{1,p(\cdot)}}\leq 1} \langle A_T\varphi_n - A_T\varphi, v\rangle_{W_D^{1,p(\cdot)}} \\ &\leq \sup_{v\in W_D^{1,p(\cdot)}(\Omega),\, \|v\|_{W^{1,p(\cdot)}}\leq 1} \int_\Omega c\sigma_2\big(|\nabla\varphi_n| + |\nabla\varphi|\big)^{p(x)-2}|\nabla(\varphi_n-\varphi)||\nabla v|\,\mathrm{d}x \\ &\leq \sup_{v\in W_D^{1,p(\cdot)}(\Omega),\, \|v\|_{W^{1,p(\cdot)}}\leq 1} c\Big\{\|\nabla(\varphi_n-\varphi)\|_{L^2(\Omega_1)}\|\nabla v\|_{L^2(\Omega_1)} \\ &\qquad + \int_{\Omega_0}\Big(|\nabla\varphi_n|^{p(x)-2} + |\nabla\varphi|^{p(x)-2}\Big)|\nabla(\varphi_n-\varphi)||\nabla v|\,\mathrm{d}x\Big\}.\end{aligned} \tag{3.6}$$

The first term in the supremum can be estimated by $c\|\nabla(\varphi_n-\varphi)\|_{L^{p(\cdot)}}\|\nabla v\|_{L^{p(\cdot)}}$. For the second term we use the generalized Hölder inequality with three factors (see Lemma A.1)

$$\int_\Omega |f(x)g(x)h(x)|\,\mathrm{d}x \leq c_{r,r',r''}\|f\|_{L^{r(\cdot)}}\|g\|_{L^{r'(\cdot)}}\|h\|_{L^{r''(\cdot)}}$$

for every $f \in L^{r(\cdot)}(\Omega)$, $g \in L^{r'(\cdot)}(\Omega)$, $h \in L^{r''(\cdot)}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} + \frac{1}{r''} = 1$. We set $f = |\nabla\varphi_n|^{p(\cdot)-2}$ or $f = |\nabla\varphi|^{p(\cdot)-2}$, $g = |\nabla(\varphi_n-\varphi)|$, and $h = |\nabla v|$, $r(\cdot) = \frac{p(\cdot)}{p(\cdot)-2}$, $r'(\cdot) =$

$r''(\cdot) = p(\cdot)$. Note that $\rho_{r(\cdot)}(|\nabla\varphi|^{p(\cdot)-2}) = \rho_{p(\cdot)}(|\nabla\varphi|)$, which is finite for φ and φ_n, too. Thus, for all $v \in W_D^{1,p(\cdot)}(\Omega)$ we find

$$\begin{aligned}\langle A_T\varphi_n - A_T\varphi, v\rangle_{W_D^{1,p(\cdot)}} \le \Big[1 + \big\||\nabla\varphi_n|^{p(\cdot)-2}\big\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(\Omega_0)} + \big\||\nabla\varphi|^{p(\cdot)-2}\big\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(\Omega_0)}\Big] \\ \times c\,\|\nabla(\varphi_n-\varphi)\|_{L^{p(\cdot)}}\|\nabla v\|_{L^{p(\cdot)}}.\end{aligned} \tag{3.7}$$

For $w = \varphi$ or φ_n, (2.3) ensures that on Ω_0 and for p_{0-} defined in (2.11)

$$\begin{aligned}\big\||\nabla w|^{p(\cdot)-2}\big\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(\Omega_0)} &\le \max\Big\{\rho_{\frac{p(\cdot)}{p(\cdot)-2}}(|\nabla w|^{p(\cdot)-2})^{\frac{p_{0-}-2}{p_{0-}}}, \rho_{\frac{p(\cdot)}{p(\cdot)-2}}(|\nabla w|^{p(\cdot)-2})^{\frac{p_+-2}{p_+}}\Big\} \\ &= \max\Big\{\rho_{p(\cdot)}(|\nabla w|)^{\frac{p_{0-}-2}{p_{0-}}}, \rho_{p(\cdot)}(|\nabla w|)^{\frac{p_+-2}{p_+}}\Big\}\end{aligned}$$

which, due to (2.3), can be estimated from above by terms which are uniformly bounded for $w = \varphi$ or φ_n for the sequence $\varphi_n - \varphi \to 0$ in $W_D^{1,p(\cdot)}(\Omega)$.

Here, we applied [5, Lemma 3.2.6] and the fact that due to the definition of the norm in $W^{1,p(\cdot)}(\Omega)$ and (2.3) we have the estimate $\|\nabla v\|_{L^{p(\cdot)}} \le c\|v\|_{W^{1,p(\cdot)}(\Omega)}$. Thus, (3.6) and (3.7) ensure that $\|A_T\varphi_n - A_T\varphi\|_{W_D^{1,p(\cdot)}(\Omega)^*} \to 0$ as $\varphi_n \to \varphi$ in $W_D^{1,p(\cdot)}(\Omega)$.

Next, we prove the coercivity of A_T. We apply (2.8) to estimate

$$\begin{aligned}&\langle A_T\varphi, \varphi - \varphi^D\rangle_{W_D^{1,p(\cdot)}} \\ &\ge \int_\Omega \sigma_0 F(T)2^{2-p(x)}|\nabla(\varphi-\varphi^D)|^{p(x)}\,\mathrm{d}x - \int_\Omega \sigma_0 F(T)|\nabla\varphi^D|^{p(x)-1}|\nabla(\varphi-\varphi^D)|\,\mathrm{d}x \\ &\ge \int_\Omega \sigma_1 2^{2-p_+}|\nabla(\varphi-\varphi^D)|^{p(x)}\,\mathrm{d}x - c\big\||\nabla\varphi^D|^{p(\cdot)-1}\big\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}}\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}} \\ &\ge c_0\rho_{p(\cdot)}(|\nabla(\varphi-\varphi^D)|) - c\big\||\nabla\varphi^D|^{p(\cdot)-1}\big\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}}\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}}.\end{aligned} \tag{3.8}$$

The factor $\||\nabla\varphi^D|^{p(\cdot)-1}\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}}$ in the last term is bounded since $\varphi^D \in W^{1,\infty}(\Omega)$ and (2.3) holds. Since by assumption (A1) $\mathrm{mes}(\Gamma_D) > 0$, the seminorm $\|\nabla(\cdot)\|_{L^{p(\cdot)}}$ is an equivalent norm on $W_D^{1,p(\cdot)}(\Omega)$, compare Corollary A.1. According to (2.3) we can estimate $\rho_{p(\cdot)}(|\nabla(\varphi-\varphi^D)|)$ from below, either by $\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}}^{p_+}$ or $\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}}^{p_-}$. Note that both exponents are greater than 1. Dividing the previous estimate (3.8) by $\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}}$ the right hand side goes to $+\infty$ if $\|\nabla(\varphi-\varphi^D)\|_{L^{p(\cdot)}} \to \infty$ which guarantees that the operator A_T is coercive.

In summary, the main theorem of monotone operators (see [12, 21]) ensures the existence of a solution to (3.3).

3. Uniqueness. To show the uniqueness of the solution to (3.3) we assume that we had two solutions φ_1 and φ_2 with $\varphi_i - \varphi^D \in W_D^{1,p(\cdot)}(\Omega)$, $i = 1, 2$. Testing the equation (3.3) for both solutions with $\varphi_1 - \varphi_2 \in W_D^{1,p(\cdot)}(\Omega)$ and using (2.8) gives

$$\begin{aligned} 0 = \langle A_T\varphi_1 - A_T\varphi_2, \varphi_1 - \varphi_2\rangle_{W_D^{1,p(\cdot)}} \\ &= \int_\Omega \sigma_0 F(T)\big(|\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1 - |\nabla\varphi_2|^{p(x)-2}\nabla\varphi_2\big)\cdot\nabla(\varphi_1-\varphi_2)\,\mathrm{d}x \\ &\geq \int_\Omega \sigma_1 2^{2-p(x)}|\nabla(\varphi_1-\varphi_2)|^{p(x)}\,\mathrm{d}x \end{aligned}$$

which ensures $\nabla\varphi_1 = \nabla\varphi_2$ a.e. in Ω. Since $\varphi_1 - \varphi_2 \in W_D^{1,p(\cdot)}(\Omega)$, the uniqueness of the solution to (3.3) follows. $\square$

The next step is to establish higher regularity of the weak solution φ to (3.3) for given $T \in \mathcal{T}$, namely $\varphi \in W^{1,p(\cdot)s^*}(\Omega)$ with a uniform $s^* > 1$, as well as to verify global upper and lower bounds for all arbitrarily given $T \in \mathcal{T}$.

Theorem 3.1 *We assume (A1) and (A2). Then there exist a constant $s^* > 1$, $p^*(\cdot) = p(\cdot)s^* \in \mathcal{P}(\Omega)$ and a $c_{p^*} > 0$ depending only on the data (Ω, φ^D, $\underline{\sigma_0}$, $\overline{\sigma_0}$, T_a, β, p_-, and p_+) but not on $T \in \mathcal{T}$ such that the solution φ to* (3.3) *belongs to $W^{1,p^*(\cdot)}(\Omega)$ with*

$$\rho_{p^*(\cdot)}(|\nabla\varphi|) + \rho_{p^*(\cdot)}(\varphi) \leq c_{p^*}$$

uniformly for all given functions $T \in \mathcal{T}$.

In the case that p is constant, Theorem 3.1 in [8] ensures the desired result. The general case with spatially varying p is proven in Section 4.

Corollary 3.1 *We assume (A1) and (A2). Then for $s^* > 1$ from Theorem 3.1 there exist constants c_{s^*}, $c_\infty > 0$ depending only on the data (Ω, φ^D, $\underline{\sigma_0}$, $\overline{\sigma_0}$, T_a, β, p_-, and p_+) such that for all given functions $T \in \mathcal{T}$ the solution φ to* (3.3) *fulfills the estimates*

$$\big\|(1-\eta)\sigma_0 F(T)|\nabla\varphi|^{p(\cdot)}\big\|_{L^{s^*}} \leq c_{s^*} \quad \textit{and} \quad \max_{x\in\overline{\Omega}}|\varphi(x)| \leq c_\infty.$$

Proof. According to Theorem 3.1 we know that $\rho_{p^*(\cdot)}(|\nabla\varphi|) \leq c_{p^*}$ which together with Assumption (A2) and (3.2) ensures that the expression $(1-\eta)\sigma(x,T,|\nabla\varphi|)|\nabla\varphi|^2$ belongs to $L^{s^*}(\Omega)$. In particular, the norm can be estimated by

$$\big\|(1-\eta)\sigma_0 F(T)|\nabla\varphi|^{p(\cdot)}\big\|_{L^{s^*}} \leq \sigma_2\big\||\nabla\varphi|^{p(\cdot)}\big\|_{L^{s^*}} = \sigma_2\rho_{p^*(\cdot)}(|\nabla\varphi|)^{\frac{1}{s^*}} \leq \sigma_2(c_{p^*})^{\frac{1}{s^*}} =: c_{s^*}.$$

By Theorem 3.1, φ belongs to $W^{1,p^*(\cdot)}(\Omega)$. Since $p^*_- := \operatorname{ess\,inf}_{x\in\Omega} p^* = s^* \operatorname{ess\,inf}_{x\in\Omega} p > 2$ and because of the continuous embeddings of the spaces $W^{1,p^*(\cdot)}(\Omega) \hookrightarrow W^{1,p^*_-}(\Omega) \hookrightarrow C(\overline{\Omega})$ the estimate (3.4) from Lemma 3.1 is satisfied for all $x \in \Omega$. $\square$

3.2 The coupled $p(x)$-Laplace thermistor problem

To tackle the complete $p(x)$-Laplace thermistor problem in (2.1), we introduce the operator $A : (\varphi^D + W_D^{1,s^*p(\cdot)}(\Omega)) \times (H^1(\Omega) \cap L^\infty(\Omega)) \to (W_D^{1,p(\cdot)}(\Omega))^* \times H^1(\Omega)^*$ by

$$\begin{aligned}\langle A(\varphi,T),(\overline{\varphi},\overline{T})\rangle := &\int_\Omega \big\{\sigma(x,T,|\nabla\varphi|)\nabla\varphi\cdot\nabla\overline{\varphi} + \lambda(x)\nabla T\cdot\nabla\overline{T}\big\}\mathrm{d}x \\ &- \int_\Omega (1-\eta(j,T))\sigma(x,T,|\nabla\varphi|)|\nabla\varphi|^2\overline{T}\,\mathrm{d}x \\ &+ \int_\Gamma \kappa(T-T_a)\overline{T}\,\mathrm{d}\Gamma \qquad \forall\,\overline{\varphi}\in W_D^{1,p(\cdot)}(\Omega),\ \forall\,\overline{T}\in H^1(\Omega)\end{aligned} \tag{3.9}$$

and look for solutions to Problem (P)

$$A(\varphi,T) = 0, \quad \varphi \in \varphi^D + W_D^{1,s^*p(\cdot)}(\Omega), \quad T \in H^1(\Omega)\cap L^\infty(\Omega) \tag{P}$$

which correspond to the weak solutions to the system (2.1a) – (2.1e).

Theorem 3.2 (Bounds) *We assume (A1) and (A2). Then there exist positive constants c_{p^*}, c_{q^*}, c_∞ and an exponent $q^* > 2$ such that any weak solution (φ,T) to Problem (P) fulfills*

$$\rho_{p^*(\cdot)}(|\nabla\varphi|) + \rho_{p^*(\cdot)}(\varphi) \le c_{p^*}, \quad \max_{x\in\overline{\Omega}}|\varphi(x)| \le c_\infty,$$
$$\|T\|_{W^{1,q^*}} \le c_{q^*}, \quad T_a \le T(x) \le c_\infty \quad \textit{for all } x\in\Omega.$$

Proof. 1. For the lower bound of the temperature we test (P) by $-(0,(T-T_a)^-)$ and obtain

$$\int_\Omega \lambda|\nabla(T-T_a)^-|^2\,\mathrm{d}x + \int_\Gamma \kappa((T-T_a)^-)^2\,\mathrm{d}\Gamma \le 0$$

which by (A2) ensures that $T\in\mathcal{T}$.

2. If (φ,T) is a solution to Problem (P) then φ is a solution to (3.3) for T, and the estimates for the component φ of the solution to (P) result from Lemma 3.1 and Theorem 3.1.

3. According to Corollary 3.1, the Joule heat term in the right-hand side of the heat equation, $(1-\eta)\sigma(x,T,|\nabla\varphi|)|\nabla\varphi|^2$, belongs to $L^{s^*}(\Omega)$ and its L^{s^*}-norm can be estimated by c_{s^*}. We use regularity results for second order elliptic equations with non-smooth data in the case $n=2$. According to [15, Theorem 1] there is a $\widetilde{q}>2$ such that the strongly monotone Lipschitz continuous operator $B:\ H^1(\Omega)\to H^1(\Omega)^*$,

$$\langle BT,w\rangle := \int_\Omega(\lambda\nabla T\cdot\nabla w + Tw)\,\mathrm{d}x, \quad w\in H^1(\Omega),$$

maps $W^{1,q}(\Omega)$ into and onto $W^{-1,q}(\Omega)$ for all $q\in[2,\widetilde{q}]$. Here, $W^{-1,q}(\Omega)$ means $W^{1,q'}(\Omega)^*$ with $\frac{1}{q}+\frac{1}{q'}=1$. Next we define $q^*\in(2,\widetilde{q}]$ by

$$q^* := \begin{cases}\widetilde{q} & \text{if } \dfrac{s^*}{s^*-1}\in\Big[1,\dfrac{2\widetilde{q}}{\widetilde{q}-2}\Big] \\[2ex] \dfrac{2s^*}{2-s^*} & \text{if } \dfrac{s^*}{s^*-1} > \dfrac{2\widetilde{q}}{\widetilde{q}-2}\end{cases}, \qquad \frac{1}{q^*}+\frac{1}{(q^*)'}=1.$$

This definition guarantees that $L^{s^*}(\Omega) \hookrightarrow W^{-1,q^*}(\Omega) = W^{1,(q^*)'}(\Omega)^*$. Remark 13 in [15] then ensures W^{1,q^*}-estimates for solutions to problems of the form $BT = R(T)$, where R is any mapping from $W^{1,2}(\Omega)$ into $W^{-1,q^*}(\Omega)$. For our problem under consideration we use

$$\langle R(T), w\rangle := \int_\Omega \Big((1-\eta)\sigma(x,T,\nabla\varphi)|\nabla\varphi|^2 + T\Big) w \,\mathrm{d}x + \int_\Gamma \kappa(T_a - T) w \,\mathrm{d}\Gamma,$$

for $w \in W^{1,(q^*)'}(\Omega)$. Thus, we find $c_{q^*} > 0$ such that $T \in W^{1,q^*}(\Omega)$ and $\|T\|_{W^{1,q^*}} \le c_{q^*}$.

4. The continuous embedding of $W^{1,q^*}(\Omega)$ into $C(\overline{\Omega})$ supplies the pointwise lower and upper bound of the temperature distribution T which sharpens the result of Step 1. □

Let us mention that according to the proof of Lemma 3.1, the upper and lower bounds of the electrostatic potential φ of any solution to (P) are given by the upper and lower bound of the Dirichlet function φ^D, respectively. The continous embedding $W^{1,p^*_-}(\Omega) \hookrightarrow C^{0,\alpha_1}(\overline{\Omega})$ for $p^*_- > 2$ and $0 < \alpha_1 < (p^*_- - 2)/p^*_-$ and $W^{1,q^*}(\Omega) \hookrightarrow C^{0,\alpha_2}(\overline{\Omega})$ for $q^* > 2$ and $0 < \alpha_2 < (q^* - 2)/q^*$ in two spatial dimensions ensures the following regularity result for solutions to (P).

Corollary 3.2 *We assume (A1) and (A2). Then any solution (φ, T) to (P) is Hölder continuous.*

The following main result establishes the existence of weak solutions to the coupled $p(x)$-Laplace thermistor system (2.1).

Theorem 3.3 (Existence of solutions) *We assume (A1) and (A2). Moreover, let $\eta \in L^\infty(\Omega)$ satisfy $\eta(x) \in [0,1]$ for a.a. $x \in \Omega$. Then there exists at least one solution to Problem (P).*

Proof. 1. We intend to use Schauder's fixed point theorem. We fix some q° satisfying $2 < q^\circ < q^*$, with q^* being the exponent from Theorem 3.2, and denote by $c_{q^\circ} = c_{q^*,q^\circ} c_{q^*} > 0$ the product of the embedding constant $c_{q^*,q^\circ} > 0$ of the continuous embedding $W^{1,q^*}(\Omega) \hookrightarrow W^{1,q^\circ}(\Omega)$ and of the constant c_{q^*} from Theorem 3.2. We work with the bounded, closed, convex, nonempty set

$$\mathcal{M} := \big\{T \in W^{1,q^\circ}(\Omega) : \|T\|_{W^{1,q^\circ}} \le c_{q^\circ},\ T \ge T_a\big\}.$$

We consider the following mapping $\mathcal{Q} : \mathcal{M} \mapsto \mathcal{M}$. For given $\widetilde{T} \in \mathcal{M}$ we solve problem (3.3), see Lemma 3.1, and get a unique solution $\varphi \in W^{1,p^*(\cdot)}(\Omega)$, see Theorem 3.1. Corollary 3.1 ensures that $(1-\eta)\sigma(x, \widetilde{T}, |\nabla\varphi|)|\nabla\varphi|^2 \in L^{s^*}(\Omega)$. Now we find the unique solution T of the heat flow equation with the right hand side $(1-\eta)\sigma(x, \widetilde{T}, |\nabla\varphi|)|\nabla\varphi|^2 \in L^{s^*}(\Omega) \subset H^1(\Omega)^*$, where $s^* > 1$ is the exponent from Theorem 3.1. This is possible since the corresponding operator $L : H^1(\Omega) \to H^1(\Omega)^*$,

$$\langle LT, w\rangle = \int_\Omega \lambda \nabla T \cdot \nabla w \,\mathrm{d}x + \int_\Gamma \kappa T w \,\mathrm{d}\Gamma, \quad w \in H^1(\Omega),$$

is Lipschitz continuous and strongly monotone from $H^1(\Omega)$ to $H^1(\Omega)^*$ (compare Assumption (A2)), which proves the solvability. The higher regularity of the solution is guaranteed by the regularity result of Gröger for second order elliptic equations with non-smooth data in the case $n = 2$ (see [15]). Note that the Joule heat term belongs to $L^{s^*}(\Omega) \subset W^{1,(q^*)'}(\Omega)^*$. Arguing as in Step 3 of the proof of Theorem 3.2 we find that $\|T\|_{W^{1,q^*}} \le c_{q^*}$. The continuous embedding $W^{1,q^*}(\Omega) \hookrightarrow W^{1,q^\circ}(\Omega)$ gives $\|T\|_{W^{1,q^\circ}} \le c_{q^*,q^\circ}\|T\|_{W^{1,q^*}} \le c_{q^\circ}$. Moreover, $T \ge T_a$ is verified similar to the proof of Theorem 3.2. By this procedure we define a mapping $\mathcal{Q} : \mathcal{M} \to \mathcal{M}$ with $T := \mathcal{Q}(\widetilde{T})$. To apply Schauder's fixed point theorem, we show that $\mathcal{Q} : \mathcal{M} \to \mathcal{M}$ is continuous as well as compact.

2. We start with the continuity: Let $\widetilde{T}_n \to \widetilde{T}$ in $W^{1,q^\circ}(\Omega)$ and $\varphi_n, \varphi \in W^{1,p(\cdot)}(\Omega)$ be the corresponding solutions to the current-flow equation in (3.3). The continuity is proved by four convergence results: First, we show that $\varphi_n \to \varphi$ in $W^{1,p(\cdot)}(\Omega)$, then $\varphi_n \to \varphi$ in $W^{1,p_\theta(\cdot)}(\Omega)$, where $p_\theta = \theta p + (1-\theta)p^*$ and $\theta \in (0,1)$ arbitrary. Next, we prove $T_n \to T$ in $H^1(\Omega)$, and finally $T_n \to T$ in $W^{1,q^\circ}(\Omega)$.

The test of (3.3) by $\varphi_n - \varphi \in W^{1,p(\cdot)}_D(\Omega)$ gives

$$\begin{aligned}\int_\Omega \sigma_0 F(\widetilde{T}_n)\Big(|\nabla\varphi_n|^{p(x)-2}\nabla\varphi_n - |\nabla\varphi|^{p(x)-2}\nabla\varphi\Big)\cdot\nabla(\varphi_n-\varphi)\,\mathrm{d}x \\ = \int_\Omega \sigma_0\big(F(\widetilde{T}) - F(\widetilde{T}_n)\big)|\nabla\varphi|^{p(x)-2}\nabla\varphi\cdot\nabla(\varphi_n-\varphi)\,\mathrm{d}x.\end{aligned} \tag{3.10}$$

In (3.10) we use (2.8), $\sigma_0 F(\cdot,\widetilde{T}_n) \ge \sigma_1$, the definition of $\rho_{p(\cdot)}$, the Lipschitz continuity of F in T for arguments $T \ge T_a$ and Hölder's inequality to obtain

$$\begin{aligned}\sigma_1\rho_{p(\cdot)}(|\nabla(\varphi-\varphi_n)|) &= \sigma_1\int_\Omega |\nabla(\varphi-\varphi_n)|^{p(x)}\,\mathrm{d}x \\ &\le c\int_\Omega |\widetilde{T}-\widetilde{T}_n||\nabla\varphi|^{p(x)-1}|\nabla(\varphi_n-\varphi)|\,\mathrm{d}x \\ &\le c\,\|\widetilde{T}-\widetilde{T}_n\|_{L^\infty}\int_\Omega |\nabla\varphi|^{p(x)-1}|\nabla(\varphi_n-\varphi)|\,\mathrm{d}x \\ &\le c\,\|\widetilde{T}-\widetilde{T}_n\|_{L^\infty}\||\nabla\varphi|^{p(\cdot)-1}\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}}\|\nabla(\varphi_n-\varphi)\|_{L^{p(\cdot)}}.\end{aligned} \tag{3.11}$$

According to (2.3) and Lemma 3.1 we have

$$\begin{aligned}\||\nabla\varphi|^{p(\cdot)-1}\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}} &\le \max\Big\{\rho_{p(\cdot)}(|\nabla\varphi|)^{\frac{p_--1}{p_-}}, \rho_{p(\cdot)}(|\nabla\varphi|)^{\frac{p_+-1}{p_+}}\Big\} \le \max\Big\{c_{int}^{\frac{p_--1}{p_-}}, c_{int}^{\frac{p_+-1}{p_+}}\Big\}, \\ \|\nabla(\varphi-\varphi_n)\|_{L^{p(\cdot)}} &\le \max\Big\{\rho_{p(\cdot)}(|\nabla(\varphi-\varphi_n)|)^{\frac{1}{p_-}}, \rho_{p(\cdot)}(|\nabla(\varphi-\varphi_n)|)^{\frac{1}{p_+}}\Big\}.\end{aligned}$$

The continuous embedding $W^{1,q^\circ}(\Omega) \hookrightarrow L^\infty(\Omega)$ ensures $\|\widetilde{T}-\widetilde{T}_n\|_{L^\infty} \to 0$ as $n \to \infty$. Therefore, we conclude from (3.11) the convergences

$$\rho_{p(\cdot)}(|\nabla(\varphi-\varphi_n)|) \to 0, \quad \|\nabla(\varphi-\varphi_n)\|_{L^{p(\cdot)}} \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Here, we additionally used (2.4). Since $\varphi_n - \varphi \in W^{1,p(\cdot)}_D(\Omega)$, we obtain $\|\varphi_n - \varphi\|_{W^{1,p(\cdot)}} \to 0$, see Lemma A.2.

Let be $p_\theta \in \mathcal{P}(\Omega)$ with $p_\theta(x) = \theta p(x) + (1-\theta)p^*(x)$ for $\theta \in (0,1)$. Then by Hölder's inequality

$$\rho_{p_\theta(\cdot)}(v) = \int_\Omega |v|^{\theta p(x)+(1-\theta)p^*(x)}\,\mathrm{d}x \le c\big\|v^{\theta p(\cdot)}\big\|_{L^{\frac{1}{\theta}}}\big\|v^{(1-\theta)p^*(\cdot)}\big\|_{L^{\frac{1}{1-\theta}}} = c\big\|v^{p(\cdot)}\big\|_{L^1}^{\theta}\big\|v^{p^*(\cdot)}\big\|_{L^1}^{1-\theta}$$
$$= c\,\rho_{p(\cdot)}(v)^{\theta}\rho_{p^*(\cdot)}(v)^{1-\theta}$$

and similar for the gradient

$$\rho_{p_\theta(\cdot)}(|\nabla v|) \le c\,\rho_{p(\cdot)}(|\nabla v|)^{\theta}\rho_{p^*(\cdot)}(|\nabla v|)^{1-\theta}.$$

Using these inequalities for $v = \varphi_n - \varphi$, taking into account the bounds from Theorem 3.2, the convergence $\varphi_n \to \varphi$ in $W^{1,p(\cdot)}(\Omega)$, and the relation (2.4) we obtain

$$\varphi_n \to \varphi \quad \text{in } W^{1,p_\theta(\cdot)}(\Omega) \quad \text{for all } p_\theta < p^*. \tag{3.13}$$

Let T_n and T denote the solutions to the heat flow equation with the arguments $(\widetilde{T}_n, \varphi_n)$ and $(\widetilde{T}, \varphi)$ in the Joule heat term, respectively. We test these equations by $T_n - T$. Taking into account Assumption (A2), we find

$$\begin{aligned}
&\|T_n - T\|_{H^1}^2 \\
&\le c\int_\Omega \Big(F(\widetilde{T}_n)|\nabla\varphi_n|^{p(x)} - F(\widetilde{T})|\nabla\varphi|^{p(x)}\Big)|T_n - T|\,\mathrm{d}x \\
&\le c\int_\Omega \Big(F(\widetilde{T}_n)\big||\nabla\varphi_n|^{p(x)} - |\nabla\varphi|^{p(x)}\big| + |\nabla\varphi|^{p(x)}|F(\widetilde{T}_n) - F(\widetilde{T})|\Big)|T_n - T|\,\mathrm{d}x.
\end{aligned} \tag{3.14}$$

From the vector inequality (see [20, p. 379] or [4, Chap. 4, p. 257])

$$\big||z_1|^{p-2}z_1 - |z_2|^{p-2}z_2\big| \le 3(p-1)|z_1 - z_2|(|z_1| + |z_2|)^{p-2} \qquad \text{for } p \ge 2, \quad z_1,\, z_2 \in \mathbb{R}^n$$

we derive

$$\begin{aligned}
\big||z_1|^p - |z_2|^p\big| &= \big||z_1|^{p-2}z_1\cdot z_1 - |z_2|^{p-2}z_2\cdot z_2\big| \\
&\le |z_1|\big||z_1|^{p-2}z_1 - |z_2|^{p-2}z_2\big| + \big||z_2|^{p-2}z_2\big||z_1 - z_2| \\
&\le c(p)|z_1 - z_2|(|z_1| + |z_2|)^{p-1}.
\end{aligned}$$

Applying this inequality with the maximal $c_* = \max_{x\in\Omega} c(p(x))$, $p(x) \in [p_-, p_+]$, and using the boundedness and the Lipschitz continuity of F for arguments greater or equal to T_a we continue the estimate (3.14) by

$$\begin{aligned}
&\|T_n - T\|_{H^1}^2 \\
&\le c\int_\Omega \Big(c|\nabla(\varphi_n - \varphi)|\big(|\nabla\varphi_n| + |\nabla\varphi|\big)^{p(x)-1} + |\nabla\varphi|^{p(x)}|\widetilde{T}_n - \widetilde{T}|\Big)|T_n - T|\,\mathrm{d}x \\
&\le c\|\nabla(\varphi_n - \varphi)\|_{L^{p_\theta(\cdot)}}\Big(\big\||\nabla\varphi_n|^{p(\cdot)-1}\big\|_{L^{\frac{p^*(\cdot)}{p(\cdot)-1}}} + \big\||\nabla\varphi|^{p(\cdot)-1}\big\|_{L^{\frac{p^*(\cdot)}{p(\cdot)-1}}}\Big)\|T_n - T\|_{L^{q(\theta;\cdot)}} \\
&\quad + c\big\||\nabla\varphi|^{p(\cdot)}\big\|_{L^{s^*}}\|\widetilde{T}_n - \widetilde{T}\|_{L^\infty}\|T_n - T\|_{L^{\widetilde{q}}},
\end{aligned} \tag{3.15}$$

where s^* and p^* are defined in Theorem 3.1 and

$$p_\theta(x) \in (p(x), p^*(x)), \qquad \frac{1}{p_\theta(x)} + \frac{p(x)-1}{p^*(x)} + \frac{1}{q(\theta;x)} = 1, \quad \text{and} \quad \frac{1}{\widetilde{q}} + \frac{1}{s^*} = 1.$$

Since $\frac{1}{p_\theta(x)}+\frac{p(x)-1}{p^*(x)} \le \frac{p(x)}{p_\theta(x)}$ we can choose $\theta \in (0,1)$ such that $\operatorname{ess\,sup}_{x\in\Omega}\big(\frac{1}{p_\theta(x)}+\frac{p(x)-1}{p^*(x)}\big) < 1$ and a corresponding $q(\theta;\cdot) \in \mathcal{P}(\Omega)$. Additionally, we find some q satisfying $q(\theta;x) \le q < \infty$ a.e. in Ω such that $\|T_n - T\|_{L^{q(\theta;\cdot)}} \le c\|T_n - T\|_{L^q} \le c(q)\|T_n - T\|_{H^1}$. By (2.3) and Theorem 3.1 we estimate

$$\big\||\nabla\varphi|^{p(\cdot)-1}\big\|_{L^{\frac{p^*(\cdot)}{p(\cdot)-1}}} \le \max\Big\{\rho_{p^*(\cdot)}(|\nabla\varphi|)^{\operatorname{ess\,sup}\frac{p-1}{p^*}}, \rho_{p^*(\cdot)}(|\nabla\varphi|)^{\operatorname{ess\,inf}\frac{p-1}{p^*}}\Big\} \le c,$$

and similar for φ_n. For the treatment of the last line in (3.15), note that by Theorem 3.1 $\||\nabla\varphi|^{p(\cdot)}\|_{L^{s^*}}^{s^*} \le c_{p^*}$ and $\|T_n - T\|_{L^{\widetilde{q}}} \le c(\widetilde{q})\|T_n - T\|_{H^1}$. Following all these arguments for the terms in (3.15) we arrive at

$$\|T_n - T\|_{H^1}^2 \le c\Big\{\|\varphi_n - \varphi\|_{W^{1,p_\theta(\cdot)}} + \|\widetilde{T}_n - \widetilde{T}\|_{L^\infty}\Big\}\|T_n - T\|_{H^1}.$$

Dividing by $\|T_n - T\|_{H^1}$ and taking into account that $\|\widetilde{T}_n - \widetilde{T}\|_{L^\infty} \to 0$ as $n \to \infty$ as well as (3.13) we finally obtain

$$\|T_n - T\|_{H^1} \to 0.$$

Since $\|T_n\|_{W^{1,q^*}}, \|T\|_{W^{1,q^*}} \le c_{q^*}$ this convergence implies by interpolation arguments the convergence $T_n \to T$ in $W^{1,q^\circ}(\Omega)$ for $2 < q^\circ < q^*$.

3. To show that $\mathcal{Q}$ is compact we start with any sequence $(\widetilde{T}_n)$, $\widetilde{T}_n \in \mathcal{M}$. Since $\mathcal{M}$ is bounded in $W^{1,q^\circ}(\Omega)$ and $W^{1,q^\circ}(\Omega)$ is compactly embedded in $L^\infty(\Omega)$ we find a $\widetilde{T} \in L^\infty(\Omega)$ and a subsequence (also denoted by $(\widetilde{T}_n)$) such that $\widetilde{T}_n \to \widetilde{T}$ in $L^\infty(\Omega)$. Therefore, we can argue as in Step 2 of the proof to verify that $\mathcal{Q}\widetilde{T}_n \to \mathcal{Q}\widetilde{T}$ in $W^{1,q^\circ}(\Omega)$.

Thus, we can apply Schauder's fixed point theorem, which proves the theorem. □

4 Proof of the higher regularity of the potential

Here, we prove Theorem 3.1 by deriving the higher integrability of $\nabla(\varphi-\varphi^D)$. The higher regularity of φ then results from the regularity assumption for φ^D in (A2) and the L^∞-estimate for φ. We proceed in several steps: First, we localize the problem to squares using the model situations (a) – (h) in Fig. 2. Then, we derive Caccioppoli-type inequalities near the boundary (Lemma 4.1) and use reflection arguments to extend the estimates from half squares to full squares (Lemma 4.2). The Caccioppoli-type inequalities for interior squares can be derived similarly and are hence only shortly mentioned (Lemma 4.3). Finally, we establish the higher integrability of the gradient by applying a Gehring-type lemma (Subsection 4.5).

4.1 Localization

As before, we denote by $C_1(0) \subset \mathbb{R}^2$ the unit square centered at 0 with side length 2 and by $C_1^+(0)$ its upper half. For $x^0 \in \partial\Omega$ let $\Phi_{x^0} : U_{x^0} \cap \Omega \to C_1^+(0)$ and for $x^0 \in \Omega$ let $\Phi_{x^0} : U_{x^0} \cap \Omega \to C_1(0)$ be bi-Lipschitz transformations with $\Phi_{x^0}(x) = y$, which exist due (A1). Let

$$\overline{\Omega} \subset \bigcup_{i=1}^{N} \Phi_{x_i^0}^{-1}(C_{\frac{1}{24}}(0)) \cup \bigcup_{i=N+1}^{M} \Phi_{x_i^0}^{-1}(C_{\frac{1}{24}}(0))$$

be a finite open covering of $\overline{\Omega}$. If $x^0 \in \partial\Omega$ is such that $\Gamma_D \cap U_{x^0} \neq \emptyset$, we denote $\widehat{\Gamma}_D = \Phi_{x^0}(\Gamma_D \cap U_{x^0})$ the localized part of the Dirichlet boundary.

Moreover, we have assumed that p takes at most two different constant values p_A, p_B in the set $\Phi^{-1}_{x_i^0}(C_1(0))$, which we always assume to satisfy $p_A < p_B$. We find constants $0 < \underline{\gamma} \leq \overline{\gamma} < \infty$ such that for

$$\gamma_i(y) := |\det D\Phi^{-1}_{x_i^0}(y)|$$

it holds $\gamma_i \in L^\infty(C_1(0))$ and $\underline{\gamma} \leq \gamma_i(y) \leq \overline{\gamma}$ almost everywhere. For the center points x_i^0, $i = 1, \ldots, M$ we introduce

$$\begin{aligned} v := \varphi \circ \Phi^{-1}_{x_i^0}, \quad v^D := \varphi^D \circ \Phi^{-1}_{x_i^0}, \quad \overline{v} := \overline{\varphi} \circ \Phi^{-1}_{x_i^0}, \\ \widehat{\sigma}(y) := \sigma\big(\Phi^{-1}_{x_i^0}(y), T(\Phi^{-1}_{x_i^0}(y))\big), \quad \widehat{p}(y) := p\big(\Phi^{-1}_{x_i^0}(y)\big) \end{aligned}$$

and neglect the dependency on x_i^0 of the transformation $\Phi_{x_i^0}$.

In the following, we concentrate on the covering of the boundary. For $i = 1, \ldots, N$, we have $x_i^0 \in \partial\Omega$. Let x^0 be one of them. According to the transformation formula, we obtain for the solution to problem (3.3) and for test functions $\overline{\varphi} \in W^{1,p(\cdot)}_D(\Omega)$, having their support in $U_{x^0} \cap (\Omega \cup \Gamma_N)$, that

$$\begin{aligned} 0 &= \int_{U_{x^0} \cap \Omega} \sigma_0(x) F(T(x)) |\nabla\varphi(x)|^{p(x)-2} \nabla\varphi(x) \cdot \nabla\overline{\varphi}(x)\, \mathrm{d}x \\ &= \int_{C_1^+(0)} \sigma_0\big(\Phi^{-1}(y)\big) F\big(T(\Phi^{-1}(y))\big) \big|\nabla_x\varphi\big(\Phi^{-1}(y)\big)\big|^{p(\Phi^{-1}(y))-2} \\ &\qquad \nabla_x\varphi\big(\Phi^{-1}(y)\big) \cdot \nabla_x\overline{\varphi}\big(\Phi^{-1}(y)\big)\, |\det D\Phi^{-1}(y)|\, \mathrm{d}y \end{aligned}$$

which, for $H(y) = D\Phi|_{\Phi^{-1}(y)}$, gives us the "localized" relation

$$0 = \int_{C_1^+(0)} \widehat{\sigma}(y)\gamma(y) \big|\nabla_y v(y) H(y)\big|^{\widehat{p}(y)-2} \big(\nabla_y v(y) H(y)\big) \cdot \big(\nabla_y \overline{v}(y) H(y)\big)\, \mathrm{d}y. \tag{4.1}$$

Note that $0 < \sigma_1\underline{\gamma} \leq \widehat{\sigma}\gamma \leq \sigma_2\overline{\gamma}$ a.e. in $C_1^+(0)$. Moreover, we introduce the notation

$$\begin{aligned} \underline{\pi} &:= \min_{i=1,\ldots,M} \Big\{ \min\big\{\epsilon^{\frac{p_-}{2}}, \epsilon^{\frac{p_+}{2}}\big\} \,:\, \epsilon \text{ is the smallest eigenvalue of } D\Phi^{\mathsf{T}}_{x_i^0} D\Phi_{x_i^0} \text{ on } U_{x_i^0} \cap \Omega \Big\}, \\ \overline{\pi} &:= \max_{i=1,\ldots,M} \Big\{ \max\big\{\epsilon_n^{\frac{p_-}{2}}, \epsilon_n^{\frac{p_+}{2}}\big\} \,:\, \epsilon_n \text{ is the maximal norm of } D\Phi^{\mathsf{T}}_{x_i^0} D\Phi_{x_i^0} \text{ on } U_{x_i^0} \cap \Omega \Big\}. \end{aligned}$$

We have to discuss the different model cases for $C_1(0)$ depicted in Fig. 2. Note that the situations (a), (c), (e) and (g) can be treated already by the methods provided in [8]. Therefore, our discussion contains new ideas for the cases (d), (f) and (h) where the corresponding $x^0 \in \partial\Omega$. The "interior" case (b) with $x^0 \in \Omega$ follows from similar arguments, hence, it is only briefly discussed.

4.2 Caccioppoli-type inequalities near the boundary

We consider additional squares $C_s(y^0)$ with smaller side length $0 < 2s < 2$ and center at y^0. In particular, for a given $y^0 \in C^+_{1/4}(0)$ and $0 < r < \frac{1}{4}$ we have $C_{3r}(y^0) \subset C_1(0)$. We will often abbreviate $C_r(y^0)$ with C_r and $C^+_r(y^0) := \{y \in C_r(y^0) : y_2 > 0\}$ with C^+_r, respectively. By (A1), $\widehat{p}$ takes at most two different values (denoted $p_A < p_B$) in $C_1(0)$. For a given y^0, we denote by C^+_{Ar} (resp. C^+_{A3r}) the part of $C^+_r(y^0)$ (resp. $C^+_{3r}(y^0)$), where $\widehat{p}$ takes the value p_A, and $m^+_{Ar}(v)$ (resp. $m^+_{A3r}(v)$) stands for the mean value of a function v on C^+_{Ar} (resp. C^+_{A3r}). For p_B, we proceed analogously.

Lemma 4.1 *We assume (A1) and (A2) and suppose that $x^0 \in \partial\Omega$ and that $\Phi_{x^0} : U_{x^0} \cap \Omega \to C^+_1(0)$ is the corresponding bi-Lipschitzian map leading to one of the cases (c)-(h) in Fig. 2. Let φ be the solution to (3.3) for any $T \in \mathcal{T}$ and $v = \varphi \circ \Phi^{-1}_{x^0}$ the corresponding localized part of φ, $v_0 = \varphi \circ \Phi^{-1}_{x^0} - \varphi^D \circ \Phi^{-1}_{x^0}$. Let $y^0 \in C^+_{1/4}(0)$ and $0 < r < \frac{1}{4}$. Then there exists a constant $\widetilde{c}_1 > 0$ independent of y^0, r and the involved T such that*

$$\begin{aligned}\int_{C^+_{\frac{r}{2}}} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y \le \widetilde{c}_1 \int_{C^+_{3r}} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y \\ + \frac{\widetilde{c}_1}{r^{p_A}}\Big(\int_{C^+_{3r}} |\nabla v_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} + \frac{\widetilde{c}_1}{r^{p_B}}\Big(\int_{C^+_{B3r}} |\nabla v_0|^{\widetilde{p}_B}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}},\end{aligned} \tag{4.2}$$

where $\widetilde{p}_A = 2p_A/(p_A{+}2)$ and $\widetilde{p}_B = 2p_B/(p_B{+}2)$, respectively.

Proof. We fix an arbitrary $y^0 \in C^+_{1/4}(0)$ and consider $0 < r < \frac{1}{4}$. Moreover, we take t and s such that $\frac{r}{2} \le t < s \le r$. We work with cut-off functions $\xi \in C^1(\mathbb{R}^2; [0,1])$ fulfilling

$$\xi|_{C_t} = 1, \quad \xi|_{\mathbb{R}^2\setminus C_s} = 0, \quad |\nabla\xi| \le \frac{\theta}{s-t}, \tag{4.3}$$

where $\theta \ge 1$ does not depend on t and s. For v_0 as above we have

$$\nabla(v_0\xi) = \xi\nabla v_0 + v_0\nabla\xi \quad \text{and} \quad |\nabla v| \le |\nabla v_0| + |\nabla v^D|. \tag{4.4}$$

Depending on the position of $C^+_r(y_0)$ and $\widehat{p}$ we consider different test functions for the localized current-flow equation (4.1). In particular, we choose $\overline{v} = (v_0 - k)\xi$, where $k \in \mathbb{R}$ is a constant to be fixed. Assuming that $\overline{v}$ is an admissible test function we can use it to test (4.1) to obtain with (4.4)

$$0 = \int_{C^+_s} \widehat{\sigma}(y)\gamma(y)\big|\nabla v H(y)\big|^{\widehat{p}(y)-2}\big(\nabla v H(y)\big) \cdot \Big[\xi\nabla v_0 + (v_0{-}k)\nabla\xi\Big] H(y)\,\mathrm{d}y. \tag{4.5}$$

This identity, together with the estimate in (2.8) for $z_1 = \nabla v(y)H(y)$ and $z_2 = \nabla v^D H(y)$ as well as the definition of $\underline{\pi}$ and $\overline{\pi}$, gives a constant $c_1 > 0$ such that

$$\begin{aligned} c_1 \int_{C_s^+} |\nabla v_0|^{\widehat{p}(y)} \xi \,\mathrm{d}y &\le \int_{C_s^+} \xi\, \widehat{\sigma}\gamma \Big(\big|\nabla v H\big|^{\widehat{p}(y)-2} \nabla v - \big|\nabla v^D H\big|^{\widehat{p}(y)-2} \nabla v^D \Big) H \cdot \nabla v_0 H \,\mathrm{d}y \\ &= - \int_{C_s^+} \widehat{\sigma}\gamma \Big((v_0 - k) \big|\nabla v H\big|^{\widehat{p}(y)-2} (\nabla v H) \cdot (\nabla \xi H) \\ &\qquad + \big|\nabla v^D H\big|^{\widehat{p}(y)-2} (\nabla v^D H) \cdot (\nabla v_0 H) \xi \Big) \,\mathrm{d}y \\ &\le \int_{C_s^+} \Big(c |\nabla v|^{\widehat{p}(y)-1} \frac{|v_0 - k|}{s-t} + \frac{c_1}{2} \xi |\nabla v_0|^{\widehat{p}(y)} + c \xi |\nabla v^D|^{\widehat{p}(y)} \Big) \mathrm{d}y, \end{aligned}$$

where we have used (4.3) and Young's inequality for the last line. Exploiting that $\xi = 1$ in C_t^+ and $\xi \le 1$, we restrict to the smaller domain C_t^+ in the left-hand side and finally arrive with Hölder's and Young's inequality at

$$\int_{C_t^+} |\nabla v_0|^{\widehat{p}(y)} \,\mathrm{d}y \le \frac{1}{2} \int_{C_s^+} |\nabla v_0|^{\widehat{p}(y)} \,\mathrm{d}y + c \int_{C_s^+} |\nabla v^D|^{\widehat{p}(y)} \,\mathrm{d}y + c \int_{C_s^+} \Big(\frac{|v_0 - k|}{s-t} \Big)^{\widehat{p}(y)} \mathrm{d}y. \tag{4.6}$$

<u>Case 1:</u> If $C_r^+ = C_{Ar}^+$ and $\overline{C_r^+} \cap \widehat{\Gamma}_D = \emptyset$ we set $k = m_{Ar}^+(v_0)$ and obtain with (4.6) and $\widehat{p}(y) = p_A$ for $y \in C_r^+$ the estimate

$$\begin{aligned} &\int_{C_t^+} |\nabla v_0|^{\widehat{p}(y)} \,\mathrm{d}y \\ &\quad \le \frac{1}{2} \int_{C_s^+} |\nabla v_0|^{\widehat{p}(y)} \,\mathrm{d}y + c \int_{C_s^+} |\nabla v^D|^{\widehat{p}(y)} \,\mathrm{d}y + c \int_{C_{Ar}^+} \Big(\frac{|v_0 - m_{Ar}^+(v_0)|}{s-t} \Big)^{p_A} \mathrm{d}y \\ &\quad \le \frac{1}{2} \int_{C_s^+} |\nabla v_0|^{\widehat{p}(y)} \,\mathrm{d}y + c \int_{C_{3r}^+} |\nabla v^D|^{\widehat{p}(y)} \,\mathrm{d}y + \frac{c}{(s-t)^{p_A}} \Big(\int_{C_{A3r}^+} |\nabla v_0|^{\widetilde{p}_A} \,\mathrm{d}y \Big)^{\frac{p_A}{\widetilde{p}_A}}. \end{aligned} \tag{4.7}$$

In the last line we used the Poincaré-Sobolev inequality in Lemma A.4 on C_{Ar}^+ and for the exponent $\widetilde{p}_A = \frac{2p_A}{2+p_A}$ (with a uniform embedding constant, note that C_{Ar}^+ contains at least a $2r \times r$ rectangle). In particular, we have that

$$\widehat{C} := \max \big\{ \max\{ C_{\mathrm{PS},p_i}, C_{\mathrm{PF},p_i} \},\ i = 1, \ldots, m \big\} \qquad \text{is finite} \tag{4.8}$$

(for $C_{\mathrm{PS},p}$ and $C_{\mathrm{PF},p}$ from Lemma A.4 and Lemma A.3, respectively, and p_i being the constant value of p on the subdomain $\Omega_i \subset \Omega$). Finally, we have enlarged the integration domain to C_{A3r}^+. Putting

$$Z(t) = \int_{C_t^+} |\nabla v_0|^{\widehat{p}(y)} \,\mathrm{d}y, \quad W_1 = c \Big(\int_{C_{A3r}^+} |\nabla v_0|^{\widetilde{p}_A} \,\mathrm{d}y \Big)^{\frac{p_A}{\widetilde{p}_A}}, \quad Y = c \int_{C_{3r}^+} |\nabla v^D|^{\widehat{p}(y)} \,\mathrm{d}y$$

(note that due to (3.5) in Lemma 3.1 and (A2) W_1 and Y are finite), $W_2 = 0$, $R = r$, $\rho = \frac{r}{2}$, $\mu_1 = p_A$, $\mu_2 = 1$, and $\iota = \frac{1}{2}$ we can apply Lemma A.5 to obtain from (4.7) that

$$\int_{C_{\frac{r}{2}}^+} |\nabla v_0|^{\widehat{p}(y)} \,\mathrm{d}y \le c \int_{C_{3r}^+} |\nabla v^D|^{\widehat{p}(y)} \,\mathrm{d}y + \frac{c}{r^{p_A}} \Big(\int_{C_{A3r}^+} |\nabla v_0|^{\widetilde{p}_A} \,\mathrm{d}y \Big)^{\frac{p_A}{\widetilde{p}_A}}. \tag{4.9}$$

Case 2: If $C_r^+ = C_{Ar}^+$ and $\overline{C_r^+} \cap \widehat{\Gamma}_D \neq \emptyset$ we use $k = 0$ and derive from (4.6) the estimate

$$\begin{aligned}
&\int_{C_t^+} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y \\
&\quad\le \frac{1}{2}\int_{C_s^+} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y + c\int_{C_s^+} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y + c\int_{C_{A3r}^+} \Big(\frac{|v_0|}{s-t}\Big)^{p_A}\mathrm{d}y \\
&\quad\le \frac{1}{2}\int_{C_s^+} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y + c\int_{C_{3r}^+} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y + \frac{c}{(s-t)^{p_A}}\Big(\int_{C_{A3r}^+} |\nabla v_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}}.
\end{aligned} \tag{4.10}$$

In the last line we used Poincaré-Friedrichs inequality in Lemma A.3 on C_{A3r}^+ and for the exponent $\widetilde{p}_A$ (with a uniform embedding constant, note (4.8) and that C_{A3r}^+ contains at least a $r \times 3r$ rectangle with Dirichlet boundary of length r). With the same meaning of the quantities as in Case 1 we obtain by Lemma A.5 from (4.10)

$$\int_{C_{\frac{r}{2}}^+} |\nabla v|^{\widehat{p}(y)}\,\mathrm{d}y \le c\int_{C_{3r}^+} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y + \frac{c}{r^{p_A}}\Big(\int_{C_{A3r}^+} |\nabla v_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}}. \tag{4.11}$$

The estimates for Case 1 and 2 can be done analogously for the situation $C_r^+ = C_{Br}^+$ using $\widetilde{p}_B = \frac{2p_B}{2+p_B}$.

Case 3: If $|C_{Ar}^+|, |C_{Br}^+| > 0$ and $\overline{C_{3r}^+} \cap \widehat{\Gamma}_D = \emptyset$ we set $k = m_{B3r}^+(v_0)$ to obtain from (4.6)

$$\begin{aligned}
\int_{C_t^+} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y &\le \frac{1}{2}\int_{C_s^+} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y + c\int_{C_s^+} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y \\
&\quad + c\int_{C_{B3r}^+}\Big(\frac{|v_0 - m_{B3r}^+(v_0)|}{s-t}\Big)^{p_B}\mathrm{d}y + \int_{C_{A3r}^+}\Big(\frac{|v_0 - m_{B3r}^+(v_0)|}{t-s}\Big)^{p_A}\mathrm{d}y.
\end{aligned} \tag{4.12}$$

For the first term on the last line, we apply the Poincaré-Sobolev inequality in Lemma A.4 on C_{B3r}^+ and for the exponent $\widetilde{p}_B = \frac{2p_B}{2+p_B}$ (with a uniform embedding constant, note (4.8) and that C_{B3r}^+ contains at least a $r \times 3r$ rectangle). Moreover, the second term in the last line is estimated by the corresponding integral over C_{3r}^+, note that $p_A < p_B$. Indeed, by triangle inequality we have

$$\big\|v_0 - m_{B3r}^+(v_0)\big\|_{L^{p_A}(C_{3r}^+)} \le \big\|v_0 - m_{3r}^+(v_0)\big\|_{L^{p_A}(C_{3r}^+)} + \big\|m_{3r}^+(v_0) - m_{B3r}^+(v_0)\big\|_{L^{p_A}(C_{3r}^+)},$$

where the last term can be estimated as follows

$$\begin{aligned}
&\big\|m_{3r}^+(v_0) - m_{B3r}^+(v_0)\big\|_{L^{p_A}(C_{3r}^+)} = \big|m_{3r}^+(v_0) - m_{B3r}^+(v_0)\big|\|1\|_{L^{p_A}(C_{3r}^+)} \\
&\le \frac{1}{|C_{B3r}^+|}\big\|v_0 - m_{3r}^+(v_0)\big\|_{L^1(C_{B3r}^+)}\|1\|_{L^{p_A}(C_{3r}^+)} \\
&\le \frac{1}{|C_{B3r}^+|}\big\|v_0 - m_{3r}^+(v_0)\big\|_{L^{p_A}(C_{B3r}^+)}\|1\|_{L^{p_A'}(C_{B3r}^+)}\|1\|_{L^{p_A}(C_{3r}^+)} \\
&\le \frac{1}{|C_{B3r}^+|}\big\|v_0 - m_{3r}^+(v_0)\big\|_{L^{p_A}(C_{3r}^+)}\|1\|_{L^{p_A'}(C_{3r}^+)}\|1\|_{L^{p_A}(C_{3r}^+)},
\end{aligned}$$

where $\frac{1}{p_A} + \frac{1}{p'_A} = 1$. With this, we find

$$\begin{aligned}\|v_0 - m^+_{B3r}(v_0)\|_{L^{p_A}(C^+_{3r})} &\le \Big(1 + \frac{|C^+_{3r}|}{|C^+_{B3r}|}\Big)\|v_0 - m^+_{3r}(v_0)\|_{L^{p_A}(C^+_{3r})} \\ &\le \frac{2|C^+_{3r}|}{|C^+_{B3r}|}\|v_0 - m^+_{3r}(v_0)\|_{L^{p_A}(C^+_{3r})}.\end{aligned}$$

Therefore, we apply the Poincaré-Sobolev inequality Lemma A.4 now on C^+_{3r} and (4.8) to obtain for the second term in the last line of (4.12)

$$\|v_0 - m^+_{B3r}(v_0)\|^{p_A}_{L^{p_A}(C^+_{3r})} \le \Big[\frac{2|C^+_{3r}|}{|C^+_{B3r}|}\Big]^{p_+}\|v_0 - m^+_{3r}(v_0)\|^{p_A}_{L^{p_A}(C^+_{3r})} \le c\|\nabla v_0\|^{p_A}_{L^{\widetilde{p}_A}(C^+_{3r})},$$

where the constant in front of the right-hand side is chosen independently of r and $\widehat{p}$. In summary, we continue the estimate in (4.12) by

$$\begin{aligned}\int_{C^+_t} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y &\le \frac{1}{2}\int_{C^+_s} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y + c\int_{C^+_{3r}} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y \\ &\quad + \frac{c}{(s-t)^{p_B}}\Big(\int_{C^+_{B3r}} |\nabla v_0|^{\widetilde{p}_B}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}} + \frac{c}{(s-t)^{p_A}}\Big(\int_{C^+_{3r}} |\nabla v_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}}.\end{aligned} \tag{4.13}$$

Since $v_0 \in W^{1,\widetilde{p}_B}(C^+_{B3r})$ and $v_0 \in W^{1,\widetilde{p}_A}(C^+_{3r})$ we set $\mu_1 = p_B$, $\mu_2 = p_A$,

$$W_1 := c\Big(\int_{C^+_{B3r}} |\nabla v_0|^{\widetilde{p}_B}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}}, \quad W_2 := c\Big(\int_{C^+_{3r}} |\nabla v_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}},$$

and keep the other quantities as in the Cases 1 and 2 to apply Lemma A.5. This leads to

$$\begin{aligned}\int_{C^+_{\frac{r}{2}}} |\nabla v_0|^{\widehat{p}(y)}\,dy &\le c\int_{C^+_{3r}} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y + \frac{c}{r^{p_B}}\Big(\int_{C^+_{B3r}} |\nabla v_0|^{\widetilde{p}_B}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}} \\ &\quad + \frac{c}{r^{p_A}}\Big(\int_{C^+_{3r}} |\nabla v_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}}.\end{aligned} \tag{4.14}$$

Case 4: If $|C^+_{Ar}|, |C^+_{Br}| > 0$ and $\overline{C^+_{3r}} \cap \widehat{\Gamma}_D \neq \emptyset$ we use $k = 0$ in (4.6) to obtain

$$\begin{aligned}\int_{C^+_t} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y &\le \frac{1}{2}\int_{C^+_s} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y + c\int_{C^+_s} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y \\ &\quad + c\int_{C^+_{B3r}} \Big(\frac{|v_0|}{s-t}\Big)^{p_B}\mathrm{d}y + c\int_{C^+_{A3r}} \Big(\frac{|v_0|}{s-t}\Big)^{p_A}\mathrm{d}y.\end{aligned} \tag{4.15}$$

For the first term in the second line, we applied the Poincaré-Friedrichs inequality in Lemma A.3 on C^+_{B3r} and for the exponent $\widetilde{p}_B = \frac{2p_B}{2+p_B}$ (with a uniform embedding constant, note (4.8) and that $C^+_{B3r} \subset C^+_{3r}$ contains at least a $r \times 3r$ rectangle with a Dirichlet boundary of length r). Finally, the second term in the last line of (4.15) is estimated by the corresponding integral over C^+_{3r}, and then the Poincaré-Friedrichs inequality Lemma A.3

on C_{3r}^+ with $p_A < p_B$ is applied. The constant is uniform since C_{3r}^+ contains at least a Dirichlet boundary part of length r and (4.8). In summary, we proceed with the estimate in (4.15) by

$$\begin{aligned}\int_{C_t^+} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y \le \frac{1}{2}\int_{C_s^+} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y + c\int_{C_{3r}^+} |\nabla v^D|^{\widehat{p}(y)}\,\mathrm{d}y \\ + \frac{c}{(s-t)^{p_B}}\Big(\int_{C_{B3r}^+} |\nabla v_0|^{\widetilde{p}_B}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}} + \frac{c}{(s-t)^{p_A}}\Big(\int_{C_{3r}^+} |\nabla v_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}}\end{aligned} \tag{4.16}$$

which is exactly the same as in (4.13) of Case 3. Therefore, with the same arguments as for Case 3, we obtain the estimate (4.14) also for Case 4.

The estimates in the Cases 1 to 4 cover all situations (c) to (h) in Fig. 2. Hence, by adding all (non-negative) terms on the right-hand sides of (4.9) and (4.14) and taking also for the Cases 1 and 2 the terms for p_B into account, we end up with (4.2). $\square$

4.3 Reflection

We extend the estimates from Lemma 4.1 to full squares $C_{r/2}(y^0)$ and $C_{3r}(y^0)$, respectively. To do this, we expand functions v from $C_1^+(0)$ onto $C_1^-(0)$ by reflection at the hyperplane $\{y \in \mathbb{R}^2 : y_2 = 0\}$. Defining

$$\widetilde{v}(y) := \begin{cases} v(y_1, y_2), & \text{if } y \in C_1^+(0), \\ v(y_1, -y_2), & \text{if } y \in C_1^-(0), \end{cases} \tag{4.17}$$

and extending the Dirichlet function v^D and the exponent $\widehat{p}$ by the same procedure to $\widetilde{v}^D$ and $\widetilde{\widehat{p}}$, respectively, gives $\widetilde{v} \in W^{1,\widetilde{\widehat{p}}(\cdot)}(C_1(0))$ provided that $v \in W^{1,\widehat{p}(\cdot)}(C_1^+(0))$. We work with $\widetilde{v}_0 = \widetilde{v} - \widetilde{v}^D$.

Lemma 4.2 *Let the assumptions of Lemma 4.1 be fulfilled, and let $y^0 \in C_{1/4}(0)$ and $0 < r < \frac{1}{4}$. Then there exists a constant $\widetilde{c}_2 > 0$ independent of y^0, r and the involved T such that*

$$\begin{aligned}\int_{C_{\frac{r}{2}}} |\nabla \widetilde{v}_0|^{\widetilde{\widehat{p}}(y)}\,\mathrm{d}y \le \widetilde{c}_2 \int_{C_{3r}} |\widetilde{v}^D|^{\widetilde{\widehat{p}}(y)}\,\mathrm{d}y \\ + \frac{\widetilde{c}_2}{r^{p_A}}\Big(\int_{C_{3r}} |\nabla \widetilde{v}_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} + \frac{\widetilde{c}_2}{r^{p_B}}\Big(\int_{C_{B3r}} |\nabla \widetilde{v}_0|^{\widetilde{p}_B}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}}.\end{aligned} \tag{4.18}$$

Proof. We follow the ideas in [8] and discuss separately the following two cases.
Case A: $C_{3r}(y^0) \cap \{y \in \mathbb{R}^2 : y_2 = 0\} \neq \emptyset$:
i) In case of $y_2^0 > 0$ we use the estimate

$$\int_{C_{\frac{r}{2}}(y^0)} |\nabla \widetilde{v}_0|^{\widetilde{\widehat{p}}(y)}\,\mathrm{d}y \le 2\int_{C^+_{\frac{r}{2}}(y^0)} |\nabla \widetilde{v}_0|^{\widetilde{\widehat{p}}(y)}\,\mathrm{d}y = 2\int_{C^+_{\frac{r}{2}}(y^0)} |\nabla v_0|^{\widehat{p}(y)}\,\mathrm{d}y,$$

apply Lemma 4.1 and enlarge the integration domains from $C_{3r}^+(y^0)$ to $C_{3r}(y^0)$, from $C_{A3r}^+(y^0)$ to $C_{A3r}(y^0)$, from $C_{B3r}^+(y^0)$ to $C_{B3r}(y^0)$ and change the integrands to the corresponding prolongated quantities to verify the desired estimate of Lemma 4.2.
ii) If $y_2^0 < 0$ we find for $\overline{y}^0 = (y_1^0, -y_2^0)$ that

$$\int_{C_{\frac{r}{2}}(y^0)} |\nabla \widetilde{v}_0|^{\widetilde{\widehat{p}}(y)}\, \mathrm{d}y = \int_{C_{\frac{r}{2}}(\overline{y}^0)} |\nabla \widetilde{v}_0|^{\widetilde{\widehat{p}}(y)}\, \mathrm{d}y \le 2 \int_{C_{\frac{r}{2}}^+(\overline{y}^0)} |\nabla v_0|^{\widehat{p}(y)}\, \mathrm{d}y.$$

Next we exploit Lemma 4.1 and with the estimate

$$\int_{C_{i3r}^+(\overline{y}^0)} |w|^{s_i}\, \mathrm{d}y \le \int_{C_{i3r}(\overline{y}^0)} |\widetilde{w}|^{s_i}\, \mathrm{d}y = \int_{C_{i3r}(y^0)} |\widetilde{w}|^{s_i}\, \mathrm{d}y$$

for integrands $w = \nabla v_0$, ∇v^D and $s_i > 0$, $i = 1, \dots, m$, we arrive at the desired result.

Case B: $C_{3r}(y^0) \cap \{y \in \mathbb{R}^2 : y_2 = 0\} = \emptyset$:
i) If $y_2^0 > 0$ then $C_{r/2}^+(y^0) = C_{r/2}(y^0)$ and $C_{3r}^+(y^0) = C_{3r}(y^0)$ and $\widetilde{v}_0 = v_0$. Therefore we can directly apply the result of Lemma 4.1.
ii) In case of $y_2^0 < 0$ we find for $\overline{y}^0 = (y_1^0, -y_2^0)$ that $C_{r/2}(\overline{y}^0)$, $C_{3r}(\overline{y}^0) \subset \{y \in \mathbb{R}^2 : y_2 > 0\}$ which ensures

$$\int_{C_{\frac{r}{2}}(y^0)} |\nabla \widetilde{v}_0|^{\widetilde{\widehat{p}}(y)}\, \mathrm{d}y = \int_{C_{\frac{r}{2}}(\overline{y}^0)} |\nabla \widetilde{v}_0|^{\widetilde{\widehat{p}}(y)}\, \mathrm{d}y = \int_{C_{\frac{r}{2}}(\overline{y}^0)} |\nabla v_0|^{\widehat{p}(y)}\, \mathrm{d}y.$$

Thus, again Lemma 4.1 and arguments as in Case A ii) give the desired estimate. This finishes the proof. □

4.4 Caccioppoli-type inequalities for interior squares

Lemma 4.3 *We assume (A1) and (A2) and suppose that $x^0 \in \Omega$ and that $\Phi_{x^0} : U_{x^0} \cap \Omega \to C_1(0)$ is the corresponding bi-Lipschitzian map producing the cases (a) or (b) of Fig. 2. Let φ be the solution to* (3.3) *for any $T \in \mathcal{T}$ and $v = \varphi \circ \Phi_{x^0}^{-1}$ the corresponding localized part of φ, $v_0 = \varphi \circ \Phi_{x^0}^{-1} - \varphi^D \circ \Phi_{x^0}^{-1}$. Let $y^0 \in C_{1/4}(0)$ and $0 < r < \frac{1}{4}$. Then there exists a constant $\widetilde{c}_3 > 0$ independent of y^0, r, and the involved T such that*

$$\begin{aligned}\int_{C_{\frac{r}{2}}} |\nabla v_0|^{\widehat{p}(y)}\, \mathrm{d}y &\le \widetilde{c}_3 \int_{C_{3r}} |\nabla v^D|^{\widehat{p}(y)}\, \mathrm{d}y \\ &\quad + \frac{\widetilde{c}_3}{r^{p_A}} \Big(\int_{C_{3r}} |\nabla v_0|^{\widetilde{p}_A}\, \mathrm{d}y \Big)^{\frac{p_A}{\widetilde{p}_A}} + \frac{\widetilde{c}_3}{r^{p_B}} \Big(\int_{C_{B3r}} |\nabla v_0|^{\widetilde{p}_B}\, \mathrm{d}y \Big)^{\frac{p_B}{\widetilde{p}_B}}.\end{aligned} \tag{4.19}$$

Proof. We work with the cut-off functions introduced in (4.3) and use the test function $v_0 - m_{Ar}(v_0)$ for case (a) and $v_0 - m_{B3r}(v_0)$ for case (b) of Fig. 2, where $m_{Ar}(v_0)$ and $m_{B3r}(v_0)$ denote the means of v_0 on $C_{Ar}(y^0)$ and $C_{B3r}(y^0)$, respectively. We can follow all the arguments in the proof of Lemma 4.1, Case 1 and 3 substituting the sets $C_t^+(y^0)$, $C_s^+(y^0)$, $C_{r/2}^+(y^0)$, $C_{3r}^+(y^0)$, $C_{A3r}^+(y^0)$, $C_{B3r}^+(y^0)$, $C_1^+(0)$ by the corresponding sets $C_t(y^0)$, $C_s(y^0)$, $C_{r/2}(y^0)$, $C_{3r}(y^0)$, $C_{A3r}(y^0)$, $C_{B3r}(y^0)$, $C_1(0)$. Having in mind that the uniform bound for the Poincaré-Sobolev embedding result in Lemma A.4 covers also this situation, we obtain the result. □

4.5 Higher integrability of the gradient

Our aim is to apply the Giaquinta-Modica Theorem A.2 to establish the higher integrability of the gradient stated in Theorem 3.1. If $x^0 \in \partial\Omega$ let $\widetilde{v}$, $\widetilde{v}_0$ be given as in (4.17), while for $x^0 \in \Omega$ we set $\widetilde{v} = v$, $\widetilde{v}_0 = v_0$. First, we rewrite the inequalities (4.18) and (4.19) as follows: We estimate the second term in the right-hand side in (4.18) and (4.19) by

$$\frac{1}{r^{p_A}}\Big(\int_{C_{3r}} |\nabla\widetilde{v}_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} \le \frac{c}{r^{p_A}}\Big(\int_{C_{A3r}} |\nabla\widetilde{v}_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} + \frac{c}{r^{p_A}}\Big(\int_{C_{B3r}} |\nabla\widetilde{v}_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}}, \quad (4.20)$$

where the last term can be estimated by

$$\begin{aligned}
\frac{1}{r^{p_A}}\Big(\int_{C_{B3r}} |\nabla\widetilde{v}_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} &\le \frac{1}{r^{p_A}}\Big(\big\||\nabla\widetilde{v}_0|^{\widetilde{p}_A}\big\|_{L^{\frac{p_A}{\widetilde{p}_A}}(C_{B3r})}\|1\|_{L^{\frac{p_A}{p_A-\widetilde{p}_A}}(C_{B3r})}\Big)^{\frac{p_A}{\widetilde{p}_A}} \\
&= \frac{1}{r^{p_A}}\Big(\|\nabla\widetilde{v}_0\|^{\widetilde{p}_A}_{L^{p_A}(C_{B3r})}\|1\|_{L^{\frac{p_A}{p_A-\widetilde{p}_A}}(C_{B3r})}\Big)^{\frac{p_A}{\widetilde{p}_A}} \\
&\le c\|\nabla\widetilde{v}_0\|^{p_A}_{L^{p_A}(C_{B3r})}\, r^{\frac{2p_A-\widetilde{p}_A(2+p_A)}{\widetilde{p}_A}} \le c\|\nabla\widetilde{v}_0\|^{p_A}_{L^{p_A}(C_{B3r})} \\
&= c\int_{C_{B3r}} |\nabla\widetilde{v}_0|^{p_A}\,\mathrm{d}y
\end{aligned} \quad (4.21)$$

since $2p_A - \widetilde{p}_A(2+p_A) = 0$. Denoting by χ_C the characteristic function of a set $C \subset C_1(0)$, we introduce the functions

$$\begin{aligned}
g(y) &:= \chi_{C_{A1}(0)}(y)\big|\nabla\widetilde{v}_0(y)\big|^{p_A} + \chi_{C_{B1}(0)}(y)\big|\nabla\widetilde{v}_0(y)\big|^{p_B}, \\
h(y) &:= c\Big(\chi_{C_{B1}(0)}(y)\big|\nabla\widetilde{v}_0(y)\big|^{p_A} + \big|\nabla v^D(y)\big|^{\widehat{p}(y)}\Big).
\end{aligned} \quad (4.22)$$

With this, the left hand side in (4.18) and (4.19) can be written as $r^2 \fint_{C_{r/2}} g\,\mathrm{d}y$. Moreover, dividing the inequality in (4.20) by r^2, we have for the first term in the right-hand side

$$\frac{1}{r^{2+p_A}}\Big(\int_{C_{A3r}} |\nabla\widetilde{v}_0|^{\widetilde{p}_A}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} \le \frac{1}{r^{2+p_A}}\Big(\int_{C_{3r}} g^{\frac{\widetilde{p}_A}{p_A}}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} \le c\Big(\fint_{C_{3r}} g^{\frac{\widetilde{p}_A}{p_A}}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}},$$

where we have used again that $2p_A - \widetilde{p}_A(2+p_A) = 0$. The last term in (4.18) and (4.19) can be estimated in a similar way using that $2p_B - \widetilde{p}_B(2+p_B) = 0$. Finally, we can estimate the right-hand side in (4.21) and the first term in the right-hand side of (4.18) and (4.19) via h to obtain

$$\fint_{C_{r/2}} g\,\mathrm{d}y \le c\Big(\fint_{C_{3r}} g^{\frac{\widetilde{p}_A}{p_A}}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} + c\Big(\fint_{C_{3r}} g^{\frac{\widetilde{p}_B}{p_B}}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}} + c\fint_{C_{3r}} h(y)\,\mathrm{d}y. \quad (4.23)$$

Let $\omega := \frac{2+p_+}{2} > 1$ such that $\omega\frac{\widetilde{p}_A}{p_A}, \omega\frac{\widetilde{p}_B}{p_B} \ge 1$ and define $\widetilde{g} := g^{1/\omega}$. We can write

$$\fint_{C_{r/2}} \widetilde{g}^\omega\,\mathrm{d}y = \frac{1}{r^2}\|\widetilde{g}\|^\omega_{L^\omega(C_{r/2})}, \quad \Big(\fint_{C_{3r}} g^{\frac{\widetilde{p}_B}{p_B}}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}} = \Big(\frac{1}{36r^2}\Big)^{\frac{p_B}{\widetilde{p}_B}}\|\widetilde{g}\|^\omega_{L^{\omega\frac{\widetilde{p}_B}{p_B}}(C_{3r})},$$

and similar for p_A. Since $p_A > p_B$ we have $\frac{\widetilde{p}_A}{p_A} > \frac{\widetilde{p}_B}{p_B}$ and can estimate

$$\begin{aligned}\|\widetilde{g}\|^{\omega}_{L^{\omega\frac{\widetilde{p}_B}{p_B}}(C_{3r})} &= \Big(\int_{C_{3r}} \widetilde{g}^{\omega\frac{\widetilde{p}_B}{p_B}}\,\mathrm{d}y\Big)^{\frac{p_B}{\widetilde{p}_B}} \le \Big[\|\widetilde{g}^{\omega\frac{\widetilde{p}_B}{p_B}}\|_{L^{\frac{p_B\widetilde{p}_A}{\widetilde{p}_B p_A}}(C_{3r})}\|1\|_{L^{\frac{p_B\widetilde{p}_A}{p_B\widetilde{p}_A-\widetilde{p}_B p_A}}(C_{3r})}\Big]^{\frac{p_B}{\widetilde{p}_B}}\\ &\le \|\widetilde{g}\|^{\omega}_{L^{\omega\frac{\widetilde{p}_A}{p_A}}(C_{3r})}\|1\|^{\frac{p_B}{\widetilde{p}_B}}_{L^{\frac{p_B\widetilde{p}_A}{p_B\widetilde{p}_A-\widetilde{p}_B p_A}}(C_{3r})}.\end{aligned}$$

Moreover, an elementary calculation shows that

$$\Big(\frac{1}{36r^2}\Big)^{\frac{p_B}{\widetilde{p}_B}}\|1\|^{\frac{p_B}{\widetilde{p}_B}}_{L^{\frac{p_B\widetilde{p}_A}{p_B\widetilde{p}_A-\widetilde{p}_B p_A}}(C_{3r})} = \Big(\frac{1}{36r^2}\Big)^{\frac{p_A}{\widetilde{p}_A}}.$$

Concluding, this enables us to estimate the second term in the right-hand side of (4.23) so that we end up with

$$⨍_{C_{r/2}} g\,\mathrm{d}y \le c\Big\{\Big(⨍_{C_{3r}} g^{\frac{\widetilde{p}_A}{p_A}}\,\mathrm{d}y\Big)^{\frac{p_A}{\widetilde{p}_A}} + ⨍_{C_{3r}} h(y)\,\mathrm{d}y\Big\} \quad \forall\, y^0 \in C_{\frac14}(0),\ \forall\, r \in (0,\tfrac14). \tag{4.24}$$

We are going to apply the Giaquinta-Modica result in Theorem A.2. For this, we set $Q_R := C_{\frac14}(0)$ and $a := \frac{\widetilde{p}_A}{p_A}$ and take g and h as in (4.22). Since by assumption (A2) $\varphi^D \in W^{1,\infty}(\Omega)$, and we supposed $p_A < p_B$ there is some $b > 1$ such that $h \in L^b(C_{\frac14}(0))$. Then, (4.24) guarantees the assumptions of Theorem A.2 for all $Q \subset \widetilde{Q} \subset Q_R$, where $\widetilde{Q}$ has six times the diameter of Q. Thus, the Giaquinta-Modica Theorem A.2 and Remark A.1 yield an exponent $s^* > 1$ and a constant $c > 0$ such that $g = |\nabla\widetilde{v}_0|^{\widehat{p}(\cdot)} \in L^{s^*}(C_{\frac{1}{24}}(0))$ and

$$⨍_{C_{\frac{1}{24}}(0)} |\nabla\widetilde{v}_0|^{\widehat{p}(y)s^*}\,\mathrm{d}y \le c\Big\{\Big(⨍_{C_{\frac14}(0)} |\nabla\widetilde{v}_0|^{\widehat{p}(y)}\,\mathrm{d}y\Big)^{s^*} + ⨍_{C_{\frac14}(0)} h(y)^{s^*}\,\mathrm{d}y\Big\}.$$

This ensures

$$\int_{C_{\frac{1}{24}}(0)}|\nabla\widetilde{v}_0|^{\widehat{p}(y)s^*}\mathrm{d}y \le c\Big\{\Big(\int_{C_{\frac14}(0)}|\nabla\widetilde{v}_0|^{\widehat{p}(y)}\,\mathrm{d}y\Big)^{s^*} + \int_{C_{\frac14}(0)}\big(\chi_{C_{B1}}(y)|\nabla\widetilde{v}_0|^{p_A} + |\nabla v^D|^{\widehat{p}(y)}\big)^{s^*}\mathrm{d}y\Big\}.$$

If $x^0 \in \partial\Omega$, restriction to the upper half square and back transformation by means of $\Phi^{-1}_{x^0}$ (or only back transformation by means of $\Phi^{-1}_{x^0}$ if $x^0 \in \Omega$) leads to

$$\begin{aligned}&\int_{\Phi^{-1}_{x^0}(C_{\frac{1}{24}})\cap\Omega} |\nabla\varphi_0|^{p(x)s^*}\,\mathrm{d}x\\ &\le c\Big\{\Big(\int_{\Phi^{-1}_{x^0}(C_{\frac14})\cap\Omega} |\nabla\varphi_0|^{p(x)}\,\mathrm{d}x\Big)^{s^*} + \int_{\Phi^{-1}_{x^0}(C_{\frac14})\cap\Omega} \big(\chi_{\Omega_B}(x)|\nabla\varphi_0|^{p_A} + |\nabla\varphi^D|^{p(x)}\big)^{s^*}\,\mathrm{d}x\Big\},\end{aligned}$$

where $\varphi_0 = \varphi - \varphi^D$ and χ_{Ω_B} is the characteristic function of the set $\{x \in \Omega : p(x) = p_B\}$. This finishes the proof of Theorem 3.1, since by (A1) there exists a finite number of sets $\Phi^{-1}_{x^0}(C_{\frac{1}{24}})$ which cover Ω.

A Appendix

Lemma A.1 (Generalized Hölder inequality) *Let $r(\cdot), r'(\cdot), r''(\cdot) \in \mathcal{P}(\Omega)$ be variable exponents with $r_-, r'_-, r''_- > 1$, $r_+, r'_+, r''_+ < \infty$, and $\frac{1}{r(x)} + \frac{1}{r'(x)} + \frac{1}{r''(x)} = 1$ a.e. in Ω. Then*

$$\int_\Omega |f(x)g(x)h(x)|\,dx \le c(r_-, r'_-, r''_-)\|f\|_{L^{r(\cdot)}}\|g\|_{L^{r'(\cdot)}}\|h\|_{L^{r''(\cdot)}}$$

for every $f \in L^{r(\cdot)}(\Omega)$, $g \in L^{r'(\cdot)}(\Omega)$, $h \in L^{r''(\cdot)}(\Omega)$.

Proof. We generalize the proof given in [17] to the situation of three factors. We suppose that $\|f\|_{L^{r(\cdot)}}, \|g\|_{L^{r'(\cdot)}}, \|h\|_{L^{r''(\cdot)}} \ne 0$ and set pointwise $a = f(x)/\|f\|_{L^{r(\cdot)}}$, $b = g(x)/\|g\|_{L^{r'(\cdot)}}$, $c = h(x)/\|h\|_{L^{r''(\cdot)}}$ and use the inequality

$$abc \le \frac{a^r}{r} + \frac{b^{r'}}{r'} + \frac{c^{r''}}{r''},$$

integrate over Ω and apply (2.2) to find

$$\begin{aligned}&\int_\Omega \frac{|f(x)g(x)h(x)|}{\|f\|_{L^{r(\cdot)}}\|g\|_{L^{r'(\cdot)}}\|h\|_{L^{r''(\cdot)}}}\,\mathrm{d}x \\ &\le \operatorname*{ess\,sup}_\Omega \frac{1}{r(x)}\rho_{r(\cdot)}\Big(\frac{f}{\|f\|_{L^{r(\cdot)}}}\Big) + \operatorname*{ess\,sup}_\Omega \frac{1}{r'(x)}\rho_{r'(\cdot)}\Big(\frac{g}{\|g\|_{L^{r'(\cdot)}}}\Big) \\ &\quad + \operatorname*{ess\,sup}_\Omega \frac{1}{r''(x)}\rho_{r''(\cdot)}\Big(\frac{h}{\|h\|_{L^{r''(\cdot)}}}\Big) \le \frac{1}{r_-} + \frac{1}{r'_-} + \frac{1}{r''_-}.\end{aligned}$$

Then multiplication by $\|f\|_{L^{r(\cdot)}}\|g\|_{L^{r'(\cdot)}}\|h\|_{L^{r''(\cdot)}}$ gives the desired result. □

Theorem A.1 (Poincaré-Friedrichs inequality, [2, Theorem 5.4.3]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitzian domain and $p \in [1, \infty)$. Then for every $w \in W^{1,p}(\Omega)$ with boundary value zero on a set $\Gamma_0 \subset \partial\Omega$ of positive measure, we have $\|w\|_{L^p(\Omega)} \le C\|\nabla w\|_{L^p(\Omega)}$, where the constant $C > 0$ depends only on n, p, Γ_0 and Ω.*

The previous result is formulated for C^1 domains in [2], however, all arguments in the proof work also for Lipschitzian domains.

Lemma A.2 (Poincaré inequality) *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitzian domain with $mes\,(\Gamma_D) > 0$ and let $p \in \mathcal{P}(\Omega)$ a variable exponent with $2 \le p_- \le p_+ < \infty$. Then there is a constant $c > 0$ such that*

$$\|v\|_{L^{p(\cdot)}} \le c\|\nabla v\|_{L^{p(\cdot)}} \qquad \text{for all } v \in W_D^{1,p(\cdot)}(\Omega).$$

Proof. This lemma generalizes some assertion of [5, Theorem 8.2.18] to Sobolev functions with mixed boundary conditions with zero boundary values on a part Γ_D of the boundary with positive measure. Note that in the situation $n = 2$ and $p_- \ge 2$ for $u \in W_D^{1,p(\cdot)}(\Omega)$ we can make use of the following embedding results

$$W_D^{1,p(\cdot)}(\Omega) \hookrightarrow W_D^{1,p_-}(\Omega) \hookrightarrow L^{p_+}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

comp. [5, Lemma 8.1.8], classical Poincaré-Sobolev inequality for the constant exponents $p_- \geq 2$ and $p_+ < \infty$, Poincaré-Friedrichs inequality Theorem A.1 for p^-, and [17, Theorem 2.8]. Therefore we find with changing constants $c > 0$ depending on Ω, p_- and p_+ that

$$\|v\|_{L^{p(\cdot)}} \leq c\|v\|_{L^{p_+}} \leq c\|v\|_{W^{1,p_-}} \leq c\|\nabla v\|_{L^{p_-}} \leq c\|\nabla v\|_{L^{p(\cdot)}} \quad \forall v \in W_D^{1,p(\cdot)}(\Omega)$$

which gives the desired estimate. $\square$

From the definition of the $W^{1,p(\cdot)}(\Omega)$-norm in (2.5) and the Poincaré inequality Lemma A.2 directly follows

Corollary A.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitzian domain with mes$(\Gamma_D) > 0$ and let $p \in \mathcal{P}(\Omega)$ with $2 \leq p_- \leq p_+ < \infty$. Then $\|\nabla\cdot\|_{L^{p(\cdot)}}$ is an equivalent norm on $W_D^{1,p(\cdot)}(\Omega)$,*

$$c_1\|\nabla v\|_{L^{p(\cdot)}} \leq \|v\|_{W^{1,p(\cdot)}} \leq c_2\|\nabla v\|_{L^{p(\cdot)}} \quad \forall v \in W_D^{1,p(\cdot)}(\Omega).$$

Lemma A.3 (Uniform Poincaré-Friedrichs type inequality) *Let $y^0 \in C^+_{\frac{1}{4}}(0)$, $\rho \in (0, \frac{1}{4}]$. Let $G \subset (y^0 + [-3\rho, 3\rho]^2) \cap [-1,1] \times [0,1]$ be an axis parallel rectangle with length $(a_0 + a_1)\rho$ and hight $a_2\rho$, where $a_0\rho = \mathrm{mes}(\overline{G} \cap \{y \in \mathbb{R}^2 : y_1 \geq 0, y_2 = 0\})$ is the length of the Dirichlet boundary and $a_1\rho = \mathrm{mes}(\overline{G} \cap \{y \in \mathbb{R}^2 : y_1 < 0, y_2 = 0\})$. Additionally we assume that a_0, $a_2 \in [1,6]$ and $a_1 \in [0,6]$. Let $p \in [p_-, p_+]$ be constant and $\widetilde{p} = \frac{2p}{p+2}$. Then there is a constant $C_{\mathrm{PF},p} > 0$ such that*

$$\|w\|^p_{L^p(G)} \leq C_{\mathrm{PF},p}\|\nabla w\|^p_{L^{\widetilde{p}}(G)^2} \quad \forall w \in W_D^{1,\widetilde{p}}(G)$$

for all G with admissible y^0, ρ, a_0, a_1, a_2.

Proof. For $\rho \in (0, \frac{1}{4}]$ and $y^0 \in C_{\frac{1}{4}}(0)$ and a_0, $a_2 \in [1,6]$, $a_1 \in [0,6]$ we introduce the affine transformation $V^D_{\rho,y^0,a_0,a_1,a_2} : C^+_1(0) \to G$ by

$$V^D_{\rho,y^0,a_0,a_1,a_2}(z) = \begin{pmatrix} \frac{a_0+a_1}{2}\rho & 0 \\ 0 & a_2\rho \end{pmatrix} z + b^D_{\rho,y^0,a_0,a_1,a_2}$$

with a suited translation $b^D_{\rho,y^0,a_0,a_1,a_2}$. Note that

$$|\det DV^D_{\rho,y^0,a_0,a_1,a_2}| = \frac{(a_0 + a_1)a_2}{2}\rho^2.$$

Let $p \in [p_-, p_+]$ be fixed. Then the Poincaré-Friedrichs inequality Theorem A.1 for the embedding $W^{1,p}(C^+_1(0)) \hookrightarrow L^p(C^+_1(0))$ with the Dirichlet boundary part $\Gamma_0 := [2/3, 1] \times \{0\}$ can be applied.

By construction, for functions w on G with zero Dirichlet values on $\widehat{\Gamma}_D$ the function $\widehat{w}(z) := w(V^D_{\rho,y^0,a_0,a_1,a_2}(z))$ has zero Dirichlet values at least on $\Gamma_0 = [2/3, 1] \times \{0\}$. Applying the transformation formula, the classical Sobolev embedding result $W^{1,\widetilde{p}}(C^+_1(0)) \hookrightarrow$

$L^p(C_1^+(0))$, Theorem A.1 and back transformation we obtain with (from estimate to estimate changing) constants $c>0$ not depending on G

$$\begin{aligned}\|w\|^p_{L^p(G)} &= \int_{C_1^+(0)} |\widehat{w}(z)|^p \,|\det DV^D_{\rho,y^0 a_0,,a_1,a_2}|\,\mathrm{d}z\\ &\le c\rho^2\Big(\int_{C_1^+(0)}\Big(|\widehat{w}(z)|^{\widetilde{p}}+|\nabla_z\widehat{w}(z)|^{\widetilde{p}}\Big)\,\mathrm{d}z\Big)^{\frac{p}{\widetilde{p}}}\\ &\le c\rho^2\Big(\int_{C_1^+(0)}|\nabla_z\widehat{w}(z)|^{\widetilde{p}}\,\mathrm{d}z\Big)^{\frac{p}{\widetilde{p}}} \le c\|\nabla w\|^p_{L^{\widetilde{p}}(G)}. \qquad \square\end{aligned}$$

Lemma A.4 (Uniform Poincaré-Sobolev type inequality) *Let $y^0 \in C_{\frac14}(0)$ and $\rho \in (0,\frac14]$. Let $G \subset y^0+[-3\rho,3\rho]^2 \subset [-1,1]^2$ be an axis parallel rectangle with side lengths $a_1\rho$ and $a_2\rho$, a_1, $a_2\in[1,6]$. Moreover, let $p\in[p_-,p_+]$ be constant and $\widetilde{p}=\frac{2p}{p+2}$. Then there is a constant $C_{\mathrm{PS},p}>1$ such that*

$$\|w-m_G(w)\|^p_{L^p(G)} \le C_{\mathrm{PS},p}\|\nabla w\|^p_{L^{\widetilde{p}}(G)^2} \quad \forall w\in W^{1,\widetilde{p}}(G), \quad m_G(w)=\frac{1}{|G|}\int_G w(y)\,dy$$

for all G with admissible y^0, ρ, a_1, a_2.

Proof. For all $p\in[p_-,p_+]$, the classical Poincaré-Sobolev inequality for the embedding $W^{1,\widetilde{p}}(C_1(0)) \hookrightarrow L^p(C_1(0))$ gives an embedding constant c_p which continuously depends on p, and $C^0_{\mathrm{PS}} := \max_{p\in[p_-,p_+]} c_p^p$ is finite.

For $\rho\in(0,\frac14]$ and $y^0\in C_{\frac14}(0)$ and a_1, $a_2\in[1,6]$ we introduce the affine transformation $V_{\rho,y^0,a_1,a_2}: C_1(0)\to G$ by

$$V_{\rho,y^0,a_1,a_2}(z) = \begin{pmatrix} \frac{a_1}{2}\rho & 0 \\ 0 & \frac{a_2}{2}\rho \end{pmatrix} z + b_{\rho,y^0,a_1,a_2}$$

with a suited translation b_{ρ,y^0,a_1,a_2}. Note that

$$|\det DV_{\rho,y^0,a_1,a_2}| = \frac{a_1a_2}{4}\rho^2, \qquad \frac{|\det DV_{\rho,y^0,a_1,a_2}|}{|G|} = \frac{a_1a_2\rho^2}{4a_1a_2\rho^2}=\frac14=\frac{1}{|C_1(0)|}.$$

Therefore, the mean value $m_G(w)$ can be expressed by $m_G(w)=m_{C_1(0)}(\widehat{w})$ with $\widehat{w}(z)=w(V_{\rho,y^0,a_1,a_2}(z))$. Using the transformation formula, the classical Sobolev embedding result $W^{1,\widetilde{p}}(C_1(0))\hookrightarrow L^p(C_1^+(0))$ for the function $\widehat{w}-m_{C_1(0)}(\widehat{w})$, the classical Poincaré inequality and back transformation we obtain with (from estimate to estimate changing) constants $c>0$ not depending on G

$$\begin{aligned}\big\|w-m_G(w)\big\|^p_{L^p(G)} &= \int_{C_1(0)}\big|\widehat{w}(z)-m_{C_1(0)}(\widehat{w})\big|^p\,|\det DV_{\rho,y^0,a_1,a_2}|\,\mathrm{d}z\\ &\le c\rho^2\Big(\int_{C_1(0)}\Big(\big|\widehat{w}(z)-m_{C_1(0)}(\widehat{w})\big|^{\widetilde{p}}+|\nabla_z\widehat{w}(z)|^{\widetilde{p}}\Big)\,\mathrm{d}z\Big)^{\frac{p}{\widetilde{p}}}\\ &\le c\rho^2\Big(\int_{C_1(0)}|\nabla_z\widehat{w}(z)|^{\widetilde{p}}\,\mathrm{d}z\Big)^{\frac{p}{\widetilde{p}}} \le c\|\nabla w\|^p_{L^{\widetilde{p}}(G)}. \qquad \square\end{aligned}$$

Lemma A.5 *Let $Z(t)$ be a bounded non-negative function on the interval $[\rho, R]$. Let for all $\rho \le t < s \le R$ the inequality*

$$Z(t) \le \Big[W_1(s-t)^{-\mu_1} + W_2(s-t)^{-\mu_2} + Y\Big] + \iota Z(s) \tag{A.1}$$

with W_1, W_2, $Y \ge 0$ and $\mu_1 > \mu_2 > 0$ and $0 < \iota < 1$ be fulfilled. Then

$$Z(\rho) \le c(\mu_1, \iota)\Big[W_1(R-\rho)^{-\mu_1} + W_2(R-\rho)^{-\mu_2} + Y\Big].$$

This lemma is taken from [13, Lemma 6.1]. Some form of a generalized Gehring lemma is

Theorem A.2 (Giaquinta and Modica, Theorem 6.6 in [13]) *Let be $g, h \in L^1(Q_R)$ with $g, h \ge 0$ a.e. and assume that for every pair of concentric cubes $Q \subset \widetilde{Q} \subset\subset Q_R$ where $\widetilde{Q}$ has the double diameter of Q, we have for some constant $\omega > 0$*

$$\fint_Q g \, \mathrm{d}x \le \omega \Big\{ \Big(\fint_{\widetilde{Q}} g^a \, \mathrm{d}x \Big)^{\frac{1}{a}} + \fint_{\widetilde{Q}} h \, \mathrm{d}x \Big\},$$

with $0 < a < 1$. Let the function $h \in L^b(Q_R)$ for some $b > 1$. Then there exist constants $c > 0$ and $s^* > 1$ such that g belongs to $L^{s^*}(Q_{R/2})$ and

$$\fint_{Q_{R/2}} g^{s^*} \, \mathrm{d}x \le c \Big\{ \Big(\fint_{Q_R} g \, \mathrm{d}x \Big)^{s^*} + \fint_{Q_R} h^{s^*} \, \mathrm{d}x \Big\}. \tag{A.2}$$

Remark A.1 *As already mentioned in [8], an inspection of the proof of [13, Theorem 6.6] ensures that the result in the spirit of Theorem A.2 remains valid if $\widetilde{Q}$ has six times the diameter of Q. However, in this case the resulting estimate in* (A.2) *is obtained with $g \in L^{s^*}(Q_{R/6})$ and the integration domain in the left-hand side of* (A.2) *has to be $Q_{R/6}$ instead of $Q_{R/2}$.*

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