Asymptotic Behavior of the Solutions to a Landau-Ginzburg System with Viscosity for Martensitic Phase Transitions in Shape Memory Alloys

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Abstract

In this paper, we investigate the system of partial differential equations governing the dynamics of martensitic phase transitions in shape memory alloys under the presence of a (possibly small) viscous stress. The corresponding free energy is assumed in Landau-Ginzburg form and nonconvex as function of the order parameter. Results concerning the asymptotic behavior of the solution as time tends to infinity are proved, and the compactness of the orbit is shown.

1 Introduction

In the present paper, we study the asymptotic behavior of the solutions to a system that arises in the thermomechanical developments in a one-dimensional heat-conducting viscous solid of constant mass density ϱ (assumed to be normalized to unity, i.e. $\varrho = 1$). The solid is subjected to heating and loading. We think of metallic solids that not only respond to a change of the strain ε by a (possibly nonlinear) elastic stress $\sigma = \sigma(\varepsilon)$, but also to a change of the curvature of their metallic lattice by a couple stress $\mu = \mu(\varepsilon_x)$.
We assume that the Helmholtz free energy density F is a potential of Landau-Ginzburg form, i.e.

$$F = F(\varepsilon, \varepsilon_x, \theta) \tag{1.1}$$

where θ denotes the absolute temperature. To cover systems modelling first-order stress-induced and temperature-induced solid-solid phase transitions accompanied by hysteresis phenomena, we do not assume that F is a convex function of the order parameter ε.
A particular class of materials, in which both stress-induced and temperature-induced first-order phase transitions leading to a rather spectacular hysteretic behavior occur, are the so-called *shape memory alloys*. In these materials the metallic lattice is deformed by shear, and the assumption of a constant density is justified. The shape memory effect itself is due to martensitic phase transitions between different configurations of the crystal lattice, namely austenite and martensitic twins. For an account of the physical properties of shape memory alloys, we refer the reader to chapter 5 in the monograph [4]. In a series of papers (cf., for instance, [7], [8]), Falk has proposed a Laudau-Ginzburg theory that uses the shear strain ε as order parameter in order to explain the occurrence of the martensitic transitions in shape memory alloys. In this connection, we also refer to the works of Müller (cf. [1], [14]).
The simplest form for the free energy density F that accounts quite well for the experimentally observed behavior and that takes couple stresses into account is (see Falk [7], [8]) given by

$$F(\varepsilon, \varepsilon_x, \theta) = F_0(\theta) + F_1(\varepsilon)\theta + F_2(\varepsilon) + \frac{\delta}{2}\varepsilon_x^2 \,, \tag{1.2}$$

where

$$F_1(\varepsilon) = \alpha_1 \varepsilon^2, \quad F_2(\varepsilon) = \alpha_3\varepsilon^6 - \alpha_2\varepsilon^4 - \alpha_1\theta_1\varepsilon^2 \,, \tag{1.3}$$

$$F_0(\theta) = -C_V\theta \log\left(\frac{\theta}{\theta_2}\right) + C_V\theta + \tilde{C} \,, \tag{1.4}$$

with positive physical constants $\theta_1, \delta, \alpha_1, \alpha_2, \alpha_3, \theta_2, C_V, \tilde{C}$. The constant C_V denotes the specific heat. Observe that in the interesting range of temperatures, for θ close to θ_1, F is not a convex function of the shear strain ε. In fact, $F(\cdot, \varepsilon_x, \theta)$ may have up to three minima that correspond to the austenitic and the two martensitic phases.
We want to forecast the dynamics of the phase transitions in the one-dimensional situation. To this end, let $\Omega = (0,1)$, and, for $t > 0$, $\Omega_t = \Omega \times (0,t)$. Then the balance laws of linear momentum and internal energy read

$$u_{tt} - \sigma_x + \mu_{xx} = 0, \quad \text{in} \quad \Omega_\infty \,, \tag{1.5}$$

$$U_t + q_x - \sigma\varepsilon_t - \mu\varepsilon_{xt} = 0, \quad \text{in} \quad \Omega_\infty\,. \tag{1.6}$$

The second law of thermodynamics is expressed by the Clausius-Duhem inequality

$$S_t + \left(\frac{q}{\theta}\right)_x \geq 0, \quad \text{in} \quad \Omega_\infty\,. \tag{1.7}$$

Here, u, σ, μ, U, q, ε, S, and θ, denote displacement, shear stress, couple stress, internal energy density, heat flux, shear strain, entropy density, and absolute temperature, in that order. For one-dimensional homogeneous thermoviscoelastic materials, we have the constitutive relations

$$\varepsilon = u_x, \quad \sigma = \frac{\partial F}{\partial \varepsilon} + \gamma\varepsilon_t, \quad \mu = \frac{\partial F}{\partial \varepsilon_x}, \quad S = -\frac{\partial F}{\partial \theta}, \quad U = F + \theta S\,, \tag{1.8}$$

where $\gamma > 0$ is the viscosity. For the heat flux q, we assume Fourier's law

$$q = -k\theta_x, \tag{1.9}$$

where $k > 0$ is the heat conductivity (assumed constant). Obviously, this assumption implies the validity of (1.7), so that the second law of thermodynamics is automatically satisfied.
Inserting the constitutive relations in the balance laws (1.5)–(1.6), we obtain the system of partial differential equations

$$u_{tt} - (f_1\theta + f_2)_x - \gamma\varepsilon_{xt} + \delta u_{xxxx} = 0, \quad \text{in} \quad \Omega_\infty, \tag{1.10}$$

$$C_V\theta_t - k\theta_{xx} - f_1\theta\varepsilon_t - \gamma\varepsilon_t^2 = 0, \quad \text{in} \quad \Omega_\infty, \tag{1.11}$$

$$\varepsilon = u_x, \quad \text{in} \quad \Omega_\infty, \tag{1.12}$$

where

$$f_1 = f_1(\varepsilon) = F_1'(\varepsilon), \quad f_2 = f_2(\varepsilon) = F_2'(\varepsilon). \tag{1.13}$$

In addition, we prescribe the initial and boundary conditions

$$u|_{x=0} = \varepsilon_x|_{x=0} = 0, \quad \varepsilon|_{x=1} = (\gamma u_{xt} - \delta u_{xxx} + \sigma_1)|_{x=1} = 0, \tag{1.14}$$

with

$$\sigma_1 = f_1\theta + f_2, \tag{1.15}$$

as well as

$$\theta_x|_{x=0,1} = 0, \tag{1.16}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x) > 0, \ x \in \overline{\Omega}. \tag{1.17}$$

The physical meaning of the boundary conditions is clear; for instance, the second condition at $x = 1$ describes the stress-free situation.
Next, we employ an idea of Andrews [2] and Pego [17] to simplify the problem by introducing the *velocity potential*

$$p(x,t) = \int_1^x u_t(y,t)\,dy\,. \tag{1.18}$$

Then,

$$\varepsilon_t = p_{xx}, \quad \text{in} \quad \Omega_\infty, \tag{1.19}$$

and (1.10)–(1.11) can be rewritten as

$$p_t - \gamma p_{xx} + \delta\varepsilon_{xx} - \sigma_1 = 0, \quad \text{in} \quad \Omega_\infty, \tag{1.20}$$

$$C_V\theta_t - k\theta_{xx} - f_1\theta p_{xx} - \gamma p_{xx}^2 = 0, \quad \text{in} \quad \Omega_\infty. \tag{1.21}$$

Accordingly, the initial and boundary conditions (1.14), (1.16), (1.17) become

$$p_x|_{x=0} = p_{xxx}|_{x=0} = \varepsilon_x|_{x=0} = 0, \tag{1.22}$$

$$p|_{x=1} = p_{xx}|_{x=1} = \varepsilon|_{x=1} = 0, \tag{1.23}$$

$$\varepsilon(x,0) = \varepsilon_0 = u_{0x}, \quad p(x,0) = p_0(x) = \int_1^x u_1(y)\,dy, \quad \theta(x,0) = \theta_0, \;\; x \in \overline{\Omega}. \tag{1.24}$$

It is easy to see that if (u, v, θ) is a smooth solution to (1.10)–(1.17), then (ε, p, θ) is a smooth solution to (1.19)–(1.24), and vice versa. Therefore, it suffices to consider the problem (1.19)–(1.24). In the sequel, we assume without loss of generality that $C_V = 1$.

Before stating and proving our results, let us first recall some related results in the literature. In the case $\delta = 0$, Dafermos [5], Dafermos & Hsiao [6], Chen & Hoffmann [9], and Jiang [11], proved the global existence of a classical solution to the system of (1.10)–(1.12) with various boundary conditions for a class of solid-like materials. However, an analysis of the asymptotic behavior as $t \to \infty$ was not performed in these papers. Recently, on the basis of Dafermos [5] and Dafermos & Hsiao [6], T. Luo [13] further investigated the asymptotic behavior of smooth solutions as time tends to infinity for a special class of solid-like materials in which $e = C_V\,\theta$, $F_2 = 0$, and $\delta = 0$. Racke & Zheng [18] obtained global existence, uniqueness and the asymptotic behavior of weak solutions to (1.10)–(1.12) for $\delta = 0$ if both ends of the rod are insulated and if at least one end is stress-free.

In the case $\delta > 0$, we refer to Sprekels & Zheng [20], if $\delta > 0, \gamma = 0$, and to Hoffmann & Zochowski [10], if $\delta > 0, \gamma > 0$, for global existence and uniqueness results for Falk's Landau-Ginzburg model of shape memory alloys. However, the a priori estimates for the solution obtained in these papers depend on t, and hence the asymptotic behavior of the solution for $t \to \infty$ could not be treated there.

We also refer to the works of Andrews [2], Andrews & Ball [3], and Pego [17], for the isothermal and purely viscoelastic case.

The purpose of our contribution is to study the asymptotic behavior as $t \to \infty$ of the solutions to the system (1.19)–(1.24) and to prove the compactness of the orbit.

Next, we state the main result of this paper.

Theorem 1.1 *Suppose that* $\varepsilon_0, p_0 \in H^3$ *and* $\theta_0 \in H^1$ *are given functions that satisfy the compatibility conditions* $p_{0x}|_{x=0} = \varepsilon_{0x}|_{x=0} = 0$, $p_0|_{x=1} = p_{xx}|_{x=1} = \varepsilon_x|_{x=1} = 0$, *and suppose that* $\theta_0 > 0$ *in* $[0,1]$. *Then the following results hold.*

(i) *The problem admits a unique global solution* (ε, p, θ) *satisfying*

$$\begin{gathered}
\varepsilon \in C(I\!R^+; H^3), \quad \varepsilon_t \in C(I\!R^+; H^1) \cap L^2(I\!R^+; H^2);\\
p \in C(I\!R^+; H^3) \cap L^2(I\!R^+; H^4), \quad p_t \in C(I\!R^+; H^1) \cap L^2(I\!R^+; H^2);\\
\theta \in C(I\!R^+; H^1), \quad \theta_x \in L^2(I\!R^+; H^1), \quad \theta_t \in L^2(I\!R^+; L^2),\\
\theta(x,t) > 0, \;\; \forall\,(x,t) \in \; [0,1] \times I\!R^+.
\end{gathered} \tag{1.25}$$

(ii) *As* $t \to \infty$, *it holds*

$$\|p(\cdot,t)\|_{H^3} \to 0, \quad \|p_t(\cdot,t)\|_{H^1} \to 0, \tag{1.26}$$

$$\|\delta\varepsilon_{xx}(\cdot,t) - \sigma_1(\cdot,t)\|_{H^1} \to 0, \quad \|\varepsilon_t(\cdot,t)\|_{H^1} \to 0, \quad \|\theta_x(\cdot,t)\| \to 0. \tag{1.27}$$

(iii) *For all* $\nu > 0$,

$$\varepsilon \in C([\nu,+\infty); H^4), \quad p \in C([\nu,+\infty); H^4), \quad \theta \in C([\nu,+\infty); H^3), \tag{1.28}$$

i.e. the orbit is compact in $H^3 \times H^3 \times H^1$.

(iv)

$$(\varepsilon(\cdot,t), p(\cdot,t), \theta(\cdot,t)) \to (\overline{\varepsilon}, 0, \overline{\theta}), \quad \textit{as} \quad t \to \infty, \quad \textit{in} \quad H^3 \times H^3 \times H^1\,, \tag{1.29}$$

where $(\overline{\varepsilon},\, \overline{\theta})$ *is one of the equilibria for the corresponding stationary problem.*

The main difficulties in proving Theorem 1.1 are due to the higher degree of nonlinearity inherent in the system (1.19)–(1.21), and to the higher order derivative arising for $\delta > 0$. The presence of this higher order derivative makes the problem in two ways significantly different from the problem with $\delta = 0, \gamma > 0$: it renders the orbit compact (while discontinuities of strain will persist in the case $\delta = 0, \gamma > 0$, as shown in [18]), and the technique needed to obtain the asymptotic behavior differs considerably from that used in the case $\delta = 0, \gamma > 0$. One of the main ingredients of the proof in this paper is to bound the norms of ε, p, as well as of their derivatives, in terms of expressions of the form

$$1 + \sup_{0 \le \tau \le t} \|\theta(\tau)\|_{L^\infty}^{\alpha} + \Big(\int_0^t \|\theta_t(\tau)\|^2 \, d\tau\Big)^{\beta}, \tag{1.30}$$

where $0 \le \alpha \le \frac{3}{2}, 0 \le \beta \le \frac{1}{2}$. This makes it possible to reduce the degree of nonlinearity via interpolation techniques. To study the asymptotic behavior, we will make repeated use of a basic lemma in analysis proved in Shen & Zheng [19]. In Section 2, we will prove the uniform a priori estimates and the compactness of the orbit. In Section 3, the asymptotic behavior is investigated.
The notation in this paper will be as follows: L^p, $1 \le p \le \infty$, $W^{m,\infty}$, $m \in I\!N$, $H^1 \equiv W^{1,2}$, and $H_0^1 = W_0^{1,2}$, respectively, denote the usual Lebesgue and Sobolev spaces on (0,1). By $(\cdot,\cdot)$, we denote the inner product in L^2, and $\|\cdot\|_B$ denotes the norm in the space B. We use the abbreviation $\|\cdot\| := \|\cdot\|_{L_2}$, and $C^k(I,B)$, $k \in I\!N_0$, denotes the space of k-times continuously differentiable functions from $I \subset I\!R$ into a Banach space B. The spaces $L^p(I,B)$, $1 \le p \le \infty$, are defined analogously. Finally, ∂_t or $\frac{d}{dt}$ or a subscript t and, likewise, ∂_x or a subscript x, denote the partial derivatives with respect to t and x, respectively.

2 Uniform A Priori Estimates

The general framework to prove global existence and uniqueness of solution has been established in earlier papers, for instance in Sprekels & Zheng [20] and Hoffmann & Zochowski [10]. The setting will become more apparent soon during the derivation of uniform a priori estimates. Therefore, we can focus our attention on the study of the asymptotic behavior and on the compactness of the orbit. In order to get the asymptotic behavior of the solution as $t \to \infty$, we shall prove uniform a priori estimates on ε, p, and θ with repect to t. From now on, we will always denote by C a universal positive constant that may depend on the initial data, but not on t.

Lemma 2.1 *For any $t > 0$, the following estimates hold.*

$$\|\varepsilon(t)\| + \|\varepsilon(t)\|_{L^6} + \|p_x(t)\| + \|\varepsilon_x(t)\| + \|\theta(t)\|_{L^1} \le C, \tag{2.1}$$

$$\|p(t)\|_{L^\infty} + \|\varepsilon(t)\|_{L^\infty} \le C, \tag{2.2}$$

$$\theta(x,t) > 0, \quad \forall\,(x,t) \in [0,1] \times I\!R^+. \tag{2.3}$$

PROOF. First, applying the maximum principle to (1.21), we find that

$$\theta(x,t) > 0, \quad \forall\,(x,t) \in [0,1] \times I\!R^+. \tag{2.4}$$

Next, multiplying (1.20) by $-p_{xx}$, adding the result to (1.21), and integrating with repect to x over Ω, we arrive at

$$\frac{d}{dt}\int_0^1 (\theta + F_2(\varepsilon) + \frac{1}{2}p_x^2 + \frac{\delta}{2}\varepsilon_x^2)(t)\, dx = 0. \tag{2.5}$$

Thus,

$$\int_0^1 (\theta + F_2(\varepsilon) + \frac{1}{2}p_x^2 + \frac{\delta}{2}\varepsilon_x^2)(t)\, dx = E_1\,, \tag{2.6}$$

where E_1 is a constant depending only on the initial data.
Using Young's inequality, we see that

$$F_2(\varepsilon) \geq C_1\varepsilon^6 - C_2\,, \tag{2.7}$$

whence

$$\|\varepsilon(t)\| + \|p_x(t)\| + \|\varepsilon_x(t)\| + \|\varepsilon(t)\|_{L^6} + \|\theta(t)\|_{L^1} \leq C. \tag{2.8}$$

By virtue of the boundary conditions and of Poincare's inequality, we find

$$\|p(t)\|_{L^\infty} + \|\varepsilon(t)\|_{L^\infty} \leq C\,, \tag{2.9}$$

whence the assertion follows. □

Lemma 2.2 *For any $t > 0$, the following estimates hold.*

$$\int_0^t \int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \frac{p_{xx}^2}{\theta}\right) dx\, d\tau \leq C, \tag{2.10}$$

$$\int_0^t \|p_x(\tau)\|^2 d\tau \leq \int_0^t \|p_x(\tau)\|_{L^\infty}^2 d\tau \leq C, \quad \int_0^t \|p(\tau)\|_{L^\infty}^2 d\tau \leq C\,, \tag{2.11}$$

$$\int_0^t \|p_x(\tau)\|^{n+2} d\tau \leq C, \quad \forall \quad n \geq 0. \tag{2.12}$$

PROOF. Multiplication of (1.21) by θ^{-1} and integration with respect to x over Ω yield

$$\frac{d}{dt}\int_0^1 (\log\theta - F_1(\varepsilon))(t)\, dx \;-\; \int_0^1 \left(\frac{k\theta_x^2}{\theta^2} + \frac{\gamma p_{xx}^2}{\theta}\right)(t)\, dx = 0. \tag{2.13}$$

Since $\log\theta \leq \theta - 1$ for all $\theta > 0$, we obtain

$$\int_0^t \int_0^1 \left(\frac{k\theta_x^2}{\theta^2} + \frac{\gamma p_{xx}^2}{\theta}\right) dx\, d\tau \leq C. \tag{2.14}$$

From $p_x|_{x=0} = 0$ it follows that

$$p_x(x,t) = p_x(0,t) + \int_0^x p_{xx}(y,t)\, dy = \int_0^x p_{xx}(y,t)\, dy. \tag{2.15}$$

Hence,

$$\begin{aligned}
\int_0^t \|p_x(\tau)\|_{L^\infty}^2 d\tau &\leq \int_0^t \Big(\int_0^1 |p_{xx}(x,\tau)|\, dx\Big)^2 d\tau \\
&\leq \int_0^t \left(\int_0^1 \sqrt{\theta}\,\frac{|p_{xx}|}{\sqrt{\theta}}\, dx\right)^2 d\tau \leq \int_0^t \Big(\int_0^1 \theta\, dx\Big)\Big(\int_0^1 \frac{p_{xx}^2}{\theta}\, dx\Big)\, d\tau \\
&\leq C\int_0^t \int_0^1 \frac{p_{xx}^2}{\theta}\, dx\, d\tau \leq C.
\end{aligned} \tag{2.16}$$

Thus,

$$\int_0^t \|p_x(\tau)\|^2 d\tau \leq \int_0^t \|p_x(\tau)\|_{L^\infty}^2 d\tau \leq C. \tag{2.17}$$

Combining (2.11) with (2.8), a simple induction yields that to any $n \in I\!N$ there is some $C = C(n)$ such that

$$\int_0^t \|p_x(\tau)\|^{n+2} d\tau \leq C\,. \tag{2.18}$$

The proof of the assertion is complete. □

In the sequel we will see that (2.18) is very useful for reducing the degree of nonlinearity. To get further estimates, we will now derive estimates for the derivatives of the norms of ε, p by expressions of the form (1.30).

Lemma 2.3 *For any $t > 0$, the following estimates hold.*

$$\int_0^t (\|\varepsilon_t(\tau)\|^2 + \|p_{xx}(\tau)\|^2)\, d\tau \le C \sup_{0\le\tau\le t} \|\theta(\tau)\|_{L^\infty}, \tag{2.19}$$

$$\int_0^t \|\theta_x(\tau)\|^2 d\tau \le C \sup_{0\le\tau\le t} \|\theta(\tau)\|^2_{L^\infty}. \tag{2.20}$$

PROOF. Using Lemma 2.2, we obtain

$$\begin{aligned}\int_0^t \|p_{xx}(\tau)\|^2 d\tau &= \int_0^t \left\| \sqrt{\theta}\frac{p_{xx}}{\sqrt{\theta}}(\tau)\right\|^2 d\tau \\ &\le \sup_{0\le\tau\le t} \|\theta(\tau)\|_{L^\infty} \int_0^t \left\|\frac{p_{xx}}{\sqrt{\theta}}(\tau)\right\|^2 d\tau \\ &\le C \sup_{0\le\tau\le t} \|\theta(\tau)\|_{L^\infty}. \end{aligned}\tag{2.21}$$

Similarly, we have

$$\int_0^t \|\theta_x(\tau)\|^2 d\tau \le C \sup_{0\le\tau\le t} \|\theta(\tau)\|^2_{L^\infty}. \tag{2.22}$$

The proof is complete. □

We can now show further estimates.

Lemma 2.4 *For any $t > 0$ the following estimates hold.*

$$\begin{aligned}&\|p_{xt}(t)\|^2 + \|p_{xxx}(t)\|^2 + \int_0^t (\|p_{xxt}(\tau)\|^2 + \|\varepsilon_{tt}(\tau)\|^2)\, d\tau \\ &\le C\left(1 + \sup_{0\le\tau\le t}\|\theta(\tau)\|^3_{L^\infty} + \int_0^t \|\theta_t(\tau)\|^2\, d\tau\right),\end{aligned}\tag{2.23}$$

$$\begin{aligned}&\|\varepsilon_{xt}(t)\|^2 + \int_0^t (\|p_{xxxx}(\tau)\|^2 + \|\varepsilon_{xxt}(\tau)\|^2)\, d\tau \\ &\le C\left(1 + \sup_{0\le\tau\le t}\|\theta(\tau)\|^3_{L^\infty} + \int_0^t \|\theta_t(\tau)\|^2\, d\tau\right).\end{aligned}\tag{2.24}$$

PROOF. First, differentiating (1.20) with respect to t, multiplying the result by $-\varepsilon_{tt}$, and integrating with repect to x over Ω, we obtain

$$\begin{aligned} 0 &= (p_{tt}(t), -p_{xxt}(t)) + \gamma\|\varepsilon_{tt}(t)\|^2 + (\delta\varepsilon_{xt}(t), \varepsilon_{xtt}(t)) + \int_0^1 \sigma_{1t}(t)\,\varepsilon_{tt}(t)\, dx \\ &= (p_{xtt}(t), p_{xt}(t)) + \gamma\|\varepsilon_{tt}(t)\|^2 + \delta(\varepsilon_{xt}(t), \varepsilon_{xtt}(t)) \\ &\quad + \int_0^1 (f_1'(\varepsilon)\,\varepsilon_t\,\theta + f_2'(\varepsilon)\,\varepsilon_t + f_1(\varepsilon)\,\theta_t)(t)\,\varepsilon_{tt}(t)\, dx. \end{aligned}\tag{2.25}$$

Combination with (2.9) yields

$$\frac{1}{2}\frac{d}{dt}(\|p_{xt}(t)\|^2 + \delta\|\varepsilon_{xt}(t)\|^2) + \gamma\|\varepsilon_{tt}(t)\|^2 \le \frac{\gamma}{2}\|\varepsilon_{tt}(t)\|^2 + C\int_0^1 (\theta^2\varepsilon_t^2 + \varepsilon_t^2 + \theta_t^2)(t)\, dx. \tag{2.26}$$

Integrating (2.26) with respect to t and applying Lemma 2.3, we arrive at

$$\begin{aligned}&\|p_{xt}(t)\|^2 + \|\varepsilon_{xt}(t)\|^2 + \int_0^t \|\varepsilon_{tt}(\tau)\|^2 d\tau \\ &\le C + C\int_0^t (\|\theta(\tau)\,\varepsilon_t(\tau)\|^2 + \|\varepsilon_t(\tau)\|^2 + \|\theta_t(\tau)\|^2)\, d\tau \\ &\le C + C\sup_{0\le\tau\le t}\|\theta(\tau)\|^2_{L^\infty}\int_0^t \|\varepsilon_t(\tau)\|^2\, d\tau + C\int_0^t (\|\varepsilon_t(\tau)\|^2 + \|\theta_t(\tau)\|^2)\, d\tau \\ &\le C\Big(1 + \sup_{0\le\tau\le t}\|\theta(\tau)\|^3_{L^\infty} + \int_0^t \|\theta_t(\tau)\|^2 d\tau\Big).\end{aligned}\tag{2.27}$$

Here, we have used Young's inequality in the form $a \le Ca^3 + C'$.
Next, we differentiate (1.20) with respect to t, then multiply by ε_{xxt}, and integrate the result with respect to x over Ω, to obtain

$$
\begin{aligned}
0 &= (p_{tt}(t), \varepsilon_{xxt}(t)) - \gamma(\varepsilon_{tt}(t), \varepsilon_{xxt}(t)) + \delta\|\varepsilon_{xxt}(t)\|^2 - \int_0^1 \varepsilon_{xxt}(t)\,\sigma_{1t}(t)\,dx \\
&= (p_{xxtt}(t), \varepsilon_t(t)) + \gamma(\varepsilon_{xtt}(t), \varepsilon_{xt}(t)) + \delta\|\varepsilon_{xxt}(t)\|^2 - \int_0^1 \varepsilon_{xxt}(t)\,\sigma_{1t}(t)\,dx \\
&= \frac{d}{dt}(p_{xxt}(t), \varepsilon_t(t)) - \|p_{xxt}(t)\|^2 + \frac{\gamma}{2}\frac{d}{dt}\|\varepsilon_{xt}(t)\|^2 + \delta\|\varepsilon_{xxt}(t)\|^2 \\
&\quad - \int_0^1 \varepsilon_{xxt}(t)\,\sigma_{1t}(t)\,dx. \qquad (2.28)
\end{aligned}
$$

However, by integration by parts, we have

$$
(p_{xxt}(t), \varepsilon_t(t)) = -(p_{xt}(t), \varepsilon_{xt}(t)). \qquad (2.29)
$$

Combining this with (2.28), and using (2.23) and Young's inequality, we find

$$
\begin{aligned}
&\frac{\gamma}{4}\|\varepsilon_{xt}(t)\|^2 + \delta\int_0^t \|\varepsilon_{xxt}(\tau)\|^2 d\tau \\
&\le C + \frac{\delta}{2}\int_0^t \|\varepsilon_{xxt}(\tau)\|^2 d\tau + C\left(\|p_{xt}(t)\|^2 + \int_0^t (\|\sigma_{1t}(\tau)\|^2 + \|p_{xxt}(\tau)\|^2)\,d\tau\right) \\
&\le C\left(1 + \sup_{0\le\tau\le t}\|\theta(\tau)\|^3_{L^\infty} + \int_0^t \|\theta_t(\tau)\|^2 d\tau\right) + \frac{\delta}{2}\int_0^t \|\varepsilon_{xxt}(\tau)\|^2 d\tau\,. \qquad (2.30)
\end{aligned}
$$

The proof of the lemma is complete. □
In the sequel, we will find that the above lemma plays a crucial role in reducing the degree of nonlinearity.

Lemma 2.5 *For any $t > 0$, the following estimates hold.*

$$
\|\theta_x(t)\|^2 + \int_0^t \|\theta_t(\tau)\|^2 d\tau \le C. \qquad (2.31)
$$

$$
\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^\infty} \le C. \qquad (2.32)
$$

PROOF. Multiplying (1.21) by θ_t and integrating with repect to x over Ω, we obtain

$$
\begin{aligned}
&\frac{k}{2}\frac{d}{dt}\|\theta_x(t)\|^2 + \|\theta_t(t)\|^2 = \int_0^1 (f_1(\varepsilon)\,\theta\,\theta_t\,p_{xx} + \gamma\,\theta_t\,p_{xx}^2)(t)\,dx \\
&\le C\left(\|\theta(t)\,p_{xx}(t)\|\,\|\theta_t(t)\| + \left(\int_0^1 p_{xx}^4(t)\,dx\right)^{\frac12}\|\theta_t(t)\|\right) \\
&\le C\left(\|\theta(t)\|^{\frac12}_{L^\infty}\|p_{xx}(t)\|_{L^\infty}\left(\int_0^1 \theta(t)\,dx\right)^{\frac12}\|\theta_t(t)\| + \|p_{xx}(t)\|^2_{L^4}\|\theta_t(t)\|\right). \qquad (2.33)
\end{aligned}
$$

Therefore, integration with respect to t yields

$$
\begin{aligned}
\|\theta_x(t)\|^2 + \int_0^t \|\theta_t(\tau)\|^2 d\tau &\le C\left(\sup_{0\le\tau\le t}\|\theta(\tau)\|^{\frac12}_{L^\infty}\left(\int_0^t \|p_{xx}(\tau)\|^2_{L^\infty}d\tau\right)^{\frac12}\left(\int_0^t \|\theta_t(\tau)\|^2 d\tau\right)^{\frac12}\right. \\
&\quad \left. + \left(\int_0^t \|p_{xx}(\tau)\|^4_{L^4}d\tau\right)^{\frac12}\left(\int_0^t \|\theta_t(\tau)\|^2 d\tau\right)^{\frac12} + 1\right) \\
&= C\,(I_1 + I_2 + 1)\,. \qquad (2.34)
\end{aligned}
$$

We now estimate the terms I_1, I_2. By virtue of Nirenberg's inequality and the boundary conditions, we obtain

$$
\|p_{xx}(t)\|_{L^\infty} \le C\,\|p_{xxxx}(t)\|^{\frac12}\,\|p_x(t)\|^{\frac12}, \qquad (2.35)
$$

$$\|p_{xx}(t)\|_{L^4} \le C\,\|p_{xxxx}(t)\|^{\frac{5}{12}}\,\|p_x(t)\|^{\frac{7}{12}}. \tag{2.36}$$

Hence,

$$\begin{aligned}
I_1 &= C\left(\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^\infty}\int_0^t\|\theta_t(\tau)\|^2d\tau\int_0^t\|p_{xx}(\tau)\|^2_{L^\infty}d\tau\right)^{\frac{1}{2}}\\
&\le C\left(\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^\infty}\int_0^t\|\theta_t(\tau)\|^2d\tau\int_0^t\|p_{xxxx}(\tau)\|\,\|p_x(\tau)\|\,d\tau\right)^{\frac{1}{2}}\\
&\le C\left(\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^\infty}\int_0^t\|\theta_t(\tau)\|^2d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|p_{xxxx}(\tau)\|^2d\tau\int_0^t\|p_x(\tau)\|^2d\tau\right)^{\frac{1}{4}}.
\end{aligned} \tag{2.37}$$

Using Lemma 2.2, Lemma 2.4, and Young's inequality, we conclude that

$$\begin{aligned}
I_1 &\le C\left(\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^\infty}\int_0^t\|\theta_t(\tau)\|^2d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|p_{xxxx}(\tau)\|^2d\tau\right)^{\frac{1}{4}}\\
&\le C\left(\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^\infty}\int_0^t\|\theta_t(\tau)\|^2d\tau\right)^{\frac{1}{2}}\left(1+\sup_{0\le\tau\le t}\|\theta(\tau)\|^3_{L^\infty}+\int_0^t\|\theta_t(\tau)\|^2d\tau\right)^{\frac{1}{4}}\\
&\le \frac{1}{4}\int_0^t\|\theta_t(\tau)\|^2d\tau + C\left(1+\sup_{0\le\tau\le t}\|\theta(\tau)\|^{\frac{5}{2}}_{L^\infty}\right).
\end{aligned} \tag{2.38}$$

Next, owing to Schwarz's inequality and (2.36), we have

$$\begin{aligned}
I_2 &= C\left(\int_0^t\|\theta_t(\tau)\|^2d\tau\int_0^t\|p_{xx}(\tau)\|^4_{L^4}d\tau\right)^{\frac{1}{2}}\\
&\le C\left(\int_0^t\|\theta_t(\tau)\|^2d\tau\int_0^t\|p_{xxxx}(\tau)\|^{\frac{5}{3}}\,\|p_x(\tau)\|^{\frac{7}{3}}d\tau\right)^{\frac{1}{2}}\\
&\le C\left(\int_0^t\|\theta_t(\tau)\|^2d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|p_{xxxx}(\tau)\|^2d\tau\right)^{\frac{5}{12}}\left(\int_0^t\|p_x(\tau)\|^{14}d\tau\right)^{\frac{1}{12}}.
\end{aligned} \tag{2.39}$$

Applying (2.18) with $n=12$ and Lemma 2.4, we get

$$\begin{aligned}
I_2 &\le C\left(\int_0^t\|\theta_t(\tau)\|^2d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|p_{xxxx}(\tau)\|^2d\tau\right)^{\frac{5}{12}}\\
&\le C\left(\int_0^t\|\theta_t(\tau)\|^2d\tau\right)^{\frac{1}{2}}\left(1+\sup_{0\le\tau\le t}\|\theta(\tau)\|^3_{L^\infty}+\int_0^t\|\theta_t(\tau)\|^2d\tau\right)^{\frac{5}{12}}\\
&\le \frac{1}{4}\int_0^t\|\theta_t(\tau)\|^2d\tau + C\sup_{0\le\tau\le t}\|\theta(\tau)\|^{\frac{5}{2}}_{L^\infty}+C.
\end{aligned} \tag{2.40}$$

Owing to Nirenberg's inequality and (2.1), we have

$$\|\theta(t)\|_{L^\infty} \le C\,\|\theta_x(t)\|^{\frac{2}{3}}\,\|\theta(t)\|^{\frac{1}{3}}_{L^1}+C\,\|\theta(t)\|_{L^1} \le C\,\|\theta_x(t)\|^{\frac{2}{3}}+C. \tag{2.41}$$

Combining (2.38)–(2.40) with (2.34) and (2.41), and applying Young's inequality, we find

$$\begin{aligned}
\|\theta_x(t)\|^2+\int_0^t\|\theta_t(\tau)\|^2d\tau &\le \frac{1}{2}\int_0^t\|\theta_t(\tau)\|^2d\tau + C\Big(1+\sup_{0\le\tau\le t}\|\theta_x(\tau)\|^{\frac{5}{3}}\Big)\\
&\le C+\frac{1}{2}\left(\int_0^t\|\theta_t(\tau)\|^2d\tau+\sup_{0\le\tau\le t}\|\theta_x(\tau)\|^2\right).
\end{aligned} \tag{2.42}$$

Taking the supremum with respect to t in (2.42) yields

$$\sup_{0\le\tau\le t}\|\theta_x(\tau)\|^2+\int_0^t\|\theta_t(\tau)\|^2d\tau \le \frac{1}{2}\Big(\int_0^t\|\theta_t(\tau)\|^2d\tau+\sup_{0\le\tau\le t}\|\theta_x(\tau)\|^2\Big)+C. \tag{2.43}$$

Hence,

$$\sup_{0\le\tau\le t}\|\theta_x(\tau)\|^2 + \int_0^t \|\theta_t(\tau)\|^2 d\tau \le C. \tag{2.44}$$

Thus, using (2.41),

$$\sup_{0\le\tau\le t}\|\theta(\tau)\|^2_{L^\infty} \le C\,, \tag{2.45}$$

which concludes the proof of the assertion. □

Combining the Lemmas 2.3 to 2.5, and using the system equations, we immediately conclude that

Lemma 2.6 *For any $t > 0$, the following estimates hold.*

$$\int_0^t (\|p_{xx}(\tau)\|^2 + \|\varepsilon_t(\tau)\|^2 + \|\theta_x(\tau)\|^2_{H^1})\, d\tau \le C, \tag{2.46}$$

$$\int_0^t (\|p_{xxt}(\tau)\|^2 + \|\varepsilon_{tt}(\tau)\|^2 + \|p_{xxxx}(\tau)\|^2 + \|\varepsilon_{xxt}(\tau)\|^2)\, d\tau \le C, \tag{2.47}$$

$$\|p_{xt}(t)\|^2 + \|p_{xxx}(t)\|^2 + \|\varepsilon_{xt}(t)\|^2 + \|\varepsilon_{xxx}(t)\|^2 \le C. \tag{2.48}$$

Lemma 2.7 *For any $t > 0$, the following estimates hold.*

$$\int_0^t (\|p_t(\tau)\|^2 + \|p_t(\tau)\|^2_{L^\infty} + \|p_{xt}(\tau)\|^2 + \|p_{xt}(\tau)\|^2_{L^\infty})\, d\tau \le C, \tag{2.49}$$

$$\int_0^t (\|\delta\varepsilon_{xx}(\tau) - \sigma_1(\tau)\|^2 d\tau + \|(\delta\varepsilon_{xx} - \sigma_1)_t(\tau)\|^2)\, d\tau \le C, \tag{2.50}$$

$$\int_0^t (\|p_{xx}(\tau)\|^2_{L^\infty} + \|p_{xxx}(\tau)\|^2 + \|p_{xxx}(\tau)\|^2_{L^\infty} + \|p_{tt}(\tau)\|^2)\, d\tau \le C, \tag{2.51}$$

$$\|p_t(t)\|^2 + \|p_{xx}(t)\|^2 + \|p_t(t)\|^2_{L^\infty} + \|p_x(t)\|^2_{L^\infty} + \|p_{xx}(t)\|^2_{L^\infty} \le C. \tag{2.52}$$

PROOF. These estimates can easily be derived from the system equations and from the Lemmas 2.5 and 2.6. □

Now we proceed to investigate the compactness of the orbit of the solution for $t > 0$ in $H^3 \times H^3 \times H^1$. For the time being, we assume that the initial data are so smooth that the solution will have enough smoothness to carry out the following argument; if the initial data belonged just to $H^3 \times H^3 \times H^1$, we could approximate them by smooth functions and then pass to the limit.

Differentiating (1.20) twice with respect to t, we find that

$$p_{ttt} - \gamma p_{xxtt} + \delta\varepsilon_{xxtt} - \sigma_{1tt} = 0\,. \tag{2.53}$$

A straightforward calculation yields

$$\sigma_{1tt} = f_1'(\varepsilon)\,\varepsilon_{tt}\,\theta + 2\,f_1'(\varepsilon)\,\varepsilon_t\,\theta_t + f_1(\varepsilon)\,\theta_{tt} + f_2''(\varepsilon)\,\varepsilon_t^2 + f_2'(\varepsilon)\,\varepsilon_{tt}. \tag{2.54}$$

Multiplying (2.53) by p_{tt} and integrating with respect to x over Ω, we find

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\|p_{tt}(t)\|^2 - \gamma(p_{xxtt}(t), p_{tt}(t)) + \delta(\varepsilon_{xxtt}(t), p_{tt}(t)) - (\sigma_{1tt}(t), p_{tt}(t)) \\ &= \frac{1}{2}\frac{d}{dt}\|p_{tt}(t)\|^2 + \gamma\|p_{xtt}(t)\|^2 + \delta(\varepsilon_{tt}(t), p_{xxtt}(t)) - (\sigma_{1tt}(t), p_{tt}(t)) \\ &= \frac{1}{2}\frac{d}{dt}\left(\|p_{tt}(t)\|^2 + \delta\|\varepsilon_{tt}(t)\|^2\right) + \gamma\|p_{xtt}(t)\|^2 - (\sigma_{1tt}(t), p_{tt}(t))\,. \end{aligned} \tag{2.55}$$

Multiplying (2.55) by t^2 and using (2.32), as well as the Lemmas 2.6 and 2.7, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(t^2\|p_{tt}(t)\|^2+t^2\delta\|\varepsilon_{tt}\|^2\right)-t\left(\|p_{tt}(t)\|^2+\delta\|\varepsilon_{tt}(t)\|^2\right)+\gamma t^2\|p_{xtt}(t)\|^2\\
\le\ & t^2\|p_{tt}(t)\|^2+Ct^2\|\sigma_{1tt}(t)\|^2\\
\le\ & t^2\|p_{tt}(t)\|^2+Ct^2(\|\varepsilon_{tt}(t)\|^2+\|\theta_t(t)\|^2+\|\theta_{tt}(t)\|^2+\|\varepsilon_t(t)\|^2)\,. \qquad (2.56)
\end{aligned}$$

Hence, it follows from (2.31), (2.46), and (2.47), that

$$t^2(\|p_{tt}(t)\|^2+\delta\|\varepsilon_{tt}(t)\|^2)+\int_0^t\tau^2\|p_{xtt}(\tau)\|^2d\tau\le C_1+Ct^2+C\int_0^t\tau^2\|\theta_{tt}(\tau)\|^2d\tau\,, \qquad (2.57)$$

where $C_1=C(\|\varepsilon_0\|_{H^3},\|p_0\|_{H^3},\|\theta_0\|_{H^1})$.
On the other hand, differentiating (1.21) with respect to t, we get

$$\theta_{tt}-k\theta_{xxt}-(f_1(\varepsilon)\theta p_{xx}+\gamma p_{xx}^2)_t=0. \qquad (2.58)$$

Multiplying by θ_{tt} and integrating with respect to x, we arrive at

$$\begin{aligned}
\frac{k}{2}\frac{d}{dt}\|\theta_{xt}(t)\|^2+\|\theta_{tt}(t)\|^2 &\le \frac{1}{2}\|\theta_{tt}(t)\|^2+\frac{1}{2}\|(f_1(\varepsilon)\theta p_{xx}+\gamma p_{xx}^2)_t(t)\|^2\\
&\le \frac{1}{2}\|\theta_{tt}(t)\|^2+C\left(\|p_{xx}(t)\|^2+\|\theta_t(t)\|^2+\|p_{xxt}(t)\|^2\right). \qquad (2.59)
\end{aligned}$$

Multiplication of (2.59) by t^2 yields

$$\frac{k}{2}\frac{d}{dt}(t^2\|\theta_{xt}(t)\|^2)-kt\|\theta_{xt}(t)\|^2+\frac{t^2}{2}\|\theta_{tt}(t)\|^2\le Ct^2(\|p_{xx}(t)\|^2+\|\theta_t(t)\|^2+\|p_{xxt}(t)\|^2). \qquad (2.60)$$

In order to estimate $\int_0^t\tau\|\theta_{xt}(\tau)\|^2d\tau$, we multiply (2.58) by θ_t and then integrate with respect to x over Ω, to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta_t(t)\|^2+k\|\theta_{xt}(t)\|^2 &\le \frac{1}{2}\|\theta_t(t)\|^2+\frac{1}{2}\|(f_1(\varepsilon)\theta p_{xx}+\gamma p_{xx}^2)_t(t)\|^2\\
&\le \frac{1}{2}\|\theta_t(t)\|^2+C\left(\|\varepsilon_t(t)\|^2+\|\theta_t(t)\|^2+\|p_{xxt}(t)\|^2\right). \qquad (2.61)
\end{aligned}$$

Multiplying (2.61) by t, we find

$$\frac{1}{2}\frac{d}{dt}(t\|\theta_t(t)\|^2)+kt\|\theta_{tx}(t)\|^2\le C\left(\|\theta_t(t)\|^2+t\|\theta_t(t)\|^2+t\left(\|\varepsilon_t(t)\|^2+\|\theta_t(t)\|^2+\|p_{xxt}(t)\|^2\right)\right). \qquad (2.62)$$

Therefore,

$$t\|\theta_t(t)\|^2+\int_0^t\tau\|\theta_{xt}(\tau)\|^2d\tau\le Ct+C_2\,, \qquad (2.63)$$

where $C_2=C(\|\varepsilon_0\|_{H^3},\|p_0\|_{H^3},\|\theta_0\|_{H^1})$.
Combination of (2.63) with (2.60) yields

$$\int_0^t\tau^2\|\theta_{tt}(\tau)\|^2d\tau\le C_3+Ct^2\,, \qquad (2.64)$$

with $C_3=C(\|\varepsilon_0\|_{H^3},\|p_0\|_{H^3},\|\theta_0\|_{H^1})$.
Thus, it follows from (2.57) that

$$\|p_{tt}(t)\|^2+\|\varepsilon_{tt}(t)\|^2\le C_4t^{-2}+C. \qquad (2.65)$$

Also, using (2.63) and (2.60),

$$\|\theta_t(t)\|^2\le C+C_4t^{-1},\quad \|\theta_{xt}(t)\|^2\le C_4t^{-2}+C\,, \qquad (2.66)$$

with C_4 depending only on $\|\varepsilon_0\|_{H^3}, \|p_0\|_{H^3}, \|\theta_0\|_{H^1}$.
Thus, it easily follows from the equations (1.19) to (1.21) that for any initial data in $H^3 \times H^3 \times H^1$ it holds

$$(\varepsilon(\cdot,t), p(\cdot,t), \theta(\cdot,t)) \in H^4 \times H^4 \times H^3\,, \quad \forall\, t > 0\,. \tag{2.67}$$

Moreover, we can infer from the Lemmas 2.5 to 2.7, and from (2.55), (2.59) and (2.61), that for any $\nu > 0$ the triple (ε, p, θ) is bounded in $C([\nu, +\infty); H^4 \times H^4 \times H^3)$. From this the compactness of the orbit in $H^3 \times H^3 \times H^1$ follows. □

3 Asymptotic Behavior

In this section, we will prove the results on the asymptotic behavior of the solution given in Theorem 1.1. In the sequel, a convergence symbol " $\to$ " is always to be understood as $t \to \infty$. We will make use of the following basic lemma from Shen & Zheng [19]:

Lemma 3.1 *Suppose that y and h are nonnegative functions on $(0,\infty)$ such that y' is locally integrable and such that y, h satisfy*

$$\forall t \geq 0: \qquad y'(t) \leq A_1 y^2(t) + A_2 + h(t), \tag{3.1}$$

$$\forall T > 0: \qquad \int_0^T y(\tau)d\tau \leq A_3, \quad \int_0^T h(\tau)d\tau \leq A_4, \tag{3.2}$$

where A_1, A_2, A_3, A_4 denote positive constants which are independent of t and T. Then, for any $r > 0$,

$$\forall t \geq 0: \quad y(t+r) \leq \left(\frac{A_3}{r} + A_2 r + A_4\right) e^{A_1 A_2}. \tag{3.3}$$

Moreover,

$$\lim_{t\to\infty} y(t) = 0. \tag{3.4}$$

Lemma 3.2 *It holds*

$$\|p(t)\|_{H^3} \to 0, \quad \|p_t(t)\|_{H^1} \to 0, \tag{3.5}$$

$$\|\varepsilon_t(t)\|_{H^1} \to 0, \quad \|(\delta\varepsilon_{xx} - \sigma_1)(t)\|_{H^1} \to 0, \tag{3.6}$$

$$\|u_t(t)\|_{H^2} \to 0. \tag{3.7}$$

Proof. It follows from (2.26) and (2.32) that

$$\begin{aligned} & \frac{d}{dt}(\|p_{xt}(t)\|^2 + \delta\|\varepsilon_{xt}(t)\|^2) + \gamma\|\varepsilon_{tt}(t)\|^2 \\ \leq\ & C\,(\|\theta(t)\,\varepsilon_t(t)\|^2 + \|\varepsilon_t(t)\|^2 + \|\theta_t(t)\|^2) \\ \leq\ & C\,(\|\varepsilon_t(t)\|^2 + \|\theta_t(t)\|^2). \end{aligned} \tag{3.8}$$

Combining (3.8) with (2.51), (2.46), (2.49), (2.31), and applying Lemma 3.1, we arrive at

$$\|p_{xt}(t)\|^2 + \|\varepsilon_{xt}(t)\|^2 \to 0. \tag{3.9}$$

Hence, $\|p_{xxx}(t)\|^2 \to 0$, and thus $\|u_t\|_{H^2} \to 0$.
Next, we differentiate (1.20) with respect to t, then multiply by $\delta\varepsilon_{xx} - \sigma_1$ and integrate with respect to x over Ω. It follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta\varepsilon_{xx}(t) - \sigma_1(t)\|^2 &= -(p_{tt}(t) - \gamma\varepsilon_{tt}(t), \delta\varepsilon_{xx}(t) - \sigma_1(t)) \\ &\leq \frac{1}{2}\|\delta\varepsilon_{xx}(t) - \sigma_1(t)\|^2 + C(\|p_{tt}(t)\|^2 + \|\varepsilon_{tt}(t)\|^2). \end{aligned} \tag{3.10}$$

Combining (3.10) with (2.50), (2.51), (2.47), and applying Lemma 3.1, we conclude that

$$\|\delta\varepsilon_{xx}(t)-\sigma_1(t)\|^2\to 0. \tag{3.11}$$

From (1.20) and (3.9), we also get

$$\|(\delta\varepsilon_{xx}-\sigma_1)_x(t)\|^2\to 0. \tag{3.12}$$

The assertions of Lemma 3.2 now follow from the above estimates and from Poincare's inequality. □

Lemma 3.3 *It holds*

$$\|\theta_x(t)\|\to 0\,. \tag{3.13}$$

Proof. We multiply (1.21) by θ_t and integrate with respect to x over Ω to get

$$\begin{aligned}\frac{k}{2}\frac{d}{dt}\|\theta_x(t)\|^2+\|\theta_t(t)\|^2 &= \int_0^1\Big(\gamma\, p_{xx}^2\,\theta_t\;+\;f_1(\varepsilon)\,\theta\,\theta_t\,p_{xx}\Big)(t)\,dx\\ &\le \frac{1}{2}\|\theta_t(t)\|^2+\|\theta(t)\,p_{xx}(t)\|^2+\|p_{xx}^2(t)\|^2. \end{aligned}\tag{3.14}$$

Combining (3.14) with (2.32) and (2.52), we see that

$$k\frac{d}{dt}\|\theta_x(t)\|^2+\|\theta_t(t)\|^2\le C\|p_{xx}(t)\|^2. \tag{3.15}$$

Hence, we can infer from (2.46) and Lemma 3.1 that

$$\|\theta_x(t)\|^2\to 0\,,$$

which concludes the proof. □

Concerning the convergence of ε,u,θ, we have the following result.

Lemma 3.4 *It holds*

$$(\varepsilon(\cdot,t),p(\cdot,t),\theta(\cdot,t))\to(\overline{\varepsilon},0,\overline{\theta}),\quad in\;\; H^3\times H^3\times H^1\,, \tag{3.16}$$

$$u(\cdot,t)\to\overline{u}\,,\quad in\quad H^4\,,\quad with\;\overline{u}(x)=\int_0^x\overline{\varepsilon}(y)\,dy,\quad\forall\,x\in[0,1]\,, \tag{3.17}$$

where $(\overline{\varepsilon},\overline{\theta})$ is one of the equilibria for the corresponding stationary problem,

$$\delta\varepsilon_{xx}-f_1(\varepsilon)\theta-f_2(\varepsilon)=0, \tag{3.18}$$

$$\varepsilon_x|_{x=0}=0,\quad \varepsilon|_{x=1}=0, \tag{3.19}$$

$$\theta=Const., \tag{3.20}$$

$$\int_0^1\Big(\theta+F_2(\varepsilon)+\frac{\delta}{2}\varepsilon_x^2\Big)\,dx=E_1. \tag{3.21}$$

Proof. It is easy to see from (2.4) and (2.12) that, for any $0<\nu<1$,

$$\frac{d}{dt}\int_0^1\Big(\theta-\nu\,\log\theta+F_2(\varepsilon)+\nu F_1(\varepsilon)+\frac{1}{2}p_x^2+\frac{\delta}{2}\varepsilon_x^2\Big)(t)\,dx+\nu\int_0^1\left(\frac{k\theta_x^2}{\theta^2}+\frac{\gamma p_{xx}^2}{\theta}\right)(t)\,dx=0. \tag{3.22}$$

Thus the system (1.19)–(1.21) has a Lyapunov function of the form

$$\int_0^1\Big(\theta-\nu\,\log\theta+F_2(\varepsilon)+\nu F_1(\varepsilon)+\frac{1}{2}p_x^2+\frac{\delta}{2}\varepsilon_x^2\Big)(t)\,dx\,.$$

Since the orbit is compact, as proved in previous section, it follows from the standard theory of dynamical systems that the ω-limit set is connected, compact and consists of equilibria. Since the corresponding stationary problem admits only a finite number of solutions (see Zhou [22], and also Luckhaus & Zheng [12], Novick-Cohen & Zheng [16], Zheng [21]), (3.16) follows. In view of the boundary condition $u|_{x=0}=0$, we also get (3.17). Therefore, the proof is complete. □

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