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On a long range segregation model

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Abstract. In this work we study the properties of segregation processes modeled by a family of equations

$$L(u_i)(x) = u_i(x)\, F_i(u_1, \ldots, u_K)(x) \qquad i = 1, \ldots, K$$

where $F_i(u_1, \ldots, u_K)(x)$ is a non-local factor that takes into consideration the values of the functions u_j's in a full neighborhood of x. We consider as a model problem

$$\Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{i \neq j} H(u_j^\varepsilon)(x)$$

where ε is a small parameter and $H(u_j^\varepsilon)(x)$ is for instance

$$H(u_j^\varepsilon)(x) = \int_{\mathcal{B}_1(x)} u_j^\varepsilon(y)\, \mathrm{d}y$$

or

$$H(u_j^\varepsilon)(x) = \sup_{y \in \mathcal{B}_1(x)} u_j^\varepsilon(y).$$

Here the set $\mathcal{B}_1(x)$ is the unit ball centered at x with respect to a smooth, uniformly convex norm ρ of $\mathbb{R}^n$. Heuristically, this will force the populations to stay at ρ-distance 1, one from each other, as $\varepsilon \to 0$.

1. Introduction

Segregation phenomena occur in many areas of mathematics and science: from equipartition problems in geometry, to social and biological processes (cells, bacteria, ants, mammals), to finance (sellers and buyers). There is a large body of literature and in connection to our work, we would like to refer to [1, 2, 4, 5, 6, 8, 7, 9, 10, 11, 12, 13, 14, 15, 18, 20] and the references therein. They study a family of models arising from different applications whose main two ingredients are: in the absence of competition species follow a "propagation" equation involving diffusion, transport, birth-death, etc, but when two species overlap, their growth is mutually inhibited by competition, consumption of resources, etc. The simplest form of such models consists, for species σ_i with spatial density u_i, on a system of equations

$$L(u_i) = u_i\, F_i(u_1, \ldots, u_K).$$

The operator L quantifies diffusion, transport, etc, while the term $u_i\, F_i$ does attrition of u_i from competition with the remaining species.

In these models, the interaction is punctual, i.e. $u_i(x)$ interacts with the remaining densities also at position x. There are many processes, though where the growth of σ_i at x is inhibited by the populations σ_j in a full area surrounding x.

The purpose of this work is a first attempt to study the properties of such a segregation process. Basically, we consider a family of equations,

$$L(u_i)(x) = u_i(x)\, F_i(u_1, \dots, u_K)(x)$$

where $F_i(u_1, \dots, u_K)(x)$ is now a non-local factor that takes into consideration the values of u_j in a full neighborhood of x. Given the previous discussion a possible model problem would be the system

$$\Delta u_i^\varepsilon(x) = \frac{1}{\varepsilon^2} u_i^\varepsilon(x) \sum_{i \neq j} H(u_j^\varepsilon)(x), \quad i = 1, \dots, K$$

where ε is a small parameter and $H(u_j^\varepsilon)(x)$ is a non-local operator, for instance

$$H(u_j^\varepsilon)(x) = \int_{B_1(x)} u_j^\varepsilon(y)\, \mathrm{d}y$$

or

$$H(u_j^\varepsilon)(x) = \sup_{y \in B_1(x)} u_j^\varepsilon(y)\ .$$

Heuristically, this will force the populations to stay at distance 1, one from each other as ε tends to 0.

We will consider instead of the unit ball in the Euclidean norm $B_1(x)$, the translation at x of a general smooth, uniformly convex, bounded, symmetric with respect to the origin set, $\mathcal{B}$. The set $\mathcal{B}$ defines a smooth, uniformly convex norm ρ in $\mathbb{R}^n$.

Let us note that there is some similarity also with the Lasry-Lions model of price formation (see [3, 17]) where selling and buying prices are separated by a gap due to transaction cost.

2. Notation and statement of the problem

Let $\mathcal{B}$ be an open bounded domain of $\mathbb{R}^n$, convex, symmetric with respect to the origin and with smooth boundary. Then $\mathcal{B}$ can be represented as the unit ball of a norm $\rho : \mathbb{R}^n \to \mathbb{R}$,

$\rho \in C^\infty(\mathbb{R}^n \setminus \{0\})$, called the defining function of $\mathcal{B}$, i.e.,

$$\mathcal{B} = \{x \in \mathbb{R}^n \mid \rho(x) < 1\} .$$

We assume that $\mathcal{B}$ is uniformly convex, i.e., there exists $0 < a \leq A$ such that in $\mathbb{R}^n \setminus \{0\}$

$$aI_n \leq D^2\left(\frac{1}{2}\rho^2\right) \leq AI_n \ , \tag{2.1}$$

where I_n is the $n \times n$ identity matrix. In what follows we denote

$$\mathcal{B}_r := \{y \in \mathbb{R}^n \mid \rho(y) < r\} ,$$

$$\mathcal{B}_r(x) := \{y \in \mathbb{R}^n \mid \rho(x-y) < r\} .$$

So through the paper we will always refer to the Euclidean ball as B and to the ρ-ball as $\mathcal{B}$. For a given closed set K, let

$$d_\rho(\cdot, K) = \inf_{y \in K} \rho(\cdot - y)$$

be the distance function from K associated to ρ. Then there exist $c_1, c_2 > 0$ such that

$$c_1 d(\cdot, K) \leq d_\rho(\cdot, K) \leq c_2 d(\cdot, K) \ , \tag{2.2}$$

where $d(\cdot, K)$ is the distance function associated to the Euclidian norm $|\cdot|$ of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We will denote by $(\partial\Omega)_1$ the ρ-strip of size 1 around $\partial\Omega$ in the complement of Ω defined by

$$(\partial\Omega)_1 := \{x \in \Omega^c : d_\rho(x, \partial\Omega) \leq 1\} \ .$$

For $i = 1, \ldots, K$, let f_i be non-negative Hölder continuous functions defined on $(\partial\Omega)_1$ with supports at ρ-distance greater or equal than 1, one from each other:

$$d_\rho(\operatorname{supp} f_i, \operatorname{supp} f_j) \geq 1 \ , \quad \text{for} \ \ i \neq j \ . \tag{2.3}$$

We will consider the following system of equations: for $i = 1, \ldots, K$

$$\begin{cases} \Delta u_i^\varepsilon(x) = \dfrac{1}{\varepsilon^2} u_i^\varepsilon(x) \displaystyle\sum_{j \neq i} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\ u_i^\varepsilon = f_i & \text{on } (\partial\Omega)_1. \end{cases}$$

The functional $H(u_j)(x)$ depends only on the restriction of u_j to $\mathcal{B}_1(x)$.

We will consider, for simplicity,

$$H(w)(x) = \int_{\mathcal{B}_1(x)} w^p(y)\varphi\big(\rho(x-y)\big)\mathrm{d}y, \qquad 1 \le p < \infty \tag{2.4}$$

or

$$H(w)\big(x\big) = \sup_{\mathcal{B}_1(x)} w \tag{2.5}$$

with φ a strictly positive smooth function of ρ, with at most polynomial decay at $\partial\mathcal{B}_1$:

$$\varphi(\rho) \ge C(1-\rho)^q, \quad q \ge 0. \tag{2.6}$$

In rest of the paper, when we refer to consider $u_1^\varepsilon, \dots, u_K^\varepsilon$*, viscosity solutions of the problem* (2.7)*, we mean that* $u_1^\varepsilon, \dots, u_K^\varepsilon$ *are continuous functions that satisfy in the viscosity sense the system of equations*

$$\begin{cases} \Delta u_i^\varepsilon(x) = \dfrac{1}{\varepsilon^2} u_i^\varepsilon(x) \displaystyle\sum_{j \ne i} H(u_j^\varepsilon)(x) & \text{in } \Omega, \\ u_i^\varepsilon = f_i & \text{on } (\partial\Omega)_1, \end{cases} \tag{2.7}$$

under the hypothesis that $\varepsilon > 0$*,* Ω *is a bounded Lipschitz domain of* $\mathbb{R}^n$*,* f_i *are non-negative Hölder continuous functions defined on* $(\partial\Omega)_1$ *satisfying (2.3),* H *is either of the form (2.4) or (2.5) and (2.6) holds.*

3. Main results

For the reader's convenience we present our main results below:

Existence (Theorem 4.1):

There exist continuous functions $u_1^\varepsilon, \dots, u_K^\varepsilon$*, depending on the parameter* ε*, viscosity solutions of the problem* (2.7)*.*

Limit problem (Corollary 5.6):

There exists a subsequence $(\vec{u})^{\varepsilon_m}$ *converging locally uniformly, as* $\varepsilon \to 0$*, to a function* $\vec{u} = (u_1, \dots, u_K)$*, satisfying the following properties:*

i) *the u_i's are Lipschitz continuous in Ω and have supports at distance at least 1, one from each other, i.e.*

$$u_i \equiv 0 \quad \textit{in the set} \quad \{x \in \Omega \,|\, d_\rho(x, \textit{supp } u_j) \le 1\} \quad \textit{for any } j \neq i.$$

ii) $\Delta u_i = 0$ *when* $u_i > 0$.

Semiconvexity of the free boundary (Corollary 6.2):

If $x_0 \in \partial\{u_i > 0\}$ there is an exterior tangent ρ-ball of radius 1 at x_0.

Hausdorff measure of the free boundary (Corollary 6.3):

The set $\partial\{u_i > 0\}$ has finite $(n-1)$-dimensional Hausdorff measure.

Sharp characterization of the interfaces (Theorem 7.1):

The supports of the limit functions are at distance exactly 1, one from each other, i.e, if $x_0 \in \partial\{u_i > 0\} \cap \Omega$, then there exists $j \neq i$ such that

$$\overline{\mathcal{B}_1(x_0)} \cap \partial\{u_j > 0\} \neq \emptyset \ .$$

Classification of singular points in dimension 2 (Theorem 8.2, Corollary 8.3, Corollary 8.16):

For $i \neq j$, let $x_0 \in \partial\{u_i > 0\}$ and $y_0 \in \partial\{u_j > 0\}$ be points such that $\{u_i > 0\}$ has an angle θ_i at x_0, $\{u_j > 0\}$ has an angle θ_j at y_0 and $\rho(x_0 - y_0) = 1$. Then we have

$$\theta_i = \theta_j.$$

In the case of 2 populations, singular points, i.e. points where the free boundaries have corners, are finite. Moreover under additional monotonicity assumptions on the boundary data, the sets $\partial\{u_i > 0\}$, $i = 1, 2$, are of class C^1.

Free boundary condition (Theorem 9.1):

In any dimension, if we have 2 populations, H is defined as in (2.4) *with $\varphi \equiv 1$, $p = 1$ and $\mathcal{B}_1(x) = B_1(x)$ the Euclidian ball, and if $0 \in \partial\{u_1 > 0\}$, $e_n \in \partial\{u_2 > 0\}$ and $\varkappa_i$*

denote the principal curvatures, we have the following relation on the normal derivatives of u_1 *and* u_2*:*

$$\frac{u^1_\nu(0)}{u^2_\nu(e_n)} = \prod_{\substack{i=1\\ \varkappa_i(0)\neq 0}}^{n-1} \frac{\varkappa_i(0)}{\varkappa_i(e_n)} \quad \text{if } \varkappa_i(0)\neq 0 \text{ for some } i=1,\dots,n-1,$$

and

$$u^1_\nu(0) = u^2_\nu(e_n) \quad \text{if } \varkappa_i(0) = 0 \text{ for any } i=1,\dots,n-1.$$

4. Existence of solutions

This proof follows the same steps as in [19] and it is written below for the reader's convenience.

Theorem 4.1. *There exist continuous functions* $u_1^\varepsilon,\dots,u_K^\varepsilon$*, depending on the parameter* ε*, viscosity solutions of the problem* (2.7)*.*

Proof. The proof uses a fixed point result. Let B be the Banach space of bounded continuous vector-valued functions defined on the domain Ω with the norm

$$\|(u_1,u_2,\dots,u_K)\|_B = \max_i \left(\sup_{x\in\Omega}|u_i(x)|\right).$$

For $i=1,\dots,K$, let ϕ_i be the solutions of

$$\begin{cases} \Delta\phi_i = 0 & \text{in } \Omega,\\ \phi_i = f_i & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Let Θ be the the subset of bounded continuous functions in Ω, that satisfy prescribed boundary data, and are bounded from above and from below as stated below:

$$\Theta = \Big\{(u_1,u_2,\dots,u_K)\,|\,u_i:\Omega\to\mathbb{R} \text{ is continuous, } 0\le u_i\le\phi_i \text{ in } \Omega,\ u_i=f_i \text{ on } (\partial\Omega)_1\Big\}.$$

Notice that Θ is a closed and convex subset of B. Let T^ε be the operator that is defined on Θ in the following way: $T^\varepsilon\big((u_1,u_2,\dots,u_K)\big) := (v_1^\varepsilon, v_2^\varepsilon,\dots,v_K^\varepsilon)$ if for any $i=1,\dots,K$, v_i^ε is solution to

$$
\begin{cases} \Delta(v_i^\varepsilon)(x) = \dfrac{1}{\varepsilon^2} v_i^\varepsilon(x) \displaystyle\sum_{j\neq i} H(u_j)(x) & \text{in } \Omega, \\ \\ v_i^\varepsilon = f_i & \text{on } (\partial\Omega)_1, \end{cases} \tag{4.2}
$$

where u_j, $j \neq i$ are given. Observe that if T^ε has a fixed point

$$
T^\varepsilon\big((u_1^\varepsilon, u_2^\varepsilon, \ldots, u_K^\varepsilon)\big) = (u_1^\varepsilon, u_2^\varepsilon, \ldots, u_K^\varepsilon)
$$

then $(u_1^\varepsilon, u_2^\varepsilon, \ldots, u_K^\varepsilon)$ is a solution of problem (2.7).

In order for T^ε to have a fixed point, we need to prove that it satisfies the hypothesis of the Schauder fixed point Theorem, see [16]:

(1) $T^\varepsilon(\Theta) \subset \Theta$:

Classical existence results guarantee the existence of smooth solutions $(v_1^\varepsilon, v_2^\varepsilon, \ldots, v_K^\varepsilon)$ of the equations (4.2). Remark that v_i^ε is subsolution of $\Delta u = 0$ in Ω, therefore the comparison principle implies

$$
v_i^\varepsilon \leq \phi_i \quad \text{in } \Omega.
$$

Since in addition the f_i's are non-negative, again from the comparison principle we have

$$
v_i^\varepsilon \geq 0 \quad \text{in } \Omega.
$$

We conclude that $T^\varepsilon\big((u_1, u_2, \ldots, u_K)\big) \in \Theta$.

(2) T^ε *is continuous*:

Let us assume that $((u_1)_m, \ldots, (u_K)_m) \to (u_1, \ldots, u_K)$ in B meaning that when m tends to $+\infty$,

$$
\max_{1\leq i\leq K} \|(u_i)_m - u_i\|_{L^\infty} \to 0 \ .
$$

We need to prove that for each fixed $\varepsilon > 0$

$$
\|T^\varepsilon\big((u_1)_m, \ldots, (u_K)_m\big) - T^\varepsilon(u_1, \ldots, u_K)\|_B \to 0
$$

when $m \to +\infty$. Let

$$
T^\varepsilon\big((u_1)_m, \ldots, (u_K)_m\big) = \big((v_1^\varepsilon)_m, \ldots, (v_K^\varepsilon)_m\big),
$$

then if we prove that there exists a constant C_ε independent of m, so that we have the estimate, for $i = 1, \dots, K$

$$\|(v_i^\varepsilon)_m - v_i^\varepsilon\|_{L^\infty} \le C_\varepsilon \max_j \|(u_j)_m - u_j\|_{L^\infty},$$

the result follows. For all $x \in \Omega$ and for fixed i, let ω_m be the function

$$\omega_m(x) = (v_i^\varepsilon)_m(x) - v_i^\varepsilon(x) \ ,$$

and suppose that there exists $y \in \Omega$ such that

$$\omega_m(y) > r^2 D \max_j \|(u_j)_m - u_j\|_{L^\infty} \ , \tag{4.3}$$

for some large $D > 0$, where r is such that $\Omega \subset B_r$, and B_r is the ball centered at 0 of radius r in the Euclidean norm. We want to prove that this is impossible if D is sufficiently large. Let h_m be the concave radially symmetric function

$$h_m(x) = \gamma\big(r^2 - |x|^2\big) \ ,$$

with $\gamma = D \max_j \|(u_j)_m - u_j\|_{L^\infty}$. Observe that:

(a) $h_m(x) = 0$ on ∂B_r;

(b) $h_m(x) \le r^2 D \max_j \|(u_j)_m - u_j\|_{L^\infty}$ for all x in B_r;

(b) $0 = \omega_m(x) \le h_m(x)$ on $\partial\Omega$, since $(v_i^\varepsilon)_m$ and v_i^ε are solutions with the same boundary data.

Since we are assuming (4.3), there exists a negative minimum of $h_m - \omega_m$ in Ω. Let $x_0 \in \Omega$ be a point where the minimum value of $h_m - \omega_m$ is attained. Then

$$h_m(x_0) - \omega_m(x_0) < 0 \quad \text{and} \quad \Delta(h_m - \omega_m)(x_0) \ge 0.$$

Moreover,

$$\begin{aligned}
\Delta\omega_m &= \Delta\big((v_i^\varepsilon)_m\big) - \Delta v_i^\varepsilon \\
&\ge \frac{1}{\varepsilon^2}\Bigg(((v_i^\varepsilon)_m - v_i^\varepsilon)\sum_{j\ne i} H((u_j)_m) - v_i^\varepsilon \sum_{j\ne i} \left(H(u_j) - H((u_j)_m)\right)\Bigg) \\
&\ge \frac{1}{\varepsilon^2}\Bigg(((v_i^\varepsilon)_m - v_i^\varepsilon)\sum_{j\ne i} H((u_j)_m) - v_i^\varepsilon (K-1) C \,\|(u_j)_m - u_j\|_{L^\infty(\Omega)}\Bigg)
\end{aligned}$$

adding and subtracting $\frac{1}{\varepsilon^2} v_i^\varepsilon \sum_{j\neq i} H((u_j)_m)$, where C depends on the f_j's and φ. Then

$$
\begin{aligned}
0 &\leq \Delta(h_m - \omega_m)(x_0) \\
&\leq -2\gamma n - \frac{1}{\varepsilon^2}\Big(((v_i^\varepsilon)_m - v_i^\varepsilon)(x_0) \sum_{j\neq i} H((u_j)_m)(x_0) \\
&- v_i^\varepsilon(x_0)(K-1)C \left\|(u_j)_m - u_j\right\|_{L^\infty} \Big) \\
&\leq -2nD \max_j \|(u_j)_m - u_j\|_{L^\infty} + \frac{1}{\varepsilon^2} v_i^\varepsilon(x_0)(K-1)C \left\|(u_j)_m - u_j\right\|_{L^\infty} \\
&\leq -2nD \max_j \|(u_j)_m - u_j\|_{L^\infty} + \frac{\widetilde{C}}{\varepsilon^2} \left\|(u_j)_m - u_j\right\|_{L^\infty}
\end{aligned}
$$

because $0 < h_m(x_0) < \omega_m(x_0) = \big((v_i^\varepsilon)_m - v_i^\varepsilon\big)(x_0)$ and $\sum_{j\neq i} H((u_j)_m)(x_0) \geq 0$ and so

$$
-\frac{1}{\varepsilon^2}\big((v_i^\varepsilon)_m - v_i^\varepsilon\big)(x_0) \sum_{j\neq i} H((u_j)_m)(x_0) \leq 0 \ .
$$

Taking $D = D_\varepsilon > \frac{\widetilde{C}}{2n\varepsilon^2}$, we obtain that

$$
0 \leq \Delta(h_m - \omega_m)(x_0) < 0
$$

which is a contradiction.

(3) $T(\Theta)$ *is precompact*:

This is a consequence of the fact that the solutions to (4.2) are Hölder continuous on $\overline{\Omega}$ and the subset of Θ of Hölder continuous functions on $\overline{\Omega}$ is precompact in Θ.

This concludes the proof of the theorem. □

5. Uniform in ε Lipschitz estimates

In this section we will prove uniform in ε Lipschitz estimates that will imply the convergence, up to subsequence, of the solution $(u_1^\varepsilon, u_2^\varepsilon, \ldots, u_K^\varepsilon)$ of (2.7) to a limit function $(u_1, \ldots, u_K)$ as $\varepsilon \to 0$. We will show that the functions u_i's are Lipschitz continuous in Ω and harmonic inside their support. Moreover, $u_i \equiv 0$ in the ρ-strip of size 1 of the support of u_j for any $j \neq i$, i.e., the supports of the limit functions are at distance at least 1, one from each other. We start by proving general properties of subsolutions of uniform elliptic equations.

Lemma 5.1. *Let:*

a) ω *be a subharmonic function in* $\mathcal{B}_1$*, such that*

a$_1$) $\omega \le 1$ *in* $\mathcal{B}_1$*;*

a$_2$) $\omega(0) = m > 0$.

b) *Let* D_0 *be a smooth convex set with bounded curvatures*

$$\varkappa_i(\partial D_0) \le C_0, \quad i = 1, \dots, n-1$$

(like $\mathcal{B}_1$ *above).*

Then, there exists a universal $\tau = \tau(C_0, n)$ *such that, if the distance* $d_\rho(D_0, 0) \le \tau m$*, then*

$$\sup_{\partial D_0 \cap \mathcal{B}_1} \omega \ge \frac{m}{2}.$$

Proof. Let h be harmonic in $\mathcal{B}_1 \setminus D_0$ and such that

$$\begin{cases} h = 1 \;\text{ on }\; (\partial \mathcal{B}_1) \setminus D_0 \\ h = \frac{m}{2} \;\text{ on }\; (\partial D_0) \cap \mathcal{B}_1 \;. \end{cases}$$

Then, h grows linearly away from ∂D_0 in $\mathcal{B}_{\frac{1}{2}}$. If τ is small enough, then

$$h(0) < m.$$

Therefore we must have $\sup_{(\partial D_0 \cap \mathcal{B}_1)} \omega \ge \frac{m}{2}$, otherwise the comparison principle would imply $\omega(x) \le h(x)$ in $\mathcal{B}_1 \setminus D_0$, which is a contradiction at $x = 0$. □

Remark. If we replace Δu by a uniformly elliptic equation: $a_{ij} D_{ij} u$ or div $a_{ij} D_j u$ and D_0 by a Lipschitz domain with a uniformly interior cone condition, the same result holds with $d_\rho(D_0, 0) = \tau m^\mu$ (μ large) instead of τm. (This follows from a-priori estimates for equations with bounded measurable coefficients.)

Lemma 5.2. *Let* ω *be a positive subsolution of a uniformly elliptic equation,* ($\lambda^2 I \le a_{ij} \le \Lambda^2 I$)

$$a_{ij} D_{ij} \omega \ge \theta^2 \omega \qquad \textit{in } \mathcal{B}_r.$$

Then there exist $c, C > 0$ such that

$$\frac{\omega(0)}{\sup\limits_{\mathcal{B}_r} \omega} \leq C\mathrm{e}^{-c\theta r}.$$

Proof. The function

$$g(x) = \sum_{i=1}^{n} \cosh\left(\frac{\theta}{\Lambda} x_i\right)$$

is a supersolution of the equation $a_{ij} D_{ij} u = \theta^2 u$. Moreover, using the convexity of the exponential function, it is easy to check that it satisfies

$$g(x) \geq C_1 e^{c\theta r} \quad \text{for any } x \in \partial\mathcal{B}_r.$$

Then, the comparison principle implies

$$\frac{\omega(x)}{\sup\limits_{\mathcal{B}_r} \omega} \leq \frac{g(x)}{C_1 e^{c\theta r}} \quad \text{for any } x \in \mathcal{B}_r.$$

The result follows taking $x = 0$. □

The next lemma says that if u_i^ε attains a positive value σ at some interior point, then all the other functions u_j^ε, $j \neq i$, goes to zero exponentially in a ρ-ball of radius $1 + c\sigma$ around that point.

Lemma 5.3. *Let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be viscosity solution of the problem (2.7). For $i = 1, \ldots, K$, $\sigma, r > 0$, let*

$$\Gamma_i^{\sigma,r} := \{y \in \Omega \,:\, d_\rho(y, \operatorname{supp} f_i) \geq 2r,\, u_i^\varepsilon = \sigma\}$$

and

$$m := \frac{\sigma}{\sup_{\partial\Omega} f_i}.$$

Then, in the sets

$$A_{i,j}^{\sigma,r} := \left\{x \in \Omega \,:\, d_\rho(x, \Gamma_i^{\sigma,r}) \leq 1 + \frac{\tau m r}{2},\, d_\rho(x, \operatorname{supp} f_j) \geq \frac{\tau m r}{4}\right\}$$

(where τ is given by Lemma 5.1), we have

$$u_j^\varepsilon \leq C e^{-\frac{c\sigma^\alpha r^\beta}{\varepsilon}}, \quad \textit{for } j \neq i,$$

for some positive α and β depending on the structure of H (p and q).

Proof. Let $\overline{x} \in A_{i,j}^{\sigma,r}$. We want to show that for $j \neq i$, we have

$$\Delta u_j^\varepsilon \geq \frac{C\sigma^{\overline{\alpha}} r^{\overline{\beta}}}{\varepsilon^2} u_j^\varepsilon \quad \text{in } \mathcal{B}_{\frac{\tau m r}{4}}(\overline{x}) \tag{5.1}$$

for some $\overline{\alpha}, \overline{\beta} > 0$. Let us prove it for $\overline{x}$ such that $d_\rho(\overline{x}, \Gamma_i^{\sigma,r}) = 1 + \frac{\tau m r}{2}$, which is the hardest case. The ball $\mathcal{B}_1(\overline{x})$ is at distance $\frac{\tau m r}{2}$ from a point $y \in \Gamma_i^{\sigma,r}$. Remark that since $\mathcal{B}_{2r}(y) \cap \operatorname{supp} f_i = \emptyset$, the function u_i^ε, which is eventually equal to zero in $\mathcal{B}_{2r}(y) \cap \Omega^c$, satisfies $\Delta u_i^\varepsilon \geq 0$ in $\mathcal{B}_{2r}(y)$. Moreover, since u_i^ε is subharmonic in Ω, it attains its maximum at the boundary of Ω, so that $u_i^\varepsilon / \sup_{\partial\Omega} f_i \leq 1$ in Ω. Hence, from Lemma 4.1 applied to the function $v(x) := u_i^\varepsilon(y + rx)/\sup_{\partial\Omega} f_i$ with $m = \frac{\sigma}{\sup_{\partial\Omega} f_i}$ and $D_0 = \mathcal{B}_{\frac{1}{r} - \frac{\tau m}{2}}\left(\frac{\overline{x} - y}{r}\right)$, there is a point z in $\partial\mathcal{B}_{1 - \frac{\tau m r}{2}}(\overline{x}) \cap \mathcal{B}_r(y)$, such that $u_i^\varepsilon(z) \geq \sigma/2$. Remark that if $x \in \mathcal{B}_{\frac{\tau m r}{4}}(\overline{x})$ then

$$\mathcal{B}_1(x) \supset \mathcal{B}_{\frac{\tau m r}{4}}(z)$$

(since $d_\rho(x, z) \leq d_\rho(x, \overline{x}) + d_\rho(\overline{x}, z) \leq \frac{\tau m r}{4} + 1 - \frac{\tau m r}{2} = 1 - \frac{\tau m r}{4}$).

Let us first consider the case H defined as in (2.5). Then for any $x \in \mathcal{B}_{\frac{\tau m r}{4}}(\overline{x})$ we have

$$H(u_i^\varepsilon)(x) = \sup_{\mathcal{B}_1(x)} u_i^\varepsilon \geq u_i^\varepsilon(z) \geq \frac{\sigma}{2}$$

and we get (5.1) with $\overline{\alpha} = 1$ and $\overline{\beta} = 0$. Remark that since $d_\rho(\overline{x}, \operatorname{supp} f_j) \geq \frac{\tau m r}{4}$, the ball $\mathcal{B}_{\frac{\tau m r}{4}}(\overline{x})$ does not intersect the support of f_j.

Next, let us turn to the case H defined as in (2.4). Remark that since $z \in \mathcal{B}_r(y)$ and $d_\rho(y, \operatorname{supp} f_i) \geq 2r$, we have $\mathcal{B}_r(z) \cap \operatorname{supp} f_i = \emptyset$ and therefore the function u_i^ε, which is eventually equal to zero in $\mathcal{B}_r(z) \cap \Omega^c$, satisfies $\Delta u_i^\varepsilon \geq 0$ in $\mathcal{B}_r(z)$. This implies that $(u_i^\varepsilon)^p$ is subharmonic in $\mathcal{B}_r(z)$ and by the mean value inequality

$$⨍_{B_s(z)} (u_i^\varepsilon)^p dx \geq \left(\frac{\sigma}{2}\right)^p \tag{5.2}$$

in any Euclidian ball $B_s(z) \subset \mathcal{B}_r(z)$, for any $p \geq 1$. Since d_ρ and the Euclidian distance are equivalent, there is an $s \sim \tau m r$ such that

$$B_s(z) \subset \mathcal{B}_{\frac{\tau m r}{8}}(z) \subset \mathcal{B}_{\frac{\tau m r}{4}}(z) \subset \mathcal{B}_1(x). \tag{5.3}$$

Moreover, if $y \in B_s(z)$ and $x \in \mathcal{B}_{\frac{\tau m r}{4}}(\overline{x})$, then

$$\rho(y-x) \leq \rho(y-z) + \rho(z-\overline{x}) + \rho(\overline{x}-x) \leq \frac{\tau m r}{8} + \left(1 - \frac{\tau m r}{2}\right) + \frac{\tau m r}{4} = 1 - \frac{\tau m r}{8},$$

that is

$$1 - \rho(y-x) \geq \frac{\tau m r}{8}. \tag{5.4}$$

Hence, using (5.3), (2.6), (5.4) and (5.2), for all $x \in \mathcal{B}_{\frac{\tau m r}{4}}(\overline{x})$ we get

$$\begin{aligned} H(u_i^\varepsilon)(x) &= \int_{\mathcal{B}_1(x)} (u_i^\varepsilon)^p(y)\varphi(\rho(y-x))dy \\ &\geq \int_{B_s(z)} (u_i^\varepsilon)^p(y) C(1-\rho(y-x))^q dy \\ &\geq \int_{B_s(z)} (u_i^\varepsilon)^p(y) C \left(\frac{\tau m r}{8}\right)^q dy \\ &\geq C\sigma^{\overline{\alpha}} r^{\overline{\beta}} \end{aligned}$$

where $\overline{\alpha}$ and $\overline{\beta}$ depend on p, q and on the dimension n. This implies (5.1).

Now, by Lemma 4.2 we get

$$u_j^\varepsilon(\overline{x}) \leq C e^{-\frac{c\sigma^\alpha r^\beta}{\varepsilon}}$$

for $\alpha = \frac{\overline{\alpha}}{2} + 1$ and $\beta = \frac{\overline{\beta}}{2} + 1$, and the lemma is proven.

□

Corollary 5.4. *Let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be viscosity solution of the problem (2.7). Let y be a point in Ω such that*

$$u_i^\varepsilon(y) = \sigma, \quad d_\rho(y, \operatorname{supp} f_j) \geq 1 + \tau m r, \quad i \neq j \quad \text{and} \quad d_\rho(y, supp(f_i)) \geq 2r,$$

where $m = \frac{\sigma}{\sup_{\partial\Omega} f_i}$ and for $\varepsilon \leq \sigma^\theta$ for some large θ, and $r \geq \sigma^\gamma$ for some small $\gamma > 0$. Then there exists a universal constant $C_0 > 0$ such that in $\mathcal{B}_{\frac{\tau m r}{4}}(y)$ we have

$$|\nabla u_i^\varepsilon| \leq C_0$$

and

$$\Delta u_i^\varepsilon \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly.}$$

Proof. We want to estimate $\Delta u_i^\varepsilon(z)$, for $z \in \mathcal{B}_{\frac{\tau m r}{2}}(y)$. In order to do that, we need to estimate $H(u_j^\varepsilon)(z)$ for $j \neq i$. But $H(u_j^\varepsilon)(z)$ involves points x at ρ-distance 1 from z. Let x be such that $d_\rho(x,z) \leq 1$, then $d_\rho(x,y) \leq 1 + \frac{\tau m r}{2}$. Moreover, since $d_\rho(y, \operatorname{supp} f_j) \geq 1 + \tau m r$, we have $d_\rho(x, \operatorname{supp} f_j) \geq \frac{\tau m r}{2}$. Hence, by Lemma 5.3, for any $j \neq i$

$$u_j^\varepsilon(x) \leq C e^{-\frac{c\sigma^\alpha r^\beta}{\varepsilon}}.$$

It follows that for $z \in \mathcal{B}_{\frac{\tau m r}{2}}(y)$

$$0 \leq \Delta u_i^\varepsilon(z) \leq u_i^\varepsilon(z) \frac{C e^{-\frac{c\sigma^\alpha r^\beta}{\varepsilon}}}{\varepsilon^2}.$$

If we normalize the ball $\mathcal{B}_{\frac{\tau m r}{2}}(y)$ in a Lipschitz fashion

$$\overline{u}_i^\varepsilon(\overline{z}) := 2 \frac{u_i^\varepsilon\left(\frac{\tau m r}{2}\overline{z}\right)}{\tau m r},$$

we have

$$0 \leq \Delta \overline{u}_i^\varepsilon \leq C\sigma r \frac{e^{-\frac{c\sigma^\alpha r^\beta}{\varepsilon}}}{\varepsilon^2} \quad \text{in } \mathcal{B}_1(y).$$

In particular, if $\varepsilon \leq \sigma^{4\alpha}$ and $r^\beta \geq \sigma^\alpha$, we have

$$0 \leq \Delta \overline{u}_i^\varepsilon \leq C \frac{e^{-c\varepsilon^{-\frac{1}{2}}}}{\varepsilon^2} \leq C_0 \quad \text{in } \mathcal{B}_1(y).$$

It follows that $|\nabla \overline{u}_i^\varepsilon|$ is universally bounded in $\mathcal{B}_{\frac{1}{2}}(y)$ and therefore $|\nabla u_i^\varepsilon|$ is universally bounded in $\mathcal{B}_{\frac{\tau m r}{4}}(y)$.

Further, $\Delta u_i^\varepsilon(z)$ converges uniformly to zero, as $\varepsilon \to 0$ in the ball $\mathcal{B}_{\frac{\tau m r}{2}}(y)$. $\square$

The next lemma says that in a ρ-strip of size 1 of the support of the f_j's, the function u_i^ε, $i \neq j$, decays to 0 exponentially.

Lemma 5.5. *Let $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ be viscosity solution of the problem (2.7). For $j = 1, \dots, K$, $\sigma > 0$, let $\overline{\Gamma}_j^\sigma := \{f_j \geq \sigma\} \subset \Omega^c$. Then on the sets*

$$\{x \in \Omega \,:\, d_\rho(x, \overline{\Gamma}_j^\sigma) \leq 1 - r\}, \quad 0 < r < 1$$

we have

$$u_i^\varepsilon \leq C e^{-\frac{c\sigma^\alpha r^\beta}{\varepsilon}}, \quad \textit{for } i \neq j,$$

for some positive α and β depending on the structure of H (p and q) and the modulus of continuity of f_j.

Proof. Let $\overline{x} \in \Omega$ and $y \in \overline{\Gamma}_j^\sigma$ be such that $d_\rho(\overline{x}, y) \leq 1 - r$. We want to estimate $H(u_j^\varepsilon)(x)$, for any $x \in \mathcal{B}_{\frac{r}{2}}(\overline{x})$. Let $x \in \mathcal{B}_{\frac{r}{2}}(\overline{x})$, then

$$d_\rho(x, y) \leq 1 - \frac{r}{2}. \tag{5.5}$$

Let us first consider the case H defined as in (2.5). We have

$$H(u_j^\varepsilon)(x) = \sup_{\mathcal{B}_1(x)} u_j^\varepsilon \geq f_j(y) \geq \sigma.$$

Next, let us turn to the case H defined as in (2.4). Let $r_0 := \min\{\sigma^\gamma, r/4\}$, for some γ depending on the modulus of continuity of f_j (which is Hölder continuous), then $f_j \geq \sigma/2$ in the set $\mathcal{B}_{r_0}(y) \cap (\partial\Omega)_1$. Moreover, remark that from (5.5) and $r_0 \leq r/4$, we have

$$\mathcal{B}_{r_0}(y) \cap (\partial\Omega)_1 \subset \mathcal{B}_{\frac{r}{4}}(y) \subset \mathcal{B}_{\frac{r}{2}}(y) \subset \mathcal{B}_1(x),$$

and for any $z \in \mathcal{B}_{r_0}(y) \cap (\partial\Omega)_1$

$$\rho(x - z) \leq \rho(x - y) + \rho(y - z) \leq 1 - \frac{r}{2} + r_0 \leq 1 - \frac{r}{4}.$$

Therefore, using in addition (2.6), we get

$$\begin{aligned}
H(u_j^\varepsilon)(x) &= \int_{\mathcal{B}_1(x)} (u_j^\varepsilon)^p(z)\varphi(\rho(x - z))dz \\
&\geq \int_{\mathcal{B}_{r_0}(y) \cap (\partial\Omega)_1} (u_j^\varepsilon)^p(z)(1 - \rho(x - z))^q dz \\
&\geq \int_{\mathcal{B}_{r_0}(y) \cap (\partial\Omega)_1} (f_j)^p(z) C \left(\frac{r}{4}\right)^q dz \\
&\geq C\sigma^p r_0^{\overline{\beta}},
\end{aligned}$$

where $\overline{\beta}$ depends on q and on the dimension n.

Now, remark that assumption (2.3) guarantees that $\mathcal{B}_{\frac{r}{2}}(\overline{x}) \cap \mathrm{supp} f_i = \emptyset$. Then, for H defined as in (2.4) or (2.5), the function u_i^ε, $i \neq j$, eventually extended to zero outside Ω, is subsolution of

$$\Delta u_i^\varepsilon \geq u_i^\varepsilon \frac{C \sigma^p r_0^{\overline{\beta}}}{\varepsilon^2}$$

in $B_{\frac{r}{2}}(\overline{x})$, where $p = 1$ and $\overline{\beta} = 0$ in the case (2.5). The conclusion follows as in Lemma 5.3. $\square$

The following corollary is a consequence of Lemma 5.3, Corollary 5.4 and Lemma 5.5.

Corollary 5.6. *Let $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$ be viscosity solution of the problem (2.7). Then we have*

a) *There exist $\widetilde{\theta}$ such that function*

$$(\widetilde{u_i^\varepsilon}) := \left(u_i^\varepsilon - \varepsilon^{1/\widetilde{\theta}}\right)^+$$

is locally uniformly Lipschitz in Ω independently of ε.

b) *The function $u_i^\varepsilon \to 0$ as $\varepsilon \to 0$ in the set*

$$\{x \in \Omega \,|\, d_\rho(x, supp f_j) \leq 1\} \quad \textit{for any } j \neq i.$$

c) *Let $\vec{u} = (u_1, \ldots, u_K)$ be the (local uniform) limit of a convergent subsequence $(\vec{u})^{\varepsilon_m}$. Then:*

i) *the u_i's are Lipschitz continuous in Ω and have disjoint supports, in particular*

$$u_i \equiv 0 \quad \textit{in the set} \quad \{x \in \Omega \,|\, d_\rho(x, \mathrm{supp}\, u_j) \leq 1\} \quad \textit{for any } j \neq i.$$

ii) $\Delta u_i = 0$ *when* $u_i > 0$.

6. A semiconvexity property of the free boundaries

Let $(u_1, \ldots, u_K)$ be the limit of a convergent subsequence of $(u_1^\varepsilon, \ldots, u_K^\varepsilon)$, whose existence is guaranteed by Corollary 5.6. For $i = 1, \ldots, K$, let us denote

$$S(u_i) := \{u_i > 0\}. \tag{6.1}$$

(In the next sections this set will be represented by S_i.) Then the sets $S(u_i)$ have the following semiconvexity property:

Lemma 6.1. *Given $S(u_i)$ consider*

$$T(u_i) = \{x \in \Omega \,:\, d_\rho(x, S(u_i)) \geq 1\}$$

and

$$S^*(u_i) = \{x \in \Omega \,:\, d_\rho(x, T(u_i)) > 1\}$$

Then $S^*(u_i) = S(u_i)$.

Proof. We have that $S^*(u_i) \supset S(u_i)$. To prove the other inclusion consider

$$S_\sigma(u_i) := \{u_i > \sigma\} \ ,$$

$$T_\sigma(u_i) := \{x \in \Omega : d_\rho(x, S_\sigma(u_i)) \geq 1\}$$

and

$$S^*_\sigma(u_i) := \{x \in \Omega \,:\, d_\rho(x, T_\sigma(u_i)) > 1\}.$$

Notice that, the union of ρ-balls centered at points in $S_\sigma(u_i)$ coincides with the union of ρ-balls centered at points in $S^*_\sigma(u_i)$, i.e.

a) $(T_\sigma(u_i))^c = \cup \mathcal{B}_1(x)$ for $x \in S_\sigma(u_i)$ and

b) $(T_\sigma(u_i))^c = \cup \mathcal{B}_1(x)$ for $x \in S^*_\sigma(u_i)$.

If $x \in S_\sigma(u_i)$, from (b) of Corollary 5.6 we have that $d_\rho(x, \text{supp} f_j) > 1$ for $j \neq i$, and the uniform convergence of u_i^ε to u_i and Lemma 5.3 imply that $u_j^\varepsilon \leq Ce^{-\frac{c\sigma^\alpha r^\beta}{\varepsilon}}$ in $\mathcal{B}_1(x)$, where $2r = \min\{d_\rho(x, \text{supp} f_i), C(d_\rho(x, \text{supp} f_j) - 1)\}$. Now, the set where u_j^ε decays is the same if we had considered $x \in S^*_\sigma(u_i)$, since that from (a) and (b) we have

$$\cup_{x \in S_\sigma(u_i)} \mathcal{B}_1(x) = \cup_{x \in S^*_\sigma(u_i)} \mathcal{B}_1(x).$$

Therefore $\frac{H(u_j^\varepsilon)}{\varepsilon^2}$ goes to zero as ε goes to zero in $S^*_\sigma(u_i)$. It follows that $\Delta u_i \equiv 0$ in $S^*_\sigma(u_i)$, if $S^*_\sigma(u_i)$ is not empty. Now, from the inclusion $S_\sigma(u_i) \subset S^*_\sigma(u_i)$ we infer that $u_i \not\equiv 0$ in $S^*_\sigma(u_i)$, since in addition u_i is harmonic and non-negative in $S^*_\sigma(u_i)$, the strong maximum principle implies that $u_i > 0$ in all $S^*_\sigma(u_i)$, that is $S^*_\sigma \subset S(u_i)$. We pass to the limit on σ. $\square$

From the properties of the distance function used in the proof of Lemma 6.1 we can conclude that the sets $S(u_i)$ have a tangent ρ-ball of radius 1 from outside at any point of the boundary, as stated in the following corollary.

Corollary 6.2. *If $x_0 \in \partial S(u_i)$ there is an exterior tangent ball, $\mathcal{B}_1(y)$ at x_0, in the sense that for $x \in \mathcal{B}_1(y) \cap \mathcal{B}_1(x_0)$, all $u_j(x) \equiv 0$ (including u_i).*

Corollary 6.3. *The set $\partial S(u_i)$ has finite $(n-1)$-dimensional Hausdorff measure.*

Proof. From Corollary 6.2 and (2.2), at any point $x \in \partial S(u_i)$ there is an Euclidian ball $B_{d_0}(y) \subset (S(u_i))^c$ tangent to $S(u_i)$ at x, for some d_0 independent of x. Therefore the Euclidian distance function $d(x) := d(x, S(u_i))$ is locally $C^{1,1}$ in the set $\{0 < d < d_0/2\}$ and satisfies

$$\Delta d = -\sum_{l=1}^{n-1} \frac{1}{\frac{1}{\varkappa_l} - d} \geq -\frac{n-1}{d_0 - d},$$

where $\varkappa_l$, $l = 1, \ldots, n-1$, are the principal curvatures of $\partial S(u_i)$. Then, for any euclidian ball $B_R(y)$, $y \in \partial S(u_i)$ and $0 < \sigma < d_0/2$, we have

$$\begin{aligned}
-\frac{2(n-1)}{d_0}\mathrm{vol}(\{\sigma < d < d_0/2\} \cap B^E_R(y)) &\leq \int_{\{\sigma<d<d_0/2\}\cap B^E_R(y)} -\frac{n-1}{d_0 - d} \\
&\leq \int_{\{\sigma<d<d_0/2\}\cap B^E_R(y)} \Delta d \\
&= \int_{\partial(\{\sigma<d<d_0/2\}\cap B^E_R(y))} D_\nu d \\
&\leq \mathrm{Area}(\partial\{d < d_0/2\} \cap B^E_R(y)) - \mathrm{Area}(\partial\{d > \sigma\} \cap B^E_R(y)) \\
&\quad + \mathrm{Area}(\partial B^E_R(y)).
\end{aligned}$$

We infer that

$$\mathrm{Area}(\partial\{d > \sigma\} \cap B^E_R(y)) \leq \mathrm{Area}(\partial\{d < d_0/2\} \cap B^E_R(y)) + C\mathrm{vol}(B^E_R(y)) + \mathrm{Area}(\partial B^E_R(y)).$$

The sets $\partial\{d > \sigma\}$ have therefore uniform bounded measure and converge uniformly to $\partial\{d > 0\}$ as $\sigma \to 0$. Therefore

$$\text{Area}(\partial\{d > 0\} \cap B_R^E(y)) \leq \text{Area}(\partial\{d < d_0/2\} \cap B_R^E(y)) + C\text{vol}(B_r^E(y)) + \text{Area}(\partial B_R^E(y)).$$

□

7. A sharp characterization of the interfaces

In Section 5 we proved that the supports of the limit functions u_i's are at distance at least 1, one from each other (see Corollary 5.6). In this section we will prove that they are exactly at distance 1, as shown in the following theorem.

Theorem 7.1. *Assume $p = 1$ in (2.4). Let $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ be viscosity solution of the problem (2.7) and $(u_1, \dots, u_K)$ the limit as $\varepsilon \to 0$ of a convergent subsequence. Let $x_0 \in \partial\{u_i > 0\} \cap \Omega$, then there exists $j \neq i$ such that*

$$\overline{\mathcal{B}_1(x_0)} \cap \partial\{u_j > 0\} \neq \emptyset \; . \tag{7.1}$$

Proof. First of all, remark that from (b) in Corollary 5.6, we have that $d_\rho(x_0, \text{supp} f_j) \geq 1$ for any $j \neq i$. If there is a j such that $d_\rho(x_0, \text{supp} f_j) = 1$, then (7.1) is obviously true. Therefore, we can assume that $d_\rho(x_0, \text{supp} f_j) > 1$ for any $j \neq i$.

We divide the proof in two cases.

a) $H(u)(x) = \displaystyle\int_{\mathcal{B}_1(x)} u(y)\varphi\big(\rho(x-y)\big)\,\mathrm{d}y$

and

b) $H(u)(x) = \displaystyle\sup_{y \in \mathcal{B}_1(x)} u(y)$.

Proof of case a): Let $S(u_i) = \{u_i > 0\}$ as in (6.1). Let $\mathcal{B}_S$ be a small ρ-ball centered at $x_0 \in \partial S(u_i)$. Then, as a measure, as $\varepsilon \to 0$, up to subsequence

$$\Delta u_i^\varepsilon\big|_{\mathcal{B}_S(x_0)} \longrightarrow \Delta u_i\big|_{\mathcal{B}_S(x_0)}$$

(that has strictly positive mass, since u_i is not harmonic in $\mathcal{B}_S(x_0)$).

We bound by below

$$\int_{\mathcal{B}_{1+S}(x_0)} \sum_{j\neq i} \Delta u_j^\varepsilon dx \quad \text{by} \quad \int_{B_S(x_0)} \Delta u_i^\varepsilon dx.$$

Indeed

$$\begin{aligned}
\varepsilon^2 \int_{\mathcal{B}_{1+S}(x_0)} \sum_{j\neq i} \Delta u_j^\varepsilon(x) dx &\geq \sum_{j\neq i} \int_{\mathcal{B}_{1+S}(x_0)} \int_{\mathcal{B}_1(x)} u_j^\varepsilon(x) \varphi\big(\rho(x-y)\big) u_i^\varepsilon(y) \mathrm{d}y\, \mathrm{d}x \\
&\geq \sum_{j\neq i} \int_{\mathcal{B}_S(x_0)} \int_{\mathcal{B}_1(y)} u_j^\varepsilon(x) \varphi\big(\rho(x-y)\big) u_i^\varepsilon(y) \mathrm{d}x\, \mathrm{d}y \\
&= \varepsilon^2 \int_{\mathcal{B}_S(x_0)} \Delta u_i^\varepsilon(y) dy,
\end{aligned}$$

since the domain of the first integral on the right-hand side is $\mathcal{B}_{1+S}(x_0) \times \mathcal{B}_{2+S}(x_0)$ and the domain of the second integral on the right-hand side is $\mathcal{B}_{1+S}(x_0) \times \mathcal{B}_S(x_0)$.

Therefore, for any positive S, taking the limit in ε we get

$$\int_{\mathcal{B}_{1+S}(x_0)} \sum_{j\neq i} \Delta u_j \geq \int_{\mathcal{B}_S(x_0)} \Delta u_i > 0$$

which implies that there exists $j \neq i$ such that u_j cannot be identical equal to zero in $\mathcal{B}_{1+S}(x_0)$. Since S small is arbitrary, the result follows.

The case b) is more involved. It is enough to prove the theorem for a point x_0 for which $\partial S(u_i)$ has a tangent ρ-ball from inside, since such points are dense on $\partial S(u_i)$ (by the semiconvexity property of $\partial S(u_i)$). We may assume $x_0 = 0$. Let y_0 be such that $\mathcal{B}_\mu(y_0) \subset S(u_i)$ and $0 \in \partial \mathcal{B}_\mu(y_0)$. By Corollary 6.2 we know that there exists a ρ-ball $\mathcal{B}_1(y_1)$ such that $\mathcal{B}_1(y_1) \cap S(u_i) = \emptyset$ and $0 \in \partial \mathcal{B}_1(y_1)$.

Let us first prove two claims.

Claim 1: There exists $\mu' < \mu$ and $C_1 > 0$ such that in the annulus $\{\mu' < \rho(x - y_0) < \mu\}$ we have

$$u_i(x) \geq C_1 d_\rho\big(x, \partial \mathcal{B}_\mu(y_0)\big)\ .$$

Since any ρ-ball $\mathcal{B}$ satisfies the uniform interior ball condition, for any point $\bar{x} \in \partial \mathcal{B}_\mu(y_0)$ there exists an Euclidian ball $B_{R_0}(z_0)$ of radius R_0 independent of $\bar{x}$ contained in $\mathcal{B}_\mu(y_0)$ and tangent to $\partial\mathcal{B}_\mu(y_0)$ at $\bar{x}$. Let $m>0$ be the infimum of u_i on the set $\{x\in\mathcal{B}_\mu(y_0)\mid d(x,\partial\mathcal{B}_\mu(y_0))\geq R_0/2\}$, where d is the Euclidian distance function, and let ϕ be the solution of

$$\begin{cases}\Delta\phi=0 & \text{in} \quad \left\{\dfrac{R_0}{2}<|x-z_0|<R_0\right\}\\ \phi=0 & \text{on} \quad \partial B_{R_0}(z_0)\\ \phi=m & \text{on} \quad \partial B_{\frac{R_0}{2}}(z_0)\end{cases}$$

i.e., for $n\geq 3$,

$$\phi(x)=C(n)m\Big(\frac{R_0^{n-2}}{|x-z_0|^{n-2}}-1\Big)\ .$$

Since u_i is harmonic in $\mathcal{B}_\mu(y_0)$ and $u_i\geq\phi$ on $\partial B_{R_0}(z_0)\cup\partial B_{\frac{R_0}{2}}(z_0)$, by comparison principle $u_i\geq\phi$ in $\{\frac{R_0}{2}<|x-z_0|<R_0\}$. In particular, for any $x\in\{\frac{R_0}{2}<|x-z_0|<R_0\}$ and belonging to the segment between z_0 and $\bar{x}$, using that ϕ is convex in the radial direction,

$$\frac{\partial\phi}{\partial\nu_i}|_{\partial B_{R_0}(z_0)}=\frac{C(n)(n-2)m}{R_0}$$

where ν_i is the interior normal at $\partial B_{R_0}(z_0)$, and (2.2), we get

$$u_i(x)\geq\frac{C(n)(n-2)m}{R_0}d(x,\partial B_{R_0}(z_0))=C(n,R_0)md(x,\partial\mathcal{B}_\mu(y_0))\geq C_1d_\rho(x,\partial\mathcal{B}_\mu(y_0))\ .$$

Therefore, letting $\bar{x}$ vary in $\partial\mathcal{B}_\mu(y_0)$ we get

$$u_i(x)\geq C_1d_\rho(x,\partial\mathcal{B}_\mu(y_0))\quad\text{for any } x\in\mathcal{B}_\mu(y_0)\text{ with } d(x,\partial\mathcal{B}_\mu(y_0))\leq\frac{R_0}{2}.$$

Using (2.2), Claim 1 follows.

Claim 2: There exist $\delta>0$ and $C_2>0$ such that in $\mathcal{B}_{1+\delta}(y_1)$ we have

$$u_i(x)\leq C_2d_\rho(x,\partial\mathcal{B}_1(y_1))\ .$$

Again using barriers, the fact that u_i is subharmonic in Ω and that $\mathcal{B}_1$ satisfies the interior uniform ball condition, Claim 2 follows.

Next, let $e_0=y_0/\rho(y_0)$ and fix $\sigma<\mu$ so small that $\mathcal{B}_\sigma(\sigma e_0)\subset\{\mu'<\rho(x-y_0)<\mu\}\cap\mathcal{B}_{1+\delta}(y_1)$. For $r\in[\sigma-v,\sigma+v]$ and small $v<\sigma$, let us define

$$\underline{u}_i^\varepsilon:=\inf_{\partial\mathcal{B}_r(\sigma e_0)}u_i^\varepsilon\quad\text{and}\quad\underline{u}_i:=\inf_{\partial\mathcal{B}_r(\sigma e_0)}u_i\ .$$

Since for $r \in [\sigma, \sigma + \upsilon]$, $\partial\mathcal{B}_r(\sigma e_0) \cap (S(u_i))^c \neq \emptyset$ and $u_i \equiv 0$ on $(S(u_i))^c$, we have

$$\underline{u}_i = 0 \quad \text{for} \quad r \in [\sigma, \sigma + \upsilon] \ . \tag{7.2}$$

By Claim 1, we know that in $B_\sigma(\sigma e_0)$ we have

$$\begin{aligned} u_i(x) &\geq C_1 d_\rho(x, \partial\mathcal{B}_\mu(y_0)) \\ &\geq C_1 d_\rho(x, \partial\mathcal{B}_\sigma(\sigma e_0)) \\ &= C_1(\sigma - \rho(x - \sigma e_0)). \end{aligned}$$

We deduce that for $r \in [\sigma - \upsilon, \sigma]$

$$\underline{u}_i = \inf_{\partial\mathcal{B}_r(\sigma e_0)} u_i \geq \inf_{\partial\mathcal{B}_r(\sigma e_0)} C_1(\sigma - \rho(x - \sigma e_0)) = C_1(\sigma - r).$$

From the previous inequality and (7.2), we infer that

$$\underline{u}_i \geq C_1(\sigma - r)^+, \quad r \in [\sigma - \upsilon, \sigma + \upsilon]. \tag{7.3}$$

Similarly, using Claim 2 and making a Taylor expansion of $\rho(y_1 - x)$ and $\rho(\sigma e_0 - x)$ at 0, in $\mathcal{B}_\sigma(\sigma e_0)$ we have

$$\begin{aligned} u_i(x) &\leq C_2 d_\rho(x, \partial\mathcal{B}_1(y_1)) \\ &= C_2(\rho(y_1 - x) - 1) \\ &= C_2(-\nabla\rho(y_1) \cdot x + o(|x|)) \\ &= C_2(\nabla\rho(\sigma e_0) \cdot x + o(|x|)) \\ &= C_2(\sigma - \rho(\sigma e_0 - x)) + o(|x|). \end{aligned} \tag{7.4}$$

Here we have used that $\nabla\rho(y_1) = -\nabla\rho(\sigma e_0)$ being the balls $\mathcal{B}_1(y_1)$ and $\mathcal{B}_\sigma(\sigma e_0)$ tangent at 0. Remark that for $r \in [\sigma - \upsilon, \sigma]$, from (2.2) we have

$$\inf_{\partial\mathcal{B}_r(\sigma e_0)} |x| \leq \inf_{\partial\mathcal{B}_r(\sigma e_0)} c_2\rho(x) = c_2 d_\rho(\partial\mathcal{B}_r(\sigma e_0), 0) = c_2(\sigma - r).$$

Therefore, from (7.4), for $r \in [\sigma - \upsilon, \sigma]$ we have

$$
\begin{aligned}
\underline{u}_i &= \inf_{\partial\mathcal{B}_r(\sigma e_0)} u_i(x) \\
&\le \inf_{\partial\mathcal{B}_r(\sigma e_0)} C_2(\sigma - \rho(x - \sigma e_0)) + o(|x|) \\
&= \inf_{\partial\mathcal{B}_r(\sigma e_0)} C_2(\sigma - r) + o(|x|) \\
&\le \widetilde{C}_2(\sigma - r).
\end{aligned}
$$

From the previous inequalities and (7.2), we infer that

$$
\underline{u}_i \le \widetilde{C}_2(\sigma - r)^+, \quad r \in [\sigma - \upsilon, \sigma + \upsilon]. \tag{7.5}
$$

Next, for $j \neq i$, $r \in [\sigma - \upsilon, \sigma + \upsilon]$, let us define

$$
\bar{u}_j^\varepsilon := \sup_{\partial\mathcal{B}_{1+r}(\sigma e_0)} u_j^\varepsilon \quad \text{and} \quad \bar{u}_j := \sup_{\partial\mathcal{B}_{1+r}(\sigma e_0)} u_j \ .
$$

The functions $\underline{u}_i^\varepsilon$ and $\bar{u}_j^\varepsilon$ are respectively solutions of

$$
\begin{aligned}
\Delta_r \underline{u}_i^\varepsilon &\le \frac{1}{\varepsilon^2} \underline{u}_i^\varepsilon \sum_{i \neq j} \sup_{\mathcal{B}_1(\underline{z}_r^i)} u_j^\varepsilon \\
\Delta_r \overline{u}_j^\varepsilon &\ge \frac{1}{\varepsilon^2} \overline{u}_j^\varepsilon \sup_{\mathcal{B}_1(\bar{z}_r^j)} u_i^\varepsilon
\end{aligned} \tag{7.6}
$$

where

$$
\Delta_r u = u_{rr} - \frac{(n-1)}{r} u_r = \frac{1}{r^{n-1}} \ \frac{\partial}{\partial r}\Big(r^{n-1} \frac{\partial u}{\partial r} \Big)
$$

and $\underline{z}_r^i$ and $\bar{z}_r^j$ are respectively the points where the infimum of u_i^ε on $\partial\mathcal{B}_r(\sigma e_0)$ and the supremum of u_j^ε on $\partial\mathcal{B}_{1+r}(\sigma e_0)$ are attained. Note that in spherical coordinates

$$
\Delta u = \Delta_r u + \Delta_\theta u
$$

and that if we are on a point where u attains a minimum value in the θ for a fixed r then $\Delta_\theta u \ge 0$ and the opposite inequality holds if we are on a maximum point. We also remark that

$$
\overline{y}_r^j := \sigma e_0 + \frac{r}{r+1} (\bar{z}_r^j - \sigma e_0) \in \partial\mathcal{B}_r(\sigma e_0) \cap \partial\mathcal{B}_1(\bar{z}_r^j) \ ,
$$

therefore

$$
\sup_{\mathcal{B}_1(\bar{z}_r^j)} u_i^\varepsilon \ge u_i^\varepsilon(\overline{y}_r^j) \ge \underline{u}_i^\varepsilon \ . \tag{7.7}
$$

Moreover, since $\mathcal{B}_1(\underline{z}_r^i) \subset \mathcal{B}_{1+r}(\sigma e_0)$ and u_j^ε is a subharmonic function, we have

$$\begin{aligned} \sup_{\mathcal{B}_1(\underline{z}_r^i)} u_j^\varepsilon &\leq \sup_{\mathcal{B}_{1+r}(\sigma e_0)} u_j^\varepsilon \\ &= \sup_{\partial\mathcal{B}_{1+r}(\sigma e_0)} u_j^\varepsilon \\ &= \bar{u}_j^\varepsilon \ . \end{aligned} \tag{7.8}$$

From (7.6), (7.7) and (7.8), we conclude that

$$\Delta_r \underline{u}_i^\varepsilon \leq \Delta_r \Big(\sum_{j \neq i} \bar{u}_j^\varepsilon \Big) \ .$$

In other words, for any $\phi \in C_c^\infty(\sigma - \upsilon, \sigma + \upsilon)$, $\phi \geq 0$, we have

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i^\varepsilon \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}\Big(\frac{1}{r^{n-1}}\phi\Big)\right) dr \leq \int_{\sigma-\upsilon}^{\sigma+\upsilon} \sum_{j\neq i} \bar{u}_j^\varepsilon \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}\Big(\frac{1}{r^{n-1}}\phi\Big)\right) dr \ .$$

Passing to the limit as $\varepsilon \to 0$ along a uniformly converging subsequence, we get

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}\Big(\frac{1}{r^{n-1}}\phi\Big)\right) dr \leq \int_{\sigma-\upsilon}^{\sigma+\upsilon} \sum_{j\neq i} \bar{u}_j \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}\Big(\frac{1}{r^{n-1}}\phi\Big)\right) dr \ .$$

The linear growth of u_i away from the free boundary given by inequalities (7.3) and (7.5), implies that $\Delta_r \underline{u}_i$ develops a Dirac mass at $r = \sigma$ and

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}\Big(\frac{1}{r^{n-1}}\phi\Big)\right) dr > 0,$$

for υ small enough. Hence, $\Delta_r(\sum_{j\neq i} \bar{u}_j)$ is a positive measure in $(\sigma - \upsilon, \sigma + \upsilon)$ and therefore there exists $j \neq i$ such that u_j cannot be identically equal to zero in the ball $\mathcal{B}_{1+\sigma}(\sigma e_0)$. Since σ small is arbitrary, the result follows. □

8. Classification of singular points in dimension 2

From the results of the previous sections we know that the solutions $u_1^\varepsilon, \ldots, u_K^\varepsilon$ of system (2.7), through a subsequence, converge as $\varepsilon \to 0$ to functions $u_1, \ldots, u_K$ which are Lipschitz continuous in Ω and harmonic inside their support. For $i = 1, \ldots, K$, let us denote the interior of the support of u_i by

$$S_i := \{u_i > 0\}$$

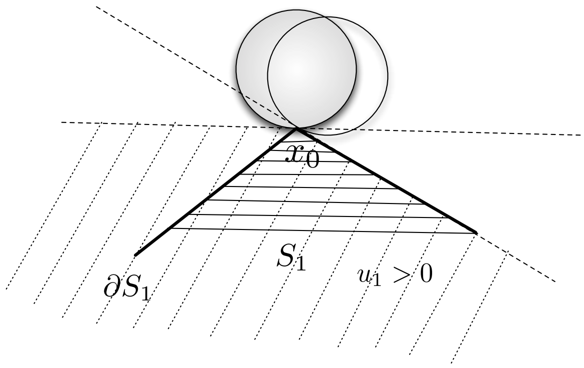

FIGURE 1. Asymptotic cone at x_0

and the union of the interior of the supports of all the other functions by

$$C_i := \cup_{j \neq i} S_j. \tag{8.1}$$

Since the sets S_i are disjoint we have $\partial C_i = \cup_{j\neq i} \partial S_j$. As a consequence of semiconvexity we know that the free boundaries ∂S_i are Lipschitz curves on the plane. Moreover, from Theorem 7.1 we know that S_i and C_i are at ρ-distance 1, therefore for any point $x \in \partial S_i$ there is a point $y \in \partial C_i$ such that $\rho(x - y) = 1$. We say that x realizes at y the distance from C_i.

Definition. A point $x \in \partial S_i$ is a singular point if it realizes the distance from C_i to at least two points in ∂C_i. We say that $x \in \partial S_i$ is a regular point if it is not singular.

Geometrically, we can describe regular and singular points as follows. Let $x \in \partial S_i$ be a singular point and $y_1, y_2 \in \partial C_i$ points where x realizes the distance from C_i. Then the balls $\mathcal{B}_1(y_1)$ and $\mathcal{B}_1(y_2)$ are tangent to ∂S_i at x. Consider the convex cone determined by the two tangent lines to the two tangent ρ-balls which does not intersect the two ρ-balls. The intersection of all cones generated by all ρ-balls of radius 1 tangent at x defines a convex asymptotic cone centered at x, see Figure 1. If $x \in \partial S_i$ is a regular point the cone at $x \in \partial S_i$ is an half-plane, because there is only one point $y \in \partial C_i$ where x realizes the distance from C_i. If $\theta \in (0, \pi]$ is the opening of the cone at x, we say that S_i has an angle θ at x.

We start by proving that positive functions which are harmonic in a cone and vanishes on its boundary, growth linearly away from the boundary of the cone and far from the vertex, with a constant of linear decay which depends on the distance from the vertex and on the opening of the cone.

Lemma 8.1. *Let $\theta_0 \in (0, \pi]$. Let $\mathcal{C}$ be the cone defined in polar coordinates by $\mathcal{C} = \{(\varrho, \theta) \mid \varrho \in [0, +\infty),\ 0 \le \theta \le \theta_0\}$. Let u be a function harmonic and positive in the interior of $\mathcal{C} \cap B_{2r_0}$, which vanishes on $\partial\mathcal{C} \cap B_{2r_0}$. Then for $r < r_0/5$ there exist $R = R(\theta_0, r)$, and c, C constants just depending on $(\theta_0, \|u\|_\infty, r_0)$, such that for any $x \in [r, 3r] \times \left[0, \frac{R}{2}\right]$ we have*

(1) $u(x) \ge c r^\alpha d(x, \partial\mathcal{C})$

(2) $u(x) \le C r^\alpha d(x, \partial\mathcal{C})$

where α is given by

$$1 + \alpha = \frac{\pi}{\theta_0}.$$

Proof. Let us introduce the function

$$v(\varrho, \theta) := \varrho^{1+\alpha} \sin((1 + \alpha)\theta).$$

Notice that v is harmonic in the interior of $\mathcal{C}$, since it is the imaginary part of the function $z^{1+\alpha}$, where $z = x + iy$, which is holomorphic in the set $\mathbb{C} \setminus (-\infty, 0]$. Moreover v is positive inside $\mathcal{C}$ and vanishes on its boundary. Since v and u have linear growth away from the boundary of $\mathcal{C}$, for $\rho \in [r_0/2, 3r_0/2]$ (far from the vertex and from ∂B_{2r_0}), and are positive inside the cone, we can find constants $c, C > 0$ depending on $\|u\|_\infty$, r_0 and θ_0, such that

$$cv \le u \le Cv \quad \text{on } \mathcal{C} \cap \partial B_{r_0}.$$

Since in addition u and v vanish on $\partial C \cap B_{r_0}$, the comparison principle implies

$$cv \le u \le Cv \quad \text{in } \mathcal{C} \cap B_{r_0}. \tag{8.2}$$

In what follows we denote by C several constants independent of r.

(1) Let $x_r := (r, 0)$, with $0 < r < r_0/5$. If $R := r \min\{1, \tan(\theta_0/2)\}$, there exists z such that the Euclidian ball $B_R(z)$ is contained in $\mathcal{C} \cap B_{r_0}$ and $x_r \in \partial B_R(z)$. Let us introduce the barrier function

$$\phi(x) := \frac{m}{\log 2} \log\left(\frac{R}{|x - z|}\right),$$

where

$$m = \inf_{\partial B_{\frac{R}{2}}(z)} u.$$

Remark that ϕ is, up to translations and multiplicative and additive constants, the fundamental solution of the Laplacian in dimension 2. Therefore ϕ satisfies

$$\begin{cases} \Delta\phi = 0 & \text{in } B_R(z) \setminus B_{\frac{R}{2}}(z) \\ \phi = 0 & \text{on } \partial B_R(z) \\ \phi = m & \text{on } \partial B_{\frac{R}{2}}(z). \end{cases}$$

Since $u \geq \phi$ on $\partial B_R(z) \cup \partial B_{\frac{R}{2}}(z)$ the comparison principle then implies

$$u \geq \phi \qquad \text{in } B_R(z) \setminus B_{\frac{R}{2}}(z).$$

If ν_i is the inner normal vector of $B_R(z)$, then for $x \in \partial B_R(z)$,

$$\frac{\partial \phi}{\partial \nu_i}(x) = \frac{m}{R \log 2},$$

and the convexity of ϕ in the radial direction gives, for any $x \in B_R(z) \setminus B_{\frac{R}{2}}(z)$ belonging to the segment between z and x_r

$$u(x) \geq \frac{m}{R \log 2} \operatorname{dist}(x, \partial B_R(z)).$$

Let us estimate m. If a point x with polar coordinates (ϱ, θ) belongs to $\partial B_{\frac{R}{2}}(z)$, then

$$\varrho \geq \frac{R}{2}$$

and

$$\theta_1 \leq \theta \leq \max\left\{\theta_0, \frac{\pi}{2}\right\} - \theta_1,$$

where θ_1 is such that

$$\tan\theta_1 = \frac{R}{2}\frac{1}{r} = \frac{1}{2}\min\left\{1, \tan\left(\frac{\theta_0}{2}\right)\right\}.$$

Therefore (8.2) implies that

$$m = \inf_{\partial B_{\frac{R}{2}}(z)} u \geq c\left(\frac{R}{2}\right)^{\alpha+1} \sin((1+\alpha)\theta_1) \geq cr^{\alpha+1}.$$

We infer that for any $x \in B_R(z) \setminus B_{\frac{R}{2}}(z)$ belonging to the segment between z and x_r

$$u(x) \geq cr^{\alpha}\mathrm{dist}(x, \partial B_R(z)) = cr^{\alpha} d(x, \partial\mathcal{C}).$$

Repeating this argument with balls tangent to $\partial\mathcal{C}$ from inside at any point in the segment $[x_r, 3x_r]$, we get (1).

(2) Consider u extended by zero outside of $\mathcal{C}$. Consider z now such that the Euclidian ball $B_R(z)$ is contained in $\mathcal{C}^c$ and $x_r = (r, 0) \in \partial B_R(z)$, where R is defined as above. Let us take now as barrier the function

$$\psi(x) := \frac{M}{\log\frac{3}{2}} \log\left(\frac{|z-x|}{R}\right),$$

where

$$M = \sup_{\partial B_{\frac{3}{2}R}(z)} u \leq Cr^{\alpha+1}.$$

Note that if a point x with polar coordinates (ϱ, θ) belongs to $\partial B_{\frac{3R}{2}}(z)$, then $\varrho \leq 5r/2$.

Like before the barrier ψ satisfies

$$\begin{cases} \Delta\psi = 0 & \text{in } B_{\frac{3R}{2}}(z) \setminus B_R(z) \\ \psi = M & \text{on } \partial B_{\frac{3R}{2}}(z) \\ \psi = 0 & \text{on } \partial B_R(z). \end{cases}$$

Using the comparison principle and the concavity of ψ in the radial direction, and then letting the tangent ball moving along the segment $[x_r, 3x_r]$, we get (2).

In Figure 2 is represented the wall of barriers used in the proof of the lemma. $\square$

Theorem 8.2. *Assume $n = 2$ and $p = 1$ in (2.4). Let $(u_1^\varepsilon, \dots, u_K^\varepsilon)$ be viscosity solution of the problem (2.7) and $(u_1, \dots, u_K)$ the limit as $\varepsilon \to 0$ of a convergent subsequence. For $i \neq j$, let $x_0 \in \partial S_i$ and $y_0 \in \partial S_j$ be points such that S_i has an angle θ_i at x_0, S_j has an angle θ_j at y_0 and $\rho(x_0 - y_0) = 1$. Then we have*

$$\theta_i = \theta_j.$$

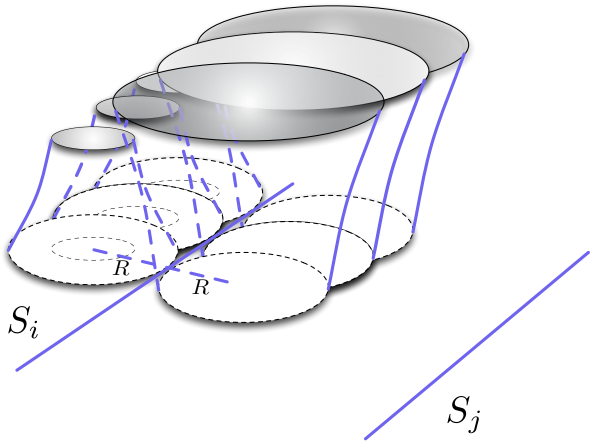

FIGURE 2. Wall of barriers

Proof. Without loss of generality we can assume that $x_0 = 0$. It suffices to show the theorem for y_0 belonging to a region that is side by side with S_i, in the sense that 0 is the limit as $h \to 0$ of regular points $x_h \in \partial S_i$ with the property that x_h realizes the distance from C_i at $y_h \in \partial S_j$ regular points with $y_h \to y_0$ as $h \to 0$. Let $\mathcal{C}$ be the asymptotic cone at 0. Let us suppose for simplicity that ∂S_i and ∂S_j are locally a cone around 0 and y_0 respectively. If this is not the case, we can adapt the proof by approximating the free boundaries at 0 and y_0 with cones from outside and inside. Then there exists $r_0 > 0$ such that $\partial S_i \cap B_{2r_0} = \mathcal{C} \cap B_{2r_0}$, where B_{2r_0} is the Euclidian ball centered at 0 of radius $2r_0$.

If (ϱ, θ) is a system of polar coordinates in the plane centered at zero, we may assume that $\mathcal{C}$ is the cone given by

$$\mathcal{C} = \{(\varrho, \theta) \,|\, \varrho \in [0, +\infty),\, 0 \le \theta \le \theta_i\}.$$

Let us first consider the case (2.5). Let us assume that $x_h = (2r_h, 0)$, with $r_h > 0$. We know that $r_h \to 0$ as $h \to 0$, then we can fix h so small that $r_h < r_0/5$. By Lemma 8.1 applied to $u = u_i$, we have

$$u_i(x) \ge c r_h^{\alpha} d(x, \partial S_i) \quad \text{for any } x \in [r_h, 3r_h] \times \left[0, \frac{R_h}{2}\right], \tag{8.3}$$

and

$$u_i(x) \le Cr_h^\alpha d(x, \partial S_i) \quad \text{for any } x \in [r_h, 3r_h] \times \left[0, \frac{R_h}{2}\right], \tag{8.4}$$

where

$$1 + \alpha = \frac{\pi}{\theta_i} \ge 1. \tag{8.5}$$

Now, we repeat an argument similar to the one in the proof of Theorem 7.1. We look at $\inf u_i$ in small circles of radius r that go across the free boundary of u_i and we look at $\sup u_j$ in circles of radius $r+1$ across the free boundary of u_j, then we compare the mass of the correspondent Laplacians. Precisely, there exists a small $\sigma > 0$ and $e \in S_i$ such that $\mathcal{B}_\sigma(e) \subset [r_h, 3r_h] \times [0, R_h/2]$ and $x_h \in \partial\mathcal{B}_\sigma(e)$. In particular, in $\mathcal{B}_\sigma(e)$ the function u_i satisfies (8.3) and (8.4). For $\upsilon < \sigma$ and $r \in [\sigma - \upsilon, \sigma + \upsilon]$, we define

$$\underline{u}_i := \inf_{\partial\mathcal{B}_r(e)} u_i \quad \text{and} \quad \bar{u}_j := \sup_{\partial\mathcal{B}_{1+r}(e)} u_j\ . \tag{8.6}$$

In what follows we denote by C and c several constants independent of h. For $r \in [\sigma - \upsilon, \sigma]$, by (8.3) we have

$$\underline{u}_i \ge \inf_{\partial\mathcal{B}_r(e)} cr_h^\alpha d(x, \partial S_i) \ge \inf_{\partial\mathcal{B}_r(e)} Cr_h^\alpha d_\rho(x, \partial S_i) \ge Cr_h^\alpha(\sigma - r).$$

For $r \in [\sigma, \sigma + \upsilon]$, the ball $\mathcal{B}_r(e)$ goes across ∂S_i, therefore we have $\underline{u}_i = 0$. Hence

$$\begin{aligned} &\underline{u}_i(r) \ge Cr_h^\alpha(\sigma - r) \quad \text{for } r \in [\sigma - \upsilon, \sigma] \\ &\underline{u}_i(r) = 0 \quad \text{for } r \in [\sigma, \sigma + \upsilon]. \end{aligned} \tag{8.7}$$

Next, let us study the behaviour of $\bar{u}_j$. First of all, let us show that

$$d_\rho(e, \partial S_j) = \rho(e - y_h) = 1 + \sigma. \tag{8.8}$$

Since $d_\rho(e, \partial S_i) = \sigma$ and $d_\rho(S_i, S_j) \ge 1$, it is easy to see that $d_\rho(x, \partial S_j) \ge 1 + \sigma$. The function ρ is also called a Minkowski norm and from known results about Minkowski norms, if we denote by T the Legendre transform $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(y) = \rho(y) D\rho(y)$, then T is a bijection with inverse $T^{-1}(\xi) = \rho^*(\xi) D\rho^*(\xi)$, where ρ^* is the dual norm defined by $\rho^*(\xi) := \sup\{y \cdot \xi \,|\, y \in \mathcal{B}_1\}$.

Now, the ball $\mathcal{B}_1(y_h)$ is tangent to ∂S_i at x_h and therefore is also tangent to $\mathcal{B}_\sigma(e)$ at x_h. This implies that $D\rho(e-x_h) = -D\rho(x_h-e) = D\rho(x_h-y_h)$. Consequently we have

$$\begin{aligned} e - x_h &= T^{-1}(T(e-x_h)) = T^{-1}(\sigma D\rho(e-x_h)) = T^{-1}(\sigma D\rho(x_h-y_h)) \\ &= \sigma T^{-1}(T(x_h-y_h) = \sigma(x_h-y_h). \end{aligned}$$

We infer that

$$e = x_h + \sigma(x_h - y_h) \tag{8.9}$$

and

$$\rho(e-y_h) = (1+\sigma)\rho(x_h-y_h) = 1+\sigma,$$

which proves (8.8). As a consequence $\partial\mathcal{B}_{1+r}(e)\cap S_j = \emptyset$ for $r \in [\sigma-\upsilon,\sigma)$, while if $r \in (\sigma,\sigma+\upsilon]$ then $\partial\mathcal{B}_{1+r}(e)\cap S_j \neq \emptyset$ and $\partial\mathcal{B}_{1+r}(e)$ enters inside S_j at ρ-distance at most $r-\sigma$ from the boundary of S_j. In particular we have

$$\bar{u}_j = 0 \quad \text{for } r \in [\sigma-\upsilon,\sigma]. \tag{8.10}$$

Next, if θ_j is the angle of S_j at y_0, let β be defined by

$$1+\beta = \frac{\pi}{\theta_j} \geq 1. \tag{8.11}$$

Remark that y_h is at ρ-distance $2r_h$ from y_0. Again by Lemma 8.1 applied to $u = u_j$, (after a rotation and a translation), we have the following estimate

$$u_j(x) \leq Cr_h^\beta d(x,\partial S_j) \leq Cr_h^\beta d_\rho(x,\partial S_j),$$

in a neighborhood of y_h. As a consequence, recalling in addition that the ball $\mathcal{B}_{1+r}(e)$ enters in S_j at ρ-distance $r-\sigma$ from the boundary, for $r \in [\sigma,\sigma+\upsilon]$ we get

$$\bar{u}_j = \sup_{\partial\mathcal{B}_{1+r}(e)} u_j \leq Cr_h^\beta(r-\sigma).$$

The last estimate and (8.10) imply

$$\bar{u}_j(r) \leq Cr_h^\beta(r-\sigma)^+, \quad \text{for } r \in [\sigma-\upsilon,\sigma+\upsilon]. \tag{8.12}$$

Now, we want to compare the mass of the Laplacians of $\underline{u}_i$ and $\bar{u}_j$. Defining as in (8.6)

$$\underline{u}_i^\varepsilon := \inf_{\partial\mathcal{B}_r(e)} u_i^\varepsilon, \qquad \bar{u}_k^\varepsilon := \sup_{\partial\mathcal{B}_{1+r}(e)} u_k^\varepsilon,\ k \neq i$$

and arguing as in the proof of Theorem 7.1, we see that

$$\Delta_r \underline{u}_i^\varepsilon \leq \sum_{k\neq i} \Delta_r \bar{u}_k^\varepsilon \quad \text{in } (\sigma - \upsilon, \sigma + \upsilon), \tag{8.13}$$

where $\Delta_r u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$. Remark that since x_h is a regular point of ∂S_i that realizes the distance from C_i at $y_h \in \partial S_j$, the ball $\mathcal{B}_{1+\sigma+\upsilon}(e)$ does not intersect the support of the functions u_k for $k \neq j$ and small σ. Therefore, multiplying inequality (8.13) by a positive test function $\phi \in C_c^\infty(\sigma - \upsilon, \sigma + \upsilon)$, integrating by parts in $(\sigma - \upsilon, \sigma + \upsilon)$ and passing to the limit as $\varepsilon \to 0$ along a converging subsequence, the only surviving function on the right-hand side is $\bar{u}_j$ and we get

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right)\right) dr \leq \int_{\sigma-\upsilon}^{\sigma+\upsilon} \bar{u}_j \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right)\right) dr. \tag{8.14}$$

Let us choose a function ϕ which is increasing and $(\sigma - \upsilon, \sigma)$ and decreasing in $(\sigma, \sigma + \upsilon)$ and hence with maximum at $r = \sigma$, and let us estimates the left and the right hand-side of the last inequality. Estimates (8.7) imply that $\frac{\partial \underline{u}_i}{\partial r}(\sigma^-) \leq -Cr_h^\alpha$. Therefore, for small υ we have

$$\begin{aligned}
\int_{\sigma-\upsilon}^{\sigma+\upsilon} \underline{u}_i \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right)\right) dr &= -\int_{\sigma-\upsilon}^{\sigma} \frac{\partial \underline{u}_i}{\partial r} r \frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right) dr \\
&= -\int_{\sigma-\upsilon}^{\sigma} \left(\frac{\partial \underline{u}_i}{\partial r}(\sigma^-) + o_{\sigma-r}(1)\right) r\frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right) dr \\
&\geq -\int_{\sigma-\upsilon}^{\sigma} \frac{\partial \underline{u}_i}{\partial r}(\sigma^-)\left(\frac{\partial\phi}{\partial r} - \frac{1}{r}\phi\right) dr \\
&\quad - o_\upsilon(1)\int_{\sigma-\upsilon}^{\sigma}\left(\frac{\partial\phi}{\partial r} + \frac{1}{r}\phi\right) dr \\
&\geq -\frac{\partial \underline{u}_i}{\partial r}(\sigma^-)\left[\phi(\sigma) - \phi(\sigma)\log\left(\frac{\sigma}{\sigma-\upsilon}\right)\right] \\
&\quad - o_\upsilon(1)\left[\phi(\sigma) + \phi(\sigma)\log\left(\frac{\sigma}{\sigma-\upsilon}\right)\right] \\
&\geq (Cr_h^\alpha - o_\upsilon(1))\phi(\sigma).
\end{aligned}$$

Similarly, using (8.12) and integrating by parts, we get

$$\int_{\sigma-\upsilon}^{\sigma+\upsilon} \bar{u}_j \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\phi\right)\right) dr \leq (Cr_h^\beta + o_\upsilon(1))\phi(\sigma).$$

From the previous estimates and (8.14), letting v go to 0, we obtain

$$r_h^\alpha \le C r_h^\beta,$$

and therefore, for h small enough

$$\beta \le \alpha.$$

Recalling the definitions (8.5) and (8.11) of α and β respectively, we infer that

$$\theta_i \le \theta_j.$$

Exchanging the roles of u_i and u_j, the same proof gives the opposite inequality

$$\theta_j \le \theta_i,$$

and this concludes the proof of the theorem for H defined as in (2.5).

Next, let us turn to the case (2.4). Again we compare the mass of Laplacians of u_i and u_j across the free boundaries. For $\sigma < r_h$ let us define

$$D_\sigma(x_h) := \{x \in \mathcal{B}_{r_h}(x_h) \,|\, d(x, \partial S_i) \le \sigma^2\}.$$

Then, if we denote by $(D_\sigma(x_h))_1$ and $(D_\sigma(x_h))_2$ the sets of points at ρ-distance respectively less than 1 and 2 from $D_\sigma(x_h)$, we have

$$\begin{aligned}
\varepsilon^2 \int_{D_\sigma(x_h)} \Delta u_i^\varepsilon(x)dx &= \sum_{k\neq i} \int_{D_\sigma(x_h)} \int_{\mathcal{B}_1(x)} u_i^\varepsilon(x)\varphi\big(\rho(x-y)\big)u_k^\varepsilon(y)dydx \\
&= \sum_{k\neq i} \int\int_{D_\sigma(x_h)\times(D_\sigma(x_h))_1} u_i^\varepsilon(x)\varphi\big(\rho(x-y)\big)u_k^\varepsilon(y)dxdy \\
&\le \sum_{k\neq i} \int\int_{(D_\sigma(x_h))_2\times(D_\sigma(x_h))_1} u_i^\varepsilon(x)\varphi\big(\rho(x-y)\big)u_k^\varepsilon(y)dxdy \\
&= \sum_{k\neq i} \int_{(D_\sigma(x_h))_1} \int_{\mathcal{B}_1(y)} u_i^\varepsilon(x)\varphi\big(\rho(x-y)\big)u_k^\varepsilon(y)dxdy \\
&\le \varepsilon^2 \sum_{k\neq i} \int_{(D_\sigma(x_h))_1} \Delta u_k^\varepsilon(y)dy,
\end{aligned}$$

which implies

$$\int_{D_\sigma(x_h)} \Delta u_i^\varepsilon(x)dx \le \sum_{k\neq i} \int_{(D_\sigma(x_h))_1} \Delta u_k^\varepsilon(x)dx. \tag{8.15}$$

By Lemma 8.1 the normal derivative of u_i with respect to the inner normal ν_i, at any point on the boundary $\partial\mathcal{C}$ at distance r from the vertex is greater than cr^α, then

$$\int_{D_\sigma(x_h)} \Delta u_i = \int_{\partial\mathcal{C}\cap D_\sigma(x_h)} \frac{\partial u_i}{\partial \nu_i} dA \geq c\int_{cr_h}^{Cr_h} r^\alpha dr = Cr_h^{\alpha+1}.$$

Remark that there exists $c > 0$ such that

$$(D_\sigma(x_h))_1 \cap \partial S_j \subset \mathcal{B}_{r_h+c\sigma}(y_h) \cap \partial S_j$$

therefore, for σ small enough, again from Lemma 8.1 we have

$$\int_{(D_\sigma(x_h))_1} \Delta u_j \leq Cr_h^{\beta+1}.$$

Then from the limit in ε of inequality (8.15) we obtain that

$$\beta \leq \alpha$$

and therefore

$$\theta_i \leq \theta_j.$$

Exchanging the roles of u_i and u_j we get the opposite inequality

$$\theta_j \leq \theta_i.$$

This concludes the proof of the theorem.

□

Corollary 8.3. *Assume $n = K = 2$. Assume in addition that the supports on $\partial\Omega$ of the boundary data f_1 and f_2 have a finite number of connected components. Then S_1 and S_2 have a finite number of connected components. Moreover, singular points form a finite set.*

Proof. Consider all the connected components of S_1 and S_2, S_i^j, $i = 1, 2$ and $j = 1, 2, 3, \ldots$. Remark that for any i and j

$$\partial S_i^j \cap \{x \in \partial\Omega \,:\, f_i(x) > 0\} \neq \emptyset.$$

Indeed, if not we would have $u_i = 0$ on the ∂S_i^j and $\Delta u_i \geq 0$ in S_i^j. The maximum principle then would imply $u \equiv 0$ in S_i^j, which is not possible. Moreover, since u_i is continuous up to the

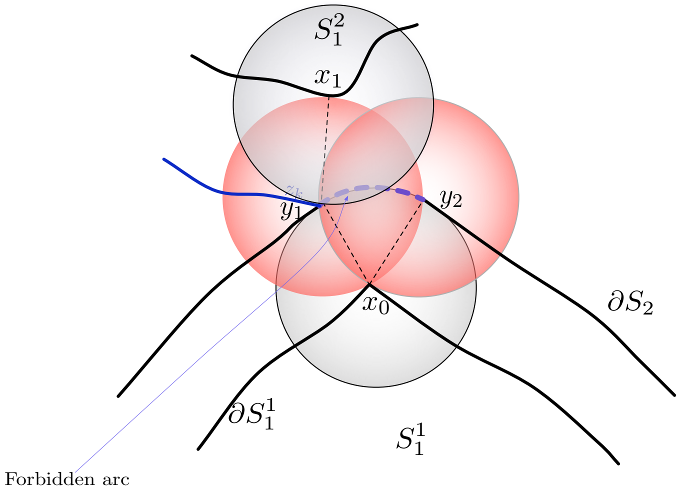

FIGURE 3. Forbidden arc

boundary of Ω, ∂S_i^j must contain one connected component of the set $\{x \in \partial\Omega \,:\, f_i(x) > 0\}$; we say that the components of S_1 and S_2 reach the boundary of Ω. This implies that the connected components of S_1 and S_2 are finite.

Next, let $x_0 \in \partial S_1$ be a singular point that realizes the distance from S_2 at $y_1, y_2 \in \partial S_2$ two different points ($y_1, y_2 \in \partial\mathcal{B}_1(x_0) \cap \partial S_2$, see Figure 3). We can choose y_1 such that $\mathcal{B}_1(x_0)$ is the limit as $k \to +\infty$ of balls $\mathcal{B}_1(x_k)$ with $x_k \in \partial S_1$, tangent to points $y_k \in \partial S_2$ with $y_k \to y_1$ and $x_k \to x_0$ as $k \to +\infty$. Theorem 8.2 implies that S_2 has an angle at y_1 and y_2 and the intersection of the arc on $\partial\mathcal{B}_1(x_0)$ between y_1 and y_2 with ∂S_2 must have empty interior. This means that near y_1 there are points on ∂S_2 outside $\overline{\mathcal{B}_1(x_0)}$. These points are at distance greater than 1 from x_0 and from any other point of ∂S_1 close to x_0 and must realize the distance from S_1 outside $\mathcal{B}_1(y_1)$, see Figure 3. Therefore if we take a sequence z_k of such points converging to y_1 and we consider the corresponding tangent balls centered at points that are in ∂S_1 where the z_k's realize the distance, we obtain a second tangent ball $\mathcal{B}_1(x_1)$ for y_1 with $x_1 \neq x_0$, i.e., y_1 is a singular point of ∂S_2.

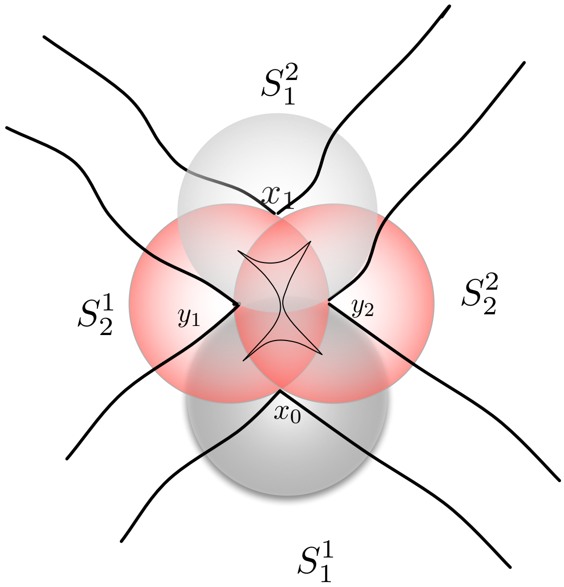

FIGURE 4. A singular point involving four components

Now, let us denote by S_1^1 the connected component of S_1 whose boundary contains x_0. Remember that since S_1 and S_2 are at ρ-distance 1, we have $u_1 \equiv 0$ in $\overline{\mathcal{B}_1(y_1)} \cup \overline{\mathcal{B}_1(y_2)}$. Remark that $\mathcal{B}_1(y_1)$ and $\mathcal{B}_1(y_2)$ must have not empty intersection, since they are tangent balls to a Lipschitz curve. Moreover, since the connected components of S_2 whose boundaries contain y_1 and y_2 must reach the boundary of Ω, they separate the components of S_1 whose boundaries contain x_0 and x_1. Therefore x_1 must belong to the boundary of different components of S_1, let us call it S_1^2. The same argument that we have used for x_1 and x_0 proves also that y_1 and y_2 must belong to the boundary of different components of S_2.

We conclude that a singular point x_0 of S_1 involves at least four different connected components and there correspond to it another singular point of S_1 belonging to a different connected component of S_1, see Figure 4. Moreover, since all the connected components of S_1 and S_2 must reach the boundary of Ω, x_1 is the only singular point of S_1^2 corresponding to a singular point of S_1^1. Since the connected component of S_1 are finite, we infer that there is a finite number of singular points on ∂S_1^1. This argument applied to any connected component of S_1 shows that singular points of S_1 are finite. In the same way we can prove that singular points of S_2 are finite and this concludes the proof of the theorem.

$\square$

Another corollary of Theorem 8.2 is the C^1-regularity of the free boundaries when $K = 2$ and under the following additional assumptions on Ω, f_1 and f_2:

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, g(x_2) \leq x_1 \leq h(x_2),\, x_2 \in [a, b]\}, \quad b - a \geq 4 \tag{8.16}$$

where

$$\begin{cases} g, h : [a, b] \to \mathbb{R} \text{ are Lipschitz functions with} \\ -m_2 \leq g \leq -m_1 \leq M_2 \leq h \leq M_1, \quad M_2 \geq -m_1 + 4; \end{cases} \tag{8.17}$$

the boundary data are such that

$$\begin{cases} f_1 \equiv 1,\, f_2 \equiv 0 \quad \text{on } \{x_1 \leq g(x_2)\}, \\ f_1 \equiv 0,\, f_2 \equiv 1 \quad \text{on } \{x_1 \geq h(x_2)\}, \\ f_1 \text{ is monotone decreasing in } x_1 \text{ on } \{x_2 \leq a\} \cup \{x_2 \geq b\}, \\ f_2 \text{ is monotone increasing in } x_1 \text{ on } \{x_2 \leq a\} \cup \{x_2 \geq b\}. \end{cases} \tag{8.18}$$

These assumptions imply that $-u_1$ and u_2 are monotone increasing in the x_1 direction. Then we have the following

Corollary 8.4. *Assume $K = n = 2$, (8.16), (8.17), (8.18) and $p = 1$ in (2.4). Then the sets ∂S_i, $i = 1, 2$, are of class C^1.*

Proof. We know that the sets ∂S_i are Lipschitz graph at ρ-distance 1, one from each other. Suppose by contradiction that ∂S_1 has an angle $\theta < \pi$ at y_0. In particular, there exist two ρ-balls of radius 1, centered at two points $z, w \in \partial S_2$ that are tangent to ∂S_1 at y_0. Then, by the monotonicity property of the u_i's and Theorem 7.1, the arc of the ρ-ball of radius 1 centered at y_0 between the points z and w must be all in ∂S_2. This means that any point inside this arc, which is a regular point of ∂S_2, is at ρ-distance 1 from the singular point $y_0 \in \partial S_1$. This contradicts Theorem 8.2.

$\square$

9. A relation between the normal derivatives at the free boundary

In this section we restrict ourself to the following case:

$$\tag{9.1} \begin{cases} K = 2 \\ H \text{ defined like in (2.4), with} \\ p = 1,\ \varphi \equiv 1 \text{ and } \rho \text{ the Euclidian norm.} \end{cases}$$

Therefore, the system (2.7) becomes

$$\Delta u_1^\varepsilon(x) = \frac{1}{\varepsilon^2} u_1^\varepsilon(x) \int_{B_1(x)} u_2^\varepsilon(y)\,\mathrm{d}y \quad \text{in } \Omega,$$

$$\Delta u_2^\varepsilon(x) = \frac{1}{\varepsilon^2} u_2^\varepsilon(x) \int_{B_1(x)} u_1^\varepsilon(y)\,\mathrm{d}y \quad \text{in } \Omega,$$

where we denote by $B_1(x)$ the Euclidian ball of radius 1 centered at x. Let (u_1, u_2) be the limit functions of a converging subsequence that we still denote $(u_1^\varepsilon, u_2^\varepsilon)$ and for $i = 1, 2$ let

$$S_i := \{u_i > 0\}.$$

From Section 7 we know that the u_i's have disjoint support and that there is a strip of width exactly one that separates S_1 and S_2. Moreover, Corollary 6.2 guarantees that at any point of the boundary of the two sets, the principal curvatures are less or equal 1. For $i = 1, 2$, let $x_i \in \partial S_i$ be such that x_1 is at distance 1 from x_2, ∂S_i is of class C^2 in a neighborhood of x_i and all the principal curvatures of ∂S_i at x_i are strictly less than 1. Without loss of generality we can assume $x_1 = 0$ and $x_2 = e_n$, where $e_n = (0, \ldots, 1)$. Let us denote by $u_\nu^1(0)$ and $u_\nu^2(e_n)$ the exterior normal derivatives of u_1 and u_2 respectively at 0 and e_n. Note that the two normals have opposite direction. We want to deduce a relation between $u_\nu^1(0)$ and $u_\nu^2(e_n)$. Let us start by recalling some basic properties about the level surfaces of the distance function to a set.

9.1. **Level surfaces of the distance function to a set. Some basic Properties.** Consider a set S and its boundary ∂S, of the class C^2. Let $\varkappa_i(x)$ be the principal curvatures of ∂S at x (outward is the positive direction). Assume that $\varkappa_i(x) < 1 - \varepsilon$. Then:

a) the distance function to S, $d_S(x) = d(x, S)$, is defined and is C^2 as long as

$$d_S(x) < 1 + \varepsilon.$$

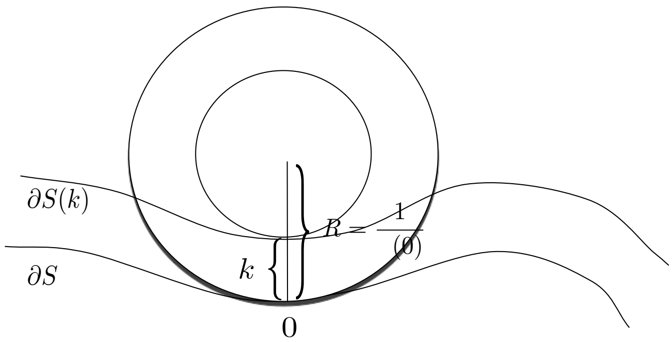

FIGURE 5. Curvatures relation

Let $S(k)$ denote the surface that is at distance k from S

$$S(k) := \{x : d_S(x) = k\},$$

then, for $k < 1 + \varepsilon$ and $x \in S(k)$, there is a unique point $x_0 \in S(0)$, such that $x = x_0 + k\nu(x_0)$ where $\nu(x_0)$ is the unit normal vector at x_0 in the positive direction. More precisely, if we denote $K := \max\{|\varkappa_i(x)| : 1 \le i \le n-1, x \in \partial S\}$ and $f(x,t) := x + t\nu(x)$, then f is a diffeomorphism between $\partial S \times (-k, k)$ and the neighborhood of ∂S, $N_k(S) = \{x + t\nu(x) : x \in \partial S, |t| < k\}$ with $k < \frac{1}{K}$.

b) for all $x_0 \in \partial S$ if we consider the linear transformation $x_t = x_0 + t\nu(x_0)$ we obtain $S(t)$. Hence, since the tangent plane for each $S(t)$ is always perpendicular to $\nu(x_0)$, the eigenvectors of the principal curvatures remain constant along the trajectories of d_S, for $d_S < 1 + \varepsilon$.

c) the curvatures of $S(k)$ satisfy, see Figure 5

$$\varkappa_i(x_0 + k\nu(x_0)) = \frac{1}{\frac{1}{\varkappa_i(x_0)} - k} = \frac{\varkappa_i(x_0)}{1 - \varkappa_i(x_0)k}, \qquad i = 1, \dots, n-1, \quad k < 1 + \varepsilon$$

for $x_0 \in \partial S$.

d) for $x_0 \in \partial S$, the ball $B_1(x_0)$ touches $S(1)$ at the point $x_0 + \nu(x_0)$, ν the outwards normal, and separates quadratically from $S(1)$.

9.2. **Free boundary condition.** Following Subsection 9.1, we denote by $\varkappa_i(0)$ the principal curvatures of ∂S_1 at 0 where outward is the positive direction and by $\varkappa_i(e_n) = \frac{\varkappa_i(0)}{1-\varkappa_i(0)}$, the principal curvatures of ∂S_2 at e_n. Remark that since the normal vectors to S_1 and S_2 respectively at 0 and e_n, have opposite directions, for $\varkappa_i(e_n)$ the inner direction of S_2 is the positive one. The main result of this section is the following:

Theorem 9.1. *Assume* (9.1)*. Let* $0 \in \partial S_1$ *and* $e_n \in \partial S_2$*. Assume that* ∂S_1 *is of class* C^2 *in* $B_{4h_0}(0)$ *and that the principal curvatures satisfy:* $\varkappa_i(0) < 1$ *for any* $i = 1, \dots, n-1$*. Then, we have the following relation:*

$$\frac{u^1_\nu(0)}{u^2_\nu(e_n)} = \prod_{\substack{i=1\\ \varkappa_i(0)\neq 0}}^{n-1} \frac{\varkappa_i(0)}{\varkappa_i(e_n)} \quad \textit{if } \varkappa_i(0) \neq 0 \textit{ for some } i = 1, \dots, n-1,$$

and

$$u^1_\nu(0) = u^2_\nu(e_n) \quad \textit{if } \varkappa_i(0) = 0 \textit{ for any } i = 1, \dots, n-1.$$

In order to prove Theorem 9.1, we first prove a lemma that relates the mass of the Laplacians of the limit functions across the interfaces. For a point x belonging to a neighborhood of ∂S_1 around 0, let us denote by $\nu(x) = \nu(x_0)$ the exterior normal vector at $x_0 \in \partial S_1$, where x_0 is the unique point such that $x = x_0 + \nu(x_0)$. From (a) in Subsection 9.1, $\nu(x)$ is well defined.

Lemma 9.2. *Under the assumptions of Theorem 9.1, for small* $h < h_0$*, let*

$$D_h := B_h(0) \cap \{x : d(x, \partial S_1) \leq h^2\}$$

and

$$E_h := \{y \in \mathbb{R}^n \,|\, y = x + \nu(x), x \in D_h\}.$$

Then

$$\int_{D_h} \Delta u_1 = \int_{E_h} \Delta u_2.$$

Proof. Remark that the surface $E_h \cap \partial S_2$ is of class C^2 for h small enough, being $\varkappa_i(0) < 1$ for $i = 1, \dots, n-1$, see Subsection 9.1. The Laplacians of the u_i's are positive measures and

$$\int_{D_h} \Delta u_1 = \lim_{\varepsilon \to 0} \int_{D_h} \Delta u_1^\varepsilon(x)\,\mathrm{d}x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{D_h} \int_{B_1(x)} u_1^\varepsilon(x) u_2^\varepsilon(y)\,\mathrm{d}y\mathrm{d}x,$$

and

$$\int_{E_h} \Delta u_2 = \lim_{\varepsilon \to 0} \int_{E_h} \Delta u_2^\varepsilon(y)\,\mathrm{d}y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{E_h} \int_{B_1(y)} u_1^\varepsilon(x) u_2^\varepsilon(y)\,\mathrm{d}x\mathrm{d}y.$$

Let s be such that $\varepsilon^{\frac{1}{4\alpha}} < s < h$, where α is given by Lemma 5.3. We split the set D_h in the following way

$$D_h = D_{h,s}^+ \cup D_{h,s}^- \cup D_{h,s},$$

where

$$D_{h,s}^+ := \{x \in D_h \,|\, d(x, \partial S_1) > s^2 \text{ and } u_1(x) > 0\},$$

$$D_{h,s}^- := \{x \in D_h \,|\, d(x, \partial S_1) > s^2 \text{ and } u_1(x) = 0\},$$

$$D_{h,s} := \{x \in D_h \,|\, d(x, \partial S_1) \le s^2\}.$$

Similarly

$$E_h = E_{h,s}^+ \cup E_{h,s}^- \cup E_{h,s},$$

where

$$E_{h,s}^+ := \{x \in E_h \,|\, d(x, \partial S_2) > s^2 \text{ and } u_2(x) > 0\},$$

$$E_{h,s}^- := \{x \in E_h \,|\, d(x, \partial S_2) > s^2 \text{ and } u_2(x) = 0\},$$

$$E_{h,s} := \{x \in E_h \,|\, d(x, \partial S_2) \le s^2\},$$

see Figure 6. Since ∂S_1 is a smooth surface around 0, and $\Delta u_1 = 0$ in S_1, we have that u_1 grows linearly away from the boundary in a neighborhood of 0. This and the uniform convergence of u_1^ε to u_1, imply that there exists $c > 0$ such that $u_1^\varepsilon(x) > cs^2$, for any $x \in D_{h,s}^+$ for ε small enough. Then, by Lemma 5.3, $u_2^\varepsilon(y) \le ae^{-\frac{b(cs^2)^\alpha}{\varepsilon}}$, ($a, b$ positive constants), for $y \in B_1(x)$ and any $x \in D_{h,s}^+$. In an analogous way, if $y \in E_{h,s}^+$, we know that for ε small enough $u_2^\varepsilon(y) > cs^2$ and by Lemma 5.3, $u_1^\varepsilon(x) \le ae^{-\frac{b(cs^2)^\alpha}{\varepsilon}}$ for $x \in B_1(y)$. Since we have chosen s such that $s^{2\alpha} > \varepsilon^{\frac{1}{2}}$, we

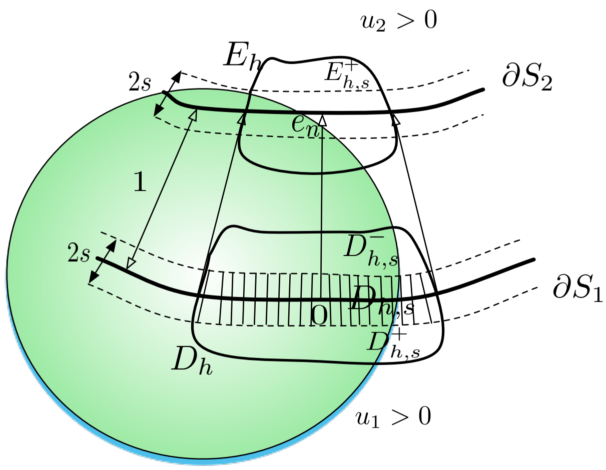

FIGURE 6. Relation between the mass of the Laplacians

have that $u_2^\varepsilon(y) = o(\varepsilon^2)$ uniformly in y, for any $y \in \cup_{x\in D^+_{h,s}} B_1(x)$ and $u_1^\varepsilon(x) = o(\varepsilon^2)$ uniformly in x, for any $x \in \cup_{y\in E^+_{h,s}} B_1(y)$. Remark that

$$D^-_{h,s} \subset \cup_{y\in E^+_{h,s}} B_1(y).$$

Therefore we have

$$\begin{aligned}
\frac{1}{\varepsilon^2}\int_{x\in D_h}\int_{y\in B_1(x)} u_1^\varepsilon(x)u_2^\varepsilon(y)\mathrm{d}y\mathrm{d}x &= \frac{1}{\varepsilon^2}\int_{x\in D^+_{h,s}}\int_{y\in B_1(x)} u_1^\varepsilon(x)\underbrace{u_2^\varepsilon(y)}_{negligible}\mathrm{d}y\mathrm{d}x \\
&+ \frac{1}{\varepsilon^2}\int_{x\in D_{h,s}}\int_{y\in B_1(x)} u_1^\varepsilon(x)u_2^\varepsilon(y)\mathrm{d}y\mathrm{d}x \\
&+ \frac{1}{\varepsilon^2}\int_{x\in D^-_{h,s}}\int_{y\in B_1(x)} \underbrace{u_1^\varepsilon(x)}_{negligible}u_2^\varepsilon(y)\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\varepsilon^2}\int_{x\in D_{h,s}}\int_{y\in B_1(x)} u_1^\varepsilon(x)u_2^\varepsilon(y)\mathrm{d}y\mathrm{d}x + o(1).
\end{aligned} \tag{9.2}$$

Analogously

$$\frac{1}{\varepsilon^2}\int_{y\in E_h}\int_{x\in B_1(y)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,\mathrm{d}x\mathrm{d}y = \frac{1}{\varepsilon^2}\int_{E_{h,s}}\int_{B_1(y)} u_1^\varepsilon(x)u_2^\varepsilon(y)\,\mathrm{d}x\mathrm{d}y + o(1). \tag{9.3}$$

Next, for fixed $x \in D_{h,s}$,we have

$$B_1(x) \cap \{y \,|\, d(y,\partial S_2) > s^2\} \subset B_{1+h}(0) \cap \{y \,|\, d(y,\partial S_2) > s^2\} \cap \{u_2 \equiv 0\}.$$

Therefore for any $y \in B_1(x) \cap \{y \mid d(y, \partial S_2) > s^2\}$, the ball $B_1(y)$ enters in $S_1 \cap B_{2h}(0)$ at distance at least s^2 from ∂S_1. Since $\partial S_1 \cap B_{4h}(0)$ is of class C^2, u_1 has linear growth away from the boundary in $\partial S_1 \cap B_{2h}(0)$ and therefore there exists a point in $B_1(y)$ where $u_1 \geq cs^2$ for some $c > 0$. Like before, Lemma 5.3 implies that $u_2^\varepsilon(y) = o(\varepsilon^2)$. We infer that

(9.4)

$$\frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in B_1(x)} u_1^\varepsilon(x) u_2^\varepsilon(y) \mathrm{d}y \mathrm{d}x = \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in B_1(x) \cap \{y \mid d(y, \partial S_2) \leq s^2\}} u_1^\varepsilon(x) u_2^\varepsilon(y) \mathrm{d}y \mathrm{d}x + o(1).$$

Finally, remark that (d) of Subsection 9.1 implies that for $x \in D_{h,s}$

$$B_1(x) \cap \{y \mid d(y, \partial S_2) \leq s^2\} \subset E_{h+cs,s} \tag{9.5}$$

for some $c > 0$. From (9.2), (9.3), (9.4) and (9.5), we get

$$\begin{aligned}
\int_{D_h} \Delta u_1^\varepsilon(x) dx &= \frac{1}{\varepsilon^2} \int_{x \in D_h} \int_{y \in B_1(x)} u_1^\varepsilon(x) u_2^\varepsilon(y) \mathrm{d}y \mathrm{d}x \\
&= \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in B_1(x) \cap \{y \mid d(y, \partial S_2) \leq s^2\}} u_1^\varepsilon(x) u_2^\varepsilon(y) \mathrm{d}y \mathrm{d}x + o(1) \\
&\leq \frac{1}{\varepsilon^2} \int_{x \in D_{h,s}} \int_{y \in E_{h+cs,s}} u_1^\varepsilon(x) u_2^\varepsilon(y) \mathrm{d}y \mathrm{d}x + o(1) \\
&\leq \frac{1}{\varepsilon^2} \int_{y \in E_{h+cs,s}} \int_{x \in B_1(y)} u_1^\varepsilon(x) u_2^\varepsilon(y) \mathrm{d}x \mathrm{d}y + o(1) \\
&= \int_{E_{h+cs}} \Delta u_2^\varepsilon(y) dy + o(1).
\end{aligned}$$

Similar computations give

$$\int_{E_h} \Delta u_2^\varepsilon(y) dy \leq \int_{D_{h+cs}} \Delta u_1^\varepsilon(x) dx + o(1).$$

Letting first ε and then s go to 0, the conclusion of the lemma follows.

□

Lemma 9.3. *Under the assumptions of Theorem 9.1, let* $\Gamma_h^1 = \partial S_1 \cap B_h(0)$ *and let* $\Gamma_h^2 = \{x + \nu(x) : x \in \Gamma_h^1\}$. *Then we have the limits*

$$\lim_{h \to 0} \frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA} = \prod_{\substack{i=1 \\ \varkappa_i(0) \neq 0}}^{n-1} \frac{\varkappa_i(0)}{\varkappa_i(e_n)} \quad \textit{if } \varkappa_i(0) \neq 0 \textit{ for some } i = 1, \dots, n-1, \tag{9.6}$$

and

$$\lim_{h\to 0}\frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA}=1 \quad \text{if } \varkappa_i(0)=0 \text{ for any } i=1,\dots,n-1. \tag{9.7}$$

Proof. Consider the diffeomorphism $f_t(x)=f(x,t)=x+t\nu(x)$. Then $\Gamma_h^2=f_1(\Gamma_h^1)$ and

$$\int_{\Gamma_h^2} dA=\int_{\Gamma_h^1}|Jf_1(x)|dA,$$

where $|Jf_1|$ is the determinant of the Jacobian of f_1. Taking as basis of the tangent space at 0 the principal directions, τ_i, then the differential of f_1 at x is given by

$$(df_1)(\tau_i)=\tau_i+(d\nu)(\tau_i)=\tau_i-\varkappa_i\tau_i.$$

So,

$$|Jf_1(x)|=\prod_{i=1}^{n-1}(1-\varkappa_i(x))$$

and

$$\frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA}=\frac{1}{\text{Area }(\Gamma_h^1)}\int_{\Gamma_h^1}\prod_{i=1}^{n-1}(1-\varkappa_i(x))dA.$$

Passing to the limit when h converges to zero, we obtain

$$\lim_{h\to 0}\frac{\int_{\Gamma_h^2} dA}{\int_{\Gamma_h^1} dA}=\prod_{i=1}^{n-1}(1-\varkappa_i(0)).$$

Now, if $\varkappa_i(0)\neq 0$ for some $i=1,\dots,n-1$, then

$$\prod_{i=1}^{n-1}(1-\varkappa_i(0))=\prod_{\substack{i=1\\ \varkappa_i(0)\neq 0}}^{n-1}(1-\varkappa_i(0))=\prod_{\substack{i=1\\ \varkappa_i(0)\neq 0}}^{n-1}\left(\frac{1-\varkappa_i(0)}{\varkappa_i(0)}\varkappa_i(0)\right)=\prod_{\substack{i=1\\ \varkappa_i(0)\neq 0}}^{n-1}\frac{\varkappa_i(0)}{\varkappa_i(e_n)},$$

and (9.6) follows.

If $\varkappa_i(0)=0$ for any $i=1,\dots,n-1$, then

$$\prod_{i=1}^{n-1}(1-\varkappa_i(0))=1$$

and we get (9.7).

□

Proof of Theorem 9.1.

Let $\Gamma_h^1 = \partial S_1 \cap D_h$ and $\Gamma_h^2 = \partial S_2 \cap E_h$. The Laplacians Δu_i, are jump measures along ∂S_i, $i = 1, 2$, and satisfy

$$\int_{D_h} \Delta u_1 = -\int_{\Gamma_h^1} u_\nu^1 \, dA \quad \text{and} \quad \int_{E_h} \Delta u_2 = -\int_{\Gamma_h^2} u_\nu^2 \, dA.$$

Then, using Lemma 9.2 we get

$$1 = \frac{\int_{D_h} \Delta u_1}{\int_{E_h} \Delta u_2} = \frac{\int_{\Gamma_h^1} u_\nu^1 \, dA}{\int_{\Gamma_h^2} u_\nu^2 \, dA},$$

and so

$$\frac{⨍_{\Gamma_h^1} u_\nu^1 \, dA}{⨍_{\Gamma_h^2} u_\nu^2 \, dA} = \frac{\int_{\Gamma_h^2} \, dA}{\int_{\Gamma_h^1} \, dA}.$$

Since, when $h \to 0$,

$$\frac{⨍_{\Gamma_h^1} u_\nu^1 \, dA}{⨍_{\Gamma_h^2} u_\nu^2 \, dA} \to \frac{u_\nu^1(0)}{u_\nu^2(e_n)},$$

by Lemma 9.3 the conclusion of Theorem 9.1 follows.

□

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