Stability of N-fronts bifurcating from a twisted heteroclinic loop and an application to the FitzHugh-Nagumo equation

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Abstract

In this article, existence and stability of N-front travelling wave solutions of partial differential equations on the real line is investigated. The N-fronts considered here arise as heteroclinic orbits bifurcating from a twisted heteroclinic loop in the underlying ordinary differential equation describing travelling wave solutions. It is proved that the N-front solutions are linearly stable provided the fronts building the twisted heteroclinic loop are linearly stable. The result is applied to travelling waves arising in the FitzHugh-Nagumo equation.

1 Introduction

In this article, existence and stability of N-front solutions of parabolic equations

$$U_t = AU + F(U, \epsilon) \qquad x \in \mathbb{R} \tag{1.1}$$

on the real line is investigated. Here, the differential operator A generates a C^0-semiflow on $BU(\mathbb{R}, \mathbb{R}^m)$ – the space of bounded, uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^m$ – and F is typically a Nemitskii operator defined on the same space. Fronts and backs are travelling wave solutions $U(\xi) = U(x + ct)$ which are asymptotically constant for $\xi \to \pm\infty$. Transforming (1.1) into a moving coordinate frame $(x, t) \mapsto (x + ct, t) = (\xi, t)$ yields

$$U_t = AU - cU_\xi + F(U, \epsilon) \qquad \xi \in \mathbb{R}. \tag{1.2}$$

Then fronts and backs of (1.1) with wave speed c correspond to equilibria of (1.2) solving

$$\begin{aligned} AU - cU_\xi + F(U, \epsilon) &= 0 \\ \lim_{\xi \to \pm\infty} U(\xi) &= U_\pm. \end{aligned} \tag{1.3}$$

Stability of a front U is often determined by the spectrum of the linearized operator

$$L(U)\,V = AV - cV_\xi + D_U F(U, \epsilon)\,V. \tag{1.4}$$

A front or back is called linearly stable if the spectrum of L is contained in the left half plane with the exception of a simple eigenvalue at zero which is inevitable due to translational invariance. Under rather general assumptions on A, linear stability implies nonlinear stability, see [Hen81] or [BJ89].

Suppose now that for $(c, \epsilon) = (c_0, \epsilon_0)$ linearly stable front and back waves do exist simultaneously. Then, upon varying $\mu := (c, \epsilon)$, other front solutions may arise. In particular,

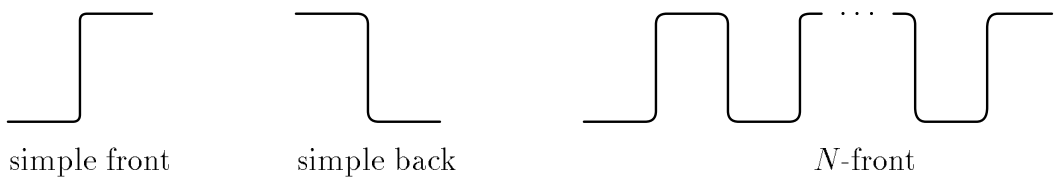

Figure 1: N-front solutions consist of $2N+1$ concatenated copies of a simple front and back.

so-called N-fronts may bifurcate which are formed by alternately concatenating $2N+1$ copies of the simple front and back, see Figure 1. A natural and interesting question is whether the bifurcating N-fronts U_N inherit the linear stability from the simple front and back. For a fairly general class of operators A, it follows from [AGJ90] that the spectrum of $L(U_N)$ is bounded to the left of the imaginary axis except for $2N+1$ eigenvalues near zero. It therefore suffices to calculate these critical eigenvalues, that is solutions (λ, V) of

$$AV - c_N V_\xi + D_U F(U_N, \epsilon_N) V = \lambda V \tag{1.5}$$

for λ close to zero, where U_N is the N-front existing for $(c, \epsilon) = (c_N, \epsilon_N)$.

Notice that the steady-state equation (1.3) and the eigenvalue problem (1.5) are ordinary differential equations in the time variable ξ. As such they can be written as first-order systems

$$\begin{aligned} \dot{u} &= f(u, \mu) \qquad\qquad \mu = (c, \epsilon) && (1.6) \\ \dot{v} &= (D_u f(u, \mu) + \lambda B)\, v, && (1.7) \end{aligned}$$

respectively. Simple fronts and backs of (1.3) correspond to heteroclinic solutions $q_1(\xi)$ and $q_2(\xi)$ of (1.6) connecting two equilibria p_1 and p_2.

In this article, we investigate the existence and stability of N-fronts (and N-backs) under the assumption that the simple heteroclinic orbits q_1 and q_2 form a twisted heteroclinic loop, see Figure 2. Under certain generic assumptions, we prove existence of N-fronts of (1.6) for any $N > 1$ and determine all eigenvalues λ of (1.7) with $|\lambda|$ small. The N-fronts are either all stable or all unstable depending only on conditions on the simple front and back solution. The proof relies on a geometric reduction of the flow onto a two-dimensional invariant manifold containing the heteroclinic loop, see [Hom93], [San93] and [San95a]. The reduction allows for a smooth linearization of the vector field near both equilibria. The existence of N-fronts is then proved using Ljapunov-Schmidt reduction for the resulting vector field in $\mathbb{R}^2$ in the spirit of [Lin90] and [San93]. Finally, the critical eigenvalues of the operator (1.5) are calculated using [San95b].

Deng [Den91a] proved the existence of N-fronts bifurcating from a twisted heteroclinic loop under the additional assumption that the stable manifolds of the relatively contractive equilibria p_1 and p_2 are one-dimensional using topological methods, see [Den91a, section 7(a)]. Shashkov [Sha92] asserts the existence of N-fronts for two-dimensional vector fields of class C^3, however, without giving a proof.

Finally, we apply the stability result to the FitzHugh-Nagumo equation

$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \epsilon(u - \gamma w). \end{aligned}$$

Deng [Den91b] showed that the hypotheses of his existence result [Den91a] are satisfied, while Yanagida [Yan89] proved that the simple front and back are both linearly stable. Nii [Nii95b] proved linear stability of the 1-front provided f is linear near both equilibria. We show that in fact all N-fronts are linearly stable. Recently, Nii (private communication) has extended his result to N-fronts under the same restrictive hypothesis on f using topological methods.

The paper is organized as follows. In section 2, we state the basic assumptions and the main results about existence and stability of N-front solutions. The existence theorem is proved in section 3, the stability result in section 4. Finally, in section 5, the application to the FitzHugh-Nagumo system is given.

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2 Main results

Consider the equation

$$\dot{u} = f(u, \mu) \qquad (u, \mu) \in \mathbb{R}^n \times \mathbb{R}^2, \tag{2.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$ is C^2. We assume that equation (2.1) possesses two hyperbolic equilibria $p_1(\mu)$ and $p_2(\mu)$ for all μ. Moreover, the spectrum of the linearized vector field at these equilibria decomposes as follows.

(H1) We assume that $\dim W^s(p_1(0), 0) = \dim W^s(p_2(0), 0)$ and

$$\sigma(D_u f(p_k(\mu), \mu)) = \sigma_k^{ss} \cup \{-\alpha_k^s(\mu), \alpha_k^u(\mu)\} \cup \sigma_k^{uu}, \quad 0 < \alpha_k^s(\mu) < \alpha_k^u(\mu)$$

hold with $\operatorname{Re}\sigma_k^{ss} < -\alpha_k^s(\mu)$, $\operatorname{Re}\sigma_k^{uu} > \alpha_k^u(\mu)$ for $k = 1, 2$ and all μ. Moreover, $-\alpha_k^s(\mu)$ and $\alpha_k^u(\mu)$ are simple eigenvalues for $k = 1, 2$. We define $\alpha_k(\mu) = \alpha_k^u(\mu)/\alpha_k^s(\mu) > 1$.

We choose coordinates such that the equilibria do not depend on μ. Suppose that for $\mu = 0$ there exist two heteroclinic orbits $q_1(t)$ and $q_2(t)$ connecting p_1 to p_2 and vice versa, that is

(**H2**) The solution $q_1(t)$ fulfills $\lim_{t\to-\infty} q_1(t) = p_1$ and $\lim_{t\to\infty} q_1(t) = p_2$ while $q_2(t)$ satisfies $\lim_{t\to-\infty} q_2(t) = p_2$ and $\lim_{t\to\infty} q_2(t) = p_1$.

Owing to hypothesis (H1), the next assumption is satisfied for generic vector fields.

(**H3**) The heteroclinic solutions $q_1(t)$ and $q_2(t)$ are non-degenerate, that is

$$\begin{aligned} T_{q_1(0)}W^u(p_1,0) \cap T_{q_1(0)}W^s(p_2,0) &= \mathbb{R}\dot{q}_1(0) \\ T_{q_2(0)}W^u(p_2,0) \cap T_{q_2(0)}W^s(p_1,0) &= \mathbb{R}\dot{q}_2(0) \end{aligned}$$

hold.

Due to (H3), there exist two unique (up to constant multiples) bounded solutions $\psi_k(t)$ of the adjoint variational equation

$$\dot{w} = -D_u f(q_k(t), 0)^* \, w$$

evaluated at $q_k(t)$ for $k = 1, 2$, respectively. As a matter of fact, they satisfy

$$\psi_k(t) \perp \Big(T_{q_k(t)}W^u(p_k, 0) + T_{q_k(t)}W^s(p_{k+1}, 0) \Big). \tag{2.2}$$

Upon changing the parameter μ, the heteroclinic solutions $q_k(t)$ should break up. That is made precise in the next hypothesis.

(**H4**) The Melnikov integrals

$$N_k := \int_{-\infty}^{\infty} \langle \psi_k(t), D_\mu f(q_k(t), 0) \rangle \, dt \in \mathbb{R}^2 \qquad\qquad k = 1, 2$$

are linearly independent (and in particular non-zero).

We need to assume that $q_k(t)$ and $\psi_k(t)$ converge along the leading directions to the equilibria and zero, respectively.

(**H5**) Assume that the limits

$$\begin{aligned} \lim_{t\to-\infty} e^{-\alpha_k^u t}\dot{q}_k(t) &=: v_k^- & \qquad \lim_{t\to\infty} e^{\alpha_{k+1}^s t}\dot{q}_k(t) &=: v_{k+1}^+ \\ \lim_{t\to-\infty} e^{-\alpha_k^s t}\psi_k(t) &=: w_k^+ & \qquad \lim_{t\to\infty} e^{\alpha_{k+1}^u t}\psi_k(t) &=: w_{k+1}^- \end{aligned}$$

are non-zero for $k = 1, 2$, see Figure 2.

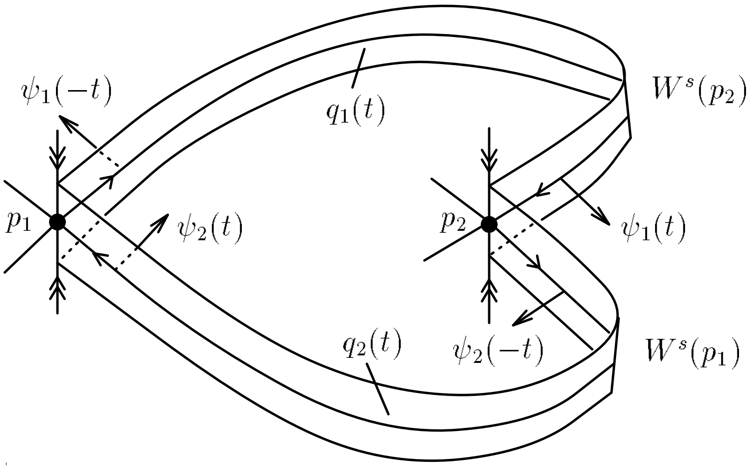

Figure 2: A twisted heteroclinic loop.

Then $v_k^{\pm}$ and $w_k^{\pm}$ are right and left eigenvectors of $D_u f(p_k, 0)$ for the eigenvalues $\alpha_k^{s,u}$. Due to (2.2), hypothesis (H5) is equivalent to the strong inclination property. Finally, we suppose that both heteroclinic orbits are twisted.

(H6) Suppose that the scalar products $\langle w_k^-, v_k^- \rangle > 0$ and $\langle w_k^+, v_k^+ \rangle > 0$ are positive for $k = 1, 2$, see Figure 2. Note that the scalar products do not vanish according to hypotheses (H1) and (H5).

Choose two sections Σ_k transverse to the vector field and placed at $q_k(0)$ for $k = 1, 2$. We call the heteroclinic solutions $q_1(t)$ and $q_2(t)$ simple fronts and backs, respectively. An N-front solution is a heteroclinic orbit connecting p_1 to p_2 and intersecting Σ_2 N-times, see Figure 3. In other words, it follows the heteroclinic loop $N+\frac{1}{2}$-times and hits the set $\Sigma_1 \cup \Sigma_2$ $2N+1$-times. Similarly, an N-back is defined connecting p_2 to p_1.

The first result is an extension of the existence theorem proved by Deng [Den91a].

Theorem 1 *Assume that (H1) – (H6) are satisfied. Then for each $N > 1$ there exists a unique curve $\mu_N(r)$ for $r \in [0, r_0)$ in parameter space such that $\mu_N(0) = 0$ and (2.1) possesses an N-front precisely for $\mu = \mu_N(r)$ for some r. The N-fronts are unique and the curve μ_N is of class C^1. See Figure 4 for the bifurcation diagram.*

Assume that $\mu_1 = 0$ and $\mu_2 = 0$ correspond to the existence of a simple front or back, respectively. Then the return times of the N-fronts with respect to the Poincaré sections

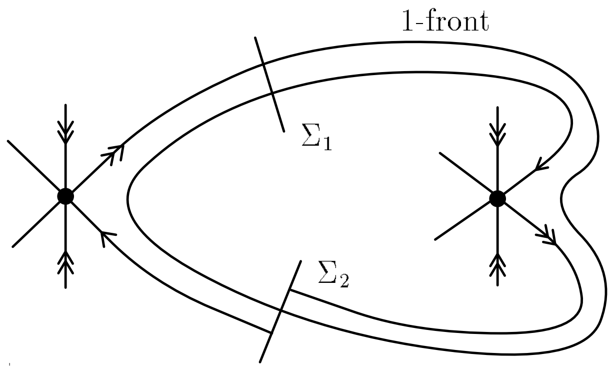

Figure 3: N-Front solutions.

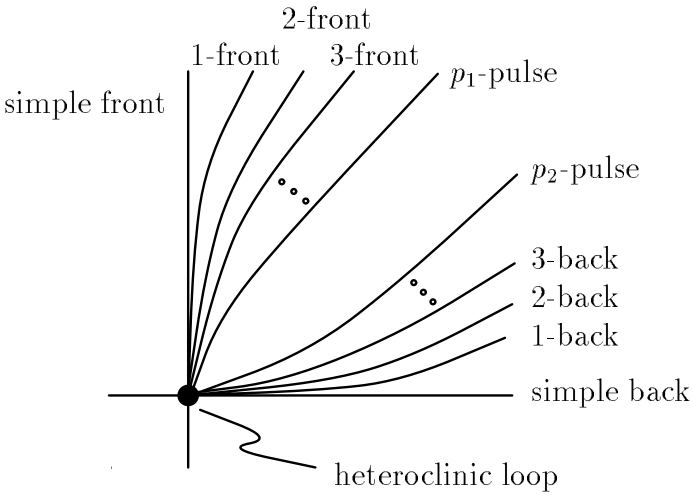

Figure 4: The bifurcation diagram.

Σ_1 and Σ_2 are given by

$$
(2.3) \qquad \begin{array}{lcll}
T_{2l+1} & = & -\frac{\alpha_2+\gamma_{l+1}}{\alpha_1^s}\ln r & \textit{time spent near } p_1 \\
T_{2l} & = & -\frac{1}{\alpha_2^s}\ln r & \textit{time spent near } p_2 \\
\mu_N & = & (r^{\alpha_2}(1+o(1)), r), &
\end{array}
$$

for $l = 0, ..., N-1$ as $r \to 0$, where the sequence γ_l is defined recursively by $\gamma_N = 0$, $\gamma_{N-1} = \alpha_1\alpha_2 - 1 > 0$ and $\gamma_{l-1} := \alpha_1\gamma_l + \gamma_{N-1} > \gamma_l$, see (3.20). Note that $\gamma_1 \to \infty$ as $N \to \infty$. Analogous results hold for N-backs.

Next we describe the bounded solutions $v \in C^1(\mathbb{R}, \mathbb{R}^n)$ of the equation

$$
(2.4) \qquad \dot{v} = \Big(D_u f(q_N(r)(t), \mu_N(r)) + \lambda B(t)\Big)v
$$

for $\lambda \in U_\delta(0) \subset \mathbb{C}$, where $q_N(r)$ denotes the N-front existing for $\mu = \mu_N(r)$. Here, B is a bounded, continuous and matrix-valued function. Equation (2.4) is a generalized eigenvalue problem of the form

$$
Lv = \lambda Bv.
$$

Generalized eigenfunctions of (2.4) corresponding to an eigenvalue λ are functions v_i satisfying

$$
Lv_i = \lambda Bv_i + Bv_{i-1}
$$

with $v_0 = 0$. The algebraic multiplicity of eigenvalues can be defined in the usual way. We assume a non-degeneracy assumption with respect to λ.

(**H7**) Suppose that the Melnikov integrals

$$
M_k := \int_{-\infty}^{\infty} \langle \psi_k(t), B(t)\,\dot{q}_k(t)\rangle \, dt \neq 0
$$

are non-zero for $k = 1, 2$, where ψ_k is chosen according to hypothesis (H6).

The next theorem – which is the main result of the present paper – describes the set of $\lambda \in U_\delta(0) \subset \mathbb{C}$ for $\delta > 0$ small for which (2.4) possesses a bounded solution v.

Theorem 2 *Suppose that the assumptions (H1) – (H7) are satisfied. Then there exists a $\delta > 0$ independent of N such that the following holds. For any $N > 1$ and $r_0 = r_0(N) > 0$ sufficiently small there exist precisely $2N+1$ solutions $(\lambda_j, v_j) \in \mathbb{C} \times C^1(\mathbb{R}, \mathbb{R}^n)$ of (2.4) with $|\lambda| < \delta$. The eigenvalues are counted with multiplicity and are given by*

$$
\begin{array}{lcl}
\lambda_{2l-1} & = & (c_{2l-1} + o(1))\, r \\
\lambda_{2l} & = & (c_{2l} + o(1))\, r^{\alpha_2+\gamma_l} \\
\lambda_{2N+1} & = & 0,
\end{array}
$$

for $l = 1, ..., N$ as $r \to 0$, where the exponents γ_l have been defined in Theorem 1.

The constants c_j are non-zero and fulfill $\operatorname{sign} c_{2l} = \operatorname{sign} M_1$ and $\operatorname{sign} c_{2l+1} = \operatorname{sign} M_2$. In particular, the eigenvalues λ_j are contained in the left half plane for $j = 1, ..., 2N$ provided $M_1, M_2 < 0$ are negative. Analogous results hold for N-backs.

The second theorem establishes stability of the N-front solutions with respect to the underlying partial differential equation, see section 5 for an example.

Notice that there exist precisely two pulses corresponding to p_1 and p_2, see Figure 4. The existence proof is implicitly contained in section 3.3. As far as their stability is concerned, the same statement as for the N-fronts holds. This follows from [Nii95a] or section 4 of the present article.

3 Existence

In order to prove existence of N-fronts, a geometric reduction onto a two-dimensional invariant manifold in phase space is employed. The manifold is diffeomorphic to an annulus. Next, a system of $2N+1$ equations is derived using Ljapunov-Schmidt reduction applied to the flow on the invariant manifold. In the final subsection, this system is being solved for using an implicit function theorem.

Throughout we assume that hypotheses (H1) to (H6) are fulfilled.

3.1 Center-manifold reduction

We have the following lemma.

Lemma 3.1 *There exists a two-dimensional, locally invariant and normally hyperbolic manifold $W^c_{hom} \subset \mathbb{R}^n$ of class $C^{1,\rho}$ jointly in (u, μ) for some $\rho > 0$. All solutions staying near the heteroclinic loop for all times and for parameter values close to zero are contained in W^c_{hom}. The manifold is homeomorphic to an annulus.*

Moreover, the flow restricted to W^c_{hom} is C^1-conjugated to the flow of an appropriate vector field $g(u, \mu)$ of class C^1 defined on $\mathbb{R}^2$. The hypotheses (H1) to (H6) are still satisfied for g and, in addition, g is linear locally near both equilibria.

Proof. The existence of W^c_{hom} is an application of [San95a, Theorem 1]. We shall verify the assumptions of that theorem using the decomposition

$$\sigma(D_u f(p_k, 0)) = \sigma_k^{ss} \cup \sigma_k^c \cup \sigma_k^{uu} \qquad \sigma_k^c = \{-\alpha_k^s, \alpha_k^u\}.$$

Then [San95a, (H1), $(\widetilde{\text{H3}})$] are satisfied due to (H1) and (H5), while [San95a, (H4)] is void. It remains to verify [San95a, $(\widetilde{\text{H2}})$] which reads

$$\begin{aligned} T_{q_1(0)}W^{uu}(p_1) \oplus T_{q_1(0)}W^{u,s,ss}(p_2) &= \mathbb{R}^n \\ T_{q_1(0)}W^{s,u,uu}(p_1) \oplus T_{q_1(0)}W^{ss}(p_2) &= \mathbb{R}^n \end{aligned}$$

and the analogous condition for $q_2(t)$. Here, $W^{u,s,ss}(p_2)$ denotes an invariant manifold tangent to the generalized eigenspace $E^{u,s,ss}$ associated with $\sigma_2^{ss} \cup \sigma_2^c$ at p_2 and similarly for $W^{s,u,uu}(p_1)$. Owing to (H1), it suffices to prove that

$$\begin{aligned} T_{q_1(0)}W^{uu}(p_1) \cap T_{q_1(0)}W^{u,s,ss}(p_2) &= \{0\} \\ T_{q_1(0)}W^{s,u,uu}(p_1) \cap T_{q_1(0)}W^{ss}(p_2) &= \{0\}. \end{aligned} \tag{3.1}$$

We have

$$T_{q_1(0)}W^{u,s,ss}(p_2) = T_{q_1(0)}W^{s}(p_2) \oplus \mathbb{R}v^u$$

for some non-zero v^u. Because of (H3) and (H5), the intersection

$$T_{q_1(0)}W^{uu}(p_1) \cap T_{q_1(0)}W^{s}(p_2) = \{0\}$$

is trivial. Therefore, if (3.1)(i) does not hold, there exists a vector $w \in T_{q_1(0)}W^s(p_2)$ such that

$$v^u + w \in T_{q_1(0)}W^{uu}(p_1) \cap T_{q_1(0)}W^{u,s,ss}(p_2).$$

Choose $q_1(0)$ close to p_2, whence $T_{q_1(0)}W^{u,s,ss}(p_2)$ is close to $E^{u,s,ss}$. Then, due to (H5), $\langle \psi_1(0), v^u \rangle \neq 0$. However, the solution $v^u(t) + w(t) \in T_{q_1(t)}W^{uu}(p_1)$ decays exponentially to zero for $t \to -\infty$, while

$$\langle \psi_1(t), v^u(t) + w(t) \rangle \overset{(2.2)}{=} \langle \psi_1(t), v^u(t) \rangle = \langle \psi_1(0), v^u(0) \rangle \neq 0$$

is independent of t as $\psi_1(t)$ solves the adjoint equation. This is a contradiction to $\psi_1(t)$ being bounded, whence

$$T_{q_1(0)}W^{uu}(p_1) \cap T_{q_1(0)}W^{u,s,ss}(p_2) = \{0\}.$$

The argument for the other equation (3.1)(ii) is similar. Thus we can apply [San95a, Theorem 1] to conclude the existence of an invariant manifold W^c_{hom}. Moreover, W^c_{hom} is homeomorphic to an annulus owing to (H6). That the flow on W^c_{hom} is C^1-conjugated to the flow of a C^1-vector field in $\mathbb{R}^2$ follows from [San95a, Section 3.5]. The statement about the smooth linearization is proved in [Hom93]. □

Hence we can restrict the analysis to a C^1-vector field g in $\mathbb{R}^2$ fulfilling $(\widetilde{\text{H1}})$, (H2) to (H6), and being linear locally near both equilibria, where hypothesis $(\widetilde{\text{H1}})$ is given by

($\widetilde{\mathbf{H1}}$) We assume that $\dim W^s(p_1, 0) = \dim W^s(p_2, 0) = 1$ and

$$\sigma(D_u f(p_k(\mu), \mu)) = \{-\alpha_k^s(\mu), \alpha_k^u(\mu)\} \qquad 0 < \alpha_k^s(\mu) < \alpha_k^u(\mu)$$

hold for $k = 1, 2$. We define $\alpha_k(\mu) = \alpha_k^u(\mu)/\alpha_k^s(\mu) > 1$.

3.2 Lin's method in $\mathbb{R}^2$

According to the last subsection, it suffices to consider a vector field

$$\dot{u} = g(u, \mu) \qquad (u, \mu) \in \mathbb{R}^2 \times \mathbb{R}^2, \tag{3.2}$$

with $g \in C^1$ such that ($\widetilde{\mathrm{H1}}$) and (H2) up to (H6) are satisfied and the flow near the equilibria p_k for $k = 1, 2$ is linear. Choose Poincaré sections Σ_k and $\tilde{\Sigma}_k$ for $k = 1, 2$ as in Figure 5. All sections are chosen inside the regions near the equilibria p_k where the flow is linear. Moreover, we shall identify the one-dimensional sections with intervals in $\mathbb{R}$ as shown in Figure 5. Next, we compute various Poincaré maps. The map from Σ_1 to Σ_2 is given by

$$\begin{array}{ccc} \Sigma_1 & \to & \Sigma_2 \\ e^{-\alpha_2^u(\mu)T} & \mapsto & e^{-\alpha_2^s(\mu)T} \end{array} \tag{3.3}$$

using that the vector field is linear. Similarly, the map from $\tilde{\Sigma}_2$ to $\tilde{\Sigma}_1$ equals

$$\begin{array}{ccc} \tilde{\Sigma}_2 & \to & \tilde{\Sigma}_1 \\ e^{-\alpha_1^u(\mu)\tau} & \mapsto & e^{-\alpha_1^s(\mu)\tau}. \end{array} \tag{3.4}$$

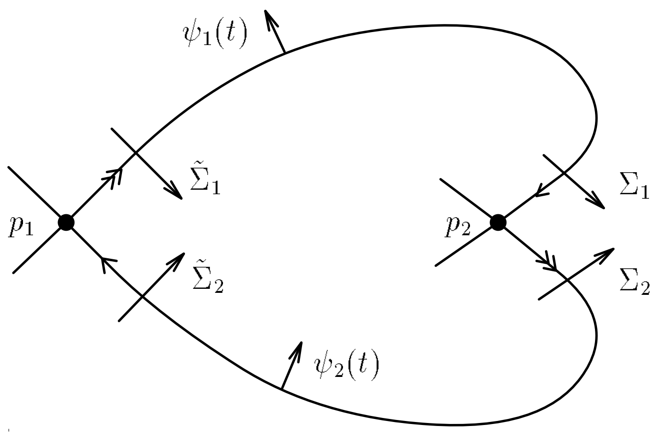

Figure 5: The choice of the sections in $\mathbb{R}^2$. The arrows denote the positive direction once sections are identified with intervals in $\mathbb{R}$.

The maps

$$\tilde{\Pi}_k(u,\mu) \; : \; \begin{array}{ccc} \tilde{\Sigma}_k & \to & \Sigma_k \\ u & \mapsto & -\Pi_k(u,\mu) - d_k(\mu) \end{array} \tag{3.5}$$

are diffeomorphisms with $\Pi_k(u,\mu) \in C^1$, $\Pi_k(0,\mu) = 0$ and $D_u\Pi_k(0,\mu) > 0$ for $k = 1,2$. The sign appearing in (3.5) is a consequence of hypothesis (H6), see Figure 5. Owing to hypothesis (H4), we may assume that $d_k(\mu) = \mu_k$ by a C^1-transformation of parameters. Indeed, $d_k(\mu)$ is the separation function measuring the distance of the one-dimensional stable and unstable manifolds of the equilibria at the section Σ_k. The integrals N_k appearing in (H4) are in fact the derivatives of $d_k(\mu)$ at $\mu = 0$ up to sign.

Summarizing the above, we obtain a map

$$\begin{array}{ccc} \Sigma_2 & \to & \Sigma_1 \\ -\Pi_2(e^{-\alpha_1^u(\mu)\tau},\mu) - \mu_2 & \mapsto & -\Pi_1(e^{-\alpha_1^s(\mu)\tau},\mu) - \mu_1. \end{array} \tag{3.6}$$

All solutions being mapped from Σ_2 to Σ_1 are captured by the above parametrization. The next step consists in formulating the Poincaré map by means of the return time with respect to the sections Σ_k instead of the one for $\tilde{\Sigma}_k$.

The times needed for initial points $u \in \tilde{\Sigma}_k$ to reach the sections Σ_k are given by functions $\Omega_k(u,\mu)$. Both functions $\Omega_k(u,\mu)$ are in C^1 and bounded uniformly in u. Thus the time T needed for the initial point

$$-\Pi_2(e^{-\alpha_1^u(\mu)\tau},\mu) - \mu_2 \in \Sigma_2$$

to reach

$$-\Pi_1(e^{-\alpha_1^s(\mu)\tau},\mu) - \mu_1 \in \Sigma_1$$

is given by

$$T = \tau + \Omega_1(e^{-\alpha_1^s(\mu)\tau},\mu) + \Omega_2(e^{-\alpha_1^u(\mu)\tau},\mu).$$

By the implicit function theorem, we can solve this equation with respect to T yielding a C^1-function $\tau(T,\mu)$, whence

$$\tau(T,\mu) = T - \Omega_1(e^{-\alpha_1^s(\mu)\,\tau(T,\mu)},\mu) - \Omega_2(e^{-\alpha_1^u(\mu)\,\tau(T,\mu)},\mu). \tag{3.7}$$

Therefore, we obtain the following lemma.

Lemma 3.2 *The Poincaré maps from Σ_1 to Σ_2 and vice versa are given by*

$$\begin{array}{ccc} \Sigma_1 & \to & \Sigma_2 \\ e^{-\alpha_2^u(\mu)T} & \mapsto & e^{-\alpha_2^s(\mu)T} \end{array} \tag{3.8}$$

and

$$
(3.9)\qquad \begin{array}{ccc} \Sigma_2 & \to & \Sigma_1 \\ -\Pi_2(e^{-\alpha_1^u(\mu)\,\tau(T,\mu)},\mu)-\mu_2 & \mapsto & -\Pi_1(e^{-\alpha_1^s(\mu)\,\tau(T,\mu)},\mu)-\mu_1, \end{array}
$$

respectively. The C^1-function $\tau(T,\mu)$ defined in (3.7) satisfies

$$
\left|\frac{d}{dT}\tau(T,\mu)-1\right| \ll 1
$$

and the maps $\Omega_k(u,\mu)$ are bounded uniformly in u. Moreover, $\Pi_k(u,\mu)\in C^1$, $\Pi_k(0,\mu)=0$ and $D_u\Pi_k(0,\mu)>0$ for $k=1,2$.

Up to this point, the construction looks pretty much like using Shilnikov variables. However, in order to describe solutions following the original heteroclinic loop several times, we shall adopt a boundary-value-point-of-view. That is, we are not going to iterate the Poincaré maps given in the previous lemma, but shall derive matching conditions in the sections.

Using Lemma 3.2, the existence of N-front solutions is equivalent to the existence of return times $T_j<\infty$ for $j=0,...,2N-1$ and parameter values μ such that

$$
(3.10)\qquad \begin{array}{rcll} e^{-\alpha_2^u(\mu)T_0} & = & -\mu_1 & \\ e^{-\alpha_2^s(\mu)T_{2j}} & = & -\Pi_2(e^{-\alpha_1^u(\mu)\,\tau(T_{2j+1},\mu)},\mu)-\mu_2 & j=0,...,N-1 \\ e^{-\alpha_2^u(\mu)T_{2j}} & = & -\Pi_1(e^{-\alpha_1^s(\mu)\,\tau(T_{2j-1},\mu)},\mu)-\mu_1 & j=1,...,N-1 \\ 0 & = & -\Pi_1(e^{-\alpha_1^s(\mu)\,\tau(T_{2N-1},\mu)},\mu)-\mu_1 & \end{array}
$$

holds. Indeed, then the various pieces of solutions defined in between the sections will fit together. Moreover, the first and last equation assert that the solution is contained in the unstable and stable manifolds of the equilibria p_1 and p_2, respectively. In fact, T_{2j+1} and T_{2j} are the times spent near the equilibria p_1 and p_2, respectively. Define

$$
(3.11)\qquad \begin{array}{rclrcl} a_{2j+1}\,s & = & e^{-\alpha_1^s(\mu)\,\tau(T_{2j+1},\mu)} & s & = & e^{-\alpha_1^s(\mu)\,\tau(T_{2N-1},\mu)} \\ a_{2j}\,r & = & e^{-\alpha_2^s(\mu)T_{2j}} & r & = & e^{-\alpha_2^s(\mu)T_0} \end{array}
$$

for $j=0,...,N-1$ such that $a_0=a_{2N-1}=1$ and $a_1,...,a_{2N-2}$ are bounded. In the new variables a_j, r and s, equation (3.10) reads

$$
(3.12)\qquad \begin{array}{rcll} \mu_1+r^{\alpha_2(\mu)} & = & 0 & \\ r+\mu_2+\Pi_2((a_1s)^{\alpha_1(\mu)},\mu) & = & 0 & \\ \Pi_1(a_{2j-1}s,\mu)+\mu_1+(a_{2j}r)^{\alpha_2(\mu)} & = & 0 & j=1,...,N-1 \\ a_{2j}r+\mu_2+\Pi_2((a_{2j+1}s)^{\alpha_1(\mu)},\mu) & = & 0 & j=1,...,N-1 \\ \mu_1+\Pi_1(s,\mu) & = & 0 & \end{array}
$$

with $\alpha_k(\mu) = \alpha_k^u(\mu)/\alpha_k^s(\mu) > 1$. Whenever (a_j, r, s) solve (3.12) such that $a_j > 0$ and $r, s > 0$, we obtain associated return times $T_j < \infty$ which solve (3.10) by using (3.11). Indeed, we have

$$\begin{aligned} \tau(T_{2j+1}, \mu) &= -\tfrac{1}{\alpha_1^s(\mu)} \ln(a_{2j+1} s) \\ T_{2j} &= -\tfrac{1}{\alpha_2^s(\mu)} \ln(a_{2j} r) \end{aligned} \tag{3.13}$$

and Lemma 3.2 implies that $\tau(T, \mu)$ is invertible with respect to T. Hence, it suffices to consider (3.12) keeping in mind that only positive solutions of this system correspond to solutions of the original problem.

3.3 Existence of N-fronts bifurcating from a twisted heteroclinic cycle

We shall solve (3.12). Note that the functions Π_1 and Π_2 are in C^1. By convention, for $\alpha > 1$, define x^α to be zero for negative values of x yielding a C^1-function, too. Then (3.12) is defined for all a_j bounded and r, s small including negative values. Throughout this subsection, the range of the index j is $j = 1, ..., N-1$.

First, solve

$$\begin{aligned} \mu_1 + r^{\alpha_2(\mu)} &= 0 \\ r^{\alpha_2(\mu)} - \Pi_1(s, \mu) &= 0 \end{aligned} \tag{3.14}$$

with respect to (μ_1, s) near $(r, s, \mu) = 0$ by the implicit function theorem using Lemma 3.2. Denote the solutions by $\mu_1(\mu_2, r)$ and $s(\mu_2, r)$ both of which are of class C^1. Observe that, owing to $\Pi_1(0, \mu) = 0$, the estimates

$$|s(\mu_2, r)|, |D_{\mu_2} s(\mu_2, r)| \leq C_\delta\, r^{\alpha_2 - \delta} \tag{3.15}$$

hold for arbitrary small positive δ. Using the ansatz $\mu_2 = \epsilon r$, the second equation in (3.12) reads

$$r + \mu_2 + \Pi_2((a_1 s)^{\alpha_1(\mu)}, \mu) = r + \epsilon r + \Pi_2\Big((a_1 s(\epsilon r, r))^{\alpha_1(\epsilon r, r)}, \mu_1(\epsilon r, r), \epsilon r\Big) = 0. \tag{3.16}$$

Here and in the following, we will be a bit sloppy concerning the dependence of $\alpha_k(\mu)$ and Π_k on ϵ and r to avoid unnecessary complicated notation. Dividing (3.16) by r yields

$$1 + \epsilon + r^{-1}\Pi_2\Big((a_1 s(\epsilon r, r))^{\alpha_1(\epsilon r, r)}, \mu_1(\epsilon r, r), \epsilon r\Big) = 0, \tag{3.17}$$

which is C^1 in (ϵ, a_1) for $r \geq 0$ owing to (3.15) and because of the fact that the dependence on ϵ is due to $\mu_2 = \epsilon r$. Using (3.15), we can solve (3.17) with respect to ϵ near $\epsilon = -1$, $r = 0$ and arbitrary bounded a_1 yielding a C^1-function

$$\epsilon = \epsilon(a_1, r) = -1 - r^{-1}\Pi_2\Big((a_1 \tilde{s}(a_1, r))^{\tilde{\alpha}_1(a_1, r)}, \tilde{\mu}_1(a_1, r), \epsilon(a_1, r) r\Big), \tag{3.18}$$

where

$$\begin{aligned}\tilde{s}(a_1,r) &= s(\epsilon(a_1,r)r,r)\\ \tilde{\alpha}_k(a_1,r) &= \alpha_k(\epsilon(a_1,r)r,r)\\ \tilde{\mu}_1(a_1,r) &= \mu_1(\epsilon(a_1,r)r,r).\end{aligned}$$

Notice that the dependence of all these functions on a_1 is due to terms of the form $\epsilon(a_1,r)r$. It remains to solve the system

$$\begin{aligned}\Pi_1\Big(a_{2j-1}\tilde{s}(a_1,r),\tilde{\mu}(a_1,r)\Big)+\tilde{\mu}_1(a_1,r)+(a_{2j}r)^{\tilde{\alpha}_2(a_1,r)} &= 0\\ a_{2j}r+\epsilon(a_1,r)r+\Pi_2\Big((a_{2j+1}\tilde{s}(a_1,r))^{\tilde{\alpha}_1(a_1,r)},\tilde{\mu}(a_1,r)\Big) &= 0\end{aligned}$$

for $j=1,...,N-1$. Dividing by $r^{\tilde{\alpha}_2(a_1,r)}$ and r, respectively, yields

$$\begin{aligned}r^{-\tilde{\alpha}_2(a_1,r)}\,\Pi_1\Big(a_{2j-1}\tilde{s}(a_1,r),\tilde{\mu}(a_1,r)\Big)-1+a_{2j}^{\tilde{\alpha}_2(a_1,r)} &= 0\\ a_{2j}+\epsilon(a_1,r)+r^{-1}\Pi_2\Big((a_{2j+1}\tilde{s}(a_1,r))^{\tilde{\alpha}_1(a_1,r)},\tilde{\mu}(a_1,r)\Big) &= 0.\end{aligned}\tag{3.19}$$

The functions

$$\begin{gathered}r^{-\tilde{\alpha}_2(a_1,r)}\,\Pi_1\Big(a_{2j-1}\tilde{s}(a_1,r),\tilde{\mu}(a_1,r)\Big)\\ r^{-1}\Pi_2\Big((a_{2j-1}\tilde{s}(a_1,r))^{\tilde{\alpha}_1(a_1,r)},\tilde{\mu}(a_1,r)\Big)\end{gathered}$$

are C^1 in (a_{2j-1},a_1) up to $r=0$ owing to (3.14) and the above comment about the dependence on a_1. Moreover, the derivative with respect to a_{2j-1} at $r=0$ equals one for the first and zero for the second function. Therefore, $a_{2j}=1$ and $a_{2j-1}=0$ for $j=1,...,N-1$ solve (3.19) with $r=0$ and we can use the implicit function theorem to obtain solutions $a_{2j}(r)$ and $a_{2j-1}(r)$ for positive r.

It remains to show that $a_{2j-1}(r)>0$ is positive for $r>0$. Define constants γ_j recursively by

$$\begin{aligned}\gamma_N &:= 0\\ \gamma_{N-1} &:= \alpha_1\alpha_2-1>0\\ \gamma_{j-1} &:= \alpha_1\gamma_j+\gamma_{N-1}>\gamma_j\end{aligned}\tag{3.20}$$

and set

$$\begin{aligned}a_{2j-1} &= b_{2j-1}\,r^{\gamma_j}\\ a_{2j} &= 1-b_{2j}\,r^{\gamma_j}\end{aligned}\tag{3.21}$$

for $j=1,...,N-1$. Let $b_{2N-1}=1$. Substituting these expressions together with (3.18) into equation (3.19) yields

$$\begin{aligned}0 &= r^{-\hat{\alpha}_2(b_1,r)}\,\Pi_1\Big(b_{2j-1}r^{\gamma_j}\hat{s}(b_1,r),\hat{\mu}(b_1,r)\Big)-1+(1-b_{2j}r^{\gamma_j})^{\hat{\alpha}_2(b_1,r)}\\ 0 &= b_{2j}r^{\gamma_j}+r^{-1}\Big(\Pi_2\Big((b_1r^{\gamma_1}\hat{s}(b_1,r))^{\hat{\alpha}_1(b_1,r)},\hat{\mu}(b_1,r)\Big)-\\ &\quad -\Pi_2\Big((b_{2j+1}r^{\gamma_{j+1}}\hat{s}(b_1,r))^{\hat{\alpha}_1(b_1,r)},\hat{\mu}(b_1,r)\Big)\Big),\end{aligned}$$

where

$$
(3.22)\qquad \begin{array}{rcl}
\hat{s}(b_1,r) &=& s(\epsilon(b_1 r^{\gamma_1},r)r,r)\\
\hat{\alpha}_k(b_1,r) &=& \alpha_k(\epsilon(b_1 r^{\gamma_1},r)r,r)\\
\hat{\mu}_1(b_1,r) &=& \mu_1(\epsilon(b_1 r^{\gamma_1},r)r,r)\\
\hat{\mu}_2(b_1,r) &=& \epsilon(b_1 r^{\gamma_1},r)r.
\end{array}
$$

Dividing these equations by r^{γ_j} reads

$$
(3.23)\qquad \begin{array}{rcl}
0 &=& r^{-(\hat{\alpha}_2(b_1,r)+\gamma_j)}\,\Pi_1\Big(b_{2j-1}r^{\gamma_j}\,\hat{s}(b_1,r),\hat{\mu}(b_1,r)\Big)+r^{-\gamma_j}\Big((1-b_{2j}r^{\gamma_j})^{\hat{\alpha}_2(b_1,r)}-1\Big)\\
0 &=& b_{2j}+r^{-(1+\gamma_j)}\Big(\Pi_2\Big((b_1 r^{\gamma_1}\,\hat{s}(b_1,r))^{\hat{\alpha}_1(b_1,r)},\hat{\mu}(b_1,r)\Big)-\\
&& -\Pi_2\Big((b_{2j+1}r^{\gamma_{j+1}}\,\hat{s}(b_1,r))^{\hat{\alpha}_1(b_1,r)},\hat{\mu}(b_1,r)\Big)\Big).
\end{array}
$$

As before, using the recursive relations (3.20), it is tedious but straightforward to see that the functions appearing in (3.23) are C^1 up to $r=0$. Moreover, for $r=0$, (3.23) boils down to

$$
(3.24)\qquad \begin{array}{rcll}
b_{2i-1}-\alpha_2\, b_{2i} &=& 0 & i=1,\ldots,N-1\\
b_{2i}-D_u\Pi_2(0,0)\,D_u\Pi_1(0,0)^{-\alpha_1}\,b_{2i+1}^{\alpha_1} &=& 0 & i=1,\ldots,N-2\\
b_{2N-2}-D_u\Pi_2(0,0)\,D_u\Pi_1(0,0)^{-\alpha_1} &=& 0 &
\end{array}
$$

owing to (3.14). It is straightforward to check that the Jacobian of (3.24) with respect to (b_j) is upper-triangular with non-zero diagonal elements. Equation (3.23) can therefore be solved near

$$
(3.25)\qquad \begin{array}{rcll}
b_{2N-2} &=& D_u\Pi_2(0,0)\,D_u\Pi_1(0,0)^{-\alpha_1} & \\
b_{2i-1} &=& \alpha_2\, b_{2i} & i=1,\ldots,N-1\\
b_{2i-2} &=& b_{2N-2}\, b_{2i-1}^{\alpha_1} & i=2,\ldots,N-1
\end{array}
$$

by invoking an implicit function theorem. This proves that

$$
(3.26)\qquad \begin{array}{rcl}
a_{2j-1} &=& (b_{2j-1}+\mathrm{o}(1))\,r^{\gamma_j}\\
a_{2j} &=& 1-(b_{2j}+\mathrm{o}(1))\,r^{\gamma_j}
\end{array}
$$

holds for $j=1,\ldots,N-1$. In particular, $a_{2j-1}(r)>0$ is positive for $r>0$ thanks to (3.25) and Lemma 3.2.

The expansion (2.3) of the return times is now an easy consequence of (3.13) and (3.26). Moreover, the claim about the ordering of the bifurcation curves in Figure 4 follows from (3.16) and (3.14).

Hence the proof of Theorem 1 is complete. $\square$

4 Stability

This section is devoted to the proof of Theorem 2. The basic technique used is Lin's method applied to the eigenvalue problem (2.4). We shall use the abstract results from [San95b] together with certain modifications needed in the present situation. As for the concrete bifurcation investigated here, we are again going to exploit the reduction to a two-dimensional invariant manifold. Finally, the eigenvalues of the resulting tridiagonal matrix are calculated.

Throughout we suppose that hypotheses (H1) to (H7) are fulfilled.

Convention. Throughout this section, we use the convention that the ranges of the indices i and j are $i = 1, ..., 2N+1$ and $j = 1, ..., 2N$ as long as stated otherwise. Moreover, we define $i \bmod 2 \in \{1, 2\}$ by convention. The Landau symbol $o(1)$ is taken with respect to $r \to 0$.

4.1 Abstract reduction of the eigenvalue problem

We consider equation (2.1) and (2.4) in $\mathbb{R}^n$ keeping in mind that the N-fronts are actually contained in the invariant C^1-manifold W^c_{hom}. We also extend the sections Σ_k for $k = 1, 2$ to sections in $\mathbb{R}^n$ without changing notation.

Any solution with initial point in Σ_k and end point in Σ_{k+1} is uniquely described by the associated return time T. In particular, any N-front $q_N(t)$ is determined by $2N$ return times T_j for $j = 0, ..., 2N-1$, see Theorem 1 and the proof in the last section. Define $u_i^{\pm}(t)$ by

$$q_N\Big(t + \sum_{j=0}^{i-2} T_j\Big) = \begin{cases} u_i^-(t) & \text{for } t \in [-\frac{1}{2}T_{i-2}, 0] \\ u_i^+(t) & \text{for } t \in [0, \frac{1}{2}T_{i-1}] \end{cases} \tag{4.1}$$

for $i = 1, ..., 2N+1$ and with $T_{-1} = T_{2N} = \infty$, see Figure 6. As $q_N(t)$ is a solution of (2.1), the functions $u_i^{\pm}$ fulfill

$$\begin{aligned} u_i^+(0) &= u_i^-(0) & i &= 1, ..., 2N+1 \\ u_j^+(\tfrac{1}{2}T_{j-1}) &= u_{j+1}^-(-\tfrac{1}{2}T_{j-1}) & j &= 1, ..., 2N. \end{aligned} \tag{4.2}$$

The eigenvalue problem (2.4)

$$\dot v = \Big(D_u f(q_N(t), \mu_N) + \lambda B(t)\Big) v \qquad t \in \mathbb{R}$$

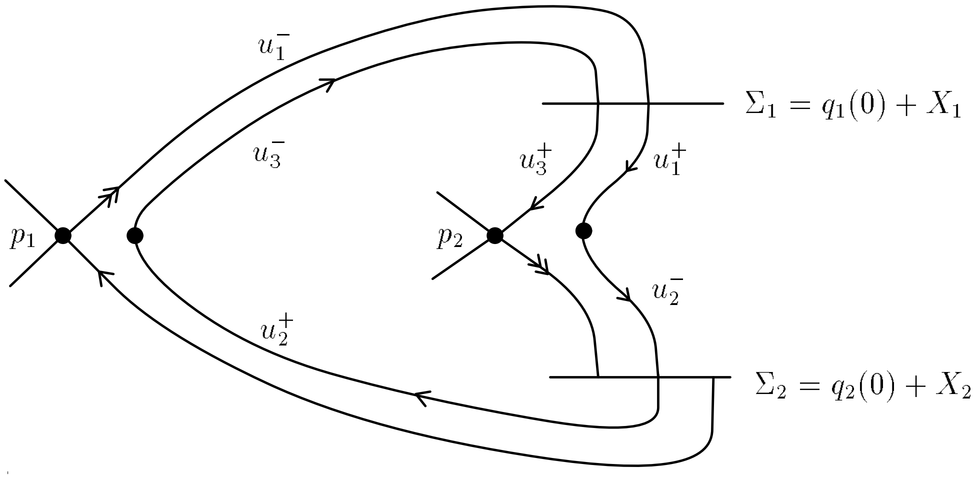

Figure 6: Description of N-Front solutions.

can be written as

$$
(4.3)\quad
\begin{aligned}
\dot{v}_i^- &= (D_u f(u_i^-(t), \mu_N) + \lambda B(t))\, v_i^- && \text{for } t \in (-\tfrac{1}{2}T_{i-2}, 0) \\
\dot{v}_i^+ &= (D_u f(u_i^+(t), \mu_N) + \lambda B(t))\, v_i^+ && \text{for } t \in (0, \tfrac{1}{2}T_{i-1}) \\
v_i^+(0) &= v_i^-(0) \\
v_j^+(\tfrac{1}{2}T_{j-1}) &= v_{j+1}^-(-\tfrac{1}{2}T_{j-1})
\end{aligned}
$$

considered as equations over the complex field. Exploiting the fact that $\dot{q}_N(t)$ solves (2.4) for $\lambda = 0$ and using (4.1), we take the ansatz

$$v_i^\pm(t) = \dot{u}_i^\pm(t)\, d_i + w_i^\pm(t),$$

with $d_i \in \mathbb{R}$. Owing to [San95b, section 3.1] and (4.2), equation (4.3) is then equivalent to

$$
(4.4)\quad
\begin{aligned}
\dot{w}_i^\pm &= (D_u f(u_i^\pm(t), \mu_N) + \lambda B(t))\, w_i^\pm + \lambda\, B(t)\, \dot{u}_i^\pm(t)\, d_i \\
&\qquad \text{for } t \in (-\tfrac{1}{2}T_{i-2}, 0) \text{ and } t \in (0, \tfrac{1}{2}T_{i-1}), \text{ respectively} \\
w_i^+(0) &= w_i^-(0) \\
w_i^\pm(0) &\in X_{i \bmod 2} \\
w_j^+(\tfrac{1}{2}T_{j-1}) &= w_{j+1}^-(-\tfrac{1}{2}T_{j-1}) + \dot{u}_{j+1}^-(-\tfrac{1}{2}T_{j-1})(d_{j+1} - d_j),
\end{aligned}
$$

where the (complexified) subspaces X_k are defined by $\Sigma_k = q_k(0) + X_k$ for $k = 1, 2$. Following [San95b], we shall investigate the system

$$
(4.5)\quad
\begin{aligned}
\dot{w}_i^\pm &= (D_u f(u_i^\pm(t), \mu_N) + \lambda B(t))\, w_i^\pm + \lambda\, B(t)\, \dot{u}_i^\pm(t)\, d_i \\
&\qquad \text{for } t \in (-\tfrac{1}{2}T_{i-2}, 0) \text{ and } t \in (0, \tfrac{1}{2}T_{i-1}), \text{ respectively} \\
w_i^+(0) - w_i^-(0) &\in \mathbb{C}T_{u_i^+(0)} W^c_{hom}(\mu_N) \cap X_{i \bmod 2} \;\cong\; \mathbb{C} \\
w_i^\pm(0) &\in X_{i \bmod 2} \\
w_j^+(\tfrac{1}{2}T_{j-1}) &= w_{j+1}^-(-\tfrac{1}{2}T_{j-1}) + \dot{u}_{j+1}^-(-\tfrac{1}{2}T_{j-1})(d_{j+1} - d_j).
\end{aligned}
$$

Define the signed distances

$$\xi_i := \langle \psi_{i \bmod 2}(0), w_i^+(0) - w_i^-(0) \rangle \in \mathbb{C}, \tag{4.6}$$

see Figure 5. Then we have the following lemma.

Lemma 4.1 *Equation (4.5) possesses a unique solution $w = W(\lambda)d$ linear in d and analytic in λ. Moreover, w solves (4.4) if and only if*

$$\xi = S(\lambda)\, d = \Big(A(r) - \lambda\,(M + o(1)) + O(|\lambda|^2) \Big)\, d = 0 \tag{4.7}$$

for some analytic, matrix-valued function $S(\lambda)$ and

$$M = diag(M_1 K_1, M_2 K_2, ..., M_1 K_1)$$

with $K_1, K_2 > 0$ positive. The matrix $A(r)$ is determined by (4.5) with $\lambda = 0$. Any solution of (2.4) with $|\lambda|$ small is given by the above function $W(\lambda)$. In particular, $d = (1, ..., 1)$ solves $S(0)\, d = 0$.

With the equivalence of (2.4) and (4.1) as well as Lemma 4.1 at hand, it therefore remains to solve the reduced equation

$$\det S(\lambda) = 0. \tag{4.8}$$

Proof. The proof of the lemma is essentially contained in [San95b], where the analysis was done for N-pulses. We will briefly mention the changes needed here.

The hypotheses (H1) and (H3) ensure that the technique developed in [San95b] works in the present context. The only difference is that the linearized flows for the heteroclinic solutions are used instead of linearizing along a single homoclinic orbit. The major change made here in comparison with [San95b] is that we allow for jumps in

$$w_i^+(0) - w_i^-(0) \in \mathbb{C} T_{u_i^+(0)} W^c_{hom}(\mu_N) \cap X_{i \bmod 2} \cong \mathbb{C}$$

compared with jumps in $\mathbb{C}\psi(0)$

$$w_i^+(0) - w_i^-(0) \in \mathbb{C}\psi_{i \bmod 2}(0),$$

where $\psi_k(t)$ are the unique bounded solutions of the adjoint equation, see section 2. However, the only property of $\mathbb{C}\psi_k(0)$ used in [San95b] is the transversality condition

$$\mathbb{R}\psi_k(0) \oplus \mathbb{R}\dot{q}_k(0) \oplus T_{q_k(0)} W^{uu}(p_k) \oplus T_{q_k(0)} W^{ss}(p_{k+1}) = \mathbb{R}^n$$

for $k = 1, 2$, see [San95b, Lemma 3.5]. The corresponding relations

$$\Big(T_{u_i^+(0)} W^c_{hom}(\mu_N) \cap X_k \Big) \oplus \mathbb{R}\dot{q}_k(0) \oplus T_{q_k(0)} W^{uu}(p_k) \oplus T_{q_k(0)} W^{ss}(p_{k+1}) = \mathbb{R}^n \qquad k = i \bmod 2$$

are satisfied. Indeed, this is a consequence of (2.2) and the proof of Lemma 3.1. The statement about the matrix M follows from [San95b, Lemma 3.6] and the above discussion. Indeed, taking the limit $r \to 0$ is equivalent to computing the matrix M by investigating the eigenvalue problem (2.4) for the primary heteroclinic orbits $q_k(t)$ for $k = 1, 2$ as $u_i \to q_{i \bmod 2}$ for $r \to 0$ in the sup-norm. The positive factors K_1 and K_2 stem from the projection of $\psi_k(0)$ onto the tangent spaces$T_{q_k(0)} W^c_{hom}$ for $k = 1, 2$. $\square$

4.2 Determining the reduced problem using center-manifolds

In order to solve (4.8)

$$\det S(\lambda) = \det\Big(A(r) - \lambda\,(M + \mathrm{o}(1)) + \mathrm{O}(|\lambda|^2)\Big) = 0,$$

we have to determine the matrix $A(r)$. By definition, with $\lambda = 0$,

$$\xi = (\langle \psi_{i \bmod 2}(0), w_i^+(0) - w_i^-(0)\rangle)_{i=1,\dots,2N+1} = A(r)\, d,$$

where $w = W(0)\, d$ solves (4.5) with $\lambda = 0$, that is

$$(4.9)\quad \begin{array}{lrcl} (i) & \dot w_i^\pm & = & D_u f(u_i^\pm, \mu_N)\, w_i^\pm \\ & & & \text{for } t \in (-\tfrac{1}{2}T_{i-2}, 0) \text{ and } t \in (0, \tfrac{1}{2}T_{i-1}), \text{ respectively} \\ (ii) & w_i^+(0) - w_i^-(0) & \in & \mathbb{C}T_{u_i^+(0)} W^c_{hom}(\mu_N) \cap X_{i \bmod 2} \\ (iii) & w_i^\pm(0) & \in & X_{i \bmod 2} \\ (iv) & w_j^+(\tfrac{1}{2}T_{j-1}) & = & w_{j+1}^-(-\tfrac{1}{2}T_{j-1}) + \dot u_{j+1}^-(-\tfrac{1}{2}T_{j-1})(d_{j+1} - d_j). \end{array}$$

Therefore, the solutions w_i have to solve the variational equation along the N-front. Because W^c_{hom} is locally invariant and C^1, its continuous tangent bundle is invariant under the linearized flow. Since $\dot u_i \in T_{q_N} W^c_{hom}$ and the jumps of w_i are required to be in $T_{q_N} W^c_{hom}$, too, we expect that the solutions $w_i \in T_{q_N} W^c_{hom}$ are contained in the tangent bundle as well. By uniqueness of w as stated in Lemma 4.1, it is therefore sufficient to prove that we can solve (4.9) with $w_i \in T_{u_i} W^c_{hom}$. Since the linearized flow is still C^0-conjugated to the linearized flow in $\mathbb{R}^2$, see Lemma 3.1, it suffices to consider (4.9) for the vector field in $\mathbb{R}^2$ investigated in section 3 - note that we do not need any differentiability further on.

Hence consider $w \in \mathbb{R}^2$ from now on. Denote the evolution of

$$\dot w = D_u f(u_i^\pm(t), \mu_N)\, w$$

by $\Phi_i^\pm(t, s)$, whence $w_i^\pm(t) = \Phi_i^\pm(t, 0)\, w_i^\pm(0)$ solves (4.9)(i) and (iii) for arbitrary $w_i^\pm(0) \in X_k$. Note that (4.9)(ii) is then satisfied, too, as the subspaces $X_k \subset \mathbb{R}^2$ are one-dimensional.

We shall solve (4.9)(iv)

$$w_j^+(\tfrac{1}{2}T_{j-1}) = w_{j+1}^-(-\tfrac{1}{2}T_{j-1}) + \dot u_{j+1}^-(-\tfrac{1}{2}T_{j-1})(d_{j+1}-d_j) \tag{4.10}$$

for given $d=(d_i)_{i=1,\dots,2N+1}$ and $j=1,\dots,2N$. Observe that these equation decouple as we can choose $w_i^\pm(0)\in X_k$ arbitrarily.

First, consider (4.10) for odd $j=2l+1$ for $l=0,\dots,N-1$. Then

$$\Phi_{2l+1}^+(t,0) = \Phi_{2l+2}^-(t,0) = \begin{pmatrix} e^{-\alpha_2^s(\mu)t} & 0 \\ 0 & e^{\alpha_2^u(\mu)t} \end{pmatrix}$$

as the flow is linear. Also,

$$\dot u_{2l+2}^-(-\tfrac{1}{2}T_{2l}) = (-\alpha_2^s(\mu)\, e^{-\frac{1}{2}\alpha_2^s(\mu)T_{2l}}, \alpha_2^u(\mu)\, e^{-\frac{1}{2}\alpha_2^u(\mu)T_{2l}})$$

and

$$\begin{aligned} w_{2l+1}^+(\tfrac{1}{2}T_{2l}) &= (0, e^{\frac{1}{2}\alpha_2^u(\mu)T_{2l}}\, w_{2l+1}^+(0)) \\ w_{2l+2}^-(-\tfrac{1}{2}T_{2l}) &= (e^{\frac{1}{2}\alpha_2^s(\mu)T_{2l}}\, w_{2l+2}^-(0), 0), \end{aligned}$$

identifying the subspaces X_k with $\mathbb{R}$ as in Figure 5. Thus, we conclude that

$$\begin{aligned} w_{2l+1}^+(0) &= \alpha_2^u(\mu)\, e^{-\alpha_2^u(\mu)T_{2l}}\,(d_{2l+2}-d_{2l+1}) &&= \mathrm{o}(r)\,(d_{2l+2}-d_{2l+1}) \\ w_{2l+2}^-(0) &= \alpha_2^s(\mu)\, e^{-\alpha_2^s(\mu)T_{2l}}\,(d_{2l+2}-d_{2l+1}) &&= \alpha_2^s\,(1+\mathrm{o}(1))\,r\,(d_{2l+2}-d_{2l+1}), \end{aligned} \tag{4.11}$$

using (3.7) and (3.26).

Next, consider (4.10) for even $j=2l$ for $l=1,\dots,N$. Then

$$\begin{aligned} \Phi_{2l}^+(t,0) &= \begin{pmatrix} e^{-\alpha_1^s(\mu)(t-\Omega_2)} & 0 \\ 0 & e^{\alpha_1^u(\mu)(t-\Omega_2)} \end{pmatrix} \Phi_{2l}^+(\Omega_2,0) \\ \Phi_{2l+1}^-(-t,0) &= \begin{pmatrix} e^{-\alpha_1^s(\mu)(-t+\Omega_1)} & 0 \\ 0 & e^{\alpha_1^u(\mu)(-t+\Omega_1)} \end{pmatrix} \Phi_{2l+1}^+(-\Omega_1,0) \end{aligned}$$

for $t>0$ large and with

$$\begin{aligned} \Omega_1 &= \Omega_1(e^{-\alpha_1^s(\mu)\,\tau(T_{2l-1},\mu)},\mu) \\ \Omega_2 &= \Omega_2(e^{-\alpha_1^u(\mu)\,\tau(T_{2l-1},\mu)},\mu), \end{aligned}$$

see section 3.2. Therefore, we obtain

$$\begin{aligned} w_{2l}^+(\tfrac{1}{2}T_{2l-1}) &= (e^{\alpha_1^s(\mu)(-\frac{1}{2}T_{2l-1}+\Omega_2)}\,\pi_{2l}^s, e^{\alpha_1^u(\mu)(\frac{1}{2}T_{2l-1}-\Omega_2)}\,\pi_{2l}^u)\, w_{2l}^+(0) \\ w_{2l+1}^-(-\tfrac{1}{2}T_{2l-1}) &= (e^{\alpha_1^s(\mu)(\frac{1}{2}T_{2l-1}-\Omega_1)}\,\pi_{2l+1}^s, e^{\alpha_1^u(\mu)(-\frac{1}{2}T_{2l-1}+\Omega_1)}\,\pi_{2l+1}^u)\, w_{2l+1}^-(0), \end{aligned}$$

for some constants π_{2l}^k, π_{2l+1}^k uniformly bounded in T_{2l-1} for $k=s,u$ such that

$$\pi_{2l}^u, \pi_{2l+1}^s < -\delta < 0 \tag{4.12}$$

for some δ owing to the sign convention for the sections - we identify the subspaces X_k with $\mathbb{R}$ in the same way as we did for Σ_k, see Figure 5. The time derivative is given by

$$\dot{u}^-_{2l+1}(-\tfrac{1}{2}T_{2l-1}) = (-\alpha^s_1(\mu)\, e^{-\alpha^s_1(\mu)(\frac{1}{2}T_{2l-1}-\Omega_2)}, \alpha^u_1(\mu)\, e^{-\alpha^u_1(\mu)(\frac{1}{2}T_{2l-1}-\Omega_1)}).$$

Thus, (4.10) reads

$$\begin{pmatrix} -e^{\alpha^s_1(\mu)(\frac{1}{2}T_{2l-1}-\Omega_1)}\,\pi^s_{2l+1} & e^{\alpha^s_1(\mu)(-\frac{1}{2}T_{2l-1}+\Omega_2)}\,\pi^s_{2l} \\ -e^{\alpha^u_1(\mu)(-\frac{1}{2}T_{2l-1}+\Omega_1)}\,\pi^u_{2l+1} & e^{\alpha^u_1(\mu)(\frac{1}{2}T_{2l-1}-\Omega_2)}\,\pi^u_{2l} \end{pmatrix} \begin{pmatrix} w^-_{2l}(0) \\ w^+_{2l+1}(0) \end{pmatrix}$$
$$= \begin{pmatrix} -\alpha^s_1(\mu)\, e^{-\alpha^s_1(\mu)(\frac{1}{2}T_{2l-1}-\Omega_2)} \\ \alpha^u_1(\mu)\, e^{-\alpha^u_1(\mu)(\frac{1}{2}T_{2l-1}-\Omega_1)} \end{pmatrix} (d_{2l+1}-d_{2l})$$

and it is straightforward to calculate that for some $\delta > 0$

$$\begin{aligned} w^+_{2l}(0) &= \alpha^u_1(\mu)\, e^{-\alpha^u_1(\mu)(T_{2l-1}-\Omega_1-\Omega_2)}\,\pi^u_{2l}\,(1+\mathrm{O}(e^{-\delta T_{2l-1}}))\,(d_{2l+1}-d_{2l}) \\ &= \alpha^u_1(\mu)\, e^{-\alpha^u_1(\mu)\,\tau(T_{2l-1})}\,\pi^u_{2l}\,(1+\mathrm{O}(e^{-\delta\tau(T_{2l-1})}))\,(d_{2l+1}-d_{2l}) \\ &= \mathrm{o}(r^{\alpha_2+\gamma_l})\,(d_{2l+1}-d_{2l}) \\ w^-_{2l+1}(0) &= \alpha^s_1(\mu)\, e^{-\alpha^s_1(\mu)(T_{2l-1}-\Omega_1-\Omega_2)}\,\pi^s_{2l+1}\,(1+\mathrm{O}(e^{-\delta T_{2l-1}}))\,(d_{2l+1}-d_{2l}) \\ &= \alpha^s_1(\mu)\, e^{-\alpha^s_1(\mu)\,\tau(T_{2l-1})}\,\pi^s_{2l+1}\,(1+\mathrm{O}(e^{-\delta\tau(T_{2l-1})}))\,(d_{2l+1}-d_{2l}) \\ &= \alpha^s_1\,(b_{2l-1}+\mathrm{o}(1))\,\pi^s_{2l+1}\,r^{\alpha_2+\gamma_l}\,(d_{2l+1}-d_{2l}), \end{aligned} \tag{4.13}$$

see again (3.7) and (3.26). It is convenient to check the signs appearing in (4.11) and (4.13) by inspecting Figure 5 and 6.

Thus, the differences of $w^{\pm}_i(0)$ for $i = 1, ..., 2N+1$ with $\lambda = 0$ are given by

$$\begin{aligned} w^+_{2l}(0) - w^-_{2l}(0) &= \mathrm{o}(r^{\alpha_2+\gamma_l})\,(d_{2l+1}-d_{2l}) - \alpha^s_2\,(1+\mathrm{o}(1))\,r\,(d_{2l}-d_{2l-1}) \\ w^+_{2l+1}(0) - w^-_{2l+1}(0) &= \mathrm{o}(r)\,(d_{2l+2}-d_{2l+1}) - \alpha^s_1\,(b_{2l-1}+\mathrm{o}(1))\,\pi^s_{2l+1}\,r^{\alpha_2+\gamma_l}\,(d_{2l+1}-d_{2l}), \end{aligned}$$

whence the jumps ξ_i read

$$\begin{aligned} \xi_{2l} &= \langle\psi_2(0), w^+_{2l}(0) - w^-_{2l}(0)\rangle \\ &= r\Big(\mathrm{o}(r^{\alpha_2+\gamma_l-1})\,(d_{2l+1}-d_{2l}) + \alpha^s_2\,(1+\mathrm{o}(1))\,(d_{2l}-d_{2l-1})\Big) \\ \xi_{2l+1} &= \langle\psi_1(0), w^+_{2l+1}(0) - w^-_{2l+1}(0)\rangle \\ &= r\Big(\mathrm{o}(1)\,(d_{2l+2}-d_{2l+1}) - \alpha^s_1\,(b_{2l-1}+\mathrm{o}(1))\,\pi^s_{2l+1}\,r^{\alpha_2+\gamma_l-1}\,(d_{2l+1}-d_{2l})\Big). \end{aligned} \tag{4.14}$$

Notice that the sign changes in the first equation since $\psi_2(0)$ points in the negative direction of X_2, see Figure 5. We rewrite (4.14) according to

$$\begin{aligned} \xi_{2l} &= r\,(-\kappa_{2l-1}\,d_{2l-1} + (\kappa_{2l-1}-\tilde{\kappa}_{2l})\,d_{2l} + \tilde{\kappa}_{2l}\,d_{2l+1}) \\ \xi_{2l+1} &= r\,(-\kappa_{2l}\,d_{2l} + (\kappa_{2l}-\tilde{\kappa}_{2l+1})\,d_{2l+1} + \tilde{\kappa}_{2l+1}\,d_{2l+2}), \end{aligned}$$

using the definitions

$$
(4.15) \qquad \begin{array}{lcl@{\qquad}cl}
\kappa_{2l-1} & := & c_{2l-1} + \mathrm{o}(1) & := & \alpha_2^s\,(1 + \mathrm{o}(1)) \\
\tilde{\kappa}_{2l-1} & := & \mathrm{o}(1) & & \\
\kappa_{2l} & := & (c_{2l} + \mathrm{o}(1))\, r^{\beta_l} & := & -\alpha_1^s\,(b_{2l-1} + \mathrm{o}(1))\,\pi_{2l+1}^s\, r^{\alpha_2+\gamma_l-1} \\
\tilde{\kappa}_{2l} & := & \mathrm{o}(r^{\beta_l}) & := & \mathrm{o}(r^{\alpha_2+\gamma_l-1})
\end{array}
$$

for $l = 1, ..., N$ and

$$\kappa_0 = \tilde{\kappa}_0 = \kappa_{2N+1} = \tilde{\kappa}_{2N+1} = 0.$$

The exponents β_l and the constants c_j fulfill

$$
(4.16) \qquad \begin{array}{ll}
\beta_l := \alpha_2 + \gamma_l - 1 & l = 1, ..., N \\
0 < \alpha_2 - 1 = \beta_N < \beta_l < \beta_{l-1} & l = 2, ..., N-1 \\
c_j > 0 & j = 1, ..., 2N,
\end{array}
$$

due to (3.20), (3.25) and (4.12).

Therefore, we end up with computing solutions of

$$
(4.17) \qquad \det\Big(r\tilde{A}(r) - M\lambda + \mathrm{O}(|\lambda|(|\lambda| + \mathrm{o}(1))\Big) = 0,
$$

where

$$M = \mathrm{diag}(M_1K_1, M_2K_2, ..., M_1K_1)$$

for some positive constants $K_1, K_2 > 0$ and

$$
(4.18) \qquad \tilde{A}(r) = \begin{pmatrix}
-\tilde{\kappa}_1 & \tilde{\kappa}_1 & & & \\
-\kappa_1 & \kappa_1 - \tilde{\kappa}_2 & \tilde{\kappa}_2 & & \\
 & -\kappa_2 & \kappa_2 - \tilde{\kappa}_3 & \tilde{\kappa}_3 & \\
 & & \ddots & \ddots & \\
 & & & -\kappa_{2N} & \kappa_{2N}
\end{pmatrix}.
$$

As we are mainly interested in stable N-front solutions, we assume sign $M_1 =$ sign $M_2 = -1$ from now on, whence, by rescaling the solutions $\psi_k(t)$, we obtain

$$M = -\,\mathrm{id}\,.$$

The other cases can be handled similarly.

4.3 Solving the reduced eigenvalue problem

Thus we shall solve (4.17). By Rouché's Theorem, there exist precisely $2N+1$ solutions of (4.17), since $S(\lambda)$ is analytic in λ and

$$\det S(\lambda) = \lambda^{2N+1} + \mathrm{o}(1)$$

near $\lambda = 0$.

One of these solutions is equal to zero

$$\lambda_{2N+1} = 0 \tag{4.19}$$

due to translational invariance. By construction, the associated eigenvector is given by $v = (1, ..., 1)$, see Lemma 4.1.

Substituting $\lambda = \nu r$ and $M = -\,\mathrm{id}$ into (4.17) and dividing by r^{2N+1} yields

$$\det\Big(\tilde{A}(r) + \nu\,(\mathrm{id} + \mathrm{o}(1))\Big) = 0. \tag{4.20}$$

There are another N eigenvalues which can be computed easily. Indeed, setting $r = 0$ in (4.20), we obtain

$$\det(\tilde{A}(0) + \nu\,\mathrm{id}) = \nu^{N+1} \prod_{l=1}^{N} (c_{2l-1} + \nu).$$

Hence, again by Rouché's Theorem, there exist precisely N solutions $\nu_{2l-1}(r)$ of (4.20) counted with multiplicity and continuous in r such that

$$\nu_{2l-1}(0) = -c_{2l-1} < 0.$$

They correspond to N eigenvalues $\lambda_{2l-1}(r)$ of (4.17) given by

$$\lambda_{2l-1}(r) = \nu_{2l-1}(r)\,r = -(c_{2l-1} + \mathrm{o}(1))\,r < 0 \qquad l = 1, ..., N. \tag{4.21}$$

It remains to calculate the remaining N eigenvalues of (4.20). The columns of the matrix $S(\nu, r) = \tilde{A}(r) + \nu\,(\mathrm{id} + \mathrm{o}(1))$ are given by

$$\begin{array}{lcll} C_1 & = & (-\tilde{\kappa}_1 + \nu, -\kappa_1, 0, ..., 0) + \mathrm{o}(1)\nu & \\ C_j & = & (0, ..., 0, \tilde{\kappa}_{j-1}, \underbrace{\kappa_{j-1} - \tilde{\kappa}_j + \nu}_{j\mathrm{th}}, -\kappa_j, 0, ..., 0) + \mathrm{o}(1)\nu & j = 2, ..., 2N \\ C_{2N+1} & = & (0, ..., 0, \tilde{\kappa}_{2N}, \kappa_{2N} + \nu) + \mathrm{o}(1)\nu, & \end{array}$$

see (4.18). Adding successively the jth column C_j to C_{j-1} for $j = 2N+1, ..., 2$ yields a matrix with columns

$$\begin{array}{lcll} C_1 & = & (\nu, ..., \nu) + \mathrm{o}(1)\nu & \\ C_j & = & (0, ..., 0, \tilde{\kappa}_{j-1}, \underbrace{\kappa_{j-1} + \nu}_{j\mathrm{th}}, \nu, ..., \nu) + \mathrm{o}(1)\nu & j = 2, ..., 2N \\ C_{2N+1} & = & (0, ..., 0, \tilde{\kappa}_{2N}, \kappa_{2N} + \nu) + \mathrm{o}(1)\nu. & \end{array}$$

Note that this transformation does not change the determinant. Moreover, recall from (4.15) that

$$\begin{array}{lllllll} \kappa_{2l-1} & = & c_{2l-1} + \mathrm{o}(1) & \quad \tilde\kappa_{2l-1} & = & \mathrm{o}(1) & = & \mathrm{o}(\kappa_{2l-1}) \\ \kappa_{2l} & = & (c_{2l} + \mathrm{o}(1))\, r^{\beta_l} & \quad \tilde\kappa_{2l} & = & \mathrm{o}(r^{\beta_l}) & = & \mathrm{o}(\kappa_{2l}) \end{array}$$

for positive constants $c_j > 0$ and exponents $\beta_l > 0$ strictly decreasing in l, see (4.16). This suggests the ansatz

$$\nu = r^{\beta_k}\, \eta$$

for fixed k with $k = 1, ..., N$. Substituting it into the matrix yields

$$\begin{array}{lll} C_1 & = & \Big[(\eta, ..., \eta) + \mathrm{o}(1)\Big]\, r^{\beta_k} \\ C_{2l} & = & \Big[(0, ..., 0, \underbrace{c_{2l-1}}_{(2l)\mathrm{th}}, 0..., 0) + \mathrm{o}(1)\Big] \\ C_{2l+1} & = & \begin{cases} \Big[(0, ..., 0, \underbrace{\eta}_{(2l+1)\mathrm{th}}, \eta, ..., \eta) + \mathrm{o}(1)\Big]\, r^{\beta_k} & l < k \\ \Big[(0, ..., 0, \underbrace{c_{2k} + \eta}_{(2k+1)\mathrm{th}}, \eta, ..., \eta) + \mathrm{o}(1)\Big]\, r^{\beta_k} & l = k \\ \Big[(0, ..., 0, \underbrace{c_{2l}}_{(2l+1)\mathrm{th}}, 0, ..., 0) + \mathrm{o}(1)\Big]\, r^{\beta_l} & l > k \end{cases} \end{array}$$

for $l = 1, ..., N$. Thus, factorizing the powers of r multiplying each column, the determinant of the matrix $S(r^{\beta_k}\eta, r)$ equals

$$\det S(r^{\beta_k}\, \eta, r) = \Big(\det \tilde S(\eta, r)\Big)\, r^{(k+1)\beta_k} \prod_{l=k+1}^{N} r^{\beta_l},$$

where the columns of $\tilde S(\eta, r)$ are given by

$$\begin{array}{lll} C_1 & = & \Big[(\eta, ..., \eta) + \mathrm{o}(1)\Big] \\ C_{2l} & = & \Big[(0, ..., 0, \underbrace{c_{2l-1}}_{(2l)\mathrm{th}}, 0..., 0) + \mathrm{o}(1)\Big] \\ C_{2l+1} & = & \begin{cases} \Big[(0, ..., 0, \underbrace{\eta}_{(2l+1)\mathrm{th}}, \eta, ..., \eta) + \mathrm{o}(1)\Big] & l < k \\ \Big[(0, ..., 0, \underbrace{c_{2k} + \eta}_{(2k+1)\mathrm{th}}, \eta, ..., \eta) + \mathrm{o}(1)\Big] & l = k \\ \Big[(0, ..., 0, \underbrace{c_{2l}}_{(2l+1)\mathrm{th}}, 0, ..., 0) + \mathrm{o}(1)\Big] & l > k. \end{cases} \end{array}$$

As we are interested in zeroes for $r > 0$, it suffices to solve

$$\det \tilde S(\eta, r) = 0. \tag{4.22}$$

This matrix, however, is upper-triangular up to terms of order o(1). Its determinant is therefore given by

$$\begin{aligned} \det \tilde{S}(\eta, r) &= \det \tilde{S}(\eta, 0) + \mathrm{o}(1) \\ &= \eta^k \Big(\prod_{l=k+1}^{N} c_{2l} \Big) (\eta + c_{2k}) \Big(\prod_{l=1}^{N} c_{2l-1} \Big) + \mathrm{o}(1). \end{aligned}$$

Again by Rouché's Theorem, there is a unique solution $\eta_{2l}(r)$ of (4.22) satisfying

$$\eta_{2l}(0) = -c_{2l}$$

for $l = 1, ..., N$. The corresponding solution $\lambda_{2l}(r)$ of (4.17) is given by

$$\lambda_{2l}(r) = \nu_{2l}(r)\, r = \eta_{2l}(r)\, r^{1+\beta_l} = -(c_{2l} + \mathrm{o}(1))\, r^{1+\beta_l} = -(c_{2l} + \mathrm{o}(1))\, r^{\alpha_2 + \gamma_l} \tag{4.23}$$

for $l = 1, ..., N$, see (4.16) for the last identity. Note that these solutions are not the same for different values of l owing to (4.16). Moreover, they converge faster to zero than the eigenvalues λ_{2l-1} obtained in (4.21).

Summarizing the facts obtained above, we have calculated $2N+1$ solutions λ_j of (4.17) appearing in (4.19), (4.21) and (4.23). According to the remark above, they are pairwise distinct, whence we have found all solutions. This proves Theorem 2. □

5 Application to the FitzHugh-Nagumo equation

Consider the FitzHugh-Nagumo equation

$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \epsilon(u - \gamma w) \end{aligned} \tag{5.1}$$

for $x \in \mathbb{R}$ with $f(u) = u(1-u)(u-a)$ and $a \in (0, \frac{1}{2})$ fixed. This equation is a simplification of the Hodgkin-Huxley equation modelling the propagation of impulses in nerve axons. Being interested in travelling waves $(u, w)(x, t) = (u, w)(x + ct)$, we introduce new variables $(\xi, t) = (x + ct, t)$ in which (5.1) takes the form

$$\begin{aligned} u_t &= u_{\xi\xi} - cu_\xi + f(u) - w \\ w_t &= -cw_\xi + \epsilon(u - \gamma w). \end{aligned} \tag{5.2}$$

The existence of fronts travelling with wave speed c boils down to investigating heteroclinic orbits of the ordinary differential equation

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= cv - f(u) + w \\ \dot{w} &= \tfrac{\epsilon}{c}(u - \gamma w), \end{aligned} \tag{5.3}$$

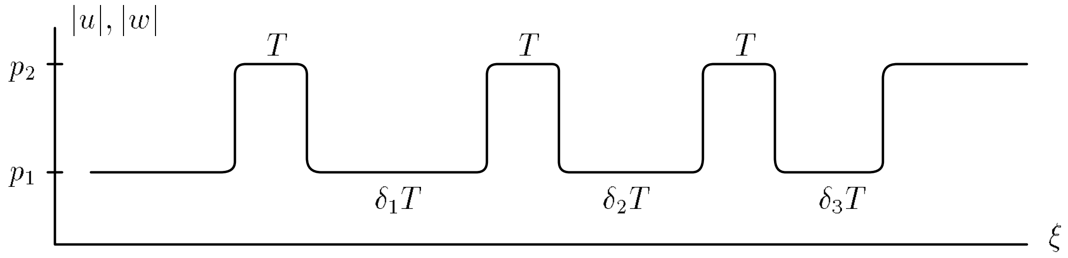

Figure 7: The N-front wave solution for $N = 3$. The distances of the layers are given by T and $\delta_j T = \frac{\alpha_2^u + \alpha_2^s \gamma_j}{\alpha_1^s} T$ with $\gamma_j > 0$ strictly decreasing in j, see Theorem 1.

which is the steady-state equation corresponding to (5.2). Here $\dot{} = d/d\xi$. Linearized stability of equilibria (u, w) of (5.2) is determined by the spectrum of the linear operator

$$L(U, W) = \begin{pmatrix} U_{\xi\xi} - cU_\xi + D_u f(u)U - W \\ -cW_\xi + \epsilon(U - \gamma W) \end{pmatrix}. \tag{5.4}$$

In particular, eigenvalues λ with corresponding eigenfunction (U, W) of L are given by bounded solutions of

$$\begin{aligned} \dot{U} &= V \\ \dot{V} &= cV - D_u f(u)U + W + \lambda U \\ \dot{W} &= \tfrac{\epsilon}{c}(U - \gamma W) - \tfrac{\lambda}{c} W. \end{aligned} \tag{5.5}$$

Deng proved in [Den91b] that there is a curve $(\gamma(\epsilon), c(\epsilon))$ for all $\epsilon > 0$ sufficiently small such that the FitzHugh-Nagumo equation (5.3) possesses as twisted heteroclinic loop for these values of parameters. In particular, he concluded the existence of N-fronts for any $N > 1$ using his result [Den91a]. Theorem 1 of the present article provides the distance of the layers, see Figure 7. Yanagida proved in [Yan89] that the simple fronts $q_1(t)$ and $q_2(t)$ building the heteroclinic loop are linearly stable with respect to the partial differential equation, that is the spectrum of the linearized operator (5.4) is contained in the left half plane except for a simple eigenvalue at zero. Finally, Nii [Nii95b] proved that the 1-fronts are linearly stable, too, using topological methods - however, he had to assume that the flow of (5.3) is linear near both equilibria. The next result asserts that in fact all N-fronts are linearly stable and provides asymptotic expansions of the critical eigenvalues.

Theorem 3 *The N-fronts (and N-backs) of (5.1) proved to exist by Deng [Den91b] are linearly stable for all N. The $2N+1$ critical eigenvalues near zero are given by Theorem 2.*

Note that linear stability implies nonlinear stability by [BJ89].

Proof. We shall use Theorem 2 to conclude linear stability of the N-fronts. First note that the hypotheses (H1) – (H6) needed in that theorem are fulfilled by [Den91b]. Moreover, by the results in [AGJ90] and the stability of the simple fronts proved in [Yan89], it is sufficient to calculate eigenvalues of the linearized operator (5.4) near zero, see for example [Nii95b] for a discussion. Indeed, the spectrum of (5.4) does not contain eigenvalues with non-negative real part and large modulus, see [Eva75]. Comparing the eigenvalue problem (5.5) and the travelling wave equation (5.3) with equation (2.1) and (2.4), we see that they are of the same form by taking B according to

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{c} \end{pmatrix}.$$

Hence it suffices to prove that the Melnikov integrals

$$\int_{-\infty}^{\infty} \langle \psi_j(t), B\dot{q}_j(t) \rangle \, dt < 0 \tag{5.6}$$

are negative, where $\psi_j(t)$ are chosen according to hypothesis (H6), see Figures 2 or 8. Indeed, then the statement of the theorem follows immediately from Theorem 2.

In order to do so, notice that for any solution (u, v, w) of (5.3)

$$B \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{u} \\ -\frac{1}{c}\dot{w} \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ -\frac{\epsilon}{c^2}(u - \gamma w) \end{pmatrix} = D_c F(u, v, w, c)$$

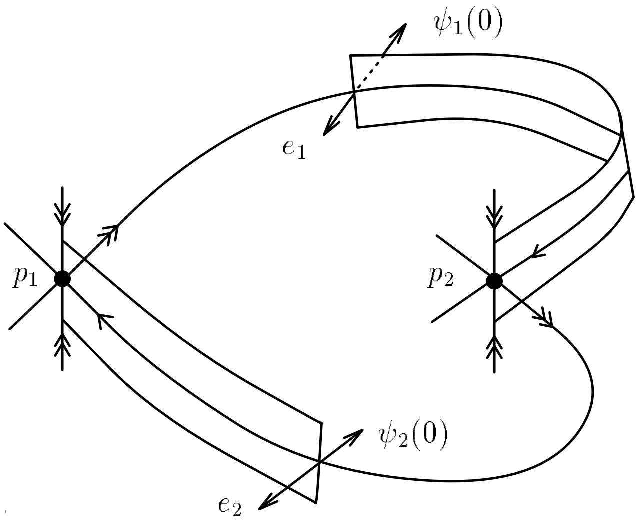

Figure 8: Conventions used by Deng and the present article.

holds, where F denotes the right-hand side of (5.3). In particular, we obtain

$$\int_{-\infty}^{\infty} \langle \psi_j(t), B\dot{q}_j(t) \rangle \, dt = \int_{-\infty}^{\infty} \langle \psi_j(t), D_c F(q_j(t), c) \rangle \, dt. \tag{5.7}$$

The second integral in the above formula is the derivative with respect to c of the signed distance of unstable and unstable manifolds measured in the direction $\psi_j(0)$, that is

$$\int_{-\infty}^{\infty} \langle \psi_j(t), D_c F(q_j(t), c) \rangle \, dt = \frac{d}{dc} \langle \psi_j(0), p_j^u(c) - p_{j+1}^s(c) \rangle, \tag{5.8}$$

where $p_j^u(c) \in W^u(p_j, c)$ and $p_j^s(c) \in W^s(p_j, c)$, see for example [Kok88], [Lin90] or [Den91b]. The last quantity appearing in (5.8) has been computed in [Den91b]. What is actually computed therein, is

$$\frac{d}{dc} Q_j = \frac{d}{dc} \langle e_j, p_{j+1}^s(c) - p_j^u(c) \rangle < 0, \tag{5.9}$$

see [Den91b, (3.1)] for the definition and [Den91b, (5.3a),(5.4a)] for the actual computation. Moreover, the vectors e_j appearing in (5.9) above are chosen in [Den91b, pages 1641 and 1644] such that

$$e_j = -\psi_j(0), \tag{5.10}$$

see Figure 8. Summarizing, we obtain from (5.7) and (5.8) that the Melnikov integrals

$$\begin{aligned} \int_{-\infty}^{\infty} \langle \psi_j(t), B\dot{q}_j(t) \rangle \, dt &\overset{(5.7),(5.8)}{=} \frac{d}{dc} \langle \psi_j(0), p_j^u(c) - p_{j+1}^s(c) \rangle \\ &\overset{(5.10)}{=} \frac{d}{dc} \langle -e_j(0), p_j^u(c) - p_{j+1}^s(c) \rangle \\ &\overset{(5.9)}{=} \frac{d}{dc} Q_j \; < \; 0 \end{aligned}$$

are indeed negative. Thus the theorem is proved. □

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