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Boundary behavior of nonlocal minimal surfaces

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Abstract. We consider the behavior of the nonlocal minimal surfaces in the vicinity of the boundary. By a series of detailed examples, we show that nonlocal minimal surfaces may stick at the boundary of the domain, even when the domain is smooth and convex. This is a purely nonlocal phenomenon, and it is in sharp contrast with the boundary properties of the classical minimal surfaces.

In particular, we show stickiness phenomena to half-balls when the datum outside the ball is a small half-ring and to the side of a two-dimensional box when the oscillation between the datum on the right and on the left is large enough.

When the fractional parameter is small, the sticking effects may become more and more evident. Moreover, we show that lines in the plane are unstable at the boundary: namely, small compactly supported perturbations of lines cause the minimizers in a slab to stick at the boundary, by a quantity that is proportional to a power of the perturbation.

In all the examples, we present concrete estimates on the stickiness phenomena. Also, we construct a family of compactly supported barriers which can have independent interest.

1. Introduction

It is well known (see e.g. [16, 14]) that the classical minimal surfaces do not stick at the boundary. Namely, if Ω is a convex domain and E is a set that minimizes the perimeter among its competitors in Ω, then ∂E is transverse to $\partial\Omega$ at their intersection points.

In this paper we show that the situation for the nonlocal minimal surfaces is completely different. Indeed, we prove that nonlocal interactions can favor stickiness at the boundary for minimizers of a fractional perimeter.

The mathematical framework in which we work was introduced in [6] and is the following. Given $s \in (0, 1/2)$ and an open set $\Omega \subseteq \mathbb{R}^n$, we define the s-perimeter of a set $E \subseteq \mathbb{R}^n$ in Ω as

$$\mathrm{Per}_s(E,\Omega) := L(E\cap\Omega, E^c) + L(\Omega\setminus E, E\setminus\Omega),$$

where $E^c := \mathbb{R}^n \setminus E$ and, for any disjoint sets F and G, we use the notation

$$L(F,G) := \iint_{F\times G} \frac{dx\,dy}{|x-y|^{n+2s}}.$$

We say that E is s-minimal in Ω if $\mathrm{Per}_s(E,\Omega) < +\infty$ and $\mathrm{Per}_s(E,\Omega) \leqslant \mathrm{Per}_s(F,\Omega)$ among all the sets F which coincide with E outside Ω.

Problems related to the s-perimeter naturally arise in several fields, such as the motion by nonlocal mean curvature and the nonlocal Allen-Cahn equation, see e.g. [7, 21]. Also, the s-perimeter can be seen as a fractional interpolation between the classical perimeter (corresponding to the case $s \to 1/2$) and the Lebesgue measure (corresponding to the case $s \to 0$), see e.g. [18, 3, 8, 1, 12].

The field of nonlocal minimal surfaces is rich of open problems and surprising examples (see e.g. [11]) and the interior regularity theory of the nonlocal minimal surfaces has been established in the plane and when the fractional parameter is close enough to 1/2 (see [9, 22]), but, as far as we know, the boundary behavior of the nonlocal minimal surfaces has not been studied till now.

We show in this paper that the boundary datum is not, in general, attained continuously. Indeed, nonlocal minimal surfaces may stick at the boundary and then detach from the boundary in a $C^{1,\frac{1}{2}+s}$-fashion. We will give concrete examples of this stickiness phenomenon with explicit (and somehow optimal) estimates. In particular, we will present stickiness phenomena to half-balls, when the domain is a ball and the datum is a small half-ring, and to the sides of a two-dimensional box, when the datum is small on one side and large on the other side.

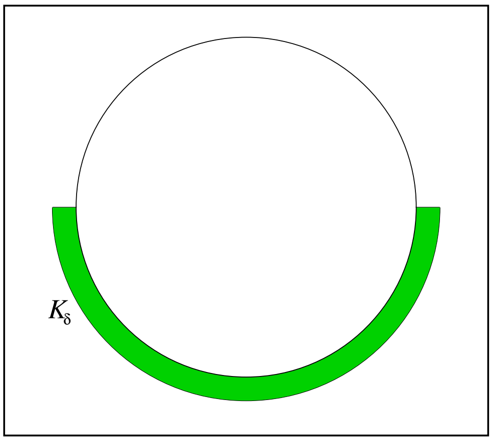

FIGURE 1. The stickiness property in Theorem 1.1.

Moreover, we study how small perturbations with compact support may affect the boundary behavior of a given nonlocal minimal surface. Quite surprisingly, these perturbations may produce stickiness effects even in the case of flat objects and in low dimension. For instance, adding a small perturbation to a half-space in the plane produces a sticking effect, with the size of the sticked portion proportional to a power of the size of the perturbation. We now present and discuss these results in further detail.

Stickiness to half-balls. For any $\delta > 0$, we let

$$K_\delta := \big(B_{1+\delta} \setminus B_1\big) \cap \{x_n < 0\}. \tag{1.1}$$

We define E_δ to be the set minimizing $\operatorname{Per}_s(E, B_1)$ among all the sets E such that $E \setminus B_1 = K_\delta$.

Notice that, in the local setting, the minimizer of the perimeter functional that takes K_δ as boundary value at ∂B_1 is the flat set $B_1 \cap \{x_n < 0\}$ (independently of δ). The picture changes dramatically in the nonlocal framework, since in this case the nonlocal minimizers stick at ∂B_1 if δ is suitably small, see Figure 1. The formal statement of this feature is the following:

Theorem 1.1. *There exists $\delta_o > 0$, depending on s and n, such that for any $\delta \in (0, \delta_o]$ we have that*

$$E_\delta = K_\delta.$$

Stickiness to the sides of a box. Given a large $M > 1$ we consider the s-minimal set E_M in $(-1, 1) \times \mathbb{R}$ with datum outside $(-1, 1) \times \mathbb{R}$ given by the jump

$$\begin{aligned} & J_M := J_M^- \cup J_M^+, \\ \text{where} \quad & J_M^- := (-\infty, -1] \times (-\infty, -M) \\ \text{and} \quad & J_M^+ := [1, +\infty) \times (-\infty, M). \end{aligned} \tag{1.2}$$

We prove that, if M is large enough, the minimal set E_M sticks at the boundary (see Figure 2). Moreover, the stickiness region gets close to the origin, up to a power of M. The precise result is the following:

Theorem 1.2. *There exist $M_o > 0$ and $C_o \geqslant C_o' > 0$, depending on s, such that if $M \geqslant M_o$ then*

$$\begin{aligned} & [-1, 1) \times [C_o M^{\frac{1+2s}{2+2s}},\, M] \subseteq E_M^c \\ \textit{and} \qquad & (-1, 1] \times [-M,\, -C_o M^{\frac{1+2s}{2+2s}}] \subseteq E_M. \end{aligned}$$

Also, the exponent $\frac{1+2s}{2+2s}$ above is optimal. For instance, if either $[-1, 1) \times [bM^{\frac{1+2s}{2+2s}},\, M] \subseteq E_M^c$ or $(-1, 1] \times [-M,\, -C_o M^{\frac{1+2s}{2+2s}}] \subseteq E_M$ for some $b \geqslant 0$, then $b \geqslant C_o'$.

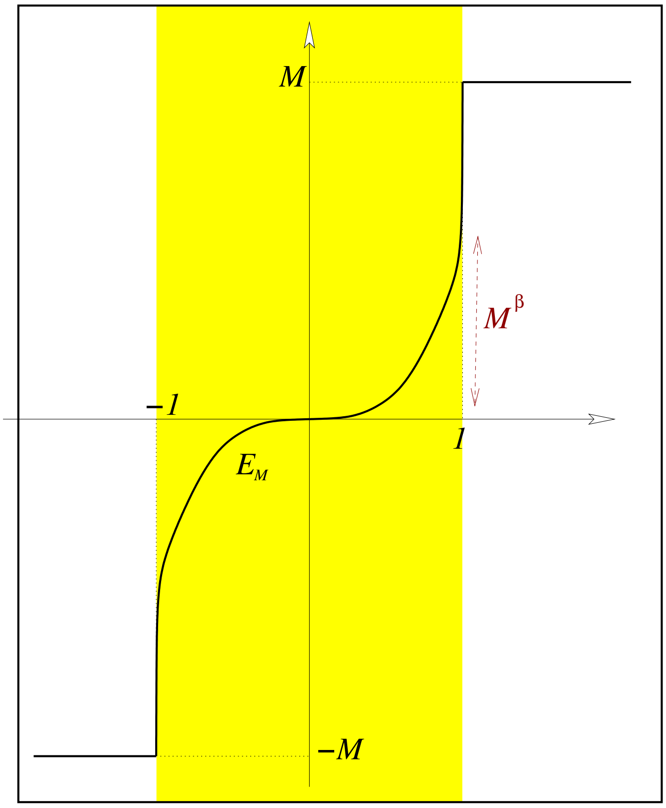

FIGURE 2. The stickiness property in Theorem 1.2, with $\beta := \frac{1+2s}{2+2s}$.

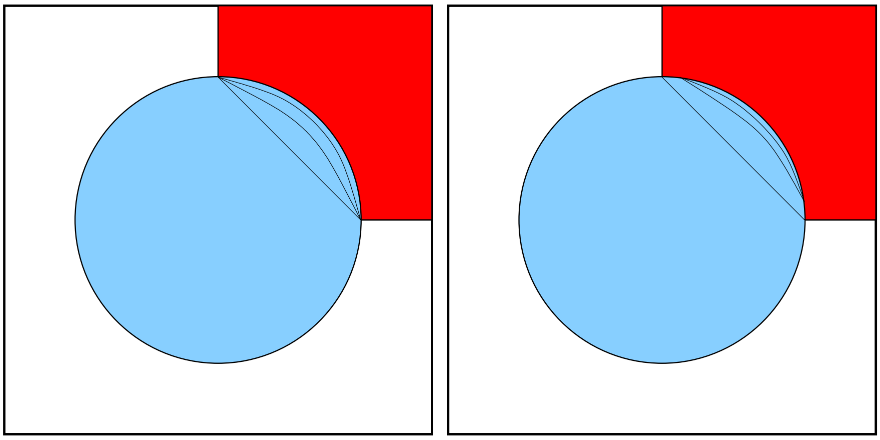

FIGURE 3. The stickiness property in Theorem 1.3.

Stickiness as $s \to 0^+$. The stickiness properties of nonlocal minimal surfaces are a purely nonlocal phenomenon and they become more evident for small values of s. To provide a confirming example, we consider the boundary value given by a sector in $\mathbb{R}^2$ outside B_1, i.e. we define

$$\Sigma := \{(x,y) \in \mathbb{R}^2 \setminus B_1 \text{ s.t. } x > 0 \text{ and } y > 0\}. \tag{1.3}$$

We show that as $s \to 0^+$ the s-minimal set in B_1 with datum Σ sticks to Σ, and, more precisely, this stickiness already occurs for a small $s_o > 0$ (see Figure 3).

Theorem 1.3. *Let E_s be the s-minimizer of $\mathrm{Per}_s(E, B_1)$ among all the sets E such that $E \setminus B_1 = \Sigma$.*

Then, there exists $s_o > 0$ such that for any $s \in (0, s_o]$ we have that $E_s = \Sigma$.

Instability of the flat fractional minimal surfaces. Rather surprisingly, one of our results states that the flat lines are "unstable" fractional minimal surfaces, in the sense that an arbitrarily small and

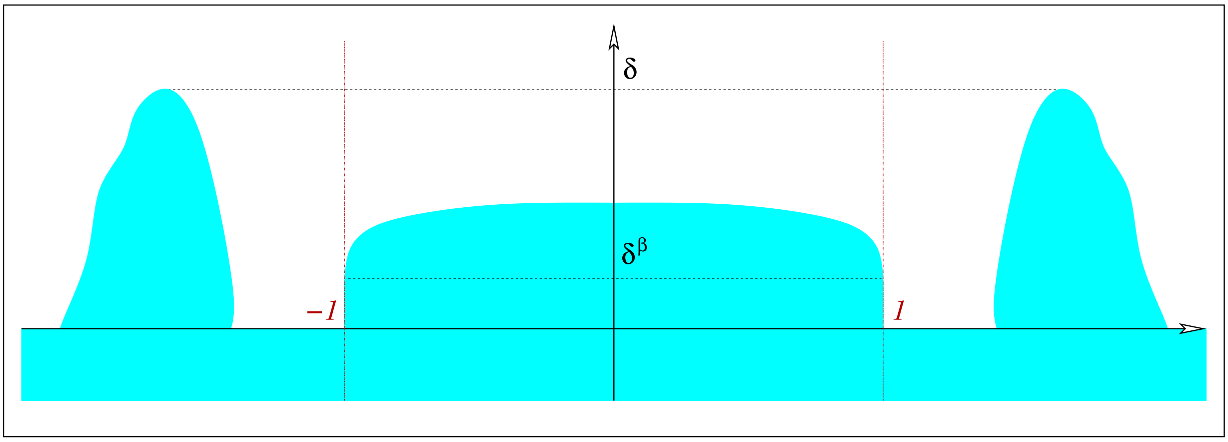

FIGURE 4. The stickiness/instability property in Theorem 1.4, with $\beta := \frac{2+\varepsilon_0}{1-2s}$.

compactly supported perturbation can cause a boundary stickiness phenomenon. We are also able to give a quantification of the size of the stickiness in terms of the size of the perturbation: namely the size of the stickiness is bounded from below by the size of the perturbation to the power $\frac{2+\varepsilon_0}{1-2s}$, for any fixed ε_0 arbitrarily small (see Figure 4). We observe that this power tends to $+\infty$ as $s \to 1/2$, which is consistent with the fact that classical minimal surfaces do not stick. The precise result that we obtain is the following:

Theorem 1.4. *Fix $\varepsilon_0 > 0$ arbitrarily small. Then, there exists $\delta_0 > 0$, possibly depending on ε_0, such that for any $\delta \in (0, \delta_0]$ the following statement holds true.*

Assume that $F \supset H \cup F_- \cup F_+$, where $H := \mathbb{R} \times (-\infty, 0)$, $F_- := (-3, -2) \times [0, \delta)$ and $F_+ := (2, 3) \times [0, \delta)$. Let E be the s-minimal set in $(-1, 1) \times \mathbb{R}$ among all the sets that coincide with F outside $(-1, 1) \times \mathbb{R}$. Then

$$E \supseteq (-1, 1) \times (-\infty, \delta^{\frac{2+\varepsilon_0}{1-2s}}].$$

The proof of Theorem 1.4 is rather delicate and it is based on the construction of suitable auxiliary barriers, which we believe are interesting in themselves. These barriers are used to detach a portion of the set in a neighborhood of the origin and their construction relies on some compensations of nonlocal integral terms. As a matter of fact, the compactly supported barriers are obtained by glueing other auxiliary barriers with polynomial growth (the latter barriers are somehow "self-sustaining solutions" and can be seen as the geometric counterparts of the s-harmonic function x_+^s).

Though quite surprising at a first glance, the sticking effects that we present in this paper have some (at least vague) heuristic explanations. Indeed, first of all, the contribution to the fractional mean curvature which comes from far may bend a nonlocal minimal surface towards the boundary of the domain: then, the points in the vicinity of the domain may end up receiving a contribution which is incompatible with the vanishing of the fractional mean curvature, due to some transverse intersection between the datum and the domain itself, thus forcing these points to stick at the boundary.

Another heuristic explanation of the stickiness phenomenon comes from the different fractional scalings that the problem exhibits at different scales. On the one hand, vanishing of the fractional mean curvature corresponds to a s-harmonicity property (i.e. a harmonicity with respect to the fractional operator $(-\Delta)^s$) for the characteristic function of the s-minimal set, with $s \in (0, 1/2)$. If the boundary of the set is the graph of a smooth function u, this gives an equation for u whose linearization corresponds to $(-\Delta)^{\frac{1}{2}+s}$, which would correspond, roughly speaking, to a regularity theory of order $C^{\frac{1}{2}+s}$ at the boundary. On the other hand, nonlocal minimal surfaces detach from free boundaries in a $C^{1,\frac{1}{2}+s}$-fashion (see [5]), which suggests that the linearized equation of the graph is not a good approximation for the boundary behavior.

The rest of the paper is organized as follows. In Section 2, we discuss the case of the stickiness to a half-ball and we prove Theorem 1.1. Then, Section 3 considers the case of a two-dimensional box with high oscillating datum, providing the proof of Theorem 1.2. The asymptotics as $s \to 0$ is presented in Section 4.

The second part of the paper is devoted to the proof of Theorem 1.4. In particular, Sections 5, 6 and 7 are devoted to the construction of the auxiliary barriers. More precisely, in Section 5 we construct barriers with a linear growth, by superposing straight lines with slowly varying slopes; then, in Section 6, we glue the barrier with linear growth with a power-like function (this is needed to obtain sharper estimates on the size of the glueing) and in Section 7 we adapt this construction to build barriers that are compactly supported.

This will allow us to prove Theorem 1.4 in Section 8. The paper ends with an appendix that contains a simple, but general, symmetry property, and an alternative proof of an integral identity.

2. Stickiness to half-balls

This section is devoted to the analysis of the stickiness phenomena to the half-ball, caused by a small half-ring as external datum. The main goal of this part is to prove Theorem 1.1. For this, we take K_δ as in (1.1), i.e.

$$K_\delta := \big(B_{1+\delta} \setminus B_1\big) \cap \{x_n < 0\}$$

and E_δ to be the set minimizing $\operatorname{Per}_s(E, B_1)$ among all the sets E such that $E \setminus B_1 = K_\delta$.

We make some auxiliary observations. First of all, we check that the s-perimeter of K_δ (and then of the minimizer) must be small if so is δ:

Lemma 2.1. *For any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $\delta \in (0, \delta_\varepsilon]$ we have that*

$$\operatorname{Per}_s(K_\delta, B_1) \leqslant \varepsilon.$$

Proof. We have

$$\operatorname{Per}_s(K_\delta, B_1) = L(B_1, K_\delta) \leqslant \iint_{B_1\times\left(B_{1+\delta}\setminus B_1\right)} \frac{dx\,dy}{|x-y|^{n+2s}}.$$

Now we observe that

$$(0,+\infty) \ni \iint_{B_1\times\left(B_2\setminus B_1\right)} \frac{dx\,dy}{|x-y|^{n+2s}} = \lim_{\delta\to 0^+} \iint_{B_1\times\left(B_2\setminus B_{1+\delta}\right)} \frac{dx\,dy}{|x-y|^{n+2s}}. \tag{2.1}$$

Indeed, the first integral in (2.1) is finite, see for instance Lemma 11 in [8] (applied here with $\varepsilon := 1$, $\Omega := B_2$ and $F := B_1$). As a consequence of (2.1), for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $\delta \in (0, \delta_\varepsilon]$ we have

$$\left| \iint_{B_1\times\left(B_2\setminus B_1\right)} \frac{dx\,dy}{|x-y|^{n+2s}} - \iint_{B_1\times\left(B_2\setminus B_{1+\delta}\right)} \frac{dx\,dy}{|x-y|^{n+2s}} \right| \leqslant \varepsilon,$$

which gives the desired result. □

Next result proves that the boundary of the minimal set E_δ can only lie in the neighborhood of ∂B_1, if δ is small enough. More precisely:

Lemma 2.2. *For any $\varepsilon \in (0, 1)$ there exists $\delta_\varepsilon > 0$ such that for any $\delta \in (0, \delta_\varepsilon]$ we have that*

$$(\partial E_\delta) \cap B_{1-\varepsilon} = \varnothing.$$

Proof. We observe that it is enough to prove the desired claim for small ε (since this would imply the claim for bigger ε). The proof is by contradiction. Suppose that there exists $p \in (\partial E_\delta) \cap B_{1-\varepsilon}$. Then $B_{\varepsilon/2}(p) \subset B_1$ and so, by the Clean Ball Condition (see Corollary 4.3 in [6]), there exist p_1, $p_2 \in B_1$ such that

$$B_{c\varepsilon}(p_1) \subset E \cap B_{\varepsilon/2}(p) \qquad \text{and} \qquad B_{c\varepsilon}(p_2) \subset E^c \cap B_{\varepsilon/2}(p),$$

for a suitable constant $c > 0$. In particular, both $B_{c\varepsilon}(p_1)$ and $B_{c\varepsilon}(p_2)$ lie inside B_1, and if $x \in B_{c\varepsilon}(p_1)$ and $y \in B_{c\varepsilon}(p_2)$ then $|x - y| \leqslant \varepsilon$. As a consequence

$$\operatorname{Per}_s(E_\delta, B_1) \geqslant L\big(B_{c\varepsilon}(p_1), B_{c\varepsilon}(p_2)\big) \geqslant \frac{|B_{c\varepsilon}(p_1)|\,|B_{c\varepsilon}(p_2)|}{\varepsilon^{n+2s}} = c_o \varepsilon^{n-2s},$$

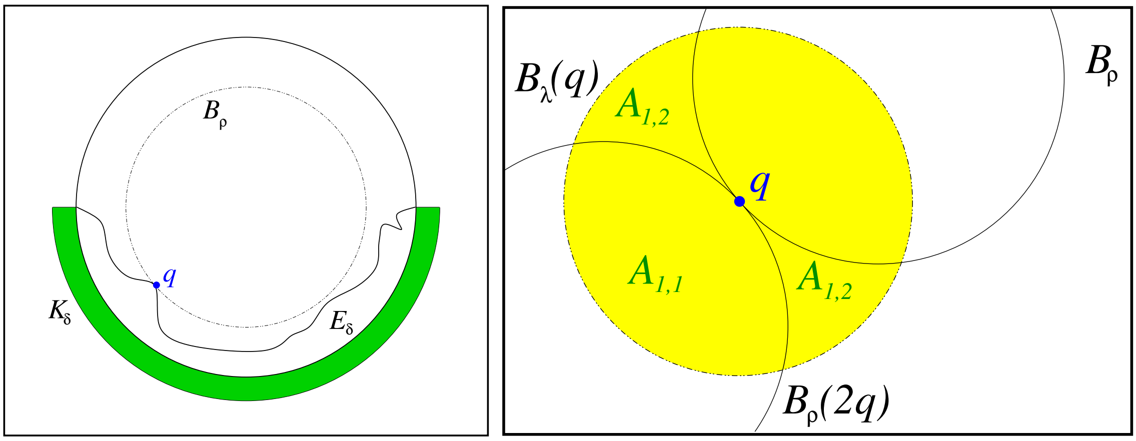

FIGURE 5. Touching the set E_δ coming from the origin.

for some $c_o > 0$. On the other hand, by Lemma 2.1 (used here with ε^n in the place of ε), we have that $\mathrm{Per}_s(E_\delta, B_1) \leqslant \mathrm{Per}_s(K_\delta, B_1) \leqslant \varepsilon^n$ provided that δ is suitably small with respect to ε. As a consequence, we obtain that $\varepsilon^n \geqslant c_o \varepsilon^{n-2s}$, which is a contradiction if ε is small enough. $\square$

The statement of Lemma 2.2 can be better specified, as follows:

Corollary 2.3. *For any $\varepsilon \in (0,1)$ there exists $\delta_\varepsilon > 0$ such that for any $\delta \in (0, \delta_\varepsilon]$ we have that*

$$E_\delta \cap B_{1-\varepsilon} = \varnothing.$$

Proof. Without loss of generality, we may suppose that $\varepsilon \in (0, 1/2)$. The proof is by contradiction. Suppose that $E_\delta \cap B_{1-\varepsilon} \neq \varnothing$. Then, by Lemma 2.2, we have that $B_{1-\varepsilon} \subseteq E_\delta$. Moreover, if we set

$$H := \big(B_2 \setminus B_1\big) \cap \{x_n > 0\},$$

we have that $H \subseteq E_\delta^c$. As a consequence,

$$\mathrm{Per}_s(E_\delta, B_1) \geqslant L(B_{1-\varepsilon}, H) \geqslant L(B_{1/2}, H) \geqslant c,$$

for some $c > 0$. This is in contradiction with Lemma 2.1 and so it proves the desired result. $\square$

With this, we are in the position of completing the proof of Theorem 1.1:

Proof of Theorem 1.1. We need to show that $E_\delta \cap B_1 = \varnothing$. By contradiction, suppose not. Then there exists

$$p \in E_\delta \cap B_1. \tag{2.2}$$

By Corollary 2.3, we know that

$$B_r \subset E_\delta^c \text{ if } r \in (0, 1-\varepsilon). \tag{2.3}$$

We enlarge r till B_r hits ∂E_δ. That is, by (2.2), there exists $\rho \in [1-\varepsilon, 1)$ such that $B_\rho \subset E_\delta^c$ and there exists $q \in (\partial B_\rho) \cap (\partial E_\delta)$ (see Figure 5).

Therefore, using the Euler-Lagrange equation in the viscosity sense (see Theorem 5.1 in [6]), we conclude that

$$\int_{\mathbb{R}^n} \frac{\chi_{E_\delta^c}(y) - \chi_{E_\delta}(y)}{|q-y|^{n+2s}}\, dy \leqslant 0. \tag{2.4}$$

By (2.3), we know that

$$E_\delta \subseteq (B_1 \setminus B_\rho) \cup K_\delta \subseteq B_{1+\delta} \setminus B_\rho$$

and so

$$\int_{\mathbb{R}^n} \frac{\chi_{E_\delta}(y) - \chi_{E_\delta^c}(y)}{|q-y|^{n+2s}} \leqslant \int_{B_{1+\delta}\setminus B_\rho} \frac{dy}{|q-y|^{n+2s}} - \int_{B_\rho} \frac{dy}{|q-y|^{n+2s}}. \tag{2.5}$$

In addition, if $y \in B_{1/2}$, then $|q-y| \leqslant |q|+|y| < 2$ and so

$$\int_{B_{1/2}} \frac{dy}{|q-y|^{n+2s}} \geqslant \tilde{c}, \tag{2.6}$$

for some $\tilde{c} > 0$.

Now we define $\lambda := (\varepsilon+\delta)^{\frac{1}{2(n+2s)}}$. We notice that λ is small if so are ε and δ, and so $B_\lambda(q) \subset B^c_{1/2}$. Then, formula (2.6) gives that

$$\int_{B_\rho} \frac{dy}{|q-y|^{n+2s}} \geqslant \tilde{c} + \int_{B_\lambda(q)\cap B_\rho} \frac{dy}{|q-y|^{n+2s}}.$$

This, (2.4) and (2.5) give that

$$\int_{B_{1+\delta}\setminus B_\rho} \frac{dy}{|q-y|^{n+2s}} - \int_{B_\lambda(q)\cap B_\rho} \frac{dy}{|q-y|^{n+2s}} \geqslant \tilde{c}. \tag{2.7}$$

Now we define

$$A_1 := \big(B_{1+\delta}\setminus B_\rho\big) \cap B_\lambda(q) \quad \text{and} \quad A_2 := \big(B_{1+\delta}\setminus B_\rho\big) \setminus B_\lambda(q).$$

We notice that

$$\int_{A_2} \frac{dy}{|q-y|^{n+2s}} \leqslant \frac{|A_2|}{\lambda^{n+2s}} \leqslant \frac{|B_{1+\delta}\setminus B_\rho|}{\lambda^{n+2s}} \leqslant \frac{C\,(\varepsilon+\delta)}{\lambda^{n+2s}} = C\,\sqrt{\varepsilon+\delta},$$

for some $C>0$. Hence, (2.7) becomes

$$\int_{A_1} \frac{dy}{|q-y|^{n+2s}} - \int_{B_\lambda(q)\cap B_\rho} \frac{dy}{|q-y|^{n+2s}} \geqslant \frac{\tilde{c}}{2}. \tag{2.8}$$

Now we set

$$A_{1,1} := A_1 \cap B_\rho(2q) \quad \text{and} \quad A_{1,2} := A_1 \setminus B_\rho(2q),$$

see again Figure 5. We remark that $B_\rho(2q)$ is tangent to B_ρ at the point q, and $A_{1,1} \subseteq B_\lambda(q)\cap B_\rho(2q)$. Therefore, by symmetry

$$\int_{A_{1,1}} \frac{dy}{|q-y|^{n+2s}} \leqslant \int_{B_\lambda(q)\cap B_\rho(2q)} \frac{dy}{|q-y|^{n+2s}} = \int_{B_\lambda(q)\cap B_\rho} \frac{dy}{|q-y|^{n+2s}}. \tag{2.9}$$

Now we observe that $A_{1,2}$ is trapped between B_ρ and $B_\rho(2q)$, and it lies in $B_\lambda(q)$ therefore (see e.g. Lemma 3.1 in [13])

$$\int_{A_{1,2}} \frac{dy}{|q-y|^{n+2s}} \leqslant C\rho^{-2s}\lambda^{1-2s} \leqslant C\lambda^{1-2s} = C\,(\varepsilon+\delta)^{\frac{1-2s}{2(n+2s)}},$$

up to renaming constants.

The latter estimate and (2.9) give

$$\int_{A_1} \frac{dy}{|q-y|^{n+2s}} \leqslant \int_{B_\lambda(q)\cap B_\rho} \frac{dy}{|q-y|^{n+2s}} + C\,(\varepsilon+\delta)^{\frac{1-2s}{2(n+2s)}}.$$

By inserting this information into (2.8), we obtain $2C\,(\varepsilon+\delta)^{\frac{1-2s}{2(n+2s)}} \geqslant \tilde{c}$, which leads to a contradiction by choosing ε small enough (and thus $\delta \leqslant \delta_\varepsilon$ small). □

3. Stickiness to the sides of a box

In this section, we discuss the stickiness properties to the sides of a box with high oscillatory external data and we prove Theorem 1.2. To this goal, we recall that the set J_M has been defined in (1.2) and E_M is the s-minimal set in $(-1,1)\times\mathbb{R}$ with datum outside $(-1,1)\times\mathbb{R}$ equal to J_M.

We first establish an easier version of Theorem 1.2, in which the sticking size is proved to be at least of the order of the oscillation (then, a refined estimate will lead to the proof of Theorem 1.2).

Proposition 3.1. *There exist $M_o > 0$, $c_o \in (0,1)$, depending on s, such that if $M \geqslant M_o$ then*

$$[-1,1) \times [c_o M, M] \subseteq E_M^c \tag{3.1}$$

$$\text{and} \qquad (-1,1] \times [-M, -c_o M] \subseteq E_M. \tag{3.2}$$

Proof. We denote coordinates in $\mathbb{R}^2$ by $x = (x_1, x_2)$. We take $\varepsilon_o > 0$, to be chosen conveniently small in the sequel. Let $t \in [0, \varepsilon_o^2]$. We considers balls of radius $\varepsilon_o M$ with center lying on the straight line $\{x_2 = (1-t)M\}$. The idea of the proof is to slide a ball of this type from left to right till we touch ∂E_M. We will show that the touching point can only occur along the boundary $\{x_1 = 1\}$. Hence, by varying $t \in [0, \varepsilon_o^2]$, we obtain that $[-1,1) \times [(1-\varepsilon_o^2)M, M]$ is contained in E_M^c. This would complete the proof of (3.1) (and the proof of (3.2) is similar).

The details of the proof of (3.1) are the following. We fix $t \in [0, \varepsilon_o^2]$. If $x_1 < -M-2$, then the ball $B_{\varepsilon_o M}(x_1, (1-t)M)$ lies in $(-\infty, -2) \times \mathbb{R}$, and so its closure is contained in E_M^c. Hence, we consider $\ell \geqslant -M-2$ such that $\overline{B_{\varepsilon_o M}(\ell, (1-t)M)} \subseteq E_M^c$ for any $x_1 < \ell$ and there exists $q = (q_1, q_2) \in (\partial E_M) \cap (\partial B_{\varepsilon_o M}(\ell, (1-t)M))$. The proof of (3.1) is complete if we show that

$$q_1 \geqslant 1. \tag{3.3}$$

To prove this, we argue by contradiction. If not, then $q_1 \in [-1, 1)$, therefore, by the Euler-Lagrange inequality (see Theorem 5.1 in [6]),

$$\int_{\mathbb{R}^2} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|q-y|^{2+2s}}\, dy \leqslant 0. \tag{3.4}$$

Now we denote by $z := (\ell, (1-t)M))$ the center of the touching ball. We also consider the extremal point of the touching ball on the right, that we denote by $p := z + (\varepsilon_o M, 0)$. We claim that

$$|q_2 - p_2| \leqslant 8\sqrt{\varepsilon_o M}. \tag{3.5}$$

To prove this, we observe that, by construction, both q and p lie in $[-1,1] \times \mathbb{R}$, hence $|q_1|$, $|p_1| \leqslant 1$, consequently

$$|q_1 - p_1| \leqslant 2. \tag{3.6}$$

Also, both q and p lie on the boundary of the touching ball, namely $|q-z| = \varepsilon_o M = |p-z|$, therefore

$$\begin{aligned} 0 = |q-z|^2 - |p-z|^2 &= |q|^2 - 2q\cdot z - |p|^2 + 2p \cdot z = (q-p)\cdot(q+p-2z) \\ &= (q_1-p_1)(q_1+p_1-2z_1) + (q_2-p_2)(q_2+p_2-2z_2) \\ &= (q_1-p_1)(q_1-p_1+2\varepsilon_o M) + (q_2-p_2)(q_2-p_2) \\ &\geqslant -4(1+\varepsilon_o M) + |q_2-p_2|^2. \end{aligned}$$

This establishes (3.5), provided that M is large enough (possibly in dependence of ε_o).

Now we consider the symmetric ball to the touching ball, with respect to the touching point q. That is, we define $\bar{z} := z + 2(q-z)$ and consider the ball $B_{\varepsilon_o M}(\bar{z})$. We remark that

$$B_{\varepsilon_o M}(z) \text{ and } B_{\varepsilon_o M}(\bar{z}) \text{ are tangent to each other at } q. \tag{3.7}$$

We also claim that

$$B_{\varepsilon_o M}(\bar{z}) \cap \left\{x_2 > \bar{z}_2 + 2\varepsilon_o^2 M\right\} \subseteq \{x_2 > M\}. \tag{3.8}$$

To prove this, we observe that

$$-\varepsilon_o^2 M - 16\sqrt{\varepsilon_o M} + 2\varepsilon_o^2 M = \varepsilon_o^2 M \left(1 - \frac{16}{\varepsilon_o^{3/2}\sqrt{M}}\right) > 0,$$

if M is large enough. Hence, recalling (3.5),

$$\begin{aligned} \bar{z}_2 + 2\varepsilon_o^2 M = z_2 + 2(q_2 - z_2) + 2\varepsilon_o^2 M &= (1-t)M + 2(q_2-p_2) + 2\varepsilon_o^2 M \\ &\geqslant (1-\varepsilon_o^2)M - 16\sqrt{\varepsilon_o M} + 2\varepsilon_o^2 M > M. \end{aligned}$$

This proves (3.8).

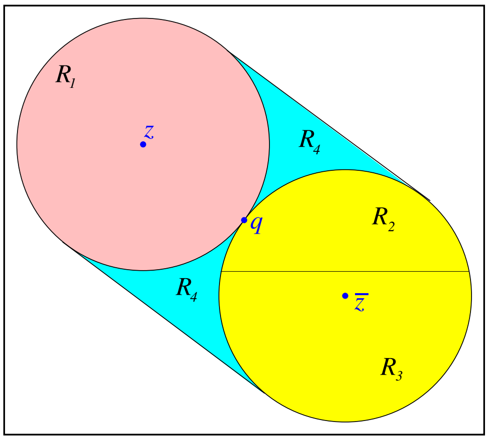

FIGURE 6. The partition of the plane needed for the proof of Proposition 3.1.

Now we decompose $\mathbb{R}^2$ into five nonoverlapping regions. Namely, we consider

$$\begin{aligned} & R_1 := B_{\varepsilon_o M}(z), \\ & R_2 := B_{\varepsilon_o M}(\bar z) \cap \left\{x_2 > \bar z_2 + 2\varepsilon_o^2 M\right\} \\ \text{and} \qquad & R_3 := B_{\varepsilon_o M}(\bar z) \cap \left\{x_2 \leqslant \bar z_2 + 2\varepsilon_o^2 M\right\}. \end{aligned}$$

Then we define $D := B_{\varepsilon_o M}(z) \cup B_{\varepsilon_o M}(\bar z)$, K the convex hull of D and $R_4 := K \setminus D$. Finally, we set $R_5 := \mathbb{R}^2 \setminus K$ and consider the partition of $\mathbb{R}^2$ given by the regions $R_1, \dots, R_5$.

We consider the contribution to the integral in (3.4) given by these regions. The regions R_1, R_2 and R_3 will be considered together: namely, $R_1 \subseteq E_M^c$, and, by (3.8), also $R_2 \subseteq E_M^c$. Therefore, by symmetry

$$\int_{R_1\cup R_2\cup R_3} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \geqslant \int_{R_1\cup R_2} \frac{dy}{|q-y|^{2+2s}} - \int_{R_3} \frac{dy}{|q-y|^{2+2s}} = \int_{R_2} \frac{2\,dy}{|q-y|^{2+2s}}. \tag{3.9}$$

Now, for $y \in R_2$, we consider the change of variable $\tilde y = T(y) := (y-q)/(\varepsilon_o M)$. We have that

$$\begin{aligned} T(R_2) \;&=\; B_1\left(\frac{q-z}{\varepsilon_o M}\right) \cap \left\{\tilde y_2 > \frac{q_2 - z_2}{\varepsilon_o M} + 2\varepsilon_o\right\} \\ &\supseteq\; B_1\left(\frac{q-z}{\varepsilon_o M}\right) \cap \left\{\tilde y_2 > 3\varepsilon_o\right\}, \end{aligned} \tag{3.10}$$

where we used again (3.5) in the last inclusion (provided that ε_o is sufficiently small and M is sufficiently large, possibly in dependence of ε_o).

Now we claim that

$$B_{\varepsilon_o}(5\varepsilon_o, 5\varepsilon_o) \subseteq B_1\left(\frac{q-z}{\varepsilon_o M}\right) \cap \left\{\tilde y_2 > 3\varepsilon_o\right\}. \tag{3.11}$$

To prove this, it is enough to take $\eta \in B_{\varepsilon_o}$ and show that

$$(5\varepsilon_o, 5\varepsilon_o) + \eta \in B_1\left(\frac{q-z}{\varepsilon_o M}\right). \tag{3.12}$$

For this, we use (3.5) to observe that

$$|q_1 - z_1|^2 = |q-z|^2 - |q_2 - z_2|^2 \geqslant (\varepsilon_o M)^2 - 64\varepsilon_o M. \tag{3.13}$$

Moreover, by (3.6),

$$q_1 - z_1 = q_1 - p_1 + p_1 - z_1 = q_1 - p_1 + \varepsilon_o M \geqslant \varepsilon_o M - 2 > 0.$$

Hence, (3.13) gives that

$$\frac{q_1 - z_1}{\varepsilon_o M} = \frac{|q_1 - z_1|}{\varepsilon_o M} \geqslant \sqrt{1 - \frac{64}{\varepsilon_o M}} \geqslant 1 - \frac{128}{\varepsilon_o M},$$

if M is large enough. In particular

$$\frac{q_1-z_1}{\varepsilon_o M}-5\varepsilon_o-\eta_1\geqslant 1-\frac{128}{\varepsilon_o M}-6\varepsilon_o\geqslant\frac{1}{2}-\frac{128}{\varepsilon_o M}>0,$$

provided that ε_o is small enough and M large enough (possibly depending on ε_o). Therefore

$$\left|\frac{q_1-z_1}{\varepsilon_o M}-5\varepsilon_o-\eta_1\right|=\frac{q_1-z_1}{\varepsilon_o M}-5\varepsilon_o-\eta_1\leqslant\frac{|q_1-z_1|}{\varepsilon_o M}-4\varepsilon_o\leqslant 1-4\varepsilon_o.$$

In addition, by (3.5),

$$\left|\frac{q_2-z_2}{\varepsilon_o M}-5\varepsilon_o-\eta_2\right|\leqslant\frac{|q_2-z_2|}{\varepsilon_o M}+6\varepsilon_o\leqslant 7\varepsilon_o.$$

Therefore

$$\begin{aligned}\left|\frac{q-z}{\varepsilon_o M}-(5\varepsilon_o,5\varepsilon_o)-\eta\right|^2&\leqslant(1-4\varepsilon_o)^2+(7\varepsilon_o)^2\\&=1-8\varepsilon_o+16\varepsilon_o^2+49\varepsilon_o^2<1\end{aligned}$$

if ε_o is small enough. This establishes (3.12) and therefore (3.11).

From (3.10) and (3.11), we see that

$$T(R_2)\supseteq B_{\varepsilon_o}(5\varepsilon_o,5\varepsilon_o)$$

and then

$$\int_{R_2}\frac{dy}{|q-y|^{2+2s}}=\frac{1}{(\varepsilon_o M)^{2s}}\int_{T(R_2)}\frac{d\tilde y}{|\tilde y|^{2+2s}}\geqslant\frac{1}{(\varepsilon_o M)^{2s}}\int_{B_{\varepsilon_o}(5\varepsilon_o,5\varepsilon_o)}\frac{d\tilde y}{|\tilde y|^{2+2s}}.\tag{3.14}$$

Now, if $\tilde y\in B_{\varepsilon_o}(5\varepsilon_o,5\varepsilon_o)$ then $|\tilde y|\leqslant\varepsilon_o+|(5\varepsilon_o,5\varepsilon_o)|\leqslant 10\varepsilon_o$, and then (3.14) gives that

$$\int_{R_2}\frac{dy}{|q-y|^{2+2s}}\geqslant\frac{\tilde c}{\varepsilon_o^{4s}M^{2s}},$$

for some $\tilde c>0$. By inserting this into (3.9) we conclude that

$$\int_{R_1\cup R_2\cup R_3}\frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy\geqslant\frac{\tilde c}{\varepsilon_o^{4s}M^{2s}}.\tag{3.15}$$

Moreover (see e.g. Lemma 3.1 in [13] with $R:=\varepsilon_o M$ and $\lambda:=1$), we see that

$$\left|\int_{R_4}\frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy\right|\leqslant\int_{R_4}\frac{dy}{|q-y|^{2+2s}}\leqslant\frac{C}{\varepsilon_o^{2s}M^{2s}},\tag{3.16}$$

for some $C>0$. Furthermore, the distance from q to any point of R_5 is at least $\varepsilon_o M$, therefore $R_5\subseteq\mathbb{R}^2\setminus B_{\varepsilon_o M}(q)$, and

$$\left|\int_{R_5}\frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy\right|\leqslant\int_{\mathbb{R}^2\setminus B_{\varepsilon_o M}(q)}\frac{dy}{|q-y|^{2+2s}}=\frac{\tilde C}{\varepsilon_o^{2s}M^{2s}},$$

for some $\tilde C>0$.

By combining the latter estimate with (3.15) and (3.16), we obtain that

$$\int_{\mathbb{R}^2}\frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy\geqslant\frac{1}{\varepsilon_o^{2s}M^{2s}}\left(\frac{\tilde c}{\varepsilon_o^{2s}}-C-\tilde C\right)>0,$$

provided that ε_o is suitably small. This estimate is in contradiction with (3.4) and therefore the proof of (3.3) is complete. □

The result in Proposition 3.1 can be refined. Namely, not only the optimal set E_M in Proposition 3.1 sticks for an amount of order M is a box of side M, but it sticks up to an order of $M^{\frac{1+2s}{2+2s}}$ from the origin, as the following Proposition 3.2 points out. As a matter of fact, the exponent $\frac{1+2s}{2+2s}$ is sharp, as we will prove in the subsequent Proposition 3.3.

Proposition 3.2. *There exist M_o, $C_o > 0$, depending on s, such that if $M \geqslant M_o$ then*

$$[-1,1) \times [C_o M^{\frac{1+2s}{2+2s}},\, M] \subseteq E_M^c \tag{3.17}$$

$$\text{and} \qquad (-1,1] \times [-M,\, -C_o M^{\frac{1+2s}{2+2s}}] \subseteq E_M. \tag{3.18}$$

Proof. We let $\beta := \frac{1+2s}{2+2s}$. We focus on the proof of (3.17) (the proof of (3.18) is similar). The proof is based on a sliding method: we will consider a suitable surface and we slide it from left to right in order to "clean" the portion of space $[-1,1) \times [C_o M^\beta, M]$. As a matter of fact, by Proposition 3.1, it is enough to take care of $[-1,1) \times [C_o M^\beta,\, c_o M]$, with $c_o \in (0,1)$.

For this we fix any

$$t \in [C_o M^\beta,\, c_o M] \tag{3.19}$$

and, for any $\mu \in \mathbb{R}$, we define

$$S_\mu := B_{M^{2\beta}}(\mu - M^{2\beta}, t) \cap \{|x_2 - t| < 4M^\beta\}.$$

Notice that if $\mu < -1$ then

$$S_\mu \subseteq (-\infty, -1) \times \{|x_2 - t| < 4M^\beta\} \subseteq E_M^c.$$

Therefore we increase μ till S_μ touches ∂E_M. This value of μ will be fixed from now on. We observe that Proposition 3.2 is proved if we show that $\mu = 1$. So we assume by contradiction that $\mu \in [-1,1)$. By construction, we have that

$$S_\mu \subseteq E_M^c \tag{3.20}$$

and there exists $q \in (\partial S_\mu) \cap (\partial E_M)$, with $q_1 \in [-1,1)$. We claim that

$$|q_2 - t| \leqslant 2M^\beta. \tag{3.21}$$

To prove this, we observe that $|q_1 - \mu + M^{2\beta}| \geqslant M^{2\beta} - |q_1| - |\mu| \geqslant M^{2\beta} - 2$. Moreover, $q \in \partial S_\mu \subseteq \overline{B_{M^{2\beta}}(\mu - M^{2\beta}, t)}$, therefore

$$M^{4\beta} \geqslant \left|q - (\mu - M^{2\beta}, t)\right|^2 \geqslant (M^{2\beta} - 2)^2 + |q_2 - t|^2 \geqslant M^{4\beta} - 4M^{2\beta} + |q_2 - t|^2,$$

from which we obtain (3.21).

Now, using the Euler-Lagrange equation in the viscosity sense (see Theorem 5.1 in [6]), we see that

$$\int_{\mathbb{R}^n} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|q-y|^{2+2s}}\, dy \leqslant 0. \tag{3.22}$$

We first estimate the contribution to the integral above coming from $B_{M^\beta}(q)$. For this, we consider the symmetric point of $z := (\mu - M^{2\beta}, t)$ with respect to q, namely we set $z' := z + 2(q - z)$. We also consider the ball $B' := B_{M^{2\beta}}(z')$. Notice that $B_{M^{2\beta}}(z)$ and B' are tangent one to the other at q. We define $A_1 := B_{M^{2\beta}}(z) \cap B_{M^\beta}(q)$, $A_2 := B' \cap B_{M^\beta}(q)$ and $A_3 := B_{M^\beta}(q) \setminus (A_1 \cup A_2)$. Hence (see e.g. Lemma 3.1 in [13], used here with $R := M^{2\beta}$ and $\lambda := M^{-\beta}$), we obtain that

$$\left| \int_{A_3} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|q-y|^{2+2s}}\, dy \right| \leqslant \int_{A_3} \frac{dy}{|q-y|^{2+2s}} \leqslant C M^{-\beta(1+2s)}. \tag{3.23}$$

Now we observe that

$$A_1 \subseteq E_M^c. \tag{3.24}$$

For this, let $y \in A_1$. Then $|y - q| < M^\beta$. Therefore, recalling (3.21),

$$|y_2 - t| \leqslant |y_2 - q_2| + |q_2 - t| < M^\beta + 2M^\beta < 4M^\beta.$$

Since also $y \in B_{M^{2\beta}}(z)$, we obtain that $y \in S_\mu$. Then we use (3.20) and we finish the proof of (3.24).

Then, we use (3.24) and a symmetry argument to see that

$$\int_{A_1 \cup A_2} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|q-y|^{2+2s}}\, dy = \int_{A_1} \frac{dy}{|q-y|^{2+2s}} + \int_{A_2} \frac{\chi_{E_M^c}(y) - \chi_{E_M}(y)}{|q-y|^{2+2s}}\, dy \geqslant 0.$$

This and (3.23) give that

$$\int_{B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \geqslant -CM^{-\beta(1+2s)}.$$

Consequently, by (3.22),

$$\int_{\mathbb{R}^n\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \leqslant -\int_{B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \leqslant CM^{-\beta(1+2s)}. \tag{3.25}$$

Now we observe that

$$\{|x_1-q_1|\leqslant 16\}\setminus B_{M^\beta}(q) \subseteq \{|x_1-q_1|\leqslant 16\}\times\left\{|x_2-q_2|\geqslant \frac{M^\beta}{2}\right\}. \tag{3.26}$$

To prove this, let $y\in\{|x_1-q_1|\leqslant 16\}\setminus B_{M^\beta}(q)$ and suppose, by contradiction, that $|y_2-q_2|<M^\beta/2$. Then

$$|y-q|^2 \leqslant 16^2+\frac{M^{2\beta}}{4} < M^{2\beta}.$$

This would say that $y\in B_{M^\beta}(q)$, which is a contradiction, and so (3.26) is proved.

By (3.26), we obtain that

$$\begin{aligned}\left|\int_{\{|x_1-q_1|\leqslant 16\}\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy\right| &\leqslant \int_{\{|x_1-q_1|\leqslant 16\}\times\left\{|x_2-q_2|\geqslant \frac{M^\beta}{2}\right\}} \frac{dy}{|q-y|^{2+2s}}\\ &\leqslant \int_{q_1-16}^{q_1+16}\left(\int_{\{|q_2-y_2|\geqslant M^\beta/2\}} \frac{dy_2}{|q_2-y_2|^{2+2s}}\right)dy_1 = CM^{-\beta(1+2s)},\end{aligned}$$

for some $C>0$.

From this and (3.25), we obtain that

$$\int_{\{|x_1-q_1|>16\}\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \leqslant CM^{-\beta(1+2s)}, \tag{3.27}$$

up to renaming $C>0$.

Now we define $H_1:=\{x_1-q_1<-16\}$ and $H_2:=\{x_1-q_1>16\}$. Notice that $H_1\subseteq\{x_1<-15\}$ and $H_2\subseteq\{x_1>15\}$. Therefore $H_1\cap\{x_2>-M\}\subseteq E_M^c$, $H_1\cap\{x_2<-M\}\subseteq E_M$, $H_2\cap\{x_2>M\}\subseteq E_M^c$ and $H_2\cap\{x_2<M\}\subseteq E_M$.

Then, we define, for any $i\in\{1,2\}$,

$$\begin{aligned} H_{i,1} &:= H_i\cap\{x_2>2q_2+M\},\\ H_{i,2} &:= H_i\cap\{x_2\in(M,2q_2+M]\},\\ H_{i,3} &:= H_i\cap\{x_2\in[-M,M]\},\\ H_{i,4} &:= H_i\cap\{x_2<-M\},\end{aligned}$$

see Figure 7.

By construction, $H_{i,1}\subseteq E_M^c$ and $H_{i,4}\subseteq E_M$, therefore, by up/down symmetry,

$$\int_{(H_{1,1}\cup H_{1,4})\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy = 0 = \int_{(H_{2,1}\cup H_{2,4})\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy. \tag{3.28}$$

Moreover, $H_{1,3}\subseteq E_M^c$ and $H_{2,3}\subseteq E_M$, therefore, by left/right symmetry,

$$\int_{(H_{1,3}\cup H_{2,3})\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy = 0. \tag{3.29}$$

Finally, we point out that $H_{1,2}\cup H_{2,2}\subseteq E_M^c$ and (recalling (3.21) and (3.19)) that

$$B_{M^\beta}(q)\subseteq\{x_2<q_2+M^\beta\}\subseteq\{x_2<t+3M^\beta\}\subseteq\{x_2<M\}.$$

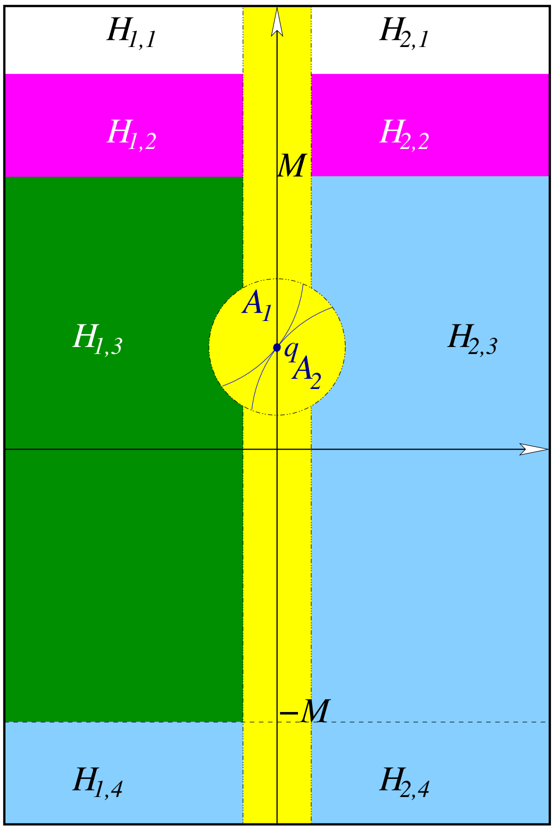

FIGURE 7. The geometry involved in the proof of Proposition 3.2.

Therefore

$$\int_{(H_{1,2}\cup H_{2,2})\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy = \int_{H_{1,2}\cup H_{2,2}} \frac{dy}{|q-y|^{2+2s}} \tag{3.30}$$
$$\geqslant \int_{\{y_1-q_1\in(16,16+M),\ y_2\in(M,2q_2+M)\}} \frac{dy}{|q-y|^{2+2s}}.$$

Now we observe that if $y_1-q_1\in(16,16+M)$ and $y_2\in(M,2q_2+M)$, then $|q-y|\leqslant CM$, for some $C>0$. Then (3.30) implies that

$$\int_{(H_{1,2}\cup H_{2,2})\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \geqslant cq_2M^{-1-2s},$$

for some $c>0$. As a consequence of (3.21) and (3.19), we also know that $q_2\geqslant t-2M^\beta\geqslant (C_o-2)M^\beta\geqslant C_oM^\beta/2$, if C_o is taken suitably large. Hence we obtain

$$\int_{(H_{1,2}\cup H_{2,2})\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \geqslant cC_oM^{\beta-1-2s}, \tag{3.31}$$

up to renaming $c>0$. Now we observe that

$$\beta-1-2s=\frac{(1+2s)(1-2-2s)}{2+2s}=-\beta(1+2s),$$

so we can write (3.31) as

$$\int_{(H_{1,2}\cup H_{2,2})\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \geqslant cC_oM^{-\beta(1+2s)}.$$

This, together with (3.28) and (3.29), gives that

$$\int_{\{|x_1-q_1|>16\}\setminus B_{M^\beta}(q)} \frac{\chi_{E_M^c}(y)-\chi_{E_M}(y)}{|q-y|^{2+2s}}\,dy \geqslant cC_oM^{-\beta(1+2s)}.$$

By comparing this inequality with (3.27), we obtain that

$$cC_oM^{-\beta(1+2s)} \leqslant CM^{-\beta(1+2s)},$$

which is a contradiction if C_o is large enough. This completes the proof of Proposition 3.2. $\square$

As a counterpart of Proposition 3.2, we show that the stickiness to the boundary of the domain does not get too close to the origin, as next result points out:

Proposition 3.3. *In the setting of Proposition 3.2, suppose that*

$$[-1,1)\times[bM^{\frac{1+2s}{2+2s}},\,M]\subseteq E_M^c, \tag{3.32}$$

with $p=(1,bM^{\frac{1+2s}{2+2s}})\in\partial E_M$, for some $b\geqslant 0$. Then $b\geqslant C_o$, for some $C_o>0$, only depending on s, provided that M is large enough.

Proof. For short, we set $\beta:=\frac{1+2s}{2+2s}$. We remark that

$$1-\frac{\beta}{1+2s}=1-\frac{1}{2+2s}=\beta. \tag{3.33}$$

We argue by contradiction, supposing that

$$b\leqslant C_o \tag{3.34}$$

for some $C_o\in(0,1)$ that we can take conveniently small in the sequel. By Lemma A.1 (used here with $T(x):=-x$), we have that E_M is odd with respect to the origin. This and (3.32) give that

$$(-1,1]\times[-M,\,-bM^\beta]\subseteq E_M. \tag{3.35}$$

Now we let $L:=M-bM^\beta$ and we consider the cube Q of side $2L$ that has the point p on its left side, namely

$$Q:=(1,1+2L)\times(M-2L,M).$$

Notice that

$$Q\subseteq E_M, \tag{3.36}$$

by the boundary datum of the problem. We also take the symmetric reflection of Q with respect to $\{x_1=1\}$, that is we set

$$Q':=(1-2L,1)\times(M-2L,M).$$

We also set

$$G:=(-1,1)\times(-3bM^\beta-2,\,-bM^\beta-1).$$

We claim that

$$G\subseteq Q'. \tag{3.37}$$

Indeed, if $x_1\in(-1,1)$ and $x_2\in(-3bM^\beta-2,\,-bM^\beta-1)$, then

$$1-2L=1-2M+2bM^\beta\leqslant 1-2M+2M^\beta<-1<x_1,$$

since M is large. Also

$$M-2L=-M+2bM^\beta<-3bM^\beta-2<x_2,$$

using again that M is large. Accordingly, $x_1\in(1-2L,1)$ and $x_2\in(M-2L,M)$, which proves (3.37).

Now we claim that

$$G\subseteq(-1,1]\times[-M,\,-bM^\beta]. \tag{3.38}$$

Indeed, if $x_2\in(-3bM^\beta-2,\,-bM^\beta-1)$, then

$$-M<-3M^\beta-2\leqslant-3bM^\beta-2<x_2,$$

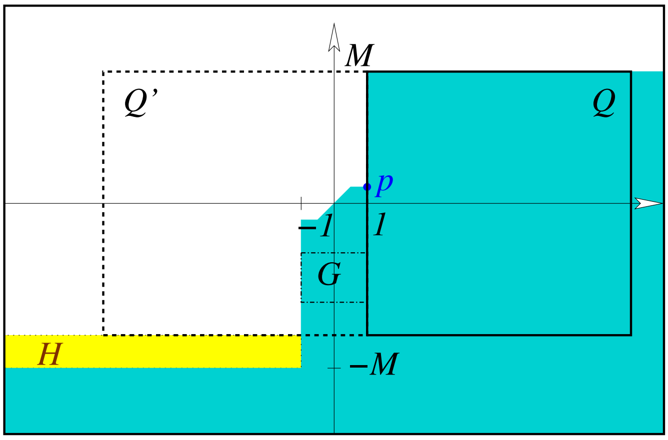

FIGURE 8. The geometry involved in the proof of Proposition 3.3.

for large M, and so $x_2 \in [-M, -bM^\beta]$, which proves (3.38).

From (3.35), (3.37) and (3.38), we obtain that

$$G \subseteq Q' \cap E_M.$$

Using this and (3.36), by a symmetry argument we conclude that

$$\int_{Q\cup Q'} \frac{\chi_{E_M}(y) - \chi_{E_M^c}(y)}{|p-y|^{2+2s}}\,dy \geqslant \int_G \frac{dy}{|p-y|^{2+2s}}. \tag{3.39}$$

Now we recall that $p = (p_1, p_2) = (1, bM^\beta)$ and we observe that if $y \in G$ then

$$\begin{aligned} |p_2 - y_2| &= |bM^\beta - y_2| \geqslant |y_2| - bM^\beta \\ &\geqslant bM^\beta + 1 - bM^\beta = 1 \geqslant \frac{|p_1| + |y_1|}{2} \geqslant \frac{|p_1 - y_1|}{2}. \end{aligned}$$

Hence, $|p - y| \leqslant C|p_2 - y_2|$, for some $C > 0$ and thus (3.39) and the substitution $t := p_2 - y_2$ give

$$\begin{aligned} &\int_{Q\cup Q'} \frac{\chi_{E_M}(y) - \chi_{E_M^c}(y)}{|p-y|^{2+2s}}\,dy \geqslant C\int_G \frac{dy}{|p_2-y_2|^{2+2s}} \\ &= C\int_{-3bM^\beta-2}^{-bM^\beta-1} \frac{dy_2}{|p_2-y_2|^{2+2s}} = C\int_{2bM^\beta+1}^{4bM^\beta+2} \frac{dt}{t^{2+2s}} = \frac{C}{(2bM^\beta+1)^{1+2s}}, \end{aligned} \tag{3.40}$$

up to renaming C.

Now we define

$$H := (-\infty, -1) \times (-M, M - 2L),$$

see Figure 8. By construction, $H \subseteq E_M^c$. We notice that the portion on the right of Q all belongs to E_M, while the portion on the left of Q' all belongs to E_M^c, that is

$$\begin{aligned} &(-\infty, 1-2L) \times (M-2L, M) \subseteq E_M^c \\ \text{and}\quad &(1+2L, +\infty) \times (M-2L, M) \subseteq E_M. \end{aligned}$$

Therefore, by symmetry, these contributions cancel and we have

$$\int_{\mathbb{R}^2\setminus(Q\cup Q')} \frac{\chi_{E_M}(y) - \chi_{E_M^c}(y)}{|p-y|^{2+2s}}\,dy = \int_{\{x_2>M\}\cup\{x_2<M-2L\}} \frac{\chi_{E_M}(y) - \chi_{E_M^c}(y)}{|p-y|^{2+2s}}\,dy. \tag{3.41}$$

Now we observe that $\{x_2 > M\} \subseteq E_M^c$ and $\{x_2 < M - 2L\} \setminus H \subseteq E_M$, therefore, by symmetry,

$$\int_{\{x_2>M\}\cup\{x_2<M-2L\}} \frac{\chi_{E_M}(y) - \chi_{E_M^c}(y)}{|p-y|^{2+2s}}\,dy = -2\int_H \frac{dy}{|p-y|^{2+2s}}. \tag{3.42}$$

Now we observe that if $y \in H$ then $|y_2| \geqslant 2L - M$ and so

$$|y_2 - p_2| \geqslant 2L - M - bM^\beta = M - 3bM^\beta \geqslant M - 3M^\beta \geqslant \frac{M}{2}$$

if M is large enough. Therefore

$$\begin{aligned}
\int_H \frac{dy}{|p-y|^{2+2s}} &\leqslant C \int_{-\infty}^{1} \left(\int_{-M}^{M-2L} \frac{dy_2}{\left(|p_1 - y_1|^2 + M^2\right)^{\frac{2+2s}{2}}} \right) dy_1 \\
&= C\,(M-L) \int_{-\infty}^{1} \frac{dy_1}{\left(|1-y_1|^2 + M^2\right)^{\frac{2+2s}{2}}} \\
&\leqslant C\,(M-L) \left(\int_{-\infty}^{-M} \frac{dy_1}{|1-y_1|^{2+2s}} + \int_{-M}^{1} \frac{dy_1}{M^{2+2s}} \right) \\
&\leqslant C\,(M-L)\,M^{-1-2s} = CbM^{\beta-1-2s} \leqslant CM^{\beta-1-2s},
\end{aligned}$$

for some $C > 0$ (possibly varying from line to line). Using this, (3.41) and (3.42), we obtain that

$$\int_{\mathbb{R}^2 \setminus (Q \cup Q')} \frac{\chi_{E_M}(y) - \chi_{E_M^c}(y)}{|p-y|^{2+2s}}\, dy = -2 \int_H \frac{dy}{|p-y|^{2+2s}} \geqslant -CM^{\beta-1-2s}, \tag{3.43}$$

up to renaming C.

Now we use the Euler-Lagrange equation in the viscosity sense at p and we obtain that

$$\int_{\mathbb{R}^2} \frac{\chi_{E_M}(y) - \chi_{E_M^c}(y)}{|p-y|^{2+2s}}\, dy \leqslant 0.$$

Combining this with (3.40) and (3.43), we obtain

$$0 \geqslant \frac{C}{(2bM^\beta + 1)^{1+2s}} - CM^{\beta-1-2s}.$$

That is, up to renaming constants,

$$(2bM^\beta + 1)^{1+2s} \geqslant c_* M^{1+2s-\beta},$$

for some $c_* > 0$. Using this and (3.33), we conclude that

$$2bM^\beta + 1 \geqslant c_*^{\frac{1}{1+2s}} M^{1-\frac{\beta}{1+2s}} = c_o M^\beta.$$

Now we multiply by $M^{-\beta}$ and we take M large enough, such that $M^{-\beta} \leqslant c_o/2$, so we obtain

$$2b \geqslant -M^{-\beta} + c_o \geqslant \frac{c_o}{2}.$$

This is in contradiction with (3.34), if we choose C_o small enough. ☐

As a combination of Propositions 3.2 and 3.3, we have the optimal statement in Theorem 1.2.

4. Stickiness as $s \to 0^+$

This section contains the asymptotic properties as $s \to 0$ and the proof of Theorem 1.3. For this, we recall that Σ has been defined in (1.3) as

$$\Sigma := \{(x, y) \in \mathbb{R}^2 \setminus B_1 \text{ s.t. } x > 0 \text{ and } y > 0\}$$

and E_s is the s-minimizer in B_1 with datum Σ outside B_1.

Proof of Theorem 1.3. First, we show that

$$E_s \subseteq \{x+y=1\}. \tag{4.1}$$

To prove it, we slide the half-plane $h_t := \{x+y \leqslant t\}$. If $t \leqslant -3$, we have that h_t lies below $\Sigma \cup B_1$ and so $h_t \subseteq E_s^c$. Then we increase t until h_{t^*} intersects E_s, with $t_* \in [-3,1]$. Notice that (4.1) is proved if we show that

$$t_* = 1. \tag{4.2}$$

We prove this arguing by contradiction. If not, there exists $p \in B_1 \cap (\partial E_s) \cap \{x+y=t_*\}$. Hence, using the Euler-Lagrange equation in the viscosity sense (see Theorem 5.1 in [6]) and the fact that $h_{t_*} \subseteq E_s^c$, we obtain

$$0 \geqslant \int_{\mathbb{R}^2} \frac{\chi_{E_s^c}(y) - \chi_{E_s}(y)}{|p-y|^{2+2s}}\, dy \geqslant \int_{\mathbb{R}^2} \frac{\chi_{h_{t_*}}(y) - \chi_{h_{t_*}^c}(y)}{|p-y|^{2+2s}}\, dy = 0.$$

This shows that h_{t_*} must coincide with E_s^c. This is impossible, since E_s is not a half-plane outside B_1. Hence, we have proved (4.2) and so (4.1).

By (4.1), we get that $B_{\sqrt{2}/2} \subseteq E_s^c$. So we can enlarge $r \in [\sqrt{2}/2, 1]$ till B_r touches E_s. We remark that Theorem 1.3 is proved if we show that this touching property only occurs at $r=1$.

Thus, we argue by contradiction and we suppose that there exists

$$r \in [\sqrt{2}/2, 1) \tag{4.3}$$

such that $B_r \subseteq E_s^c$ and there exists $q \in (\partial B_r) \cap (\partial E_s)$. Then, by the Euler-Lagrange equation, we have that

$$\int_{\mathbb{R}^2} \frac{\chi_{E_s^c}(y) - \chi_{E_s}(y)}{|q-y|^{2+2s}}\, dy \leqslant 0. \tag{4.4}$$

By construction,

$$E_s \subseteq \{(x,y) \in \mathbb{R}^2 \setminus B_r \text{ s.t. } x>0 \text{ and } y>0\}. \tag{4.5}$$

Also, $0 < q_1, q_2 < 1$. Then we consider the translation by q: namely we define $F_s := E_s - q$. It follows from (4.5) that

$$F_s \subseteq \{(x,y) \in \mathbb{R}^2 \setminus B_r(-q) \text{ s.t. } x>-1 \text{ and } y>-1\}. \tag{4.6}$$

Also, by (4.4),

$$\int_{\mathbb{R}^2} \frac{\chi_{F_s^c}(y) - \chi_{F_s}(y)}{|y|^{2+2s}}\, dy \leqslant 0. \tag{4.7}$$

Now we define $D_r := B_r(q) \cup B_r(-q)$ and we let K_r be the convex hull of D_r. Notice that

$$B_r \subseteq K_r. \tag{4.8}$$

We also define $P_r := K_r \setminus D_r$. Since $B_r(-q) \subseteq F_s^c$, by symmetry we obtain that

$$\int_{D_r} \frac{\chi_{F_s^c}(y) - \chi_{F_s}(y)}{|y|^{2+2s}}\, dy \geqslant 0. \tag{4.9}$$

Moreover (see Lemma 3.1 in [13], used here with $\lambda := 1$) and (4.3),

$$\left| \int_{P_r} \frac{\chi_{F_s^c}(y) - \chi_{F_s}(y)}{|y|^{2+2s}}\, dy \right| \leqslant \frac{C_1 r^{-2s}}{1-2s} \leqslant \frac{C_2}{1-2s},$$

for suitable positive constants C_1 and C_2 that do not depend on s. Using this, (4.7) and (4.9) we obtain that

$$\begin{aligned} 0 &\geqslant \int_{\mathbb{R}^2 \setminus K_r} \frac{\chi_{F_s^c}(y) - \chi_{F_s}(y)}{|y|^{2+2s}}\, dy + \int_{D_r} \frac{\chi_{F_s^c}(y) - \chi_{F_s}(y)}{|y|^{2+2s}}\, dy + \int_{P_r} \frac{\chi_{F_s^c}(y) - \chi_{F_s}(y)}{|y|^{2+2s}}\, dy \\ &\geqslant \int_{\mathbb{R}^2 \setminus K_r} \frac{\chi_{F_s^c}(y) - \chi_{F_s}(y)}{|y|^{2+2s}}\, dy - \frac{C_2}{1-2s}. \end{aligned} \tag{4.10}$$

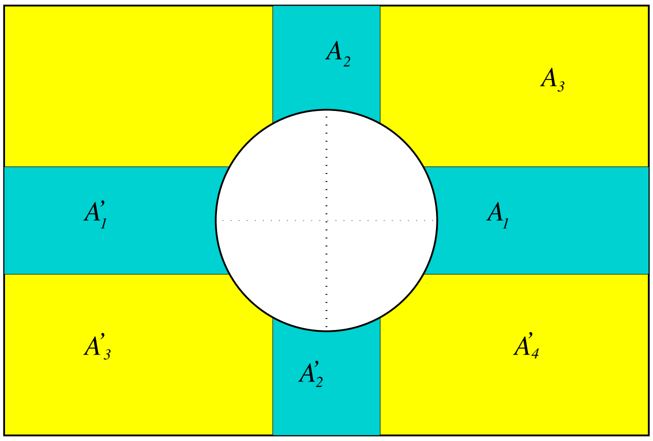

FIGURE 9. The partition of the plane needed for the proof of Theorem 1.3.

Moreover, recalling (4.8) (and using again (4.3)), we have that

$$\left| \int_{B_{2r}\setminus K_r} \frac{\chi_{F_s^c}(y)-\chi_{F_s}(y)}{|y|^{2+2s}}\,dy \right| \leqslant \int_{B_{2r}\setminus B_r} \frac{dy}{|y|^{2+2s}} \leqslant C_3,$$

for some $C_3>0$ that does not depend on s. Hence (4.10) gives

$$0 \geqslant \int_{\mathbb{R}^2\setminus B_{2r}} \frac{\chi_{F_s^c}(y)-\chi_{F_s}(y)}{|y|^{2+2s}}\,dy - C_3 - \frac{C_2}{1-2s}. \tag{4.11}$$

Now we observe that $B_r(-q)\subseteq B_{2r}$, since $|q|=r$. Consequently, recalling (4.6),

$$F_s\setminus B_{2r}\subseteq\{(x,y)\in\mathbb{R}^2\setminus B_{2r} \text{ s.t. } x>-1 \text{ and } y>-1\}.$$

That is, $F_s\setminus B_{2r}\subseteq A_1\cup A_2\cup A_3$, where

$$\begin{aligned}
&A_1 := \{(x,y)\in\mathbb{R}^2\setminus B_{2r} \text{ s.t. } x>0 \text{ and } y\in(-1,1)\},\\
&A_2 := \{(x,y)\in\mathbb{R}^2\setminus B_{2r} \text{ s.t. } x\in(-1,1) \text{ and } y>0\}\\
\text{and}\qquad &A_3 := \{(x,y)\in\mathbb{R}^2\setminus B_{2r} \text{ s.t. } x\geqslant 1 \text{ and } y\geqslant 1\}.
\end{aligned}$$

On the other hand, $F_s^c\setminus B_{2r}\supseteq A'_1\cup A'_2\cup A'_3\cup A'_4$, where

$$\begin{aligned}
&A'_1 := \{(x,y)\in\mathbb{R}^2\setminus B_{2r} \text{ s.t. } x<0 \text{ and } y\in(-1,1)\},\\
&A'_2 := \{(x,y)\in\mathbb{R}^2\setminus B_{2r} \text{ s.t. } x\in(-1,1) \text{ and } y<0\},\\
&A'_3 := \{(x,y)\in\mathbb{R}^2\setminus B_{2r} \text{ s.t. } x\leqslant -1 \text{ and } y\leqslant -1\},\\
\text{and}\qquad &A'_4 := \{(x,y)\in\mathbb{R}^2\setminus B_{2r} \text{ s.t. } x\geqslant 1 \text{ and } y\leqslant -1\},
\end{aligned}$$

see Figure 9. After simplifying A_1 with A'_1, A_2 with A'_2 and A_3 with A'_3, we obtain

$$\int_{\mathbb{R}^2\setminus B_{2r}} \frac{\chi_{F_s^c}(y)-\chi_{F_s}(y)}{|y|^{2+2s}}\,dy \geqslant \int_{A'_4} \frac{dy}{|y|^{2+2s}}. \tag{4.12}$$

Notice now that A'_4 contains a cone with positive constant opening with vertex at the origin, therefore

$$\int_{A'_4} \frac{dy}{|y|^{2+2s}} \geqslant c_1 \int_{2r}^{+\infty} \frac{d\rho}{\rho^{1+2s}} = \frac{c_2}{s\,r^{2s}} \geqslant \frac{c_3}{s},$$

where we have used again (4.3), and the positive constants c_1, c_2 and c_3 do not depend on s. The latter estimate and (4.12) give that

$$\int_{\mathbb{R}^2\setminus B_{2r}} \frac{\chi_{F_s^c}(y)-\chi_{F_s}(y)}{|y|^{2+2s}}\,dy \geqslant \frac{c_3}{s}.$$

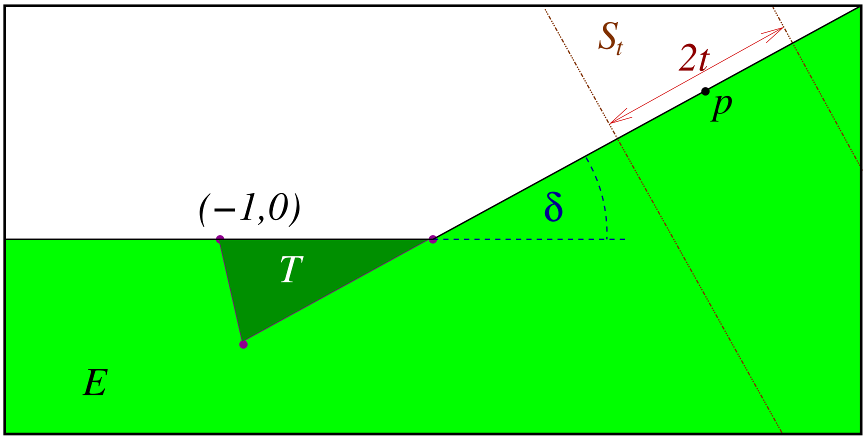

FIGURE 10. The proof of Lemma 5.1.

Therefore, recalling (4.11),

$$0 \geqslant \frac{c_3}{s} - C_3 - \frac{C_2}{1-2s}.$$

This is a contradiction if $s \in (0, s_o)$ and s_o is small enough. Hence, we have completed the proof of Theorem 1.3. □

5. Construction of barriers that are piecewise linear

This part of the paper is devoted to the proof of Theorem 1.4. The argument will rely on the construction of a series of barriers, and the proof of Theorem 1.4 will be completed in Section 8.

In this section, we construct barriers in the plane, which are subsolutions of the fractional curvature equation when $\{x_1 > 0\}$, which possess a "vertical" portion along $\{x_1 = 0\}$ and which are built by joining linear functions whose slope becomes arbitrarily close to being horizontal (a precise statement will be given in Proposition 5.3). For this scope, we start with a simple auxiliary observation to bound explicitly from below the fractional curvature of an angle:

Lemma 5.1. *Let $\ell \geqslant 0$,*

$$\begin{aligned} & E_1 := (-\infty, 0] \times (-\infty, 0) \\ & E_2 := \{\ell x_2 - x_1 < 0, \quad x_1 > 0\} \\ \text{and} \qquad & E := E_1 \cup E_2. \end{aligned}$$

Then, for any $p = (p_1, p_2) \in \partial E$ with $p_2 > 0$,

$$\int_{\mathbb{R}^2} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant \frac{c(\ell)}{|p|^{2s}}, \tag{5.1}$$

for a suitable nonincreasing function $c : [0, +\infty) \to (0, 1)$.

More precisely, for large ℓ, one has that $c(\ell) \sim \bar{c}\ell^{-1}$, for some $\bar{c} > 0$.

Proof. Let $\delta := \arctan(1/\ell) \in \big(0, \frac{\pi}{2}\big]$. By scaling, it is enough to prove (5.1) when

$$|p| = 4. \tag{5.2}$$

Now, for any $t > 0$, let S_t be the slab with boundary orthogonal to the straight line $\{\ell x_2 - x_1 = 0\}$ of width $2t$, having p on its symmetry axis (see Figure 10). For small t, the slab S_t does not contain the origin, thus, the "upper" half of the slab is contained in E^c, while the "lower" half of the slab is contained in E, namely

$$\int_{S_t} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y-p|^{2+2s}}\, dy = 0.$$

Enlarging t, the "lower" half of the slab is always contained in E. As for the "upper" half, we have that the triangle T with vertices $(0,0)$, $(-\cos\delta,-\sin\delta)$, $(-1,0)$ lies in E. Notice that

$$|T| = \frac{\sin\delta}{2}.$$

Also, if $y \in T$ then $|y| \leqslant 2$ and so, recalling (5.2),

$$|y-p| \leqslant |p| + 2 \leqslant 2|p|.$$

Consequently,

$$\int_T \frac{\chi_E(y) - \chi_{E^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{|T|}{2^{2+2s}\,|p|^{2+2s}} = \frac{\sin\delta}{2^{7+2s}\,|p|^{2s}},$$

which gives the desired result. $\square$

The next result is the building block needed to construct a barrier iteratively. Roughly speaking, next result says that we can tilt a straight line towards infinity by estimating precisely the effect of this modification on the fractional curvature.

Lemma 5.2. *Let $\ell \geqslant q \geqslant 0$ and $\delta := \arctan(1/\ell) \in \big(0, \frac{\pi}{2}\big]$. Let $e := (\ell - q, 1)$.*

Let $\bar\tau \in C^\infty_0(B_1(e))$ with $\bar\tau = 1$ in $B_{1/2}(e)$.

Let $\tau_o \in C^\infty(\mathbb{R})$ be such that

$$\tau_o(t) = 1 \text{ if } t \in \left[\tfrac{\delta}{2}, \tfrac{3\delta}{2}\right] \text{ and } \tau_o(t) = 0 \text{ if } t \in \mathbb{R}\setminus\left[\tfrac{\delta}{4}, \tfrac{7\delta}{4}\right]. \tag{5.3}$$

For any $x \in \mathbb{R}^2$, let also $\alpha(x) \in [0, 2\pi)$ be the angle between the vector $x - e$ and the x_1-axis. Let

$$\tau(x) := \big(1 - \bar\tau(x)\big)\,\tau_o\big(\alpha(x)\big). \tag{5.4}$$

For any $\theta \in \mathbb{R}$, let R_θ be the clockwise rotation by an angle θ, i.e.

$$R_\theta(x) = R_\theta(x_1,x_2) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Let also

$$\Psi_\theta(x) := R_{\tau(x)\theta}\, x.$$

Let $E \subset \mathbb{R}^2$ be an epigraph such that

$$\begin{aligned} & E \cap \{x_1 < 0\} = (-\infty, 0) \times (-\infty, 0), \\ & E \supseteq \mathbb{R} \times (-\infty, 0), \\ & E \cap \{x_2 > 1\} = \{\ell x_2 - x_1 - q < 0\} \cap \{x_2 > 1\} \\ \text{and } & E \cap \{x_1 > \ell - q\} = \{\ell x_2 - x_1 - q < 0\} \cap \{x_1 > \ell - q\}. \end{aligned}$$

Assume that, for any $p \in \partial E \cap \{x_2 > 0\}$,

$$\int_{\mathbb{R}^2} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c}{|p|^{2s}} \tag{5.5}$$

for some $c \in (0,1)$.

Then, there exist nonincreasing functions $\phi : [0,+\infty) \to (0,1)$ and $c_o : [0,+\infty) \to (0,c)$ such that for any $\theta \in [0, \phi(\ell)]$ the following claim holds true. Let $F := \Psi_\theta(E)$. Then, for any $p \in (\partial F) \cap \{x_2 > 0\}$,

$$\int_{\mathbb{R}^2} \frac{\chi_F(y) - \chi_{F^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c_o(\ell)}{|p|^{2s}}. \tag{5.6}$$

More precisely, for large ℓ, one has that $c_o(\ell) \sim \bar c \min\{c, \ell^{-1}\}$, for some $\bar c > 0$.

Proof. First we point out that

$$|\nabla\alpha(x)| \leqslant \frac{C}{|x-e|}, \tag{5.7}$$

for some $C>0$. Indeed, $\alpha(x)$ is identified by the two conditions

$$|x-e|\cos\alpha(x) = |x_1-\ell+q| \tag{5.8}$$

and $|x-e|\sin\alpha(x) = |x_2-1|$. Assume also that $\sin^2\alpha(x) \geqslant 1/2$ (the case $\cos^2\alpha(x) \geqslant 1/2$ is similar). Then we differentiate the relation (5.8) and we obtain

$$\frac{x-e}{|x-e|}\cos\alpha(x) - |x-e|\sin\alpha(x)\,\nabla\alpha(x) = \frac{x_1-\ell+q}{|x_1-\ell+q|}\,(1,0).$$

Therefore

$$\frac{\sqrt{2}}{2}|x-e|\,|\nabla\alpha(x)| \leqslant |x-e|\,|\sin\alpha(x)|\,|\nabla\alpha(x)| = \left|\frac{x-e}{|x-e|}\cos\alpha(x) - \frac{x_1-\ell+q}{|x_1-\ell+q|}\,(1,0)\right| \leqslant 2,$$

which proves (5.7).

Similarly, taking one more derivative, one sees that

$$|D^2\alpha(x)| \leqslant \frac{C}{|x-e|^2}. \tag{5.9}$$

Now, by (5.4) and (5.7),

$$|\nabla\tau(x)| \leqslant C\left(\chi_{B_1(e)\setminus B_{1/2}(e)}(x) + \frac{\chi_{\mathbb{R}^2\setminus B_{1/2}(e)}(x)}{|x-e|}\right). \tag{5.10}$$

Using (5.9), one also obtains that

$$|D^2\tau(x)| \leqslant C\left(\chi_{B_1(e)\setminus B_{1/2}(e)}(x) + \frac{\chi_{\mathbb{R}^2\setminus B_{1/2}(e)}(x)}{|x-e|}\right). \tag{5.11}$$

Let now

$$\Phi_\theta(x) := \Psi_\theta(x) - x = \begin{pmatrix}\cos(\tau(x)\theta)-1 & \sin(\tau(x)\theta)\\ -\sin(\tau(x)\theta) & \cos(\tau(x)\theta)-1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

We claim that

$$|D\Phi_\theta(x)| \leqslant C\,(1+\ell)\,\theta, \tag{5.12}$$

for some $C>0$. To prove it, we consider the first coordinate of $\Phi_\theta(x)$, which is

$$\big(\cos(\tau(x)\theta)-1\big)\,x_1 + \sin(\tau(x)\theta)\,x_2, \tag{5.13}$$

since the computation with the second coordinate is similar. We bound the derivative of (5.13) by

$$\big|\cos(\tau(x)\theta)-1\big| + \big|\sin(\tau(x)\theta)\big| + \theta\Big(\big|\sin(\tau(x)\theta)\big| + \big|\cos(\tau(x)\theta)\big|\Big)|\nabla\tau(x)|\,|x|. \tag{5.14}$$

Thus, we bound $\big|\cos(\tau(x)\theta)-1\big| \leqslant C\theta^2$ and $\big|\sin(\tau(x)\theta)\big| \leqslant C\theta$ and we make use of (5.10), to estimate the quantity in (5.14) by

$$C\theta\left(1 + \frac{\chi_{\mathbb{R}^2\setminus B_{1/2}(e)}(x)\ |x|}{|x-e|}\right). \tag{5.15}$$

Now we observe that $|e| = \sqrt{(\ell-q)^2+1} \leqslant \sqrt{\ell^2+1}$, therefore

$$|x| \leqslant |x-e| + \sqrt{\ell^2+1}$$

and so, if $|x-e| \geqslant 1/2$,

$$\frac{|x|}{|x-e|} \leqslant 1 + 2\sqrt{\ell^2+1}.$$

By inserting this information into (5.15) we bound the first coordinate of $\Phi_\theta(x)$ by $C\,(1+\ell)\,\theta$. This proves (5.12).

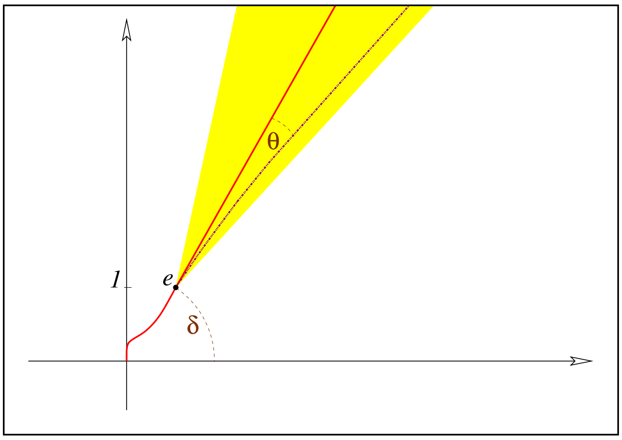

FIGURE 11. The diffeomorphism of $\mathbb{R}^2$ in Lemma 5.2.

Similarly, making use of (5.11), one sees that

$$|D^2\Phi_\theta(x)| \leqslant C\,(1+\ell)\,\theta. \tag{5.16}$$

Notice also that, for any fixed $x\in\mathbb{R}^2$, we have that

$$|\Psi_\theta(x)| = |R_{\tau(x)\theta}\,x| = |x|, \tag{5.17}$$

therefore

$$\lim_{|x|\to+\infty} |\Psi_\theta(x)| = +\infty.$$

From this, (5.12), and the Global Inverse Function Theorem (see e.g. Corollary 4.3 in [19]), we obtain that Ψ_θ is a global diffeomorphism of $\mathbb{R}^2$, see Figure 11.

As a consequence, using (5.12), (5.16) and the curvature estimates for diffeomorphisms (see Theorem 1.1 in [10]), we conclude that

$$\int_{\mathbb{R}^2} \frac{\chi_F(y)-\chi_{F^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \int_{\mathbb{R}^2} \frac{\chi_E(y)-\chi_{E^c}(y)}{|y-q|^{2+2s}}\,dy - C\,(1+\ell)\,\theta, \tag{5.18}$$

with $q:=\Psi_\theta^{-1}(p)$, for any $p\in(\partial F)\cap\{x_2>0\}$.

Now we claim that

$$\text{if } p\in\{x_2>0\} \text{ then } \Psi_\theta^{-1}(p)\in\{x_2>0\}. \tag{5.19}$$

Suppose, by contradiction, that $\Psi_\theta^{-1}(p)\in\{x_2\leqslant 0\}$. Notice that τ vanishes in $\{x_2\leqslant 0\}$, therefore Ψ_θ is the identity in $\{x_2\leqslant 0\}$. As a consequence $p=\Psi_\theta(\Psi_\theta^{-1}(p))=\Psi_\theta^{-1}(p)\in\{x_2\leqslant 0\}$. This is a contradiction with our assumptions and so it proves (5.19).

Using (5.5), (5.17), (5.18) and (5.19), we have that

$$\int_{\mathbb{R}^2} \frac{\chi_F(y)-\chi_{F^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c}{|q|^{2s}} - C\,(1+\ell)\,\theta = \frac{c}{|p|^{2s}} - C\,(1+\ell)\,\theta \geqslant \frac{c}{2\,|p|^{2s}}, \tag{5.20}$$

with $q:=\Psi_\theta^{-1}(p)$, for any $p\in(\partial F)\cap\{x_2>0\}\cap B_{r_\theta}$, where $r_\theta := \left(\frac{c}{2C\,(1+\ell)\,\theta}\right)^{\frac{1}{2s}}$ (we stress that r_θ is large, for small θ, according to the statement of Lemma 5.2).

Now we take $p\in\big((\partial F)\cap\{x_2>0\}\big)\setminus B_{r_\theta}$ and we observe that $((\partial F)\setminus B_{r_\theta})\cap\{x_2>0\}$ coincides with a straight line of the form $\lambda:=\{\ell_\theta x_2 - x_1 - q_\theta = 0\}$, with $\ell_\theta\geqslant\ell$, $|\ell_\theta-\ell|$ as close to zero as we wish for small θ, and $q_\theta:=\ell_\theta-\ell+q$. The intersections of the straight line λ with $\{x_2=8\}$ and $\{x_2=0\}$ occur at points $x_1=8\ell_\theta-q_\theta$ and $x_1=-q_\theta$, respectively.

Hence, we consider the triangle T with vertices $(8\ell_\theta-q_\theta,0)$, $(8\ell_\theta-q_\theta,8)$ and $(-q_\theta,0)$. We observe that $|T|=32\ell_\theta\leqslant 32(1+\ell)$, for small θ. Moreover, if $y\in T$, then $|y|\leqslant C(1+\ell_\theta+q_\theta)\leqslant C(1+\ell)$, up to renaming constants. Therefore, if $p \in B^c_{r_\theta}$ and $y \in T$,

$$|y-p| \geqslant |p| - C(1+\ell) \geqslant \frac{|p|}{2},$$

if θ is small. Consequently,

$$\int_T \frac{dy}{|y-p|^{2+2s}} \leqslant \frac{C(1+\ell)}{|p|^{2+2s}} \leqslant \frac{C(1+\ell)}{r_\theta^2\,|p|^{2s}}. \tag{5.21}$$

Now we define $\tilde F := F \cup T$. By Lemma 5.1,

$$\int_{\mathbb{R}^2} \frac{\chi_{\tilde F}(y) - \chi_{\tilde F^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c(\ell_\theta)}{|p-(-q_\theta,0)|^{2s}}.$$

Using that $\ell_\theta \leqslant \frac{3\ell}{2}$ and that $c(\cdot)$ is nonincreasing, we see that $c(\ell_\theta) \geqslant c\left(\frac{3\ell}{2}\right)$. Moreover,

$$|p-(-q_\theta,0)| \leqslant |p| + q_\theta \leqslant |p| + \ell + 1 \leqslant 2|p|,$$

so we obtain that

$$\int_{\mathbb{R}^2} \frac{\chi_{\tilde F}(y) - \chi_{\tilde F^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c\left(\frac{3\ell}{2}\right)}{2^{2s}\,|p|^{2s}}.$$

Exploiting this and (5.21), we obtain that, for any $p \in \big((\partial F) \cap \{x_2 > 0\}\big) \setminus B_{r_\theta}$,

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\chi_F(y) - \chi_{F^c}(y)}{|y-p|^{2+2s}}\,dy &\geqslant \int_{\mathbb{R}^2} \frac{\chi_{\tilde F}(y) - \chi_{\tilde F^c}(y)}{|y-p|^{2+2s}}\,dy - \int_T \frac{dy}{|y-p|^{2+2s}} \\ &\geqslant \frac{c\left(\frac{3\ell}{2}\right)}{2^{2s}\,|p|^{2s}} - \frac{C(1+\ell)}{r_\theta^2\,|p|^{2s}} \geqslant \frac{c\left(\frac{3\ell}{2}\right)}{2^{1+2s}\,|p|^{2s}}, \end{aligned} \tag{5.22}$$

for small θ. Then, (5.6) follows by combining (5.20) and (5.22). ☐

By iterating Lemma 5.2 we can construct the following barrier:

Proposition 5.3. *Fix $K \geqslant 0$. Then there exist $a_K \in (0,1)$, $\ell_K \geqslant K$, $q_K \geqslant 0$, $c_K \in (0,1)$, a continuous function $u_K : [0,+\infty) \to [0,+\infty)$ and a set $E_K \subset \mathbb{R}^2$ with $(\partial E_K) \cap \{x_2 > 0\}$ of class $C^{1,1}$ and such that:*

- *$u_K(x_2) = \ell_K\, x_2 - q_K$ for any $x_2 \in [1,+\infty)$,*
- *we have that*

$$\begin{aligned} &E_K \cap \{x_1 < 0\} = (-\infty,0) \times (-\infty,0),\\ &E_K \supseteq \mathbb{R} \times (-\infty,0),\\ &E_K \supseteq (0,+\infty) \times (-\infty,a_K],\\ &E_K \cap \{x_2 > 1\} = \{x_1 > u_K(x_2), \quad x_2 > 1\},\\ &E_K \cap \{x_1 > \ell_K - q_K\} = \{x_1 > u_K(x_2), \quad x_1 > \ell_K - q_K\} \end{aligned}$$

  and

$$\int_{\mathbb{R}^2} \frac{\chi_{E_K}(y) - \chi_{E_K^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c_K}{|p|^{2s}},$$

  for any $p \in (\partial E_K) \cap \{x_2 > 0\}$.

More precisely, for large K, one has that $c_K \sim \bar c \ell_K^{-1}$, for some $\bar c > 0$.

Moreover, one can also prescribe that

$$q_K \leqslant K^{-1}. \tag{5.23}$$

Proof. We apply Lemma 5.2 iteratively for a large (but finite) number of times, see Figure 12.

We start with $u_0 := 0$ and $E_0 := \mathbb{R}^2 \setminus \{x_1 \leqslant 0 \leqslant x_2\}$. By Lemma 5.1 (used here with $\ell := 0$) we know that

$$\int_{\mathbb{R}^2} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c}{|p|^{2s}},$$

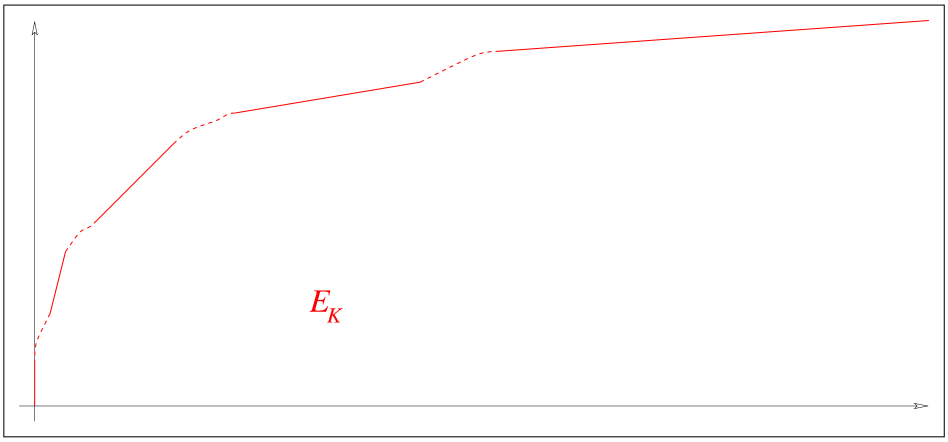

FIGURE 12. The barrier of Proposition 5.3.

for some $c > 0$. Then we apply Lemma 5.2 and we construct a set E_1 whose boundary coincides with $\{x_2 = 0\}$ when $\{x_1 < 0\}$ and with a straight line $\{\ell_1 x_2 - x_1 - q_1 = 0\}$ when $\{x_2 > 4\}$, whose fractional curvature satisfies the desired estimate (as a matter of fact, we can take the new slope ℓ_1 as the one obtained by $\phi(0)$ in Lemma 5.2, thus $\ell_1 > 0$).

Then we scale E_1 by a factor $\frac{1}{2}$ and we apply once again Lemma 5.2, obtaining a set E_2 whose boundary coincides with $\{x_2 = 0\}$ when $\{x_1 < 0\}$ and with a straight line $\{\ell_2 x_2 - x_1 - q_2 = 0\}$ when $\{x_2 > 4\}$, whose fractional curvature satisfies the desired estimate. Notice again that ℓ_2 is obtained in Lemma 5.2 by rotating clockwise the straight line of slope ℓ_1 by an angle $\phi(\ell_1) > 0$, hence $\ell_2 > \ell_1$.

Iterating this procedure, we obtain a sequence of increasing slopes ℓ_j and sets E_j satisfying the desired geometric properties. We stress that, for large j, the slope ℓ_j must become larger than the quantity K fixed in the statement of Proposition 5.3. Indeed, if not, say if $\ell_j \leqslant \ell_\star$ for some $\ell_\star > 0$, at each step of the iteration we could rotate the straight line by an angle of size larger than $\phi(\ell_\star)$, which is a fixed positive quantity (recall that ϕ in Lemma 5.2 is nonincreasing): hence repeating this argument many times we would make the slope become bigger than $\ell_\star$, that is a contradiction.

Thus, we can define j_o to be the first j for which $\ell_j \geqslant K$. The set E_{j_o} obtained in this way satisfies the desired properties, with the possible exception of (5.23). So, to obtain (5.23), we may suppose that $q_{j_o} > K^{-1}$, otherwise we are done, and we scale the picture once again by a factor $\mu := K^{-1} q_{j_o}^{-1} \in (0, 1)$. In this way, the geometric properties of the set and the estimates on the fractional curvature are preserved, but the line $\{\ell_{j_o} x_2 - x_1 - q_{j_o} = 0\}$ is transformed into the line $\{\ell_{j_o} x_2 - x_1 - \tilde{q}_{j_o} = 0\}$, with $\tilde{q}_{j_o} := \mu q_{j_o}$. By construction, we have that $\tilde{q}_{j_o} = K^{-1}$, which gives (5.23). $\square$

6. Construction of barriers which grow like $x_1^{\frac{1}{2}+s+\varepsilon_0}$

In this section, we construct barriers in the plane, which are subsolutions of the fractional curvature equation when $\{x_1 > 0\}$, which possess a "vertical" portion along $\{x_1 = 0\}$ and which grow like $x_1^{\frac{1}{2}+s+\varepsilon_0}$ at infinity (here, $\varepsilon_0 > 0$ is arbitrarily small). This is a refinement of the barrier constructed in Proposition 5.3, which grows linearly (with almost horizontal slope). Roughly speaking, the difference with Proposition 5.3 is that the results obtained there have nice scaling properties and an elementary geometry (since the barrier constructed there is basically the junction of a finite number of straight lines) but do not possess an optimal growth at infinity. As a matter of fact, the power obtained here at infinity is dictated by the growth of the functions that are harmonic with respect to the fractional Laplacian $(-\Delta)^{\gamma_0}$, where

$$\gamma_0 := \frac{1}{2} + s. \tag{6.1}$$

As a matter of fact, this procedure provides a good approximation of the fractional mean curvature equation at points with nearly horizontal tangent. Namely, we set

$$\gamma := \frac{1}{2} + s + \varepsilon_0 = \gamma_0 + \varepsilon_0 \in \left(\frac{1}{2}, 1\right).$$

We will use the fact that $\gamma > \gamma_0$ to construct a subsolution of the γ_0-fractional Laplace equation. More precisely, the main formula we need in this framework is the following:

Lemma 6.1. *Let $\varepsilon_0 \in (0, 1-\gamma_0)$. We have that*

$$\frac{1}{2}\int_{\mathbb{R}} \frac{(1+t)_+^{\gamma} + (1-t)_+^{\gamma} - 2}{|t|^{2+2s}}\, dt \geqslant c_\star \varepsilon_0,$$

for some $c_\star > 0$.

Proof. Let $r \geqslant 0$. By a Taylor expansion at $r = 1$, we have that

$$r^{\frac{\gamma}{\gamma_0}} = 1 + \frac{\gamma\,(r-1)}{\gamma_0} + \frac{\gamma\,(\gamma-\gamma_0)\,\xi^{\frac{\gamma}{\gamma_0}-2}\,(r-1)^2}{\gamma_0^2},$$

for some ξ on the segment joining r to 1. In particular, $\xi \leqslant 1 + r$. Using this with $r := (1 \pm t)_+^{\gamma_0}$, we obtain

$$(1 \pm t)_+^{\gamma} = 1 + \frac{\gamma\left((1\pm t)_+^{\gamma_0} - 1\right)}{\gamma_0} + \frac{\gamma\,(\gamma-\gamma_0)\,\xi^{\frac{\gamma}{\gamma_0}-2}\left((1\pm t)_+^{\gamma_0} - 1\right)^2}{\gamma_0^2},$$

for some $\xi \in [0, 2 + |t|]$. Consequently, since

$$\frac{\gamma}{\gamma_0} - 2 = \frac{\varepsilon_0}{\gamma_0} - 1 < 0$$

we obtain that

$$\xi^{\frac{\gamma}{\gamma_0}-2} \geqslant (2 + |t|)^{\frac{\gamma}{\gamma_0}-2} \geqslant (2 + |t|)^{-2}.$$

Accordingly,

$$(1 \pm t)_+^{\gamma} \geqslant 1 + \frac{\gamma\left((1\pm t)_+^{\gamma_0} - 1\right)}{\gamma_0} + \frac{\gamma\,(\gamma-\gamma_0)\left((1\pm t)_+^{\gamma_0} - 1\right)^2}{\gamma_0^2\,(2+|t|)^2},$$

and so

$$(1+t)_+^{\gamma} + (1-t)_+^{\gamma} - 2 \geqslant \frac{\gamma\left((1+t)_+^{\gamma_0} + (1-t)_+^{\gamma_0} - 2\right)}{\gamma_0} + \frac{\gamma\,(\gamma-\gamma_0)\left[\left((1+t)_+^{\gamma_0} - 1\right)^2 + \left((1-t)_+^{\gamma_0} - 1\right)^2\right]}{\gamma_0^2\,(2+|t|)^2}.$$

Hence, we set

$$\phi(t) := \frac{\left((1+t)_+^{\gamma_0} - 1\right)^2 + \left((1-t)_+^{\gamma_0} - 1\right)^2}{|t|^{2s}\,(2+|t|)^2},$$

we use that $\gamma = \gamma_0 + \varepsilon_0 > \gamma_0$ and we conclude that

$$\int_{\mathbb{R}} \frac{(1+t)_+^{\gamma} + (1-t)_+^{\gamma} - 2}{|t|^{2+2s}}\, dt \geqslant \frac{\gamma}{\gamma_0}\int_{\mathbb{R}} \frac{(1+t)_+^{\gamma_0} + (1-t)_+^{\gamma_0} - 2}{|t|^{2+2s}}\, dt + \frac{\varepsilon_0}{\gamma_0}\int_{\mathbb{R}} \phi(t)\, dt. \tag{6.2}$$

Also, we know (see e.g. [15]) that $(-\Delta)^{\gamma_0} t_+^{\gamma_0} = 0$ for any $t > 0$, therefore, using this formula at $t = 1$ and noticing that $1 + 2\gamma_0 = 2s$, we see that

$$\int_{\mathbb{R}} \frac{(1+t)_+^{\gamma_0} + (1-t)_+^{\gamma_0} - 2}{|t|^{2+2s}}\, dt = 0.$$

Using this and (6.2), we obtain

$$\int_{\mathbb{R}} \frac{(1+t)_+^{\gamma} + (1-t)_+^{\gamma} - 2}{|t|^{2+2s}}\, dt \geqslant \frac{\varepsilon_0}{\gamma_0}\int_{\mathbb{R}} \phi(t)\, dt,$$

which implies the desired result. $\square$

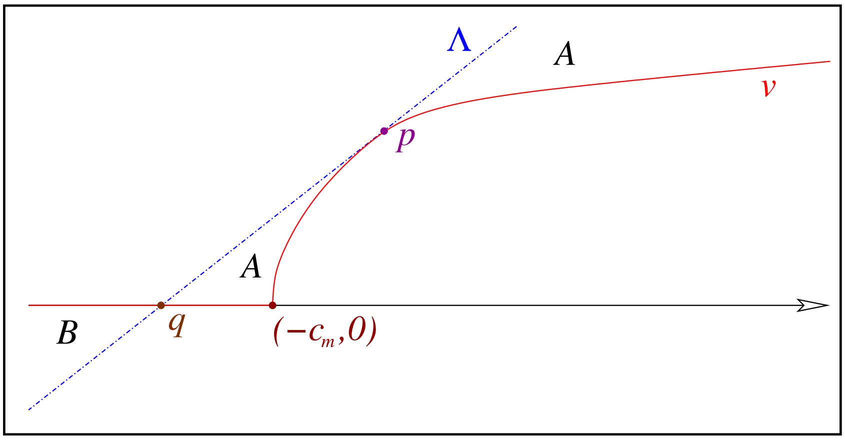

FIGURE 13. The sets involved in Section 6.

Throughout this section, we will consider m and ε_0 (to be taken appropriately small in the sequel, namely $\varepsilon_0>0$ can be fixed as small as one wishes, and then $m>0$ is taken to be small possibly in dependence of ε_0) and $c_m\in\mathbb{R}$, and let

$$v(x_1):=\frac{m\,(x_1+c_m)_+^\gamma}{\gamma}. \tag{6.3}$$

The parameter c_m will be conveniently chosen in the sequel, see in particular the following formula (6.16), but for the moment it is free. Also, given $p:=(p_1,p_2)$ with $p_1\geqslant 1-c_m$ and $p_2=v(p_1)$, we consider the tangent line at v through p, namely

$$\Lambda(x_1):=v'(p_1)(x_1-p_1)+v(p_1)=m\,(p_1+c_m)^{\gamma-1}(x_1-p_1)+\frac{m\,(p_1+c_m)^\gamma}{\gamma}. \tag{6.4}$$

We observe that the tangent line above meets the x_1-axis at the point $q=(q_1,0)$, with

$$q_1:=p_1-\frac{v(p_1)}{v'(p_1)}=p_1-\frac{p_1+c_m}{\gamma}. \tag{6.5}$$

We also consider the region A which lies above the graph of v and below the graph of Λ and the region B which lies above the graph of Λ and below the x_1-axis, see Figure 13. More explicitly, we have

$$\begin{aligned} & A:=\{(x_1,x_2)\text{ s.t. } x_1>q_1 \text{ and } v(x_1)<x_2<\Lambda(x_1)\} \\ \text{and}\quad & B:=\{(x_1,x_2)\text{ s.t. } x_1<q_1 \text{ and } \Lambda(x_1)<x_2<0\}. \end{aligned} \tag{6.6}$$

The first technical result that we need is the following:

Lemma 6.2. *Let $\varepsilon_0\in(0,1-\gamma_0)$. There exist $c,\,c'\in(0,1)$ such that if $m\in(0,c\varepsilon_0]$ then*

$$\int_B\frac{dy}{|y-p|^{2+2s}}-\int_A\frac{dy}{|y-p|^{2+2s}}\geqslant\frac{c'\,\varepsilon_0\,m}{(p_1+c_m)^{\frac12+s-\varepsilon_0}}, \tag{6.7}$$

for any $p:=(p_1,p_2)$ with $p_1\geqslant 1-c_m$ and $p_2=v(p_1)$.

Proof. First of all, we observe that $|y-p|\geqslant|y_1-p_1|$, therefore

$$\int_A\frac{dy}{|y-p|^{2+2s}}\leqslant\int_A\frac{dy}{|y_1-p_1|^{2+2s}}=\int_{q_1}^{+\infty}\frac{\Lambda(y_1)-v(y_1)}{|y_1-p_1|^{2+2s}}\,dy_1=:H. \tag{6.8}$$

Recalling (6.3) and (6.4), we have that

$$\begin{aligned} H &= \int_{q_1}^{+\infty}\frac{m\,(p_1+c_m)^{\gamma-1}(y_1-p_1)+\gamma^{-1}m\,(p_1+c_m)_+^\gamma-\gamma^{-1}m\,(y_1+c_m)_+^\gamma}{|y_1-p_1|^{2+2s}}\,dy_1 \\ &= \frac{m\,(p_1+c_m)^\gamma}{\gamma}\int_{q_1}^{+\infty}\frac{\gamma\,(p_1+c_m)^{-1}(y_1-p_1)+1-(p_1+c_m)^{-\gamma}(y_1+c_m)_+^\gamma}{|y_1-p_1|^{2+2s}}\,dy_1. \end{aligned}$$

Now we recall (6.5) and use the change of variable from the variable y_1 to the variable t given by

$$y_1 + c_m = (p_1 + c_m)(t + 1). \tag{6.9}$$

In this way, we obtain that

$$H = \frac{m}{\gamma\,(p_1 + c_m)^{1+2s-\gamma}} \int_{-\frac{1}{\gamma}}^{+\infty} \frac{\gamma t + 1 - (t+1)_+^\gamma}{|t|^{2+2s}}\,dt = \frac{C_A\,m}{(p_1 + c_m)^{1+2s-\gamma}},$$

where

$$C_A := \int_{-\frac{1}{\gamma}}^{+\infty} \frac{\gamma t + 1 - (t+1)_+^\gamma}{|t|^{2+2s}}\,dt.$$

Therefore, recalling (6.8), we conclude that

$$\int_A \frac{dy}{|y-p|^{2+2s}} \leqslant \frac{C_A\,m}{(p_1 + c_m)^{1+2s-\gamma}}. \tag{6.10}$$

Now we claim that

$$\text{if } y \in B, \text{ then } |y_2 - p_2| \leqslant m|y_1 - p_1|. \tag{6.11}$$

To prove this, we take $y \in B$. Then $\Lambda(y_1) < y_2 < 0$, therefore, since $p_2 \geqslant 0$, we have

$$|y_2 - p_2| = p_2 - y_2 \leqslant p_2 - \Lambda(y_1) = v(p_1) - \Big(v'(p_1)(y_1 - p_1) + v(p_1)\Big) \leqslant m(p_1 + c_m)^{\gamma-1}|y_1 - p_1|.$$

Now we have that $p_1 + c_m \geqslant 1$, by our assumptions. Hence, since $\gamma - 1 < 0$, we conclude that $|y_2 - p_2| \leqslant m|y_1 - p_1|$, thus proving (6.11).

As a consequence of (6.11), we have that if $y \in B$ then $|y - p| \leqslant (1 + Cm)|y_1 - p_1|$, for some $C > 0$, and therefore

$$\int_B \frac{dy}{|y-p|^{2+2s}} \geqslant (1 - Cm) \int_B \frac{dy}{|y_1 - p_1|^{2+2s}} = (1 - Cm)\,I, \tag{6.12}$$

up to renaming $C > 0$, where

$$I := \int_B \frac{dy}{|y_1 - p_1|^{2+2s}} = \int_{-\infty}^{q_1} \frac{-\Lambda(y_1)}{|y_1 - p_1|^{2+2s}}\,dy_1.$$

Recalling the definition of H in (6.8), we have that

$$J := H - I = \int_{q_1}^{+\infty} \frac{\Lambda(y_1) - v(y_1)}{|y_1 - p_1|^{2+2s}}\,dy_1 + \int_{-\infty}^{q_1} \frac{\Lambda(y_1)}{|y_1 - p_1|^{2+2s}}\,dy_1.$$

Accordingly, since $v(y_1) = 0$ if $y_1 \leqslant q_1$, we obtain that

$$J = \int_{-\infty}^{+\infty} \frac{\Lambda(y_1) - v(y_1)}{|y_1 - p_1|^{2+2s}}\,dy_1 = \int_{-\infty}^{+\infty} \frac{v(p_1) - v(y_1)}{|y_1 - p_1|^{2+2s}}\,dy_1,$$

where we have used (6.4) in the last identity and the integrals are taken in the principal value sense. Hence, we use (6.3) and the substitution in (6.9), and we conclude that

$$J = \frac{m}{\gamma} \int_{-\infty}^{+\infty} \frac{(p_1 + c_m)^\gamma - (y_1 + c_m)_+^\gamma}{|y_1 - p_1|^{2+2s}}\,dy_1 = \frac{m}{\gamma(p_1 + c_1)^{1+2s-\gamma}} \int_{-\infty}^{+\infty} \frac{1 - (t+1)_+^\gamma}{|t|^{2+2s}}\,dt = -\frac{C_B\,m}{(p_1 + c_m)^{\frac{1}{2}+s-\varepsilon_0}},$$

where

$$C_B := \int_{-\infty}^{+\infty} \frac{(t+1)_+^\gamma - 1}{|t|^{2+2s}}\,dt.$$

From Lemma 6.1, we have that $C_B \geqslant c_\star \varepsilon_0$, for some $c_\star > 0$. As a consequence,

$$I = H - J = \frac{(C_A + C_B)\,m}{(p_1 + c_m)^{\frac{1}{2}+s-\varepsilon_0}} \geqslant \frac{(C_A + c_\star \varepsilon_0)\,m}{(p_1 + c_m)^{\frac{1}{2}+s-\varepsilon_0}},$$

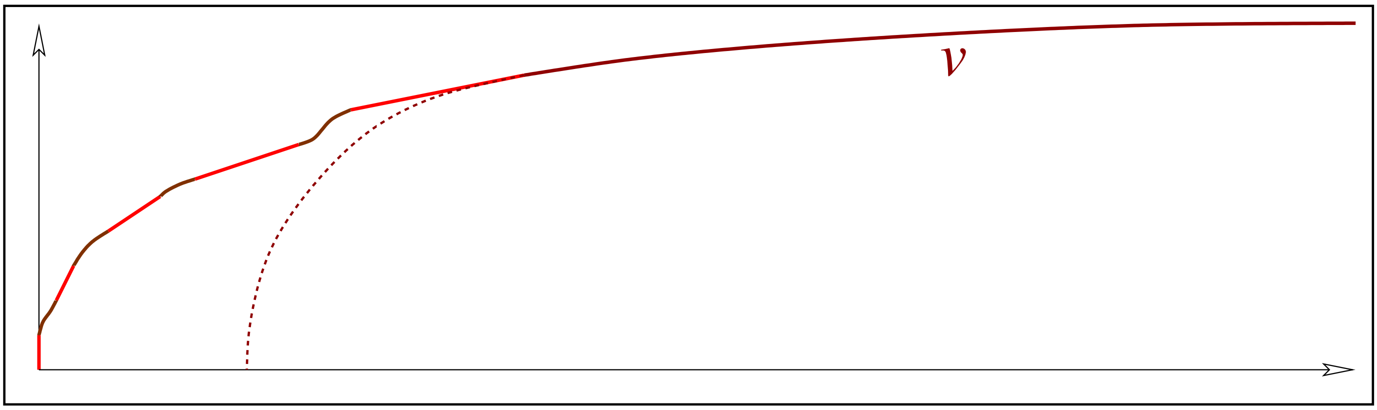

FIGURE 14. The barrier constructed in Proposition 6.3.

and so, by (6.12)

$$\int_B \frac{dy}{|y-p|^{2+2s}} \geqslant \frac{(1-Cm)(C_A+c_\star\varepsilon_0)\,m}{(p_1+c_m)^{\frac{1}{2}+s-\varepsilon_0}}.$$

Putting together this and (6.10), we obtain that

$$\int_B \frac{dy}{|y-p|^{2+2s}} - \int_A \frac{dy}{|y-p|^{2+2s}} \geqslant \frac{\big[(1-Cm)(C_A+c_\star\varepsilon_0)-C_A\big]\,m}{(p_1+c_m)^{\frac{1}{2}+s-\varepsilon_0}},$$

which implies the desired result. $\square$

Now we are in the position of improving the behavior at infinity of the barrier constructed in Proposition 5.3. The idea is to "glue" the barrier of Proposition 5.3 with the graph of the "right" power function at infinity. The construction is sketched in Figure 14 and the precise result obtained is the following:

Proposition 6.3. *Let $\varepsilon_0 \in (0, 1-\gamma_0)$. There exists $c > 0$ such that if $m \in (0, c\varepsilon_0]$, then the following statement holds.*

There exist $a_m > 0$, $d_m > 1 > \alpha_m > 0$, $c_m \in \mathbb{R}$ and a set $E_m \subset \mathbb{R}^2$ with $(\partial E_m) \cap \{x_2 > 0\}$ of class $C^{1,1}$ and such that:

$$\begin{aligned}
&E_m \cap \{x_1 < 0\} = (-\infty, 0) \times (-\infty, 0),\\
&E_m \supseteq \mathbb{R} \times (-\infty, 0),\\
&E_m \supseteq (0,+\infty) \times (-\infty, a_m],\\
&E_m \cap \{\alpha_m \leqslant x_1 \leqslant d_m\} = \{x_2 < v'(d_m)(x_1-d_m) + v(d_m),\ \alpha_m \leqslant x_1 \leqslant d_m\}\\
\text{and}\quad &E_m \cap \{x_1 > d_m\} = \{x_2 < v(x_1),\ x_1 > d_m\},
\end{aligned}$$

where v was introduced in (6.3). Moreover, there exist $c' \in (0,1)$ and $N > 1$ such that

$$\int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c'\,\varepsilon_0\,m}{|p|^{\frac{1}{2}+s-\varepsilon_0}}, \tag{6.13}$$

for any $p \in (\partial E_m) \cap \{x_1 > \frac{d_m}{N}\}$, and

$$\int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\,dy \geqslant \frac{c'\,m}{d_m^{1-\gamma}\,|p|^{2s}}, \tag{6.14}$$

for any $p \in (\partial E_m) \cap \{x_1 \in \big(0, \frac{d_m}{N}\big]\}$.

Proof. We use Proposition 5.3 with a large K. In this way, we may suppose that $\ell_K \geqslant K$ is as large as we wish, while $q_K \leqslant K^{-1}$ is as small as we wish. We fix $N > 0$, to be chosen appropriately large (independently on K) and we set

$$\begin{aligned} & d_m := N^2 \\ \text{and} \quad & m := \ell_K^{-1}\big(\gamma\,(d_m + q_K)\big)^{1-\gamma}. \end{aligned} \tag{6.15}$$

We stress that $m > 0$ is small when K is large, since

$$m \leqslant K^{-1}\big(\gamma\,(N^2 + K^{-1})\big)^{1-\gamma},$$

that is small when K is large (much larger than the fixed N). Hence Proposition 5.3 provides a set, say F_m, whose boundary agrees with a straight line λ_m of the form $x_2 = \ell_K^{-1}(x_1 + q_K)$ when $x_1 \geqslant \alpha_m$, for suitable $q_K \in [0, K^{-1}]$ and $\alpha_m > 0$.

Now we join such a straight line with the function v defined in (6.3), at the point $(d_m, v(d_m))$, with $\beta_m := d_m - \alpha_m$ suitably large. To this goal, we define

$$c_m := (\gamma - 1)d_m + \gamma q_K. \tag{6.16}$$

Notice that

$$d_m + c_m = \gamma(d_m + q_K). \tag{6.17}$$

This and (6.15) give that

$$v(d_m) = \frac{m\,(d_m + c_m)_+^\gamma}{\gamma} = \frac{m\,\big(\gamma(d_m + q_K)\big)^\gamma}{\gamma} = \ell_K^{-1}\big(\gamma\,(d_m + q_K)\big)^{1-\gamma} \cdot \frac{\big(\gamma(d_m + q_K)\big)^\gamma}{\gamma} = \ell_K^{-1}(d_m + q_K),$$

which says that v meets the straight line λ_m at the point $(d_m, v(d_m))$.

Also, by (6.15) and (6.17), we see that

$$v'(d_m) = m(d_m + c_m)^{\gamma-1} = \ell_K^{-1}\big(\gamma\,(d_m + q_K)\big)^{1-\gamma} \cdot \big(\gamma\,(d_m + q_K)\big)^{\gamma-1} = \ell_K^{-1},$$

therefore v and λ_m have the same slope at the meeting point $(d_m, v(d_m))$. Therefore, the set E_m which coincides with F_m when $\{x_1 \leqslant d_m\}$ and with the subgraph of v when $\{x_1 > d_m\}$ satisfy the geometric properties listed in the statement of Proposition 6.3, and it only remains to prove (6.13) and (6.14).

For this scope, we first consider the case in which $p_1 \geqslant d_m$. Then, we take Λ as in (6.4) and A and B as in (6.6). Let also T be the subgraph of Λ. Then, by symmetry

$$\int_{\mathbb{R}^2} \frac{\chi_T(y) - \chi_{T^c}(y)}{|y - p|^{2+2s}}\, dy = 0.$$

Notice that $T \setminus E_m \subseteq A$ and $E_m \setminus T \supseteq B$, therefore

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y - p|^{2+2s}}\, dy \\ = \; & \int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y) - \chi_T(y) + \chi_{T^c}(y)}{|y - p|^{2+2s}}\, dy \\ = \; & 2\int_{\mathbb{R}^2} \frac{\chi_{E_m \setminus T}(y) - \chi_{T \setminus E_m}(y)}{|y - p|^{2+2s}}\, dy \\ \geqslant \; & 2\int_{\mathbb{R}^2} \frac{\chi_B(y) - \chi_A(y)}{|y - p|^{2+2s}}\, dy \\ = \; & 2\left(\int_B \frac{dy}{|y - p|^{2+2s}} - \int_A \frac{dy}{|y - p|^{2+2s}}\right). \end{aligned}$$

Notice also that

$$1 - c_m = 1 - (\gamma - 1)d_m - \gamma q_K \leqslant 1 - \gamma d_m + d_m \leqslant d_m \tag{6.18}$$

thanks to (6.16) and (6.15). Hence, in this case, $p_1 \geqslant d_m \geqslant 1 - c_m$, and so the assumptions of Lemma 6.2 are fulfilled. Therefore, by (6.7),

$$\int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant \frac{c'\, \varepsilon_0\, m}{(p_1 + c_m)^{\frac{1}{2}+s-\varepsilon_0}}, \tag{6.19}$$

for some $c' > 0$. Now we notice that, by (6.16) and (6.15),

$$p_1 + c_m = p_1 + (\gamma - 1)d_m + \gamma q_K \leqslant 2p_1 \leqslant 2|p|.$$

Using this and (6.19), we see that (6.13) holds true in this case.

Hence, it remains to prove (6.13) and (6.14) when $p_1 \in (0, d_m)$. In this case, we use that, by Proposition 5.3,

$$\int_{\mathbb{R}^2} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant \frac{\bar c}{\ell_K\, |p|^{2s}},$$

for some $\bar c > 0$. Also $F_m \setminus E_m$ coincides with the portion comprised above the graph of v and below the straight line λ_m, that is

$$G := \{x_1 > d_m,\ v(x_1) < x_2 < v'(d_m)(x_1 - d_m) + v(d_m)\},$$

while $E_m \setminus F_m$ is empty. Therefore

$$\begin{aligned} \frac{\bar c}{\ell_K\, |p|^{2s}} - \int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\, dy &\leqslant \int_{\mathbb{R}^2} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y) - \chi_{E_m}(y) + \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\, dy \\ &= 2\int_G \frac{dy}{|y-p|^{2+2s}} \leqslant 2\int_G \frac{dy}{|y_1 - p_1|^{2+2s}} \\ &= 2\int_{d_m}^{+\infty} \frac{v'(d_m)(y_1 - d_m) + v(d_m) - v(y_1)}{|y_1 - p_1|^{2+2s}}\, dy_1. \end{aligned} \tag{6.20}$$

Now, we distinguish the cases $p_1 \in \big(0, \frac{d_m}{N}\big)$ and $p_1 \in \big[\frac{d_m}{N}, d_m\big)$.

If $p_1 \in \big(0, \frac{d_m}{N}\big)$, we use (6.20) and observe that $v(y_1) \geqslant v(d_m)$ if $y_1 \geqslant d_m$, to conclude that

$$\begin{aligned} \frac{\bar c}{\ell_K\, |p|^{2s}} - \int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\, dy &\leqslant 2v'(d_m) \int_{d_m}^{+\infty} \frac{y_1 - d_m}{|y_1 - p_1|^{2+2s}}\, dy_1 \\ &\leqslant 2v'(d_m) \int_{d_m}^{+\infty} \frac{dy_1}{(y_1 - p_1)^{1+2s}} \leqslant \frac{Cm\, (d_m + c_m)^{\gamma - 1}}{(d_m - p_1)^{2s}} \\ &\leqslant \frac{Cm\, (d_m + c_m)^{\gamma-1}}{d_m^{2s}}, \end{aligned}$$

up to renaming constants. Therefore, recalling (6.15) and (6.17),

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\, dy &\geqslant \frac{\bar c\, m}{\big(\gamma\,(d_m + q_K)\big)^{1-\gamma}\, |p|^{2s}} - \frac{Cm}{(d_m + c_m)^{1-\gamma}\, d_m^{2s}} \\ &= \frac{m}{(d_m + c_m)^{1-\gamma}} \left(\frac{\bar c}{|p|^{2s}} - \frac{C}{d_m^{2s}} \right). \end{aligned} \tag{6.21}$$

Now we observe that, when $p_1 \leqslant \frac{d_m}{N}$, we have that $p_2 \leqslant 1 + \ell_K^{-1}\big(\frac{d_m}{N} + q_K\big) \leqslant 2 + \frac{d_m}{N} \leqslant \frac{d_m}{N^{1/2}}$, and so $|p| \leqslant \frac{d_m}{N^{1/4}}$. Therefore

$$\frac{C}{d_m^{2s}} \leqslant \frac{C}{N^{s/2}\, |p|^{2s}} \leqslant \frac{\bar c}{2\, |p|^{2s}},$$

if N is large enough (independently on m and K). This and (6.21) imply that

$$\int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant \frac{m\, \bar c}{2(d_m + c_m)^{1-\gamma}|p|^{2s}}.$$

By recalling (6.17), we see that the latter estimate implies (6.14) in this case.

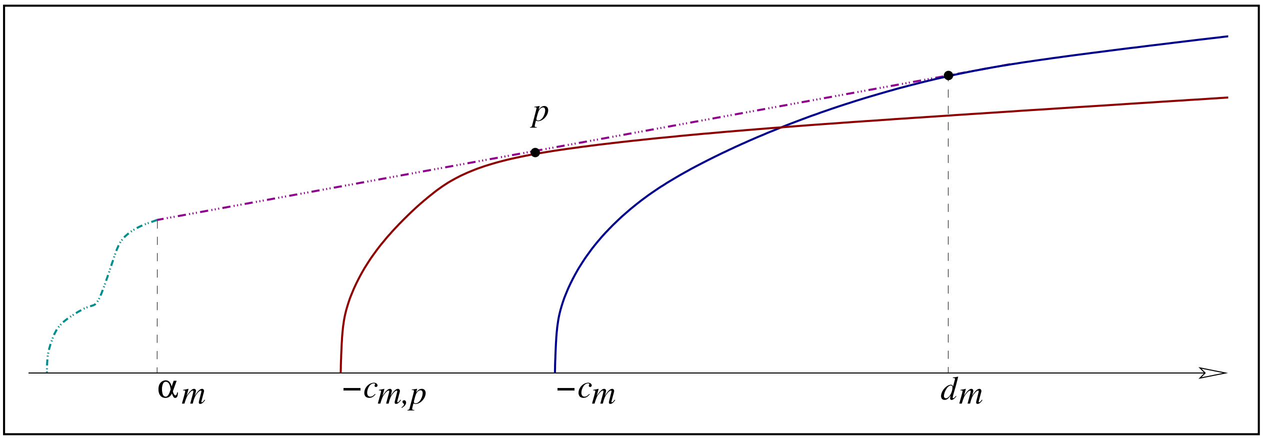

FIGURE 15. A power-like function tangent at $p \in \partial E_m$, with $p_1 \in \left[\frac{d_m}{N}, d_m\right)$.

It remains to prove (6.13) when $p_1 \in \left[\frac{d_m}{N}, d_m\right)$. In this case, we argue like this. For any $p = (p_1, p_2) \in \partial F_m$ with $p_1 \in \left[\frac{d_m}{N}, d_m\right)$, we have $p_2 = v'(d_m)(p_1 - d_m) + v(d_m)$, and we define v_p the power function whose graph passes through p and tangent to the line $\{x_2 = v'(d_m)(x_1 - d_m) + v(d_m)\}$ at p, see Figure 15. Explicitly, we define

$$\begin{aligned} & v_p(x_1) := \frac{m_p\,(x_1 + c_{m,p})_+^\gamma}{\gamma}, \\ \text{with} \qquad & m_p := (\gamma p_2)^{1-\gamma} m^\gamma (d_m + c_m)^{\gamma(\gamma-1)} \\ \text{and} \qquad & c_{m,p} := \frac{\gamma p_2}{m\,(d_m + c_m)^{\gamma-1}} - p_1. \end{aligned}$$

We remark that

$$v_p(p_1) = p_2 \text{ and } v_p'(p_1) = v'(d_m).$$

Since $p_2 < v(d_m) = m\,\gamma^{-1}(d_m + c_m)^\gamma$, we have that

$$m_p < \left(m\,(d_m + c_m)^\gamma\right)^{1-\gamma} m^\gamma (d_m + c_m)^{\gamma(\gamma-1)} = m. \tag{6.22}$$

Moreover $p_2 = v'(d_m)(p_1 - d_m) + v(d_m) = m\,(d_m + c_m)^{\gamma-1}(p_1 - d_m) + m\,\gamma^{-1}(d_m + c_m)^\gamma$, therefore

$$\begin{aligned} c_{m,p} &= \frac{\gamma m\,(d_m + c_m)^{\gamma-1}(p_1 - d_m) + m\,(d_m + c_m)^\gamma}{m\,(d_m + c_m)^{\gamma-1}} - p_1 \\ &= \gamma(p_1 - d_m) + d_m + c_m - p_1 = (1-\gamma)(d_m - p_1) + c_m. \end{aligned} \tag{6.23}$$

Hence, since $p_1 < d_m$,

$$c_{m,p} > c_m. \tag{6.24}$$

Also, from (6.16) and (6.23),

$$c_{m,p} = (1-\gamma)(d_m - p_1) + (\gamma - 1)d_m + \gamma q_K = -(1-\gamma)\,p_1 + \gamma q_K. \tag{6.25}$$

Therefore, since $p_1 \geqslant \frac{d_m}{N} = N$,

$$c_{m,p} \leqslant -(1-\gamma)\,N + \gamma q_K \leqslant -(1-\gamma)\,N + 1 < -1 \leqslant -\alpha_m, \tag{6.26}$$

provided that N is large enough.

Furthermore, using again (6.25),

$$p_1 + c_{m,p} = \gamma p_1 + \gamma q_K \geqslant \frac{\gamma d_m}{N} = \gamma N \geqslant 1. \tag{6.27}$$

In addition,

$$m_p(p_1 + c_{m,p})_+^{\gamma-1} = v_p'(p_1) = v'(d_m) = m(d_m + c_m)_+^{\gamma-1},$$

therefore, by (6.17) and (6.25),

$$\begin{aligned}\frac{m_p}{m}&=\frac{(p_1+c_{m,p})_+^{1-\gamma}}{(d_m+c_m)_+^{1-\gamma}}=\frac{(\gamma p_1+\gamma q_K)_+^{1-\gamma}}{(\gamma(d_m+q_K))_+^{1-\gamma}}=\frac{(p_1+q_K)_+^{1-\gamma}}{(d_m+q_K)_+^{1-\gamma}}\\&\geqslant\frac{\left(\frac{d_m}{N}\right)_+^{1-\gamma}}{(2d_m)_+^{1-\gamma}}=\frac{1}{(2N)^{1-\gamma}}.\end{aligned}\tag{6.28}$$

Now we claim that

$$\text{if } x_1\geqslant d_m, \text{ then } v_p(x_1)\leqslant v(x_1).\tag{6.29}$$

To prove this, we use (6.18) and (6.24) to see that

$$x_1+c_{m,p}\geqslant x_1+c_m\geqslant d_m+c_m\geqslant 1,$$

therefore

$$\psi(x_1):=\gamma\left(v_p(x_1)-v(x_1)\right)=m_p\,(x_1+c_{m,p})^\gamma-m\,(x_1+c_m)^\gamma.$$

Also, v_p is concave, therefore

$$\begin{aligned}v_p(d_m)&\leqslant v_p(p_1)+v_p'(p_1)(d_m-p_1)=p_2+v'(d_m)(d_m-p_1)\\&=v'(d_m)(p_1-d_m)+v(d_m)+v'(d_m)(d_m-p_1)=v(d_m).\end{aligned}$$

As a consequence, $\psi(d_m)\leqslant 0$. Moreover, for any $x_1\geqslant d_m$,

$$\psi'(x_1)=m_p\gamma\,(x_1+c_{m,p})^{\gamma-1}-m\gamma\,(x_1+c_m)^{\gamma-1}\leqslant m\gamma\left[(x_1+c_{m,p})^{\gamma-1}-(x_1+c_m)^{\gamma-1}\right]\leqslant 0,$$

thanks to (6.22) and (6.24). From these considerations, we obtain that $\psi\leqslant 0$ in $[d_m,+\infty)$, which proves (6.29).

Also, by concavity,

$$\begin{aligned}&\text{if } x_1\in[-c_{m,p},d_m], \text{ then}\\&v_p(x_1)\leqslant v_p'(p_1)(x_1-p_1)+v_p(p_1)=v'(d_m)(x_1-p_1)+p_2=v'(d_m)(x_1-d_m)+v(d_m).\end{aligned}\tag{6.30}$$

Now we claim that

$$\text{the subgraph of } v_p \text{ is contained in } E_m.\tag{6.31}$$

To check this, let $x=(x_1,x_2)$ be such that $x_2<v_p(x_1)$. Then, if $x_1<-c_{m,p}$ then $v_p(x_1)=0$ and so (6.31) plainly follows. If $x_1\in[-c_{m,p},d_m]$, then (6.31) is implied by (6.26) and (6.30). Finally, if $x_1>d_m$, then (6.31) is a consequence of (6.29).

Hence, we define $S:=\{x_2<v_p(x_1)\}$, we use (6.31) and Lemma 6.2 (which can be exploited in this framework with the power-like function v_p, thanks to (6.27)) and we obtain that

$$\int_{\mathbb{R}^2}\frac{\chi_{E_m}(y)-\chi_{E_m^c}(y)}{|y-p|^{2+2s}}\,dy\geqslant\int_{\mathbb{R}^2}\frac{\chi_S(y)-\chi_{S^c}(y)}{|y-p|^{2+2s}}\,dy\geqslant\frac{c'\,\varepsilon_0\,m_p}{(p_1+c_{m,p})^{\frac12+s-\varepsilon_0}},\tag{6.32}$$

for some $c'>0$. Now we recall (6.25) and we see that $p_1+c_{m,p}\leqslant p_1\leqslant|p|$. Using this and (6.28) (notice that N has now been fixed), we obtain (6.13) if $p_1\in\left[\frac{d_m}{N},d_m\right)$ as a consequence of (6.32).

This completes the proof of (6.13) in all cases and finishes the proof of Proposition 6.3. $\square$

7. Construction of compactly supported barriers

In this section, we construct a suitable barrier for the fractional mean curvature equation in the plane which is flat and horizontal outside a vertical slab, and whose geometric properties inside the slab are under control. Roughly speaking, we will take the barrier constructed in Proposition 6.3 and a reflected version of it and join it smoothly in the middle. The effect of this surgery is negligible at the points of the barrier that are near the horizontal part, and give a bounded contribution in the middle.

This barrier is described in Figure 16 and the precise result obtained is the following:

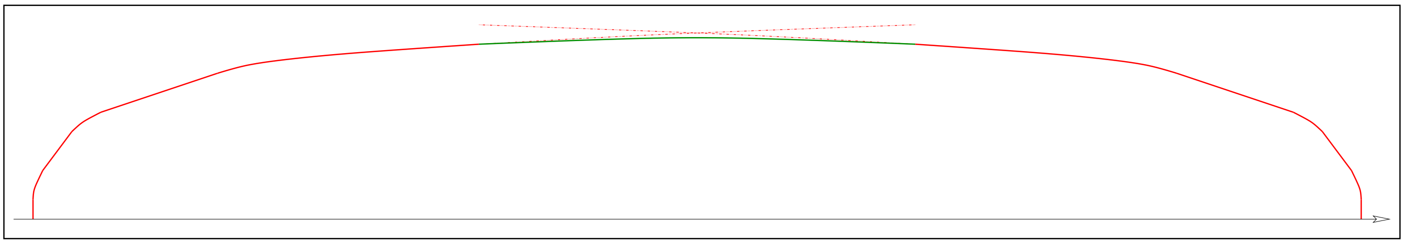

FIGURE 16. The barrier constructed in Proposition 7.1.

Proposition 7.1. *Let $\varepsilon_0 \in (0, 1-\gamma_0)$. There exists $m_{\varepsilon_0} > 0$ such that if $m \in (0, m_{\varepsilon_0}]$ then the following statement holds.*

There exist $a_m > 0$, $L_m > A_m > d_m > 1$, $c_m \in \mathbb{R}$, $C_\star > 0$ and a set $F_m \subset \mathbb{R}^2$ with $(\partial F_m) \cap \{x_2 > 0\}$ of class $C^{1,1}$ and such that:

$$\begin{aligned} & F_m \cap \{x_1 < 0\} = (-\infty, 0) \times (-\infty, 0), \\ & F_m \supseteq \mathbb{R} \times (-\infty, 0), \\ & F_m \supseteq (0, L_m + 1) \times (-\infty, a_m], \\ & F_m \subseteq \{x_2 \leqslant C_\star m\, L_m^{\frac{1}{2}+s+\varepsilon_0}\} \\ \textit{and} \qquad & F_m \cap \{d_m < x_1 < L_m\} = \{x_2 < v(x_1),\ d_m < x_1 < L_m\}, \end{aligned}$$

where v was introduced in (6.3). In addition, one can suppose that

$$L_m = 10 A_m \geqslant 2 + m^{-1} + e^{\frac{1}{a_m}}. \tag{7.1}$$

Moreover, the set F_m is even symmetric with respect to the vertical axis $\{x_1 = L_m + 1\}$, and there exists $C' > 0$ such that

$$\int_{\mathbb{R}^2} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant 0, \tag{7.2}$$

for any $p \in (\partial F_m) \cap \{x_1 \in (0, A_m)\}$, and

$$\int_{\mathbb{R}^2} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant -\frac{C' m^{2s}}{L_m^{\frac{1}{2}+s-\varepsilon_0}}, \tag{7.3}$$

for any $p \in (\partial F_m) \cap \{x_1 \in [A_m, L_m + 1]\}$.

Proof. We let E_m be the set constructed in Proposition 6.3. Let E'_m be the even reflection of E_m with respect to the vertical axis $\{x_1 = L_m + 1\}$. We take a smooth function $w : [L_m, L_m + 2] \to [v(L_m), Cm L_m^\gamma]$ that is even with respect to $\{x_1 = L_m + 1\}$, with $w(L_m) = v(L_m)$ and such that its derivatives agree with the ones of v at the point L_m. The set F_m is then defined as

$$\big(E_m \cap \{x_1 \leqslant L_m\}\big) \cup \big\{x_2 < w(x_1),\ x_1 \in (L_m, L_m + 2)\big\} \cup \big(E'_m \cap \{x_1 \geqslant L_m + 2\}\big).$$

For completeness, let us describe the above function w explicitly. One takes an odd function $\tau \in C^\infty(\mathbb{R}, [-1, 1])$ such that $\tau = -1$ in $(-\infty, -1]$ and $\tau = 1$ in $[1, +\infty)$ and defines w by

$$w(x_1) := \frac{\big(1 - \tau(x_1 - L_m - 1)\big)\, v(x_1) + \big(1 + \tau(x_1 - L_m - 1)\big)\, v(2L_m + 2 - x_1)}{2}.$$

Then $w(L_m + 1 + x_1) = w(L_m + 1 - x_1)$, hence w is even with respect to $\{x_1 = L_m + 1\}$. The set F_m has the desired geometric properties, so it remains to prove (7.2) and (7.3). For this, we take $L_m = 10A_m$ appropriately large. In particular, we suppose that $L_m \geqslant c_m + 2A_m$, and therefore, for any $y_1 \in [L_m, +\infty)$ and $p_1 \in (0, A_m)$ we have that $y_1 + c_m \leqslant 2(y_1 - p_1)$, and so, by (6.3),

$$v(y_1) = \frac{m\,(y_1 + c_m)_+^\gamma}{\gamma} \leqslant \frac{2^\gamma m\,(y_1 - p_1)^\gamma}{\gamma}.$$

We also notice that $E_m \setminus F_m \subseteq \{x_1 > L_m,\ 0 < x_2 < v(x_1)\}$. Therefore, for every $p \in (\partial F_m) \cap \{x_1 \in (0, A_m)\}$,

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\chi_{E_m}(y) - \chi_{E_m^c}(y)}{|y-p|^{2+2s}}\, dy - \int_{\mathbb{R}^2} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y)}{|y-p|^{2+2s}}\, dy \\
&\quad = 2 \int_{\mathbb{R}^2} \frac{\chi_{E_m \setminus F_m}(y) - \chi_{F_m \setminus E_m}(y)}{|y-p|^{2+2s}}\, dy \leqslant 2 \int_{E_m \setminus F_m} \frac{dy}{|y-p|^{2+2s}} \\
&\quad \leqslant 2 \int_{L_m}^{+\infty} \frac{v(y_1)\, dy_1}{(y_1 - p_1)^{2+2s}} \leqslant Cm \int_{L_m}^{+\infty} (y_1 - p_1)^{\gamma - 2 - 2s} = \frac{Cm}{(L_m - p_1)^{1+2s-\gamma}} \\
&\quad \leqslant \frac{Cm}{L_m^{1+2s-\gamma}} = \frac{Cm}{L_m^{\frac{1}{2}+s-\varepsilon_0}},
\end{aligned}
\tag{7.4}
$$

up to changing the names of the constant $C > 0$ line after line. Hence, recalling (6.13),

$$
\int_{\mathbb{R}^2} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant \frac{c'\, \varepsilon_0\, m}{|p|^{\frac{1}{2}+s-\varepsilon_0}} - \frac{Cm}{L_m^{\frac{1}{2}+s-\varepsilon_0}} \geqslant \frac{c'\, \varepsilon_0\, m}{2\, |p|^{\frac{1}{2}+s-\varepsilon_0}}
$$

for any $p \in (\partial F_m) \cap \{x_1 \in \left(\frac{d_m}{N}, A_m\right)\}$, as long as L_m is large enough (possibly in dependence of $\sup_{q_1 \in (0,A_m)} |q|$). This establishes (7.2) if $p \in (\partial F_m) \cap \{x_1 \in \left(\frac{d_m}{N}, A_m\right)\}$.

If instead $p \in (\partial F_m) \cap \{x_1 \in \left(0, \frac{d_m}{N}\right]\}$, we use (7.4) and (6.14) to obtain that

$$
\int_{\mathbb{R}^2} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant \frac{c'\, m}{d_m^{1-\gamma}\, |p|^{2s}} - \frac{Cm}{L_m^{\frac{1}{2}+s-\varepsilon_0}} \geqslant \frac{c'\, N^{2s}\, m}{d_m^{1-\gamma}\, d_m^{2s}} - \frac{Cm}{L_m^{\frac{1}{2}+s-\varepsilon_0}} = \frac{c'\, N^{2s}\, m}{d_m^{\frac{1}{2}+s-\varepsilon_0}} - \frac{Cm}{L_m^{\frac{1}{2}+s-\varepsilon_0}} \geqslant 0,
$$

as long as L_m is large enough, and this proves (7.2) also in this case.

Now we prove (7.3). For this, we take $p \in (\partial F_m) \cap \{x_1 \in [A_m, L_m + 1)\}$. By (6.3), the curvature of F_m at p is bounded (in absolute value) by $CmL_m^{\gamma-2}$. Hence (see Lemma 3.1 in [13], applied here with $\lambda := L_m^{\gamma-1}$ and $R := m^{-1}L^{2-\gamma}$, so that $\lambda R = \frac{L_m}{m}$, and canceling the contribution coming from the tangent line) one obtains that

$$
\left| \int_{B_{\frac{L_m}{m}}(p)} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y)}{|y-p|^{2+2s}}\, dy \right| \leqslant C \big(L_m^{\gamma-1}\big)^{1-2s} \big(m^{-1} L_m^{2-\gamma}\big)^{-2s} = Cm^{2s} L_m^{\gamma-1-2s} = \frac{Cm^{2s}}{L_m^{\frac{1}{2}+s-\varepsilon_0}},
\tag{7.5}
$$

for some $C > 0$, possibly varying from step to step.

Moreover, to compute the contribution coming from outside $B_{\frac{L_m}{m}}(p)$, we can compare the set F_m with the horizontal line passing through p. Notice indeed that $F_m \setminus B_{\frac{L_m}{m}}(p) = \{x_2 < 0\} \setminus B_{\frac{L_m}{m}}(p)$. Thus, since p_2 is controlled by CmL_m^{γ}

$$
\begin{aligned}
\left| \int_{\mathbb{R}^2 \setminus B_{\frac{L_m}{m}}(p)} \frac{\chi_{F_m}(y) - \chi_{F_m^c}(y)}{|y-p|^{2+2s}}\, dy \right| &\leqslant 2 \int_{\{0 < y_2 < CmL_m^{\gamma}\} \setminus B_{\frac{L_m}{m}}(p)} \frac{dy}{|y-p|^{2+2s}} \\
&\leqslant CmL_m^{\gamma} \int_{\{|y_1 - p_1| \geqslant L_m\}} \frac{dy_1}{|y_1 - p_1|^{2+2s}} = CmL_m^{\gamma-1-2s} = \frac{Cm}{L_m^{\frac{1}{2}+s-\varepsilon_0}}.
\end{aligned}
$$

up to renaming $C > 0$. This and (7.5) imply (7.3), as desired. ☐

By scaling Proposition 7.1, one obtains the following result:

Corollary 7.2. *Fix $\varepsilon_0 > 0$ arbitrarily small. There exist an infinitesimal sequence of positive δ's and sets $H_\delta \subseteq \mathbb{R}^2$, with $(\partial H_\delta) \cap \{x_2 > 0\}$ of class $C^{1,1}$, that are even symmetric with respect to the axis $\{x_1 = 0\}$ and satisfy the following properties:*

$$
\begin{aligned}
& H_\delta \cap \{x_1 < -1\} = (-\infty, -1) \times (-\infty, 0), \\
& H_\delta \supseteq \mathbb{R} \times (-\infty, 0), \\
& H_\delta \supseteq (-1, 1) \times (-\infty, \delta^{\frac{2+\varepsilon_0}{1-2s}}] \\
\text{and} \quad & H_\delta \subseteq \{x_2 \leqslant \delta\}.
\end{aligned}
$$

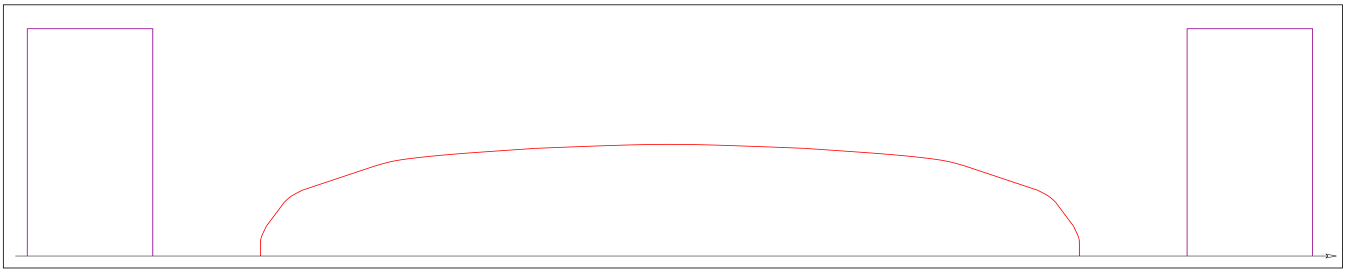

Figure 17. The barrier constructed in Proposition 7.3.

Moreover,

$$\int_{\mathbb{R}^2} \frac{\chi_{H_\delta}(y) - \chi_{H_\delta^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant 0, \tag{7.6}$$

for any $p \in (\partial H_\delta) \cap \{x_1 \in \big(-1, -1 + \frac{1}{100}\big)\}$ *and*

$$\int_{\mathbb{R}^2} \frac{\chi_{H_\delta}(y) - \chi_{H_\delta^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant -\delta, \tag{7.7}$$

for any $p \in (\partial H_\delta) \cap \{x_1 \in \big[-1 + \frac{1}{100},\ 0\big]\}$.

Proof. We scale the set F_m constructed in Proposition 7.1 by a factor of order $\frac{1}{L_m}$ (then we also translate to the left by a horizontal vector of length 1) and take $\delta := \frac{1}{L_m^{\frac{1}{2}-s-\varepsilon_0}}$. Notice that δ is infinitesimal, due to (7.1). Also, the estimates in (7.6) and (7.7) follow from the ones in (7.2) and (7.3), since the fractional curvature scales by a factor proportional to L_m^{2s}.

We also remark that the vertical stickiness of F_m in Proposition 7.1 was bounded from below by a_m, and $L_m \geqslant e^{\frac{1}{a_m}}$, by (7.1). As a consequence, by scaling, the vertical stickiness of H_δ here is bounded by an order of $\frac{a_m}{L_m} \geqslant \frac{1}{L_m \log L_m}$. This quantity is in turn bounded by an order of $\frac{\delta^{\frac{2}{1-2s-2\varepsilon_0}}}{|\log \delta|}$, which we can bound by $\delta^{\frac{2+\varepsilon_0}{1-2s}}$, up to renaming ε_0. $\square$

We observe that while in (7.6) we obtained that the fractional mean curvature of the set is nonnegative near $\{x_1 = \pm 1\}$, from (7.7) we can only say that the fractional mean curvature of the set near $\{x_1 = 0\}$ is controlled by a small negative quantity (and this cannot be improved, since at the points in which the set reaches its highest level the fractional mean curvature must be negative). By adding an additional small contribution to the set in $\{|x_1| \in (2,3)\}$, we can obtain a complete subsolution, i.e. a set whose fractional mean curvature is nonnegative. Such subsolution has the important geometric feature that the points along $\{x_1 = 0\}$ detach from $\{x_2 = 0\}$, see Figure 17. The precise statement goes as follows:

Proposition 7.3. *Fix* $\varepsilon_0 > 0$ *arbitrarily small. There exist* $C > 0$*, an infinitesimal sequence of positive* δ*'s and sets* $E_\delta \subseteq \mathbb{R}^2$*, with* $(\partial E_\delta) \cap \big((-\frac{3}{2}, \frac{3}{2}) \times (0, +\infty)\big)$ *of class* $C^{1,1}$*, that are even symmetric with respect to the axis* $\{x_1 = 0\}$ *and satisfy the following properties:*

$$\begin{aligned}
& E_\delta \cap \{x_1 \in (-\infty,-3) \cup (-2,-1)\} = \big((-\infty,-3) \cup (-2,-1)\big) \times (-\infty, 0), \\
& E_\delta \cap \{x_1 \in [-3,-2]\} = [-3,-2] \times (-\infty, C\delta), \\
& E_\delta \supseteq \mathbb{R} \times (-\infty, 0), \\
& E_\delta \supseteq (-1,1) \times (-\infty, \delta^{\frac{2+\varepsilon_0}{1-2s}}] \\
\text{and} \qquad & E_\delta \cap \{|x_1| \leqslant 1\} \subseteq \{x_2 \leqslant \delta\}.
\end{aligned}$$

Moreover, for any $p \in (\partial E_\delta) \cap \{|x_1| < 1\}$,

$$\int_{\mathbb{R}^2} \frac{\chi_{E_\delta}(y) - \chi_{E_\delta^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant 0. \tag{7.8}$$

Proof. Let H_δ be as in Corollary 7.2. We define $E_\delta := H_\delta \cup F_- \cup F_+$, where $F_- := (-3,-2) \times [0, C\delta)$ and $F_+ := (2,3) \times [0, C\delta)$. Then E_δ satisfies all the desired geometric properties, and $E_\delta \supset H_\delta$. Therefore, when $p \in (\partial E_\delta) \cap \{|x_1| \in \big(1 - \frac{1}{100}, 1\big)\}$, we have that (7.8) follows from (7.6). Moreover, when $p \in (\partial E_\delta) \cap \{|x_1| \leqslant 1 - \frac{1}{100}\}$, we have that (7.8) follows from (7.7) and the fact that $|F_+| = |F_-| = C\delta$ (and one can choose $C > 0$ conveniently large). $\square$

Remark 7.4. Concerning the statement of Proposition 7.3, by (7.8) (see in addition Lemma 3.3 in [13]), we also obtain that

$$\int_{\mathbb{R}^2} \frac{\chi_{E_\delta}(y) - \chi_{E_\delta^c}(y)}{|y-p|^{2+2s}}\, dy \geqslant 0 \tag{7.9}$$

for any $p \in \overline{(\partial E_\delta) \cap \{|x_1| < 1\}}$.

8. Instability of the flat fractional minimal surfaces

With the barrier constructed in Proposition 7.3 we are now in the position of proving Theorem 1.4. For this, we will take E and F as in the statement of Theorem 1.4.

Proof of Theorem 1.4. Let E_δ be as in Proposition 7.3. The idea is to slide E_δ (or, more precisely, $E_{\frac{\delta}{C}}$) from below. Namely, for any $t \geqslant 0$ we consider the set $E(t) := E_{\frac{\delta}{C}} - te_2$. For large t, we have that $E(t) \subseteq E$. So we take the smallest $t \geqslant 0$ for which such inclusion holds. We observe that Theorem 1.4 would be proved if we show that such t equals to 0.

Then suppose, by contradiction, that

$$t > 0. \tag{8.1}$$

By construction,

$$E(t) \subseteq E \tag{8.2}$$

and there exists a contact point between the two sets. From the data outside $[-1, 1] \times \mathbb{R}$, we have that all the contact points must lie in $[-1, 1] \times \mathbb{R}$.

Furthermore,

$$\text{no contact point can occur in } (-1,1) \times \mathbb{R}. \tag{8.3}$$

To check this, suppose that there exists $p = (p_1, p_2) \in (\partial E(t)) \cap (\partial E)$ with $|p_1| < 1$. Then, using the Euler-Lagrange equation in the viscosity sense for E (see Theorem 5.1 in [6]) and (7.8) we have that

$$\int_{\mathbb{R}^2} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y-p|^{2+2s}}\, dy \leqslant 0 \leqslant \int_{\mathbb{R}^2} \frac{\chi_{E(t)}(y) - \chi_{E^c(t)}(y)}{|y-p|^{2+2s}}\, dy.$$

Also, the opposite inequality holds, thanks to (8.2), and therefore $E(t)$ and E must coincide. This would give that $t = 0$, against our assumption. This proves (8.3).

As a consequence, we have that all the contact points lie on $\{\pm 1\} \times \mathbb{R}$. Since both $\partial E(t)$ and ∂E are closed set, we can take the contact point with lower vertical coordinate along $\{x_1 = \pm 1\}$, and we denote it by $x_o^\pm = (\pm 1, x_{o,2}^\pm)$.

Now, for any $k \in \mathbb{N}$ (to be taken as large as we wish) and any $h \in [0, 1/k]$ we consider the ball of small radius $r > 0$ (smaller than the radius of curvature of $E(t)$) centered on the line $\{x_2 = x_{o,2}^\pm + h\}$ and we slide such ball to the left (towards $\{x_1 = -1\}$) or to the right (towards $\{x_1 = 1\}$) till it touches either $\partial E \cap \{|x_1| < 1\}$ or $\{x_1 = \pm 1\}$, see Figure 18.

We claim that there exists a sequence $k \to +\infty$ for which there exists $h_k \in [0, 1/k]$ such that the sliding of this ball (either to the right or to the left) touches $\partial E \cap \{|x_1| < 1\}$. Indeed, if not, we have that ∂E, near $\{x_1 = \pm 1\}$, stays above $\{x_2 = x_{o,2}^\pm + \alpha\}$, for some $\alpha > 0$. But this would imply that we can keep sliding $E(t)$ a little more upwards, in contradiction with the minimality of t.

Therefore, we can assume that, for a suitable sequence $k \to +\infty$, we have that there exist points $x_k = (x_{k,1}, x_{k,2}) \in (\partial E) \cap \{|x_1| < 1\}$ with $x_{k,2} = x_{o,2}^\pm + h_k$ and $h_k \in [0, 1/k]$. By construction, the points x_k must lie outside $E(t)$, hence, if r is small enough, we have that $|x_{k,1}| \to 1$ as $k \to +\infty$.

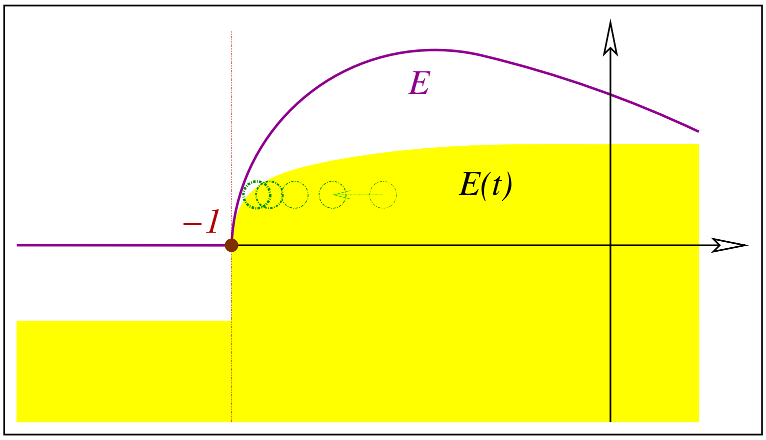

FIGURE 18. Sliding the balls from the barriers towards $\partial E \cap \{|x_1| < 1\}$.

Hence, we assume that $x_k \in (\partial E) \cap \{|x_1| < 1\}$ and $x_k \to x_o := x_o^-$ as $k \to +\infty$ (the case in which $x_k \to x_o^+$ is completely analogous). Then, by the Euler-Lagrange equation at the points x_k (see Lemma 3.4 in [13]), we obtain that

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_o - y|^{n+2s}}\, dy \leqslant 0. \tag{8.4}$$

On the other hand, by (7.9),

$$\int_{\mathbb{R}^2} \frac{\chi_{E(t)}(y) - \chi_{E^c(t)}(y)}{|x_o - y|^{2+2s}}\, dy \geqslant 0. \tag{8.5}$$

Combining (8.2), (8.4) and (8.5), it follows that $E(t) = E$. Thus, from the values of E_δ and E outside $\{|x_1| \leqslant 1\}$, we conclude that $t = 0$. This is in contradiction with (8.1) and so the desired result is proved. ☐

APPENDIX A. SYMMETRY PROPERTIES AND A VARIATION ON THE PROOF OF LEMMA 6.1

Here we prove that the minimizers inherit the symmetry properties of the boundary data:

Lemma A.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, with $T(\Omega) = \Omega$. Assume that there exists $N \in \mathbb{N}$ such that $T^N(x) = x$ for every $x \in \Omega$.*

Let $E \subseteq \mathbb{R}^n$ be such that $T(E) = E$. Let E_* be the s-minimal set in a domain Ω among all the sets F such that $F \setminus \Omega = E \setminus \Omega$. Then $T(E_*) = E_*$.

Proof. We let

$$\mathscr{F}(u) := \frac{1}{2} \iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy.$$

We observe that

$$\mathscr{F}(\chi_E) = \operatorname{Per}_s(E, \Omega).$$

Moreover, by Lemma 3 on page 685 in [20], we have that

$$\mathscr{F}(\min\{u, v\}) + \mathscr{F}(\max\{u, v\}) \leqslant \mathscr{F}(u) + \mathscr{F}(v),$$

and the equality holds if and only if either $u(x) \leqslant v(x)$ or $v(x) \leqslant u(x)$ for any $x \in \Omega$.

We use the observations above with $u := \chi_{E_*}$ and $v := \chi_{T(E_*)}$. Notice that, in this case, $\min\{u, v\} = \chi_{E_* \cap T(E_*)}$ and $\max\{u, v\} = \chi_{E_* \cup T(E_*)}$. Hence, we obtain

$$\operatorname{Per}_s\big(E_* \cap T(E_*), \Omega\big) + \operatorname{Per}_s\big(E_* \cup T(E_*), \Omega\big) \leqslant \operatorname{Per}_s(E_*, \Omega) + \operatorname{Per}_s(T(E_*), \Omega), \tag{A.1}$$

and the equality holds if and only if either $\chi_{E_*}(x) \leqslant \chi_{T(E_*)}(x)$ or $\chi_{T(E_*)}(x) \leqslant \chi_{E_*}(x)$ for any $x \in \Omega$, that is, if and only if

$$\text{either } E_* \cap \Omega \subseteq T(E_*) \cap \Omega \text{ or } T(E_*) \cap \Omega \subseteq E_* \cap \Omega. \tag{A.2}$$

Now we observe that

$$\begin{aligned}
\mathrm{Per}_s\big(T(E_*),\Omega\big) &= L\big(T(E_*)\cap\Omega,\mathbb{R}^n\setminus T(E_*)\big)+L\big(\Omega\setminus T(E_*),T(E_*)\setminus\Omega\big)\\
&= L\big(T(E_*)\cap T(\Omega),\mathbb{R}^n\setminus T(E_*)\big)+L\big(T(\Omega)\setminus T(E_*),T(E_*)\setminus T(\Omega)\big)\\
&= L\big(T(E_*\cap\Omega),T(\mathbb{R}^n\setminus E_*)\big)+L\big(T(\Omega\setminus E_*),T(E_*\setminus\Omega)\big)\\
&= L(E_*\cap\Omega,\mathbb{R}^n\setminus E_*)+L(\Omega\setminus E_*,E_*\setminus\Omega)\\
&= \mathrm{Per}_s(E_*,\Omega).
\end{aligned}$$

Substituting this in (A.1), we obtain that

$$\mathrm{Per}_s\big(E_*\cap T(E_*),\Omega\big)+\,\mathrm{Per}_s\big(E_*\cup T(E_*),\Omega\big)\leqslant 2\,\mathrm{Per}_s(E_*,\Omega). \tag{A.3}$$

On the other hand,

$$T(E_*)\setminus\Omega=T(E_*)\setminus T(\Omega)=T(E_*\setminus\Omega)=T(E\setminus\Omega)=T(E)\setminus\Omega=E\setminus\Omega. \tag{A.4}$$

This says that $E_*\cap T(E_*)$ and $E_*\cup T(E_*)$ are admissible competitors for E_* and therefore

$$\mathrm{Per}_s(E_*,\Omega)\leqslant\,\mathrm{Per}_s\big(E_*\cap T(E_*),\Omega\big)\quad\text{and}\quad\mathrm{Per}_s(E_*,\Omega)\leqslant\,\mathrm{Per}_s\big(E_*\cup T(E_*),\Omega\big).$$

This implies that the equality holds in (A.3), and so in (A.1).

Therefore, (A.2) holds true. So we suppose that $E_*\cap\Omega\subseteq T(E_*)\cap\Omega$ (the case in which $T(E_*)\cap\Omega\subseteq E_*\cap\Omega$ can be dealt with in a similar way). Then we have that $E_*\cap\Omega\subseteq T(E_*\cap\Omega)$. By applying T, we obtain $T(E_*\cap\Omega)\subseteq T^2(E_*\cap\Omega)$, and so, iterating the procedure

$$E_*\cap\Omega\subseteq T(E_*\cap\Omega)\subseteq\cdots\subseteq T^{N-1}(E_*\cap\Omega)\subseteq T^N(E_*\cap\Omega)=E_*\cap\Omega.$$

This shows that $E_*\cap\Omega=T(E_*\cap\Omega)$, that is $E_*\cap\Omega=T(E_*)\cap\Omega$.

Also, by (A.4), $E_*\setminus\Omega=T(E_*)\setminus\Omega$. Therefore $E_*=T(E_*)$, as desired. □

Now we give a different (and more general) proof of Lemma 6.1, according to the following result:

Lemma A.2. *Let σ, $\sigma_0\in(0,1)$, with $\sigma<2\sigma_0$. Then, for any $t>0$, we have*

$$(-\Delta)^{\sigma_0}t_+^{\sigma}=-4\,\Gamma(1+\sigma)\,\Gamma(2\sigma_0-\sigma)\,\sin\big(\pi(\sigma-\sigma_0)\big)t^{\sigma-2\sigma_0}, \tag{A.5}$$

where Γ is the gamma function.

In particular,

- *if $\sigma=\sigma_0$, then, for any $t>0$,*
$$(-\Delta)^{\sigma}t_+^{\sigma}=0,$$
- *if $\sigma>\sigma_0$, then for any $t>0$,*
$$(-\Delta)^{\sigma_0}t_+^{\sigma}<0,$$
- *if $\sigma<\sigma_0$, then for any $t>0$,*
$$(-\Delta)^{\sigma_0}t_+^{\sigma}>0.$$

Proof. The proof is a modification of an argument given in [4]. In order to prove Lemma A.2, we will use the Fourier transform of $|t|^q$ in the sense of distribution, where $q\in\mathbb{C}\setminus\mathbb{Z}$. Namely (see e.g. Lemma 2.23 on page 38 of [17])

$$\mathscr{F}(|t|^q)=C_q\,|\xi|^{-1-q}, \tag{A.6}$$

with

$$C_q:=-2\Gamma(1+q)\,\sin\frac{\pi q}{2}. \tag{A.7}$$

Notice that the map $\mathbb{R}\ni t\mapsto|t|^q$ is even, and so we can rewrite (A.6) as

$$\mathscr{F}^{-1}(|\xi|^q)=(2\pi)^{-1}C_q\,|t|^{-1-q}. \tag{A.8}$$

Moreover,

$$|t|^{\sigma}+\frac{1}{\sigma+1}\partial_t|t|^{\sigma+1}=2t_+^{\sigma}.$$

Therefore, taking the Fourier transform and using (A.6) with $q:=\sigma$ and $q:=\sigma+1$, we obtain that

$$\begin{aligned} 2\mathscr{F}(t_+^\sigma) &= \mathscr{F}(|t|^\sigma)+\frac{1}{\sigma+1}\mathscr{F}\big(\partial_t|t|^{\sigma+1}\big)\\ &= \mathscr{F}(|t|^\sigma)+\frac{2i\xi}{\sigma+1}\mathscr{F}(|t|^{\sigma+1})\\ &= C_\sigma\,|\xi|^{-1-\sigma}+\frac{2i\xi}{\sigma+1}C_{\sigma+1}\,|\xi|^{-2-\sigma}. \end{aligned}$$

So, multiplying the equality above by $|\xi|^{2\sigma_0}$, we obtain that

$$2|\xi|^{2\sigma_0}\mathscr{F}(t_+^\sigma)=C_\sigma\,|\xi|^{2\sigma_0-\sigma-1}+\frac{2i\xi}{\sigma+1}C_{\sigma+1}\,|\xi|^{2\sigma_0-\sigma-2},$$

and so

$$2\mathscr{F}^{-1}\Big(|\xi|^{2\sigma_0}\mathscr{F}(t_+^\sigma)\Big)=C_\sigma\,\mathscr{F}^{-1}(|\xi|^{2\sigma_0-\sigma-1})+\frac{2C_{\sigma+1}\,i}{\sigma+1}\mathscr{F}^{-1}(\xi)*\mathscr{F}^{-1}(|\xi|^{2\sigma_0-\sigma-2}) \tag{A.9}$$

Now we claim that, for any test function g,

$$\big(\mathscr{F}^{-1}(\xi)*g\big)\,(t)=-i\partial_t g(t). \tag{A.10}$$

Indeed,

$$\begin{aligned} &\big(\mathscr{F}^{-1}(\xi)*g\big)\,(t)=\mathscr{F}^{-1}\left(\xi\mathscr{F}g(\xi)\right)(t)\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}dy\int_{\mathbb{R}}d\xi\,e^{iy\cdot(t-\xi)}\,y\,g(\xi)=-\frac{1}{2\pi i}\int_{\mathbb{R}}dy\int_{\mathbb{R}}d\xi\,\partial_\xi e^{iy\cdot(t-\xi)}\,g(\xi)\\ &=\frac{1}{2\pi i}\int_{\mathbb{R}}dy\int_{\mathbb{R}}d\xi\,e^{iy\cdot(t-\xi)}\,\partial_\xi g(\xi)=\frac{1}{2\pi i}\int_{\mathbb{R}}dy\,e^{iy\cdot t}\,\mathscr{F}\big(\partial_\xi g\big)(y)\\ &=\frac{1}{i}\mathscr{F}^{-1}\left(\mathscr{F}(\partial_\xi g)\right)(t)=-i\,\partial_\xi g(t), \end{aligned}$$

which shows (A.10).

Using (A.10) into (A.9), we obtain that

$$\begin{aligned} 2\mathscr{F}^{-1}\Big(|\xi|^{2\sigma_0}\mathscr{F}(t_+^\sigma)\Big) &= C_\sigma\,\mathscr{F}^{-1}(|\xi|^{2\sigma_0-\sigma-1})-\frac{C_{\sigma+1}\,i}{\sigma+1}\cdot i\,\partial_t\mathscr{F}^{-1}(|\xi|^{2\sigma_0-\sigma-2})\\ &= C_\sigma\,\mathscr{F}^{-1}(|\xi|^{2\sigma_0-\sigma-1})+\frac{C_{\sigma+1}}{\sigma+1}\partial_t\mathscr{F}^{-1}(|\xi|^{2\sigma_0-\sigma-2}). \end{aligned}$$

As a consequence, exploiting (A.8) with $q:=2\sigma_0-\sigma-1$ and $q:=2\sigma_0-\sigma-2$, we have that

$$\begin{aligned} 2\mathscr{F}^{-1}\Big(|\xi|^{2\sigma_0}\mathscr{F}(t_+^\sigma)\Big) &= C_\sigma\,C_{2\sigma_0-\sigma-1}\,|t|^{\sigma-2\sigma_0}+\frac{C_{\sigma+1}\,C_{2\sigma_0-\sigma-2}}{\sigma+1}\partial_t|t|^{\sigma-2\sigma_0+1}\\ &= C_\sigma\,C_{2\sigma_0-\sigma-1}\,|t|^{\sigma-2\sigma_0}+\frac{\sigma-2\sigma_0+1}{\sigma+1}\cdot C_{\sigma+1}\,C_{2\sigma_0-\sigma-2}\,t\,|t|^{\sigma-2\sigma_0-1}. \end{aligned}$$

This gives that, for $t>0$,

$$2\mathscr{F}^{-1}\Big(|\xi|^{2\sigma_0}\mathscr{F}(t_+^\sigma)\Big)=\left(C_\sigma\,C_{2\sigma_0-\sigma-1}+\frac{\sigma-2\sigma_0+1}{\sigma+1}\cdot C_{\sigma+1}\,C_{2\sigma_0-\sigma-2}\right)t^{\sigma-2\sigma_0}.$$

So we obtain that, up to a dimensional constant, for any $t>0$,

$$(-\Delta)^{\sigma_0}(t_+^\sigma)=\left(C_\sigma\,C_{2\sigma_0-\sigma-1}+\frac{\sigma-2\sigma_0+1}{\sigma+1}\cdot C_{\sigma+1}\,C_{2\sigma_0-\sigma-2}\right)t^{\sigma-2\sigma_0}. \tag{A.11}$$

Now, we observe that

$$C_\sigma\,C_{2\sigma_0-\sigma-1}=4\,\Gamma(1+\sigma)\,\Gamma(2\sigma_0-\sigma)\,\sin\left(\frac{\pi}{2}\sigma\right)\,\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right). \tag{A.12}$$

Moreover,

$$\Gamma(2+\sigma)=(1+\sigma)\Gamma(1+\sigma)\;\text{ and }\;\Gamma(2\sigma_0-\sigma)=(2\sigma_0-\sigma-1)\Gamma(2\sigma_0-\sigma-1).$$

As a consequence, recalling (A.7) and (A.12),

$$\begin{aligned}
&\frac{\sigma-2\sigma_0+1}{\sigma+1}\cdot C_{\sigma+1}\,C_{2\sigma_0-\sigma-2}\\
=\;&\frac{\sigma-2\sigma_0+1}{\sigma+1}\,4\,\Gamma(2+\sigma)\,\Gamma(2\sigma_0-\sigma-1)\,\sin\Big(\frac{\pi}{2}(\sigma+1)\Big)\,\sin\Big(\frac{\pi}{2}(2\sigma_0-\sigma-2)\Big)\\
=\;&-4\,\Gamma(1+\sigma)\,\Gamma(2\sigma_0-\sigma)\,\sin\Big(\frac{\pi}{2}(\sigma+1)\Big)\,\sin\Big(\frac{\pi}{2}(2\sigma_0-\sigma-2)\Big)\\
=\;&-C_\sigma\,C_{2\sigma_0-\sigma-1}\cdot\frac{\sin\left(\frac{\pi}{2}(\sigma+1)\right)}{\sin\left(\frac{\pi}{2}\sigma\right)}\cdot\frac{\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-2)\right)}{\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}.
\end{aligned}$$

Plugging this into (A.11), we get

$$(-\Delta)^{\sigma_0}(t_+^\sigma)=C_\sigma\,C_{2\sigma_0-\sigma-1}\left(1-\frac{\sin\left(\frac{\pi}{2}(\sigma+1)\right)}{\sin\left(\frac{\pi}{2}\sigma\right)}\cdot\frac{\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-2)\right)}{\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}\right)t^{\sigma-2\sigma_0}.$$

Now, by elementary trigonometry, we see that

$$\sin\Big(\frac{\pi}{2}(\sigma+1)\Big)=\cos\Big(\frac{\pi}{2}\sigma\Big)\quad\text{and}\quad\sin\Big(\frac{\pi}{2}(2\sigma_0-\sigma-2)\Big)=-\cos\Big(\frac{\pi}{2}(2\sigma_0-\sigma-1)\Big).$$

Therefore,

$$\begin{aligned}
&1-\frac{\sin\left(\frac{\pi}{2}(\sigma+1)\right)}{\sin\left(\frac{\pi}{2}\sigma\right)}\cdot\frac{\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-2)\right)}{\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}\\
=\;&1+\frac{\cos\left(\frac{\pi}{2}\sigma\right)}{\sin\left(\frac{\pi}{2}\sigma\right)}\cdot\frac{\cos\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}{\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}\\
=\;&\frac{\cos\left(\frac{\pi}{2}\sigma\right)}{\sin\left(\frac{\pi}{2}\sigma\right)}\left[\frac{\sin\left(\frac{\pi}{2}\sigma\right)}{\cos\left(\frac{\pi}{2}\sigma\right)}+\frac{\cos\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}{\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}\right]\\
=\;&\frac{\cos\left(\frac{\pi}{2}\sigma\right)}{\sin\left(\frac{\pi}{2}\sigma\right)}\cdot\frac{\sin\left(\frac{\pi}{2}\sigma\right)\,\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)+\cos\left(\frac{\pi}{2}\sigma\right)\,\cos\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}{\cos\left(\frac{\pi}{2}\sigma\right)\,\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}\\
=\;&\frac{\cos\left(\frac{\pi}{2}\sigma\right)}{\sin\left(\frac{\pi}{2}\sigma\right)}\cdot\frac{\cos\left(\pi(\sigma-\sigma_0)+\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\sigma\right)\,\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}\\
=\;&-\frac{\cos\left(\frac{\pi}{2}\sigma\right)}{\sin\left(\frac{\pi}{2}\sigma\right)}\cdot\frac{\sin\left(\pi(\sigma-\sigma_0)\right)}{\cos\left(\frac{\pi}{2}\sigma\right)\,\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}\\
=\;&-\frac{\sin\left(\pi(\sigma-\sigma_0)\right)}{\sin\left(\frac{\pi}{2}\sigma\right)\,\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}.
\end{aligned}$$

Accordingly, up to a dimensional constant,

$$(-\Delta)^{\sigma_0}(t_+^\sigma)=-C_\sigma\,C_{2\sigma_0-\sigma-1}\frac{\sin\left(\pi(\sigma-\sigma_0)\right)}{\sin\left(\frac{\pi}{2}\sigma\right)\,\sin\left(\frac{\pi}{2}(2\sigma_0-\sigma-1)\right)}t^{\sigma-2\sigma_0}.$$

So, recalling (A.12), we obtain that, for any $t>0$,

$$(-\Delta)^{\sigma_0}(t_+^\sigma)=-4\,\Gamma(1+\sigma)\,\Gamma(2\sigma_0-\sigma)\,\sin\left(\pi(\sigma-\sigma_0)\right),$$

which shows (A.5).

We finish the proof of Lemma A.2 by noticing that

- if $\sigma=\sigma_0$, then $\sin\left(\pi(\sigma-\sigma_0)\right)=0$,
- if $\sigma>\sigma_0$, then $\sin\left(\pi(\sigma-\sigma_0)\right)>0$,
- if $\sigma<\sigma_0$, then $\sin\left(\pi(\sigma-\sigma_0)\right)<0$.

This implies the desired result. ∎

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