

**Weierstraß-Institut**  
**für Angewandte Analysis und Stochastik**  
**Leibniz-Institut im Forschungsverbund Berlin e. V.**

Preprint

ISSN 2198-5855

**A note on the extremal process**  
**of the supercritical Gaussian Free Field**

Alberto Chiarini<sup>1</sup>, Alessandra Cipriani<sup>2</sup>, Rajat Subhra Hazra<sup>3</sup>

submitted: May 26, 2015

<sup>1</sup> Technische Universität Berlin  
MA 766  
Straße des 17. Juni 136  
10623 Berlin  
Germany  
E-Mail: [chiarini@math.tu-berlin.de](mailto:chiarini@math.tu-berlin.de)

<sup>2</sup> Weierstrass Institute  
Mohrenstraße 39  
10117 Berlin  
Germany  
E-Mail: [Alessandra.Cipriani@wias-berlin.de](mailto:Alessandra.Cipriani@wias-berlin.de)

<sup>3</sup> Theoretical Statistics and Mathematics Unit  
Indian Statistical Institute  
203 B.T. Road  
Kolkata, 700108  
India  
E-Mail: [rajatmaths@gmail.com](mailto:rajatmaths@gmail.com)

No. 2111  
Berlin 2015



---

2010 *Mathematics Subject Classification.* 60G15, 60G30, 60G55, 60G57, 60G70, 82B41.

*Key words and phrases.* Gaussian Free Field, Poisson approximation, point processes, extremal process, Stein-Chen method.

The first author's research is supported by RTG 1845.

Edited by  
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)  
Leibniz-Institut im Forschungsverbund Berlin e. V.  
Mohrenstraße 39  
10117 Berlin  
Germany

Fax: +49 30 20372-303  
E-Mail: [preprint@wias-berlin.de](mailto:preprint@wias-berlin.de)  
World Wide Web: <http://www.wias-berlin.de/>

ABSTRACT. We consider both the infinite-volume discrete Gaussian Free Field (DGFF) and the DGFF with zero boundary conditions outside a finite box in dimension larger or equal to 3. We show that the associated extremal process converges to a Poisson point process. The result follows from an application of the Stein-Chen method from Arratia et al. (1989).

## 1. INTRODUCTION

In this article we study the behavior of the extremal process of the DGFF in dimension larger or equal to 3. This extends the result presented in Chiarini et al. (2015) in which the convergence of the rescaled maximum of the infinite-volume DGFF and the 0-boundary condition field was shown. It was proved there that the field belongs to the maximal domain of attraction of the Gumbel distribution; hence, a natural question that arises is that of describing more precisely its extremal points. In dimension 2, this was carried out by Biskup and Louidor (2013, 2014) complementing a result of Bramson et al. (2013) on the convergence of the maximum; namely, the characterization of the limiting point process with a random mean measure yields as by-product an integral representation of the maximum. The extremes of the DGFF in dimension 2 have deep connections with those of Branching Brownian Motion (Aïdékon et al. (2013), Arguin et al. (2011, 2012, 2013)). These works showed that the limiting point process is a randomly shifted decorated Poisson point process, and we refer to Subag and Zeitouni (2015) for structural details. In  $d \geq 3$ , one does not get a non-trivial decoration but instead a Poisson point process analogous to the extremal process of independent Gaussian random variables. To be more precise, we let  $E := [0, 1]^d \times (-\infty, +\infty]$  and  $V_N := [0, n-1]^d \cap \mathbb{Z}^d$  the hypercube of volume  $N = n^d$ . Let  $(\varphi_\alpha)_{\alpha \in \mathbb{Z}^d}$  be the infinite-volume DGFF, that is a centered Gaussian field on the square lattice with covariance  $g(\cdot, \cdot)$ , where  $g$  is the Green's function of the simple random walk. We define the following sequence of point processes on  $E$ :

$$\eta_n(\cdot) := \sum_{\alpha \in V_N} \varepsilon_{\left(\frac{\alpha}{n}, \frac{\varphi_\alpha - b_N}{a_N}\right)}(\cdot) \quad (1)$$

where  $\varepsilon_x(\cdot)$ ,  $x \in E$ , is the point measure that gives mass one to a set containing  $x$  and zero otherwise, and

$$b_N := \sqrt{g(0)} \left[ \sqrt{2 \log N} - \frac{\log \log N + \log(4\pi)}{2\sqrt{2 \log N}} \right], \quad a_N := g(0)(b_N)^{-1}. \quad (2)$$

Here  $g(0)$  denotes the variance of the DGFF. Our main result is

**Theorem 1.** *For the sequence of point processes  $\eta_n$  defined in (1) we have that*

$$\eta_n \xrightarrow{d} \eta,$$

as  $n \rightarrow +\infty$ , where  $\eta$  is a Poisson random measure on  $E$  with intensity measure given by  $d t \otimes (e^{-z} dz)$  where  $d t \otimes dz$  is the Lebesgue measure on  $E$ , and  $\xrightarrow{d}$  is the convergence in distribution on  $\mathcal{M}_p(E)^a$ .

The proof is based on the application of the two-moment method of Arratia et al. (1989) that allows us to compare the extremal process of the DGFF and a Poisson point process with the same mean measure. To prove that the two processes converge, we will exploit a classical theorem by Kallenberg.

<sup>a</sup> $\mathcal{M}_p(E)$  denotes the set of (Radon) point measures on  $E$  endowed with the topology of vague convergence.

It is natural then to consider also convergence for the DGFF  $(\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$  with zero boundary conditions outside  $V_N$ . For the sequences of point measures

$$\rho_n(\cdot) := \sum_{\alpha \in V_N} \varepsilon_{\left(\frac{\alpha}{n}, \frac{\psi_\alpha - b_N}{a_N}\right)}(\cdot) \quad (3)$$

we establish the following Theorem:

**Theorem 2.** *For the sequence of point processes  $\rho_n$  defined in (3) we have that*

$$\rho_n \xrightarrow{d} \eta,$$

as  $n \rightarrow +\infty$  in  $\mathcal{M}_p(E)$ , where  $\eta$  is as in Theorem 1.

The convergence is shown by reducing ourselves to check the conditions of Kallenberg's Theorem on the bulk of  $V_N$ , where we have a good control on the drift of the conditioned field, and then by showing that the process on the whole of  $V_N$  and on the bulk are close as  $n$  becomes large.

The outline of the paper is as follows. In Section 2 we will recall the definition of DGFF and the Stein-Chen method, while Section 3 and Section 4 are devoted to the proofs of Theorems 1 and 2 respectively.

## 2. PRELIMINARIES

**2.1. The DGFF.** Let  $d \geq 3$  and denote with  $\|\cdot\|$  the  $\ell_\infty$ -norm on  $\mathbb{Z}^d$ . Let  $\psi = (\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$  be a discrete Gaussian Free Field with zero boundary conditions outside  $\Lambda \subset \mathbb{Z}^d$ . On the space  $\Omega := \mathbb{R}^{\mathbb{Z}^d}$  endowed with its product topology, its law  $\tilde{\mathbb{P}}_\Lambda$  can be explicitly written as

$$\tilde{\mathbb{P}}_\Lambda(d\psi) = \frac{1}{Z_\Lambda} \exp\left(-\frac{1}{2d} \sum_{\alpha, \beta \in \mathbb{Z}^d: \|\alpha - \beta\|=1} (\psi_\alpha - \psi_\beta)^2\right) \prod_{\alpha \in \Lambda} d\psi_\alpha \prod_{\alpha \in \mathbb{Z}^d \setminus \Lambda} \varepsilon_0(\psi_\alpha).$$

In other words  $\psi_\alpha = 0$   $\tilde{\mathbb{P}}_\Lambda$ -a. s. if  $\alpha \in \mathbb{Z}^d \setminus \Lambda$ , and  $(\psi_\alpha)_{\alpha \in \Lambda}$  is a multivariate Gaussian random variable with mean zero and covariance  $(g_\Lambda(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}^d}$ , where  $g_\Lambda$  is the Green's function of the discrete Laplacian problem with Dirichlet boundary conditions outside  $\Lambda$ . For a thorough review on the model the reader can refer for example to Sznitman (2012). It is known (Georgii, 1988, Chapter 13) that the finite-volume measure  $\psi$  admits an infinite-volume limit as  $\Lambda \uparrow \mathbb{Z}^d$  in the weak topology of probability measures. This field will be denoted as  $\varphi = (\varphi_\alpha)_{\alpha \in \mathbb{Z}^d}$ . It is a centered Gaussian field with covariance matrix  $g(\alpha, \beta)$  for  $\alpha, \beta \in \mathbb{Z}^d$ . With a slight abuse of notation, we write  $g(\alpha - \beta)$  for  $g(0, \alpha - \beta)$  and also  $g_\Lambda(\alpha) = g_\Lambda(\alpha, \alpha)$ .  $g$  admits a so-called random walk representation: if  $\mathbb{P}_\alpha$  denotes the law of a simple random walk  $S$  started at  $\alpha \in \mathbb{Z}^d$ , then

$$g(\alpha, \beta) = \mathbb{E}_\alpha \left[ \sum_{n \geq 0} \mathbb{1}_{\{S_n = \beta\}} \right].$$

In particular this gives  $g(0) < +\infty$  for  $d \geq 3$ . A comparison of the covariances in the infinite and finite-volume is possible in the *bulk* of  $V_N$ : for  $\delta > 0$  this is defined as

$$V_N^\delta := \left\{ \alpha \in V_N : \|\alpha - \beta\| > \delta n, \forall \beta \in \mathbb{Z}^d \setminus V_N \right\}. \quad (4)$$

In order to compare covariances in the finite and infinite-volume field, we recall the following Lemma, whose proof is presented in Chiarini et al. (2015, Lemma 7)).

**Lemma 3.** For any  $\delta > 0$  and  $\alpha, \beta \in V_N^\delta$  one has

$$g(\alpha, \beta) - C_d \left( \delta N^{1/d} \right)^{2-d} \leq g_{V_N}(\alpha, \beta) \leq g(\alpha, \beta). \quad (5)$$

In particular we have,  $g_{V_N}(\alpha) = g(0) \left( 1 + O \left( N^{(2-d)/d} \right) \right)$  uniformly for  $\alpha \in V_N^\delta$ .

**2.2. The Stein-Chen method.** As main tool of this article we will use (and restate here) a theorem from Arratia et al. (1989). Consider a sequence of Bernoulli random variables  $(X_\alpha)_{\alpha \in \mathcal{I}}$  where  $X_\alpha \sim Be(p_\alpha)$  and  $\mathcal{I}$  is some index set. For each  $\alpha$  we define a subset  $B_\alpha \subseteq \mathcal{I}$  which we consider a “neighborhood” of dependence for the variable  $X_\alpha$ , such that  $X_\alpha$  is nearly independent from  $X_\beta$  if  $\beta \in \mathcal{I} \setminus B_\alpha$ . Set

$$\begin{aligned} b_1 &:= \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in B_\alpha} p_\alpha p_\beta, \\ b_2 &:= \sum_{\alpha \in \mathcal{I}} \sum_{\alpha \neq \beta \in B_\alpha} \mathbb{E} [X_\alpha X_\beta], \\ b_3 &:= \sum_{\alpha \in \mathcal{I}} \mathbb{E} [|\mathbb{E} [X_\alpha - p_\alpha | \mathcal{H}_1]|] \end{aligned}$$

where

$$\mathcal{H}_1 := \sigma (X_\beta : \beta \in \mathcal{I} \setminus B_\alpha).$$

**Theorem 4** (Arratia et al. (1989, Theorem 2)). Let  $\mathcal{I}$  be an index set. Partition the index set  $\mathcal{I}$  into disjoint non-empty sets  $\mathcal{I}_1, \dots, \mathcal{I}_k$ . For any  $\alpha \in \mathcal{I}$ , let  $(X_\alpha)_{\alpha \in \mathcal{I}}$  be a dependent Bernoulli process with parameter  $p_\alpha$ . Let  $(Y_\alpha)_{\alpha \in \mathcal{I}}$  be independent Poisson random variables with intensity  $p_\alpha$ . Also let

$$W_j := \sum_{\alpha \in \mathcal{I}_j} X_\alpha \quad \text{and} \quad Z_j := \sum_{\alpha \in \mathcal{I}_j} Y_\alpha \quad \text{and} \quad \lambda_j := \mathbb{E}[W_j] = \mathbb{E}[Z_j].$$

Then

$$\|\mathcal{L}(W_1, \dots, W_k) - \mathcal{L}(Z_1, \dots, Z_k)\|_{TV} \leq 2 \min \left\{ 1, 1.4 (\min \lambda_j)^{-1/2} \right\} (2b_1 + 2b_2 + b_3) \quad (6)$$

where  $\|\cdot\|_{TV}$  denotes the total variation distance and  $\mathcal{L}(W_1, \dots, W_k)$  denotes the joint law of these random variables.

### 3. PROOF OF THEOREM 1: THE INFINITE-VOLUME CASE

*Proof.* We recall that  $E = [0, 1]^d \times (-\infty, +\infty]$  and  $V_N = [0, n-1]^d \cap \mathbb{Z}^d$ . To show the convergence of  $\eta_n$  to  $\eta$ , we will exploit Kallenberg’s theorem (Kallenberg, 1983, Theorem 4.7). According to it, we need to verify the following conditions:

i) for any  $A$ , a bounded rectangle<sup>b</sup> in  $[0, 1]^d$ , and  $R = (x, y] \subset (-\infty, +\infty]$

$$\mathbb{E}[\eta_n(A \times (x, y])] \rightarrow \mathbb{E}[\eta(A \times (x, y])] = |A|(e^{-x} - e^{-y}).$$

We adopt the convention  $e^{-\infty} = 0$  and the notation  $|A|$  for the Lebesgue measure of  $A$ .

<sup>b</sup>A bounded rectangle has the form  $J_1 \times \dots \times J_d$  with  $J_i = [0, 1] \cap (a_i, b_i]$ ,  $a_i, b_i \in \mathbb{R}$  for all  $1 \leq i \leq d$ .

ii) For all  $k \geq 1$ , and  $A_1, A_2, \dots, A_k$  disjoint rectangles in  $[0, 1]^d$  and  $R_1, R_2, \dots, R_k$ , each of which is a finite union of disjoint intervals of the type  $(x, y] \subset (-\infty, +\infty]$ ,

$$\begin{aligned} & \mathbb{P}(\eta_n(A_1 \times R_1) = 0, \dots, \eta_n(A_k \times R_k) = 0) \\ & \rightarrow \mathbb{P}(\eta(A_1 \times R_1) = 0, \dots, \eta(A_k \times R_k) = 0) = \exp\left(-\sum_{j=1}^k |A_j| \omega(R_j)\right) \end{aligned} \quad (7)$$

where  $\omega(dz) := e^{-z} dz$ .

Let us denote by  $u_N(z) := a_N z + b_N$ . The first condition follows by Mills ratio

$$\left(1 - \frac{1}{t^2}\right) \frac{e^{-t^2/2}}{\sqrt{2\pi t}} \leq \mathbb{P}(\mathcal{N}(0, 1) > t) \leq \frac{e^{-t^2/2}}{\sqrt{2\pi t}}, \quad t > 0. \quad (8)$$

More precisely

$$\begin{aligned} \mathbb{E}[\eta_n(A \times (x, y))] &= \sum_{\alpha \in nA \cap V_N} \mathbb{P}(\varphi_\alpha \in (u_N(x), u_N(y))) \\ &\leq \sum_{\alpha \in nA \cap V_N} \left( \frac{e^{-\frac{u_N(x)^2}{2g(0)}}}{\sqrt{2\pi} u_N(x)} - \frac{e^{-\frac{u_N(y)^2}{2g(0)}}}{\sqrt{2\pi} u_N(y)} \left(1 - \frac{1}{u_N(y)^2}\right) \right) \\ &\leq |nA \cap V_N| \left( \frac{e^{-x+o(1)}}{N} - \frac{e^{-y+o(1)}}{N} \left(1 - \frac{1}{2g(0) \log N(1+o(1))}\right) \right) \\ &\rightarrow |A|(e^{-x} - e^{-y}). \end{aligned} \quad (9)$$

Similarly, one can plug in (9) the reverse bounds of (8) to prove the lower bound, and thus condition i).

To show ii), we need a few more details. Let  $k \geq 1$ ,  $A_1, \dots, A_k$  and  $R_1, \dots, R_k$  be as in the assumptions. Let us denote by  $\mathcal{I}_j = nA_j \cap V_N$  and  $\mathcal{I} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_k$ . For  $\alpha \in \mathcal{I}_j$  define

$$X_\alpha := \mathbb{1}_{\left\{\frac{\varphi_\alpha - b_N}{a_N} \in R_j\right\}}$$

and  $p_\alpha := \mathbb{P}\left(\frac{\varphi_\alpha - b_N}{a_N} \in R_j\right)$ . Choose now a small  $\epsilon > 0$  and fix the neighborhood of dependence  $B_\alpha := B\left(\alpha, (\log N)^{2+2\epsilon}\right) \cap \mathcal{I}$  for  $\alpha \in \mathcal{I}$ . Let  $W_j := \sum_{\alpha \in \mathcal{I}_j} X_\alpha$  and  $Z_j$  be as in Theorem 4.

By the simple observation that

$$\mathbb{P}(\eta_n(A_1 \times R_1) = 0, \dots, \eta_n(A_k \times R_k) = 0) = \mathbb{P}(W_1 = 0, \dots, W_k = 0),$$

to prove the convergence (7), we can use Theorem 4 and show that the error bound on the RHS of (6) goes to 0.

First we bound  $b_1$  as follows. By definition of  $R_1, R_2, \dots, R_k$ , there exists  $z \in \mathbb{R}$  such that  $R_j \subset (z, +\infty]$  for  $1 \leq j \leq k$ . Hence for any  $1 \leq j \leq k$ , for any  $\alpha \in \mathcal{I}_j$  we have that

$$p_\alpha = \mathbb{P}\left(\frac{\varphi_\alpha - b_N}{a_N} \in R_j\right) \leq \mathbb{P}(\varphi_\alpha > u_N(z)) \stackrel{(8)}{\leq} \frac{e^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi} u_N(z)} \sqrt{g(0)}.$$

The bound is independent of  $\alpha$  and  $j$ , therefore for some  $C > 0$

$$b_1 \leq CN(\log N)^{d(2+2\epsilon)} e^{-2z} N^{-2} \rightarrow 0. \quad (11)$$

For  $b_2$  note that it was shown in Chiarini et al. (2015) that for  $z \in \mathbb{R}$  and  $\alpha \neq \beta \in V_N$

$$P(\varphi_\alpha > u_N(z), \varphi_\beta > u_N(z)) \leq \frac{(2-\kappa)^{3/2}}{\kappa^{1/2}} N^{-2/(2-\kappa)} \max \left\{ e^{-2z} \mathbb{1}_{\{z \leq 0\}}, e^{-2z/(2-\kappa)} \mathbb{1}_{\{z > 0\}} \right\}. \quad (12)$$

Here we have introduced  $\kappa := \mathbb{P}_0(\tilde{H}_0 = +\infty) \in (0, 1)$  and  $\tilde{H}_0 = \inf \{n \geq 1 : S_n = 0\}$ . Observe that for any  $1 \leq j \leq k$ ,  $\alpha \in \mathcal{I}$  and  $\beta \in B_\alpha$  one has

$$E[X_\alpha X_\beta] \leq P(\varphi_\alpha > u_N(z), \varphi_\beta > u_N(z))$$

so that by (12) we can find some constant  $C' > 0$  such that

$$b_2 \leq C' N^{-\kappa/(2-\kappa)} (\log N)^{d(2+2\epsilon)} \max \left\{ e^{-2z} \mathbb{1}_{\{z \leq 0\}}, e^{-2z/(2-\kappa)} \mathbb{1}_{\{z > 0\}} \right\} \rightarrow 0.$$

Finally we need to handle  $b_3$ . From Section 2.2 we set for  $\alpha \in \mathcal{I}$ ,  $\mathcal{H}_1 := \sigma(X_\beta : \beta \in \mathcal{I} \setminus B_\alpha)$  and we define  $\mathcal{H}_2 := \sigma(\varphi_\beta : \beta \in \mathcal{I} \setminus B_\alpha)$ . We observe that

$$b_3 = \sum_{\alpha \in \mathcal{I}} E[|E[X_\alpha - p_\alpha | \mathcal{H}_1]|] \leq \sum_{\alpha \in \mathcal{I}} E[|E[X_\alpha | \mathcal{H}_2] - p_\alpha|]$$

since  $\mathcal{H}_1 \subseteq \mathcal{H}_2$  and using the tower property of the conditional expectation. Now denote by  $U_\alpha := \mathbb{Z}^d \setminus (\mathcal{I} \setminus B_\alpha)$ . Let us abbreviate  $u_N(R_j) := \{u_N(y) : y \in R_j\}$ . Then for  $\alpha \in I_j$  and  $1 \leq j \leq k$ , by the Markov property of the DGFF (Rodriguez and Sznitman, 2013, Lemma 1.2) we have that

$$E[X_\alpha | \mathcal{H}_2] = \tilde{P}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) \quad \text{P - a. s.}$$

where  $(\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$  is a Gaussian Free Field with zero boundary conditions outside  $U_\alpha$  and

$$\mu_\alpha = \sum_{\beta \in \mathcal{I} \setminus B_\alpha} \mathbb{P}_\alpha \left( H_{\mathcal{I} \setminus B_\alpha} < +\infty, S_{H_{\mathcal{I} \setminus B_\alpha}} = \beta \right) \varphi_\beta.$$

Here  $H_\Lambda := \inf \{n \geq 0 : S_n \in \Lambda\}$ ,  $\Lambda \subset \mathbb{Z}^d$ . Now as in Chiarini et al. (2015) one can show, using the Markov property, that

$$\text{Var}[\mu_\alpha] \leq \sup_{\beta \in \mathcal{I} \setminus B_\alpha} g(\alpha, \beta) \leq \frac{c}{(\log N)^{2(1+\epsilon)(d-2)}}$$

for some  $c > 0$ . Hence we get that there exists a constant  $c' > 0$  (independent of  $\alpha$  and  $j$ ) such that

$$P\left(|\mu_\alpha| > (u_N(z))^{-1-\epsilon}\right) \leq c' \exp\left(-(\log N)^{(2d-5)(1+\epsilon)}\right). \quad (13)$$

Recalling that  $R_j \subset (z, +\infty]$  for all  $1 \leq j \leq k$ , this immediately shows that for  $d \geq 3$

$$\sum_{j=1}^k \sum_{\alpha \in \mathcal{I}_j} E \left[ \left| \tilde{P}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) - p_\alpha \right| \mathbb{1}_{\{|\mu_\alpha| > (u_N(z))^{-1-\epsilon}\}} \right] \rightarrow 0.$$

So to show that  $b_3 \rightarrow 0$  we are left with proving

$$\sum_{j=1}^k \sum_{\alpha \in \mathcal{I}_j} E \left[ \left| \tilde{P}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) - p_\alpha \right| \mathbb{1}_{\{|\mu_\alpha| \leq (u_N(z))^{-1-\epsilon}\}} \right] \rightarrow 0. \quad (14)$$

We now focus on the term inside the summation. For this, first we write  $R_j = \bigcup_{l=1}^m (w_l, r_l]$  with  $-\infty < w_1 < r_1 < w_2 < \dots < r_m \leq +\infty$  for some  $m \geq 1$ . Hence, we can expand the difference in the absolute value of (14) as follows:

$$\begin{aligned}
& \left( p_\alpha - \tilde{\mathbb{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) \right) \\
&= \sum_{l=1}^m \left( \mathbb{P}(\varphi_\alpha \in (u_N(w_l), u_N(r_l)]) - \tilde{\mathbb{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in (u_N(w_l), u_N(r_l))) \right) \\
&= \sum_{l=1}^m \left( \mathbb{P}(\varphi_\alpha > u_N(w_l)) - \tilde{\mathbb{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w_l)) \right) \\
&\quad - \sum_{l=1}^m \left( \mathbb{P}(\varphi_\alpha > u_N(r_l)) - \tilde{\mathbb{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(r_l)) \right) \tag{15}
\end{aligned}$$

(if  $r_l = +\infty$  for some  $l$ , we conventionally set  $\mathbb{P}(\varphi_\alpha > u_N(r_l)) = 0$  and similarly for the other summand). Using the triangular inequality in (14), it turns out that to finish it is enough to show that for an arbitrary  $w \in \mathbb{R}$ ,

$$\sum_{\alpha \in \mathcal{I}} \mathbb{E} \left[ \left| \tilde{\mathbb{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w)) - \mathbb{P}(\varphi_\alpha > u_N(w)) \right| \mathbb{1}_{\{|\mu_\alpha| \leq (u_N(z))^{-1-\epsilon}\}} \right] \rightarrow 0. \tag{16}$$

For this, first we show that on  $\mathcal{Q} := \left\{ \mathbb{P}(\varphi_\alpha > u_N(w)) > \tilde{\mathbb{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w)) \right\}$

$$\sum_{\alpha \in \mathcal{I}} \mathbb{E} \left[ \left( \mathbb{P}(\varphi_\alpha > u_N(w)) - \tilde{\mathbb{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w)) \right) \mathbb{1}_{\{|\mu_\alpha| \leq (u_N(z))^{-1-\epsilon}\}} \mathbb{1}_{\mathcal{Q}} \right] \rightarrow 0. \tag{17}$$

This follows from the same estimates of  $T_{1,2}$  and Claim 6 of Chiarini et al. (2015). Indeed on  $\mathcal{Q} \cap \left\{ |\mu_\alpha| \leq (u_N(z))^{-1-\epsilon} \right\}$

$$\begin{aligned}
& \sum_{\alpha \in \mathcal{I}} \left( \mathbb{P}(\varphi_\alpha > u_N(w)) - \mathbb{P}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w)) \right) \\
& \leq \sum_{\alpha \in \mathcal{I}} \frac{\sqrt{g(0)} e^{-\frac{u_N(w)^2}{2g(0)}}}{\sqrt{2\pi} u_N(w)} \left( 1 - (1 + o(1)) \left( \frac{\sqrt{g_{U_\alpha}(\alpha)} u_N(w) e^{\left(1 - \frac{g(0)}{g_{U_\alpha}(\alpha)}\right) \frac{u_N(w)^2}{2g(0)} + o(1)}}{\sqrt{g(0)} u_N(w) (1 + o(1))} \right) \right) \\
& \leq CN \frac{\sqrt{g(0)} e^{-\frac{u_N(w)^2}{2g(0)}}}{\sqrt{2\pi} u_N(w)} o(1) = o(1).
\end{aligned}$$

Similarly one can show that on the complementary event  $\mathcal{Q}^c$  (recall (17) for the definition of  $\mathcal{Q}$ )

$$\sum_{\alpha \in \mathcal{I}} \mathbb{E} \left[ \left( \tilde{\mathbb{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w)) - \mathbb{P}(\varphi_\alpha > u_N(w)) \right) \mathbb{1}_{\{|\mu_\alpha| \leq (u_N(z))^{-1-\epsilon}\}} \mathbb{1}_{\mathcal{Q}^c} \right] = o(1).$$

This shows that  $b_3 \rightarrow 0$ . Hence from Theorem 4 it follows that

$$\left| \mathbb{P}(W_1 = 0, \dots, W_k = 0) - \prod_{j=1}^k \mathbb{P}(Z_j = 0) \right| = o(1),$$



having used the independence of the  $Z_j$ 's. Notice that by definition  $Z_j$  is a Poisson random variable with intensity  $\sum_{\alpha \in \mathcal{I}_j} P((\varphi_\alpha - b_N)/a_N \in R_j)$ . Decomposing  $R_j$  as a union of finite intervals and using Mills ratio, similarly to the argument leading to (10), one has

$$P(Z_j = 0) \rightarrow \exp(-|A_j|\omega(R_j))$$

(recall  $\omega(R_j) = \int_{R_j} e^{-z} dz$ ). Hence it follows that

$$\prod_{j=1}^k P(Z_j = 0) \rightarrow \exp\left(-\sum_{j=1}^k |A_j|\omega(R_j)\right), \quad (18)$$

which completes the proof of ii) and therefore of Theorem 1.  $\square$

#### 4. PROOF OF THEOREM 2: THE FINITE-VOLUME CASE

We will now show the theorem for the field with zero boundary conditions. As remarked in the Introduction, since on the bulk defined in (4) we have a good control on the conditioned field, we will first prove convergence therein, and then we will use a converging-together theorem to achieve the final limit. We will first need some notation used throughout the Section: first, we consider  $(\psi_\alpha)_{\alpha \in V_N}$  with law  $\tilde{P}_N := \tilde{P}_{V_N}$ . We also use the shortcut  $g_N(\cdot, \cdot) = g_{V_N}(\cdot, \cdot)$ . We will need the notation  $\mathcal{C}_K^+(E)$  for the set of positive, continuous and compactly supported functions on  $E = [0, 1]^d \times (-\infty, +\infty]$ .

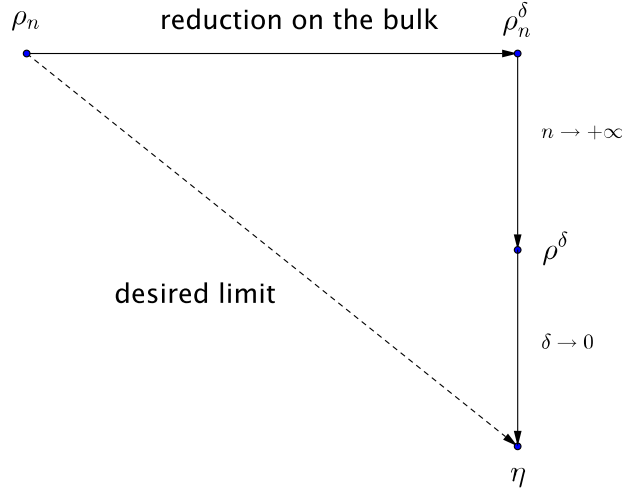


FIGURE 1. Sketch of the proof of Theorem 2

We first begin with a lemma on the point process convergence on bulk. Define a point process on  $E$  by

$$\rho_n^\delta(\cdot) = \sum_{\alpha \in V_N^\delta} \varepsilon_{\left(\frac{\alpha}{n}, \frac{\psi_\alpha - b_N}{a_N}\right)}(\cdot). \quad (19)$$

**Lemma 5.** *Let  $\delta > 0$ . On  $\mathcal{M}_p(E)$ ,  $\rho_n^\delta \xrightarrow{d} \rho^\delta$  where  $\rho^\delta$  is a Poisson random measure with intensity  $d t_{|_{[\delta, 1-\delta]^d}} \otimes (e^{-x} dx)^c$ .*

<sup>c</sup> $d t_{|_{[\delta, 1-\delta]^d}}$  is the restriction of the Lebesgue measure to  $[\delta, 1-\delta]^d$ .

*Proof.* We will show i) and ii) of Page 3 (and from which we will borrow the notation starting from now).

i) We begin with an upper bound on  $\tilde{\mathbb{E}}_N [\rho_n^\delta(A \times (x, y))]$ :

$$\begin{aligned}
& \sum_{\alpha \in nA \cap V_N^\delta} \tilde{\mathbb{P}}_N(\psi_\alpha > u_N(x)) - \tilde{\mathbb{P}}_N(\psi_\alpha > u_N(y)) \\
& \stackrel{(8)}{\leq} \sum_{\alpha \in nA \cap V_N^\delta} \frac{e^{-\frac{u_N(x)^2}{2g_N(\alpha)}}}{\sqrt{2\pi u_N(x)}} \sqrt{g_N(\alpha)} - \frac{e^{-\frac{u_N(y)^2}{2g_N(\alpha)}}}{\sqrt{2\pi u_N(y)}} \sqrt{g_N(\alpha)} (1 + o(1)) \\
& \stackrel{\text{Lemma 3}}{=} \sum_{\alpha \in nA \cap V_N^\delta} \frac{e^{-\frac{u_N(x)^2}{2g(0)(1+c_n)}}}{\sqrt{2\pi u_N(x)}} \sqrt{g(0)(1+c_n)} - \frac{e^{-\frac{u_N(y)^2}{2g(0)(1+c_n)}}}{\sqrt{2\pi u_N(y)}} \sqrt{g(0)(1+c_n)} \\
& \xrightarrow{n \rightarrow +\infty} (e^{-x} - e^{-y}) \mathbb{1}_{A \cap [\delta, 1-\delta]^d}. \tag{20}
\end{aligned}$$

We stress that in the second step the error term  $c_n := O(n^{2-d})$  coming from Lemma 3 guarantees the convergence in the last line. The lower bound follows similarly.

ii) To show the second condition we again use Theorem 4. Let  $A_1, \dots, A_k$  and  $R_1, \dots, R_k$  be as in proof of Theorem 1. Let  $\mathcal{I}_j := nA_j \cap V_N^\delta$  and  $\mathcal{I} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_k$ . For  $\epsilon > 0$  we are setting  $B_\alpha := B\left(\alpha, (\log N)^{2(1+\epsilon)}\right) \cap \mathcal{I}$ . Note that, albeit slightly different, we are using the same notations for the neighborhood of dependence and the index sets of Section 3, but no confusion should arise. Observe that there exists  $z \in \mathbb{R}$  such that for all  $1 \leq j \leq k$ ,  $R_j \subset (z, \infty]$ ; we have

$$p_\alpha = \tilde{\mathbb{P}}_N\left(\frac{\psi_\alpha - b_N}{a_N} \in u_N(R_j)\right) \leq \tilde{\mathbb{P}}_N(\psi_\alpha > u_N(z)) \stackrel{(8)}{\leq} \frac{e^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi u_N(z)}} \sqrt{g(0)}$$

where we have also used the fact that  $g_N(\alpha) \leq g(0)$ . The bound on  $b_1$  (cf. Theorem 4) follows exactly as in (11) and yields that, for some  $C > 0$ ,

$$b_1 \leq CN(\log N)^{d(2+2\epsilon)} e^{-2z} N^{-2} \rightarrow 0.$$

The calculation of  $b_2$  can be performed similarly using the covariance matrix of the vector  $(\psi_\alpha, \psi_\beta)$ ,  $\alpha \neq \beta \in V_N^\delta$  and Lemma 3. This gives that for some  $C, C' > 0$  independent of  $\alpha, \beta \in V_N^\delta$

$$\begin{aligned}
b_2 & \leq \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in B_\alpha} \frac{C}{\log N} \exp\left(-\frac{u_N(z)^2}{g(0) + g(\alpha - \beta)} \left(1 + O\left(N^{(2-d)/d}\right)\right)\right) \\
& \leq C' N^{-\kappa/(2-\kappa)} (\log N)^{2d(1+\epsilon)} \max\left\{e^{-2z} \mathbb{1}_{\{z \leq 0\}}, e^{-2z/(2-\kappa)} \mathbb{1}_{\{z > 0\}}\right\} \rightarrow 0
\end{aligned}$$

(cf. Chiarini et al. (2015)). We will now pass to  $b_3$ . We repeat our choice of  $\mathcal{H}_1 = \sigma(X_\beta : \beta \in \mathcal{I} \setminus B_\alpha)$  and  $\mathcal{H}_2 = \sigma(\psi_\beta : \beta \in \mathcal{I} \setminus B_\alpha)$  so that  $b_3$  becomes

$$\sum_{j=1}^k \sum_{\alpha \in \mathcal{I}_j} \tilde{\mathbb{E}}_N \left[ \left| \tilde{\mathbb{E}}_N [X_\alpha - p_\alpha | \mathcal{H}_1] \right| \right] \leq \sum_{j=1}^k \sum_{\alpha \in \mathcal{I}_j} \tilde{\mathbb{E}}_N \left[ \left| \tilde{\mathbb{E}}_N [X_\alpha | \mathcal{H}_2] - p_\alpha \right| \right].$$

We define  $U_\alpha := V_N \setminus (\mathcal{I} \setminus B_\alpha)$ . By the Markov property of the DGFF

$$\tilde{\mathbb{E}}_N [X_\alpha | \mathcal{H}_2] = \tilde{\mathbb{P}}_{U_\alpha}(\xi_\alpha + h_\alpha \in u_N(R_j)) \tilde{\mathbb{P}}_N - a. s. \tag{21}$$

for  $(\xi_\alpha)_{\alpha \in \mathbb{Z}^d}$  a DGFF with law  $\tilde{\mathbb{P}}_{U_\alpha}$  and  $(h_\alpha)_{\alpha \in \mathbb{Z}^d}$  is independent of  $\xi$ . From Chiarini et al. (2015) we can see that, for  $\alpha \in V_N^\delta$  and  $N$  large enough such that  $B(\alpha, (\log N)^{2(1+\epsilon)}) \subsetneq V_N$ ,

$$\begin{aligned} \text{Var}[h_\alpha] &= \sum_{\beta \in \mathcal{I} \setminus B_\alpha} \mathbb{P}_\alpha \left( H_{\mathcal{I} \setminus B_\alpha} < +\infty, S_{H_{\mathcal{I} \setminus B_\alpha}} = \beta \right) g_N(\alpha, \beta) \\ &\leq \sup_{\beta \in \mathcal{I} \setminus B_\alpha} g_N(\alpha, \beta) \leq \frac{c}{(\log N)^{2(1+\epsilon)(d-2)}}. \end{aligned}$$

This yields

$$\sum_{j=1}^k \sum_{\alpha \in \mathcal{I}_j} \tilde{\mathbb{E}}_N \left[ \left| \tilde{\mathbb{P}}_{U_\alpha}(\xi_\alpha + h_\alpha) > u_N(R_j) \right| - p_\alpha \right] \mathbb{1}_{\{|h_\alpha| > (u_N(z))^{-1-\epsilon}\}} \rightarrow 0. \quad (22)$$

It then suffices to show

$$\sum_{j=1}^k \sum_{\alpha \in \mathcal{I}_j} \tilde{\mathbb{E}}_N \left[ \left| \tilde{\mathbb{P}}_{U_\alpha}(\xi_\alpha + h_\alpha) > u_N(R_j) \right| - p_\alpha \right] \mathbb{1}_{\{|h_\alpha| \leq (u_N(z))^{-1-\epsilon}\}} \rightarrow 0. \quad (23)$$

One sees that the breaking up (15) can be performed also here replacing  $\varphi_\alpha$  and  $\psi_\alpha$  (with their laws) with  $\tilde{\varphi}_\alpha$  and  $\tilde{\psi}_\alpha$  (with their laws) respectively, and  $\mu_\alpha$  with  $h_\alpha$ . Accordingly, it is enough to show that

$$\sum_{\alpha \in \mathcal{I}} \tilde{\mathbb{E}}_N \left[ \left| \tilde{\mathbb{P}}_{U_\alpha}(\xi_\alpha + h_\alpha > u_N(w)) - \tilde{\mathbb{P}}_N(\psi_\alpha > u_N(w)) \right| \mathbb{1}_{\{|h_\alpha| \leq (u_N(z))^{-1-\epsilon}\}} \right] \rightarrow 0 \quad (24)$$

for all  $w \in \mathbb{R}$ . To this aim, we choose for any  $w \in \mathbb{R}$  the event

$$\mathcal{Q}' := \left\{ \tilde{\mathbb{P}}_N(\psi_\alpha > u_N(w)) > \tilde{\mathbb{P}}_{U_\alpha}(\xi_\alpha + h_\alpha > u_N(w)) \right\}$$

and we proceed as in (17) with the help of Lemma 3 to show (24). Given this, the convergence  $b_3 \rightarrow 0$  is finally ensured. Hence we can conclude that

$$\|\mathcal{L}(W_1, \dots, W_k) - \mathcal{L}(Z_1, \dots, Z_k)\|_{TV} \rightarrow 0$$

where  $Z_j$  are i. i. d. Poisson of mean  $p_\alpha$ . By Mills ratio, as in (20) we see that

$$\mathbb{P}(Z_j = 0) \rightarrow \exp\left(-\left|A_j \cap [\delta, 1 - \delta]^d\right| \omega(R_j)\right).$$

From this it follows that the two conditions i) and ii) of Kallenberg's Theorem are satisfied, and thus we obtain the convergence to a Poisson point process with mean measure given in i).  $\square$

*Proof of Theorem 2.*  $\mathcal{M}_p(E)$  is a Polish space with metric  $d_p$ :

$$d_p(\mu, \mu') = \sum_{i \geq 1} \frac{\min\{|\mu(f_i) - \mu'(f_i)|, 1\}}{2^i}, \quad \mu, \mu' \in \mathcal{M}_p(E)$$

for a sequence of functions  $f_i \in \mathcal{C}_K^+(E)$  (cf. Resnick (1987, Section 3.3)). Therefore we are in the condition to use a converging-together theorem (Resnick, 2007, Theorem 3.5), namely to prove that  $\rho_n \xrightarrow{d} \eta$  it is enough to show the following:

- (a)  $\rho_n^\delta \xrightarrow{d} \rho^\delta$ , as  $n \rightarrow +\infty$ .
- (b)  $\rho^\delta \xrightarrow{d} \eta$  as  $\delta \rightarrow 0$ .

(c) For every  $\epsilon > 0$ ,

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} \tilde{\mathbb{P}}_N \left( d_p \left( \rho_n, \rho_n^\delta \right) > \epsilon \right) = 0. \quad (25)$$

Note that by Lemma 5, (a) is satisfied. For  $f \in \mathcal{C}_K^+(E)$ , the Laplace functional of  $\rho^\delta$  is given by (cf. Resnick (1987, Prop. 3.6))

$$\Psi_\delta(f) := \mathbb{E} \left[ \exp \left( -\rho^\delta(f) \right) \right] = \exp \left( - \int_E \left( 1 - e^{-f(t,x)} \right) dt \Big|_{[0,1-\delta]^d} e^{-x} dx \right).$$

Hence by the dominated convergence theorem we can exchange limit and expectation as  $\delta \rightarrow 0$  to obtain that

$$\Psi_\delta(f) \rightarrow \exp \left( - \int_E \left( 1 - e^{-f(t,x)} \right) dt e^{-x} dx \right)$$

and the right hand side is the Laplace functional of  $\eta$  at  $f$ . This shows (b).

Hence to complete the proof it is enough to show (25). Thanks to the definition of the metric  $d_p$  it suffices to prove that for  $f \in \mathcal{C}_K^+(E)$  and for  $\epsilon > 0$

$$\limsup_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} \tilde{\mathbb{P}}_N \left( \left| \rho_n(f) - \rho_n^\delta(f) \right| > \epsilon \right) = 0.$$

Without loss of generality assume that the support of  $f$  is contained in  $[0, 1]^d \times [z_0, +\infty)$  for some  $z_0 \in \mathbb{R}$ . Choosing  $n$  large enough such that  $u_N(z_0) > 0$  and  $g_N(\alpha) \leq g(0)$ , we obtain that

$$\begin{aligned} \tilde{\mathbb{E}}_N \left[ \left| \rho_n(f) - \rho_n^\delta(f) \right| \right] &= \tilde{\mathbb{E}}_N \left[ \sum_{\alpha \in V_N \setminus V_N^\delta} f \left( \frac{\alpha}{n}, \frac{\psi_\alpha - b_N}{a_N} \right) \mathbb{1}_{\left\{ \frac{\psi_\alpha - b_N}{a_N} > z_0 \right\}} \right] \\ &\leq \sup_{z \in E} |f(z)| \sum_{\alpha \in V_N \setminus V_N^\delta} \tilde{\mathbb{P}}_N \left( \frac{\psi_\alpha - b_N}{a_N} > z_0 \right) \stackrel{(8)}{\leq} C \sum_{\alpha \in V_N \setminus V_N^\delta} \frac{e^{-u_N(z_0)^2/g(0)}}{\sqrt{2\pi}u_N(z_0)} \sqrt{g(0)} \\ &\leq C' \left( 1 - (1 - 2\delta)^d \right) e^{-z_0} \end{aligned}$$

as  $n \rightarrow +\infty$  for some positive constants  $C, C'$ . Now letting  $\delta \rightarrow 0$  the result follows and this completes the proof.  $\square$

## REFERENCES

- E. Aïdékon, J. Berestycki, É. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probability Theory and Related Fields*, 157(1-2):405–451, 2013. ISSN 0178-8051. doi: 10.1007/s00440-012-0461-0. URL <http://dx.doi.org/10.1007/s00440-012-0461-0>.
- L.-P. Arguin, A. Bovier, and N. Kistler. Genealogy of extremal particles of branching Brownian motion. *Communications on Pure and Applied Mathematics*, 64(12):1647–1676, 2011. ISSN 1097-0312. doi: 10.1002/cpa.20387. URL <http://dx.doi.org/10.1002/cpa.20387>.
- L.-P. Arguin, A. Bovier, and N. Kistler. Poissonian statistics in the extremal process of branching brownian motion. *Ann. Appl. Probab.*, 22(4):1693–1711, 08 2012. doi: 10.1214/11-AAP809. URL <http://dx.doi.org/10.1214/11-AAP809>.

- L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probability Theory and Related Fields*, 157(3-4):535–574, 2013. ISSN 0178-8051. doi: 10.1007/s00440-012-0464-x. URL <http://dx.doi.org/10.1007/s00440-012-0464-x>.
- R. Arratia, L. Goldstein, and L. Gordon. Two Moments Suffice for Poisson Approximations: The Chen-Stein Method. *Ann. Probab.*, 17(1):9–25, 01 1989. doi: 10.1214/aop/1176991491. URL <http://dx.doi.org/10.1214/aop/1176991491>.
- M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *ArXiv e-prints*, June 2013. <http://adsabs.harvard.edu/abs/2013arXiv1306.2602B>.
- M. Biskup and O. Louidor. Conformal symmetries in the extremal process of two-dimensional discrete Gaussian Free Field. *ArXiv e-prints*, Oct. 2014. <http://adsabs.harvard.edu/abs/2014arXiv1410.4676B>.
- M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *ArXiv e-prints*, Jan. 2013. <http://arxiv.org/abs/1301.6669v4>.
- A. Chiarini, A. Cipriani, and R. S. Hazra. Extremes of the supercritical Gaussian Free Field. *arXiv preprint*, 2015. URL <http://arxiv.org/abs/1504.07819>.
- H. O. Georgii. *Gibbs measures and Phase transitions*. de Gruyter, Berlin, 1988.
- O. Kallenberg. *Random measures*. Akademie-Verlag, 1983. ISBN 9780123949608. URL <https://books.google.de/books?id=bBnvAAAAMAAJ>.
- S. Resnick. *Extreme Values, Regular Variation, and Point Processes*. Applied probability : a series of the applied probability trust. Springer, 1987. ISBN 9783540964810. URL <http://books.google.it/books?id=DXilQgAACAAJ>.
- S. Resnick. *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Number Bd. 10 in Heavy-tail Phenomena: Probabilistic and Statistical Modeling. Springer, 2007. ISBN 9780387242729. URL <https://books.google.de/books?id=p8uq2QFw9PUC>.
- P.-F. Rodriguez and A.-S. Sznitman. Phase transition and level-set percolation for the gaussian free field. *Communications in Mathematical Physics*, 320(2):571–601, 2013. ISSN 0010-3616. doi: 10.1007/s00220-012-1649-y. URL <http://dx.doi.org/10.1007/s00220-012-1649-y>.
- E. Subag and O. Zeitouni. Freezing and Decorated Poisson Point Processes. *Communications in Mathematical Physics*, 337(1):55–92, 2015. ISSN 0010-3616. doi: 10.1007/s00220-015-2303-2. URL <http://dx.doi.org/10.1007/s00220-015-2303-2>.
- A. Sznitman. *Topics in Occupation Times and Gaussian Free Fields*. Zurich Lectures in Advanced Mathematics. American Mathematical Society, 2012. ISBN 9783037191095. URL <http://books.google.ch/books?id=RnENO-nQ7TIC>.