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Extremality of the disordered state for the Ising model on general trees

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Abstract

We develop a method to study extremality of the disordered state $\mathbb{P}^\beta$ for the Ising model on a general countable tree $\mathbf{T}$. It is shown that the tail σ-field is $\mathbb{P}^\beta$-trivial as soon as β is less than the spin glass critical inverse temperature β_c^{SG}, which is determined from the relation $\tanh(\beta_c^{SG}) = 1/\sqrt{br(\mathbf{T})}$. The method is based on the FK representation of ferromagnetic systems and recursive estimates on conditional expectations of the spin at the root. Similar estimates in the context of the bit reconstruction problem on general trees were originally obtained in [EKPS] using different methods.

Key words: Countable trees, Ising model, FK representation

1. Introduction

Let $\mathbf{T}$ be a countable locally finite tree. Choose a vertex $0 \in \mathbf{T}$, which from now on will be called the root of $\mathbf{T}$. Once the root is fixed, one encounters a natural partial ordering on the set of vertices of $\mathbf{T}$. Namely, for $k, l \in \mathbf{T}$, we say that $k \prec l$ if k lies on the unique chain of edges leading from l to the root. If $k \prec l$ and k, l are nearest neighbours , then l is called a successor of k . For each $k \in \mathbf{T}$ let $b(k)$ to denote the number of the successors of k, in other words, $b(k)$ is the forward branching ratio at k. It is possible [L] to introduce an average forward branching number of $\mathbf{T}$ as well: A cutset π is a subset $\pi \in \mathbf{T}$, such that any infinite selfavoiding chain of bonds emanating from the root contains exactly one vertex from π. Then,

$$br(\mathbf{T}) \stackrel{def}{=} \inf\{\lambda > 0 : \liminf_{\pi-cutset} \sum_{l \in \pi} \lambda^{-|l|} = 0\},$$

where $|l|$ is the number of bonds in the unique chain leading from l to the root. We refer to [L] for a comprehensive discussion of this quantity.

The purpose of this work is to develop a convenient tool for studying extremality properties of the disordered state for the Ising model on $\mathbf{T}$. This question has a somewhat curious history for homogeneous trees $\mathbf{T}_d$ with the forward branching ratio $d \in \mathbb{N}$: The critical inverse temperature β_c for the Ising model on $\mathbf{T}_d$ is given [P] by ,

$$\tanh(\beta_c) = 1/d .$$

In a seminal paper [S] Spitzer constructed a family of Ising Gibbs measures, which he called Markov chains on trees, and asserted that above β_c any such Markov chain is extremal, the limit state with free boundary conditions in particular. However, Higuchi in [H] reproduced an unpublished computation of Kamae, which clearly revealed that the disordered state cannot be extremal as long as

$$\tanh(\beta) > 1/\sqrt{d} .$$

An insight into the above quantity came with the work of [CCST], where they identified

$$\tanh(\beta_c^{SG}) = 1/\sqrt{d}$$

as a spin glass critical inverse temperature in the case of the binary tree $d = 2$.

Bleher [B] tried to modify their approach in order to prove that β_c^{SG} is, in fact, a threshold for extremality for the disordered state as well, but in the latter case the situation is substantially complicated due to the loss of the independence, which is intrinsic for the spin glass picture, and Bleher's proof was eventually found to be erroneous. The corresponding mistake was corrected in [BRZ] at a price of rather tedious computations. A different method, based on the FK representation, was suggested in [I]. Here we extend the techniques of [I] to derive results on general trees. These results, however, are not as satisfactory as those obtained for the Bethe lattice. Specifically, we prove

Theorem 1. *If,*

$$\tanh(\beta) \; < \; 1/\sqrt{br(\mathbf{T})} \tag{1}$$

then the disordered state at the inverse temperature β is extremal.

As in [BRZ], our strategy to prove the above theorem is based on an observation that the tail triviality of the spin at the root is equivalent to the tail triviality of the disordered state itself, which is stated in lemma 2 below, and , most crucially, on recursive estimates on second moments of conditional expectations of the the spin at the root, formulated in the proposition 4 below, or, more precisely, on its weaker version (14). An estimate similar to (14) was originally proved in [EKPS] in the context of the bit reconstruction problem on general trees using completely different methods.

Now, a proper generalization [EKPS] of the computation in [H] shows that the extremality fails whenever

$$\tanh(\beta) \; > \; 1/\sqrt{br(\mathbf{T})}.$$

What, therefore, remains unclear for us is the exact state of affairs at the critical point β_c,

$$\tanh(\beta_c) \; = \; 1/\sqrt{br(\mathbf{T})}.$$

Never the less, we hope that our estimate (10) below already provides all the necessary data to enable a complete treatment of the critical case[1] .

The paper is organized as follows: In section 2 we describe the model and list some of its relevant properties. Section 3 is devoted to the FK arithmetics, which lies in the heart of our approach. Theorem 1 is proved in section 4.

2. The Model

Let $\mathbf{V}$ be a finite tree with a distinguished root 0 and the boundary

$$\partial\mathbf{V} \; = \; \{k \in \mathbf{V} : \; b(k) = 0\}.$$

Recall that $b(k)$ is the forward branching ratio at k with the respect to the root. The Ising Gibbs state on $\mathbf{V}$ at the inverse temperature β and with boundary condition ξ on $\partial\mathbf{V}$ is the probability distribution on $\Omega_V = \{-1,1\}^{\mathbf{V}}$, given by

$$\mathbb{P}^{\beta}_{\mathbf{V},\xi}(x) \; = \; \frac{1}{\mathbf{Z}^{\beta}_{\mathbf{V},\xi}} \exp\{\frac{\beta}{2} \sum_{|k-l|=1} x_k x_l \; + \; \beta \sum_{k\in\partial\mathbf{V}} \xi_k x_k\},$$

where the distance $|k-l|$ is defined to be the number of bonds in the chain connecting k and l and $\mathbf{Z}^{\beta}_{\mathbf{V},\xi}$ is the normalizing constant or partition function. In particular, if we pick free boundary conditions $\xi \equiv 0$, then the corresponding measure will be called the disordered state (on $\mathbf{V}$), and we shall denote it as $\mathbb{P}^{\beta}_{V}$.

For each $k \in \mathbf{V}$ let $S(k)$ to denote the set of successors of k,

$$S(k) \; = \; \{l : \; k \prec l \text{ and } |l-k| = 1\}.$$

Assume that k is such that $S(k) \subseteq \partial\mathbf{V}$. The following computation is very specific for the tree structure of $\mathbf{V}$:

$$\sum_{l\in S(k)} \sum_{x_l=\pm 1} \mathbb{P}^{\beta}_{\mathbf{V},\xi}(x) \; = \; \mathbb{P}^{\beta}{}_{\mathbf{V}\setminus S(k),\nu}(x), \tag{2}$$

[1] I was informed by Yuval Peres that the question of the tail triviality of the spin at the root in the critical case was completely resolved in his recent work with Robin Pemantle

where $\nu_r = \xi_r$ for $r \in \partial\mathbf{V} \setminus S(k)$ and,

$$\nu_k \;=\; \frac{1}{2}\sum_{l\in S(k)} \log[\frac{e^{\beta+\xi_l}+e^{-(\beta+\xi_l)}}{e^{\beta-\xi_l}+e^{\xi_l-\beta}}].$$

Note that the function on the right hand side above is an odd function of the boundary configuration ξ. Proceeding with the summation from the boundary to the root, one obtains an odd function $H = H(\xi_l, l \in \partial\mathbf{V})$, which describes the distribution of the spin at the root given boundary conditions ξ,

$$\mathbb{E}^{\beta}_{\mathbf{V},\xi} x_0 \;=\; \tanh(H(\xi)).$$

H can be viewed as the effective magnetic field at the root given ξ on $\partial\mathbf{V}$.

In particular, for the disordered state the effective magnetic field at the root is always zero. In fact, it is easy to see by means of the computation sketched above that for any connected subtree $\mathbf{V}' \subseteq \mathbf{V}$, the relativization of $\mathbb{P}^{\beta}_{\mathbf{V}}$ on $\Omega_{\mathbf{V}'}$ is precisely $\mathbb{P}^{\beta}_{\mathbf{V}'}$. Back to our infinite tree $\mathbf{T}$ we, therefore, readily obtain that all finite volume Gibbs states on connected subtrees of $\mathbf{T}$ with free boundary conditions are, in fact, relativizations of of a certain infinite volume measure on $\Omega_{\mathbf{T}}$, which we call the disordered state on $\mathbf{T}$ and denote as $\mathbb{P}^{\beta}$. Note that because of the relativization property one can drop the subindex "$\mathbf{V}$" in $\mathbb{P}^{\beta}_{\mathbf{V}}$ even when studying events from $\Omega_{\mathbf{V}}$ proper.

Our main objective here is to understand when the disordered state $\mathbb{P}^{\beta}$ is extremal. More precisely, for each cutset π set

$$\mathcal{F}^{\pi} \;=\; \sigma(\{x_l, k \prec l \text{ for some } k \in \pi\}).$$

Define the tail σ-field of $\Omega_{\mathbf{T}}$ as

$$\mathcal{F}_{\infty} \;=\; \bigcap_{\pi} \mathcal{F}^{\pi}.$$

Recall that $\mathbb{P}^{\beta}$ is extremal if and only if $\mathcal{F}_{\infty}$ is $\mathbb{P}^{\beta}$-trivial. Our investigation of the extremality of $\mathbb{P}^{\beta}$ is based on the following lemma, which we prove in the next section:

Lemma 2. $\mathbb{P}^{\beta}$ *is extremal iff,*

$$\lim_{\pi\to\infty} \mathbf{Var}^{\beta}[\mathbb{E}^{\beta}(x_o \,|\mathcal{F}_{\pi})] \;=\; 0. \tag{3}$$

By the backward martingale convergence theorem the above limit always exists. Our main effort, therefore, is to derive conditions to ensure that it is actually zero. Both the proof of the lemma and the investigation of (3) depend on a simple FK arithmetics, developed in the next section.

3. FK(Fortuin-Kasteleyn) Arithmetics

Let $(\mathbf{V}, \mathcal{E}_{\mathbf{V}})$ be a finite connected graph with the vertex set $\mathbf{V}$ and the edge set $\mathcal{E}_{\mathbf{V}}$ respectively. As before, we use $\Omega_{\mathbf{V}} = \{-1,1\}^{\mathbf{V}}$ to denote the set of spin configurations on $\mathbf{V}$. Let now $\Omega^{\mathfrak{e}}_{\mathbf{V}} = \{0,1\}^{\mathcal{E}_{\mathbf{V}}}$ to be the set of all edge (bond) configurations on $\mathcal{E}_{\mathbf{V}}$, i.e to each bond $b =< kl >\in \mathcal{E}_{\mathbf{V}}$ we assign a number $n(b)$, which is either 1, and in this case we say that b is open, or it is zero, and then we say that b is closed. Thus, each edge configuration $n \in \Omega^{\mathfrak{e}}_{\mathbf{V}}$ splits $\mathbf{V}$ into disjoint union of maximal connected components, where we call two vertices k and l connected, if there is a chain of open bonds leading from k to l. Let $\rfloor(n)$ to denote the number of these components. Consider now a joint probability distribution μ on $\Omega_{\mathbf{V}} \times \Omega^{\mathfrak{e}}_{\mathbf{V}}$, given by:

$$\mu(x,n) \;=\; \frac{1}{\mathbf{Z}(\beta)} \prod_{n(<kl>)=1} q\delta(x_k - x_l) \prod_{n(<kl>)=0} (1-q).$$

Then [ACCN], the site marginal of μ is precisely $\mathbb{P}^{\beta}_{\mathbf{V}}$ with β and q related via: $\beta = -1/2\log(1-q)$ or $q = 1 - e^{-2\beta}$. On the other hand the edge marginal of μ is given by:

$$\mathbb{Q}^{\beta}_{\mathbf{V}}(n) \;=\; \frac{1}{\mathbf{Z}(\beta)} 2^{\rfloor(n)} q^{\sum n(b)} (1-q)^{\sum(1-n(b))}.$$

The disordered state $\mathbb{P}^{\beta}_{\mathbf{V}}$ has [ACCN], then , the following convenient representation: Choose maximal connected clusters according to $\mathbb{Q}^{\beta}_{\mathbf{V}}$ and paint each cluster independently into ± 1 with probability $1/2$ each.

In general $\mathbb{Q}^{\beta}_{\mathbf{V}}$ is a nonlocal dependent percolation. If, however, $\mathbf{V}$ is a tree, then things are greatly simplified [CCST]. Indeed, in the latter case the number of maximal connected components of $n \in \Omega^{\mathfrak{e}}_{\mathbf{V}}$ equals to one plus the number of closed bonds of n, i.e.

$$\rfloor(n) \;=\; 1 \;+\; \sum_{b\in\mathcal{E}_{\mathbf{V}}} (1-n(b)).$$

Consequently,

$$\mathbb{Q}^{\beta}_{\mathbf{V}}(n) \;=\; \frac{2}{\mathbf{Z}(\beta)} q^{\sum n(b)} (2(1-q))^{\sum(1-n(b))} \;=\; p^{\sum n(b)}(1-p)^{\sum(1-n(b))},$$

where

$$p \;=\; \frac{q}{2-q} \;=\; \tanh(\beta).$$

Therefore, in the case of trees the random cluster measure $\mathbb{Q}^{\beta}_{\mathbf{V}}$ is just the usual Bernoulli bond percolation at p given above.

Let us summarize the above discussion in the form of the following algorithm of construction of $\mathbb{P}^{\beta}_{\mathbf{V}}$ on finite subtrees $\mathbf{V} \subset \mathbf{T}$:

Step1. Set $p = \tanh(\beta)$ and consider an independent Bernoulli percolation on $\mathcal{E}_{\mathbf{V}}$, i.e. assign to each bond configuration $n \in \Omega^{\mathfrak{e}}_{\mathbf{V}}$ the probability

$$\mathbb{Q}^{\beta}_{\mathbf{V}}(n) \;=\; p^{\sum_{b\in\mathcal{E}_{\mathbf{V}}} n(b)}(1-p)^{\sum_{b\in\mathcal{E}_{\mathbf{V}}}(1-n(b))}.$$

Step2. Given a bond configuration n, paint independently each maximal connected component of $\mathbf{V}$ into $+1$ or -1 with probability $1/2$ each.

The above two-step procedure can be, using some labelling algorithm to avoid ambiguities, equally applied to construct probability measures $\mathbb{P}^{\beta}_{\mathbf{V}}$ for infinite connected subtrees $\mathbf{V} \subseteq \mathbf{T}$, in particular for $\mathbf{T}$ itself. Thus, let $\mathbb{Q}^{\beta}$ to denote the independent Bernoulli percolation measure on the edges of $\mathbf{T}$ and $\mathbb{P}^{\beta}$ to denote the corresponding measure on Ω_T, which is, of course, the infinite volume disordered state defined earlier. Note, by the way, that the relativization property of $\mathbb{P}^{\beta}$ becomes very transparent under the FK representation. Moreover, many quantities related to $\mathbb{P}^{\beta}$ admit a natural percolation interpretation:

Let $<\bullet>^{\beta}$ to denote the expectation under $\mathbb{P}^{\beta}$. Then,

$$< x_k x_l >^{\beta} \;=\; \mathbb{Q}^{\beta}\{k \text{ is connected to } l\}.$$

Another important example is provided by the following computation:

For any finite subset $A \subset \mathbf{T}$ set

$$\mathbf{x}_A \;=\; \prod_{j\in A} x_j.$$

Then, if $|A|$ is odd,

$$< \mathbf{x}_A >^{\beta} = 0 \,. \tag{4}$$

If $|A|$ is even, let us say that a configuration $n \in \{0,1\}^{\mathcal{E}}$ splits A evenly, if there is even number of vertices of A in each maximal connected component of n. Then,

$$< \mathbf{x}_A >^{\beta} = \mathbb{Q}^{\beta}(n : \ n \text{ splits } A \text{ evenly}) \tag{5}$$

Formulas (4) and (5) are immediate consequences of the FK representation and the relativization property of $\mathbb{P}^{\beta}$. Indeed, let $\mathcal{T}(A) = (V(A), \mathcal{E}(A))$ be the minimal connected subtree which spans A. Then,

$$< \mathbf{x}_A >^{\beta} = < \mathbf{x}_A >^{\beta}_{V(A)},$$

where $< \bullet >^{\beta}_{V(A)}$ is the expectation with respect to $\mathbb{P}^{\beta}_{V(A)}$. Simple combinatoric arguments, then, imply:

$$< \mathbf{x}_A >^{\beta} \ = < \mathbf{x}_A >^{\beta}_{V(A)} \ = \ 0 \ ,$$

in the odd case, and

$$< \mathbf{x}_A >^{\beta} \ = < \mathbf{x}_A >^{\beta}_{V(A)} \ = \ \mathbb{Q}^{\beta}_{\mathcal{E}(A)}(n : \ n \text{ splits } A \text{ evenly})$$

in the even case. Since $\mathbb{Q}^{\beta}_{\mathcal{E}(A)}$ is the relativization of $\mathbb{Q}^{\beta}$, (5) follows.

Proposition 3 (I). *Let two disjoint finite subsets $A, B \subset V$ have edge disjoint minimal spanning trees, i.e $\mathcal{E}(A) \cap \mathcal{E}(B) = \emptyset$.*

a) If both $|A|$ and $|B|$ are even, then

$$< \mathbf{x}_A \mathbf{x}_B >^{\beta} = < \mathbf{x}_A >^{\beta} < \mathbf{x}_B >^{\beta} . \tag{6}$$

b) If both $|A|$ and $|B|$ are odd, then for any site j, which lies on the unique chain connecting $V(A)$ to $V(B)$,

$$< \mathbf{x}_A \mathbf{x}_B >^{\beta} = < \mathbf{x}_A x_j >^{\beta} < \mathbf{x}_B x_j >^{\beta} . \tag{7}$$

Proof: Both formulas are consequences of (4) and (5) above and independence relations for Bernoulli percolation.

We turn now to the proof of lemma 2. Let $\mathbf{S}_n$ to denote the set of sites at distance n from the root. By the Markov property it will be enough to show that $\mathbb{P}^{\beta}$-a.s. for any n fixed and for all $A \subseteq \mathbf{S}_n$,

$$\lim_{\pi \to \infty} \mathbb{E}^{\beta}(\mathbf{x}_A \, | \mathcal{F}^{\pi}) \ = < \mathbf{x}_A >^{\beta}. \tag{8}$$

We shall view conditional expectations as projections on corresponding subspaces, the scalar product being defined by $< \bullet >^{\beta}$, and the rules of the game being dictated by the proposition 3. Let $\mathbf{T}_k$ to be the tree growing from the site k, $\pi_k = \mathbf{T}_k \cap \pi$ and set

$$\theta(k) \ = \ \lim_{\pi \to \infty} \mathbf{Var}^{\beta}(x_k \, | \mathcal{F}^{\pi_k}).$$

Lemma 2 asserts that $\mathbb{P}^{\beta}$ is extremal whenever $\theta(0) = 0$. Our first observation is that

$$\theta(0) = 0 \implies \theta(k) = 0 \ \forall k \in T.$$

Indeed, by the proposition 3, for any $A \subset \mathbf{T}_k$,

$$< x_0 \mathbf{x}_A >^{\beta} \ = \ p^{|k|} < x_k \mathbf{x}_A >^{\beta} \implies \theta(0) > p^{2|k|} \theta(k).$$

Let us redefine $\mathcal{F}^{\pi}$ to be the σ-algebra, generated by the spins from π only. Because of the Markov property the left hand side of (3) is insensitive to such an abuse of notations.

We proceed by induction. Formula (8) is precisely $\theta(0) = 0$ for $n = 0$ and assume that (8) is true for $n = 1, 2, ..., N - 1$.

Define ϕ to be a parity function on finite sets, i.e $\phi(A) = \pm 1$ depending on whether $|A|$ is even or odd respectively. Let $\mathcal{I}_N = \{-1,1\}^{\mathbf{S}_N}$. For each cutset π far enough from the root we split $\mathcal{F}^\pi$ into a direct sum of subspaces

$$\mathcal{F}^\pi = \bigoplus_{r \in \mathcal{I}_N} \mathcal{F}^\pi_{r,N}, \tag{9}$$

where $\mathcal{F}^\pi_{r,N}$ is spanned by monomials $\mathbf{x}_A,\ A \subseteq \pi$, such that $\phi(A \cap \mathbf{T}_k) = r(k)$ for all $k \in \mathbf{S}_N$. By the virtue of the proposition 3, for any $\mathbf{x}_A \in \mathcal{F}^\pi_{r,N}$ and $\mathbf{x}_B \in \mathcal{F}^\pi_{q,N}$,

$$< \mathbf{x}_A \mathbf{x}_B >^\beta \ = < \mathbf{x}_A \mathbf{x}_B \prod_{k: r(k)q(k) = -1} x_k \mathbb{E}^\beta (x_k \,| \mathcal{F}^{\pi_k}) >^\beta$$

Consequently, if $\theta(0)$, and hence all $\theta(k)$, equals to zero, then

$$\lim_{\pi \to \infty} \cos(\mathcal{F}^\pi_{r,N}, \mathcal{F}^\pi_{q,N}) \ = \ \delta(r - q).$$

Therefore, whenever $\theta(0) = 0$, one can choose a constant $c < \infty$ such that for any $u = \sum_{r \in \mathcal{I}_N} u_r \in \mathcal{F}^\pi$,

$$\|u\| \ \leq \ 1 \implies \max\{\|u_r\|,\ r \in \mathcal{I}_N\} \ \leq \ c$$

independently of π. We used $\| \bullet \|$ above to denote the norm defined by the scalar product $< \bullet >^\beta$.

Now let $A \subseteq \mathbf{S}_N$ and define $\tilde{A}$ via

$$\tilde{A} \ = \ \{k \in \mathbf{S}_{N-1} : \ \phi(S(k) \cap A) = -1\}.$$

Note that, in a view of the proposition 3,

$$< \mathbf{x}_A >^\beta \ = \ p^{|A|} < \mathbf{x}_{\tilde{A}} >^\beta.$$

Pick now a $u \in \mathcal{F}^\pi$ with $\|u\| \leq 1$ and let $u = \sum u_r$ be its decomposition with respect to (9). If $r \in \mathcal{I}_N$ is such that $r(k) = -1$ for some $k \in A$, then by the proposition 3,

$$| < u_r \mathbf{x}_A >^\beta | \ = \ | < u_r \mathbf{x}_{A \setminus k} \mathbb{E}^\beta (x_k \,| \mathcal{F}^{\pi_k}) >^\beta | \ \leq \ c \| \mathbb{E}^\beta (x_k \,| \mathcal{F}^{\pi_k}) \|,$$

and

$$| < u_r >^\beta | \ = \ | < u_r x_k \mathbb{E}^\beta (x_k \,| \mathcal{F}^{\pi_k}) >^\beta | \ \leq \ c \| \mathbb{E}^\beta (x_k \,| \mathcal{F}^{\pi_k}) \|.$$

Otherwise, if $r(k) = 1$ for all $k \in A$,

$$< u_r \mathbf{x}_A >^\beta \ = \ p^{|A|} < u_r \mathbf{x}_{\tilde{A}} >^\beta \ = \ p^{|A|} < u_r >^\beta < \mathbf{x}_{\tilde{A}} >^\beta \ + \ o(1),$$

where the second equality follows from the induction assumption.

Thus,

$$\lim_{\pi \to \infty} \max_{\|u\| = 1} | < u \mathbf{x}_A >^\beta \ - \ < u >^\beta < \mathbf{x}_A >^\beta | \ = \ 0$$

and (8) follows.

4. Proof of Theorem 1

Let $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$, where $\mathbf{V}_1$ and $\mathbf{V}_2$ are two finite edge disjoint trees growing from the common root, i.e $\mathbf{V}_1 \cap \mathbf{V}_2 = 0$. Also let $A_i = \partial \mathbf{V}_i, i = 1,2$ and let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}$ be σ-algebras generated by the spins from A_1, A_2 and $\partial \mathbf{V} = A_1 \vee A_2$ respectively. For fixed boundary configurations $\xi_i \in \Omega_i \stackrel{def}{=} \{-1,1\}^{A_i},\ i = 1,2$, set

$$g_i(\xi_i) \ = \ \mathbb{E}^\beta (x_0 \,| \mathcal{F}_i) \ \textit{and} \ g(\xi_1, \xi_2) \ = \ \mathbb{E}^\beta (x_0 \,| \mathcal{F})$$

Proposition 4 (I). *There exists a positive constant α such that,*

$$\|g\|^2 \le \|g_1\|^2 + \|g_2\|^2 - \alpha\|g_1\|^2\|g_2\|^2 \tag{10}$$

Moreover, one may pick α to be

$$\alpha = [\cosh(\frac{1}{2}\log\frac{2-(1-p)^{b(0)}}{(1-p)^{b(0)}})]^{-2}, \tag{11}$$

where $b(0)$ is the branching ratio at the root in $\mathbf{V}$.

Proof : Note, first of all, that

$$g = g_1 + g_2 - g_1 g_2 g. \tag{12}$$

Indeed, let $\mathbf{x}_B \in \mathcal{F}$. Then $\mathbf{x}_B = \mathbf{x}_{B_1}\mathbf{x}_{B_2}$, where $\mathbf{x}_{B_i} \in \mathcal{F}_i$; $i = 1,2$. There are only two symmetric possibilities to consider: either $|B_1|$ is odd and $|B_2|$ is even or the other way around. So let us assume that $|B_1|$ is odd. Then, using Proposition 3, one obtains:

$$<\mathbf{x}_B g>^\beta =<\mathbf{x}_{B_1}x_0>^\beta <\mathbf{x}_{B_2}>^\beta =<\mathbf{x}_{B_1}g_1>^\beta <\mathbf{x}_{B_2}>^\beta =<\mathbf{x}_B g_1>^\beta$$

In the same fashion,

$$<\mathbf{x}_B g_2>^\beta = <g_1\mathbf{x}_{B_1}>^\beta <x_0 g_2\mathbf{x}_{B_2}>^\beta =<g_1g_2g\mathbf{x}_B>^\beta$$

and (12) follows. Consequently, multiplying both sides of (12) by x_0, we obtain:

$$\|g\|^2 = \|g_1\|^2 + \|g_2\|^2 - <g^2g_1g_2>^\beta.$$

Thus, it remain to show that for some $\alpha > 0$,

$$<g^2g_1g_2>^\beta \ \ge \ \alpha <g_1g_2>^\beta.$$

According to the discussion in section 2,

$$g_i(\xi_i) = \tanh(H_1(\xi_i)) \ ; \ i = 1,2;$$

and

$$g(\xi_1,\xi_2) = \tanh(H_1(\xi_i) + H_2(\xi_2)),$$

where H_1, H_2 are the effective magnetic fields at 0 given boundary conditions on A_1 and A_2 respectively.

Next, define $\Omega_{i,+}$ and $\Omega_{i,-}$; $i = 1,2$, via

$$\Omega_{i,+(-)} = \{\xi \in \Omega_i : \ H_i(\xi) > (<) 0\},$$

Using the $\pm$ symmetry of the model , we obtain:

$$<g^2g_1g_2>^\beta = 2\sum_{g_i\in\Omega_{i,+}} (\mathbb{P}^\beta(x|_{A_1}=\xi_1, x|_{A_2}=\xi_2)\tanh^2(H_1(\xi_1)+H_2(\xi_2)) -$$

$$- \mathbb{P}^\beta(x|_{A_1}=\xi_1, x|_{A_2}=-\xi_2)\tanh^2(H_1(\xi_1)-H_2(\xi_2)))g_1(\xi_1)g_2(\xi_2).$$

Now, it is easy to see that the probabilities above can be represented as follows: For any $\xi_1 \in \Omega_1$ and $\xi_2 \in \Omega_2$,

$$\mathbb{P}^\beta(x|_{A_1}=\xi_1, x|_{A_2}=\xi_2) = \frac{1}{\mathbf{Z}}\mathbf{Z}_1(\xi_1)\mathbf{Z}_2(\xi_2)\cosh(H_1(\xi_1)+H_2(\xi_2)),$$

where $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are positive even functions of boundary configurations and $\mathbf{Z}$ is the corresponding partition function. Therefore,

$$<g^2g_1g_2>^\beta = 2\sum_{\xi_i\in\Omega_{i,+}} g_1(\xi_1)g_2(\xi_2)\frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}}\mathbf{U}(\xi_1,\xi_2),$$

where

$$\mathbf{U}(\xi_1,\xi_2) = \frac{\sinh^2(H_1(\xi_1)+H_2(\xi_2))}{\cosh(H_1(\xi_1)+H_2(\xi_2))} - \frac{\sinh^2(H_1(\xi_1)-H_2(\xi_2))}{\cosh(H_1(\xi_1)-H_2(\xi_2))}.$$

After a small computation one infers that

$$\mathbf{U}(\xi_1, \xi_2) \geq \frac{\cosh(H_1(\xi_1) + H_2(\xi_2)) - \cosh(H_1(\xi_1) - H_2(\xi_2))}{\cosh(H_1(\xi_1) + H_2(\xi_2))\cosh(H_1(\xi_1) - H_2(\xi_2))}. \qquad (13)$$

However, due to the monotonicity of the expectation of the spin at the root in boundary conditions, one may estimate:

$$\tanh(H_1(\xi_1) + H_2(\xi_2)) \leq 1 - (1-p)^{b(0)},$$

where the right hand side above is the expectation at the root conditioned on the event that all the spins in $S(0)$ are up. Consequently, the inverse of the denominator in (13) can be bounded above by α, specified in (11). Thus,

$$< g^2 g_1 g_2 >^\beta \geq 2\alpha \sum_{\xi_i \in \Omega_{i,+}} g_1 g_2 \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\mathbf{Z}} (\cosh(H_1 + H_2) - \cosh(H_1 - H_2)) =$$

$$= \alpha < g_1 g_2 >^\beta = \alpha \|g_1\|^2 \|g_2\|^2,$$

and the proof of the Proposition 4 is concluded.

It is very easy now to prove the extremality of $\mathbb{P}^\beta$ in the case $\tanh(\beta) < 1/\sqrt{br(\mathbf{T})}$. Indeed, for each $k \in \mathbf{T}$ one can view $\mathbf{T}_k$ as the union of $b(k)$ disjoint trees with a common root k. Therefore, using (10) $(b(k)-1)-$ times and taking into account that for any $l \in S(k)$ and any cutset π,

$$\mathbb{E}^\beta(x_k \,|\mathcal{F}^{\pi_l}) = p\mathbb{E}^\beta(x_l \,|\mathcal{F}^{\pi_l}),$$

we obtain that for all $k \in \mathbf{T}$,

$$\theta(k) \leq p^2 \sum_{l \in S(k)} \theta(l),$$

where $\theta(\bullet)$ was defined in section 3. Consequently [2], for each cutset π,

$$\theta(0) \leq \sum_{k \in \pi} p^{2|k|}\theta(k) \leq \sum_{k \in \pi} p^{2|k|}. \qquad (14)$$

Taking a $\liminf$ as $\pi \to \infty$, we obtain that $\theta(0) = 0$, which , by lemma 2, implies extremality.

Remark: Note that we did not use (10) in its full strength to prove the theorem for noncritical β. As in the case of Bethe lattices the negative term in the right hand side of (10) should become crucial, though, for a treatment of the critical case $\tanh(\beta) = 1/\sqrt{br(\mathbf{T})}$. For example, if $\mathbf{T}$ is spherically symmetric, i.e if $b(k)$ depends only to the distance from k to the root; $b(k) = b(|k|)$, then also $\theta(k) = \theta(|k|)$, and (10) is easily seen to yield:

$$\theta(n) \leq b(n)p^2\theta(n+1)(1 - p^2\frac{b(n)-1}{b(n)}\theta(n+1)), \; n = 1, 2, \dots .$$

Consequently, setting $M_n = p^{2(n+1)} \prod_{k \leq n} b(k)$, we obtain:

$$\theta(0) \leq M_n \theta(n+1) \prod_0^n (1 - p^2 \frac{b(k)-1}{b(k)} \theta(k+1)).$$

Therefore, $\mathbb{P}^\beta$ is clearly extremal as soon as

$$\limsup_{n \to \infty} M_n < \infty, \qquad (15)$$

[2] see [EKPS] for a similar inequality, derived via different methods

However, the (15) above should not be the right criteria for extremality in the spherically symmetric case. The right criteria is likely to be provided by the computation in the spirit of [H], which reveals that if

$$\sum_{n=1}^{\infty} \frac{1}{M_n} < \infty,$$

then the disordered state is not extremal. We failed to close this gap while preparing this report.

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