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Hölder estimates for parabolic operators on domains with rough boundary

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Abstract

In this paper we investigate linear parabolic, second-order boundary value problems with mixed boundary conditions on rough domains. Assuming only boundedness/ellipticity on the coefficient function and very mild conditions on the geometry of the domain – including a very weak compatibility condition between the Dirichlet boundary part and its complement – we prove Hölder continuity of the solution in space and time.

1 Introduction

This paper is concerned with parabolic initial-boundary value problems including mixed boundary conditions of the type

$$\begin{aligned} u'(t,x) - \operatorname{div}(\mu(x)\nabla u)(t,x) &= f(t,x), && \text{in } (0,T)\times\Omega, \\ u(t,x) &= 0, && \text{on } (0,T)\times D, \\ \mu(x)(\nabla u)(t,x)\cdot\nu(x) &= 0, && \text{on } (0,T)\times\Upsilon, \\ u(0,x) &= u_0, && \text{in } \Omega, \end{aligned} \tag{1}$$

where $D\subset\partial\Omega$ and $\Upsilon=\partial\Omega\setminus D$ are Dirichlet and Neumann boundary parts for the domain $\Omega\subset\mathbb{R}^d$ with outer normal vectors ν and $d\geq 2$. We show that both for all $f\in L^s((0,T);L^p(\Omega))$, with $p\in(\frac{d}{2},\infty)$ and for all $f\in L^s((0,T);W_D^{-1,q}(\Omega))$, with $q\in(d,\infty)$, and s sufficiently large, the problem is well-posed and there exists a $\beta>0$ such that the solution satisfies $u\in C^\beta((0,T)\times\Omega)$, that is the solution is Hölder continuous in space and time.

Hölder continuity is one of the classical features in the theory for parabolic equations, where we refer to the initial work of Nash and Moser, [Nas], [Mos2], [Mos1] and to the extensive theory for parabolic initial-boundary value problems developed in the monograph [LSU]. One of the main reasons for proving Hölder estimates is their usefulness in the investigation of nonlinear problems. In [LSU, p. 9] the domain Ω is assumed to be 'piecewise C^1 with nonzero interior angles'. In the standard work of Lieberman [Lie], the domain is assumed to be Lipschitz. The main novelty of our results lies in reducing the assumptions on the parts D and Υ of the boundary $\partial\Omega$ to include rough settings. For the Dirichlet part D, we merely require the outer volume condition (see e.g. [KS, Chapter II Theorem B.4]), which is classical for the elliptic pure Dirichlet problem. In particular, the domain may be rough in that the inner volume condition is not required. The second achievement is that we can considerably weaken the conditions on the relative boundary of the Dirichlet and Neumann boundary part in that we replace the geometrical condition established in [Grö] (compare also [GKR] [HiR], [Gri1], [Gri2]) by a measure theoretic one. Roughly speaking, it states that in balls around points in the intersection $D\cap\overline{\Upsilon}$, the Dirichlet boundary part is not rare (in a certain quantitative sense) with respect to the boundary measure, see (2) below. This reflects the fact that Hölder continuity for the elliptic Dirichlet problem also requires only a measure theoretic assumption [Sta]. Our framework is thus much broader than the classical one and allows for interesting new cases. In particular, the Dirichlet boundary part need not be (part of) a continuous boundary in the sense of

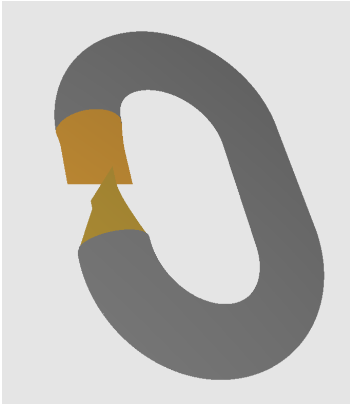

Figure 1: The light coloured part of the boundary carries Dirichlet boundary conditions, the dark coloured part has Neumann boundary conditions.

[Gri, Definition 1.2.1.1] and the domain is not required to 'lie on one side of the Dirichlet boundary part', see Figure 1.

Under these more general assumptions on the geometry, we essentially reproduce the classical parabolic Hölder theory in [LSU], however, in our case, in the standard Hölder spaces, but with the coefficients independent of time. This rests on the fact that our prescribed Dirichlet data are identically zero, whereas in [LSU], more general data are admitted.

Our paper is an extension of the results provided in [ER2], where the geometric setting was developed and where Hölder regularity for the elliptic problem as well as Gaussian Hölder kernel bounds on the semigroup were proved. We show that space-time Hölder continuity for the parabolic problem with $f \in L^s((0,T); L^p(\Omega))$ follows essentially by employing maximal parabolic regularity and interpolation.

The second main point of this paper is to study the case $f \in L^s((0,T); W_D^{-1,q}(\Omega))$ and provide a similar result. A motivation for including distributional right-hand-sides is given at the beginning of Section 3 and we refer to Section 2 for a precise definition of $W_D^{-1,q}(\Omega)$. The method of proof transfers from the L^p-setting to the $W_D^{-1,q}$-setting due to an abstract relation of the fractional powers of the elliptic operator considered in L^p and in $W_D^{-1,q}$, respectively, see Lemma 4.7. We prove this property under slightly less general assumptions on D and on the coefficient function μ, cf. Assumptions 4.1 and 4.3.

The outline of the paper is as follows. In Section 2 we provide basic definitions, the main assumptions and preliminary results. In Section 3 we study problem (1) in $L^s((0,T); L^p(\Omega))$ and prove that solutions are Hölder continuous in space and time. In Section 4 a similar result is proved for $f \in L^s((0,T); W_D^{-1,q}(\Omega))$.

2 Preliminary results

Fix $d \in \{2, 3, \ldots\}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let D be a closed subset of $\partial\Omega$. We define

$$C_D^\infty(\Omega) := \{w|_\Omega : w \in C_c^\infty(\mathbb{R}^d) \text{ and } \operatorname{supp} w \cap D = \emptyset\}.$$

Note that if $D = \partial\Omega$, then $C_{\partial\Omega}^\infty(\Omega) = C_c^\infty(\Omega)$. For all $p \in [1, \infty)$, we denote the closure of $C_D^\infty(\Omega)$ in $W^{1,p}(\Omega)$ by $W_D^{1,p}(\Omega)$, where $W^{1,p}(\Omega)$ is the usual complex Sobolev space of order 1. If $p \in (1, \infty]$, then the space $W_D^{-1,p}(\Omega)$ is the anti-dual of $W_D^{1,p'}(\Omega)$ in $L^p(\Omega)$, where p' is the conjugate index for p, so $\frac{1}{p} + \frac{1}{p'} = 1$. The domain Ω remains fixed throughout the paper, and hence we omit Ω in the notation of all function spaces. For example, we write L^p instead of $L^p(\Omega)$.

We always assume that the coefficient function $\mu \colon \Omega \to \mathbb{R}^{d \times d}$ is bounded, measurable and satisfies the ellipticity condition, that is, there exists a $\underline{\mu} > 0$ such that

$$\operatorname{Re} \sum_{i,j=1}^{d} \mu_{ij}(x)\, \xi_i\, \overline{\xi_j} \geq \underline{\mu}\, |\xi|^2$$

for almost all $x \in \Omega$ and for all $\xi \in \mathbb{C}^d$, where $\mu_{ij}(x)$ denote the matrix coefficients of $\mu(x)$ in Euclidean coordinates. We define the sesquilinear form $\mathfrak{l} \colon W_D^{1,2} \times W_D^{1,2} \to \mathbb{C}$ by

$$\mathfrak{l}(u, v) = \int_\Omega \sum_{i,j=1}^{d} \mu_{ij}\, (\partial_i u)\, \overline{(\partial_j v)}.$$

Then $\mathfrak{l}$ is closed and sectorial. Next we define $\mathcal{A} \colon W_D^{1,2} \to W_D^{-1,2}$ by

$$\langle \mathcal{A}u, v \rangle = \mathfrak{l}(u, v).$$

If $q \in (2, \infty)$, then define the operator $\mathcal{A}_q \colon \operatorname{Dom}(\mathcal{A}_q) \to W_D^{-1,q}$ by

$$\operatorname{Dom}(\mathcal{A}_q) = \{u \in W_D^{1,2} : \mathcal{A}u \in W_D^{-1,q}\}$$

and $\mathcal{A}_q u = \mathcal{A}u$ for all $u \in \operatorname{Dom}(\mathcal{A}_q)$. We consider $\mathcal{A}_q$ as an unbounded operator in $W_D^{-1,q}$. Similarly, let A be the m-sectorial operator associated with $\mathfrak{l}$ in L^2.

Remark 2.1. If Ω satisfies suitable regularity conditions, then the elements $u \in \operatorname{Dom}(A)$ satisfy the conditions $u|_D = 0$ in the sense of traces and $\nu \cdot (\mu \nabla u) = 0$ on $\partial\Omega \setminus D$ in a generalized sense, cf. [Cia, Chapter 1.2], [GGZ, Chapter II.2]) or [Lio, Chapter 3.3.2]. Thus, the operator A realizes mixed boundary conditions and provides solutions for (1) in a generalized sense.

We call D the Dirichlet (boundary) part and

$$\Upsilon := \partial\Omega \setminus D$$

the Neumann (boundary) part of $\partial\Omega$.

It is easy to see that the form $\mathfrak{l}$ satisfies the Beurling–Deny criteria (see [EMR, Corollary 2.17], [HKR, Section 2.3] or [Ouh, Section 4.3]). Hence the semigroup S generated by

$-A$ extends consistently to a contraction semigroup S^p in $L^p(\Omega)$ for all $p \in [1,\infty]$ and S^p is a C_0-semigroup for all $p \in [1,\infty)$. Let $-A_p$ denote the generator of S^p. If $p \in (2,\infty)$ then $\mathrm{Dom}(A_p) = \{u \in \mathrm{Dom}(A) \cap L_p(\Omega) : Au \in L_p(\Omega)\}$ and if $p \in [1,2)$ then A_p is the closure of A. If no confusion is possible, then we write $S = S^p$.

We denote by

$$E = \{x = (\tilde{x}, x_d) : -1 < x_d < 1 \text{ and } \|\tilde{x}\|_{\mathbb{R}^{d-1}} < 1\}$$

the open cylinder in $\mathbb{R}^d$. Its lower half is denoted by $E^- = \{x \in E : x_d < 0\}$ and

$$P = E \cap \{x \in \mathbb{R}^d : x_d = 0\}$$

is its midplate. Furthermore, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$ let $B^n_R(x)$ denote the ball in $\mathbb{R}^n$ with radius R and centre x. By $\mathcal{H}_n$ we denote the n-dimensional Hausdorff measure. We denote the volume of a measurable subset $F \subset \mathbb{R}^d$ by $|F|$ and the volume of a measurable subset $F \subset \mathbb{R}^{d-1}$ by $\mathrm{mes}_{d-1}(F)$.

Let $\Omega \subset \mathbb{R}^d$ be open, $M \subset \partial\Omega$ and $\alpha \in (0,1]$. Then, following [KS, Definition II.C.1] and [LSU, Section 1.1], we say that M is of class $(\mathbf{A}_\alpha)$ if

$$|B^d_R(x) \setminus \Omega| \geq \alpha\, |B^d_R(x)|$$

for all $R \in (0,1]$ and $x \in M$. It is not hard to see that the boundary of any Lipschitz domain is of class $(\mathbf{A}_\alpha)$ for a suitable $\alpha > 0$. Finally, let us recall the concept of a positive operator, cf. [Tri, Subsection 1.14.1].

Definition 2.2. A densely defined operator B on a Banach space X is called *positive*, if there is a $c > 0$ such that

$$\|(B+\lambda)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{1+\lambda}$$

for all $\lambda \in [0,\infty)$.

We next introduce three assumptions on the domain Ω and its boundary $\partial\Omega$. Recall that $\Upsilon = \partial\Omega \setminus D$ is the Neumann part of $\partial\Omega$.

Assumption 2.3. For all $x \in \overline{\Upsilon}$ there is an open neighbourhood U_x and a bi-Lipschitz map ϕ_x from a neighbourhood of $\overline{U_x}$ onto an open subset of $\mathbb{R}^d$, such that $\phi_x(U_x) = E$, $\phi_x(\Omega \cap U_x) = E^-$, $\phi_x(\partial\Omega \cap U_x) = P$ and $\phi_x(x) = 0$.

Assumption 2.4. There is an $\alpha > 0$ such that the set D is of class $(\mathbf{A}_\alpha)$.

Assumption 2.5. Let $\partial\Upsilon$ be the boundary of Υ in $\partial\Omega$. For all $x \in \partial\Upsilon$, there are $c_0 \in (0,1)$ and $c_1 > 0$ such that

$$\mathrm{mes}_{d-1}\{\tilde{z} \in B^{d-1}_R(\tilde{y}) : \mathrm{dist}(\tilde{z}, \phi_x(\Upsilon \cap U_x)) > c_0\, R\} \geq c_1\, R^{d-1} \tag{2}$$

for all $R \in (0,1]$ and $\tilde{y} \in \mathbb{R}^{d-1}$ with $(\tilde{y}, 0) \in \phi_x(\partial\Upsilon \cap U_x)$, where U_x and ϕ_x are as in Assumption 2.3.

We would like to remark on two consequences of these assumptions.

Remark 2.6. Assumptions 2.3 and 2.4 exclude the presence of cracks in Ω as these cracks would include boundary points which satisfy neither the $(\mathbf{A}_\alpha)$-condition nor do they allow for a Lipschitz chart satisfying Assumption 2.3.

Remark 2.7. Assumption 2.5 implies the 'lower bound' in the Ahlfors–David condition (cf. [JW, Chapter II]), i.e. there is a $\check{c}_1 > 0$ such that

$$\mathcal{H}_{d-1}(D \cap B_R^d(x)) \geq \check{c}_1 R^{d-1}$$

for all $x \in D$ and $R \in (0,1]$. See also [ER2, Lemma 5.4].

In the sequel we collect results of foregoing papers which will enable us to prove parabolic Hölder estimates. The following result was shown in [ER2, Theorem 1.1].

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and D a closed subset of the boundary $\partial\Omega$. Suppose that Assumptions 2.3, 2.4 and 2.5 are valid. Then for all $q \in (d,\infty)$ there exists a $\kappa > 0$ such that $\mathrm{Dom}(\mathcal{A}_q) \subset C^\kappa$.*

The next theorem concerns properties of A_p and the semigroup S generated by $-A$.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and D a closed subset of the boundary $\partial\Omega$. Suppose that Assumption 2.3 holds true. Then one has the following.*

(a) *For all $p \in (1,\infty)$ and $\lambda_0 > 0$ the operator $A_p + \lambda_0$ has a bounded H^∞-calculus. In particular, $A_p + \lambda_0$ is a positive operator with bounded imaginary powers.*

(b) *If $p \in (1,\infty)$, then A_p has maximal parabolic $L^s((0,T);L^p)$-regularity.*

(c) *The semigroup S has a kernel K satisfying Gaussian upper bounds. Stronger: for all $\omega > 0$ there are $b, c > 0$ such that*

$$|K_t(x,y)| \leq c\, t^{-d/2}\, e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t} \tag{3}$$

for all $x, y \in \Omega$ and $t > 0$.

(d) *For all $\omega > 0$ there exists a $c > 0$ such that*

$$\|S_t\|_{\mathcal{L}(L^p,L^r)} \leq ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}e^{\omega t} \tag{4}$$

for all $t \in (0,\infty)$ and $p, r \in [1,\infty]$ with $p \leq r$.

Proof. Since S^p is a contraction semigroup, Statement (a) follows from [Cow, LX, LeM] and Statement (b) from [Lam].

By [ER1, Theorem 3.1] there are $b, c, \omega > 0$ such that (3) is valid for all $x, y \in \Omega$ and $t > 0$. Hence there are $c, \omega > 0$ such that (4) is valid for all $t \in (0,\infty)$ and $p, r \in [1,\infty]$ with $p \leq r$. Since S is a contraction semigroup on L^2, the bounds on $\|S_t\|_{\mathcal{L}(L^2,L^\infty)}$ can be improved by using [Ouh, Lemma 6.5] and there exists a $c > 0$ such that $\|S_t\|_{\mathcal{L}(L^2,L^\infty)} \leq ct^{-d/4}(1+t)^{d/4}$ for all $t > 0$. Duality gives that there exists a $c > 0$ such that

$$\|S_t\|_{\mathcal{L}(L^1,L^\infty)} \leq ct^{-d/2}(1+t)^{d/2} \leq c\varepsilon^{-d/2}t^{-d/2}e^{\varepsilon dt/2}$$

for all $t > 0$ and $\varepsilon \in (0,1]$. Since $|K_t(x,y)| \leq \|S_t\|_{\mathcal{L}(L^1,L^\infty)}^{1-\varepsilon}|K_t(x,y)|^\varepsilon$ the Gaussian bounds of [ER1, Theorem 3.1] give Statement (c). Then Statement (d) follows directly from Statement (c). $\square$

The last two statements in the following theorem are corollaries to the results in [ER2].

Theorem 2.10. *In addition to the assumptions of Theorem* 2.9*, suppose that Assumptions* 2.4 *and* 2.5 *are valid. Then one has the following.*

(a) *The kernel K of the semigroup S satisfies Gaussian Hölder kernel bounds, i.e. there are $\kappa_*, b, c, \omega > 0$ such that*

$$|K_t(x,y) - K_t(x',y')| \le c\, t^{-d/2} \Big(\frac{|x-x'| + |y-y'|}{t^{1/2}} \Big)^{\kappa_*} e^{-b\frac{|x-y|^2}{t}} \, e^{\omega t} \tag{5}$$

for all $x,x',y,y' \in \Omega$ and $t > 0$ with $|x-x'| + |y-y'| \le t^{1/2}$.

(b) *There exists a $c > 0$ such that*

$$\|S_t\|_{\mathcal{L}(L^p, C^\kappa)} \le c\, t^{-\frac{d}{2p} - \frac{\kappa}{2}} e^{\omega t}$$

for all $p \in [1,\infty]$, $\kappa \in (0,\kappa_*]$ and $t \in (0,\infty)$, where κ_* and ω are as in (a)*.*

(c) *Let κ_* be as in* (a)*. Then*

$$\mathrm{Dom}\big(A_p^\theta\big) \hookrightarrow C^\kappa.$$

for all $p \in [1,\infty)$, $\kappa \in (0,\kappa_*]$ and $\theta \in (\frac{d}{2p} + \frac{\kappa}{2}, \infty)$.

Proof. Statement (a) was shown in [ER2, Theorem 7.5]. We next show Statement (b). Let $u \in L^p$ and let $x, x' \in \Omega$ with $0 < |x - x'| \le 1$. Let $t > 0$. We consider two cases.
Case 1. Suppose that $|x-x'| \le t^{1/2}$.
Then (5) implies that

$$\begin{aligned} |(S_t u)(x) - (S_t u)(x')| &\le \int_\Omega |K_t(x,y) - K_t(x',y)| \, |u(y)| \, dy \\ &\le c \Big(\frac{|x-x'|}{t^{1/2}} \Big)^\kappa \int_\Omega t^{-d/2} e^{-b\frac{|x-y|^2}{t}} e^{\omega t} |u(y)| \, dy \\ &\le c \Big(\frac{|x-x'|}{t^{1/2}} \Big)^\kappa t^{-\frac{d}{2p}} e^{\omega t} \|u\|_{L^p}, \end{aligned}$$

where the last step follows from the Hölder inequality.
Case 2. Suppose that $|x - x'| \ge t^{1/2}$.
Then trivially,

$$|(S_t u)(x) - (S_t u)(x')| \le 2 \|S_t u\|_{L^\infty} \le 2c t^{-\frac{d}{2p}} e^{\omega t} \|u\|_{L^p} \le 2c \Big(\frac{|x-x'|}{t^{1/2}} \Big)^\kappa t^{-\frac{d}{2p}} e^{\omega t} \|u\|_{L^p},$$

where in the second step (4) was used with $r = \infty$.

A combination of both cases implies Statement (b).

Statement (c) follows from Statement (b) and the integral representation

$$B^{-\theta} = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta - 1} e^{-tB} \, dt,$$

(see [Paz, (2.6.9)]), applied to $B = A_p + \omega + 1$. We obtain $\mathrm{Dom}((A_p + \omega + 1)^\theta) \hookrightarrow C^\kappa$. But $\mathrm{Dom}((A_p + \omega + 1)^\theta) = \mathrm{Dom}\big(A_p^\theta\big)$ with equivalent norms. □

Remark 2.11. By Theorem 2.9(a) the operator $A_p + 1$ admits bounded imaginary powers. Hence

$$\mathrm{Dom}(A_p^\theta) = [L^p, \mathrm{Dom}(A_p + 1)]_\theta$$

by [Tri, Theorem 1.15.3], if in addition $\theta < 1$. So $[L^p, \mathrm{Dom}(A_p)]_\theta \hookrightarrow C^\kappa$ by Theorem 2.10(c).

3 Hölder regularity for parabolic problems in L^p

We interpret parabolic problems of the form (1) as the abstract Cauchy problems associated to A. Our first theorem is the following.

Theorem 3.1. *Adopt the notation and assumptions as in Theorem* 2.10. *Let* κ_* *be as in Theorem* 2.10(a). *Let* $T > 0$ *and write* $J = (0, T)$. *Let* $p \in (\frac{d}{2}, \infty)$, $\kappa \in (0, \kappa_*]$, $\theta \in (0, 1)$ *and* $s \in (1, \infty)$. *Suppose that*

$$\frac{d}{2p} + \frac{\kappa}{2} < \theta < 1 - \frac{1}{s}.$$

Then there are $c > 0$ *and* $\beta \in (0, 1)$ *such that the following is valid. Let* $f \in L^s(J; L^p)$ *and* $u_0 \in X_{s,p} := (L^p, \mathrm{Dom}(A_p))_{1-\frac{1}{s},s}$. *Then any solution* u *of the equation*

$$u' + Au = f \in L^s(J; L^p), \quad u(0) = u_0, \tag{6}$$

belongs to $C^\beta(J; C^\kappa)$ *and*

$$\|u\|_{C^\beta(J;C^\kappa)} \le c(\|f\|_{L^s(J;L^p)} + \|u_0\|_{X_{s,p}}),$$

where $\beta = 1 - \frac{1}{s} - \theta$.

Note that

$$C^\beta(J; C^\kappa) \subset C^{\min(\beta,\kappa)}(J \times \Omega).$$

In preparation for the proof of this theorem, we first recall the notion of maximal parabolic regularity.

Definition 3.2. Let $s \in (1, \infty)$ and let X be a Banach space. Assume that B is a densely defined closed operator in X. Let $T > 0$ and set $J = (0, T)$. We say that B satisfies *maximal parabolic* $L^s(J; X)$ *regularity*, if there is an isomorphism which maps every $f \in L^s(J; X)$ to the unique function $u \in W^{1,s}(J; X) \cap L^s(J; \mathrm{Dom}(B))$ satisfying

$$u' + Bu = f, \qquad u(0) = 0.$$

Remark 3.3. We recall the following results associated to Definition 3.2.

- The property of maximal parabolic $L^s(J; X)$ regularity of an operator B is independent of the summability index $s \in (1, \infty)$ and the choice of T for the interval J, cf. [Dor]. We will say for short that B admits maximal parabolic regularity on X.

- If an operator satisfies maximal parabolic regularity on a Banach space X, then its negative generates an analytic semigroup on X, cf. [Dor]. In particular, a suitable left half-plane belongs to its resolvent set.

- Let X be a Banach space and let $s \in (1, \infty)$ and $T > 0$. Set $J = (0, T)$. Let B be an operator in X which admits maximal parabolic regularity. Then there exists a $c > 0$ such that for all $f \in L^s(J; X)$ and $u_0 \in (X, \mathrm{Dom}(B))_{1-\frac{1}{s},s}$ there exists a unique $u \in W^{1,s}(J; X) \cap L^s(J; \mathrm{Dom}(B))$ such that

$$u' + Bu = f, \qquad u(0) = u_0.$$

Moreover,

$$\|u\|_{W^{1,s}(J;X)\cap L^s(J;\mathrm{Dom}(B))} \le c(\|f\|_{L^s(J;X)} + \|u_0\|_{(X,\mathrm{Dom}(B))_{1-\frac{1}{s},s}}),$$

cf. [Ama3, Proposition 2.1 (i)⇒(iii)].

The space of maximal parabolic regularity allows for the following embedding results.

Lemma 3.4. *Let X, Y be Banach spaces and assume that Y is continuously embedded into X. Let $T > 0$ and set $J = (0,T)$.*

(a) *If $s \in (1,\infty)$, then*

$$W^{1,s}(J;X) \cap L^s(J;Y) \hookrightarrow C(\overline{J};(X,Y)_{1-\frac{1}{s},s}).$$

(b) *If $s \in (1,\infty)$ and $\theta \in (0, 1-\frac{1}{s})$, then*

$$W^{1,s}(J;X) \cap L^s(J;Y) \hookrightarrow C^\beta(J;(X,Y)_{\theta,1}),$$

where $\beta = 1 - \frac{1}{s} - \theta$.

Proof. The first part of the lemma is proved in [Ama1, Chapter III, Theorem 4.10.2].

In order to prove (b), we first note that

$$\begin{aligned}
\|w(t_1) - w(t_2)\|_X = \|\int_{t_1}^{t_2} w'(t)\,dt\|_X &\le \int_{t_1}^{t_2} \|w'(t)\|_X\,dt \\
&\le \Big(\int_{t_1}^{t_2} \|w'(t)\|_X^s\,dt\Big)^{1/s} |t_1 - t_2|^{1-1/s} \\
&\le \Big(\int_J \|w'(t)\|_X^s\,dt\Big)^{1/s} |t_1 - t_2|^{1-1/s} \\
&\le \|w\|_{W^{1,s}(J;X)} |t_1 - t_2|^{1-1/s}
\end{aligned}$$

for all $w \in W^{1,s}(J;X)$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Moreover, since $0 < \theta < 1 - \frac{1}{s}$, the reiteration theorem [Tri, Theorem 1.10.2] gives

$$(X,Y)_{\theta,1} = (X,(X,Y)_{1-\frac{1}{s},s})_{\lambda,1},$$

where $\lambda := \frac{\theta}{1-\frac{1}{s}} < 1$. Then $\beta = (1-\lambda)(1-\frac{1}{s})$ and

$$\begin{aligned}
\frac{\|w(t_1) - w(t_2)\|_{(X,Y)_{\theta,1}}}{|t_1 - t_2|^\beta} &\le \frac{\|w(t_1) - w(t_2)\|_X^{1-\lambda}}{|t_1 - t_2|^\beta} \Big(\sum_{j=1}^2 \|w(t_j)\|_{(X,Y)_{1-\frac{1}{s},s}}\Big)^\lambda \\
&\le \Big(\frac{\|w(t_1) - w(t_2)\|_X}{|t_1 - t_2|^{1-\frac{1}{s}}}\Big)^{1-\lambda} 2\big(\sup_{t\in J} \|w(t)\|_{(X,Y)_{1-\frac{1}{s},s}}\big)^\lambda \\
&\le 2c^\lambda \|w\|_{W^{1,s}(J;X)\cap L^s(J;Y)},
\end{aligned}$$

where c is the norm of the inclusion in Statement (a). □

Remark 3.5. As (a), also (b) in Lemma 3.4 is known, cf. [Ama2, Theorem 3], but our proof is elementary.

Proof of Theorem 3.1. By Theorem 2.9(b) the operator A_p has maximal parabolic regularity in L^p. Therefore the third point in Remark 3.3 gives that the solution u of (6) belongs to the space $W^{1,s}(J; L^p) \cap L^s(J; \mathrm{Dom}(A_p))$ with the estimate

$$\|u\|_{W^{1,s}(J,L^p)\cap L^s(J;\mathrm{Dom}(A_p))} \le c(\|f\|_{L^s(J;L^p)} + \|u_0\|_{X_{s,p}})$$

for a suitable $c > 0$. Putting $X := L^p$ and $Y := \mathrm{Dom}(A_p)$, Lemma 3.4(b) gives $u \in C^\beta(J; (L^p, \mathrm{Dom}(A_p))_{\theta,1})$ including the estimate

$$\|u\|_{C^\beta(J;(L^p,\mathrm{Dom}(A_p))_{\theta,1})} \le c_1 \|u\|_{W^{1,s}(J,L^p)\cap L^s(J;\mathrm{Dom}(A_p))} \le c\, c_1(\|f\|_{L^r(J;L^p)} + \|u_0\|_{X_{s,p}}) \qquad (7)$$

for a suitable $c_1 > 0$. Since $A_p + 1$ is a positive operator on L^p, we have the continuous embedding

$$(L^p, \mathrm{Dom}(A_p))_{\theta,1} = (L^p, \mathrm{Dom}(A_p + 1))_{\theta,1} \hookrightarrow \mathrm{Dom}\big((A_p + 1)^\theta\big) = \mathrm{Dom}(A_p^\theta)$$

by [Tri, Theorem 1.15.2(d)]. This, combined with (7) and Theorem 2.10(c), gives the claim. $\square$

4 Hölder regularity for parabolic problems in $W_D^{-1,q}$

The treatment of parabolic equations in L^p spaces is quite common; let us therefore start this section with some motivation for the consideration of parabolic equations in $W_D^{-1,q}$. If the right hand side of the equation (considered at any time point) has a Lebesgue density in the domain and if the boundary condition is either homogeneous or purely Dirichlet, then, e.g. L^p spaces are adequate. Naturally, spaces of type $W^{-1,q}$ come into play when the right-hand side is given by a distributional object, as e.g. surface charge densities or thermal sources, concentrated on a $(d-1)$-dimensional surface. These spaces may also be adequate for studying inhomogeneous Neumann boundary conditions, see [Lio, Chapter 3.2], for example, if the right-hand-side in the first equation in (1) is given by $f \in L^s(J; L^{m_0})$ and 0 on the right hand side of the third equation in (1) is replaced by a function $g \in L^s(J; L^{m_1}(\Upsilon))$ with suitable $m_0(d,q), m_1(d,q) \in [1,\infty)$, one can define $F \in L^s(J; W_D^{-1,q})$ by

$$F(t)(\phi) = \int_\Omega f(t)\phi + \int_\Upsilon g(t)\, \phi|_\Upsilon, \quad \text{for all } \phi \in W_D^{1,q'},$$

and choose F as the right-hand-side in the abstract Cauchy problem (see (9) below). Note also that in general, one cannot replace the condition $f \in L^s(J; W_D^{-1,q})$, where $q \in (d,\infty)$, by $f \in L^s(J; W_D^{-1,2})$, because this would not necessarily yield the regularity which is needed in particular for the treatment of non-linear problems. The aim of this section is to show that for all $q \in (d,\infty)$ the solutions of the parabolic problem in $W_D^{-1,q}$ are Hölder continuous in space and time.

In order to state the main result of this section, we must introduce additional assumptions on D and μ.

Assumption 4.1. Either $D = \emptyset$ or D satisfies the *Ahlfors–David condition*: There are constants $c_0, c_1 > 0$ and $r_{AD} > 0$, such that

$$c_0 R^{d-1} \le \mathcal{H}_{d-1}(D \cap B_R^d(x)) \le c_1 R^{d-1} \tag{8}$$

for all $x \in D$ and $R \in (0, r_{AD}]$.

Remark 4.2. Assumption 4.1 means the following.

(a) The set D is a $(d-1)$-set in the sense of Jonsson/Wallin [JW, Chapter II].

(b) On the set $\partial\Omega \cap \big(\bigcup_{x \in \partial\Upsilon} U_x\big)$, the measure $\mathcal{H}_{d-1}$ equals the surface measure σ which can be constructed via the bi-Lipschitz charts ϕ_x given in Assumption 2.3, cf. [EG, Subsection 3.3.4 C] or [HaR, Section 3]. In particular, (8) implies that $\sigma\big(D \cap \big(\bigcup_{x \in \partial\Upsilon} U_x\big)\big) > 0$, if $\partial\Omega \ne D \ne \emptyset$.

Assumption 4.3. $\mathrm{Dom}\big((A+1)\big)^{\frac{1}{2}} = W_D^{1,2}$.

Remark 4.4. Assumption 4.3 is *not* known for arbitrary non-symmetric coefficient functions under our general assumptions on the geometry of Ω and D. But many special cases are available:

(a) If Assumption 4.3 is satisfied for some coefficient function μ, then it is also true for the adjoint coefficient function, cf. [Kat, Theorems 1 and 2].

(b) Assumption 4.3 is always fulfilled if the coefficient function μ takes its values in the set of real *symmetric* $d \times d$-matrices.

(c) For results on non-symmetric coefficient functions, see [AKM]. By a recent result in [EHT, Theorem 4.1], Assumption 4.3 is valid in our geometric setting, if the domain Ω itself is a d-set, cf. [JW, Chapter II].

Let us now state the second main result of this paper.

Theorem 4.5. *Adopt the notation and assumptions as in Theorem* 2.10 *and, in addition, adopt Assumptions* 4.1 *and* 4.3. *Let* κ_* *be as in Theorem* 2.10(a). *Moreover, let* $q \in (d, \infty)$, $\kappa \in (0, \kappa_*]$, $\theta \in (0,1)$ *and* $s \in (1, \infty)$ *be such that* $\frac{d}{2q} + \frac{\kappa}{2} + \frac{1}{2} < \theta < 1 - \frac{1}{s}$. *Then there exists a* $c > 0$ *such that the following is valid. Let* $f \in L^s(J; W_D^{-1,q})$ *and* $u_0 \in X_{s,-1,q} := (W_D^{-1,q}, \mathrm{Dom}(\mathcal{A}_q))_{1-\frac{1}{s},s}$. *Then any solution* u *of the equation*

$$u' + \mathcal{A}_q u = f, \quad u(0) = u_0, \tag{9}$$

belongs to $C^\beta(J; C^\kappa)$ *and*

$$\|u\|_{C^\beta(J;C^\kappa)} \le c(\|f\|_{L^s(J;W_D^{-1,q})} + \|u_0\|_{X_{s,-1,q}}),$$

where $\beta = 1 - \frac{1}{s} - \theta$.

For the proof of this theorem, we need some additional results from [ABHR, Section 11].

Theorem 4.6. *Adopt Assumptions* 2.3, 4.1 *and* 4.3. *Let* $q \in [2, \infty)$. *Then one has the following.*

(a) $\mathcal{A}_q + 1$ *is a positive operator in* $W_D^{-1,q}$.

(b) $(\mathcal{A}_q + 1)^{-1/2}$ *provides a topological isomorphism between* $W_D^{-1,q}$ *and* L^q.

(c) $\mathcal{A}_q$ *admits maximal parabolic regularity in* $W_D^{-1,q}$.

We exploit Theorem 4.6 for the proofs of the following lemmas.

Lemma 4.7. *Adopt the notation and assumptions as in Theorem* 2.10 *and, in addition, adopt Assumptions* 4.1 *and* 4.3. *If* $\theta \in (\frac{1}{2}, 1)$, $\varsigma \in [1, \infty]$ *and* $q \in (2, \infty)$ *then*

$$(W_D^{-1,q}, \mathrm{Dom}(\mathcal{A}_q))_{\theta,\varsigma} = (L^q, \mathrm{Dom}(A_q))_{\theta-\frac{1}{2},\varsigma}.$$

Proof. Theorem 4.6(b) gives $\mathrm{Dom}((\mathcal{A}_q + 1)^{1/2}) = L^q$, which implies that $\mathrm{Dom}(\mathcal{A}_q + 1) = \mathrm{Dom}((A_q + 1)^{1/2})$. By Theorems 2.10(b) and 4.6(a) both operators, $\mathcal{A}_q + 1$ and $A_q + 1$, are positive in $W_D^{-1,q}$ and L^q, respectively. By [Tri, Subsection 1.10.1 and Theorem 1.15.2(d)] the space $\mathrm{Dom}(\mathcal{A}_q + 1)^{1/2}$ belongs to the class $J(\frac{1}{2}) \cap K(\frac{1}{2})$ between the spaces $W_D^{-1,q}$ and $\mathrm{Dom}(\mathcal{A}_q)$ and the space $\mathrm{Dom}(A_q + 1)^{1/2}$ belongs to the class $J(\frac{1}{2}) \cap K(\frac{1}{2})$ between the spaces L^q and $\mathrm{Dom}(A_q + 1)$. Therefore the reiteration theorem for real interpolation [Tri, Theorem 1.10.2] gives

$$\begin{aligned}
(W_D^{-1,q}, \mathrm{Dom}(\mathcal{A}_q))_{\theta,\varsigma} &= (\mathrm{Dom}((\mathcal{A}_q + 1)^{1/2}), \mathrm{Dom}(\mathcal{A}_q + 1))_{2\theta-1,\varsigma} \\
&= (L^q, \mathrm{Dom}((A_q + 1)^{1/2}))_{2\theta-1,\varsigma} \\
&= (L^q, \mathrm{Dom}(A_q + 1))_{\theta-\frac{1}{2},\varsigma}
\end{aligned}$$

as requested. □

Lemma 4.8. *Adopt the notation and assumptions as in Theorem* 2.10 *and, in addition, adopt Assumptions* 4.1 *and* 4.3. *Let* κ_* *be as in Theorem* 2.10(a), $\kappa \in (0, \kappa_*]$ *and* $\theta \in (0, 1)$ *with* $\theta > \frac{d}{2q} + \frac{\kappa}{2} + \frac{1}{2}$. *Then*

$$(W_D^{-1,q}, \mathrm{Dom}(\mathcal{A}_q))_{\theta,1} \hookrightarrow C^\kappa.$$

Proof. If follows from Lemma 4.7 and [Tri, Theorem 1.15.2(d)] that

$$(W_D^{-1,q}, \mathrm{Dom}(\mathcal{A}_q))_{\theta,1} = (L^q, \mathrm{Dom}(A_q + 1))_{\theta-\frac{1}{2},1} \subset \mathrm{Dom}((A_q + 1)^{\theta-\frac{1}{2}}).$$

Now an application of Theorem2.10(c) gives the claim. □

Proof of Theorem 4.5. By Theorem 4.6(c), the solution satisfies $u \in W^{1,s}(J, W_D^{-1,q}) \cap L^s(J; \mathrm{Dom}(\mathcal{A}_q))$ and there is a suitable $c > 0$ such that

$$\|u\|_{W^{1,s}(J,W_D^{-1,q}) \cap L^s(J;\mathrm{Dom}(\mathcal{A}_q))} \le c(\|f\|_{L^s(J;W_D^{-1,q})} + \|u_0\|_{X_{s,-1,q}}).$$

Next Lemma 3.4 gives

$$W^{1,s}(J; W_D^{-1,q}) \cap L^s(J; \mathrm{Dom}(\mathcal{A}_q)) \hookrightarrow C^\beta(J; (W_D^{-1,q}, \mathrm{Dom}(\mathcal{A}_q))_{\theta,1}).$$

Then the theorem is a consequence of Lemma 4.8. □

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