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Outer limit of subdifferentials and calmness moduli in linear and nonlinear programming

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Abstract

With a common background and motivation, the main contributions of this paper are developed in two different directions. Firstly, we are concerned with functions which are the maximum of a finite amount of continuously differentiable functions of n real variables, paying attention to the case of polyhedral functions. For these max-functions, we obtain some results about outer limits of subdifferentials, which are applied to derive an upper bound for the calmness modulus of nonlinear systems. When confined to the convex case, in addition, a lower bound on this modulus is also obtained. Secondly, by means of a KKT index set approach, we are also able to provide a point-based formula for the calmness modulus of the argmin mapping of linear programming problems without any uniqueness assumption on the optimal set. This formula still provides a lower bound in linear semi-infinite programming. Illustrative examples are given.

1 Introduction

The present paper was initially motivated by the problem of computing the *calmness modulus* of linear programs having optimal sets which are not a singleton. In relation to this problem, the immediate antecedents are gathered in [2], [3] and [4], where the assumption of the uniqueness of nominal optimal solution is essential. To this respect, we advance that an exact formula for the aimed modulus is obtained in Section 4 and that it is given in terms of the calmness moduli of certain sub-level multifunctions which are nothing else but feasible set mappings.

In the context of finite linear systems, the computation of the calmness modulus for feasible set mappings is dealt in [5] , where an operative expression (exclusively in terms of the nominal data) for this modulus is provided. With respect to this subject, the present work presents some extensions to the setting of $\mathcal{C}^1$-systems, where the constraints are described by continuously differentiable (sometimes convex) functions.

According to Theorem 1 below, the key ingredient in the computation (or estimation) of the calmness modulus for a $\mathcal{C}^1$-system at some feasible point is the outer limit of subdifferentials, by approaching this point from outside the feasible set, of a certain max-function associated with the system. Besides this original motivation and its application to calmness moduli, the problem of analyzing this outer limit is of independent interest, and it is tackled in the present paper in two stages: firstly, in the particular case of polyhedral functions and, in a second step, in the more general context of continuously differentiable functions. The reader is addressed to [2, Theorem 3.1] for a direct antecedent to this problem, when confined to the convex case (non necessarily differentiable).

The results about outer limits of subdifferentials obtained in the current work are applied to derive an upper bound on the calmness modulus of the feasible set mapping associated with a parameterized $\mathcal{C}^1$-system, under *right-hand-side* (RHS) *perturbations.* If, additionally, functions defining the constraints are convex, then we also derive a lower bound on the aimed calmness modulus. These results are inspired by the known exact formula for linear systems, which is recalled in Theorem 2 for completeness purposes. In this case of finite linear systems, it is well-known that the feasible set mapping is always calm at any point of its (polyhedral) graph as a consequence of a classical result by Robinson [21].

The paper also deals with the calmness of the *optimal set mapping* (also called *argmin mapping*), $\mathcal{S}$, in the framework of linear problems with *canonical perturbations*; i.e., where perturbations fall on the objective function coefficient vector and on the RHS of the constraints. The same result by Robinson ensures that mapping $\mathcal{S}$ is always calm at any point of its graph, since the KKT conditions allow us to express the graph of $\mathcal{S}$ as a finite union of polyhedral sets. This is no longer the case in the framework of perturbations of all data. In relation to this last framework, [3, Theorem 4.1] establishes a characterization for the calmness of the corresponding argmin mapping (by combining two results from the seminal paper [22]), and provides an operative upper bound for the corresponding calmness modulus, assuming the uniqueness of optimal nominal solution.

Comprehensive studies on calmness and other variational properties for generic multifunctions can be traced out from the monographs [7, 16, 20, 23]; see also [9, 12, 17, 15] in relation to the calmness of constraint systems in the context of RHS perturbations; where calmness translates into the existence of a *local error bound* for the corresponding supremum function (see [1], [8] and [18]). Other subdifferential approaches to calmness/local error bounds can be found in [11, 14].

The structure of the paper is as follows: Section 2 provides the necessary notation, definitions and preliminary results. Section 3 gathers the announced results on outer limits of subdifferentials of max-functions under different assumptions. It is divided into three subsections. The first one deals with the particular case of a polyhedral function where an exact formula is provided, while the second is focused on the nonlinear case. The third subsection provides the application to the estimation of the calmness modulus for the feasible set mapping in the context of $\mathcal{C}^1$-systems mentioned above, paying attention to the particular case of convex $\mathcal{C}^1$-systems. In Section 4, by means of a KKT index set approach, we provide an operative expression for the calmness modulus of $\mathcal{S}$ at a given nominal parameter in the case when the nominal optimal set does not necessarily reduce itself to a singleton. Moreover, we prove that this expression still remains as a lower bound in the semi-infinite continuous case, when the index set T is assumed to be a compact Hausdorff space and all the constraints' coefficients are continuous functions (with respect to the index) on T. The reader is addressed to [10, Chapter 10] for details about stability in this semi-infinite setting. Illustrative examples are provided in order to show that, in this general case (without uniqueness of nominal optimal solution), the referred expression may be strictly smaller than the upper bound given in [4, Theorem 7]. We finish the paper with a section of conclusions.

2 Preliminaries

In this section we introduce some notation, definitions and preliminary results which are needed later on. Given $A \subset \mathbb{R}^k$, we denote by $\mathrm{conv}A$ and $\mathrm{cone}A$ the *convex hull* and the *conical convex hull* of A, respectively. It is assumed that $\mathrm{cone}A$ always contains the zero-vector 0_k, in particular $\mathrm{cone}(\emptyset) = \{0_k\}$. If A is a subset of any topological space, $\mathrm{int}A$, $\mathrm{cl}A$ and $\mathrm{bd}A$ stand, respectively, for the interior, the closure, and the boundary of A. If $\|\cdot\|$ is any norm in $\mathbb{R}^k$, its corresponding *dual norm* is denoted by $\|\cdot\|_*$, i.e., $\|u\|_* = \max_{\|x\|\leq 1} |u'x|$.

In the next paragraphs we recall some definitions related to a generic mapping $\mathcal{M} : Y \rightrightarrows X$ between metric spaces (with distances denoted indistinctly by d). $\mathcal{M}$ is said to be *calm* at $(\overline{y}, \overline{x}) \in \mathrm{gph}\mathcal{M}$ (the graph of $\mathcal{M}$) if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d\left(x, \mathcal{M}\left(\overline{y}\right)\right) \leq \kappa d\left(y, \overline{y}\right) \tag{1}$$

whenever $x \in \mathcal{M}(y) \cap U$ and $y \in V$; where, as usual, $d(x, \Omega)$ is defined as $\inf\{d(x,z) \mid z \in \Omega\}$ for $\Omega \subset \mathbb{R}^n$, and $d(x, \emptyset) := +\infty$.

It is well-known that the calmness of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ is equivalent to the *metric subregularity* of the inverse multifunction $\mathcal{M}^{-1}$ at $(\overline{x}, \overline{y})$ (see, for instance, [7, Theorem 3H.3 and Exercise 3H.4]), which reads as follows: there exist a constant $\kappa \geq 0$ and a (possibly smaller) neighborhood U of $\overline{x}$ such that

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d\left(\overline{y}, \mathcal{M}^{-1}(x)\right), \text{ for all } x \in U. \tag{2}$$

The infimum of those $\kappa \geq 0$ for which (1) –or (2)– holds (for some associated neighborhoods) is called the *calmness modulus* of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ and is denoted by $\mathrm{clm}\mathcal{M}(\overline{y}, \overline{x})$. The case $\mathrm{clm}\mathcal{M}(\overline{y}, \overline{x}) = +\infty$ corresponds to the one in which $\mathcal{M}$ is not calm at $(\overline{y}, \overline{x})$.

2.1 Preliminaries on the feasible set mapping

We consider the parametrized $\mathcal{C}^1$- system

$$\sigma(b) := \{f_i(x) \leq b_i, \text{ for all } i = 1, ..., m\}, \tag{3}$$

and the associated feasible set mapping $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n \mid f_i(x) \leq b_i, \text{ for all } i = 1, ..., m\}, \tag{4}$$

where $f_i \in \mathcal{C}^1(\mathbb{R}^n)$ and $b_i \in \mathbb{R}$, $i = 1, ...m$. In this setting, $b \equiv (b_i)_{i=1,...m}$ is the parameter to be perturbed. The space of variables of the system, $\mathbb{R}^n$, is equipped with an arbitrary norm, while our *parameter space*, $\mathbb{R}^m$, is endowed with the supremum norm $\|b\|_\infty := \max_{i=1,...,m} |b_i|$, $b \in \mathbb{R}^m$.

Associated with system (3), we consider the max-function

$$g := \max_{1,...,m} g_i, \text{ where } g_i(x) = f_i(x) - b_i,\ i = 1, ..., m. \tag{5}$$

Throughout the paper, we appeal to the *set of active indices* at $x \in \mathcal{F}(b)$, denoted by $T_b(x)$ and defined as

$$T_b(x) := \{i \in \{1, ..., m\} \mid g_i(x) = 0\}.$$

If $T_b(x) = \emptyset$, x is a Slater point of $\sigma(b)$, and in this case one trivially has $\mathrm{clm}\mathcal{F}(b, x) = 0$. So, along the paper we assume that our nominal solution $\overline{x} \in \mathcal{F}(\overline{b})$ satisfies $T_{\overline{b}}(\overline{x}) \neq \emptyset$, or, equivalently,

$$g(\overline{x}) = 0.$$

The following theorem constitutes our starting point in the estimation of $\mathrm{clm}\mathcal{F}(\overline{b}, \overline{x})$. Statement (i) in this theorem comes from [8, Prop. 1, Prop. 11, Prop. 5 (ii)], whereas (ii) follows directly from [18, Theorem 1]. In it, we have taken into account the well-known relationship between $\mathrm{clm}\mathcal{F}(\overline{b}, \overline{x})$ and the *error bound modulus* of g at $\overline{x}$, specifically

$$\mathrm{clm}\mathcal{F}(\overline{b}, \overline{x}) = [\mathrm{Er}\ g(\overline{x})]^{-1}, \tag{6}$$

and the easily verifiable fact that

$$\liminf_{x \to \overline{x},\ g(x) > 0} d_*(0_n, \partial g(x)) = d_*\left(0_n, \limsup_{x \to \overline{x},\ g(x) > 0} \partial g(x)\right), \tag{7}$$

where d_* stands for the distance in $\mathbb{R}^n$ associated with $\|\cdot\|_*$ and ∂g represents the Clarke subdifferential of g.

Theorem 1 *Let* $(\overline{b}, \overline{x}) \in \mathrm{gph}\mathcal{F}$ *such that* $g(\overline{x}) = 0$. *Then:*

(i) *We have*

$$\mathrm{clm}\mathcal{F}(\overline{b}, \overline{x}) \leq \left[d_* \left(0_n, \limsup_{x \to \overline{x},\; g(x)>0} \partial g(x) \right) \right]^{-1}; \tag{8}$$

(ii) *If, additionally, functions* f_i *in* (5), $i = 1, ..., m$, *are convex, then*

$$\mathrm{clm}\mathcal{F}(\overline{b}, \overline{x}) = \left[d_* \left(0_n, \limsup_{x \to \overline{x},\; g(x)>0} \partial g(x) \right) \right]^{-1}. \tag{9}$$

Remark 1 In relation to the previous theorem let us comment that:

(i) With respect statement (i), [8, Prop. 1, Prop. 11, Prop. 5 (ii)] refers to the Fréchet subdifferential $\hat{\partial}$. In principle, from that results in [8] we deduce

$$\mathrm{Er}\, g(\bar{x}) \geq \liminf_{x \to \bar{x}, g(x)>0} d_* \left(0_n, \hat{\partial} g(x) \right).$$

However, in our case we may replace $\hat{\partial}$ by the Clarke subdifferential, ∂, as consequence of the Clarke regularity of g (see the beginning of Section 3), and then, taking also (6) and (7) into account, we obtain inequality (8).

(ii) Equality (9) is held under convexity, even without differentiability assumptions on the f_i's (see again [18, Theorem 1]), in which case, ∂g stands for the usual subdifferential of convex analysis.

Our next step is to obtain estimations for $\mathrm{clm}\mathcal{F}(\overline{b}, \overline{x})$ which only involve the nominal data. Having the previous theorem in mind, as advanced in Section 1, a way of tackling this problem consists of analyzing the outer limit inside. The next theorem, dealing with the case of linear systems, provides a motivation for some results of the following section (Theorems 4 and 5).

The following theorem deals with the linear case, in which the f_i's are given by

$$f_i(x) := a_i' x, \; i = 1, ..., m,$$

where $a_i \in \mathbb{R}^n$, $1, ..., m$, are fixed. Here, any vector $y \in \mathbb{R}^n$ is regarded as a column-vector, and y' denotes its transpose (hence $y'x$ stands for the usual inner product). In order to emphasize the difference between the linear and nonlinear contexts, the feasible set mapping in the particular case of linear systems will be denoted by $\mathcal{F}_a$; specifically,

$$\mathcal{F}_a(b) := \{ x \in \mathbb{R}^n \mid a_i' x \leq b_i, \text{ for all } i = 1, ..., m \}. \tag{10}$$

From now on, $\mathcal{D}_{\overline{b}}(\overline{x})$ denotes the family of all subsets $D \subset T_{\overline{b}}(\overline{x})$ such that system

$$\left\{ \begin{array}{ll} a_i' d = 1, & i \in D, \\ a_i' d < 1 & i \in T_{\overline{b}}(\overline{x}) \setminus D \end{array} \right\} \tag{11}$$

is consistent (in the variable $d \in \mathbb{R}^n$). In other words, $D \in \mathcal{D}_{\overline{b}}(\overline{x})$ if there exists a hyperplane containing $\{a_i, i \in D\}$ such that

$$\{0_n\} \cup \{a_i, i \in T_{\overline{b}}(\overline{x}) \setminus D\}$$

lies on one of the open half-spaces determined by this hyperplane.

Theorem 2 [5, Theorem 4]*Given* $\left(\overline{b},\overline{x}\right)\in\operatorname{gph}\mathcal{F}$, *we have*

$$\operatorname{clm}\mathcal{F}_a(\overline{b},\overline{x})=\left(\min_{D\in\mathcal{D}_{\overline{b}}(\overline{x})}d_*\left(0_n,\operatorname{conv}\left\{a_i,\ i\in D\right\}\right)\right)^{-1}.$$

2.2 Preliminaries on the argmin mapping

We consider the optimal set mapping $\mathcal{S}:\mathbb{R}^n\times\mathbb{R}^m\rightrightarrows\mathbb{R}^n$ in the linear framework under canonical perturbations, which is given by

$$\mathcal{S}\left(c,b\right):=\arg\min\{c'x\mid x\in\mathcal{F}_a\left(b\right)\}. \tag{12}$$

The parameter space, $\mathbb{R}^n\times\mathbb{R}^m$, is endowed with the norm

$$\|(c,b)\|:=\max\left\{\|c\|_*,\|b\|_\infty\right\},\ \ (c,b)\in\mathbb{R}^n\times\mathbb{R}^m. \tag{13}$$

The next theorem comes directly from [4, Theorem 7] and constitutes our starting point of Section 4. In it, associated with a given $\left((\overline{c},\overline{b}),\overline{x}\right)\in\operatorname{gph}\mathcal{S}$, we appeal to the following family of index subsets associated with the Karush-Kuhn-Tucker (KKT) conditions (hereafter referred to as *KKT index sets*):

$$\mathcal{K}_{\overline{c},\overline{b}}\left(\overline{x}\right)=\left\{D\subset T_{\overline{b}}\left(\overline{x}\right):|D|\leq n\text{ and }-\overline{c}\in\operatorname{cone}\left\{a_i,\ i\in D\right\}\right\},$$

where $|D|$ stands for the cardinality of D and condition $|D|\leq n$ comes from Carathéodory's Theorem. For any $D\in\mathcal{K}_{\overline{c},\overline{b}}\left(\overline{x}\right)$ we consider the mapping $\mathcal{L}_D:\mathbb{R}^m\times\mathbb{R}^D\rightrightarrows\mathbb{R}^n$ given by

$$\mathcal{L}_D\left(b,d\right):=\left\{x\in\mathbb{R}^n\mid a_i'x\leq b_i,\ i=1,...,m;\ -a_i'x\leq d_i,\ i\in D\right\}. \tag{14}$$

Observe that all preliminary results for the feasible set mappings $\mathcal{F}_a$ may be specified for $\mathcal{L}_D$, which is nothing else but the feasible set mapping associated with an enlarged system.

Theorem 3 [4, Theorem 7] *Let* $\left(\overline{c},\overline{b}\right)\in\mathbb{R}^n\times\mathbb{R}^m$. *Then*

$$\operatorname{clm}\mathcal{S}(\left(\overline{c},\overline{b}\right),\overline{x})=\operatorname{clm}\mathcal{S}_{\overline{c}}\left(\overline{b},\overline{x}\right)\leq\max_{D\in\mathcal{K}_{\overline{c},\overline{b}}(\overline{x})}\operatorname{clm}\mathcal{L}_D\left(\left(\overline{b},-\overline{b}_D\right),\overline{x}\right), \tag{15}$$

where $\overline{b}_D$ *stands for* $\left(\overline{b}_i\right)_{i\in D}$ *and* $\mathcal{S}_{\overline{c}}\left(b\right):=\mathcal{S}\left(\overline{c},b\right)$ for $b\in\mathbb{R}^m$.

Remark 2 Corollary 8 in [4] shows that (15) holds as an equality under the additional assumption that $\mathcal{S}\left(\overline{c},\overline{b}\right)=\{\overline{x}\}$.

3 Outer limits of subdifferentials and calmness modulus of differentiable convex systems

The present section is divided into three subsections. The first one, inspired by Theorem 2 (having also Theorem 1 in mind), establishes an exact expression for the outer limit of subdifferentials of polyhedral functions. The second is focussed on the more general case of max-functions in a nonlinear differentiable framework. The third applies some previous results to obtain estimations of the aimed $\operatorname{clm}\mathcal{F}(\overline{b},\overline{x})$ for convex and nonlinear systems.

We consider the max-function $g : \mathbb{R}^n \to \mathbb{R}$

$$g(x) := \max_{i=1,\ldots,m} g_i(x),$$

where the g_i's are continuously differentiable on $\mathbb{R}^n$. As a consequence, g is a regular function in the sense of Clarke (see, for instance, [23, Examples 10.24(e) and 10.25(a)]), and we have

$$\partial g(x) = \operatorname{conv}\{\nabla g_i(x) : i \in I(x)\}$$

(see [6, 2.3.12]), where

$$I(x) := \{i = 1, ..., m : g_i(x) = g(x)\}.$$

Note that always $I(x) \neq \emptyset$.

Lemma 1 *For each $x \in \mathbb{R}^n$ there exists $\varepsilon_x > 0$ such that*

$$I(z) \subset I(x) \text{ whenever } \|z - x\| < \varepsilon_x.$$

Proof. Reasoning by contradiction, suppose that there exist a sequence (z_k) such that $z_k \to x$ and an associated $i_k \in I(z_k) \backslash I(x)$. Since $i_k \in \{1, ..., m\}$, which is a finite set, there will exist $i_0 \in \{1, ..., m\}$ such that $i_k = i_0$ infinitely many times. If we restrict ourselves to the corresponding subsequence (without relabeling), we can write by continuity,

$$g_i(x) = \lim_{k\to\infty} g_i(z_k) = \lim_{k\to\infty} g(z_k) = g(x),$$

and we get the contradiction $i_0 \in I(x)$. ■

Finally, inspired by $\mathcal{D}_{\overline{b}}(\overline{x})$ (see (11)) we define the family $\mathcal{D}(x)$ formed by all subsets of indices $D \subset I(x)$ such that the system

$$\left\{ \begin{array}{ll} \nabla g_i(x)' d = 1, & i \in D, \\ \nabla g_i(x)' d < 1 & i \in I(x) \setminus D \end{array} \right\} \tag{16}$$

is consistent in the variable $d \in \mathbb{R}^n$.

3.1 Outer limits of subdifferentials of polyhedral functions

This subsection deals with the particular case when the g_i's are affine functions; i.e.,

$$g_i(x) := a_i' x - b_i, \ i = 1, ..., m,$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ are fixed. In this case, the corresponding max-function

$$g(x) := \max_{i=1,\ldots,m} a_i' x - b_i, \tag{17}$$

is a *polyhedral function*, and its subdifferential (in the sense of Clarke, which in this case coincides with the usual subdifferential of convex analysis) writes as

$$\partial g(x) = \operatorname{conv}\{a_i \mid i \in I(x)\}. \tag{18}$$

Theorem 4 *Let g be defined in (17). We have*

$$\limsup_{x\to\overline{x},\ g(x)>g(\overline{x})} \partial g\left(x\right) = \bigcup_{D\in\mathcal{D}(\overline{x})} \operatorname{conv}\left\{a_i,\ i\in D\right\}.$$

Proof. First, let us prove the '$\supset$' inclusion. Pick any $D\in\mathcal{D}\left(\overline{x}\right)$ and consider $d\in\mathbb{R}^n$ such that (16) fulfills. Then, for any $\alpha>0$ one has

$$\left.\begin{array}{ll} g_i(\overline{x}+\alpha d)=a_i'\left(\overline{x}+\alpha d\right)-b_i=g_i(\overline{x})+\alpha=g(\overline{x})+\alpha & \text{for } i\in D \\ g_i(\overline{x}+\alpha d)=a_i'\left(\overline{x}+\alpha d\right)-b_i<g_i(\overline{x})+\alpha=g(\overline{x})+\alpha & \text{for } i\in I\left(\overline{x}\right)\setminus D \end{array}\right\}. \tag{19}$$

Suppose $0<\alpha\left\|d\right\|<\varepsilon_{\overline{x}}$ (defined in Lemma 1). Then, Lemma 1 together with (19) ensure $g\left(\overline{x}+\alpha d\right)=g(\overline{x})+\alpha$ and

$$\partial g\left(\overline{x}+\alpha d\right)=\operatorname{conv}\left\{a_i,\ i\in D\right\}.$$

Therefore

$$\operatorname{conv}\left\{a_i,\ i\in D\right\}=\limsup_{\alpha\to 0^+}\partial g\left(\overline{x}+\alpha d\right)\subset\limsup_{x\to\overline{x},\ g(x)>g(\overline{x})}\partial g\left(x\right).$$

In order to prove the '$\subset$' inclusion, take any $u\in\limsup_{x\to\overline{x},\ g(x)>g(\overline{x})}\partial g\left(x\right)$. Let us write $u=\lim_{k\to\infty}u_k$ with $u_k\in\partial g\left(x_k\right), g\left(x_k\right)>g\left(\overline{x}\right)$ (for all $k\in\mathbb{N}$) and $x_k\to\overline{x}$ (without loss of generality we assume $\left\|x_k-\overline{x}\right\|<\varepsilon_{\overline{x}}$ for all k). Then the sequence $\left(I\left(x_k\right)\right)_{k\in\mathbb{N}}$ has a constant subsequence because $I\left(x_k\right)\subset\left\{1,...,m\right\}$ for all k. Accordingly, let us assume without loss of generality that

$$I\left(x_k\right)=D\subset I\left(\overline{x}\right)\ \text{ for all } k\in\mathbb{N},$$

where the last inclusion comes from Lemma 1. Since $\partial g\left(x_k\right)=\operatorname{conv}\left\{a_i,\ i\in D\right\}$ is a compact set (independent on k), we obtain

$$u\in\operatorname{conv}\left\{a_i,\ i\in D\right\}. \tag{20}$$

Pick any particular $k\in\mathbb{N}$ and define

$$d:=\frac{x_k-\overline{x}}{g\left(x_k\right)-g\left(\overline{x}\right)}.$$

Then, we have

$$a_i'd=\frac{a_i'x_k-a_i'\overline{x}}{g\left(x_k\right)-g\left(\overline{x}\right)}=\frac{g_i\left(x_k\right)-g_i\left(\overline{x}\right)}{g\left(x_k\right)-g\left(\overline{x}\right)}\left\{\begin{array}{ll}=1 & \text{for all } i\in D,\\ <1 & \text{for all } i\in I\left(\overline{x}\right)\setminus D.\end{array}\right.$$

Accordingly $D\in\mathcal{D}\left(\overline{x}\right)$, and the proof ends by appealing to (20). ■

Remark 3 Since (16) clearly implies that, for each $D\in\mathcal{D}\left(\overline{x}\right)$, $\operatorname{conv}\{a_i,\ i\in D\}$ is contained in a supporting hyperplane to $\operatorname{conv}\left\{a_i,\ i\in I\left(\overline{x}\right)\right\}$, it follows that

$$\begin{aligned}\limsup_{x\to\overline{x},\ g(x)>g(\overline{x})}\partial g\left(x\right) &= \bigcup_{D\in\mathcal{D}(\overline{x})}\operatorname{conv}\left\{a_i,\ i\in D\right\}\\ &\subset \operatorname{bd}\operatorname{conv}\left\{a_i,\ i\in I\left(\overline{x}\right)\right\})=\operatorname{bd}\partial g\left(\overline{x}\right).\end{aligned}$$

(The last equality comes from [2, Theorem 3.1]).

The next example shows that the previous inclusion may be strict.

Example 1 (see [5, Example 4]) Consider the system (in $\mathbb{R}^2$ endowed with the Euclidean norm)

$$\{x_1 \leq b_1,\ x_2 \leq b_2,\ x_1 + x_2 \leq b_3\},$$

and the nominal data $\overline{b} = 0_3$ and $\overline{x} = 0_2$. The associated supremum function is given by

$$g(x) = \max\{x_1,\ x_2,\ x_1 + x_2\},$$

and accordingly

$$\operatorname{bd}\operatorname{conv}\{a_i,\ i \in I(\overline{x})\} = \operatorname{conv}\{a_1,\ a_2\} \cup \operatorname{conv}\{a_1,\ a_3\} \cup \operatorname{conv}\{a_2,\ a_3\}.$$

However,

$$\bigcup_{D \in \mathcal{D}(\overline{x})} \operatorname{conv}\{a_i,\ i \in D\} = \operatorname{conv}\{a_1,\ a_3\} \cup \operatorname{conv}\{a_2,\ a_3\}.$$

3.2 Extensions to the nonlinear differentiable case

In the following theorem, associated with a fixed point $\overline{x} \in \mathbb{R}^n$, we appeal to the new family of subsets of indices

$$\mathcal{D}_{AI}(\overline{x}) \subset \mathcal{D}(\overline{x}), \tag{21}$$

formed by all $D \in \mathcal{D}(\overline{x})$ such that $\{\nabla g_i(\overline{x}),\ i \in D\}$ is affinely independent. Moreover, the theorem appeals several times to an standard argument of differential calculus which is isolated in the following lemma, where we use a dot standing for derivatives (recall that we are using the prime for transposition).

Lemma 2 *Let $h : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and consider $\alpha :]-a, a[\to \mathbb{R}^n$, with $a > 0$, such that*

$$\nabla h(\alpha(0))' \dot{\alpha}(0) > 0,$$

then, for any $\{t_k\} \downarrow 0$, there exists k_0 verifying

$$h(\alpha(t_k)) > h(\alpha(0)),\ \textit{for } k \geq k_0.$$

Proof. Consider the real function given by

$$\gamma(t) := h(\alpha(t)),\ -a < t < a.$$

We have that

$$\gamma(t) = \gamma(0) + \dot{\gamma}(0)\, t + o(t) = h(\alpha(0)) + t\left(\nabla h(\alpha(0))' \dot{\alpha}(0) + \frac{o(t)}{t}\right),$$

which entails the existence of $\varepsilon > 0$ such that

$$h(\alpha(t)) = \gamma(t) > \gamma(0) = h(\alpha(0)),\ \text{whenever } 0 < t < \varepsilon.$$

Then, the statement of the current lemma follows straightforwardly. ∎

Theorem 5 *Let* $g(x) := \max\limits_{i=1,...,m} g_i(x)$, *with* $g_i : \mathbb{R}^n \to \mathbb{R}$ *continuously differentiable for all* i, *and let* $\overline{x} \in \mathbb{R}^n$. *We have:*

(i) $\bigcup_{D\in\mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\} \subset \limsup\limits_{x\to\overline{x},\ g(x)>g(\overline{x})} \partial g(x)$;

(ii) $\limsup\limits_{x\to\overline{x},\ x\neq\overline{x}} \partial g(x) \subset \operatorname{bd} \partial g(\overline{x})$.

Moreover, the converse inclusion of (ii) *also holds if, for all supporting hyperplane* H *to* $\partial g(\bar{x})$, *we have that* $\{\nabla g_i(\bar{x}), i \in I(\bar{x})\} \cap H$ *is affinely independent.*

Proof. (i) Take any $D \in \mathcal{D}_{AI}(\overline{x})$.. For the sake of simplicity, let us assume that $D := \{1, 2, ..., i_0\}$ ($i_0 \leq n+1$ because of the definition of $\mathcal{D}_{AI}(\overline{x})$) and consider the system of equations

$$\{h_i(x) := g_{i+1}(x) - g_1(x) = 0,\ i = 1, ..., i_0 - 1\}. \tag{22}$$

Since $D \in \mathcal{D}_{AI}(\overline{x})$, there exists $d \in \mathbb{R}^n$ such that

$$\nabla g_i(\overline{x})' d = 1,\ i = 1, ..., i_0, \tag{23}$$

which entails

$$\nabla h_i(\overline{x})' d = 0,\ i = 1, ..., i_0 - 1.$$

Moreover, vectors $\nabla h_i(\overline{x})$, $i = 1, ..., i_0 - 1$, are linearly independent and $d \neq 0_n$ because of (23), which actually entails $i_0 \leq n$.

If we write system (22) in the vectorial form

$$h(x) = 0_{i_0-1}, \tag{24}$$

observe that $\overline{x}$ is a regular point of the surface S defined by (24). Moreover, if we denote by $\nabla h(\overline{x})$ the matrix whose columns are $\nabla h_1(\overline{x}), ..., \nabla h_{i_0-1}(\overline{x})$, we have

$$\nabla h(\overline{x})' d = 0_{i_0-1}.$$

Then, there exists a differentiable curve α such that the arc

$$\{\alpha(t),\ -a < t < a\} \subset S,\ (a > 0) \tag{25}$$

verifies

$$\alpha(0) = \overline{x} \text{ and } \overset{\bullet}{\alpha}(0) = d.$$

(See, e.g. [19, P. 325]).

Let us consider a sequence of scalars $0 < t_k < a$, $k \in \mathbb{N}$ such that $t_k \to 0$ and define

$$x_k := \alpha(t_k), \text{ for all } k.$$

From (25), we have

$$g_1(x_k) = ... = g_{i_0}(x_k),\ k \in \mathbb{N}. \tag{26}$$

Let $j \in I(\overline{x}) \diagdown D$ and consider the function

$$h_j := g_1 - g_j.$$

Observe that

$$\nabla h_j (\alpha (0))' \overset{\bullet}{\alpha} (0) = \nabla h_j (\overline{x})' d > 0,$$

which entails (by Lemma 2), for k sufficiently large,

$$h_j (x_k) = g_1 (x_k) - g_j (x_k) > 0.$$

By the previous inequality and taking (26) and Lemma 1 into account, we have that,

$$I (x_k) = D, \text{ for } k \text{ large enough,} \tag{27}$$

and then

$$\partial g (x_k) = \operatorname{conv}\{\nabla g_i (x_k) , \ i \in D\}.$$

Moreover, for $i \in D$, again applying Lemma 2, we have that

$$g_i (x_k) > g_i (\overline{x}) , \text{ for } k \text{ large enough,}$$

since $\nabla g_i (\overline{x})' d = 1 > 0$. Then,

$$g (x_k) > g (\overline{x}) , \text{ for } k \text{ large enough}$$

(recall (27)).

Finally,

$$\lim_{k\to\infty} \partial g (x_k) = \lim_{k\to\infty} \operatorname{conv} \{\nabla g_i (x_k) , \ i \in D\} = \operatorname{conv} \{\nabla g_i (\overline{x}) , \ i \in D\}$$

(with the limits being understood in the Painlevé-Kuratowski sense), yielding the aimed inclusion

$$\operatorname{conv} \{\nabla g_i (\overline{x}) , \ i \in D\} \subset \limsup_{x\to\overline{x}, \ g(x)>g(\overline{x})} \partial g (x) .$$

(ii) Let $u \in \limsup_{x\to\bar{x},x\neq\bar{x}} \partial g (x)$ be arbitrary. By definition, there exists a sequence $(x_k, u_k) \to (\bar{x}, u)$ such that $x_k \neq \bar{x}$ and $u_k \in \partial g (x_k)$. After passing to a subsequence which we do not relabel, we may assume without loss of generality that there exists an index set $I \subseteq I (\bar{x})$ and a vector $d \neq 0_n$ such that

$$I (x_k) = I \ \forall k \quad \text{and} \quad \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \to d.$$

Here, the first statement follows from the fact that $I (x_k) \subseteq I (\bar{x})$ for k large enough (recall Lemma 1) and that there exist only finitely many subsets of $I (\bar{x})$. From

$$u_k \in \partial g (x_k) = \operatorname{conv} \{\nabla g_i (x_k) \,|i \in I (x_k)\} = \operatorname{conv} \{\nabla g_i (x_k) \,|i \in I\}$$

it follows that $u_k = \sum_{i\in I} \lambda_i^k \nabla g_i (x_k)$ for some sequence $\lambda^k \in \mathbb{R}_+^{|I|}$ satisfying $\sum_{i\in I} \lambda_i^k = 1$ for all k. By compactness of the standard simplex in $\mathbb{R}^{|I|}$ we may assume again without loss of generality that, upon passing to another subsequence which we do not relabel, there exists some $\bar{\lambda} \in \mathbb{R}_+^{|I|}$ satisfying $\sum_{i\in I} \bar{\lambda}_i = 1$ such that $\lambda^k \to \bar{\lambda}$. Consequently,

$$u_k \to \sum_{i\in I} \bar{\lambda}_i \nabla g_i (\bar{x}) = u$$

showing that

$$u \in \operatorname{conv} \{\nabla g_i (\bar{x}) , \ i \in I\} . \tag{28}$$

Next, we prove the following relation involving the vector d introduced above:

$$\nabla g_i(\bar{x})' d \geq \nabla g_j(\bar{x})' d \quad \forall i \in I \ \forall j \in I(\bar{x}). \tag{29}$$

Indeed, if (29) were not true, then there existed $i \in I$ and $j \in I(\bar{x})$ such that $\nabla g_i(\bar{x})' d < \nabla g_j(\bar{x})' d$. Observe that $I \subseteq I(\bar{x})$ implies that $g_i(\bar{x}) = g_j(\bar{x})$. As $x_k \neq \bar{x}$, we get that

$$\begin{aligned}
\frac{g_i(x_k) - g_j(x_k)}{\|x_k - \bar{x}\|} &= \frac{g_i(x_k) - g_i(\bar{x})}{\|x_k - \bar{x}\|} - \frac{g_j(x_k) - g_j(\bar{x})}{\|x_k - \bar{x}\|} \\
&= (\nabla g_i(\bar{x}) - \nabla g_j(\bar{x}))' \left(\frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \right) + \frac{o(\|x_k - \bar{x}\|)}{\|x_k - \bar{x}\|},
\end{aligned}$$

where

$$\frac{o(\|x_k - \bar{x}\|)}{\|x_k - \bar{x}\|} \to 0.$$

It follows that

$$\frac{g_i(x_k) - g_j(x_k)}{\|x_k - \bar{x}\|} \to \nabla g_i(\bar{x})' d - \nabla g_j(\bar{x})' d < 0.$$

Consequently, $g_i(x_k) < g_j(x_k)$ for k large enough which entails the contradiction $i \notin I(x_k) = I$.

Now, (29) means that for all $i \in I$

$$\begin{aligned}
\nabla g_i(\bar{x}) &\in \arg\max \{z'd \mid z \in \{\nabla g_j(\bar{x}),\ j \in I(\bar{x})\}\} \\
&= \arg\max \{z'd \mid z \in \operatorname{conv} \{\nabla g_j(\bar{x}),\ j \in I(\bar{x})\}\} \\
&= \arg\max \{z'd \mid z \in \partial g(\bar{x})\} =: A.
\end{aligned}$$

Now, since $d \neq 0_n$, one has that $A \subseteq \operatorname{bd} \partial g(\bar{x})$. On the other hand, A is convex by convexity of $\partial g(\bar{x})$. Therefore, the proven relation $\nabla g_i(\bar{x}) \in A$ for all $i \in I$ along with (28) imply the desired relation

$$u \in \operatorname{conv} \{\nabla g_i(\bar{x}),\ i \in I\} \subseteq A \subseteq \operatorname{bd} \partial g(\bar{x}).$$

The following paragraphs are devoted to establish the equality

$$\limsup_{x \to \overline{x},\ x \neq \overline{x}} \partial g(x) = \operatorname{bd} \partial g(\overline{x})$$

under the following condition: "for all supporting hyperplane H to $\partial g(\bar{x})$, we have that $\{\nabla g_i(\bar{x}), i \in I(\bar{x})\} \cap H$ is affinely independent". So, we have to prove the remaining inclusion "$\supset$". To this aim, take any $u \in \operatorname{bd} \partial g(\bar{x})$ and let us show the existence $\{x^k\} \subset \mathbb{R}^n$ converging to $\overline{x}$, with $x^k \neq \overline{x}$ for all k, such that

$$u \in \lim_{k \to \infty} \partial g\left(x^k\right).$$

Since $u \in \operatorname{bd} \partial g(\bar{x})$, there exists a supporting hyperplane H to $\partial g(\bar{x})$ at u; so, we can write $H = \{z \in \mathbb{R}^n : z'd = \delta\}$, with $0_n \neq d \in \mathbb{R}^n$, $\delta \in \mathbb{R}$,

$$u'd = \delta \text{ and } w'd \leq \delta,\ \forall w \in \partial g(\bar{x}) = \operatorname{conv} \{\nabla g_i(\bar{x}),\ i \in I(\bar{x})\}. \tag{30}$$

Let $I \subset I(\bar{x})$ be such that $\{\nabla g_i(\bar{x}),\ i \in I(\bar{x})\} \cap H = \{\nabla g_i(\bar{x}), i \in I\}$; in other words,

$$\nabla g_i(\overline{x})' d = \delta \text{ for all } i \in I, \text{ and } \nabla g_i(\overline{x})' d < \delta \text{ when } i \in I(\overline{x}) \setminus I. \tag{31}$$

Then, one easily checks that

$$u \in \operatorname{conv}\{\nabla g_i(\bar{x}), i \in I\}.$$

In fact, if we write

$$u = \sum_{i \in I(\overline{x})} \lambda_i \nabla g_i(\overline{x})$$

for some $\lambda \in \mathbb{R}_+^{|I|}$ with $\sum_{i \in I} \lambda_i = 1$, we have

$$\delta = u'd = \sum_{i \in I} \lambda_i \nabla g_i(\overline{x})' d + \sum_{i \in I(\overline{x}) \setminus I} \lambda_i \nabla g_i(\overline{x})' d,$$

which implies $\lambda_i = 0$ for all $i \in I(\overline{x}) \setminus I$, as consequence of (31).

By the current assumption, $\{\nabla g_i(\bar{x}),\ i \in I\}$ is affinely independent and, by simplicity, we may assume $I = \{1, ..., i_0\}$ ($i_0 \leq n$ since $\dim H = n-1$). Then, from

$$g_i(\overline{x}) = g_1(\overline{x}), \text{ and } (\nabla g_i(\bar{x}) - \nabla g_1(\bar{x}))' d = 0, \text{ for all } i \in I,$$

by proceeding as in the proof of condition (i) above, we can establish the existence of a differentiable curve α such that

$$g_i(\alpha(t)) - g_1(\alpha(t)) = 0, \text{ whenever } -a < t < a \ (a > 0),$$

and

$$\alpha(0) = \overline{x},\ \dot{\alpha}(0) = d.$$

Again, let us consider a sequence of scalars $0 < t_k < a$, $k \in \mathbb{N}$, such that $t_k \to 0$ and define

$$x_k := \alpha(t_k), \text{ for all } k.$$

Since $\dot{\alpha}(0) = d \neq 0_n$, we may assume that $x_k \neq \overline{x}$ for all k. Then, as in the proof of condition (i), we have for k large enough

$$\begin{array}{rcl} g_1(x_k) & = & ... = g_{i_0}(x_k), \\ g_j(x_k) & < & g_1(x_k),\ j \in I(\overline{x}) \setminus \{1, ..., i_0\}, \end{array} \tag{32}$$

which yields (taking also Lemma 1 into account)

$$I(x_k) = I, \text{ and so } \partial g(x_k) = \operatorname{conv}\{\nabla g_i(x_k), i \in I\}.$$

Consequently,

$$u = \sum_{i \in I} \lambda_i \nabla g_i(\overline{x}) = \lim_{k \to \infty} \sum_{i \in I} \lambda_i \nabla g_i(x_k) \in \lim_{k \to \infty} \partial g(x_k),$$

which finishes the proof. ■

Remark 4 Recall that [2, Theorem 3.1] shows that condition (ii) in the previous theorem also holds as equality in the case when function g is convex, without differentiability assumptions, in which case, ∂g represents the usual subdifferential of convex analysis.

3.3 Application to the calmness modulus of $\mathcal{C}^1$- systems

As immediate consequence of Theorems 1 and 5 we obtain the following result which provides an upper bound for the calmness modulus of $\mathcal{C}^1$- systems and, in addition, a lower bound in the case of $\mathcal{C}^1$- convex systems.

Corollary 1 *Let us consider the $\mathcal{C}^1$- system introduced in (3). Let $\mathcal{F}$ be the associated feasible set mapping and g the max-function (5), i.e.,*

$$\mathcal{F}(b) := \{f_i(x) \leq b_i, \text{ for all } i = 1, ..., m\}$$

and

$$g(x) := \max_{i=1,\ldots,m} f_i(x) - b_i, \; x \in \mathbb{R}^n.$$

If $\left(\overline{b}, \overline{x}\right) \in \operatorname{gph}\mathcal{F}$ is such that $g(\overline{x}) = 0$, then:

(i) *We have*

$$\operatorname{clm}\mathcal{F}(\overline{b}, \overline{x}) \leq \left(d_*\left(0_n, \operatorname{bd}\partial g(\overline{x})\right)\right)^{-1};$$

(ii) *If, additionally, functions f_i, $i = 1, ..., m$, are convex, then*

$$\left(\min_{D \in \mathcal{D}_{AI}(\overline{x})} d_*\left(0_n, \operatorname{conv}\left\{\nabla f_i(\overline{x}), \; i \in D\right\}\right)\right)^{-1} \leq \operatorname{clm}\mathcal{F}(\overline{b}, \overline{x}).$$

Proof. (i) The proof comes straightforwardly from Theorem 1(i) and taking the following chain of inclusions into account :

$$\limsup_{x \to \overline{x}, \; g(x) > 0} \partial g(x) \subset \limsup_{x \to \overline{x}, \; x \neq \overline{x}} \partial g(x) \subset \operatorname{bd}\partial g(\overline{x}).$$

(The first inclusion is trivial, while the second comes from Theorem 5(ii) .)

(ii) is a direct consequence of Theorem 1(ii) and Theorem 5(i) . ■

Both inequalities in the previous corollary can be either satisfied as equalities or not, as the following examples show. In the first example, both inequalities are equalities indeed.

Example 2 Let us consider the system

$$\sigma(b) := \left\{ \begin{array}{c} g_1(x) := 2x_1^2 + x_2^2 + 4x_1 + 2x_2 \leq b_1, \\ g_2(x) := x_1^2 + x_2^2 - 4x_1 \leq b_2 \end{array} \right\}.$$

and take the point $\overline{x} = 0_2$ and the fixed parameter $\overline{b} = 0_2$. The associated max-function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(x) = \max\{g_1(x), \; g_2(x)\}.$$

It is immediate that

$$\bigcup_{D \in \mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\left\{\nabla g_i(\overline{x}), \; i \in D\right\} = \operatorname{conv}\left\{(-4, 0), (4, 2)\right\} = \partial g(\overline{x}),$$

which entails (according to Theorem 5)

$$\limsup_{x \to \overline{x}, \; g(x) > 0} \partial g(x) = \operatorname{conv}\left\{(-4, 0), (4, 2)\right\}.$$

So, all distances in the previous corollary coincide and, in fact,

$$\operatorname{clm}\mathcal{F}(\overline{b}, \overline{x}) := \left[d_*\left(0_n, \operatorname{conv}\left\{(-4, 0), (4, 2)\right\}\right)\right]^{-1}.$$

In the following example, the first inequality in the previous corollary is strictly satisfied, while the second is an equality.

Example 3 Consider the system

$$\sigma(b) := \left\{ \begin{array}{c} g_1(x) := 2x_1^2 + x_2^2 + 4x_1 + 2x_2 \leq b_1, \\ g_2(x) := x_1^2 + x_2^2 - 4x_1 \leq b_2, \\ g_3(x) := -x_1 \leq b_3, \end{array} \right\}$$

and take $\overline{x} = 0_2$ and $\overline{b} = 0_3$. Now, the associated max-function $g : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$g(x) = \max\{g_1(x),\ g_2(x),\ g_3(x)\}.$$

In this case, we easily see that:

$$\bigcup_{D \in \mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\} = \operatorname{conv}\{(-4,0),(4,2)\}.$$

Moreover, let us see that

$$\limsup_{x \to \overline{x},\ g(x) > 0} \partial g(x) = \operatorname{conv}\{(-4,0),(4,2)\}.$$

Observe that $g(x) > 0$ implies $g(x) = \max\{g_1(x), g_2(x)\} > g_3(x)$. In fact, one can easily check that $g_2(x) \leq g_3(x)$ yields $g_3(x) \leq 0$ and, then, if simultaneously $g_1(x), g_2(x) \leq g_3(x)$, we have $g(x) \leq 0$. As a consequence of that,

$$\begin{array}{rcl} \limsup\limits_{x \to \overline{x},\ g(x) > 0} \partial g(x) & \subset & \limsup\limits_{x \to \overline{x},\ g(x) > g_3(x)} \partial g(x) \\ & \subset & \lim\limits_{x \to \overline{x}} \operatorname{conv}\{\nabla g_1(x), \nabla g_2(x)\} = \operatorname{conv}\{(-4,0),(4,2)\}. \end{array}$$

Then,

$$\operatorname{clm}\mathcal{F}(\overline{b},\overline{x}) = \left[d_*\left(0_n, \operatorname{conv}\{(-4,0),(4,2)\}\right)\right]^{-1}.$$

However,

$$\operatorname{bd}\partial g(\overline{x}) = \operatorname{bdconv}\{(-4,0),(4,2),(-1,0)\},$$

and then,

$$\operatorname{clm}\mathcal{F}(\overline{b},\overline{x}) < \left[d_*\left(0_n, \operatorname{bdconv}\{(-4,0),(4,2),(-1,0)\}\right)\right]^{-1}.$$

Remark 5 Let us remark that in the last two examples, if we approach $\overline{x}$ by *directional sequences*, i.e. sequences of the type $x^k = \overline{x} + t_k u$ with $u \neq 0_2$ and $t_k \to 0$, and we represent this directional convergence with the symbol $x \xrightarrow{d} \overline{x}$, we shall obtain only the extreme points of the sets generated by arbitrary convergent sequences. More specifically, in both Examples 2 and 3,

$$\limsup_{x \xrightarrow{d} \overline{x},\ g(x) > 0} \partial g(x) = \{(-4,0),(4,2)\} \subsetneq \operatorname{bd}\partial g(\overline{x}).$$

This observation corresponds to the statement in Theorem 6.3.6 in Chapter VI of [13].

In the following example the second inequality in Corollary 1 is strict, while the first is in fact an equality.

Example 4 Consider the system

$$\sigma(b) := \left\{ \begin{array}{l} x_1^2 + x_2^2 \leq b_1, \\ x_1 + x_2 \leq b_2, \end{array} \right\}$$

and take $\overline{x} = 0_2$ and $\overline{b} = 0_2$.

In this case, $\{2\}$ is the only element in $\mathcal{D}_{AI}(\overline{x})$. So,

$$\bigcup_{D \in \mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i \in D\} = \{(1,1)\}.$$

However, one obtains from [2, Theorem 3.1] (observing that $g(x) > 0 \Leftrightarrow x \neq \overline{x}$ in this example)

$$\limsup_{x \to \overline{x},\ g(x) > 0} \partial g(x) = \limsup_{x \to \overline{x},\ x \neq \overline{x}} \partial g(x) = \operatorname{bd} \partial g(\overline{x}) = \operatorname{conv}\{(0,0),(1,1)\},$$

and then $\operatorname{clm}\mathcal{F}(\overline{b}, \overline{x}) = +\infty$..

Remark 6 If we modify g_1 in the previous example by adding $\frac{1}{2}(x_1 + x_2)$, then one still has $\mathcal{D}_{AI}(\overline{x}) = \{\{2\}\}$ and, by approaching 0_2 by points of the circumference $x_1^2 + x_2^2 = \frac{1}{2}(x_1 + x_2)$ different from 0_2 one checks

$$\limsup_{x \to \overline{x},\ g(x) > 0} \partial g(x) = \operatorname{bd} \partial g(\overline{x}) = \operatorname{conv}\left\{(\tfrac{1}{2}, \tfrac{1}{2}), (1,1)\right\},$$

and then $\operatorname{clm}\mathcal{F}(\overline{b}, \overline{x}) = \left\|(\frac{1}{2}, \frac{1}{2})\right\|_*^{-1}$.

4 Computing the calmness modulus of the argmin mapping for linear programs

In this section, a suitable Karush-Kuhn-Tucker (KKT) index set approach will allow us to derive the exact calmness modulus of $\mathcal{S}$, defined in (12), at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \operatorname{gph}\mathcal{S}$ under non-uniqueness assumptions; i.e., without assuming $\mathcal{S}(\overline{c}, \overline{b}) = \{\overline{x}\}$. In our way to prove this result, we have to extend the lower bound and sharpen the upper bound given respectively in [4, Theorems 6 and 7] (see Section 2 for more details).

To start with, the next example shows that inequality in (15) may be strict when $\mathcal{S}\left(\overline{c}, \overline{b}\right)$ is not a singleton (see Remark 2) and gives a hint to sharpen such an upper bound. In Corollary 2 we will see that this sharpened upper bound is in fact the exact calmness modulus of $\mathcal{S}$.

Example 5 Consider the nominal problem (in $\mathbb{R}^2$ endowed with the Euclidean norm)

$$\begin{array}{rll} P\left(\overline{c}, \overline{b}\right): & \textit{Min}\ x_1 & \\ \textit{s.t.} & -x_1 \leq 0, & (i = 1), \\ & -x_2 \leq 0, & (i = 2), \\ & -x_1 - x_2 \leq 0, & (i = 3). \end{array}$$

Let $\overline{x} := 0_2$. By appealing to Theorem 2, applied to mappings $\mathcal{L}_D$ –which are nothing else but feasible set mappings associated with enlarged systems– at $\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right) \in \operatorname{gph} \mathcal{L}_D$ with $D \in \mathcal{K}_{\overline{c}, \overline{b}}(\overline{x})$, we obtain the following table:

$D \in \mathcal{K}_{\overline{c},\overline{b}}(\overline{x})$	$\mathrm{clm}\mathcal{L}_D\left(\left(\overline{b},-\overline{b}_D\right),\overline{x}\right)$
$\{1\},\ \{1,2\}$	$\sqrt{2}$
$\{1,3\}$	$\sqrt{5}$

Now Theorem 3 ensures $\mathrm{clm}\mathcal{S}(\left(\overline{c},\overline{b}\right),\overline{x}) \leq \sqrt{5}$.

An *ad hoc* geometrical argument could show that $\mathrm{clm}\mathcal{S}(\left(\overline{c},\overline{b}\right),\overline{x}) = \sqrt{2}$ in the previous example. The underlying idea is that those $D \in \mathcal{K}_{\overline{c},\overline{b}}(\overline{x})$ with some zero KKT multiplier λ_i in an expression $-\overline{c} = \sum_{i \in D} \lambda_i a_i$ are not relevant. In other words, the key fact consists of confining ourselves to those KKT subsets which are minimal with respect to the inclusion order, and consequently the associated multipliers are all of them nonzero. Accordingly we consider, associated with $(\left(\overline{c},\overline{b}\right),\overline{x}) \in \mathrm{gph}\mathcal{S}$, the family of *minimal KKT subsets* given by

$$\mathcal{M}_{\overline{c},\overline{b}}(\overline{x}) = \left\{D \in \mathcal{K}_{\overline{c},\overline{b}}(\overline{x}) : D \text{ is minimal for the inclusion order}\right\}.$$

Observe that in the previous example one has $\mathcal{M}_{\overline{c},\overline{b}}(\overline{x}) = \{\{1\}\}$ ..

Remark 7 In the special case $\overline{c} = 0_n$ it is easy to see (thanks to Theorem 3) that

$$\mathrm{clm}\mathcal{S}(\left(\overline{c},\overline{b}\right),\overline{x}) = \mathrm{clm}\mathcal{S}_{\overline{c}}(\overline{b},\overline{x}) = \mathrm{clm}\mathcal{F}(\overline{b},\overline{x}),$$

and we already have an expression for the latter. So, in the sequel we could assume $\overline{c} \neq 0_n$. Nevertheless, the case $\overline{c} = 0_n$ is also included in our results if we convene $\mathcal{M}_{\overline{c},\overline{b}}(\overline{x}) = \{\emptyset\}$ whenever $\overline{c} = 0_n$, and $\mathcal{L}_\emptyset := \mathcal{F}$.

Theorem 6 *Let* $\left(\overline{c},\overline{b}\right) \in \mathbb{R}^n \times \mathbb{R}^m$, *and assume* $\overline{x} \in \mathcal{S}\left(\overline{c},\overline{b}\right)$. *Then*

$$\mathrm{clm}\mathcal{S}(\left(\overline{c},\overline{b}\right),\overline{x}) \leq \max_{D \in \mathcal{M}_{\overline{c},\overline{b}}(\overline{x})} \mathrm{clm}\mathcal{L}_D\left(\left(\overline{b},-\overline{b}_D\right),\overline{x}\right). \tag{33}$$

Proof. Under the current hypotheses Theorem 3 establishes

$$\mathrm{clm}\mathcal{S}(\left(\overline{c},\overline{b}\right),\overline{x}) = \mathrm{clm}\mathcal{S}_{\overline{c}}(\overline{b},\overline{x}),$$

where $\mathcal{S}_{\overline{c}} := \mathcal{S}(\overline{c},\cdot)$; i.e., $\mathcal{S}_{\overline{c}}(b) = \mathcal{S}(\overline{c},b)$ for each $b \in \mathbb{R}^m$. Let us write

$$\mathrm{clm}\mathcal{S}_{\overline{c}}(\overline{b},\overline{x}) = \lim_{r \to \infty} \frac{d\left(x^r,\mathcal{S}\left(\overline{c},\overline{b}\right)\right)}{\left\|b^r-\overline{b}\right\|_\infty} \tag{34}$$

for some $\mathbb{R}^m \ni b^r \to \overline{b}$ (with $b^r \neq \overline{b}$ for all $r \in \mathbb{N}$) and some $\mathcal{S}_{\overline{c}}(b^r) \ni x^r \to \overline{x}$. According to the KKT conditions, take for each r a certain $D_r \subset T_{b^r}(x^r)$ with $|D_r| \leq n$ (because of Carathéodory's Theorem) such that

$$-\overline{c} \in \mathrm{cone}\,\{a_i,\ i \in D_r\}. \tag{35}$$

The finiteness of $\{1,...,m\}$ enables us assume for a suitable subsequence (denoted as the whole sequence for simplicity) that $D_r = D$ (independent of r). Then it is clear that, for such a subsequence, in (35) we may assume that all KKT multipliers are nonzero and that set D is minimal with this property. Moreover $D \subset T_{b^r}(x^r)$ for all r clearly implies $D \subset T_{\overline{b}}(\overline{x})$ by just taking limits in $a_i' x^r = b_i^r$ for each $i \in D$. Accordingly we can write

$$D \in \mathcal{M}_{\overline{c},\overline{b}}(\overline{x}).$$

Since, on the one hand, $D \subset T_{b^r}(x^r)$ clearly implies $x^r \in \mathcal{L}_D(b^r, -b^r_D)$ and, on the other hand, $\mathcal{L}_D\left(\overline{b}, -\overline{b}_D\right) \subset \mathcal{S}\left(\overline{c}, \overline{b}\right)$ (i.e., every KKT point is optimal), (34) entails, taking into account the obvious fact that

$$\left\|(b^r, b^r_D) - \left(\overline{b}, -\overline{b}_D\right)\right\|_\infty = \left\|b^r - \overline{b}\right\|_\infty$$

(the first one in $\mathbb{R}^m \times \mathbb{R}^D$ and the second in $\mathbb{R}^m$),

$$\mathrm{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x}) \leq \limsup_{r\to\infty} \frac{d\left(x^r, \mathcal{L}_D\left(\overline{b}, -\overline{b}_D\right)\right)}{\left\|(b^r, b^r_D) - \left(\overline{b}, -\overline{b}_D\right)\right\|_\infty} \leq \mathrm{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right).$$

■

Next we are going to see that the right-hand-side of (33) already stands as a lower bound on $\mathrm{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x})$ (and hence on $\mathrm{clm}\mathcal{S}(\left(\overline{c}, \overline{b}\right), \overline{x})$) in the following semi-infinite setting, which obviously includes the case when T is finite:

- T is a compact Hausdorff space,
- The given function $a \equiv (a_t)_{t\in T}$ belongs to $C(T, \mathbb{R}^n)$,
- Parameter $b \equiv (b_t)_{t\in T}$ belongs to $C(T, \mathbb{R})$.

Hereafter in this section let us assume the previous framework. The rest of notation remains unchanged, but adapted to the new setting. Theorem 6 in [4] shows that the last term in (15), i.e., $\max_{D\in\mathcal{K}_{\overline{c},\overline{b}}(\overline{x})} \mathrm{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right)$, is a lower bound on $\mathrm{clm}\mathcal{S}_{\overline{c}}(\overline{b}, \overline{x})$ in this new setting when we also assume: (i) $\mathcal{S}(\overline{c}, \overline{b}) = \{\overline{x}\}$, (ii) the Slater constraint qualification at the nominal parameter $\overline{b}$ (i.e., the existence of some $\widehat{x} \in \mathbb{R}^n$ such that $a_t'\widehat{x} < \overline{b}_t$ for all $t \in T$). In Theorem 7 below we show that the (possibly) sharper upper bound $\max_{D\in\mathcal{M}_{\overline{c},\overline{b}}(\overline{x})} \mathrm{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right)$ also stands as a lower bound without assuming neither (i) nor (ii).

For any $D \in \mathcal{M}_{\overline{c},\overline{b}}(\overline{x})$, we consider the supremum function, $f_D : \mathbb{R}^n \to \mathbb{R}$ given by

$$\begin{aligned} f_D(x) &:= \sup\left\{\langle a_t, x\rangle - \overline{b}_t,\ t \in T;\ -\langle a_t, x\rangle + \overline{b}_t,\ t \in D\right\} \\ &= \sup\left\{\langle a_t, x\rangle - \overline{b}_t,\ t \in T \setminus D;\ \left|\langle a_t, x\rangle - \overline{b}_t\right|,\ t \in D\right\}, \end{aligned}$$

Observe that

$$\mathcal{L}_D\left(\overline{b}, -\overline{b}_D\right) = [f_D = 0] \subset \mathcal{S}\left(\overline{c}, \overline{b}\right) \text{ for all } D \in \mathcal{M}_{\overline{c},\overline{b}}(\overline{x}). \tag{36}$$

Let us also observe that, as a direct consequence of Theorem 1,

$$\mathrm{clm}\mathcal{L}_D\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right) = \limsup_{\substack{x\to\overline{x} \\ f_D(x)>0}} \frac{1}{d_*\left(0_n, \partial f_D(x)\right)}. \tag{37}$$

Proposition 1 *In our current semi-infinite setting, let* $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \mathrm{gph}\mathcal{S}$. *Then*

$$\mathcal{L}_D\left(\overline{b}, -\overline{b}_D\right) = \mathcal{S}\left(\overline{c}, \overline{b}\right), \textit{ for all } D \in \mathcal{M}_{\overline{c},\overline{b}}(\overline{x}).$$

Proof. We only have to prove the inclusion "$\supset$"(recall (36)). Reasoning by contradiction, assume the existence of a certain $\overline{D} \in \mathcal{M}_{\overline{c},\overline{b}}(\overline{x})$ and some $\widehat{x} \in \mathcal{S}\left(\overline{c}, \overline{b}\right) \setminus \mathcal{L}_{\overline{D}}\left(\overline{b}, -\overline{b}_{\overline{D}}\right)$. Observe that, since $\widehat{x}$ is feasible for $P\left(\overline{c}, \overline{b}\right)$, we have

$$a_t'(\overline{x} - \widehat{x}) = \overline{b}_t - a_t'\widehat{x} \geq 0, \text{ for all } t \in \overline{D}, \tag{38}$$

while condition $\widehat{x} \notin \mathcal{L}_{\overline{D}}\left(\bar{b}, -\bar{b}_{\overline{D}}\right)$ yields

$$a_t'(\overline{x} - \widehat{x}) = \overline{b}_t - a_t'\widehat{x} > 0, \text{ for some } t \in \overline{D}. \tag{39}$$

Moreover, $\overline{D} \in \mathcal{M}_{\bar{c},\bar{b}}\left(\overline{x}\right)$ entails the existence of a $\lambda_t > 0,$ for each $t \in \overline{D}$ such that

$$-\overline{c} = \sum_{t\in\overline{D}} \lambda_t a_t.$$

Then

$$-\overline{c}'(\overline{x} - \widehat{x}) = \sum_{t\in\overline{D}} \lambda_t a_t'(\overline{x} - \widehat{x}).$$

Observe that $\overline{c}'(\overline{x} - \widehat{x}) = 0$ (since $\overline{x}, \widehat{x} \in \mathcal{S}\left(\bar{c}, \bar{b}\right)$), which, according to (38), yields

$$\lambda_t a_t'(\overline{x} - \widehat{x}) = 0, \text{ for all } t \in \overline{D}.$$

Then, applying (39) we attain the contradiction (with the minimality condition of $\overline{D}$)

$$\lambda_t = 0, \text{ for some } t \in \overline{D}.$$

■

Theorem 7 *In our current semi-infinite setting, let* $\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) \in \mathrm{gph}\mathcal{S}$. *Then*

$$\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) \geq \mathrm{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right) \geq \sup_{D\in\mathcal{M}_{\bar{c},\bar{b}}(\overline{x})} \mathrm{clm}\mathcal{L}_D\left(\left(\bar{b}, -\bar{b}_D\right), \overline{x}\right).$$

Proof. The first inequality comes directly from the definition of calmness modulus.

Now, we are going to prove the second inequality in the non-trivial case $\bar{c} \neq 0_n$ (see Remark 7). Take any $\overline{D} \in \mathcal{M}_{\bar{c},\bar{b}}\left(\overline{x}\right)$ and let us see that $\mathrm{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right) \geq \mathrm{clm}\mathcal{L}_{\overline{D}}\left(\left(\bar{b}, -\bar{b}_{\overline{D}}\right), \overline{x}\right)$. From (37), we can write

$$\mathrm{clm}\mathcal{L}_{\overline{D}}\left(\left(\bar{b}, -\bar{b}_{\overline{D}}\right), \overline{x}\right) = \lim_{r\to+\infty} \frac{1}{\|u^r\|_*},$$

for a certain sequence $\{u^r\}_{r\in\mathbb{N}}$ verifying $u^r \in \partial f_{\overline{D}}\left(x^r\right)$, for all r, where $\{x^r\}_{r\in\mathbb{N}}$ is such that

$$\lim_{r\to+\infty} x^r = \overline{x} \text{ and } f_{\overline{D}}\left(x^r\right) > 0, \text{ for all } r.$$

In particular, $x^r \notin \mathcal{L}_{\overline{D}}\left(\bar{b}, -\bar{b}_D\right)$, for all r, and then, applying Proposition 1,

$$x^r \notin \mathcal{S}_{\bar{c}}\left(\bar{b}\right), \text{ for all } r.$$

For each r, let $\widetilde{x}^r \in \mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right)$ a best approximation of x^r in $\mathcal{S}_{\bar{c}}\left(\bar{b}\right)$; i.e.,

$$\|x^r - \widetilde{x}^r\| = d\left(x^r, \mathcal{S}_{\bar{c}}(\bar{b})\right), \text{ for all } r.$$

We have, for each r,

$$\|x^r - \widetilde{x}^r\| \, \|u^r\|_* \geq (u^r)'(x^r - \widetilde{x}^r) \geq f_{\overline{D}}\left(x^r\right) - f_{\overline{D}}\left(\widetilde{x}^r\right) = f_{\overline{D}}\left(x^r\right),$$

where we have appealed again to the previous proposition to ensure that $f_{\overline{D}}(\widetilde{x}^r) = 0$, for all r. Consequently,

$$\|x^r - \widetilde{x}^r\| \geq \frac{f_{\overline{D}}(x^r)}{\|u^r\|_*}, \text{ for all } r. \tag{40}$$

Now, following the same argument as is the last part of the proof of [4, Theorem 6] (just by adapting the notation) we may construct a sequence $\{b^r\} \subset C(T, \mathbb{R})$ such that

$$x^r \in \mathcal{S}_{\bar{c}}(b^r) \text{ and } \left\|b^r - \bar{b}\right\|_\infty \leq \left(1 + \frac{1}{r}\right) f_{\overline{D}}(x^r), \text{ for all } r. \tag{41}$$

Just for completeness, at this moment we write the definition of b^r. For each r,

$$b_t^r := (1 - \varphi_r(t))\, a_t' x^r + \varphi_r(t)\left(\bar{b}_t + f_{\overline{D}}(x^r)\right),$$

where $\varphi_r(t)$ is a continuous function from T to $[0,1]$ such that

$$\varphi_r(t) = \begin{cases} 0 \text{ if } t \in \overline{D}, \\ 1 \text{ if } a_t' x^r - \bar{b}_t \leq -\left(1 + \frac{1}{r}\right) f_{\overline{D}}(x^r), \end{cases}$$

whose existence is guaranteed by Urysohn's lemma.

Finally, taking (40) and (41) into account, we obtain the aimed inequality

$$\begin{aligned} \operatorname{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right) &\geq \lim_{r\to\infty} \frac{\|x^r - \widetilde{x}^r\|}{\left\|b^r - \bar{b}\right\|_\infty} \geq \lim_{r\to\infty}\left(1 + \frac{1}{r}\right)^{-1} \|u^r\|_*^{-1} \\ &= \operatorname{clm}\mathcal{L}_{\overline{D}}\left(\left(\bar{b}, -\bar{b}_D\right), \overline{x}\right). \end{aligned}$$

■

Corollary 2 *Assume that T is finite and let $\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) \in \operatorname{gph}\mathcal{S}$. Then*

$$\operatorname{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \overline{x}\right) = \operatorname{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \overline{x}\right) = \max_{D \in \mathcal{M}_{\bar{c},\bar{b}}(\overline{x})} \operatorname{clm}\mathcal{L}_D\left(\left(\bar{b}, -\bar{b}_D\right), \overline{x}\right).$$

5 Conclusions

The main contributions of this work are developed in two different directions: the analysis of certain outer limits of subdifferentials of max-functions

$$g := \max_{i=1,\ldots,m} g_i,$$

under different assumptions, and the computation of the calmness moduli for certain feasible and optimal set mappings. We point out the fact that the two different kind of results have a common starting point: the background about the calmness modulus of feasible set mappings associated with linear inequality systems.

In summary, the main contributions of the present paper are:

- With respect to outer limit of subdifferentials, in the more general case of functions g_i's being continuously differentiable, we have the chain of inclusions (Theorem 5):

$$\begin{aligned}\bigcup_{D\in\mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i\in D\} &\subset \limsup_{x\to\overline{x},\ g(x)>0} \partial g(x) \\ &\subset \limsup_{x\to\overline{x},\ x\neq\overline{x}} \partial g(x) \subset \operatorname{bd}\partial g(\overline{x}).\end{aligned} \tag{42}$$

  (Recall that last inclusion is in fact an equality when the g_i's are convex, even without differentiability assumptions).

- The same Theorem 5 provides a sufficient condition (different from convexity) under which the last inclusion in (42) becomes an equality.

- In the particular case when the g_i's are affine, we additionally have (Theorem 4):

$$\begin{aligned}\bigcup_{D\in\mathcal{D}_{AI}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i\in D\} &\subset \bigcup_{D\in\mathcal{D}(\overline{x})} \operatorname{conv}\{\nabla g_i(\overline{x}),\ i\in D\} \\ &= \limsup_{x\to\overline{x},\ g(x)>0} \partial g(x).\end{aligned}$$

  (See (16) and (21) for the definitions of $\mathcal{D}(\overline{x})$ and $\mathcal{D}_{AI}(\overline{x})$.)

Then, we apply the chain of inclusions (42) to derive estimations for the calmness modulus of the feasible set mapping

$$\mathcal{F}(b) := \{f_i(x)\leq b_i,\ \text{for all } i=1,...,m\},$$

associated with a $\mathcal{C}^1$- system. In this setting we consider $g(x) := \max_{i=1,...,m} f_i(x)-b_i$, $x\in\mathbb{R}^n$. Specifically, the main results related to $\operatorname{clm}\mathcal{F}(\overline{b},\overline{x})$ are:

- An upper bound:

$$\operatorname{clm}\mathcal{F}(\overline{b},\overline{x}) \leq \left(d_*\left(0_n, \operatorname{bd}\partial g(\overline{x})\right)\right)^{-1};$$

- A lower bound if, additionally, f_i, $i=1,...,m$, are convex:

$$\left(\min_{D\in\mathcal{D}_{AI}(\overline{x})} d_*\left(0_n, \operatorname{conv}\{\nabla f_i(\overline{x}),\ i\in D\}\right)\right)^{-1} \leq \operatorname{clm}\mathcal{F}(\overline{b},\overline{x}).$$

- In Section 4, an exact formula for the *calmness modulus of* $\mathcal{S}$, defined in (12), at $((\overline{c},\overline{b}),\overline{x})\in \operatorname{gph}\mathcal{S}$, without uniqueness assumptions, is provided . Specifically, Corollary 2 yields the exact formula:

$$\operatorname{clm}\mathcal{S}\left((\overline{c},\overline{b}),\overline{x}\right) = \sup_{D\in\mathcal{M}_{\overline{c},\overline{b}}(\overline{x})} \operatorname{clm}\mathcal{L}_D\left(\left(\overline{b},-\overline{b}_D\right),\overline{x}\right), \tag{43}$$

  where $\mathcal{L}_D$ is nothing else but a feasible set mapping (associated to a certain enlarged system). Moreover, we recall the fact that the supremum in the right hand side of (43) still constitutes a lower bound for $\operatorname{clm}\mathcal{S}\left((\overline{c},\overline{b}),\overline{x}\right)$ in the semi-infinite case, again without uniqueness assumptions.

Finally, we point out the fact that in our analysis of outer subdifferentials and calmness moduli we are always looking for conceptually tractable expressions (exact formulae or estimations), in the sense that they only involve the nominal data (nominal point and nominal parameter).

References

[1] Azé, D., Corvellec, J.-N.: Characterizations of error bounds for lower semicontinuous functions on metric spaces. ESAIM Control Optim. Calc. Var. **10**, 409–425 (2004).

[2] Cánovas, M. J., Hantoute, A., Parra, J., Toledo, F. J.: Boundary of subdifferentials and calmness moduli in linear semi-infinite optimization. Optim. Lett. (2014), published online 23 July 2014, DOI 10.1007/s11590-014-0767-1.

[3] Cánovas, M. J., Hantoute, A., Parra, J., Toledo, F. J.: Calmness of fully perturbed linear programs. Submitted to Math. Program. (2014).

[4] Cánovas, M. J., Kruger, A. Y., López, M. A., Parra, J., Thera, M. A.: Calmness modulus of linear semi-infinite programs. SIAM J. Optim. **24**, 29–48 (2014).

[5] Cánovas, M. J., López, M. A., Parra, J., Toledo, F. J.: Calmness of the feasible set mapping for linear inequality systems. Set-Valued Var. Anal. **22**, 375–389 (2014).

[6] Clarke, F. H.: *Optimization and Nonsmooth Analysis.* Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1983.

[7] Dontchev, A. L., Rockafellar, R. T.: *Implicit Functions and Solution Mappings: A View from Variational Analysis.* Springer, New York, 2009.

[8] Fabian, M. J., Henrion, R., Kruger, A., Outrata, J.: Error Bounds: Necessary and Sufficient Conditions. Set-Valued Anal. **18**, 121–149 (2010).

[9] Gfrerer, H.: First order and second order characterizations of metric subregularity and calmness of constraint set mappings. SIAM J. Optim. **21**, 1439–1474 (2011).

[10] Goberna, M. A., López, M. A.: *Linear Semi-Infinite Optimization.* John Wiley & Sons, Chichester (UK), 1998.

[11] Henrion, R., Jourani, A., Outrata, J.: On the calmness of a class of multifunctions. SIAM J. Optim. **13**, 603–618 (2002).

[12] Henrion, R., Outrata, J.: Calmness of constraint systems with applications. Math. Program. B **104**, 437–464 (2005).

[13] Hiriart-Urruty, J. B., Lemaréchal, C.: *Convex Analysis and Minimization Algorithms I*, Springer-Verlag, Berlin, 1991.

[14] Jourani, A.: Hoffman's Error Bound, Local Controllability, and Sensitivity Analysis. SIAM J. Control Optim. **38**, 947–970 (2000).

[15] Klatte, D., Thiere, G.: Error Bounds for Solutions of Linear Equations and Inequalities. Mathematical Methods of Operations Research **41**, 191–214 (1995).

[16] Klatte, D. Kummer, B.: *Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications.* Nonconvex Optim. Appl. 60. Kluwer Academic, Dordrecht, The Netherlands, 2002.

[17] Klatte, D. Kummer, B.: Optimization methods and stability of inclusions in Banach spaces. Math. Program. B **117**, 305–330 (2009).

[18] Kruger, A., Van Ngai, H., Théra, M.: Stability of error bounds for convex constraint systems in Banach spaces. SIAM J. Optim. **20**, 3280–3296 (2010).

[19] Luenberger, D. G., Ye, Y.: *Linear and Nonlinear Programming* (3rd Edition), Springer, 2008.

[20] Mordukhovich, B. S.: *Variational Analysis and Generalized Differentiation, I: Basic Theory.* Springer, Berlin (2006).

[21] Robinson, S. M.: Some continuity properties of polyhedral multifunctions. Mathematical programming at Oberwolfach (Proc. Conf., Math. Forschungsinstitut, Oberwolfach, 1979). Math. Programming Stud. **14**, 206–214 (1981).

[22] Robinson, S. M.: A characterization of stability in linear programming. Operations Research **25**, 435–447 (1977).

[23] Rockafellar, R. T., Wets, R. J-B.: *Variational Analysis.* Springer, Berlin, 1998.