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Sharp thresholds for Gibbs-non-Gibbs transition in the fuzzy Potts models with a Kac-type interaction

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Abstract

We investigate the Gibbs properties of the fuzzy Potts model on the d-dimensional torus with Kac interaction. We use a variational approach for profiles inspired by that of Fernández, den Hollander and Martínez [17] for their study of the Gibbs-non-Gibbs transitions of a dynamical Kac-Ising model on the torus. As our main result, we show that the mean-field thresholds dividing Gibbsian from non-Gibbsian behavior are sharp in the fuzzy Kac-Potts model. On the way to this result we prove a large deviation principle for color profiles with diluted total mass densities and use monotocity arguments.

1 Introduction

In previous years we have seen a number of measures describing systems with interacting components appearing in mathematical statistical mechanics which have lost the Gibbs property as a result of a transformation [12, 26, 14, 15, 9]. Such a loss is indicated by the failure of continuity of conditional probabilities at a given site, when the conditioning is varied away from this site. Interesting sources of non-Gibbsian behavior include time evolutions or deterministic transformations which reduce the complexity of the local state space. A prototypical example of a system of the second type is the fuzzy Potts model (fuzzy PM) [19, 27, 25, 21, 18, 1]. It is obtained from the ordinary PM by partitioning the local state space $\{1, 2, \dots, q\}$ into subclasses and observing the Potts distribution after identification of the spin-values inside the subclasses.

It has been noted in some cases for mean-field models [23, 21, 20] when the appropriate notion of mean-field Gibbsianness is employed, the question of continuity can be reduced to variational problems. For systems for which lattice results and mean-field results are available it turns out that these results are often in a striking parallel [24, 15]. It is an open challenge to understand this relation better.

One way to approach the relation between the lattice and mean-field is via Kac models (KM) [3, 6, 5, 10, 7, 4] in which there is a parameter which makes the interaction long-range but a spatial structure remains.

The first rigorous result relating Gibbs properties of a KM to that of a mean-field model was obtained in [17] in the case of independent time-evolutions from an initial Kac-Ising model. The relation between a spatial model and a mean-field model was set up as follows. The authors put the model on a torus in d dimensions, with spins sitting on a grid of spacing $1/n$, and looked at a single-site conditional probability in the large n-limit. The limiting object they studied then was a specification kernel giving the dependence of a single-site probability as a function of a magnetization profile. The existence of the limiting kernel was achieved using a combination of a large deviation principle (LDP) in equilibrium for the Ising model [6], a path LDP, and techniques from hydrodynamic limits. It was not possible to give sharp parameter values for the Gibbs-non-Gibbs (GnG) transition but sufficient conditions on time and initial temperature values to be non-Gibbsian could be provided.

In our present study of the fuzzy Kac-Potts model (fuzzy KPM) we ask related questions. Our main result is Theorem 2.7 where we provide precise threshold values dividing Gibbsian and non-Gibbsian behavior. To our knowledge this is the first sharp result for GnG in a Kac-model.

1.1 Strategy of proof and further results

The Hamiltonian of the KPM can be written in terms of an empirical color distribution field and we start by noting a LDP for the empirical color distribution field as the grid on the torus shrinks. The minimizers of the rate function for this LDP provide us with the equilibrium phases, and it is easy to see that the absolute minimizers must be flat (spatially homogeneous). Therefore the critical value for phase transitions in the KPM is given by the corresponding mean-field result (the Ellis-Wang Theorem [11]).

Next, to investigate the Gibbsian properties of the *fuzzy* model we analyze limiting expressions for the single-site conditional probabilities (the specification kernel). The idea to prove equality of critical parameters dividing GnG in mean-field with the corresponding critical parameters in the KPM is then to make rigorous the statement that there are no worse conditionings than spatially homogeneous conditionings. As an intermediate step we prove a LDP for color profiles for a spatially diluted KPM in Proposition 2.5. This and the corresponding non-homogeneous variational problems are interesting in their own right. We relate the specification kernel to solutions of such variational problems where the dilutions are prescribed by the conditioning profile. Finally this is supplemented by monotonicity arguments in the dilution to show sharpness of the mean-field values for the KM.

2 Model and main results

2.1 The Kac-Potts model

Let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional unit torus. For $n \in \mathbb{N}$, let $\mathbb{T}^d_n$ be the $(1/n)$-discretization of $\mathbb{T}^d$ defined by $\mathbb{T}^d_n := \Delta^d_n/n$, with $\Delta^d_n := \mathbb{Z}^d/n\mathbb{Z}^d$ the discrete torus of size n. For $n \in \mathbb{N}$, let $\Omega_n := \{1, \dots, q\}^{\Delta^d_n}$ be the set of *Potts-spin configurations* on Δ^d_n. We will call elements of $\{1, \dots, q\}$ *colors*. The energy of the configuration $\sigma := (\sigma(x))_{x\in\Delta^d_n} \in \Omega_n$ is given by the *Kac-type Hamiltonian*

$$H_n(\sigma) := -\frac{1}{2n^d} \sum_{x,y\in\Delta^d_n} J(\frac{x-y}{n}) 1_{\sigma(x)=\sigma(y)}, \qquad \sigma \in \Omega_n \tag{1}$$

where $0 \leq J \in C(\mathbb{T}^d)$ is a continuous *interaction-functions* on $\mathbb{T}^d$ which is symmetric and $J \not\equiv 0$. The Gibbs measure associated with H_n is given by

$$\mu_n(\sigma) := \frac{1}{Z_n} \exp(-\beta H_n(\sigma)), \qquad \sigma \in \Omega_n \tag{2}$$

with $\beta \in [0, \infty)$ the inverse temperature and Z_n the normalizing partition sum.

We are interested in the large n limit for μ_n and prepare the analysis by rewriting the Hamiltonian in terms of density profiles. More precisely, for $\Lambda \subset \Delta_n^d$ let $\pi_\Lambda : \Omega_n \mapsto \mathcal{P}(\mathbb{T}_n^d \times \{1, \ldots, q\}) \subset \mathcal{P}(\mathbb{T}^d \times \{1, \ldots, q\})$ be the *empirical color measure vector* or *color profiles* of σ inside the volume Λ defined by

$$\pi_\Lambda^\sigma := \frac{1}{|\Lambda|} \Big(\sum_{x \in \Lambda} 1_{\sigma(x)=1} \delta_{x/n}, \ldots, \sum_{x \in \Lambda} 1_{\sigma(x)=q} \delta_{x/n} \Big)^T$$

where δ_u is the point measure at $u \in \mathbb{T}^d$. In the sequel we use notation $\mathcal{P}_n := \mathcal{P}(\mathbb{T}_n^d \times \{1, \ldots, q\})$ and $\mathcal{P} := \mathcal{P}(\mathbb{T}^d \times \{1, \ldots, q\})$ and write $\pi_\Lambda(u)$ if we evaluate π_Λ at a site $u \in \mathbb{T}^d$ and $\pi_\Lambda[a]$ for the evaluation of π_Λ at a color $a \in \{1, \ldots, q\}$.

Let $u \in \mathbb{T}^d$, then for the *color profile perforated at* $u \in \mathbb{T}^d$ we write $\pi_n^{(u)} := \pi_{\Delta_n^d \setminus \lfloor nu \rfloor}$ where $\lfloor nu \rfloor$ denotes the lower-integer part of nu. Further we abbreviate $\mathcal{M}_n := \pi_n(\Omega_n) \subset \mathcal{P}_n$ and $\mathcal{M}_n^u := \pi_n^{(u)}(\Omega_n) \subset \mathcal{P}$ for the sets of possible profiles of mesh-size n and possible profiles of mesh-size n perforated at site u.

We equip $\mathcal{P}$ and the indicated subspaces with the weak topology, i.e. the topology corresponding to convergence of continuous functions $f \in C(\mathbb{T}^d \times \{1, \ldots, q\}, \mathbb{R}) =: \mathcal{C}$. This convergence can be metrized in the usual way (see for example [2, page 235]) by choosing a dense set of functions $(f_j)_{j \in \mathbb{N}} \subset \mathcal{C}$ and setting

$$d(\mu, \nu) := \sum_{j=1}^{\infty} 2^{-j} \frac{|\mu(f_j) - \nu(f_j)|}{1 + |\mu(f_j) - \nu(f_j)|}. \tag{3}$$

Moreover since $\mathbb{T}^d \times \{1, \ldots, q\}$ is compact and Polish also $(\mathcal{P}, d)$ is compact and Polish. Notice that $\sigma \in \Omega_n$ determines $\pi_n^\sigma \in \mathcal{P}_n$ and vice versa.

Using color profiles, we can rewrite the Hamiltonian as

$$H_n(\sigma) = -n^d \sum_{a=1}^{q} F(\pi_n^\sigma[a]) \tag{4}$$

with $F(\nu[a]) := \frac{1}{2}\langle J * \nu[a], \nu[a] \rangle = \frac{1}{2} \int \int \nu[a](du)\nu[a](dv)J(u-v)$. We will be interested in weak limits of color profiles in $\mathcal{P}$, especially those having q-dimensional Lebesgue densities of the form $\nu = \alpha\lambda = (\alpha[1]\lambda, \ldots, \alpha[q]\lambda)^T$ with $\alpha \in B$ where

$$\begin{aligned} B := \{\alpha = (\alpha[1], \ldots, \alpha[q])^T :\ & 0 \leq \alpha[a] \in L^\infty(\mathbb{T}^d, \lambda) \\ & \text{with } \sum_{a=1}^{q} \alpha[a](x) = 1 \text{ for } \lambda\text{-a.a } x \in \mathbb{T}^d\}. \end{aligned} \tag{5}$$

In what follows we will often write α instead of $\alpha\lambda$. Next we provide the LDP for the KPM.

Proposition 2.1 *The measures $\hat{\mu}_n = \mu_n \circ (\pi_n)^{-1}$ satisfy a LDP with rate n^d and ratefunction $I - \inf_{\nu \in \mathcal{P}} I(\nu)$ where*

$$I(\nu) = \begin{cases} -\beta \sum_{a=1}^{q} \langle J * \alpha[a], \alpha[a] \rangle + \langle S(\alpha|\mathrm{eq}), \lambda \rangle & \textit{if } \nu = \alpha\lambda \textit{ with } \alpha \in B \\ \infty & \textit{otherwise.} \end{cases} \tag{6}$$

and the relative entropy is given by $S(\alpha|\mathrm{eq}) = \sum_{a=1}^{q} \alpha[a] \log q\alpha[a]$.

Notice that we can rewrite the interaction part of the rate function as a punishing term for spatial inhomogeneities and a local term, i.e.

$$\begin{aligned} I(\nu) = \frac{\beta}{2} \sum_{a=1}^{q} \int du \int dv \big[\alpha[a](u) - \alpha[a](v)\big]^2 J(u-v) \\ + \langle S(\alpha|\mathrm{eq}) - \beta \sum_{a=1}^{q} J * \alpha[a]^2, \lambda \rangle. \end{aligned} \tag{7}$$

From this we see that global minimizers of I must be flat profiles where $\alpha[a](u)$ is independent of $u \in \mathbb{T}^d$. Hence the complete analysis of global minimizers (leading to the corresponding limit theorem for $\hat{\mu}_n$) is presented in the Ellis-Wang Theorem [11].

Before we state the main result about GnG of the fuzzy KPM in the next subsection, let us make the following definitions. These are the natural extensions to the Potts situation from the Ising situation in [17].

Definition 2.2 *Given any sequence $(\mu_n)_{n\in\mathbb{N}}$ with μ_n a probability measure on Ω_n for every $n \in \mathbb{N}$, define the single-spin conditional probabilities at site $u \in \mathbb{T}^d$ as*

$$\gamma_n^u(\cdot|\alpha_n^{(u)}) := \mu_n\big(\sigma(\lfloor nu \rfloor) = \cdot\, |\pi_n^{(u),\sigma} = \alpha_n^{(u)}\big) \qquad \alpha_n^{(u)} \in \mathcal{M}_n^u. \tag{8}$$

*(a) We call a color profile $\alpha \in B$ **good** for a sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ if there exists a neighborhood $\mathcal{N}_\alpha \subset B$ of α such that for all $\tilde{\alpha} \in \mathcal{N}_\alpha$ and for all $u \in \mathbb{T}^d$*

$$\gamma^u(\cdot|\tilde{\alpha}) := \lim_{n\uparrow\infty} \gamma_n^u(\cdot|\alpha_n^{(u)}) \tag{9}$$

exists for all sequences $(\alpha_n^{(u)})_{n\in\mathbb{N}}$ with $\alpha_n^{(u)} \in \mathcal{M}_n^u$ for every $n \in \mathbb{N}$ such that $\lim_{n\uparrow\infty} \alpha_n^{(u)} = \tilde{\alpha}$ in the weak sense. Moreover the limit must be independent of the choice of $(\alpha_n^{(u)})_{n\in\mathbb{N}}$.

*(b) A color profile $\alpha \in B$ is called **bad** for $(\mu_n)_{n\in\mathbb{N}}$ if it is not good for $(\mu_n)_{n\in\mathbb{N}}$.*

(c) $(\mu_n)_{n\in\mathbb{N}}$ is called Gibbs if it has no bad profiles in B.

Remarks: 1) Definition 2.2 (a) implies continuity of $\alpha \mapsto \gamma^u(\cdot|\alpha)$ in the metric $d(\cdot,\cdot)$ defined in (3) for all $u \in \mathbb{T}^d$ at good profiles.

2) For the KPM $(\mu_n)_{n\in\mathbb{N}}$ all color profiles $\alpha \in B$ are good since

$$\gamma^u(k|\alpha) = \frac{\exp(\beta(J * \alpha[k])(u))}{\sum_{l=1}^{q} \exp(\beta(J * \alpha[l])(u))} \tag{10}$$

and hence $(\mu_n)_{n\in\mathbb{N}}$ is Gibbs in the sense of Definition 2.2 (c).

3) Definition 2.2 assigns the notion of Gibbsianness to a sequence of probability measures that live on different spaces. This is different from the notion of Gibbsianness used for example in lattice systems [13, 14, 15, 16], but in that respect similar to the definition of Gibbsianness used in the mean-field setting [19, 21]. Since there is spatial dependence in our case it makes sense to call the quantity in (10) a specification kernel and α a boundary condition.

4) Definition 2.2 does not consider sequences $(\alpha_n^{(u)})_{n\in\mathbb{N}}$ whose weak limit is singular with respect to λ. But in Proposition 2.1 we saw that in the thermodynamic limit we can ignore profiles that are singular w.r.t. the Lebesgue measure or do not lie in the set B.

2.2 The fuzzy Kac Potts model

Consider the KPM under the local discretisation map $T : \{1,\dots,q\} \mapsto \{1,\dots,s\}$ where $1 < s < q$ and $\sum_{i=1}^s r_i = q$. Apply T to all sites simultaneously and consider the fuzzy Kac Potts measure $\mu_n^T := \mu_n \circ T^{-1}$.

Definition 2.3 *We call the generalized fuzzy KPM Gibbs if all profiles $\alpha \in B$ are good for the sequence μ_n^T.*

In order to determine Gibbsianness of the fuzzy KPM, similar to (8), we write for the single-site kernels

$$\gamma^u_{n,\beta,q,(r_1,\dots,r_s)}(k|\nu) := \mu_n^T(\sigma(\lfloor nu\rfloor) = k|\pi_n^{(u),\sigma} = \nu) \tag{11}$$

where β is the inverse temperature of the KPM and $\nu \in \mathcal{M}_n^u$ with s colors.

Proposition 2.4 *For each finite n and $u \in \mathbb{T}^d$ we have the representation*

$$\gamma^u_{n,\beta,q,(r_1,\dots,r_s)}(k|\nu) = \frac{r_k A^u\big(\beta_k(\nu), r_k, \Lambda_k(\nu)\big)}{\sum_{l=1}^s r_l A^u\big(\beta_l(\nu), r_l, \Lambda_l(\nu)\big)} \tag{12}$$

where $\Lambda_l(\nu) = \{x \in \Delta_n^d : \nu[l](x/n) = 1/n^d\}$, $\beta_l(\nu) = \beta|\Lambda_l(\nu)|/n^d$ and $A^u(\beta, r, M) := \mu_{M,\beta,r}\Big(\exp\big(\frac{\beta n^d}{|M|}(J * \pi_M[1])(\frac{\lfloor nu\rfloor}{n})\big)\Big)$. Here $\mu_{M,\beta,r}$ denotes the KPM in the subvolume $M \subset \Delta_n^d$ with Hamiltonian

$$H_M(\sigma) := -\frac{1}{2|M|}\sum_{x,y\in M} J(\frac{x-y}{n}) 1_{\sigma(x)=\sigma(y)},$$

inverse temperature β and r local states.

In view of Proposition 2.4 in order to determine GnG of the fuzzy model we must analyse limiting behavior of the constrained KPM $\mu_{M,\beta,r}$ and its continuity properties. The constrained model again satisfies a LDP similar to the one in Proposition 2.1 but now also the spatial structure of the level sets of the conditioning comes into play. We will say that a sequence of diluted sets $M_n \subset \Delta_n^d$ converges weakly to the Lebesgue density ρ if for all $f \in C(\mathbb{T}^d)$ we have

$$\frac{1}{n^d}\sum_{x\in M_n}\delta_{x/n}(f) = \frac{1}{n^d}\sum_{x\in M_n} f(\frac{x}{n}) \to \int du\rho(u)f(u)$$

as $n \uparrow \infty$ and write $M_n \Rightarrow \rho$.

Proposition 2.5 *(Diluted version of LDP for empirical color profiles). Consider a sequence of diluted sets $M_n \subset \Delta_n^d$ with $M_n \Rightarrow \rho$ for some Lebesgue density ρ with $N_\rho := \rho\lambda(\mathbb{T}^d) > 0$. Denote $\tilde\rho(u) := N_\rho^{-1}\rho(u)$, then the measures $\hat\mu_{M_n} := \mu_{M_n,\beta,q}\circ(\pi_{M_n})^{-1}$ satisfy a LDP with rate $|M_n|$ and ratefunction $I_{\tilde\rho} - \inf_{\nu\in\mathcal{P}} I_{\tilde\rho}(\nu)$ where*

$$I_{\tilde\rho}(\nu) = \begin{cases} -\beta\sum_{a=1}^q \langle J*\tilde\rho\alpha[a], \tilde\rho\alpha[a]\rangle + \langle S(\alpha|\mathrm{eq}), \tilde\rho\lambda\rangle & \text{if } \nu[a] = \tilde\rho\alpha[a]\lambda, \alpha\in B \\ \infty & \text{otherwise.}\end{cases} \tag{13}$$

Notice that we can replace the rate $|M_n|$ by the desired rate n^d since it is arbitrarily close to $|M_n|N_\rho^{-1}$ for large n. Similar to (7) we can rewrite $I_{\tilde\rho}$ as a sum of a punishing term for spatial inhomogeneities and a local term, i.e.

$$\begin{aligned} I_{\tilde\rho}(\nu) = \frac{\beta}{2}\sum_{a=1}^q \int du\tilde\rho(u)\int dv\tilde\rho(v)\big[\alpha[a](u) - \alpha[a](v)\big]^2 J(u-v) \\ + \int du\tilde\rho(u)\Big[-b_{\beta,\tilde\rho,J}(u)\sum_{a=1}^q \alpha[a]^2(u) + S(\alpha[\cdot](u)|\mathrm{eq})\Big] \end{aligned} \tag{14}$$

where we defined the site-dependent local temperature as

$$b_{\beta,\tilde\rho,J}(u) := \beta\int dv\tilde\rho(v)J(u-v).$$

Let us for the convenience of the reader recall the theorem from [19] about GnG for the mean-field fuzzy PM which summarizes the precise information on critical parameter values on GnG.

Theorem 2.6 *Consider the q-state mean-field PM at inverse temperature β, and let s and $r_1,\dots,r_s$ be positive integers with $1 < s < q$ and $\sum_{i=1}^s r_i = q$. Consider the limiting conditional probabilities of the corresponding mean-field fuzzy PM with spin partition $(r_1,\dots,r_s)$.*

(i) Suppose that $r_i \le 2$ for all $i = 1,\dots,s$. Then the limiting conditional probabilities are continuous functions of the empirical mean of the conditioning, for all $\beta \ge 0$.

Assume that $r_i \ge 3$ for some i and put $r* := \min\{r \ge 3, r = r_i \text{ for some } i = 1,\dots,s\}$. Denote by $\beta_c(r)$ the inverse critical temperature of the r-state mean-field PM. Then the following holds.

(ii) The limiting conditional probabilities are continuous for all $\beta < \beta_c(r*)$.

(iii) The limiting conditional probabilities are discontinuous for all $\beta \ge \beta_c(r*)$.

We now come to the main result, stating that for the fuzzy KPM the critical parameters for GnG are the same as for the mean-field fuzzy PM.

Theorem 2.7 *Consider the q-state KPM at inverse temperature β with $\int duJ(u) = 1$, and let s and $r_1, \dots, r_s$ be positive integers with $1 < s < q$ and $\sum_{i=1}^{s} r_i = q$. Consider the limiting conditional probabilities of the corresponding fuzzy KPM with spin partition $(r_1, \dots, r_s)$.*

(i) In case the parameters β and $(r_1, \dots, r_s)$ are such that the mean-field fuzzy PM is Gibbs, i.e. we are in the continuity region of Theorem 2.6, then also the fuzzy KPM is Gibbs. The specification kernel is given by

$$\lim_{n\uparrow\infty} \gamma^{u}_{n,\beta,q,(r_1,\dots,r_s)}(k|\alpha_n^{(u)}) = \frac{r_k \exp(\beta/r_k \int dv\rho_k(v)J(u-v))}{\sum_{l=1}^{s} r_l \exp(\beta/r_l \int dv\rho_l(v)J(u-v))} \tag{15}$$

where the limit as $(\alpha_n^{(u)})_{n\in\mathbb{N}}$ converge to $\alpha = (\rho_1\lambda, \dots, \rho_s\lambda)^T$ is defined in Definition 2.2 (a).

(ii) In case the parameters β and $(r_1, \dots, r_s)$ are such that the mean-field fuzzy PM is non-Gibbs, i.e. we are in the discontinuity region of Theorem 2.6, then also the fuzzy KPM is non-Gibbs.

We note that in case (i) the limiting kernels (15) are continuous functions of the conditioning α.

3 Proofs

Let us start with the proofs of the large deviation results. Notice, considering $M_n \equiv \Delta_n^d$ Proposition 2.1 is a special case of Proposition 2.5.

3.1 Proof of Proposition 2.5

For convenience we write μ_{M_n} for $\mu_{M_n,\beta,q}$. Let us proceed in two steps.

Step 1: First we derive the LDP for $J \equiv 0$. In this case our Gibbs measure μ_{M_n} is just a spatial product measure on $M_n \subset \Delta_n^d$ of the equidistribution on $\{1, \dots, q\}$. We consider the exponential moment generating function of the color profile at finite discretization n for some

$F \in \mathcal{C}$,

$$\begin{aligned}\mu_{M_n}[\exp(|M_n|\pi_{M_n}(F))] &= \mu_{M_n}[\exp(\sum_{a=1}^{q}\sum_{x\in M_n} 1_{\sigma(x)=a}F(a,\frac{x}{n}))]\\ &= \mu_{M_n}[\prod_{x\in M_n}\exp(\sum_{a=1}^{q} 1_{\sigma(x)=a}F(a,\frac{x}{n}))]\\ &= \prod_{x\in M_n}\frac{1}{q}\sum_{a=1}^{q}\exp\big(F(a,\frac{x}{n})\big).\end{aligned}$$

Due to spatial independence, we recover the important single-site logarithmic moment generating function

$$\Lambda(F(u)) := \log\frac{1}{q}\sum_{a=1}^{q}\exp(F_a(u)).$$

The limit of discretization going to zero for the logarithmic moment generating function of the color profile is given by

$$\frac{1}{|M_n|}\log\mu_{M_n}[\exp(|M_n|\pi_{M_n}(F))] = \frac{1}{|M_n|}\sum_{x\in M_n}\Lambda(F(\frac{x}{n})) \to \int du\tilde\rho(u)\Lambda(F(u)).$$

Notice that the diluted rate function

$$I_{\tilde\rho}(\nu) := \begin{cases}\langle S(\alpha|\mathrm{eq}),\tilde\rho\lambda\rangle, & \text{if } \nu=\alpha\tilde\rho\lambda \text{ with } \alpha\in B\\ \infty, & \text{otherwise}\end{cases}$$

is equivalent to

$$\Lambda^*_{\tilde\rho}(\nu) := \begin{cases}\sup_{F\in C}[\nu(F)-\int du\tilde\rho(u)\Lambda(F(u))], & \text{if } \nu=\alpha\tilde\rho\lambda \text{ with } \alpha\in B\\ \infty, & \text{otherwise.}\end{cases}$$

Indeed, by duality (see also [8, Lemma 6.2.13]) it suffices to show that for all $F \in \mathcal{C}$

$$\int du\tilde\rho(u)\Lambda(F(u)) = \sup_{\nu\in\mathcal{P}}\Big(\nu(F) - I_{\tilde\rho}(\nu)\Big). \tag{16}$$

From this we see that it suffices to take $\nu \in \mathcal{P}$ with Lebesgue density $\alpha\tilde\rho$ since the r.h.s. of (16) is equal to minus infinity otherwise. In that case we can write

$$\nu(F) - I_{\tilde\rho}(\nu) = \int du\tilde\rho(u)\Big(\langle F(u),\alpha[\cdot](u)\rangle - S(\alpha[\cdot](u)|\mathrm{eq})\Big)$$

and the supremum can be considered sitewise. Using Jensen's inequality it is easy to see that the supremum is attained in $\alpha[a](u) = \exp F_a(u)/\sum_{b=1}^{q}\exp F_b(u)$ and equation (16) is indeed satisfied. That the supremum is achieved follows by convexity (detailed arguments see for example [8, Lemma 2.6.13]). We further note that for continuous F this optimizing profile is even continuous w.r.t. the spatial variable as well.

Upper Bound: It suffices to consider $K \subset \mathcal{P}$ compact since $\hat{\mu}_{M_n}$ is exponentially tight as is takes values in the compact set $\mathcal{P}$.

We can assume without loss that $0 < \inf_{\nu \in K} I_{\tilde{\rho}}(\nu)$ and hence we can pick $0 < a < \inf_{\nu \in K} I_{\tilde{\rho}}(\nu)$. For every $\nu \in K$ there exists a $F_\nu \in C$ such that $\nu(F_\nu) - \int du \tilde{\rho}(u) \Lambda(F_\nu(u)) > a$ and the sets

$$U_\nu := \{\hat{\nu} \in \mathcal{P} : \hat{\nu}(F_\nu) - \int du \tilde{\rho}(u) \Lambda(F_\nu(u)) > a\}$$

form an open covering of K. Using the Markov inequality we can estimate

$$\begin{aligned}
&\frac{1}{|M_n|} \log \hat{\mu}_{M_n}(U_\nu) \\
&\quad = \frac{1}{|M_n|} \log \mu_{M_n}[\exp\big(|M_n| \pi_{M_n}(F_\nu)\big) > \exp\big(|M_n|(a + \int du \tilde{\rho}(u) \Lambda(F_\nu(u)))\big)] \\
&\quad \leq -a - \int du \tilde{\rho}(u) \Lambda(F_\nu(u)) + \frac{1}{|M_n|} \log \mu_{M_n}[\exp\big(|M_n| \pi_{M_n}(F_\nu)\big)]
\end{aligned}$$

and hence $\limsup_{n \uparrow \infty} \frac{1}{|M_n|} \log \hat{\mu}_{M_n}(U_\nu) \leq -a$ for all $\nu \in K$. Since K is compact it can be covered by a finite number of U_ν and thus $\limsup_{n \uparrow \infty} \frac{1}{|M_n|} \log \hat{\mu}_{M_n}(K) \leq -\inf_{\nu \in K} I_{\tilde{\rho}}(\nu)$.

Lower Bound: Let $G \subset \mathcal{P}$ be open and assume $\nu \in G$ to have a spatially flat color distribution on each of the sets C_k of a partitioning *after* normalization by the density $\tilde{\rho}(u)\lambda(du)$ w.r.t. space. More precisely this means $\frac{d\nu}{d\lambda}(u) = \sum_{k=1}^{N} \alpha_k \tilde{\rho}(u) 1_{C_k}(u)$ where $\alpha_k = (\alpha_k[a])_{a \in \{1,\dots,q\}}$ with $\sum_{a=1}^{q} \alpha_k[a] = 1$ for all k and $(C_k)_{k \in \{1,\dots,N'\}}$ a finite partition of the torus i.e. $\mathbb{T}^d = \bigcup_{k=1}^{N'} C_k$. Assume that the first $N \leq N'$ sets in this partition have strictly positive weight $\tilde{\rho}\lambda(C_k) > 0$.

There exists $\varepsilon_1 > 0$ such that $N_{\varepsilon_1}(\nu) \subset G$ and thus using the definition (3) we have

$$\begin{aligned}
\hat{\mu}_{M_n}(G) &\geq \mu_{M_n}(\pi_{M_n} \in N_{\varepsilon_1}(\nu)) = \mu_{M_n}(d(\pi_{M_n}, \nu) < \varepsilon_1) \\
&= \mu_{M_n}\Big(\sum_{j=1}^{\infty} 2^{-j} \frac{|(\pi_{M_n} - \nu)(f_j)|}{1 + |(\pi_{M_n} - \nu)(f_j)|} < \varepsilon_1\Big) \\
&\geq \mu_{M_n}\Big(\sum_{j=1}^{K(\varepsilon_1)} 2^{-j} \frac{|(\pi_{M_n} - \nu)(f_j)|}{1 + |(\pi_{M_n} - \nu)(f_j)|} < \frac{\varepsilon_1}{2}\Big)
\end{aligned}$$

where $K(\varepsilon_1)$ is large enough such that $\sum_{j=K(\varepsilon_1)+1}^{\infty} 2^{-j} < \varepsilon_1/2$. Further we can estimate

$$\begin{aligned}
\mu_{M_n}\Big(\sum_{j=1}^{K(\varepsilon_1)} 2^{-j} \frac{|(\pi_{M_n} - \nu)(f_j)|}{1 + |(\pi_{M_n} - \nu)(f_j)|} < \frac{\varepsilon_1}{2}\Big) &\geq \mu_{M_n}\Big(\bigcap_{j=1}^{K(\varepsilon_1)} \Big\{\frac{|(\pi_{M_n} - \nu)(f_j)|}{1 + |(\pi_{M_n} - \nu)(f_j)|} < \frac{\varepsilon_1}{2}\Big\}\Big) \\
&= \mu_{M_n}\Big(\bigcap_{j=1}^{K(\varepsilon_1)} \Big\{|(\pi_{M_n} - \nu)(f_j)| < \varepsilon_2\Big\}\Big)
\end{aligned}$$

where we set $\varepsilon_2 := \varepsilon_1/(2-\varepsilon_1)$. Notice for ν assumed to have a spatially flat color distribution the spatial color structure breaks down on every partition C_k and we deal with empirical measures rather then color profiles. More precisely

$$
\begin{aligned}
|(\pi^\sigma_{M_n} - \nu)(f_j)| &= |\sum_{k=1}^{N} (\pi^\sigma_{M_n} - \nu)(f_j 1_{C_k})| \\
&\leq \sum_{k=1}^{N} |\sum_{a=1}^{q} [n^{-d} \sum_{x \in M_n \cap nC_k} f_j(a, \frac{x}{n}) 1_{\sigma(x)=a} - \alpha_k[a] \int_{C_k} du \tilde\rho(u) f_j(a,u)]|
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\mu_{M_n}\Big(\bigcap_{j=1}^{K(\varepsilon_1)} \Big\{ |(\pi_{M_n} - \nu)(f_j)| < \varepsilon_2 \Big\} \Big) \\
&\geq \mu_{M_n}\Big(\bigcap_{k=1}^{N} \bigcap_{j=1}^{K(\varepsilon_1)} \Big\{ |\sum_{a=1}^{q} [n^{-d} \sum_{x \in M_n \cap nC_k} f_j(a, \frac{x}{n}) 1_{\sigma(x)=a} - \alpha_k[a] \int_{C_k} du \tilde\rho(u) f_j(a,u)]| < \frac{\varepsilon_2}{N} \Big\} \Big) \\
&= \prod_{k=1}^{N'} \mu_{M_n}\Big(\bigcap_{j=1}^{K(\varepsilon_1)} \Big\{ |\sum_{a=1}^{q} [n^{-d} \sum_{x \in M_n \cap nC_k} f_j(a, \frac{x}{n}) 1_{\sigma(x)=a} - \alpha_k[a] \int_{C_k} du \tilde\rho(u) f_j(a,u)]| < \frac{\varepsilon_2}{N} \Big\} \Big)
\end{aligned}
$$

where we used that μ is a product measure in the last line. Notice that for $k \in \{N+1, \dots, N'\}$ the events inside the μ_{M_n}-measure occur deterministically for n sufficiently large by the assumption of convergence of the density of the set M_n to zero on those C_k and hence the product restricts to the terms for $k \leq N$. For those k let us set $\varepsilon_3 := \varepsilon_2/(N \sup_{j \in \{1,\dots,J(\varepsilon_1)\}} \|f_j\|)$ and introduce the empirical measures $L^\sigma_{M_n,k}(a) := |M_n \cap nC_k|^{-1} \sum_{x \in M_n \cap nC_k} 1_{\sigma(x)=a}$. Then we can further write

$$
\begin{aligned}
&\mu_{M_n}\Big(\bigcap_{j=1}^{K(\varepsilon_1)} \Big\{ |\sum_{a=1}^{q} [|M_n|^{-1} \sum_{x \in M_n \cap nC_k} f_j(a, \frac{x}{n}) 1_{\sigma(x)=a} - \alpha_k[a] \int_{C_k} du \tilde\rho(u) f_j(a,u)]| < \frac{\varepsilon_2}{N} \Big\} \Big) \\
&\geq \mu_{M_n}\Big(\Big\{ \sum_{a=1}^{q} ||M_n|^{-1} \sum_{x \in M_n \cap nC_k} 1_{\sigma(x)=a} - \alpha_k[a] \tilde\rho \lambda(C_k)| < \varepsilon_3 \Big\} \Big) \\
&\geq \mu_{M_n}\Big(\Big\{ \sum_{a=1}^{q} |\frac{|M_n \cap nC_k|}{|M_n|} L^\sigma_{M_n,k}(a) - \alpha_k[a] \tilde\rho \lambda(C_k)| < \varepsilon_3 \Big\} \Big) \\
&\geq \mu_{M_n}\Big(\Big\{ |\frac{|M_n \cap nC_k|}{|M_n| \tilde\rho \lambda(C_k)} L^\sigma_{M_n,k}(a) - \alpha_k[a]| < \frac{\varepsilon_3}{q \tilde\rho \lambda(C_k)}, \text{ for all } a \in \{1, \dots, q\} \Big\} \Big).
\end{aligned}
$$

We set $\varepsilon_4 := \min_{k \in \{1,\dots,J(\varepsilon_1)\}} \varepsilon_3/(q \tilde\rho \lambda(C_k))$ and note that $|M_n \cap nC_k|/(|M_n| \tilde\rho \lambda(C_k)) \to 1$ as $n \uparrow \infty$. Thus we can assume n large enough such that $\max_{k \in \{1,\dots,J(\varepsilon_1)\}} ||M_n \cap nC_k|/(|M_n| \tilde\rho \lambda(C_k)) - 1| < \tilde\varepsilon < \varepsilon_4/2$. Let $\| \cdot \|_{TV}$ denote the total variational distance of probability measures on $\{1, \ldots, q\}$. Then we have

$$\begin{aligned}
&\mu_{M_n}\Big(\Big\{|\frac{|M_n \cap nC_k|}{n^d \tilde{\rho}\lambda(C_k)} L^\sigma_{M_n,k}(a) - \alpha_k[a]| < \frac{\varepsilon_3}{q\tilde{\rho}\lambda(C_k)}, \text{ for all } a \in \{1, \ldots, q\}\Big\}\Big) \\
&\geq \mu_{M_n}\Big(\Big\{|\frac{|M_n \cap nC_k|}{n^d \tilde{\rho}\lambda(C_k)} L^\sigma_{M_n,k}(a) - \alpha_k[a]| < \varepsilon_4, \text{ for all } a \in \{1, \ldots, q\}\Big\}\Big) \\
&\geq \mu_{M_n}\Big(\Big\{|L^\sigma_{M_n,k}(a) - \alpha_k[a]| < \varepsilon_4/2, \text{ for all } a \in \{1, \ldots, q\}\Big\}\Big) \\
&\geq \mu_{M_n}\Big(\Big\{\|L^\sigma_{M_n,k} - \alpha_k\|_{TV} < \varepsilon_4/4\Big\}\Big).
\end{aligned}$$

Now we are in the position to apply the lower bound estimate in Sanov's Theorem and write

$$\begin{aligned}
\liminf_{n\uparrow\infty} \frac{1}{n^d} \log \hat{\mu}_{M_n}(G) &\geq \sum_{k=1}^{N} \tilde{\rho}\lambda(C_k) \liminf_{n\uparrow\infty} \frac{1}{n^d \tilde{\rho}\lambda(C_k)} \log \mu_{M_n}\Big(\Big\{\|L^\sigma_{M_n,k} - \alpha_k\|_{TV} < \frac{\varepsilon_4}{4}\Big\}\Big) \\
&\geq - \inf_{\hat{\nu}\in M_{\varepsilon_4/4}(\nu)} \int du \tilde{\rho}(u) S(\hat{\nu}(u)|\mathrm{eq})
\end{aligned}$$

where $M_\varepsilon(\nu) := \{\hat{\nu} \in \mathcal{P} : \frac{d\hat{\nu}}{d\lambda}(u) = \sum_{k=1}^{N'} \hat{\alpha}_k \tilde{\rho}(u) 1_{C_k}(u)$ for the same partition as ν and $\max_{k\in\{1,\ldots,N'\}} ||\hat{\alpha}_k - \alpha_k\|_{TV} < \varepsilon\}$. Letting ε_4 go to zero we can replace

$$\inf_{\hat{\nu}\in M_{\varepsilon_4/4}(\nu)} \int du \tilde{\rho}(u) S(\hat{\nu}(u)|\mathrm{eq}) \quad \text{by} \quad \inf_{\hat{\nu}\in M\cap G} \int du \tilde{\rho}(u) S(\hat{\nu}(u)|\mathrm{eq})$$

where

$$M := \{\hat{\nu} \in \mathcal{P} : \frac{d\hat{\nu}}{d\lambda}(u) = \sum_{k=1}^{N'} \hat{\alpha}_k \tilde{\rho}(u) 1_{C_k}(u) \text{ for some finite partition } C_k \text{ of } \mathbb{T}^d\}.$$

So the proof of the lower bound is finished once we show

$$\inf_{\nu\in M\cap G_{\tilde{\rho}\lambda}} \int du \tilde{\rho}(u) S(\nu(u)|\mathrm{eq}) = \inf_{\nu\in G_{\tilde{\rho}\lambda}} \int du \tilde{\rho}(u) S(\nu(u)|\mathrm{eq})$$

where $G_{\tilde{\rho}\lambda}$ denotes the set of probability measures in G of the form $\alpha\tilde{\rho}\lambda$. The direction '$\geq$' is clear. For the direction '$\leq$' notice first that for every $\nu \in G_{\tilde{\rho}\lambda}$ we can find $\nu^{flat}(\nu)$ such that $\nu^{flat}(\nu) \in M \cap G_{\tilde{\rho}\lambda}$. Indeed, given any finite partition $(C_k)_{k\in\{1,\ldots,N'\}}$ of $\mathbb{T}^d$ where $\tilde{\rho}\lambda(C_k) > 0$ for $k \leq N$ and $\tilde{\rho}\lambda(C_k) = 0$ for $N < k \leq N'$, the measure $\nu^{flat}(\nu)$ with $d\nu^{flat}(\nu)/d\lambda(u) = \sum_{k=1}^{N'} \alpha_{k,\nu} \tilde{\rho}(u) 1_{C_k}(u)$ where

$$\alpha_{k,\nu}[a] := \begin{cases} \tilde{\rho}\lambda(C_k)^{-1} \int_{C_k} du \tilde{\rho}(u) \alpha[a](u), & \text{if } \tilde{\rho}\lambda(C_k) > 0 \\ 0, & \text{otherwise} \end{cases}$$

is in M. Since G is open it is enough to find a sufficiently fine partition such that $d(\nu, \nu^{flat}(\nu)) < \varepsilon$ for any $\varepsilon > 0$. For a given $\varepsilon > 0$ let again $K(\varepsilon)$ be large enough such that $\sum_{j=K(\varepsilon)+1}^{\infty} 2^{-j} < \varepsilon/2$. Consider the finitely many evaluation functions $(f_j)_{j\in\{1,\ldots,K(\varepsilon)\}}$ from the definition of the metric $d(\cdot,\cdot)$ see (3). The f_j are uniformly continuous and hence it is possible to partition the torus in such a way that for all $a \in \{1,\dots,q\}$ and $j \in \{1,\dots,K(\varepsilon)\}$ we have

$$\sup_{x\in C_k} |f_j(a,x) - \tilde{\rho}\lambda(C_k)^{-1} \int_{C_k} du\tilde{\rho}(u) f_j(a,u)| < \varepsilon$$

unless $\tilde{\rho}\lambda(C_k) = 0$. Again assuming the partition is ordered such that $\tilde{\rho}\lambda(C_k) > 0$ for $k \leq N$ and $\tilde{\rho}\lambda(C_k) = 0$ for $N < k \leq N'$, we can estimate

$$\begin{aligned}
d(\nu,\nu^{flat}(\nu)) &= \sum_{j=1}^{\infty} 2^{-j} \frac{|\nu^{flat}(\nu)(f_j) - \nu(f_j)|}{1 + |\nu^{flat}(\nu)(f_j) - \nu(f_j)|} \leq \frac{\varepsilon}{2} + \sum_{j=1}^{K(\varepsilon)} 2^{-j} |\nu^{flat}(\nu)(f_j) - \nu(f_j)| \\
&= \frac{\varepsilon}{2} + \sum_{j=1}^{K(\varepsilon)} 2^{-j} \Big| \sum_{a=1}^{q} \sum_{k=1}^{N} \big[\alpha_{k,\nu}[a] \int_{C_k} du\tilde{\rho}(u) f_j(a,u) - \int_{C_k} du\tilde{\rho}(u)\alpha[a](u) f_j(a,u)\big]\Big| \\
&= \frac{\varepsilon}{2} + \sum_{j=1}^{K(\varepsilon)} 2^{-j} \Big| \sum_{a=1}^{q} \sum_{k=1}^{N} \int_{C_k} du\tilde{\rho}(u)\alpha[a](u) \big[\tilde{\rho}\lambda(C_k)^{-1} \int_{C_k} dv\tilde{\rho}(v) f_j(a,v) - f_j(a,u)\big]\Big| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Finally notice that $u \mapsto S(\nu(u)|\mathrm{eq})$ is a convex function and thus using Jensen's inequality we have

$$\begin{aligned}
\int du\tilde{\rho}(u) S(\nu(u)|\mathrm{eq}) &\geq \sum_{k=1}^{N} \tilde{\rho}\lambda(C_k) S(\tilde{\rho}\lambda(C_k)^{-1} \int_{C_k} du\tilde{\rho}(u)\nu(u)|\mathrm{eq}) \\
&= \sum_{k=1}^{N} \tilde{\rho}\lambda(C_k) S(\alpha_{k,\nu}|\mathrm{eq}) = \int du\tilde{\rho}(u) S(\nu^{flat}(\nu)(u)|\mathrm{eq}).
\end{aligned}$$

This implies

$$\begin{aligned}
\inf_{\nu\in G_{\tilde{\rho}\lambda}} \int du\tilde{\rho}(u) S(\nu(u)|\mathrm{eq}) &\geq \inf_{\nu\in G_{\tilde{\rho}\lambda}} \int du\tilde{\rho}(u) S(\nu^{flat}(\nu)(u)|\mathrm{eq}) \\
&\geq \inf_{\nu\in M\cap G_{\tilde{\rho}\lambda}} \int du\tilde{\rho}(u) S(\nu(u)|\mathrm{eq}).
\end{aligned}$$

Step 2: Let us now consider the case with interaction, i.e. $J \not\equiv 0$. We want to employ Varadhan's Lemma ([8, Theorem 4.3.1]) to prove the LDP as in [22, Theorem 23.19]. The conditions in Varadhan's Lemma are indeed satisfied since J is bounded. $\square$

3.2 Proof of Proposition 2.4

To compute the l.h.s. of (12) write for a fuzzy configuration $\eta \in \{1,\dots,s\}^{\Delta_n^d \setminus \lfloor nu \rfloor}$ where $u \in \mathbb{T}^d$

$$
\begin{aligned}
\mu_n^T(\sigma(\lfloor nu \rfloor) = k | \sigma_{\Delta_n^d \setminus \lfloor nu \rfloor} = \eta) &= \frac{1}{Z_1(\eta)} \sum_{\xi : T(\xi) = (k,\eta)} \mu_n(\xi) \\
&= \frac{1}{Z_2(\eta)} \sum_{\xi : T(\xi) = (k,\eta)} \exp\Big(\beta n^d \sum_{a=1}^{q} F(\pi_n^\xi[a])\Big)
\end{aligned} \tag{17}
$$

where $Z_1(\eta)$ and $Z_2(\eta)$ are the appropriate normalization constants. For notational convenience we introduce the notation

$$
\pi_{n,\Lambda}^\sigma := \frac{1}{n^d}\Big(\sum_{x\in\Lambda} 1_{\sigma(x)=1}\delta_{x/n}, \dots, \sum_{x\in\Lambda} 1_{\sigma(x)=q}\delta_{x/n}\Big)^T
$$

for the color profile on $\Lambda \subset \Delta_n^d$ normalized by Δ_n^d. In the next step we separate the components in π_n corresponding to the site $\lfloor nu \rfloor$. We have

$$
\begin{aligned}
&\sum_{a=1}^{q} F(\pi_n^\xi[a]) = \frac{1}{2}\sum_{a=1}^{q} \langle J * \pi_n^\xi[a], \pi_n^\xi[a] \rangle \\
&= \frac{1}{2}\sum_{a=1}^{q} \Big(\langle J * \pi^\xi_{n,\Delta_n^d \setminus \lfloor nu \rfloor}[a], \pi^\xi_{n,\Delta_n^d \setminus \lfloor nu \rfloor}[a] \rangle + \frac{2}{n^d}(J * \pi^\xi_{n,\Delta_n^d \setminus \lfloor nu \rfloor}[a])(\frac{\lfloor nu \rfloor}{n}) 1_{\xi(\lfloor nu \rfloor)=a} \Big) \\
&= \frac{1}{2}\sum_{a : T(a) = k} \Big(\langle J * \pi^\xi_{n,\Delta_n^d \setminus \lfloor nu \rfloor}[a], \pi^\xi_{n,\Delta_n^d \setminus \lfloor nu \rfloor}[a] \rangle + \frac{2}{n^d}(J * \pi^\xi_{n,\Delta_n^d \setminus \lfloor nu \rfloor}[a])(\frac{\lfloor nu \rfloor}{n}) 1_{\xi(\lfloor nu \rfloor)=a} \Big) \\
&\qquad + \frac{1}{2}\sum_{l \neq k} \sum_{a : T(a) = l} \langle J * \pi^\xi_{n,\Delta_n^d \setminus \lfloor nu \rfloor}[a], \pi^\xi_{n,\Delta_n^d \setminus \lfloor nu \rfloor}[a] \rangle
\end{aligned}
$$

where in the last line we used that $T(\xi(\lfloor nu \rfloor)) = k$ assumed in (17). Notice, the first and the third summand in the last line do not depend on the site $\lfloor nu \rfloor$, in other words, they only depend on the boundary condition η. Hence in the conditional Gibbs measure (17) corresponding to the above expression the third summand can be shifted into the normalization constant in the denominator and the remaining two summands can be normalized using the first summand. Let us introduce the levelsets of the boundary condition $\Lambda_l(\eta) := \{x \in \Delta_n^d : \eta(x) = l\}$ then we can write

$$
\begin{aligned}
&\Big[\sum_{\xi(\lfloor nu\rfloor):T(\xi(\lfloor nu\rfloor))=k}\sum_{\xi_{\Lambda_k(\eta)}}\exp\Big(\frac{\beta n^d}{2}\sum_{a:T(a)=k}\big(\langle J*\pi^{\xi}_{n,\Lambda_k(\eta)}[a],\pi^{\xi}_{n,\Lambda_k(\eta)}[a]\rangle\\
&\qquad\qquad+\frac{2}{n^d}(J*\pi^{\xi}_{n,\Lambda_k(\eta)}[a])(\frac{\lfloor nu\rfloor}{n})1_{\xi(\lfloor nu\rfloor)=a}\big)\Big)\Big]\\
&\qquad\qquad\times\Big[\sum_{\xi_{\Lambda_k(\eta)}}\exp\Big(\frac{\beta n^d}{2}\sum_{a:T(a)=k}\langle J*\pi^{\xi}_{n,\Lambda_k(\eta)}[a],\pi^{\xi}_{n,\Lambda_k(\eta)}[a]\rangle\Big)\Big]^{-1}\\
&=\Big[\sum_{\xi(\lfloor nu\rfloor):T(\xi(\lfloor nu\rfloor))=k}\sum_{\xi_{\Lambda_k(\eta)}}\exp\Big(\sum_{a:T(a)=k}\big((\frac{\beta|\Lambda_k(\eta)|^2}{2n^d}\langle J*\pi^{\xi}_{\Lambda_k(\eta)}[a],\pi^{\xi}_{\Lambda_k(\eta),n}[a]\rangle\\
&\qquad\qquad+\frac{\beta|\Lambda_k(\eta)|}{n^d}(J*\pi^{\xi}_{\Lambda_k(\eta)}[a])(\frac{\lfloor nu\rfloor}{n})1_{\xi(\lfloor nu\rfloor)=a})\big)\Big)\Big]\\
&\qquad\qquad\times\Big[\sum_{\xi_{\Lambda_k(\eta)}}\exp\Big(\frac{\beta|\Lambda_k(\eta)|^2}{2n^d}\sum_{a:T(a)=k}\langle J*\pi^{\xi}_{\Lambda_k(\eta)}[a],\pi^{\xi}_{\Lambda_k(\eta)}[a]\rangle\Big)\Big]^{-1}\\
&=\sum_{\xi(\lfloor nu\rfloor):T(\xi(\lfloor nu\rfloor))=k}\mu_{\Lambda_k(\eta),\beta\frac{|\Lambda_k(\eta)|}{n^d},r_k}\Big[\exp\Big(\frac{\beta|\Lambda_k(\eta)|}{n^d}(J*\pi_{\Lambda_k(\eta)}[\xi(\lfloor nu\rfloor)])(\frac{\lfloor nu\rfloor}{n})\Big)\Big]\\
&=r_k\mu_{\Lambda_k(\eta),\beta\frac{|\Lambda_k(\eta)|}{n^d},r_k}\Big[\exp\Big(\frac{\beta|\Lambda_k(\eta)|}{n^d}(J*\pi_{\Lambda_k(\eta)}[1])(\frac{\lfloor nu\rfloor}{n})\Big)\Big]
\end{aligned}
$$

as required. □

3.3 Proof of Theorem 2.7

(i): First note that a given sequence of boundary conditions $(\nu_n)_{n\in\mathbb{N}}$ in the single-site specification kernel (12) is represented in the sequence of level sets $(M_k(\nu_n))_{n\in\mathbb{N}}$ and in the temperature parameters $(\beta_k(\nu_n))_{n\in\mathbb{N}}$ corresponding to the fuzzy classes $k\in\{1,\dots,s\}$. For each such fuzzy class k we have a LDP given in Proposition 2.5 with limiting dilution ρ_k and limiting temperature βN_{ρ_k} where β is assumed to be in the uniqueness region of the mean-field model. Notice that for any such ρ_k the first term in (14) is minimized by essentially flat profiles, i.e. profiles where $\alpha[\cdot](u)\equiv\alpha[\cdot]$ away from $\{u\in\mathbb{T}^d:\rho_k(u)=0\}$. This implies that the second term in $I_{\tilde\rho_k}$ becomes a non-normalized mean-field rate function with spatially homogeneous temperature parameter

$$b_k:=\int du\tilde\rho_k(u)b_{\beta N_{\rho_k},\tilde\rho_k,J}(u)=\beta\int dv\rho_k(v)\int du\tilde\rho_k(u)J(u-v)\le\beta.$$

By the Ellis-Wang Theorem [11] for the mean-field PM the minimizing profile density is thus the equidistribution, i.e. $\alpha[l](u)\equiv 1/r_k$. Consequently the components of the specification kernel (12)

$$A^u\big(\beta_k(\nu_n),r_k,\Lambda_k(\nu_n)\big)=\mu_{\Lambda_k(\nu_n),\beta,r_k}\Big(\exp\big(\frac{\beta_k(\nu_n)n^d}{|\Lambda_k(\nu_n)|}(J*\pi_{\Lambda_k(\nu_n)}[1])(\frac{\lfloor nu\rfloor}{n})\big)\Big)$$

converge to $\exp\big(\beta/r_k \int dv \rho_k(v) J(u-v)\big)$ as n goes to infinity. Hence we have the limiting kernels (15) are continuous functions of the dilution.

(ii): Let $\bar{\alpha} \in \mathcal{P}(\{1,\dots,s\})$ be a point of discontinuity of the limiting conditional probabilities of the mean-field fuzzy PM with inverse temperature β. Such a conditioning is characterized by the fact that for some fuzzy class r_k, $\beta\bar{\alpha}[k] = \beta_c(r_k)$. Here $\beta_c(r_k)$ is the critical temperature parameter where the mean-field non-normalized rate function

$$I_{MF,\bar{\alpha}[k]}(\hat{\alpha}) := -\beta\bar{\alpha}[k]\sum_{a=1}^{r_k}\hat{\alpha}[a]^2 + S(\hat{\alpha}|\mathrm{eq})$$

of the r_k-states PM shows a discontinuous (first-order) jump from uniqueness to non-uniqueness of the global minimizers (for details see [19]). Hence in every neighborhood of $\bar{\alpha}$ there exist $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{P}(\{1,\dots,s\}) \setminus \{\bar{\alpha}\}$ such that the corresponding minimizers α_1^*, α_2^* are unique and stay at a finite distance $\delta > 0$.

Notice that $\bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2$ can also be interpreted as flat density profiles ν, ν_1, ν_2, in other words as elements of $\mathcal{P}(\mathbb{T}_n^d \times \{1,\dots,s\})$ which are spatially homogeneous. Then, for any $u \in \mathbb{T}^d$ consider two sequences of boundary measures approaching ν_1 and ν_2 respectively which have unique corresponding minimizers. Inspecting the form of the specifications given in (12) we see different limits for the sequences going to ν_1 and ν_2. This proves non-Gibbsianness. $\square$

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