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The Dirichlet problem for nonlocal operators with kernels: Convex and nonconvex domains

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Abstract. We study the interior regularity of solutions to the Dirichlet problem $Lu = g$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, for anisotropic operators of fractional type

$$Lu(x) = \int_0^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\omega)}{\rho^{1+2s}}.$$

Here, a is any measure on S^{n-1} (a prototype example for L is given by the sum of one-dimensional fractional Laplacians in fixed, given directions).

When $a \in C^\infty(S^{n-1})$ and g is $C^\infty(\Omega)$, solutions are known to be C^∞ inside Ω (but not up to the boundary). However, when a is a general measure, or even when a is $L^\infty(S^{n-1})$, solutions are only known to be C^{3s} inside Ω.

We prove here that, for general measures a, solutions are $C^{1+3s-\epsilon}$ inside Ω for all $\epsilon > 0$ whenever Ω is convex. When $a \in L^\infty(S^{n-1})$, we show that the same holds in all $C^{1,1}$ domains. In particular, solutions always possess a classical first derivative.

The assumptions on the domain are sharp, since if the domain is not convex and the spectral measure is singular, we construct an explicit counterexample for which u is *not* $C^{3s+\epsilon}$ for any $\epsilon > 0$ – even if g and Ω are C^∞.

1. Introduction

Recently, a great attention in the literature has been devoted to the study of equations of elliptic type with fractional order. The leading example of the operators considered is the fractional Laplacian

$$(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy. \tag{1.1}$$

Several similarities arise between this operator and the classical Laplacian: for instance, the fractional Laplacian enjoys a "good" interior regularity theory in Hölder spaces and it has "nice" functional properties in Sobolev spaces (see e.g. [6, 11, 3]). Nevertheless, the fractional Laplacian also presents some striking difference with respect to the fractional case: for example, solutions are in general not uniformly Lipschitz continuous up to the boundary (see e.g. [10, 8]) and fractional harmonic functions are locally dense in C^k (see [4]), in sharp contrast with respect to the classical case.

A simple difference between the fractional and the classical Laplacians is also given by the fact that the classical Laplacian may be reconstructed as the sum of finitely many one-dimensional operators, namely one can write

$$\Delta = \partial_1^2 + \cdots + \partial_n^2, \tag{1.2}$$

and each ∂_i^2 is indeed the Laplacian in a given direction. This phenomenon is typical for the classical case and it has no counterpart in the fractional setting, since the operator in (1.1) cannot be reduced to a finite sets of directions.

Nevertheless, in order to study equations in anisotropic media, it is important to understand operators obtained by the superposition of fractional one-dimensional (or lower-dimensional) operators, or, more generally, by the superposition of different operators in different directions, see [7]. For this reason, we consider here the anisotropic integro-differential operator

$$Lu(x) = \int_0^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\omega)}{\rho^{1+2s}}. \tag{1.3}$$

Here a is a non-negative measure on S^{n-1} (called in jargon the "spectral measure"), and we suppose that it satisfies the following "ellipticity" assumption

$$\inf_{\varpi\in S^{n-1}} \int_{S^{n-1}} |\omega\cdot\varpi|^{2s}\, da(\omega) \geq \lambda \ \text{ and } \ \int_{S^{n-1}} da \leq \Lambda,$$

for some λ, $\Lambda > 0$. The simplest model example is when a is absolutely continuous with respect to the standard measure on S^{n-1} (that is $da(\omega) = a(\omega)\, d\mathcal{H}^{n-1}(\omega)$, for a suitable L^1 function $a : S^{n-1} \to [0, +\infty]$). In this case, thanks to the polar coordinate representation, the operator L may be written (up to a multiplicative constants) as

$$Lu(x) = \int_{\mathbb{R}^n} \big(2u(x) - u(x+y) - u(x-y)\big)\frac{a\,(y/|y|)}{|y|^{n+2s}}dy. \tag{1.4}$$

When $a \equiv 1$ in (1.4) (i.e. when $da \equiv d\mathcal{H}^{n-1}$ in (1.3)), we have the particularly famous case of the fractional Laplacian in (1.1).

In general, the role of the measure a in (1.3) is to weight differently the different spacial directions (hence we refer to it as an "anisotropy"). In particular, we can also allow the measure a in (1.3) to be a sum of Dirac's Deltas. Indeed, a quite stimulating example arises in the case in which

$$a = \sum_{i=1}^{n} \delta_{e_i} + \delta_{-e_i}, \tag{1.5}$$

where, as usual, $\{e_1, \cdots, e_n\}$ is the standard Euclidean base of $\mathbb{R}^n$: then the operator in (1.3) becomes

$$(-\partial_1^2)^s + \cdots + (-\partial_n^2)^s, \tag{1.6}$$

where $(-\partial_i^2)^s$ represents the one-dimensional fractional Laplacian in the ith coordinate direction (compare with (1.2)).

Goal of this paper is to develop a regularity theory for solutions of

$$\begin{cases} Lu &= g \ \text{ in } \Omega \\ u &= 0 \ \text{ in } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{1.7}$$

The class of solutions that we study are the weak ones, i.e. the ones that have finite (weighted) energy

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}} d\rho \int_{S^{n-1}} da(\omega) \, \frac{\big(u(x) - u(x+\rho\omega)\big)^2}{\rho^{1+2s}} \, < +\infty$$

and satisfy

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^n} dx \int_{\mathbb{R}} d\rho \int_{S^{n-1}} da(\omega) \, \frac{\big(u(x) - u(x+\rho\omega)\big)\big(\varphi(x) - \varphi(x+\rho\omega)\big)}{\rho^{1+2s}} \\ &\qquad = \int_{\mathbb{R}^n} dx \, g(x) \, \varphi(x), \end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$.

When the nonlinearity g is regular enough and the spectral measure of the operator L is $C^\infty(S^{n-1})$ it is known that solutions of (1.7) in bounded domains are, roughly speaking, smooth up to an additional order $2s$ in the derivatives: i.e. for any $\beta \in [0, +\infty)$ such that $\beta + 2s$ is not an integer, we have that $u \in C^{\beta+2s}_{\mathrm{loc}}(\Omega)$, thanks to the estimate

$$\|u\|_{C^{\beta+2s}(B_{r/2})} \leq C\left(\|g\|_{C^\beta(B_r)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right), \tag{1.8}$$

valid in every ball B_r of $\mathbb{R}^n$; see for instance Corollary 3.5 in [7] and also [11, 1]. The constant C in (1.8) depends on n, s, r, and the $C^\beta(S^{n-1})$ norm of the spectral measure.

In particular, solutions of (1.7) are $C^\infty(\Omega)$ if so is g and the measure a, but in general they are not better than $C^s(\mathbb{R}^n)$, i.e. they are smooth in the interior, but only Hölder continuous at the boundary. For instance, $(-\Delta)^s(1-|x|^2)^s_+$ is constant in B_1 and provides an example of solution which is not better than $C^s(\mathbb{R}^n)$.

In the general case of operators as in (1.3), the situation becomes quite different, due to the lack of regularity of the kernels outside the origin. In this generality, estimate (1.8) does not hold, and it gets replaced by the weaker estimate

$$\|u\|_{C^{\beta+2s}(B_{r/2})} \leq C\big(\|g\|_{C^\beta(B_r)} + \|u\|_{C^\beta(\mathbb{R}^n)}\big), \tag{1.9}$$

see Theorem 1.1 in [7].

Though estimates (1.8) and (1.9) may look similar at a first glance, the additional term $\|u\|_{C^\beta(\mathbb{R}^n)}$ in (1.9) prevents higher regularity results: namely, since u is *not* in general $C^\beta(\mathbb{R}^n)$ when $\beta > s$, it follows that (1.9) is meaningful mainly when $\beta \leq s$ and it cannot provide higher order regularity: namely, from it one can only show that $u \in C^{3s}_{\mathrm{loc}}(\Omega)$, even if one assumes $g \in C^\infty(\overline{\Omega})$.

In the light of these observations, in general, when $s < 1/3$, one does not have any control even on the first derivative of u. Nevertheless, we will prove here a higher regularity result as in (1.8), up to an exponent larger than one, by relating the differentiability properties of the solution with the geometry of the domain.

Namely, we will show that in convex domains the solution is always $C^{1+3s-\epsilon}_{\rm loc}(\Omega)$, for any measure a.

The same regularity result holds true also in possibly non-convex domains with $C^{1,1}$ boundary, provided that the measure a is bounded, i.e. if $da(\omega) = a(\omega)\, d\mathcal{H}^{n-1}(\omega)$, with $a \in L^\infty(S^{n-1})$.

In further detail, the main result that we prove is the following:

Theorem 1.1. *Let $\beta \in (0, 1+s)$ and assume that $\beta + 2s$ is not an integer. Assume that either*

$$\Omega \text{ is a convex, bounded domain,} \tag{1.10}$$

or

$$\begin{aligned} &\Omega \text{ is a bounded domain with } C^{1,1} \text{ boundary} \\ &\text{and the spectral measure } a \text{ is bounded.} \end{aligned} \tag{1.11}$$

Let u be a weak solution to (1.7), with $g \in C^\beta(\Omega)$.

Then $u \in C^{\beta+2s}_{\rm loc}(\Omega)$ and, for any $\delta > 0$,

$$\|u\|_{C^{\beta+2s}(\Omega_\delta)} \le C\, \|g\|_{C^\beta(\overline{\Omega})}, \qquad \beta \in (0, 1+s)$$

where Ω_δ is the set containing all the points in Ω that have distance larger than δ from $\partial\Omega$ and $C > 0$ depends also on Ω and δ.

We think that it is a very interesting open problem to establish whether or not a higher regularity theory holds true under the assumptions of Theorem 1.1 (for instance, it is an open question to establish if solutions are C^∞ if so are the data, or if the C^{1+3s} regularity is optimal also in this case). As far as we know, there are natural examples of smooth solutions (such as the one presented in Lemma 7.2), but a complete regularity theory only holds under additional regularity assumptions on the spectral measure (see [11, 1, 7]) and the general picture seems to be completely open.

The result of Theorem 1.1 also plays an important role in the proof of a Pohozaev type identity for anisotropic fractional operators, see [9].

At a first glance, it may also look surprising that the regularity of the solution in Theorem 1.1 depends on the shape of the domain. But indeed the convexity assumption in Theorem 1.1 cannot in general be avoided, as next result points out:

Theorem 1.2. *Let L be as in (1.6), with $n = 2$. There exists a bounded domain in $\mathbb{R}^2$ with C^∞ boundary and a solution $u \in C^s(\overline{\Omega})$ of*

$$\left\{ \begin{array}{rcll} Lu &=& 1 & \text{in } \Omega \\ u &=& 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{array} \right.$$

with $u \notin C^{3s+\epsilon}_{\rm loc}(\Omega)$, for any $\epsilon > 0$.

We remark that the result in Theorem 1.2 is special for the case of singular spectral measures (compare, for instance, with the regularity results in case of smooth spectral measures in [11, 1, 7]). In particular, the loss of interior regularity detected by Theorem 1.2 is in sharp contrast with the smooth interior regularity theory that holds true for both the classical and the fractional Laplacian.

Roughly speaking, the counterexample in Theorem 1.2 will be based on the fact that, if the domain is not convex, there are half-lines originating from an interior point that intersect tangentially the boundary of Ω: then, the singularity on $\partial\Omega$ (created by the solution itself) "propagates" inside the domain due to the nonlocal effect of the operator.

The rest of the paper is organized as follows. In Section 2 we recall the appropriate notion of weighted norms that we will use to prove Theorem 1.1: these norms are slightly unpleasant from the typographic point of view, but they have nice scaling properties and they encode the "right" behavior of the functions in the vicinity of the boundary as well.

Then, Sections 3 and 4 comprise several integral computations of geometric flavor to estimate suitably averaged weighted distance functions in the domain into consideration. More precisely, Section 3 is devoted to the case of convex domains. The estimates obtained there will be used for the proof of Theorem 1.1 in case of convex domains, where no structural assumption on the operator L is taken and therefore the integrals considered are "line integrals", as in (1.3). Section 4 is instead devoted to the case of bounded domains with $C^{1,1}$ boundary. The estimates of this part will be used in the proof of Theorem 1.1 for $C^{1,1}$ domains: since in this case the operator L is as in (1.4), the integrals considered are "spread" over $\mathbb{R}^n$. That is, roughly speaking, in Section 3 the singularity of the line integrals is compensated by the convexity of the domain, while in Section 4 is the operator L that somehow averages its effect in open regions of $\mathbb{R}^n$.

In Section 5 we compute the effect of a cutoff on the operator. Namely, when proving regularity results, it is often useful to distinguish the interior regularity from the one at the boundary (though, as shown here, in the nonlocal setting one may dramatically interact with the other). To this goal, it is sometimes desirable to localize the solution inside the domain by multiplication with a cutoff function: by performing this operation, some estimates are needed in order to control the effect of this cutoff on the operator. These estimates, in our case, are provided in Lemma 5.1.

In Section 6 we bootstrap the regularity theory obtained in order to increase, roughly speaking, the Hölder exponent by $2s$. In our framework, the model for such "improvement of regularity" result is given by Theorem 6.1, which somehow allows us to say that solutions in $C^{\alpha}_{\rm loc}$ are in fact in $C^{\alpha+2s}_{\rm loc}$, if the nonlinearity is nice enough and $\alpha < 1 + s$ (the precise statement involves weighted norms). Section 6 contains also Corollary 6.2, which is the iterative version of Theorem 6.1 and provides a very general regularity result, which in turn implies Theorem 1.1 (as a matter of fact, in

Corollary 6.2 it is not necessary to assume that g is C^β up to the boundary, but only that has finite weighted norm, and also the weighted norm of u is controlled up to the boundary).

The proof of Theorem 1.2 is contained in Section 7, where a somehow surprising counterexample will be constructed.

The paper ends with an appendix, which collects some "elementary", probably well known, but not trivial, ancillary results on the distance functions (in our setting, these results are needed for the integral computations of Section 4).

2. Regularity framework with weighted norms

To study the regularity theory up to the boundary, it is convenient to use the following notation for weighted norms. We consider the distance from a point $x \in \Omega$ to $\partial\Omega$, defined, as usual as

$$d(x) = \operatorname{dist}(x, \partial\Omega) = \inf_{q\in\partial\Omega} |x - q|.$$

We also denote

$$d(x, y) = \min\{d(x),\, d(y)\}. \tag{2.1}$$

Given $\sigma \in \mathbb{R}$ and $\alpha > 0$, we take $k \in \mathbb{N}$ and $\alpha' \in (0, 1]$ such that $\alpha = k + \alpha'$ and we let

$$\begin{aligned} [u]^{(\sigma)}_{\alpha;\Omega} &= \sup_{x\neq y\in\Omega} \left[d^{\alpha+\sigma}(x, y) \frac{|D^k u(x) - D^k u(y)|}{|x - y|^{\alpha'}} \right] \\ \text{and} \quad \|u\|^{(\sigma)}_{\alpha;\Omega} &= \sum_{j=0}^{k} \sup_{x\in\Omega} \left[d^{j+\sigma}(x)\, |D^j u(x)| \right] + [u]^{(\sigma)}_{\alpha;\Omega}. \end{aligned} \tag{2.2}$$

The advantage of these weighted norms is twofold. First of all, since we write $\alpha = k+\alpha'$ with $\alpha' \in (0, 1]$, we can comprise the usual Hölder and Lipschitz spaces C^β, $C^{1+\beta}$, $C^{2+\beta}$, etc., with $\beta \in (0, 1]$ with the same notation: notice for instance that when $\sigma = -\alpha \in (0, 1]$, the notation $[u]^{(\sigma)}_{\alpha;\Omega}$ boils down to the usual seminorm of $C^\alpha(\Omega)$. What is more, by choosing σ in the appropriate way, we can allow the derivatives of u to possibly blow up near the boundary, hence interior and boundary regularity can be proved at the same time and interplay[1] the one with the other.

The weighted norms in (2.2) enjoy a monotonicity property with respect to α, that is if $\alpha_1 \le \alpha_2$ and $\|u\|^{(\sigma)}_{\alpha_2;\Omega} < +\infty$ then also $\|u\|^{(\sigma)}_{\alpha_1;\Omega} < +\infty$. This is given by the following:

Lemma 2.1. *Let $\alpha_1 \le \alpha_2$. Then $\|u\|^{(\sigma)}_{\alpha_1;\Omega} \le C\,\|u\|^{(\sigma)}_{\alpha_2;\Omega}$, for some $C > 0$ only depending on α_1 and α_2 (and bounded uniformly when α_1 and α_2 range in a bounded set).*

[1] Though not explicitly used in this paper, we remark that an additional advantage of these weighted norms is that they usually behave nicely for semilinear equations, namely when the nonlinearity in (1.7) has the form $g(x) = f(x, u(x))$, in which f is locally Lipschitz, but u is not.

Proof. We write $\alpha_i = k_i + \alpha_i'$, for $i \in \{1,2\}$, $k_i \in \mathbb{N}$ and $\alpha_i' \in (0,1]$. We claim that

$$k_1 \leq k_2. \tag{2.3}$$

To prove it, suppose the converse: then $k_1 > k_2$ and therefore, being k_1 and k_2 integers, it follows that $k_1 \geq k_2 + 1$. Then

$$\alpha_2 = k_2 + \alpha_2' \leq k_1 + \alpha_2' - 1 \leq k_1 < k_1 + \alpha_1' = \alpha_1,$$

in contradiction with our assumptions. This proves (2.3).

Now we show that

$$d^{\alpha_1+\sigma}(x,y)\frac{|D^{k_1}u(x) - D^{k_1}u(y)|}{|x-y|^{\alpha_1'}} \leq C\,\|u\|^{(\sigma)}_{\alpha_2;\Omega}. \tag{2.4}$$

Notice that, to prove (2.4), we can suppose that

$$|x-y| \leq \frac{d(x,y)}{4}. \tag{2.5}$$

Indeed, if $|x-y| > d(x,y)/4$ we use (2.3) to see that

$$d^{k_1+\sigma}(x)|D^{k_1}u(x)| \leq \|u\|^{(\sigma)}_{\alpha_2;\Omega}$$

and therefore

$$\begin{aligned} d^{\alpha_1+\sigma}(x,y)\frac{|D^{k_1}u(x) - D^{k_1}u(y)|}{|x-y|^{\alpha_1'}} &\leq Cd^{\alpha_1+\sigma}(x,y)\frac{|D^{k_1}u(x)| + |D^{k_1}u(y)|}{d^{\alpha_1'}(x,y)} \\ &\leq Cd^{k_1+\sigma}(x,y)\left(d^{-k_1-\sigma}(x) + d^{-k_1-\sigma}(y)\right)\|u\|^{(\sigma)}_{\alpha_2;\Omega} \leq C\,\|u\|^{(\sigma)}_{\alpha_2;\Omega}, \end{aligned}$$

that shows (2.4) in this case. Hence, we reduce to prove (2.4) under the additional assumption (2.5). To this goal, we observe that (2.5) implies that

$$|d(x) - d(y)| \leq |x-y| \leq \frac{d(x,y)}{4}. \tag{2.6}$$

Now, in view of (2.3), we can distinguish two cases: either $k_1 = k_2$ or $k_1 < k_2$. If $k_1 = k_2$, then

$$\alpha_1' = \alpha_1 - k_1 = \alpha_1 - k_2 \leq \alpha_2 - k_2 = \alpha_2',$$

thus we set $d_x = d(x)$ and $d_{x,y} = d(x,y)$ for typographical convenience and we perform the following computation:

$$
\begin{aligned}
& d_{x,y}^{\alpha_1+\sigma} \frac{|D^{k_1}u(x) - D^{k_1}u(y)|}{|x-y|^{\alpha_1'}} \\
= \; & d_{x,y}^{\alpha_1+\sigma} \frac{|D^{k_1}u(x) - D^{k_1}u(y)|^{\frac{\alpha_2'-\alpha_1'}{\alpha_2'}} |D^{k_1}u(x) - D^{k_1}u(y)|^{\frac{\alpha_1'}{\alpha_2'}}}{|x-y|^{\alpha_1'}} \\
\leq \; & d_{x,y}^{\alpha_1+\sigma} d_{x,y}^{\frac{-\alpha_1'(\alpha_2+\sigma)}{\alpha_2'}} \frac{\left(|D^{k_1}u(x)| + |D^{k_1}u(y)|\right)^{\frac{\alpha_2'-\alpha_1'}{\alpha_2'}} \left(d_{x,y}^{\alpha_2+\sigma}|D^{k_1}u(x) - D^{k_1}u(y)|\right)^{\frac{\alpha_1'}{\alpha_2'}}}{|x-y|^{\alpha_1'}} \\
\leq \; & d_{x,y}^{\alpha_1+\sigma} d_{x,y}^{\frac{-\alpha_1'(\alpha_2+\sigma)}{\alpha_2'}} \frac{\left(d_x^{-k_1-\sigma}\|u\|_{\alpha_2;\Omega}^{(\sigma)} + d_y^{-k_1-\sigma}\|u\|_{\alpha_2;\Omega}^{(\sigma)}\right)^{\frac{\alpha_2'-\alpha_1'}{\alpha_2'}} \left(\|u\|_{\alpha_2;\Omega}|x-y|^{\alpha_2'}\right)^{\frac{\alpha_1'}{\alpha_2'}}}{|x-y|^{\alpha_1'}} \\
\leq \; & C\, d_{x,y}^{\alpha_1+\sigma} d_{x,y}^{-\frac{\alpha_1'(\alpha_2+\sigma)}{\alpha_2'}} d_{x,y}^{-\frac{(\alpha_2'-\alpha_1')(k_1+\sigma)}{\alpha_2'}} \|u\|_{\alpha_2;\Omega}^{(\sigma)}.
\end{aligned}
$$

Moreover, since $k_1 = k_2$,

$$
\alpha_1 + \sigma - \frac{\alpha_1'(\alpha_2+\sigma)}{\alpha_2'} - \frac{(\alpha_2'-\alpha_1')(k_1+\sigma)}{\alpha_2'} = 0,
$$

hence the latter inequality proves (2.4) when $k_1 = k_2$. Let us now consider the case $k_1 < k_2$, i.e. $k_1 + 1 \leq k_2$. Again, we can suppose that $d(x) \leq d(y)$, and then (2.5) implies that $y \in B_{d(x)/4}(x)$, and notice that this ball lies in Ω, at distance from $\partial\Omega$ bounded from below by $3d(x)/4$. As a consequence,

$$
\begin{aligned}
|D^{k_1}u(x) - D^{k_1}u(y)| &\leq \sup_{\zeta \in B_{d_x/4}(x)} |D^{k_1+1}u(\zeta)|\, |x-y| \\
&\leq C d_x^{-k_1-1-\sigma} \sup_{\zeta \in B_{d_x/4}(x)} d_\zeta^{k_1+1+\sigma} |D^{k_1+1}u(\zeta)|\, |x-y| \\
&\leq C d_{x,y}^{-k_1-1-\sigma} \|u\|_{\alpha_2;\Omega}^{(\sigma)} |x-y|.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
& d_{x,y}^{\alpha_1+\sigma} \frac{|D^{k_1}u(x) - D^{k_1}u(y)|}{|x-y|^{\alpha_1'}} \\
= \; & d_{x,y}^{\alpha_1+\sigma} \frac{|D^{k_1}u(x) - D^{k_1}u(y)|^{1-\alpha_1'} |D^{k_1}u(x) - D^{k_1}u(y)|^{\alpha_1'}}{|x-y|^{\alpha_1'}} \\
\leq \; & d_{x,y}^{\alpha_1+\sigma} \left(d_x^{-k_1-\sigma}\|u\|_{\alpha_2;\Omega}^{(\sigma)} + d_y^{-k_1-\sigma}\|u\|_{\alpha_2;\Omega}^{(\sigma)}\right)^{1-\alpha_1'} \|u\|_{\alpha_2;\Omega}^{(\sigma)} d_{x,y}^{-\alpha_1'(k_1+1+\sigma)} \left(\|u\|_{\alpha_2;\Omega}^{(\sigma)}\right)^{\alpha_1'} \\
\leq \; & d_{x,y}^{\alpha_1+\sigma} d_{x,y}^{-(k_1+\sigma)(1-\alpha_1')} d_{x,y}^{-\alpha_1'(k_1+1+\sigma)} \|u\|_{\alpha_2;\Omega}^{(\sigma)}.
\end{aligned}
$$

Since

$$
\alpha_1 + \sigma - (k_1+\sigma)(1-\alpha_1') - \alpha_1'(k_1+1+\sigma) = 0,
$$

the inequality above proves (2.4) when $k_1 < k_2$.

Having completed the proof of (2.4), we now notice that

$$\sum_{j=0}^{k_1} \sup_{x\in\Omega} \left[d^{j+\sigma}(x)\,|D^j u(x)|\right] \le \sum_{j=0}^{k_2} \sup_{x\in\Omega} \left[d^{j+\sigma}(x)\,|D^j u(x)|\right],$$

thanks to (2.3). This inequality and (2.4) imply the desired claim. □

3. Integral computations for convex sets

The goal of this section is to provide some (somehow optimal) weighted integral computations of geometric type for convex sets. Similar estimates for bounded domains with $C^{1,1}$ boundary when the spectral measure a is regular will be developed in the forthcoming Section 4. The first geometric integral computation is given by the following:

Lemma 3.1. *Let $p \in \mathbb{R}^n$, $R > 2r > 0$ and $\omega \in S^{n-1}$. Let $\Omega \subset \mathbb{R}^n$ be a convex open set, with $B_R(p) \subseteq \Omega$. Then there exists $C > 0$, possibly depending on n and s, such that*

$$\int_R^{+\infty} d\rho\, \frac{\chi_\Omega(p+\rho\omega)\,\chi_{[0,r]}\big(d(p+\rho\omega)\big)}{\rho^{1+2s}} \le CrR^{-1-2s}. \tag{3.1}$$

Proof. The idea is that the set in which the integrand is non-zero "typically" occupies a segment of length comparable to r.

More precisely, consider the half-line $\Theta = \{p+\rho\omega,\ \rho \ge 0\}$. If Θ lies inside Ω, then we are done. Indeed, in this case, by convexity, the set Ω contains the convex envelope between the ball $B_R(p)$ and the half-line Θ, which is an horizontal cylinder of radius R. In particular, in this case we have that $d(p+\rho\omega) \ge R > r$, hence $\chi_{[0,r]}\big(d(p+\rho\omega)\big) = 0$, and so the left hand side of (3.1) vanishes.

As a consequence, we may and do assume that $\Theta \cap (\partial\Omega)$ is non-void. Thus, we claim that $\Theta \cap (\partial\Omega)$ only contains a point. Indeed, suppose by contradiction that $p+\rho_1\omega$ and $p+\rho_2\omega$ belong to $\partial\Omega$, for some $\rho_2 > \rho_1 \ge 0$. In particular, there exists a sequence $q_j \in \Omega$ such that $q_j \to p+\rho_2\omega$ as $j \to +\infty$.

Then, by convexity, Ω contains the convex envelope K_j of $B_R(p)$ with q_j. Notice that K_j, as $j \to +\infty$, approaches the convex envelope of $B_R(p)$ with $p+\rho_2\omega$: therefore, for large j, the point $p+\rho_1\omega$ belongs to the interior of K_j and thus to Ω. This is a contradiction, and so we have shown that $\Theta \cap (\partial\Omega)$ consists of exactly one point, that we denote by $q_\star = p+\rho_\star\omega$, for some $\rho_\star \ge 0$.

We remark that, since $B_R(p)$ lies in Ω, we have that

$$\rho_\star \ge R. \tag{3.2}$$

Now we show that

$$\text{if } \rho \ge 0,\ p+\rho\omega \in \Omega \text{ and } d(p+\rho\omega) \in [0,r], \text{ then } \rho \in [\rho_\star(1-rR^{-1}),\ \rho_\star]. \tag{3.3}$$

Indeed, by convexity, Ω contains the interior of the convex envelope $K_\star$ of $B_R(p)$ with $q_\star$. Therefore, $d(p+\rho\omega)$ is controlled from below by the distance of $p+\rho\omega$ to $\partial K_\star$, which will be denoted by δ.

We remark that $K_\star$ has a radially symmetric conical singularity at $q_\star$. If we let β be the planar angle of such cone, by trigonometry we have that

$$\begin{aligned} &\delta = (\rho_\star - \rho)\,\sin(\beta/2) \\ \text{and} \qquad &R = \rho_\star\,\sin(\beta/2). \end{aligned}$$

Thus

$$d(p+\rho\omega) \geq \delta = \frac{(\rho_\star - \rho)\,R}{\rho_\star}. \tag{3.4}$$

So, if $d(p+\rho\omega) \in [0,r]$ we have that

$$r \geq \frac{(\rho_\star - \rho)\,R}{\rho_\star},$$

that is $\rho \geq \rho_\star(1 - rR^{-1})$, which proves (3.3).

Therefore, using (3.3) and the substitution $t = \rho_\star^{-1}\rho$, we conclude that

$$\begin{aligned} \int_R^{+\infty} d\rho\,\frac{\chi_\Omega(p+\rho\omega)\,\chi_{[0,r]}\big(d(p+\rho\omega)\big)}{\rho^{1+2s}} &\leq \int_{\rho_\star(1-rR^{-1})}^{\rho_\star} \frac{d\rho}{\rho^{1+2s}} \\ = \frac{1}{\rho_\star^{2s}} \int_{1-rR^{-1}}^{1} \frac{dt}{t^{1+2s}} &\leq \frac{C}{\rho_\star^{2s}} \left(\frac{1}{(1-rR^{-1})^{2s}} - 1 \right) \leq \frac{CrR^{-1}}{\rho_\star^{2s}}. \end{aligned}$$

This and (3.2) imply (3.1). □

Remark 3.2. We notice that the estimate in (3.1) is optimal, since it is attained when $p = 0$ and $\Omega = B_{3R}$.

Remark 3.3. The convexity assumption in Lemma 3.1 cannot be dropped. As a counterexample, let us endow $\mathbb{R}^n$ with coordinates $(x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$, let us take $p = 0$, $\omega = e_1$, and

$$\Omega = B_R \cup \big\{ |x_n| < Re^{-|x'|/R} \big\}.$$

Notice that Ω is not convex. Also, the points of the form $(\rho, 0, \cdots, 0, Re^{-\rho/R})$ belong to $\partial\Omega$, for any $\rho \geq 1$. Hence

$$d(\rho\omega) \leq Re^{-\rho/R} \leq r$$

for any $\rho \geq R\log(Rr^{-1})$. Hence

$$\int_R^{+\infty} d\rho\,\frac{\chi_\Omega(p+\rho\omega)\,\chi_{[0,r]}\big(d(p+\rho\omega)\big)}{\rho^{1+2s}} \geq \int_{R\log(Rr^{-1})}^{+\infty} \frac{d\rho}{\rho^{1+2s}} = \frac{C}{(R\log(Rr^{-1}))^{2s}}.$$

This quantity is larger than the right hand side of (3.1) when r is small, and this shows that the convexity assumption is essential in such result.

Next is a variation of Lemma 3.1. Namely, the integral on the left hand side is modified by a weight depending on the distance (and the condition $d(p+\rho\omega) \in [0, 2r]$ is dropped).

Lemma 3.4. *Let $\alpha \in [s, 1+s)$. Let $p,\, q \in \mathbb{R}^n$, $R > 0$ and $\omega \in S^{n-1}$. Let $\Omega \subset \mathbb{R}^n$ be a convex open set, with $B_R(p) \cup B_R(q) \subseteq \Omega$. Then there exists $C > 0$, possibly depending on α, n and s, such that*

$$\int_R^{+\infty} d\rho \, \frac{\chi_\Omega(p+\rho\omega)\, \chi_\Omega(q+\rho\omega)}{d^{\alpha-s}(p+\rho\omega, q+\rho\omega)\, \rho^{1+2s}} \le C R^{-s-\alpha}. \tag{3.5}$$

Proof. We let $\rho_\star = \sup\{\rho \text{ s.t. } p+\rho\omega \in \Omega\} \in [R, +\infty]$. By convexity and trigonometry (see e.g. (3.4)), we have that

$$d(p+\rho\omega) \ge \frac{(\rho_\star - \rho)\, R}{\rho_\star},$$

with the obvious limit notation that the formula above reads $d(p+\rho\omega) \ge R$ if $\rho_\star = +\infty$. Since the same formula holds for q instead of p, we have that

$$d(p+\rho\omega, q+\rho\omega) = \min\big\{ d(p+\rho\omega),\, d(q+\rho\omega)\big\} \ge \frac{(\rho_\star - \rho)\, R}{\rho_\star}.$$

Thus, the left hand side of (3.5) is bounded by

$$\int_R^{\rho_\star} d\rho \, \frac{\rho_\star^{\alpha-s}}{(\rho_\star - \rho)^{\alpha-s}\, R^{\alpha-s}\, \rho^{1+2s}} = R^{s-\alpha} \rho_\star^{-2s} \int_{\mu_o}^1 \frac{dt}{(1-t)^{\alpha-s} t^{1+2s}}, \tag{3.6}$$

where we used the substitution $t = \rho/\rho_\star$ and the notation $\mu_o = R/\rho_\star \in (0,1]$. Now, for any $\mu \in (0,1]$, we set

$$\psi(\mu) = \mu^{2s} \int_\mu^1 \frac{dt}{(1-t)^{\alpha-s} t^{1+2s}}$$

and we claim that

$$\sup_{\mu \in (0,1]} \psi(\mu) \le C, \tag{3.7}$$

for some $C > 0$. To check this, we recall that $\alpha - s < 1$ and we notice that $\psi(1) = 0$. Also,

$$\int_0^1 \frac{dt}{(1-t)^{\alpha-s} t^{1+2s}} = +\infty,$$

so we compute, by de l'Hôpital rule,

$$\lim_{\mu \to 0} \psi(\mu) = \lim_{\mu \to 0} \frac{\int_\mu^1 \frac{dt}{(1-t)^{\alpha-s} t^{1+2s}}}{\mu^{-2s}} = \lim_{\mu \to 0} \frac{\frac{1}{(1-\mu)^{\alpha-s} \mu^{1+2s}}}{2s \mu^{-1-2s}} = \frac{1}{2s}.$$

Thus ψ can be extended to a continuous function in $[0,1]$ and so (3.7) follows.

Then, we can bound (3.6) using (3.7): we obtain that the quantity in (3.6) is controlled by

$$R^{s-\alpha} \rho_\star^{-2s} \mu_o^{-2s} \psi(\mu_o) \le C R^{s-\alpha} \rho_\star^{-2s} \mu_o^{-2s} = C R^{s-\alpha} \rho_\star^{-2s} \cdot R^{-2s} \rho_\star^{2s},$$

which proves (3.5). □

Remark 3.5. We notice that the estimate in (3.5) is also optimal, since it is attained when $p = q = 0$ and $\Omega = B_{3R}$.

For further reference, we point out that Lemma 3.4 is of course valid in particular when $p = q$. In this case, its statement simplifies in the following way:

Lemma 3.6. *Let $\alpha \in [s, 1+s)$. Let $p \in \mathbb{R}^n$, $R > 0$ and $\omega \in S^{n-1}$. Let $\Omega \subset \mathbb{R}^n$ be a convex open set, with $B_R(p) \subseteq \Omega$. Then there exists $C > 0$, possibly depending on α, n and s, such that*

$$\int_R^{+\infty} d\rho \, \frac{\chi_\Omega(p+\rho\omega)}{d^{\alpha-s}(p+\rho\omega)\,\rho^{1+2s}} \leq CR^{-s-\alpha}.$$

The next integral computation is a simple, but operational, consequence of elementary geometry:

Lemma 3.7. *Let $R > 0$, $p \in B_R$, and $\omega \in S^{n-1}$. Let $\Omega \subset \mathbb{R}^n$ be a convex open set, with $B_{3R} \cap (\partial\Omega) \neq \varnothing$. Then there exists $C > 0$, possibly depending on n and s, such that*

$$\int_R^{+\infty} d\rho \, \frac{d^s(p+\rho\omega)}{\rho^{1+2s}} \leq CR^{-s}.$$

Proof. Let $q_o \in B_{3R} \cap (\partial\Omega)$. Then

$$d(p+\rho\omega) \leq |p+\rho\omega - q_o| \leq |p| + |q| + \rho \leq C(R+\rho).$$

Therefore, using the substitution $\rho = Rt$, we obtain

$$\int_R^{+\infty} d\rho \, \frac{d^s(p+\rho\omega)}{\rho^{1+2s}} \leq C\int_R^{+\infty} d\rho \, \frac{(R+\rho)^s}{\rho^{1+2s}} = CR^{-s}\int_1^{+\infty} dt \, \frac{(1+t)^s}{t^{1+2s}},$$

that gives the desired result. □

4. Integral computations for $C^{1,1}$ domains and bounded measures

In this section, we consider bounded domains with $C^{1,1}$ boundary and bounded spectral measures a and we obtain the corresponding results of Section 3 in this framework. More precisely, we obtain the results analogous to Lemmata 3.1, 3.4, 3.6 and 3.7, when the convexity assumption on the domain is replaced by a regularity assumption on the domain and the boundedness of the measure.

The counterpart of Lemma 3.1 in the setting of this section is the following:

Lemma 4.1. *Let $p \in \mathbb{R}^n$ and $R > 2r > 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary, with $B_R(p) \subseteq \Omega$. Then there exists $C > 0$, possibly depending on n, s and Ω, such that*

$$\int_{\mathbb{R}^n\setminus B_R} \frac{\chi_\Omega(p+x)\,\chi_{[0,r]}\big(d(p+x)\big)}{|x|^{n+2s}}\,dx \leq CrR^{-1-2s}. \tag{4.1}$$

Proof. First, notice that, by possibly replacing Ω with the translated domain $\Omega - p$, we may assume that $p = 0$.

Also, we can suppose that R is less then the diameter of Ω, otherwise the condition $B_R \subseteq \Omega$ cannot hold. Then, we perform the change of variables $z = \frac{x}{R}$, so that (4.1) becomes

$$\int_{\mathbb{R}^n \setminus B_1} \frac{\chi_{\Omega_R}(z)\, \chi_{[0,\bar r]}\big(\mathrm{dist}\,(z, \partial\Omega_R)\big)}{|z|^{n+2s}}\, dz \le C\bar r, \tag{4.2}$$

where we denoted $\Omega_R = \frac{1}{R}\Omega$ and $\bar r = \frac{r}{R}$. Notice that Ω_R has a bounded $C^{1,1}$ norm (uniformly in R), and in fact converges to a half-space as $R \to 0^+$. As a consequence, we can apply Proposition A.3: we obtain that there exists $\kappa_* > 0$ such that, if $\bar r \in (0, \kappa_*]$ then

$$\begin{aligned} &\Big|\big\{x \in \Omega_R \cap (B_{2^{k+1}} \setminus B_{2^k}) \text{ s.t. } \mathrm{dist}\,(x, \partial\Omega_R) \in [0, \bar r]\big\}\Big| \\ &\qquad \le C\bar r\, \mathcal{H}^{n-1}\big((\partial\Omega_R) \cap (B_{2^{k+2}} \setminus B_{2^{k-1}})\big) \end{aligned} \tag{4.3}$$

for every $k \in \mathbb{N}$. Now we estimate the latter term by scaling back to the original domain and exploiting Lemma A.4. We obtain

$$\begin{aligned} &\mathcal{H}^{n-1}\big((\partial\Omega_R) \cap (B_{2^{k+2}} \setminus B_{2^{k-1}})\big) \\ = \;& \frac{1}{R^{n-1}} \mathcal{H}^{n-1}\big((\partial\Omega) \cap (B_{2^{k+2}R} \setminus B_{2^{k-1}R})\big) \\ \le \;& \frac{C(2^{k-1}R)^{n-1}}{R^{n-1}}. \end{aligned}$$

This and (4.3) give that

$$\Big|\big\{x \in \Omega_R \cap (B_{2^{k+1}} \setminus B_{2^k}) \text{ s.t. } \mathrm{dist}\,(x, \partial\Omega_R) \in [0, \bar r]\big\}\Big| \le C2^{k(n-1)}\bar r.$$

As a consequence, if $\bar r \in (0, \kappa_*]$,

$$\begin{aligned} &\int_{B_{2^{k+1}} \setminus B_{2^k}} \frac{\chi_{\Omega_R}(z)\, \chi_{[0,\bar r]}\big(\mathrm{dist}\,(z, \partial\Omega_R)\big)}{|z|^{n+2s}}\, dz \\ \le \;& \int_{B_{2^{k+1}} \setminus B_{2^k}} \frac{\chi_{\Omega_R}(z)\, \chi_{[0,\bar r]}\big(\mathrm{dist}\,(z, \partial\Omega_R)\big)}{2^{k(n+2s)}}\, dz \\ \le \;& \frac{C\bar r}{2^{k(1+2s)}}. \end{aligned}$$

By summing over k, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_1} \frac{\chi_{\Omega_R}(z)\,\chi_{[0,\bar r]}\big(\mathrm{dist}\,(z,\partial\Omega_R)\big)}{|z|^{n+2s}}\,dz\\
\le\;& \sum_{k\ge0}\int_{B_{2^{k+1}}\setminus B_{2^k}} \frac{\chi_{\Omega_R}(z)\,\chi_{[0,\bar r]}\big(\mathrm{dist}\,(z,\partial\Omega_R)\big)}{|z|^{n+2s}}\,dz\\
\le\;& \sum_{k\ge0}\frac{C\bar r}{2^{k(1+2s)}}\\
\le\;& C\bar r,
\end{aligned}$$

up to renaming C. This gives (4.2) when $\bar r\le\kappa_*$.

If instead $\bar r>\kappa_*$, then

$$\int_{\mathbb{R}^n\setminus B_1} \frac{\chi_{\Omega_R}(z)\,\chi_{[0,\bar r]}\big(\mathrm{dist}\,(z,\partial\Omega_R)\big)}{|z|^{n+2s}}\,dz \le \int_{\mathbb{R}^n\setminus B_1}\frac{dz}{|z|^{n+2s}}\le C\le\frac{C\bar r}{\kappa_*}.$$

This shows that (4.2) also holds in this case, up to renaming constants, and this completes the proof of Lemma 4.1. $\square$

Following is the variation of Lemma 3.6 needed in the setting of this section:

Lemma 4.2. *Let $\alpha\in[s,1+s)$, $p\in\mathbb{R}^n$ and $R>0$. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary, with $B_R(p)\subseteq\Omega$. Then there exists $C>0$, possibly depending on α, n, s and Ω, such that*

$$\int_{\mathbb{R}^n\setminus B_R}\frac{\chi_\Omega(p+x)}{d^{\alpha-s}(p+x)\,|x|^{n+2s}}\,dx\le CR^{-s-\alpha}. \tag{4.4}$$

Proof. Up to a translation of the domain, we suppose that $p=0$. In addition, we can suppose that R is less then the diameter of Ω, otherwise the condition $B_R\subseteq\Omega$ cannot hold. Hence we do the change of variables $z=\frac{x}{R}$, so that (4.4) reduces to

$$\int_{\mathbb{R}^n\setminus B_1}\frac{\chi_{\Omega_R}(z)}{\mathrm{dist}^{\alpha-s}(z,\partial\Omega_R)\,|z|^{n+2s}}\,dz\le C \tag{4.5}$$

where we denoted $\Omega_R=\frac{1}{R}\Omega$. Since Ω_R has a bounded $C^{1,1}$ norm (uniformly in R, and indeed it converges to a half-space as $R\to0^+$), we can apply Corollary A.2 and obtain that there exists $\kappa_*>0$ such that, for any $t\in(0,\kappa_*]$,

$$\begin{aligned}
&\mathcal{H}^{n-1}\big(\{z\in\Omega_R\cap(B_{2^{k+1}}\setminus B_{2^k})\text{ s.t. }\mathrm{dist}(z,\partial\Omega_R)=t\}\big)\\
&\qquad\le C\,\mathcal{H}^{n-1}\big((\partial\Omega_R)\cap(B_{2^{k+2}}\setminus B_{2^{k-1}})\}\big),
\end{aligned}$$

for any $k\in\mathbb{N}$. Furthermore, by Lemma A.4,

$$\mathcal{H}^{n-1}\big((\partial\Omega_R)\cap(B_{2^{k+2}}\setminus B_{2^{k-1}})\}\big)\le C(2^{k-1})^{n-1}.$$

The latter two formulas give that

$$\mathcal{H}^{n-1}\big(\{z\in\Omega_R\cap(B_{2^{k+1}}\setminus B_{2^k})\text{ s.t. }\mathrm{dist}(z,\partial\Omega_R)=t\}\big)\le C(2^{k-1})^{n-1}.$$

Consequently, by Coarea Formula,

$$
\begin{aligned}
&\int_{\substack{\mathbb{R}^n\setminus B_1\\ \{\mathrm{dist}\le\kappa_*\}}} \frac{\chi_{\Omega_R}(z)}{\mathrm{dist}^{\alpha-s}(z,\partial\Omega_R)\,|z|^{n+2s}}\,dz\\
&\quad\le \sum_{k\ge0}\int_{\substack{B_{2^{k+1}}\setminus B_{2^k}\\ \{\mathrm{dist}\le\kappa_*\}}} \frac{\chi_{\Omega_R}(z)}{\mathrm{dist}^{\alpha-s}(z,\partial\Omega_R)\,|z|^{n+2s}}\,dz\\
&\quad\le \sum_{k\ge0}\frac{1}{2^{k(n+2s)}}\int_{\substack{B_{2^{k+1}}\setminus B_{2^k}\\ \{\mathrm{dist}\le\kappa_*\}}} \frac{\chi_{\Omega_R}(z)\,|\nabla \mathrm{dist}(z,\partial\Omega_R)|}{\mathrm{dist}^{\alpha-s}(z,\partial\Omega_R)}\,dz\\
&\quad\le \sum_{k\ge0}\frac{1}{2^{k(n+2s)}}\int_0^{\kappa_*}dt\int_{\substack{B_{2^{k+1}}\setminus B_{2^k}\\ \{\mathrm{dist}(z,\partial\Omega_R)=t\}}} d\mathcal{H}^{n-1}(z)\,\frac{1}{t^{\alpha-s}}\\
&\quad\le \sum_{k\ge0}\int_0^{\kappa_*}dt\,\frac{\mathcal{H}^{n-1}\big(\{z\in\Omega_R\cap(B_{2^{k+1}}\setminus B_{2^k})\text{ s.t. }\mathrm{dist}(z,\partial\Omega_R)=t\}\big)}{t^{\alpha-s}\,2^{k(n+2s)}} \qquad (4.6)\\
&\quad\le \sum_{k\ge0}\int_0^{\kappa_*}dt\,\frac{C(2^{k-1})^{n-1}}{t^{\alpha-s}\,2^{k(n+2s)}}\\
&\quad\le \sum_{k\ge0}\int_0^{\kappa_*}dt\,\frac{C}{t^{\alpha-s}\,2^{k(1+2s)}}\\
&\quad= \sum_{k\ge0}\frac{C\kappa_*^{1-\alpha+s}}{2^{k(1+2s)}}\\
&\quad\le C,
\end{aligned}
$$

up to renaming constants. Additionally, we have that

$$
\int_{\substack{\mathbb{R}^n\setminus B_1\\ \{\mathrm{dist}\ge\kappa_*\}}} \frac{\chi_{\Omega_R}(z)}{\mathrm{dist}^{\alpha-s}(z,\partial\Omega_R)\,|z|^{n+2s}}\,dz \le \int_{\mathbb{R}^n\setminus B_1}\frac{\chi_{\Omega_R}(z)}{\kappa_*^{\alpha-s}\,|z|^{n+2s}}\,dz\le C.
$$

This and (4.6) complete the proof of (4.5) and thus of Lemma 4.2. $\square$

Then, the result corresponding to Lemma 3.4 goes as follows:

Lemma 4.3. *Let $\alpha\in[s,1+s)$. Let p, $q\in\mathbb{R}^n$ and $R>0$. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary, with $B_R(p)\cup B_R(q)\subseteq\Omega$. Then there exists $C>0$, possibly depending on α, n, s and Ω, such that*

$$
\int_{\mathbb{R}^n\setminus B_R}\frac{\chi_\Omega(p+x)\,\chi_\Omega(q+x)}{d^{\alpha-s}(p+x,q+x)\,|x|^{n+2s}}\,dx\le CR^{-s-\alpha}.
$$

Proof. Notice that $d(p+x,q+x)$ coincides with either $d(p+x)$ or $d(q+x)$, therefore

$$
\frac{1}{d^{\alpha-s}(p+x,q+x)}\le\frac{1}{d^{\alpha-s}(p+x)}+\frac{1}{d^{\alpha-s}(q+x)}.
$$

Furthermore, $\chi_\Omega(p+x)\,\chi_\Omega(q+x) \le \chi_\Omega(p+x)$ and $\chi_\Omega(p+x)\,\chi_\Omega(q+x) \le \chi_\Omega(q+x)$. As a consequence

$$\frac{\chi_\Omega(p+x)\,\chi_\Omega(q+x)}{d^{\alpha-s}(p+x,q+x)\,|x|^{n+2s}} \le \frac{\chi_\Omega(p+x)}{d^{\alpha-s}(p+x)\,|x|^{n+2s}} + \frac{\chi_\Omega(q+x)}{d^{\alpha-s}(q+x)\,|x|^{n+2s}},$$

and so Lemma 4.3 follows from Lemma 4.2. □

Finally, here is the counterpart of Lemma 3.7:

Lemma 4.4. *Let $R > 0$ and $p \in B_R$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary, with $B_{3R} \cap (\partial\Omega) \neq \varnothing$. Then there exists $C > 0$, possibly depending on n, s and Ω, such that*

$$\int_{\mathbb{R}^n \setminus B_R} \frac{d^s(p+x)}{|x|^{n+2s}}\, dx \le CR^{-s}.$$

Proof. Let $q_o \in B_{3R} \cap (\partial\Omega)$. Then

$$d(p+x) \le |p+x-q_o| \le |p| + |x| + |q_o| \le |x| + 4R,$$

therefore

$$\int_{\mathbb{R}^n \setminus B_R} \frac{d^s(p+x)}{|x|^{n+2s}}\, dx \le \int_{\mathbb{R}^n \setminus B_R} \frac{(|x|+4R)^s}{|x|^{n+2s}}\, dx = R^{-s} \int_{\mathbb{R}^n \setminus B_1} \frac{(|y|+4)^s}{|y|^{n+2s}}\, dy = CR^{-s}.$$

□

5. A localization argument

In this section we introduce an appropriate cutoff function and we discuss its regularity properties. The goal of the cutoff procedure is, roughly speaking, to distinguish the behavior of the solutions inside the domain from the one at the boundary. For this we recall the notation in (2.1) and we give the following result:

Lemma 5.1. *Let $R > 0$, $\Omega \subset \mathbb{R}^n$ and $\alpha \in (0, 1+s)$. Assume that either (1.10) or (1.11) is satisfied and that*

$$B_{2R} \subseteq \Omega \text{ and } B_{3R} \cap (\partial\Omega) \neq \varnothing. \tag{5.1}$$

Let $w \in C^s(\mathbb{R}^n)$, with

$$w \equiv 0 \text{ in } B_R \tag{5.2}$$

$$\text{and} \qquad w \equiv 0 \text{ outside } \Omega. \tag{5.3}$$

Then

$$\|Lw\|_{L^\infty(B_{R/2})} \le C\, [w]_{C^s(\mathbb{R}^n)} R^{-s}. \tag{5.4}$$

In addition, if we assume also that

- *either $\alpha \in (0, s]$,*
- *or $\alpha \in (s, 1+s)$ and*

— *for any p, $q \in \Omega$ with $|p-q| \le R$ we have that*

(5.5)
$$\begin{cases} |w(p)-w(q)| \le \ell\,|p-q|^{\alpha}\left(d^{s-\alpha}(p,q)+R^{-\alpha}d^{s}(p,q)\right) & \text{if } \alpha\in(s,1],\\[1ex] \begin{array}{c}|\nabla w(p)| \le \ell\left(d^{s-1}(p)+R^{-1}d^{s}(p)\right) \quad \text{and}\\ |\nabla w(p)-\nabla w(q)| \le \ell\,|p-q|^{\alpha-1}\left(d^{s-\alpha}(p,q)+R^{-\alpha}d^{s}(p,q)\right)\end{array} & \text{if } \alpha\in(1,1+s),\end{cases}$$

for some $\ell > 0$,

then

$$R^{\alpha+s}[Lw]_{C^{\alpha}(B_{R/4})} \le \begin{cases} C\,[w]_{C^{s}(\mathbb{R}^n)} & \text{if } \alpha\in(0,s],\\ C\left([w]_{C^{s}(\mathbb{R}^n)}+\ell\right) & \text{if } \alpha\in(s,1+s).\end{cases} \tag{5.6}$$

Finally, if $\alpha \in (1, 1+s)$, we also have that

$$R^{1+s}\|\nabla Lw\|_{L^{\infty}(B_{R/4})} \le C\ell. \tag{5.7}$$

Proof. For simplicity, we state and prove this results for convex open sets, i.e. when (1.10) is assumed. The proof under condition (1.11) would be the same, except that one should use the results of Section 4 instead of the ones of Section 3. More explicitly, for convex open sets, in the proof of this result we will quote Lemmata 3.1, 3.6, 3.7 and 3.4: for bounded domains with $C^{1,1}$ boundary one has instead to quote Lemmata 4.1, 4.2, 4.4 and 4.3.

First of all, we prove (5.4). Fix $x \in B_{R/2}$. Then $w(x) = 0$ and $w(x+\rho\omega) = 0$ for any $\rho \in (-R/2, R/2)$, thanks to (5.2). Accordingly,

$$|w(x+\rho\omega)| = |w(x+\rho\omega)-w(x)| \le [w]_{C^{s}(\mathbb{R}^n)}\rho^{s}$$

therefore

$$Lw(x) \le 2\,[w]_{C^{s}(\mathbb{R}^n)} \int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\,\frac{1}{\rho^{1+s}} \le C\,[w]_{C^{s}(\mathbb{R}^n)}R^{-s},$$

which proves (5.4).

Now we prove (5.6). For this, we first consider the case $\alpha \in (0,1]$, and we fix x_1 and x_2 in $B_{R/4}$. Notice that if $y \in B_{R/2}$ then $w(x_1+y) = w(x_2+y) = 0$, thanks to (5.2). In particular, we have $w(x_1) = w(x_2) = 0$. As a consequence of these observations,

$$\begin{aligned} Lw(x_i) &= -\int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\,\frac{w(x_i+\rho\omega)+w(x_i-\rho\omega)}{\rho^{1+2s}}\\ &= -2\int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\,\frac{w(x_i+\rho\omega)}{\rho^{1+2s}}, \end{aligned}$$

for $i \in \{1,2\}$ (and possibly replacing $da(\omega)$ with $da(\omega)+da(-\omega)$). Therefore

$$|Lw(x_1)-Lw(x_2)| \le 2\int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\,\frac{|w(x_1+\rho\omega)-w(x_2+\rho\omega)|}{\rho^{1+2s}}. \tag{5.8}$$

So, we distinguish two cases. If $\alpha \in (0, s]$, then we obtain from (5.8) that

$$\begin{aligned}|Lw(x_1) - Lw(x_2)| &\le 2[w]_{C^s(\mathbb{R}^n)}|x_1 - x_2|^s \int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \frac{1}{\rho^{1+2s}} \\ &\le C[w]_{C^s(\mathbb{R}^n)}|x_1 - x_2|^s R^{-2s}.\end{aligned}$$

Therefore

$$\frac{|Lw(x_1) - Lw(x_2)|}{|x_1 - x_2|^\alpha} \le C[w]_{C^s(\mathbb{R}^n)}|x_1 - x_2|^{s-\alpha} R^{-2s}.$$

So, if $\alpha \in (0, s]$, the result in (5.6) follows by noticing that $|x_1 - x_2| \le |x_1| + |x_2| \le R$.

Now suppose that $\alpha \in (s, 1]$. We define $d_\star(\rho) = d(x_1 + \rho\omega) + d(x_2 + \rho\omega)$ and we write (5.8) as

$$|Lw(x_1) - Lw(x_2)| \le I_1 + I_2, \tag{5.9}$$

where $r = |x_1 - x_2|$,

$$I_1 = 2\int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \chi_{[0,6r]}\big(d_\star(\rho)\big)\, \frac{|w(x_1 + \rho\omega) - w(x_2 + \rho\omega)|}{\rho^{1+2s}}$$

$$\text{and} \qquad I_2 = 2\int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \chi_{(6r,+\infty)}\big(d_\star(\rho)\big)\, \frac{|w(x_1 + \rho\omega) - w(x_2 + \rho\omega)|}{\rho^{1+2s}}.$$

Now we estimate I_1. For this, we fix $\rho \ge 0$ such that $d_\star(\rho) \in [0, 6r]$. Thus, for any $i \in \{1, 2\}$, we have that $d(x_i + \rho\omega) \le d_\star(\rho) \le 6r$, thus, by (5.3),

$$\begin{aligned}|w(x_i + \rho\omega)| &= |w(x_i + \rho\omega)|\, \chi_\Omega(x_i + \rho\omega) \\ &\le [w]_{C^s(\mathbb{R}^n)}\, d^s(x_i + \rho\omega)\, \chi_\Omega(x_i + \rho\omega) \\ &\le C\, [w]_{C^s(\mathbb{R}^n)}\, r^s\, \chi_\Omega(x_i + \rho\omega).\end{aligned}$$

As a consequence

$$I_1 \le C\, [w]_{C^s(\mathbb{R}^n)}\, r^s \sum_{i=1}^{2} \int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \frac{\chi_\Omega(x_i + \rho\omega)\chi_{[0,6r]}\big(d(x_i + \rho\omega)\big)}{\rho^{1+2s}}. \tag{5.10}$$

This and Lemma 3.1 give that

$$I_1 \le C\, [w]_{C^s(\mathbb{R}^n)}\, r^{1+s} R^{-1-2s}, \tag{5.11}$$

for some $C > 0$.

Now we estimate I_2. For this we let $d_\star(\rho) > 6r$ and we will show that

$$\begin{aligned}&|w(x_1 + \rho\omega) - w(x_2 + \rho\omega)| \\ &\qquad \le Cr^\alpha \ell\big(d^{s-\alpha}(x_1 + \rho\omega) + R^{-\alpha} d^s(x_1 + \rho\omega)\big).\end{aligned} \tag{5.12}$$

To prove this, we observe that

$$d(x_1 + \rho\omega) \le d(x_2 + \rho\omega) + |x_1 - x_2| = d(x_2 + \rho\omega) + r.$$

Therefore

$$d_\star(\rho) = d(x_1 + \rho\omega) + d(x_2 + \rho\omega) \le 2d(x_2 + \rho\omega) + r.$$

Thus, if $d_\star(\rho) > 6r$ we have that

$$\frac{5}{12}d(x_1+\rho\omega) \le \frac{5}{12}d_\star(\rho) < \frac{d_\star(\rho)-r}{2} \le d(x_2+\rho\omega).$$

In particular

$$d(x_1+\rho\omega,\, x_2+\rho\omega) = \min\{d(x_1+\rho\omega),\, d(x_2+\rho\omega)\} \ge \frac{5}{12}d(x_1+\rho\omega).$$

Also, of course $d(x_1+\rho\omega,\, x_2+\rho\omega) \le d(x_1+\rho\omega)$. As a consequence of these observations, we can exploit (5.5) with $p = x_1+\rho\omega$ and $q = x_2+\rho\omega$, and we obtain

$$|w(x_1+\rho\omega)-w(x_2+\rho\omega)| \le \ell\,|x_1-x_2|^\alpha \left(\left(\frac{5}{12}d(x_1+\rho\omega)\right)^{s-\alpha} + R^{-\alpha}d^s(x_1+\rho\omega)\right),$$

which implies (5.12).

Having completed the proof of (5.12), we can use such formula to obtain that

$$\begin{aligned} I_2 &\le C\ell r^\alpha \int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \chi_{(r,+\infty)}\big(d(x_1+\rho\omega)\big)\, \frac{d^{s-\alpha}(x_1+\rho\omega)}{\rho^{1+2s}} \\ &\quad + C\ell r^\alpha R^{-\alpha} \int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \chi_{(r,+\infty)}\big(d(x_1+\rho\omega)\big)\, \frac{d^{s}(x_1+\rho\omega)}{\rho^{1+2s}}. \end{aligned}$$

So we use Lemma 3.6 (resp., Lemma 3.7) to bound the first (resp., the second) of the two integrals above: we obtain

$$I_2 \le C\ell r^\alpha R^{-s-\alpha}. \tag{5.13}$$

This, (5.9) and (5.11) give that

$$|Lw(x_1)-Lw(x_2)| \le Cr^\alpha\big([w]_{C^s(\mathbb{R}^n)}\, r^{1+s-\alpha}R^{-1-2s} + \ell R^{-s-\alpha}\big).$$

So, since $1+s-\alpha>0$, we have that $r^{1+s-\alpha} \le R^{1+s-\alpha}$ and so

$$|Lw(x_1)-Lw(x_2)| \le C|x_1-x_2|^\alpha\big([w]_{C^s(\mathbb{R}^n)}\, R^{-s-\alpha} + \ell R^{-s-\alpha}\big),$$

which establishes (5.6) when $\alpha \in (s,1]$.

It remains now to consider the case in which $\alpha \in (1,1+s)$. For this scope, we modify the argument above by looking at $L\partial_j w(x_1) - L\partial_j w(x_2)$, for a fixed $j \in \{1,\cdots,n\}$. In this case, formula (5.8) becomes

$$\begin{aligned} |L\partial_j w(x_1) - L\partial_j w(x_2)| &\le 2\int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \frac{|\partial_j w(x_1+\rho\omega)-\partial_j w(x_2+\rho\omega)|}{\rho^{1+2s}} \\ &\le J_1+J_2, \end{aligned}$$

where $r=|x_1-x_2|$,

$$J_1 = 2\int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \chi_{[0,6r]}\big(d_\star(\rho)\big)\, \frac{|\partial_j w(x_1+\rho\omega)-\partial_j w(x_2+\rho\omega)|}{\rho^{1+2s}}$$

and

$$J_2 = 2\int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \chi_{(6r,+\infty)}\big(d_\star(\rho)\big)\, \frac{|\partial_j w(x_1+\rho\omega)-\partial_j w(x_2+\rho\omega)|}{\rho^{1+2s}}.$$

First we estimate J_1. We fix $\rho \geq 0$ such that $d_\star(\rho) \in [0, 6r]$ and, for any $i \in \{1, 2\}$, we obtain that $d(x_i + \rho\omega) \leq d_\star(\rho) \leq 6r$. Hence, when both $x_1 + \rho\omega$ and $x_2 + \rho\omega$ belong to Ω we deduce from (5.5) that

$$\begin{aligned} & |\partial_i w(x_1 + \rho\omega) - \partial_i w(x_2 + \rho\omega)| \\ \leq\ & \ell\, |x_1 - x_2|^{\alpha-1} \big(d^{s-\alpha}(x_1 + \rho\omega, x_2 + \rho\omega) + R^{-\alpha} d^s(x_1 + \rho\omega, x_2 + \rho\omega)\big) \\ \leq\ & C\ell\, r^{\alpha-1}\, d^{s-\alpha}(x_1 + \rho\omega, x_2 + \rho\omega). \end{aligned}$$

This estimate and Lemma 3.4 imply that

$$\begin{aligned} & \int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \chi_{[0,6r]}\big(d_\star(\rho)\big)\, \frac{\chi_\Omega(x_1 + \rho\omega)\, \chi_\Omega(x_2 + \rho\omega)\, |\partial_j w(x_1 + \rho\omega) - \partial_j w(x_2 + \rho\omega)|}{\rho^{1+2s}} \\ & \quad \leq C\ell\, r^{\alpha-1} R^{-s-\alpha}. \end{aligned} \tag{5.14}$$

If instead $x_1 + \rho\omega \in \Omega$ and $x_2 + \rho\omega \notin \Omega$, up to a set of measure zero we have that $\partial_j w(x_2 + \rho\omega) = 0$ and so, by (5.5),

$$\begin{aligned} |\partial_i w(x_1 + \rho\omega) - \partial_i w(x_2 + \rho\omega)| &= |\partial_i w(x_1 + \rho\omega)| \\ &\leq \ell \big(d^{s-1}(x_1 + \rho\omega) + R^{-1} d^s(x_1 + \rho\omega)\big) \leq C\ell\, d^{s-1}(x_1 + \rho\omega) \\ &\leq C\ell\, r^{\alpha-1}\, d^{s-\alpha}(x_1 + \rho\omega) \end{aligned} \tag{5.15}$$

Notice that in the last two steps we have used the fact that $d(x_1 + \rho\omega) \leq d_\star(\rho) \leq 6r \leq 6R$ (together with $\alpha \geq 1$). Formula (5.15) and Lemma 3.6 imply that

$$\begin{aligned} & \int_{R/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \chi_{[0,6r]}\big(d_\star(\rho)\big)\, \frac{\chi_\Omega(x_1 + \rho\omega)\, \chi_{\mathbb{R}^n \setminus \Omega}(x_2 + \rho\omega)\, |\partial_j w(x_1 + \rho\omega) - \partial_j w(x_2 + \rho\omega)|}{\rho^{1+2s}} \\ & \quad \leq C\ell\, r^{\alpha-1} R^{-s-\alpha}. \end{aligned} \tag{5.16}$$

A similar estimate also holds by exchanging x_1 and x_2. Then, since also $|\partial_i w(x_1 + \rho\omega) - \partial_i w(x_2 + \rho\omega)| = 0$ if both $x_1 + \rho\omega$ and $x_2 + \rho\omega$ lie outside Ω, up to sets of null measure, we obtain from (5.14) and (5.16) that

$$J_1 \leq C\ell\, r^{\alpha-1} R^{-s-\alpha}.$$

Now we also bound J_2 by $C\ell r^{\alpha-1} R^{-s-\alpha}$. This can be obtained by repeating the argument from (5.12) to (5.13), replacing w by $\partial_i w$ (and $|x_1 - x_2|^\alpha$ by $|x_1 - x_2|^{\alpha-1}$). The estimates obtained on J_1 and J_2 prove that

$$|L\partial_j w(x_1) - L\partial_j w(x_2)| \leq C\ell\, r^{\alpha-1} R^{-s-\alpha},$$

up to renaming C, and so

$$R^{\alpha+s} [L\partial_j w]_{C^{\alpha-1}(B_{R/4})} \leq C\ell. \tag{5.17}$$

Now we observe that

$$L\partial_j w = \partial_j L w. \tag{5.18}$$

This can be proved by using (5.5) to obtain that, for any $x \in \Omega$ and $h \in \mathbb{R}$ sufficiently small,

$$\begin{aligned}
&\left|\frac{w(x+\rho\omega+he_i)-w(x+\rho\omega)}{h}-\partial_i w(x+\rho\omega)\right| \\
=\;& \left|\int_0^1 \partial_i w(x+\rho\omega+hte_i)-\partial_i w(x+\rho\omega)\,dt\right| \\
\le\;& \int_0^1 dt\ell\,|h|^{\alpha-1}\big(d^{s-\alpha}(x+\rho\omega+hte_i,x+\rho\omega)+R^{-\alpha}d^s(x+\rho\omega+hte_i,x+\rho\omega)\big),
\end{aligned}$$

and then integrating and using the same argument as above.

From (5.17) and (5.18), one completes the proof of (5.6) when $\alpha \in (1, 1+s)$.

It only remains to prove (5.7). For this, we take $\alpha \in (1, 1+s)$ and $x \in B_{R/4}$. We remark that $d(x) \in [R, 4R]$ and $w(x+\rho\omega)$ vanishes when $\rho \in (0, R/2)$ (thus so does $\partial_j w$, for any $j \in \{1, \cdots, n\}$), hence, recalling first (5.18) and then (5.5), we have

$$\begin{aligned}
&|\partial_j Lw(x)| \;=\; |L\partial_j w(x)| \\
\le\;& 2\int_{\{|\rho|>R/2\}} d\rho \int_{S^{n-1}} da(\omega)\frac{|\partial_j w(x+\rho\omega)|}{\rho^{1+2s}} \\
\le\;& C\ell \int_{\{|\rho|>R/2\}} d\rho \int_{S^{n-1}} da(\omega)\frac{\chi_\Omega(x+\rho\omega)\big(d^{s-1}(x+\rho\omega)+R^{-1}d^s(x+\rho\omega)\big)}{\rho^{1+2s}}.
\end{aligned}$$

The term with $\chi_\Omega(x+\rho\omega)\,d^{s-1}(x+\rho\omega)$ at the numerator can be estimated with $C\ell R^{s-1}$ by means of Lemma 3.6 (used here with $\alpha = 1$). The term with $d^s(x+\rho\omega)$ at the numerator can also be bounded in this way, using Lemma 3.7. From these considerations we obtain that $|\partial_j Lw(x)| \le C\ell R^{-s-1}$, which establishes (5.7). □

6. Iterative $C^{\alpha+2s}$-regularity

The cornerstone of our regularity theory is the following Theorem 6.1. Namely, we show that if the solution lies in some Hölder space, than it indeed lies in a better Hölder space (with estimates).

Theorem 6.1. *Let $\alpha \in (0, 1+s)$. Let $\Omega \subset \mathbb{R}^n$ and assume that either* (1.10) *or* (1.11) *is satisfied.*

Let $u \in C^s(\mathbb{R}^n) \cap C^\alpha_{\rm loc}(\Omega)$ be a solution to (1.7)*, with $g \in C^\alpha_{\rm loc}(\Omega)$.*

If $\alpha \in (s, 1+s)$ assume in addition that

$$\|u\|^{(-s)}_{\alpha;\Omega} < +\infty. \tag{6.1}$$

Then $u \in C^{\alpha+2s}_{\rm loc}(\Omega)$ and

$$\|u\|^{(-s)}_{\alpha+2s;\Omega} \le C\left(\|g\|^{(s)}_{\alpha;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} + \chi_{(s,1+s)}(\alpha)\,\|u\|^{(-s)}_{\alpha;\Omega}\right) \tag{6.2}$$

whenever $\alpha + 2s$ is not an integer.

Proof. To prove (6.2), we fix $p, q \in \Omega$, $p \neq q$, and we aim to show that

$$\begin{aligned}\sum_{j=0}^{k_s} \big(d^{j-s}(p)\,|D^j u(p)| + d^{j-s}(q)\,|D^j u(q)|\big) + d^{\alpha+s}(p,q)\frac{|D^{k_s}u(p) - D^{k_s}u(q)|}{|p-q|^{\alpha'_s}} \\ \leq C\left(\|g\|^{(s)}_{\alpha;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} + \chi_{(s,1+s)}(\alpha)\,\|u\|^{(-s)}_{\alpha;\Omega}\right),\end{aligned} \tag{6.3}$$

where $k_s \in \mathbb{N}$ and $\alpha_s \in (0,1]$ are such that $\alpha + 2s = k_s + \alpha'_s$. To prove this, we distinguish two cases: either $|p-q| < d(p,q)/30$ or $|p-q| \geq d(p,q)/30$.

We start with the case $|p-q| < d(p,q)/30$. Without loss of generality, by possibly exchanging p and q, we suppose that

$$d(p) \leq d(q) \tag{6.4}$$

and we set

$$R = \frac{d(p)}{3}. \tag{6.5}$$

Notice that there exists $p_\star \in (\partial\Omega) \cap (\partial B_{3R}(p))$ and

$$|p-q| < \frac{d(p,q)}{30} = \frac{d(p)}{30} = \frac{R}{10}.$$

Up to a translation, we also suppose that $p = 0$, hence

$$q \in B_{R/10}(p) = B_{R/10}, \tag{6.6}$$

$$\Omega \supseteq B_{3R}(p) = B_{3R} \tag{6.7}$$

$$\text{and} \qquad p_\star \in (\partial\Omega) \cap (\partial B_{3R}), \tag{6.8}$$

hence formula (5.1) holds true with this setting. We let $\eta_\star \in C^\infty(\mathbb{R}^n, [0,1])$ such that $\eta_\star \equiv 1$ in B_1 and $\eta_\star \equiv 0$ outside $B_{3/2}$. Let us also define $\eta(x) = \eta_\star(x/R)$.

Let us consider $\bar{u} = \eta u$, $\vartheta = 1 - \eta$ and $w = \vartheta u$. Since $\eta_\star$ is fixed once and for all, we can write, for any $\alpha' \in (0,1]$,

$$\begin{aligned}|\vartheta(x) - \vartheta(y)| = |\eta(y) - \eta(x)| &= \left|\eta_\star\left(\frac{y}{R}\right) - \eta_\star\left(\frac{x}{R}\right)\right| \\ &\leq [\eta_\star]_{C^{\alpha'}(\mathbb{R}^n)}\left|\frac{y}{R} - \frac{x}{R}\right|^{\alpha'} \leq CR^{-\alpha'}|x-y|^{\alpha'}.\end{aligned} \tag{6.9}$$

Similarly

$$\nabla\vartheta(x) = -\nabla\eta(x) = -R^{-1}\nabla\eta_\star\left(\frac{x}{R}\right) \tag{6.10}$$

and so, for any $\alpha' \in (0,1]$,

$$\begin{aligned}|\nabla\vartheta(x) - \nabla\vartheta(y)| &= |\nabla\eta(x) - \nabla\eta(y)| \\ &\leq R^{-1-\alpha'}\|\eta_\star\|_{C^{1+\alpha'}}|x-y|^{\alpha'} \leq CR^{-1-\alpha'}|x-y|^{\alpha'}.\end{aligned} \tag{6.11}$$

Notice also that $w \equiv 0$ in B_R and $w \equiv 0$ outside Ω. Our goal is to show that w satisfies the assumptions of Lemma 5.1. For this, when $\alpha \in (s, 1+s)$, we need to check condition (5.5). To this goal, we claim that, if $\alpha \in (s, 1+s)$, then

$$\begin{aligned}&\text{condition (5.5) holds true with}\\ &\ell = C\left(\|u\|^{(-s)}_{\alpha;\Omega} + [u]_{C^s(\mathbb{R}^n)}\right).\end{aligned} \tag{6.12}$$

To prove (6.12), we split the cases $\alpha \in (s, 1]$ and $\alpha \in (1, 1+s)$.

Let us first deal with the case

$$\alpha \in (s, 1]. \tag{6.13}$$

We fix x, $y \in \Omega$ with $|x-y| \le R$ and, up to interchanging x with y, we assume that $d(y) \le d(x)$. Then there exists $z \in \partial\Omega$ such that $|y-z| = d(y)$, and so

$$|u(y)| = |u(y) - u(z)| \le [u]_{C^s(\mathbb{R}^n)}|y-z|^s = [u]_{C^s(\mathbb{R}^n)}d^s(y). \tag{6.14}$$

Also, by (2.2), (6.1) and (6.13),

$$|u(x) - u(y)| \le [u]^{(-s)}_{\alpha;\Omega}\,|x-y|^\alpha d^{s-\alpha}(y).$$

Therefore, recalling also (6.9),

$$\begin{aligned}|w(x) - w(y)| &\le |\vartheta(x)|\,|u(x)-u(y)| + |u(y)|\,|\vartheta(x)-\vartheta(y)|\\ &\le C\big([u]^{(-s)}_{\alpha;\Omega}\,|x-y|^\alpha d^{s-\alpha}(y) + [u]_{C^s(\mathbb{R}^n)}d^s(y)R^{-\alpha}|x-y|^\alpha\big).\end{aligned}$$

This says that, in this case, condition (5.5) holds true, with $\ell = C\left([u]^{(-s)}_{\alpha;\Omega} + [u]_{C^s(\mathbb{R}^n)}\right)$, and this proves (6.12) when $\alpha \in (s, 1]$.

Now we prove (6.12) when $\alpha \in (1, 1+s)$. In this case, we can write

$$\alpha = 1 + \alpha', \tag{6.15}$$

with $\alpha' \in (0, s)$, hence we use (2.2) (with index $j = 1$) to deduce that in this case

$$\|u\|^{(-s)}_{\alpha;\Omega} \ge d^{1-s}(x)\,|\nabla u(x)|. \tag{6.16}$$

Hence, recalling (6.10) and (6.14),

$$\begin{aligned}|\nabla w(x)| &\le |\vartheta(x)|\,|\nabla u(x)| + |\nabla\vartheta(x)|\,|u(x)|\\ &\le C\left(\|u\|^{(-s)}_{\alpha;\Omega}\,d^{s-1}(x) + R^{-1}\,[u]_{C^s(\mathbb{R}^n)}d^s(x)\right).\end{aligned} \tag{6.17}$$

Now we take x, $y \in \Omega$, with $|x-y| \le R$, and suppose, without loss of generality that $d(x,y) = d(y) \le d(x)$. Since $\alpha' \in (0, s)$, we have that

$$|u(x) - u(y)| \le [u]_{C^s(\mathbb{R}^n)}|x-y|^s \le [u]_{C^s(\mathbb{R}^n)}R^{s-\alpha'}\,|x-y|^{\alpha'}. \tag{6.18}$$

Also, using (6.15) once again, we obtain

$$[u]^{(-s)}_{\alpha;\Omega} \ge d^{\alpha-s}(y)\frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\alpha'}}$$

and therefore, recalling also (6.9), (6.10), (6.11), (6.14) (6.16) and (6.18),

$$\begin{aligned}
&|\nabla w(x)-\nabla w(y)|\\
=\;& |\vartheta(x)\nabla u(x)+\nabla\vartheta(x)u(x)-\vartheta(y)\nabla u(y)-\nabla\vartheta(y)u(y)|\\
\le\;& |\vartheta(x)\nabla u(x)-\vartheta(x)\nabla u(y)|+|\vartheta(x)\nabla u(y)-\vartheta(y)\nabla u(y)|\\
&\quad+|\nabla\vartheta(x)u(x)-\nabla\vartheta(x)u(y)|+|\nabla\vartheta(x)u(y)-\nabla\vartheta(y)u(y)|\\
\le\;& C|\nabla u(x)-\nabla u(y)|+CR^{-\alpha'}|\nabla u(y)|\,|x-y|^{\alpha'}\\
&\quad+CR^{-1}|u(x)-u(y)|+[u]_{C^s(\mathbb{R}^n)}d^s(y)\,|\nabla\vartheta(x)-\nabla\vartheta(y)|\\
\le\;& C\,[u]^{(-s)}_{\alpha;\Omega}d^{s-\alpha}(y)|x-y|^{\alpha'}+CR^{-\alpha'}\|u\|^{(-s)}_{\alpha;\Omega}d^{s-1}(y)\,|x-y|^{\alpha'}\\
&\quad+CR^{s-\alpha'-1}[u]_{C^s(\mathbb{R}^n)}\,|x-y|^{\alpha'}+CR^{-1-\alpha'}[u]_{C^s(\mathbb{R}^n)}d^s(y)\,|x-y|^{\alpha'}.
\end{aligned}$$

By taking common factors and recalling (6.15), we obtain that $|\nabla w(x)-\nabla w(y)|$ is bounded from above by

$$\begin{aligned}
&C\left([u]_{C^s(\mathbb{R}^n)}+\|u\|^{(-s)}_{\alpha;\Omega}\right)|x-y|^{\alpha-1}\\
&\qquad\cdot\big(d^{s-\alpha}(y)+R^{1-\alpha}d^{s-1}(y)+R^{s-\alpha}+R^{-\alpha}d^s(y)\big).
\end{aligned}\tag{6.19}$$

Now we observe that

$$\frac{1}{R^{\alpha-1}d^{1-s}(y)}\le\frac{1}{d^{\alpha-s}(y)}+\frac{1}{R^{\alpha-s}}$$
$$\text{and}\qquad\frac{1}{R^{\alpha-s}}\le\frac{1}{d^{\alpha-s}(y)}+\frac{d^s(y)}{R^\alpha},$$

as can be checked by considering the cases $R\ge d(y)$ and $R<d(y)$, and recalling that here $\alpha>1>s$. Consequently,

$$R^{1-\alpha}d^{s-1}(y)+R^{s-\alpha}\le 2\big(d^{s-\alpha}(y)+R^{-\alpha}d^s(y)\big)$$

and then (6.19) yields that that $|\nabla w(x)-\nabla w(y)|$ is bounded from above by

$$C\left([u]_{C^s(\mathbb{R}^n)}+\|u\|^{(-s)}_{\alpha;\Omega}\right)|x-y|^{\alpha-1}\big(d^{s-\alpha}(y)+R^{-\alpha}d^s(y)\big).$$

This and (6.17) complete the proof of (6.12) also when $\alpha\in(1,1+s)$.

Having completed the proof of (6.12), now we estimate $\|w\|_{C^s(\mathbb{R}^n)}$. We claim that

$$[w]_{C^s(\mathbb{R}^n)}\le C\,[u]_{C^s(\mathbb{R}^n)}.\tag{6.20}$$

To check this, we fix a, $b\in\mathbb{R}^n$ and we aim to bound $|w(a)-w(b)|$. To this goal, by possibly exchanging a and b, we suppose that $|a|\ge|b|$. Now we distinguish two cases: either $|b|\ge 2R$ or $|b|<2R$. If $|b|\ge 2R$ we have that both a and b lie outside $B_{3R/2}$, and so $\vartheta(a)=\vartheta(b)=1$. Accordingly

$$|w(a)-w(b)|=|u(a)-u(b)|\le[u]_{C^s(\mathbb{R}^n)}|a-b|^s,$$

and (6.20) is proved in this case. On the other hand, if $|b|<2R$ we use (6.8) and we get

$$|u(b)|=|u(b)-u(p_\star)|\le[u]_{C^s(\mathbb{R}^n)}|b-p_\star|^s\le C[u]_{C^s(\mathbb{R}^n)}R^s.$$

Therefore, recalling (6.9) (used here with $\alpha = s$),

$$\begin{aligned} |w(b) - w(b)| &\leq |\vartheta(a)|\,|u(a) - u(b)| + |u(b)|\,|\vartheta(a) - \vartheta(b)| \\ &\leq C\Big([u]_{C^s(\mathbb{R}^n)}|a-b|^s + [u]_{C^s(\mathbb{R}^n)}R^s \cdot R^{-s}|a-b|^s\Big), \end{aligned}$$

which establishes (6.20) in this case too.

Now we take ℓ as in (6.12) when $\alpha \in (s, 1+s)$ and, for definiteness, we also set $\ell = 0$ when $\alpha \in [0, s]$: then, with this notation, we deduce by Lemma 5.1 and formula (6.20) that

$$\begin{aligned} & R^{2s}\|Lw\|_{L^\infty(B_{R/2})} \leq CR^s\,[w]_{C^s(\mathbb{R}^n)} \leq CR^s\,[u]_{C^s(\mathbb{R}^n)} \\ \text{and} \qquad & R^{\alpha+s}[Lw]_{C^\alpha(B_{R/2})} \leq C\Big([w]_{C^s(\mathbb{R}^n)} + \ell\Big) \leq C\Big([u]_{C^s(\mathbb{R}^n)} + \ell\Big), \end{aligned}$$

and, in case $\alpha \in (1, 1+s)$, also

$$R^{1+s}\|\nabla Lw\|_{L^\infty(B_{R/2})} \leq C\ell \leq C\Big([u]_{C^s(\mathbb{R}^n)} + \ell\Big).$$

As a consequence, if $\alpha = k + \alpha'$ with $k \in \mathbb{N}$ and $\alpha' \in (0, 1]$, we can write the weighted estimate

$$\sum_{j=0}^{k} R^{j+s}\|D^j Lw\|_{L^\infty(B_{R/2})} + R^{\alpha+s}[Lw]_{C^\alpha(B_{R/2})} \leq C\Big([u]_{C^s(\mathbb{R}^n)} + \ell\Big). \tag{6.21}$$

Now we show that $\bar{u} \in C^\alpha(\mathbb{R}^n)$, with

$$[\bar{u}]_{C^\alpha(\mathbb{R}^n)} \leq C\left([u]_{C^s(\mathbb{R}^n)} + \ell\right) R^{s-\alpha}. \tag{6.22}$$

To this goal, we distinguish two cases, either $\alpha \in (0, s]$ or $\alpha \in (s, 1+s)$.

Let us first deal with the case $\alpha \in (0, s]$. We let x, $y \in \mathbb{R}^n$ and we estimate $|\bar{u}(x) - \bar{u}(y)|$. If both x and y lies outside $B_{3R/2}$, then $\bar{u}(x) = \bar{u}(y) = 0$ and so $|\bar{u}(x) - \bar{u}(y)| = 0$. Thus we may assume (up to exchanging x and y) that $y \in B_{3R/2}$. So, by (6.8),

$$|u(y)| = |u(y) - u(p_\star)| \leq C[u]_{C^s(\mathbb{R}^n)}\,R^s. \tag{6.23}$$

To complete the proof of (6.22) when $\alpha \in (0, s]$, we now distinguish two sub-cases: either $|x| \geq 7R/4$ or $|x| < 7R/4$. If $|x| \geq 7R/4$, we have that $\bar{u}(x) = 0$ and

$$|x - y| \geq |x| - |y| \geq |x| - \frac{3R}{2} \geq \frac{R}{4}.$$

Consequently, we use (6.23) and we get

$$\begin{aligned} |\bar{u}(x) - \bar{u}(y)| = |\bar{u}(y)| \leq |u(y)| &\leq C[u]_{C^s(\mathbb{R}^n)}\,R^s \\ &= C[u]_{C^s(\mathbb{R}^n)}\,R^{s-\alpha}R^\alpha \leq C[u]_{C^s(\mathbb{R}^n)}\,R^{s-\alpha}|x-y|^\alpha, \end{aligned}$$

and this establishes (6.22) in this sub-case. Now we deal with the sub-case in which $|x| < 7R/4$. Then $|x - y| \leq (7R/4) + (3R/2) \leq CR$ and so

$$|u(x) - u(y)| \leq [u]_{C^s(\mathbb{R}^n)}|x-y|^s \leq C\,[u]_{C^s(\mathbb{R}^n)}|x-y|^\alpha R^{s-\alpha}.$$

Therefore, recalling (6.9) and (6.23), we deduce that

$$\begin{aligned} |\bar u(x)-\bar u(y)| &\le |\eta(x)|\,|u(x)-u(y)|+|u(y)|\,|\eta(x)-\eta(y)|\\ &\le C\,[u]_{C^s(\mathbb{R}^n)}\,R^{s-\alpha}|x-y|^{\alpha}\end{aligned}$$

and this proves (6.22) also in this sub-case.

Having completed the proof of (6.22) when $\alpha\in(0,s]$, we now deal with the case $\alpha\in(s,1+s)$. In this case, we take x, $y\in\Omega$ and we write $\alpha=k+\alpha'$, with $k\in\{0,1\}$ and $\alpha'\in(0,1]$ and

(6.24)

$$\ell\ge \|u\|^{(-s)}_{\alpha;\Omega}\ge d^{\alpha-s}(x,y)\frac{|D^ku(x)-D^ku(y)|}{|x-y|^{\alpha'}}+d^{-s}(z)\,|u(z)|+d^{k-s}(z)\,|D^ku(z)|,$$

for any $x,y,z\in\Omega$ (notice that we have use (2.2) with indices $j=0$ and $j=k$).

Also, $\bar u$ (and so $D^k\bar u$) vanishes in $\mathbb{R}^n\setminus\overline{B_{3R/2}}$, therefore, to estimate $|D^k\bar u(x)-D^k\bar u(y)|$, we may assume (up to interchanging x and y) that $y\in\overline{B_{3R/2}}$.

So we distinguish two sub-cases: either also $x\in B_{7R/4}$ or $x\in\mathbb{R}^n\setminus B_{7R/4}$.

Let us start with the sub-case $x\in B_{7R/4}$. Notice that

$$B_{7R/4}\subseteq\big\{\zeta\in\mathbb{R}^n \text{ s.t. } d(\zeta)\ge R\big\},$$

thanks to (6.7), so in particular $d(x)\ge R$ and $d(y)\ge R$, and therefore $d(x,y)\ge R$. In addition, by (6.8), we have that $d(x)+d(y)\le CR$. So, from (6.24), we obtain, for x and y as above and every $\zeta\in B_{7R/4}$,

$$\begin{aligned} &|D^ku(x)-D^ku(y)|\le \ell\, d^{s-\alpha}(x,y)\,|x-y|^{\alpha'}\le C\ell\,R^{s-\alpha}\,|x-y|^{\alpha'},\\ &|D^ku(\zeta)|\le \ell\, d^{s-k}(\zeta)\le C\ell\,R^{s-k}\\ \text{and}\quad &|u(x)|+|u(y)|\le \ell\,\big(d^s(x)+d^s(y)\big)\le C\ell\,R^s.\end{aligned}\tag{6.25}$$

We remark that we can also take $\zeta=x$ or $\zeta=y$ in (6.25) if we wish. Now we claim that

$$|u(x)-u(y)|\le \ell\,R^{s-\alpha'}\,|x-y|^{\alpha'}.\tag{6.26}$$

Indeed, if $k=0$ we have that (6.26) reduces to (6.25); if instead $k=1$, we use (6.25) to get that

$$|u(x)-u(y)|\le \sup_{\zeta\in B_{7R/4}}|\nabla u(\zeta)|\,|x-y|\le C\ell\,R^{s-1}\,|x-y|.$$

Also

$$|x-y|\le|x|+|y|\le 3R,$$

thus

$$|u(x)-u(y)|\le C\ell\,R^{s-1}\,|x-y|^{1-\alpha'}\,|x-y|^{\alpha'}\le C\ell\,R^{s-1}\,R^{1-\alpha'}\,|x-y|^{\alpha'},$$

up to changing the constants, and this proves (6.26).

Now we remark that

$$D^k\bar u=D^k(\eta u)=\frac{1}{2-k}\big(\eta D^ku+uD^k\eta\big),$$

for $k \in \{0,1\}$, therefore

$$
\begin{aligned}
&|D^k u(x) - D^k u(y)| \\
&= \frac{1}{2-k}\big|\eta(x)D^k u(x) + u(x)D^k\eta(x) - \eta(y)D^k u(y) - u(y)D^k\eta(y)\big| \\
&\le C\Big(\big|\eta(x)D^k u(x) - \eta(y)D^k u(y)\big| + \big|u(x)D^k\eta(x) - u(y)D^k\eta(y)\big|\Big) \\
&\le C\Big(|\eta(x)|\,|D^k u(x) - D^k u(y)| + |D^k u(y)|\,|\eta(x)-\eta(y)| \\
&\qquad + |u(x)|\,|D^k\eta(x) - D^k\eta(y)| + |D^k\eta(y)|\,|u(x)-u(y)|\Big).
\end{aligned}
\tag{6.27}
$$

Also, by (6.9) and (6.11)

$$
|\eta(y)-\eta(x)| \le CR^{-\alpha'}|x-y|^{\alpha'} \quad\text{and}\quad |D^k\eta(y) - D^k\eta(x)| \le CR^{-k-\alpha'}|x-y|^{\alpha'},
$$

thus, by (6.25) and (6.26),

$$
\begin{aligned}
&|\eta(x)|\,|D^k u(x) - D^k u(y)| + |D^k u(y)|\,|\eta(x)-\eta(y)| \\
&\quad + |u(x)|\,|D^k\eta(x) - D^k\eta(y)| + |D^k\eta(y)|\,|u(x)-u(y)| \\
&\qquad \le C\ell\left(R^{s-\alpha} + R^{s-k-\alpha'}\right)|x-y|^{\alpha'} \\
&\qquad = C\ell\, R^{s-\alpha}\,|x-y|^{\alpha'},
\end{aligned}
$$

up to renaming constants. Then, we insert this into (6.27) and we obtain the proof of (6.22) in the sub-case $x \in B_{7R/4}$.

Now we consider the sub-case $x \in \mathbb{R}^n \setminus B_{7R/4}$. In this case $\bar u$ (and so $D^k\bar u$) vanishes in the vicinity of x, therefore

$$
|D^k\bar u(x) - D^k\bar u(y)| = |D^k\bar u(y)| = \frac{1}{2-k}\big|\eta(y)D^k u(y) + u(y)D^k\eta(y)\big|.
$$

Therefore, by (6.10) and (6.25),

$$
|D^k\bar u(x) - D^k\bar u(y)| \le C\big(|\eta(y)|\,|D^k u(y)| + |u(y)|\,|D^k\eta(y)|\big) \le C\ell R^{s-k}. \tag{6.28}
$$

Now we use that x is outside $B_{7R/4}$ and y inside $\overline{B_{3R/2}}$ to conclude that $|x-y| \ge |x| - |y| \ge R/4$. Hence we deduce from (6.28) that

$$
|D^k\bar u(x) - D^k\bar u(y)| \le C\ell R^{s-k-\alpha'}\,R^{\alpha'} \le C\ell R^{s-\alpha}\,|x-y|^{\alpha'}.
$$

This completes the proof of (6.22) in the case $\alpha \in (s, 1+s)$. The proof of (6.22) is therefore finished.

Now we show that

$$
\|\bar u\|_{L^\infty(\mathbb{R}^n)} \le CR^s\left([u]_{C^s(\mathbb{R}^n)} + \ell\right). \tag{6.29}
$$

For this, fix $x \in \mathbb{R}^n$. If $|x| \ge 2R$, then $|\bar u(x)| = 0$ and we are done, so we may suppose that $|x| < 2R$. If $\alpha \in (0, s]$, we use the fact that $\bar u(2Re_1) = 0$ to conclude that

$$
|\bar u(x)| = |\bar u(x) - \bar u(2Re_1)| \le [\bar u]_{C^\alpha(\mathbb{R}^n)}|x - 2Re_1|^\alpha \le C[\bar u]_{C^\alpha(\mathbb{R}^n)}R^\alpha,
$$

hence, by (6.22), we see that $|\bar u(x)| \le C\big([u]_{C^s(\mathbb{R}^n)} + \ell\big) R^{s-\alpha} \cdot R^\alpha$, as desired. If instead $\alpha \in (s, 1+s)$, we use (2.2) with index $j = 0$ to see that

$$\ell \ge d^{-s}(x)\,|u(x)|.$$

Hence, since by (6.8) we have that

$$d(x) \le |x - p_\star| \le |x| + |p_\star| \le 5R,$$

we obtain $|u(x)| \le \ell\, d^s(x) \le \ell R^s$. These considerations prove (6.29).

Now we claim that, if $\alpha = k + \alpha' \in (0, 1+s)$ with $k \in \mathbb{N}$ and $\alpha' \in (0,1]$, then

$$\sum_{j=0}^{k} R^{j-s} \|D^j \bar u\|_{L^\infty(\mathbb{R}^n)} + R^{\alpha - s} [\bar u]_{C^\alpha(\mathbb{R}^n)} \le C\left([u]_{C^s(\mathbb{R}^n)} + \ell\right). \tag{6.30}$$

Indeed, if $k = 0$, formula (6.30) follows by combining (6.22) and (6.29). If instead $k > 0$, then necessarily $k = 1$, since $k \le \alpha < 1 + s < 2$. Consequently, using that $\bar u$ vanishes outside $B_{3R/2}$ and (6.22), we obtain

$$\begin{aligned} R^{k-s} \|D^k \bar u\|_{L^\infty(\mathbb{R}^n)} &= R^{1-s} \sup_{B_{3R/2}} |\nabla \bar u(x)| = R^{1-s} \sup_{B_{3R/2}} |\nabla \bar u(x) - \nabla \bar u(Re_1)| \\ &\le R^{1-s}\, [\bar u]_{C^\alpha(\mathbb{R}^n)} \sup_{B_{3R/2}} |x - Re_1|^{\alpha'} \le C R^{1-s+s-\alpha+\alpha'}\, C\left([u]_{C^s(\mathbb{R}^n)} + \ell\right) \\ &= C\left([u]_{C^s(\mathbb{R}^n)} + \ell\right). \end{aligned}$$

This, (6.22) and (6.29) then imply (6.30) also in this case.

After all this (rather technical, but useful) preliminary work, we are ready to perform the regularity theory needed in this setting. For this, we notice that $\bar u = u\eta = u(1 - \vartheta) = u - w$, therefore

$$L\bar u = g - Lw \qquad \text{in } B_{3R/2}. \tag{6.31}$$

It is now useful to scale $\bar u$, by setting $\bar u_R(x) = R^{-2s} \bar u(Rx)$. We have that $L\bar u_R(x) = L\bar u(Rx)$. Therefore by Theorem 1.1(b) in [7],

$$\|\bar u_R\|_{C^{\alpha+2s}(B_{1/4})} \le C\left(\|\bar u_R\|_{C^\alpha(\mathbb{R}^n)} + \|L\bar u_R\|_{C^\alpha(B_{1/2})}\right). \tag{6.32}$$

To scale back, we notice that, for any β, $a > 0$,

$$\begin{aligned} &\|D^j \bar u_R\|_{L^\infty(B_a)} = R^{j-2s} \|\bar u\|_{L^\infty(B_{aR})}, \qquad [\bar u_R]_{C^\beta(B_a)} = R^{\beta - 2s} [\bar u]_{C^\beta(B_{aR})}, \\ &\|D^j L\bar u_R\|_{L^\infty(B_a)} = R^j \|L\bar u\|_{L^\infty(B_{aR})} \qquad \text{and} \qquad [L\bar u_R]_{C^\beta(B_a)} = R^\beta [L\bar u]_{C^\beta(B_{aR})}. \end{aligned}$$

So, if we write $\alpha = k + \alpha'$, with $k \in \mathbb{N}$ and $\alpha' \in (0,1]$, we have that

$$\begin{aligned} \|\bar u_R\|_{C^\alpha(B_a)} &= \sum_{j=0}^{k} \|D^j \bar u_R\|_{L^\infty(B_a)} + [\bar u_R]_{C^\alpha(B_a)} \\ &= \sum_{j=0}^{k} R^{j-2s} \|D^j \bar u\|_{L^\infty(B_{aR})} + R^{\alpha - 2s} [\bar u]_{C^\alpha(B_{aR})}. \end{aligned} \tag{6.33}$$

Similarly, if $\alpha + 2s = k_s + \alpha'_s$, with $k_s \in \mathbb{N}$ and $\alpha'_s \in (0,1]$,

$$\|\bar u_R\|_{C^{\alpha+2s}(B_a)} = \sum_{j=0}^{k_s} R^{j-2s}\|D^j \bar u\|_{L^\infty(B_{aR})} + R^\alpha [\bar u]_{C^{\alpha+2s}(B_{aR})} \tag{6.34}$$

and

$$\|L\bar u_R\|_{C^{\alpha}(B_a)} = \sum_{j=0}^{k} R^{j}\|D^j L\bar u\|_{L^\infty(B_{aR})} + R^\alpha [L\bar u]_{C^{\alpha}(B_{aR})}. \tag{6.35}$$

From (6.30) and (6.33) we see that

$$R^s \|\bar u_R\|_{C^\alpha(\mathbb{R}^n)} \le C\big([u]_{C^s(\mathbb{R}^n)} + \ell\big). \tag{6.36}$$

Also, from (6.31) and (6.35),

$$\|L\bar u_R\|_{C^\alpha(B_{1/2})} \le \sum_{j=0}^{k} R^j \|D^j Lw\|_{L^\infty(B_{R/2})} + R^\alpha [Lw]_{C^\alpha(B_{R/2})} + \Gamma_g,$$

where

$$\Gamma_g = \sum_{j=0}^{k} R^j \|D^j g\|_{L^\infty(B_{R/2})} + R^\alpha [g]_{C^\alpha(B_{R/2})}.$$

Accordingly, from (6.21),

$$R^s \|L\bar u_R\|_{C^\alpha(B_{1/2})} \le C\Big([u]_{C^s(\mathbb{R}^n)} + \ell + R^s\Gamma_g\Big).$$

This, (6.32) and (6.36) give that

$$R^s \|\bar u_R\|_{C^{\alpha+2s}(B_{1/4})} \le C\Big([u]_{C^s(\mathbb{R}^n)} + \ell + R^s\Gamma_g\Big).$$

As a consequence of this and (6.34), we conclude that

$$\sum_{j=0}^{k_s} R^{j-s}\|D^j \bar u\|_{L^\infty(B_{R/4})} + R^{\alpha+s}[\bar u]_{C^{\alpha+2s}(B_{R/4})} \le C\Big([u]_{C^s(\mathbb{R}^n)} + \ell + R^s\Gamma_g\Big). \tag{6.37}$$

Now, from (6.4), (6.5) and (6.6), we have

$$d(p,q) = d(p) = 3R \;\text{ and }\; \frac{R}{10} \ge |p-q| = |q|,$$

therefore $d(q) \le |q - p_\star| \le 4R$, thanks to (6.8). This says that $d(p)$, $d(q)$ and $d(p,q)$ are all comparable to R, hence $R^s\Gamma_g \le C\|g\|^{(s)}_{\alpha;\Omega}$ and (6.37) gives

$$\begin{aligned}\sum_{j=0}^{k_s} \Big(d(p)^{j-s}|D^j\bar u(p)| + d(q)^{j-s}|D^j \bar u(q)|\Big) + d^{\alpha+s}(p,q)\,\frac{|D^{k_s}\bar u(p) - D^{k_s}\bar u(q)|}{|p-q|^{\alpha'_s}} \\ \le C\Big([u]_{C^s(\mathbb{R}^n)} + \ell + \|g\|^{(s)}_{\alpha;\Omega}\Big).\end{aligned} \tag{6.38}$$

Since $\bar{u} = u$ in $B_{R/4}$, and p and q lie in such ball, thanks to (6.6), we can replace $\bar{u}$ with u in (6.38), and this establishes (6.3) when $|p-q| < d(p,q)/30$.

Let us now suppose that $|p-q| \geq d(p,q)/30$ and let us prove (6.3) in this case. The proof of this will rely on the fact that we have already proved (6.3) when $|p-q| < d(p,q)/30$. We first check that

$$\sum_{j=0}^{k_s} d^{j-s}(p)\,|D^j u(p)| \leq C\left(\|g\|_{\alpha;\Omega}^{(s)} + \|u\|_{C^s(\mathbb{R}^n)} + \chi_{(s,1+s)}(\alpha)\,\|u\|_{\alpha;\Omega}^{(-s)}\right). \tag{6.39}$$

For this we take a sequence of points $p_j \to p$ as $j \to +\infty$. Since $d(p) > 0$, for j large we have that $|p - p_j| \leq d(p)/100$ and thus

$$d(p_j) \geq d(p) - |p-p_j| \geq \frac{99\,d(p)}{100}.$$

Therefore $d(p,p_j) \geq 99\,d(p)/100$ and thus

$$|p-p_j| \leq \frac{d(p)}{100} \leq \frac{d(p,p_j)}{99} < \frac{d(p,p_j)}{30}.$$

Since we have already proved (6.3) in this case, we can use it at the points p and p_j and conclude that

$$\begin{aligned}
&\sum_{j=0}^{k_s}\left(d^{j-s}(p)\,|D^j u(p)| + d^{j-s}(p_j)\,|D^j u(p_j)|\right) + d^{\alpha+s}(p,p_j)\frac{|D^{k_s}u(p) - D^{k_s}u(p_j)|}{|p-p_j|^{\alpha'_s}} \\
&\qquad \leq C\left(\|g\|_{\alpha;\Omega}^{(s)} + \|u\|_{C^s(\mathbb{R}^n)} + \chi_{(s,1+s)}(\alpha)\,\|u\|_{\alpha;\Omega}^{(-s)}\right),
\end{aligned}$$

which in turn implies (6.39).

The same argument used to prove (6.39) applied to the point q instead of p gives that

$$\sum_{j=0}^{k_s} d^{j-s}(q)\,|D^j u(q)| \leq C\left(\|g\|_{\alpha;\Omega}^{(s)} + \|u\|_{C^s(\mathbb{R}^n)} + \chi_{(s,1+s)}(\alpha)\,\|u\|_{\alpha;\Omega}^{(-s)}\right). \tag{6.40}$$

Now we want to prove that

$$d^{\alpha+s}(p,q)\frac{|D^{k_s}u(p) - D^{k_s}u(q)|}{|p-q|^{\alpha'_s}} \leq C\left(\|g\|_{\alpha;\Omega}^{(s)} + \|u\|_{C^s(\mathbb{R}^n)} + \chi_{(s,1+s)}(\alpha)\,\|u\|_{\alpha;\Omega}^{(-s)}\right). \tag{6.41}$$

For this, we use the condition $|p-q| \geq d(p,q)/30$ and the fact that

$$\alpha + s = \alpha + 2s - s = k_s + \alpha'_s - s$$

to realize that

$$\begin{aligned}
d^{\alpha+s}(p,q)\frac{|D^{k_s}u(p) - D^{k_s}u(q)|}{|p-q|^{\alpha'_s}} &\leq C\,d^{k_s+\alpha'_s-s}(p,q)\frac{|D^{k_s}u(p) - D^{k_s}u(q)|}{d^{\alpha'_s}(p,q)} \\
&\leq C\,d^{k_s-s}(p,q)\left(|D^{k_s}u(p)| + |D^{k_s}u(q)|\right) \\
&\leq C\,d^{k_s-s}(p)\,|D^{k_s}u(p)| + C\,d^{k_s-s}(q)\,|D^{k_s}u(q)|.
\end{aligned}$$

This estimate, together with (6.39) and (6.40) (used here with $j = k_s$), establishes (6.41).

Now, by collecting the estimates in (6.39), (6.40) and (6.41), we complete the proof of (6.3) when $|p-q| \geq d(p,q)/30$. This is the end of the proof of Lemma 6.1. □

By iterating Theorem 6.1, and using again the notation in (2.2), we obtain:

Corollary 6.2. *Let $\beta \in (0, 1+s)$. Let $\Omega \subset \mathbb{R}^n$ and assume that either* (1.10) *or* (1.11) *is satisfied.*

Let $u \in C^s(\mathbb{R}^n)$ be a solution to (1.7)*, with $g \in C^\beta_{\mathrm{loc}}(\Omega)$ and $\|g\|^{(s)}_{\beta;\Omega} < +\infty$.*

Then $u \in C^{\beta+2s}_{\mathrm{loc}}(\Omega)$ and

$$\|u\|^{(-s)}_{\beta+2s;\Omega} \leq C \left(\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} \right), \tag{6.42}$$

whenever $\beta + 2s$ is not an integer.

Proof. The rough idea is to iterate Theorem 6.1 say, starting with Hölder exponent (possibly below) s, to get $s+2s$, then $s+2s+2s$ and so on, till we reach the threshold imposed by β.

To make this argument rigorous we argue as follows. If $\beta \in (0, s]$, then we use Theorem 6.1 with $\alpha = \beta$ and we obtain (6.42). Thus, we can assume that

$$\beta \in (s, 1+s). \tag{6.43}$$

Given $s \in (0,1) \cap \mathbb{Q}$ (resp., $s \in (0,1) \setminus \mathbb{Q}$), we fix $\vartheta \in (0,s) \setminus \mathbb{Q}$ (resp., $\vartheta \in (0,s) \cap \mathbb{Q}$). By construction, for any $j \in \mathbb{N}$, we have that $\vartheta + 2js \notin \mathbb{Q}$, and so in particular

$$\vartheta + 2js \notin \mathbb{N}. \tag{6.44}$$

We remark that

$$\vartheta < s < \beta, \tag{6.45}$$

thanks to (6.43). We let $J \in \mathbb{N}$ the largest integer j for which $\vartheta + 2js \leq \beta + 2s$. By construction

$$\vartheta + 2Js \in (\beta,\, \beta + 2s]. \tag{6.46}$$

Furthermore $J \geq 1$, due to (6.45). We also denote by $C_1 > 1$ the constant appearing in (6.2) and by $C_2 > 1$ the one appearing in Lemma 2.1 (these constants were called C in those statements, but for clarity we emphasize now these constant by giving to them a special name and, without loss of generality, we can suppose that they are larger than 1). Let also $C_\star = 2(C_1 + C_2)^2$. We claim that, for any $j \in \{1, \cdots, J\}$,

$$\|u\|^{(-s)}_{\vartheta+2js;\Omega} \leq C^j_\star \left(\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} \right). \tag{6.47}$$

The proof is by induction. First, we use Theorem 6.1 with $\alpha = \vartheta \in (0,s)$: notice that $\vartheta + 2s$ is not an integer, thanks to (6.44), and therefore Theorem 6.1 yields that

$$\|u\|^{(-s)}_{\vartheta+2s;\Omega} \leq C_1 \left(\|g\|^{(s)}_{\vartheta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} \right). \tag{6.48}$$

Also, in view of (6.45) and Lemma 2.1, we have that $\|g\|^{(s)}_{\vartheta;\Omega} \le C_2\|g\|^{(s)}_{\beta;\Omega}$. By plugging this information into (6.48) and using that C_1, $C_2 > 1$, we see that (6.47) holds true for $j = 1$.

Now we perform the induction step, i.e. we suppose that (6.47) holds true for some $j \in \{1, \cdots, J-1\}$ and we prove it for $j+1$. For this, we use Theorem 6.1 with $\alpha = \vartheta + js$. We remark that $\vartheta + js + 2s \notin \mathbb{N}$, thanks to (6.44), hence Theorem 6.1 applies and it gives that

$$\|u\|^{(-s)}_{\vartheta+2(j+1)s;\Omega} \le C_1 \left(\|g\|^{(s)}_{\vartheta+2js;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} + \|u\|^{(-s)}_{\vartheta+2js;\Omega}\right). \tag{6.49}$$

Notice also that, by (6.46),

$$\vartheta + 2js \le \vartheta + 2(J-1)s = \vartheta + 2Js - 2s \le \beta + 2s - 2s = \beta.$$

This and Lemma 2.1 imply that

$$\|g\|^{(s)}_{\vartheta+2js;\Omega} \le C_2\|g\|^{(s)}_{\beta;\Omega}.$$

Accordingly, we deduce from (6.49) that

$$\|u\|^{(-s)}_{\vartheta+2(j+1)s;\Omega} \le C_1 \left(C_2\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} + \|u\|^{(-s)}_{\vartheta+2js;\Omega}\right).$$

Hence, since by inductive assumption (6.47) holds true for j,

$$\begin{aligned} \|u\|^{(-s)}_{\vartheta+2(j+1)s;\Omega} &\le C_1 \left(C_2\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} + C^j_\star \left(\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)}\right)\right) \\ &\le 2C_1\, C^j_\star \left(\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)}\right) \end{aligned}. \tag{6.50}$$

This proves (6.47) for $j+1$. The inductive step is thus completed, and we have established (6.47).

Now we observe that

$$\|u\|^{(-s)}_{\beta;\Omega} \le C_2\|u\|^{(-s)}_{\vartheta+2Js;\Omega},$$

due to (6.46) and Lemma 2.1. Thus, using (6.47) with $j = J$,

$$\|u\|^{(-s)}_{\beta;\Omega} \le C^{J+1}_\star \left(\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)}\right). \tag{6.51}$$

Now we use Theorem 6.1 for the last time, here with $\alpha = \beta$. Notice that $\beta + 2s \notin \mathbb{N}$, by the assumption of Corollary 6.2: consequently Theorem 6.1 can be exploited and we obtain

$$\|u\|^{(-s)}_{\beta+2s;\Omega} \le C^{J+1}_\star \left(\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} + \|u\|^{(-s)}_{\beta;\Omega}\right).$$

This and (6.51) imply that we can bound $\|u\|^{(-s)}_{\beta+2s;\Omega}$ by a constant times $\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)}$, and this completes the proof of Corollary 6.2. $\square$

With this, we can now complete the proof of the main result:

Proof of Theorem 1.1. By Proposition 4.6 in [7], we know that $u \in C^s(\mathbb{R}^n)$, with

$$\|u\|_{C^s(\mathbb{R}^n)} \le C\, \|g\|_{L^\infty(\Omega)}. \tag{6.52}$$

Also, if $x \in \Omega_\delta$ then $d^{-s}(x) \geq \delta$, while if $x \in \Omega$ then $d(x)$ is controlled by the diameter of Ω. From this we obtain that

$$\|u\|^{(-s)}_{\beta+2s;\Omega} \geq c_o \, \|u\|_{C^{\beta+2s}(\Omega_\delta)}$$
$$\text{and} \quad \|g\|^{(s)}_{\beta;\Omega} \leq C_o \, \|g\|_{C^\beta(\Omega)}$$

for some c_o, $C_o > 0$, possibly depending on δ and Ω. Using this, (6.52) and Corollary 6.2 we obtain

$$\begin{aligned} \|u\|_{C^{\beta+2s}(\Omega_\delta)} &\leq c_o^{-1} \|u\|^{(-s)}_{\beta+2s;\Omega} \\ &\leq c_o^{-1} \, C \left(\|g\|^{(s)}_{\beta;\Omega} + \|u\|_{C^s(\mathbb{R}^n)} \right) \\ &\leq c_o^{-1} \, C \left(C_o \, \|g\|_{C^\beta(\Omega)} + C \, \|g\|_{L^\infty(\Omega)} \right). \end{aligned}$$

The latter term is in turn bounded bounded by a constant, possibly depending on δ and Ω, times $\|g\|_{C^\beta(\Omega)}$, hence the desired result plainly follows. □

7. Constructing a counterexample

This section is devoted to the construction of the counterexample of Theorem 1.2. For this, we start with an auxiliary lemma that says, roughly speaking, that the operator L "looses $2s$ derivatives":

Lemma 7.1. *Let $\beta \in (0, 1)$ and $v \in C^{\beta+2s}(\mathbb{R}^n)$. Then $Lv \in C^\beta(B_1)$ and $[Lv]_{C^\beta(B_1)} \leq C \, [v]_{C^{\beta+2s}(\mathbb{R}^n)}$.*

Proof. Notice that by construction $\beta + 2s \in (0, 3)$. Let x, $y \in B_1$ and

$$r = |x - y| < 2.$$

Also, for any fixed $\omega \in S^{n-1}$ and $\rho \geq 0$, let

$$w_{\rho,\omega}(x) = 2v(x) - v(x + \rho\omega) - v(x - \rho\omega).$$

Then

$$\begin{aligned} & |Lv(x) - Lv(y)| \leq I_1 + I_2, \quad \text{with} \\ & I_1 = \int_0^r d\rho \int_{S^{n-1}} da(\omega) \, \frac{|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)|}{\rho^{1+2s}} \\ & \text{and} \quad I_2 = \int_r^{+\infty} d\rho \int_{S^{n-1}} da(\omega) \, \frac{|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)|}{\rho^{1+2s}}. \end{aligned}$$

To estimate I_1 and I_2, we first prove that

$$|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)| \leq \begin{cases} C \, [v]_{C^{\beta+2s}(\mathbb{R}^n)} \, |x - y|^{\beta+2s} & \text{if } \beta + 2s \in (0, 1], \\ C \, [v]_{C^{\beta+2s}(\mathbb{R}^n)} \, \rho \, |x - y|^{\beta+2s-1} & \text{if } \beta + 2s \in (1, 2], \\ C \, [v]_{C^{\beta+2s}(\mathbb{R}^n)} \, \rho^2 \, |x - y|^{\beta+2s-2} & \text{if } \beta + 2s \in (2, 3). \end{cases} \tag{7.1}$$

To prove this, let us first consider the case $\beta + 2s \in (0, 1]$. In this case, we have that

$$|v(x \pm \rho\omega) - v(y \pm \rho\omega)| \leq [v]_{C^{\beta+2s}(\mathbb{R}^n)} |x - y|^{\beta+2s}, \tag{7.2}$$

for every $\rho \geq 0$, and this implies (7.1) when $\beta + 2s \in (0, 1]$.

If $\beta + 2s \in (1, 2]$ we use the Fundamental Theorem of Calculus in the variable ρ to write

$$w_{\rho,\omega}(x) = \int_0^\rho d\tau \left[-\nabla v(x + \tau\omega) \cdot \omega + \nabla v(x - \tau\omega) \cdot \omega\right]. \tag{7.3}$$

Notice also that, for any $\tau \in \mathbb{R}$,

$$\left|\nabla v(x + \tau\omega) \cdot \omega - \nabla v(y + \tau\omega) \cdot \omega\right| \leq [v]_{C^{\beta+2s}(\mathbb{R}^n)} |x - y|^{\beta+2s-1},$$

since $\beta + 2s \in (1, 2]$. This inequality and (7.3) give that

$$\left|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)\right| \leq \int_0^\rho 2[v]_{C^{\beta+2s}(\mathbb{R}^n)} |x - y|^{\beta+2s-1} \, d\tau,$$

that establishes (7.1) in this case.

Now we deal with the case $\beta + 2s \in (2, 3)$: for this, we use the Fundamental Theorem of Calculus in the variable ρ twice in (7.3) and we see that

$$w_{\rho,\omega}(x) = \int_0^\rho d\tau \int_\tau^{-\tau} d\sigma \, D^2_{ij} v(x + \sigma\omega) \, \omega_i \omega_j.$$

Consequently

$$\begin{aligned}
\left|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)\right| &\leq \int_0^\rho d\tau \int_{-\tau}^\tau d\sigma \left|D^2_{ij} v(x + \sigma\omega) - D^2_{ij} v(y + \sigma\omega)\right| \\
&\leq \int_0^\rho d\tau \int_{-\tau}^\tau d\sigma \, [v]_{C^{\beta+2s}(\mathbb{R}^n)} |x - y|^{\beta+2s-2},
\end{aligned}$$

since $\beta + 2s \in (2, 3)$, and this establishes (7.1) in this case as well.

The proof of (7.1) is thus complete, and now we show that

$$|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)| \leq \begin{cases} C \, [v]_{C^{\beta+2s}(\mathbb{R}^n)} \, \rho^{\beta+2s} & \text{if } \beta + 2s \in (0, 1], \\ C \, [v]_{C^{\beta+2s}(\mathbb{R}^n)} \, |x - y| \, \rho^{\beta+2s-1} & \text{if } \beta + 2s \in (1, 3). \end{cases} \tag{7.4}$$

To prove (7.4), we distinguish three cases. If $\beta + 2s \in (0, 1]$, we have that (7.4) follows easily from the fact that $|v(x) - v(x \pm \rho\omega)| \leq [v]_{C^{\beta+2s}(\mathbb{R}^n)} \, \rho^{\beta+2s}$, and the same for y instead of x. If $\beta + 2s \in (1, 2]$ we notice that

$$\nabla w_{\rho,\omega}(x) = 2\nabla v(x) - \nabla v(x + \rho\omega) - \nabla v(x - \rho\omega)$$

and so the Fundamental Theorem of Calculus in the space variable gives that

$$\begin{aligned}
&w_{\rho,\omega}(x) - w_{\rho,\omega}(y) \\
&= \int_0^1 dt \, \nabla w_{\rho,\omega}(y + t(x - y)) \cdot (x - y) \\
&= \int_0^1 dt \, \Big(2\nabla v(y + t(x - y)) - \nabla v(y + t(x - y) + \rho\omega) - \nabla v(y + t(x - y) - \rho\omega)\Big) \cdot (x - y).
\end{aligned} \tag{7.5}$$

Now we notice that

$$\big|\nabla v(y+t(x-y))-\nabla v(y+t(x-y)\pm\rho\omega)\big|\le [v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{\beta+2s-1}, \tag{7.6}$$

since here $\beta+2s\in(1,2]$. By inserting (7.6) into (7.5) we obtain

$$|w_{\rho,\omega}(x)-w_{\rho,\omega}(y)|\le\int_0^1 dt\,2[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{\beta+2s-1}\,|x-y|,$$

which proves (7.4) in this case.

It remains to prove (7.4) when $\beta+2s\in(2,3)$. In this case, we apply the Fundamental Theorem of Calculus in the variable ρ once more to obtain

$$\partial_i v(y+t(x-y))-\partial_i v(y+t(x-y)\pm\rho\omega)=\pm\int_\rho^0 d\sigma\,D^2_{ij}v(y+t(x-y)\pm\sigma\omega)\,\omega_j$$

and therefore

$$\begin{aligned}
&\Big|2\partial_i v(y+t(x-y))-\partial_i v(y+t(x-y)+\rho\omega)-\partial_i v(y+t(x-y)-\rho\omega)\Big|\\
=\;&\left|\int_\rho^0 d\sigma\,\Big(D^2_{ij}v(y+t(x-y)+\sigma\omega)-D^2_{ij}v(y+t(x-y)-\sigma\omega)\Big)\,\omega_j\right|\\
\le\;&\int_0^\rho d\sigma\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}(2\sigma)^{\beta+2s-2}\\
=\;&C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{\beta+2s-1},
\end{aligned}$$

where the condition $\beta\in(2,3)$ was used. By plugging this information into (7.5), we obtain

$$|w_{\rho,\omega}(x)-w_{\rho,\omega}(y)|\le\int_0^1 dt\,C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{\beta+2s-1}\,|x-y|,$$

which establishes (7.4) also in this case.

Now we show that

$$\text{if } \beta+2s\in(0,2], \text{ then}\qquad |w_{\rho,\omega}(x)|\le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{\beta+2s}. \tag{7.7}$$

Indeed, if $\beta\in(0,1]$ we have that $|v(x)-v(x\pm\rho\omega)|\le[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{\beta+2s}$, and this implies (7.7) in this case. If instead $\beta\in(1,2]$, then we use (7.3) to see that

$$|w_{\rho,\omega}(x)|\le\int_0^\rho d\tau\,|\nabla v(x+\tau\omega)-\nabla v(x-\tau\omega)|\le\int_0^\rho d\tau\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,(2\tau)^{\beta+2s-1},$$

which gives (7.7) also in this case.

Now we claim that there exists $\kappa\in(0,\,\beta)$ such that

$$\begin{aligned}&\text{for any } \rho\in[0,r]\\ &\big|w_{\rho,\omega}(x)-w_{\rho,\omega}(y)\big|\le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{2s+\kappa}r^{\beta-\kappa}.\end{aligned} \tag{7.8}$$

To check this, we distinguish two cases. When $\beta + 2s \in (0, 2]$, we define $\kappa = \beta/2$ and use (7.7) and the assumption that $\rho \le r = |x - y|$ to conclude that

$$|w_{\rho,\omega}(x)| \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{\kappa+2s}\,\rho^{\beta-\kappa} \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{\kappa+2s}\,r^{\beta-\kappa}.$$

Since the same holds when x is replaced by y, we have that (7.8) when $\beta+2s \in (0, 2]$ follows from the above formula and the triangle inequality.

When instead $\beta + 2s \in (2, 3)$, we take $\kappa = \min\left\{\frac{\beta}{2},\, 1 - s\right\}$ and we use (7.1) and the assumption that $\rho \le r = |x - y|$ to deduce that

$$\begin{aligned}\big|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)\big| &\le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^2 r^{\beta+2s-2}\\ &= C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{2s+\kappa}\rho^{2-2s-\kappa}r^{\beta+2s-2} \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,\rho^{2s+\kappa}r^{2-2s-\kappa}r^{\beta+2s-2},\end{aligned}$$

which provides the proof of (7.8) also in this case.

Having completed these preliminary estimates, we are now in the position to estimate I_1. For this, we use (7.8) and the fact that $\kappa > 0$ to see that, for any fixed $\omega \in S^{n-1}$,

$$\int_0^r \frac{\big|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)\big|}{\rho^{1+2s}}\,d\rho \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,r^{\beta-\kappa}\int_0^r \rho^{2s+\kappa-1-2s}\,d\rho = C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,r^{\beta}.$$

As a consequence, by integrating in $\omega \in S^{n-1}$, we obtain that

$$I_1 \le C[v]_{C^{\beta+2s}(\mathbb{R}^n)} r^{\beta}. \tag{7.9}$$

Now we estimate I_2. We claim that

$$I_2 \le C[v]_{C^{\beta+2s}(\mathbb{R}^n)} r^{\beta}. \tag{7.10}$$

To prove this, we distinguish two cases. If $\beta + 2s \in (0, 1]$, we use (7.2) and the fact that $|x - y| = r$ to deduce that

$$\big|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)\big| \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,r^{\beta+2s}.$$

Therefore

$$I_2 \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,r^{\beta+2s}\int_r^{+\infty} d\rho\,\rho^{-1-2s} = C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,r^{\beta},$$

which proves (7.10) in this case.

If instead $\beta + 2s \in (1, 3)$ we use (7.4) to write

$$\big|w_{\rho,\omega}(x) - w_{\rho,\omega}(y)\big| \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,r\,\rho^{\beta+2s-1}$$

and so to obtain that

$$I_2 \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,r\int_r^{+\infty} d\rho\,\rho^{\beta+2s-1-1-2s} = C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,r\,r^{\beta-1}.$$

This proves (7.10) also in this case.

By combining (7.9) and (7.10), we conclude that

$$|Lv(x) - Lv(y)| \le I_1 + I_2 \le C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)} r^{\beta} = C\,[v]_{C^{\beta+2s}(\mathbb{R}^n)}\,|x - y|^{\beta},$$

from which the desired result easily follows. □

Now we recall a useful, explicit barrier:

Lemma 7.2. *Let $\phi(x) = (1 - |x|^2)^s_+$ and L be as in* (1.6)*. Then $L\phi(x) = c$, for every $x \in B_1$, where $c > 0$ is a suitable constant, only depending on n and s.*

Proof. Fix $x = (x', x_n) \in B_1$, and let $b = \sqrt{1 - |x'|^2}$ and $\xi = x_n/b$. Notice that $|x'|^2 + x_n^2 < 1$, thus

$$|\xi| < \frac{\sqrt{1 - |x'|^2}}{b} = 1. \tag{7.11}$$

Moreover, for any $\rho \in \mathbb{R}$,

$$\phi(x + \rho e_n) = (1 - |x'|^2 - |x_n + \rho|^2)^s_+ = (b^2 - |b\xi + \rho|^2)^s_+ = b^{2s}(1 - |\xi + b^{-1}\rho|^2)^s_+.$$

As a consequence, writing $\phi_o(\zeta) = (1 - |\zeta|^2)^s_+$, for any $\zeta \in \mathbb{R}$, and using the change of variable $t = b^{-1}\rho$,

$$\begin{aligned}
(-\partial_n^2)^s\phi(x) &= \int_{\mathbb{R}} \frac{2\phi(x) - \phi(x + \rho e_n) - \phi(x - \rho e_n)}{\rho^{1+2s}}\, d\rho \\
&= b^{2s} \int_{\mathbb{R}} \frac{2(1 - |\xi|^2)^s_+ - (1 - |\xi + b^{-1}\rho|^2)^s_+ - (1 - |\xi - b^{-1}\rho|^2)^s_+}{\rho^{1+2s}}\, d\rho \\
&= \int_{\mathbb{R}} \frac{2(1 - |\xi|^2)^s_+ - (1 - |\xi + t|^2)^s_+ - (1 - |\xi - t|^2)^s_+}{t^{1+2s}}\, dt \\
&= (-\partial_n^2)^s\phi_o(\xi).
\end{aligned}$$

Now, we point out that ϕ_o is a function of one variable, and $(-\partial_n^2)^s\phi_o = c_o$, for some $c_o > 0$, see e.g. [10]. Thus, recalling (7.11), we have that $(-\partial_n^2)^s\phi(x) = c_o$. By exchanging the roles of the variables, we obtain similarly that

$$(-\partial_1^2)^s\phi(x) = (-\partial_2^2)^s\phi(x) = \cdots = (-\partial_n^2)^s\phi(x) = c_o,$$

from which we obtain the desired result. □

With this, we can now construct our counterexample, by considering the planar domain $\Omega \supset B_4$ in Figure A.

Proof of Theorem 1.2. The fact that $u \in C^s(\Omega)$ follows from Proposition 4.6 of [7]. Now suppose, by contradiction, that $u \in C^{3s+\epsilon}_{\text{loc}}(\Omega)$. Let $\theta \in C^\infty_0(B_2)$ with $\theta \equiv 1$ in B_1, and let $v = \theta u$ and $w = u - v$. Then $v \in C^{3s+\epsilon}(\mathbb{R}^n)$ and so, by Lemma 7.1 (being B_1 there any ball in $\mathbb{R}^n$), we have that $Lv \in C^{s+\epsilon}(\mathbb{R}^n)$. Hence

$$Lw = 1 - Lv \in C^{s+\epsilon}(\mathbb{R}^n). \tag{7.12}$$

Now, we take $\eta \in (0, 1)$ and we evaluate Lw at the two points $x_1 = 0$ and $x_2 = (0, -\eta)$. We notice that $w = (1 - \theta)u$, so $w \equiv 0$ in B_1, hence $w(x_1) = w(x_2) = 0$ and

$$Lw(x_1) - Lw(x_2) = \int_{1/2}^{+\infty} d\rho \int_{S^{n-1}} da(\omega)\, \frac{w(x_2 + \rho\omega) + w(x_2 - \rho\omega) - w(\rho\omega) - w(-\rho\omega)}{\rho^{1+2s}}.$$

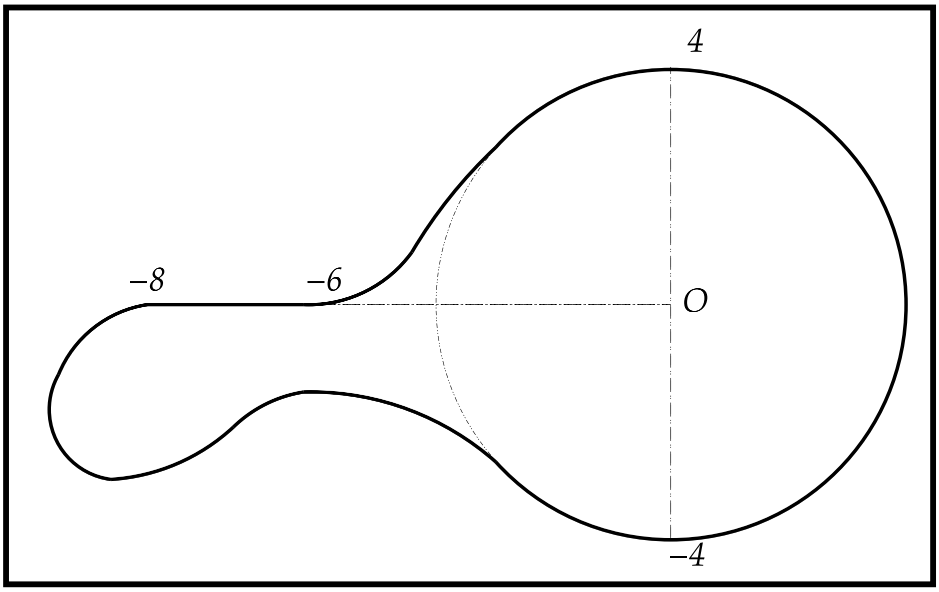

FIGURE A. The domain of Theorem 1.2.

More precisely, since L has the form (1.6), the anisotropy takes the form in (1.5) and we obtain that

$$\begin{aligned} & Lw(x_1) - Lw(x_2) \\ & = 2\sum_{i=1}^{2}\int_{\{|\rho|\geq 1/2\}} d\rho\,\frac{w(x_2+\rho e_i)-w(\rho e_i)}{|\rho|^{1+2s}} \\ & = J_1 + J_2, \end{aligned} \tag{7.13}$$

where

$$J_1 = 2\int_{\{|\rho|\geq 1/2\}} \frac{w(\rho,-\eta)-w(\rho,0)}{|\rho|^{1+2s}}\,d\rho$$

$$\text{and}\qquad J_2 = 2\int_{\{|\rho|\geq 1/2\}} \frac{w(0,\rho+\eta)-w(0,\rho)}{|\rho|^{1+2s}}\,d\rho.$$

Though the integrals J_1 and J_2 may look alike at a first glance, they are geometrically very different: indeed the integral trajectory of J_2 is transverse to the boundary (hence the interior regularity will dominate the boundary effects), while the integral trajectory of J_1 sticks at the boundary (hence it makes propagate the singularity from the boundary towards the interior). As an effect of these different geometric behaviors, we will prove that

$$J_1 \geq C^{-1}\eta^s \text{ and } |J_2| \leq C\eta^{s+\epsilon}, \tag{7.14}$$

for some $C > 1$ (here C will denote a quantity, possibly varying from line to line, which may depend on u, but which is independent of η). To prove (7.14) it is useful to observe that, since w vanishes outside Ω, the denominator in the integrands defining J_1 and J_2 is bounded uniformly and bounded uniformly away from zero. We set $\beta = 2s/(1+2s)$ and we notice that $\beta \in (0,1)$. If $\rho \in [-4, -4+\eta^\beta] \cup [4-\eta^\beta, 4]$, we use that $w \in C^s(\mathbb{R}^n)$ to obtain that

$$|w(0, \rho+\eta) - w(0,\rho)| \le C\eta^s,$$

and so

$$|J_2| \le C\eta^{\beta+s} + C \int_{\{|\rho| \in [\frac{1}{2}, 4-\eta^\beta]\}} |w(0,\rho+\eta) - w(0,\rho)| \, d\rho. \tag{7.15}$$

Also, if $|\rho| \le 4 - \eta^\beta$, we have that $d\big((0,\rho+\eta),(0,\rho)\big) \ge \eta^\beta - \eta \ge \eta^\beta/2$, if η is small enough. Hence we use Theorem 1.1(b) in [7] (applied here with $\alpha = s$) and (7.12) to see that

$$\begin{aligned} |w(0,\rho+\eta) - w(0,\rho+\eta)| &\le C \left(\|w\|_{C^s(\mathbb{R}^n)} + \|Lw\|_{C^s(\Omega)} \right) \eta^{3s} d^{-2s}\big((0,\rho+\eta),(0,\rho)\big) \\ &\le C\eta^{3s-2s\beta} \end{aligned}$$

and therefore

$$\int_{\{|\rho| \in [\frac{1}{2}, 4-\eta^\beta]\}} |w(0,\rho+\eta) - w(0,\rho)| \, d\rho \le C\eta^{3s-2s\beta}.$$

This and (7.15) imply that

$$|J_2| \le C\eta^{\beta+s} + C\eta^{3s-2s\beta} = C\eta^{\beta+s}. \tag{7.16}$$

This estimates J_2. Now we estimate J_1. To this goal, when $\rho \in [-8 - \sqrt[8]{\eta}, -8] \cup [-6, -6 - \sqrt[8]{\eta}] \cup [4 - \sqrt[8]{\eta}, 4]$ we use again that $w \in C^s(\mathbb{R}^n)$ to obtain that $|w(\rho,-\eta) - w(\rho,0)| \le C\eta$, and so

$$\left| \int_{\{\rho \in [-8-\sqrt[8]{\eta},-8] \cup [-6,-6+\sqrt[8]{\eta}] \cup [4-\sqrt[8]{\eta},4]\}} \frac{w(\rho,-\eta) - w(\rho,0)}{|\rho|^{1+2s}} \, d\rho \right| \le C\eta^{s+\frac{1}{8}}. \tag{7.17}$$

Furthermore, if $\rho \in [-20, -8 - \sqrt[8]{\eta}] \cup [-6 + \sqrt[8]{\eta}, 4 - \sqrt[8]{\eta}]$ we have that

$$d\big((\rho,-\eta),(\rho,-\eta)\big) \ge \sqrt{\eta}$$

if η is small enough, and thus, by Theorem 1.1(b) in [7], we see that

$$\begin{aligned} |w(\rho,-\eta) - w(\rho,0)| &\le C \left(\|w\|_{C^s(\mathbb{R}^n)} + \|Lw\|_{C^s(\Omega)} \right) \eta^{3s} d^{-2s}\big((0,\rho+\eta),(0,\rho)\big) \\ &\le C\eta^{2s} \end{aligned}$$

and therefore

$$\left| \int_{\{\rho \in [-20,-8-\sqrt[8]{\eta}] \cup [-6+\sqrt[8]{\eta},4-\sqrt[8]{\eta}]\}} \frac{w(\rho,-\eta) - w(\rho,0)}{|\rho|^{1+2s}} \, d\rho \right| \le C\eta^{2s}. \tag{7.18}$$

To complete the estimate on J_1, in virtue of (7.17) and (7.18), it only remains to consider the case in which $\rho \in [-8,-6]$. For this, if $\rho \in [-8,-6]$, we have that $w(\rho,0)=0$, and we claim that

$$w(\rho,-\eta) \geq c\eta^s, \tag{7.19}$$

for some $c>0$. To check this, fix $\rho \in [-8,-6]$. We notice that there exists $r_o>0$ (independent of ρ) such that the ball $B_{r_o}(\rho,-r_o)$ is tangent to $\partial\Omega$ at $(\rho,0)$. We consider the function ϕ given by Lemma 7.2 (used here with $n=2$), and we set

$$\underline{\phi}(x) = \phi_o\big(r_o^{-1}(x-(\rho,-r_o))\big) = \big(1-r_o^{-2}|x-(\rho,-r_o)|^2\big)_+^s.$$

Exploiting Lemma 7.2 and the Comparison Principle, we obtain that $u(x) \geq c\underline{\phi}(x)$ for any $x \in B_{r_o}(\rho,-r_o)$, for some $c>0$. So, choosing $x=(\rho,-\eta)$, we have that

$$|x-(\rho,-r_o)|^2 = |(0,r_o-\eta)|^2 = r_o^2+\eta^2-2r_o\eta \leq r_o^2 - r_o\eta,$$

if η is small enough. So we obtain

$$u(\rho,-\eta) \geq c\big(1-r_o^{-2}|x-(\rho,-r_o)|^2\big)_+^s \geq c\big(1-r_o^{-2}(r_o^2-r_o\eta)\big)_+^s = cr_o^{-s}\eta^s.$$

Since $u(\rho,-\eta)=w(\rho,-\eta)$, thanks to the choice of the cutoff functions, the latter formula proves (7.19), up to renaming constants.

Thus, using (7.19) (and possibly renaming c once again), we obtain that

$$\int_{\{\rho\in[-6,-8]\}} \frac{w(\rho,-\eta)-w(\rho,0)}{|\rho|^{1+2s}}\,d\rho = \int_{\{\rho\in[-6,-8]\}} \frac{w(\rho,-\eta)}{|\rho|^{1+2s}}\,d\rho \geq c\eta^s.$$

Consequently, recalling (7.17) and (7.18),

$$J_1 \geq c\eta^s - C\eta^{s+\frac{1}{8}} - C\eta^{2s},$$

that gives $J_1 \geq c\eta^s$, up to renaming constants. This and (7.16) complete the proof of (7.14).

Now, by (7.12) and (7.14), we obtain that

$$C \geq \frac{Lw(x_1)-Lw(x_2)}{|x_1-x_2|^{s+\epsilon}} \geq \eta^{-s-\epsilon}\big(J_1-|J_2|\big) \geq \eta^{-s-\epsilon}\big(C^{-1}\eta^s - C\eta^{s+\epsilon}\big) \geq \frac{C^{-1}\eta^{-\epsilon}}{2}.$$

This is a contradiction if η is sufficiently small (possibly in dependence of the fixed $\epsilon>0$). This shows that u cannot belong to $C^{3s+\epsilon}_{\rm loc}(\Omega)$ and so the construction of the counterexample in Theorem 1.2 is complete. □

Appendix A. Some basic results about the level sets of the distance function in $C^{1,1}$ domains

The goal of this appendix is to give some ancillary operational results about the distance function from the boundary of $C^{1,1}$ domains. The topic is of course of classical flavor, and the literature is rich of results in even more general settings (see e.g. [5]), but we thought it was useful to have the results needed for our scope at hand, and with proofs that do not involve any fine argument from Geometric Measure Theory.

The following result states that a $C^{1,1}$ domain satisfies an inner sphere condition, in a uniform way. This is probably well known, but we give the details for the convenience of the reader:

Lemma A.1. *Let* $\kappa, K > 0$ *and* $\Omega \subset \mathbb{R}^n$ *be such that*

$$\Omega \cap B_{2\kappa} = \{x = (x', x_n) \in B_{2\kappa} \text{ s.t. } x_n > h(x')\}, \tag{A.1}$$

for a $C^{1,1}$ *function* $h : \mathbb{R}^{n-1} \to \mathbb{R}$, *with* $\|\nabla h\|_{C^{0,1}(\mathbb{R}^n)} \leq K$.

Then, there exists $C > 0$, *only depending on* n *such that each point of* $(\partial\Omega) \cap B_\kappa$ *is touched from the interior by balls of radius*

$$r = \frac{1}{2} \min \left\{ \kappa,\ K^{-1} \right\}.$$

More explicitly, for any $p \in (\partial\Omega) \cap B_\kappa$ *there exists* $q \in \mathbb{R}^n$ *such that* $B_r(q) \subseteq \Omega \cap B_{2\kappa}$ *and* $p \in \partial B_r(q)$.

The point q *is given explicitly by the formula*

$$q = p - \frac{r\,(\nabla h(p'), -1)}{\sqrt{|\nabla h(p')|^2 + 1}}.$$

Proof. Let $p = (p', p_n) = (p', h(p')) \in (\partial\Omega) \cap B_\kappa$. By construction $|p - q|^2 = r^2$, hence

$$p \in \partial B_r(q). \tag{A.2}$$

Moreover, if $x \in B_r(q)$ then

$$|x| \leq |x - q| + |q - p| + |p| < 2r + \kappa \leq 2\kappa,$$

hence $B_r(q) \subseteq B_{2\kappa}$.

Therefore, recalling (A.1), in order to show that $B_r(q) \subseteq \Omega$, it suffices to prove that

$$B_r(q) \subseteq \{x = (x', x_n) \in \mathbb{R}^n \text{ s.t. } x_n > h(x')\} \tag{A.3}$$

To prove this, let

$$x \in B_r(q) \tag{A.4}$$

and define

$$\xi(x) = p_n + \nabla h(p') \cdot (x' - p').$$

We claim that

$$\xi(x) \leq x_n. \tag{A.5}$$

To check this, we use (A.2), (A.4) and the convexity of the ball to see that, for any $t \in (0, 1]$,

$$B_r(q) \ni p + t(x - p) = q + \frac{r\,(\nabla h(p'), -1)}{\sqrt{|\nabla h(p')|^2 + 1}} + t(x - p).$$

As a consequence

$$
\begin{aligned}
r^2 &> \left| \frac{r\,(\nabla h(p'),-1)}{\sqrt{|\nabla h(p')|^2+1}} + t(x-p) \right|^2 \\
&= r^2 + t^2|x-p|^2 + \frac{2rt\,(\nabla h(p'),-1)\cdot(x-p)}{\sqrt{|\nabla h(p')|^2+1}} \\
&= r^2 + t^2|x-p|^2 + \frac{2rt}{\sqrt{|\nabla h(p')|^2+1}}\big(\nabla h(p')\cdot(x'-p') - (x_n-p_n)\big) \\
&= r^2 + t^2|x-p|^2 + \frac{2rt}{\sqrt{|\nabla h(p')|^2+1}}\big(\xi(x) - x_n\big).
\end{aligned}
\tag{A.6}
$$

Simplifying r^2 to both the terms, multiplying by $t^{-1}\sqrt{|\nabla h(p')|^2+1}$ and taking the limit in t, we deduce that

$$
\begin{aligned}
0 \;&\geq\; \lim_{t\to 0^+} t|x-p|^2\sqrt{|\nabla h(p')|^2+1} + 2r\big(\xi(x)-x_n\big) \\
&=\; 2r\big(\xi(x)-x_n\big).
\end{aligned}
$$

This proves (A.5) and we can now continue the proof of (A.3).

To this goal, we use again (A.6), here with $t=1$, to observe that

$$
\begin{aligned}
0 \;&>\; |x-p|^2 + \frac{2r}{\sqrt{|\nabla h(p')|^2+1}}\big(\xi(x)-x_n\big) \\
&\geq\; |x'-p'|^2 + \frac{2r}{\sqrt{|\nabla h(p')|^2+1}}\big(\xi(x)-x_n\big).
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
h(x') - x_n &= h(x') - h(p') + p_n - x_n \\
&\leq \nabla h(p')\cdot(x'-p') + \|\nabla h\|_{C^{0,1}(\mathbb{R}^n)}|x'-p'|^2 + p_n - x_n \\
&\leq \xi(x) - x_n + K|x'-p'|^2 \\
&< \xi(x) - x_n + \frac{2Kr}{\sqrt{|\nabla h(p')|^2+1}}\big(x_n - \xi(x)\big) \\
&= \big(x_n - \xi(x)\big)\left(\frac{2Kr}{\sqrt{|\nabla h(p')|^2+1}} - 1\right)
\end{aligned}
\tag{A.7}
$$

Furthermore,

$$
\frac{2Kr}{\sqrt{|\nabla h(p')|^2+1}} \leq 2Kr \leq 1.
$$

By inserting this and (A.5) into (A.7) we conclude that $h(x') - x_n < 0$. This completes the proof of (A.3) and thus of Lemma A.1. $\square$

As a consequence of Lemma A.1, we obtain that the level sets of the distance function from a $C^{1,1}$ domain are locally a Lipschitz graph:

Corollary A.2. *Under the assumptions of Lemma A.1, there exists $\kappa_* \in (0, \kappa]$, $K_* \geq K > 0$ only depending on n, κ and K such that for any $t \in (0, \kappa_*]$ the level set*

$$\{x = (x', x_n) \in B_{\kappa_*} \; s.t. \; d(x) = t\} \tag{A.8}$$

lies in the graph of a Lipschitz function, whose Lipschitz seminorm is bounded by K_*.

Proof. First all all, we show that for any $x' \in \mathbb{R}^n$ with $|x'| \leq \kappa_*$ there exists a unique $x_n(x', t) \in \mathbb{R}$ such that $d(x', x_n(x', t)) = t$, i.e. the level set in (A.8) enjoys a graph property (as long as κ_* is sufficiently small).

Indeed, given x' as above, we consider the point $p = (x', h(x')) \in (\partial\Omega)$ given by the graph property of $\partial\Omega$. Notice that $d(p) = 0$. Also, by Lemma A.1 we know that a tubular neighborhood of width r lies in Ω: points on the upper boundary of this neighborhood stay at distance $r > \kappa_* \geq t$ from $\partial\Omega$. Therefore, by the continuity of the distance function, we find $\ell(x', t) \geq 0$ such that $d(p + \ell(x', t)e_n) = t$. Notice that

$$p + \ell(x', t)e_n = (p', p_n + \ell(x', t)) = (x', h(x') + \ell(x', t))$$

hence we found a point $x_n(x', t) = h(x') + \ell(x', t)$ with the desired properties.

We remark that the point $(x', x_n(x', t))$ is unique in B_{κ_*}. Indeed, suppose by contradiction that (x', x_n), $(x', x_n + \xi) \in B_{\kappa_*}$ satisfy $t = d(x', x_n) = d(x', x_n + \xi)$, with $\xi > 0$. Since the gradient of the distance function agrees with the normal ν of the projection $\pi : \Omega \to \partial\Omega$ in the vicinity of the boundary, we obtain that

$$\begin{aligned} 0 = d(x', x_n + \xi) - d(x', x_n) &= \xi \int_0^1 \partial_n d(x', x' + \tau\xi) \, d\tau \\ &= \xi \int_0^1 \nu_n(\pi(x', x' + \tau\xi)) \, d\tau. \end{aligned} \tag{A.9}$$

Notice that

$$\nu_n = \frac{1}{\sqrt{|\nabla h|^2 + 1}} \geq \frac{1}{\sqrt{K^2 + 1}},$$

thus (A.9) implies that

$$0 \geq \frac{\xi}{\sqrt{K^2 + 1}},$$

which is a contradiction, that shows the uniqueness of the value $x_n(x', t)$.

Now we show the Lipschitz property of such graph. For this we observe that it also follows from Lemma A.1 that the distance function in B_κ is semiconcave (see e.g. Proposition 2.2.2(iii) in [2]), namely there exists $C > 0$, only depending on n, κ and K, such that, for any x, $y \in B_\kappa$ and any $\lambda \in [0, 1]$,

$$\lambda d(x) + (1 - \lambda)d(y) - d(\lambda x + (1 - \lambda)y)) \leq C\lambda(1 - \lambda)|x - y|^2. \tag{A.10}$$

Our goal is now to show that, for any $x, y \in B_{\kappa_*}$, with $d(x) = d(y) = t$, we can bound $|x_n - y_n|$ by $K_*|x' - y'|$, for a suitable K_*. For this, without loss of generality, up to exchanging the roles of x and y, we may suppose that

$$x_n \geq y_n. \tag{A.11}$$

So, fixed x and y as above, we let $z = y - x$ and we obtain from (A.10) that

$$t = \lambda t + (1-\lambda)t \leq d(x + (1-\lambda)z) + C(1-\lambda)|z|^2.$$

So we set $\epsilon = 1 - \lambda \in [0, 1]$ and we obtain that

$$t \leq d(x + \epsilon z) + C\epsilon |z|^2. \tag{A.12}$$

Let $X = (X', X_n) \in \partial\Omega$ such that $t = d(x) = |x - X|$. Then

$$x = X + \frac{t\,(-\nabla h(X'), 1)}{\sqrt{|\nabla h(X')|^2 + 1}}$$

and

$$\begin{aligned} d(x + \epsilon z) &\leq |(x + \epsilon z) - X| \\ &= \left| \epsilon z + \frac{t\,(-\nabla h(X'), 1)}{\sqrt{|\nabla h(X')|^2 + 1}} \right|, \end{aligned}$$

and so

$$d^2(x + \epsilon z) \leq \epsilon^2 |z|^2 + t^2 + \frac{2\epsilon t\, z \cdot (-\nabla h(X'), 1)}{\sqrt{|\nabla h(X')|^2 + 1}}.$$

By comparing this and (A.12) we obtain

$$\begin{aligned} C^2 \epsilon^2 |z|^4 - 2\epsilon t\, C |z|^2 &= \big(t - C\epsilon |z|^2\big)^2 - t^2 \\ &\leq d^2(x + \epsilon z) - t^2 \leq \epsilon^2 |z|^2 + \frac{2\epsilon t\, z \cdot (-\nabla h(X'), 1)}{\sqrt{|\nabla h(X')|^2 + 1}}. \end{aligned}$$

We divide by $2\epsilon t$ and then take the limit as $\epsilon \to 0^+$, hence we obtain

$$-C|z|^2 \leq \frac{z \cdot (-\nabla h(X'), 1)}{\sqrt{|\nabla h(X')|^2 + 1}}. \tag{A.13}$$

Recalling (A.11), we also have that $z_n \leq 0$, and therefore (A.13) gives that

$$-C|z|^2 \leq \frac{|\nabla h(X')|\,|z'|}{\sqrt{|\nabla h(X')|^2 + 1}} + \frac{z_n}{\sqrt{|\nabla h(X')|^2 + 1}} \leq |z'| - \frac{|z_n|}{\sqrt{|\nabla h(X')|^2 + 1}}. \tag{A.14}$$

Now we observe that $|z| \leq |x| + |y| \leq 2\kappa_*$, hence

$$C|z| \leq C\kappa_* \leq \frac{1}{2\sqrt{2(K^2 + 1)}}$$

if we choose κ_* conveniently small. Thus we obtain from (A.14) that

$$\frac{|z_n|}{\sqrt{K^2+1}} \leq \frac{|z_n|}{\sqrt{|\nabla h(X')|^2+1}} \leq |z'| + C|z|^2 = |z'| + \frac{|z|}{2\sqrt{2(K^2+1)}}.$$

In addition

$$|z| = \sqrt{|z'|^2+|z_n|^2} \leq \sqrt{2\max\{|z'|^2,|z_n|^2\}} = \sqrt{2\mathrm{max}^2\{|z'|,|z_n|\}}$$
$$= \sqrt{2}\max\{|z'|,|z_n|\} \leq \sqrt{2}\big(|z'|+|z_n|\big),$$

therefore

$$\frac{|z_n|}{\sqrt{K^2+1}} \leq |z'| + \frac{|z'|}{2\sqrt{K^2+1}} + \frac{|z_n|}{2\sqrt{K^2+1}}$$

and so, by taking the latter term to the left hand side,

$$\frac{|z_n|}{2\sqrt{K^2+1}} \leq |z'| + \frac{|z'|}{2\sqrt{K^2+1}},$$

which establishes the desired Lipschitz property. □

Next is an auxiliary measure theoretic result that follows from Corollary A.2 and the Coarea Formula:

Proposition A.3. *Let $\Omega \subset \mathbb{R}^n$ and $p \in \Omega$. Assume that there exist $\kappa > 0$, $N \in \mathbb{N} \cup \{+\infty\}$ and $K > 0$ such that*

(A.15)
$\partial\Omega$ is covered by a family of balls $B_\kappa(x_i)$, with $i \in \{1, \cdots, N\}$ and $x_i \in \partial\Omega$,
with the property that $\partial\Omega \cap B_{8\kappa}(x_i)$ lies in a $C^{1,1}$ graph
whose $C^{1,1}$ seminorm is bounded by K,

for any $i \in \{1, \cdots, N\}$.

Then, there exist $\kappa_* \in (0, \kappa)$, possibly depending on κ and K, and $C > 0$, possibly depending on n, such that for any $\mu \in (0, \kappa_*]$ we have that

$$\big|\{x \in \mathbb{R}^n \text{ s.t. } p + x \in \Omega \cap A_{R_1,R_2,P} \text{ and } d(p+x) \in [0,\mu]\}\big|$$
$$\leq C\mu\,\mathcal{H}^{n-1}\big((\partial\Omega) \cap A_{R_1-\mu,R_2+\mu,P}\big),$$

for any annulus $A_{R_1,R_2,P} = B_{R_2}(P)\backslash B_{R_1}(P)$, with $P \in \mathbb{R}^n$, and $R_1, R_2 > 0$ with $R_2 - R_1 > 2\mu$.

Proof. We can assume that $\Omega \cap A_{R_1,R_2,P} \neq \varnothing$, otherwise we are done. Also, by possibly translating Ω, we can suppose that $p = 0$.

We cover $\partial\Omega$ with a finite overlapping family of balls of radius μ centered at points of $\partial\Omega$, say $B_\mu(y_j)$, with $j \in \{1, \cdots, M_\mu\}$, for some $M_\mu \in \mathbb{N}$.

Notice that each ball $B_\mu(y_j)$ is contained in some $B_{2\kappa}(x_{i_j})$: indeed, since $y_j \in \partial\Omega$, the covering property implies that there exists $i_j \in \{1, \ldots, N\}$ such that $y_j \in B_\kappa(x_{i_j})$; accordingly, if $q \in B_\mu(y_j)$, then

$$|q - x_{i_j}| \leq |q - y_j| + |y_j - x_{i_j}| < \mu + \kappa \leq 2\kappa,$$

which says that $B_\mu(y_j) \subseteq B_{2\kappa}(x_{i_j})$.

This implies that we can apply (A.15) inside each ball $B_\mu(y_j)$. As a consequence, by Corollary A.2 the level sets of the distance function in $B_\mu(y_j)$ are Lipschitz graphs with respect to the tangent hyperplane of $\partial\Omega$ at y_j, therefore

$$\mathcal{H}^{n-1}\big(\big\{x \in \Omega \cap B_\mu(y_j) \text{ s.t. } d(x) = t\big\}\big) \leq C\mu^{n-1},$$

for some $C > 0$. On the other hand $(\partial\Omega) \cap B_\mu(y_j)$ is also a $C^{1,1}$ graph with respect to the tangent hyperplane of $\partial\Omega$ at y_j and so

$$\mathcal{H}^{n-1}\big((\partial\Omega) \cap B_\mu(y_j)\big) \geq c\mu^{n-1},$$

for some $c > 0$. By comparing the latter two formulas, and possibly renaming $C > 0$, we conclude that

$$\mathcal{H}^{n-1}\big(\big\{x \in \Omega \cap B_\mu(y_j) \text{ s.t. } d(x) = t\big\}\big) \leq C\,\mathcal{H}^{n-1}\big((\partial\Omega) \cap B_\mu(y_j)\big). \tag{A.16}$$

Let us now reorder the indices in such a way the balls $B_\mu(y_1), \cdots, B_\mu(y_{L_\mu})$ intersect the annulus $A_{R_1,R_2,P}$, for some $L_\mu \in \mathbb{N}$, $L_\mu \leq M_\mu$. The finite overlapping property of the covering gives that

$$\sum_{j=1}^{L_\mu} \mathcal{H}^{n-1}\left((\partial\Omega) \cap B_\mu(y_j)\right) \leq C\,\mathcal{H}^{n-1}\left((\partial\Omega) \cap \left(\bigcup_{j=1}^{L_\mu} B_\mu(y_j)\right)\right)$$

and so, by set inclusions,

$$\sum_{j=1}^{L_\mu} \mathcal{H}^{n-1}\left((\partial\Omega) \cap B_\mu(y_j)\right) \leq C\,\mathcal{H}^{n-1}\left((\partial\Omega) \cap A_{R_1-\mu,R_2+\mu,P}\right). \tag{A.17}$$

Furthermore, the gradient of the distance function agrees with the normal of the projection in the vicinity of the boundary (hence it has modulus 1), so we use the

Coarea Formula and (A.16) to obtain that

$$\begin{aligned}
& \big|\big\{x \in \Omega \cap A_{R_1,R_2,P} \text{ s.t. } d(x) \in [0,\mu]\big\}\big| \\
\le\; & \sum_{j=1}^{L_\mu} \big|\big\{x \in \Omega \cap B_\mu(y_j) \text{ s.t. } d(x) \in [0,\mu]\big\}\big| \\
=\; & \sum_{j=1}^{L_\mu} \int_{\big\{x\in\Omega\cap B_\mu(y_j) \text{ s.t. } d(x)\in[0,\mu]\big\}} dx \\
=\; & \sum_{j=1}^{L_\mu} \int_{\big\{x\in\Omega\cap B_\mu(y_j) \text{ s.t. } d(x)\in[0,\mu]\big\}} |\nabla d(x)|\, dx \\
=\; & \sum_{j=1}^{L_\mu} \int_0^\mu \mathcal{H}^{n-1}\big(\big\{x \in \Omega \cap B_\mu(y_j) \text{ s.t. } d(x) = t\big\}\big)\, dt \\
\le\; & C \sum_{j=1}^{L_\mu} \int_0^\mu \mathcal{H}^{n-1}\big((\partial\Omega) \cap B_\mu(y_j)\big)\, dt \\
\le\; & C\mu \sum_{j=1}^{L_\mu} \mathcal{H}^{n-1}\big((\partial\Omega) \cap B_\mu(y_j)\big).
\end{aligned}$$

This and (A.17) imply the desired result. □

When condition (A.15) is fulfilled, it is also possible to control the surface of $\partial\Omega$ inside an annulus with the "correct power" of the size of the annulus itself. A precise statement goes as follows:

Lemma A.4. *Let $\Omega \subset \mathbb{R}^n$ and assume that there exist $\kappa > 0$, $N \in \mathbb{N} \cup \{+\infty\}$ and $K > 0$ such that $\partial\Omega$ is covered by a family of balls $B_\kappa(x_i)$, with $i \in \{1, \cdots, N\}$ and $x_i \in \partial\Omega$, with the property that $\partial\Omega \cap B_{8\kappa}(x_i)$ lies in a $C^{1,1}$ graph whose $C^{1,1}$ seminorm is bounded by K, for any $i \in \{1, \cdots, N\}$.*

Suppose also that Ω is bounded, with diameter less than D. Then, there exists $C > 0$, possibly depending on κ, K and D, such that

$$\mathcal{H}^{n-1}\big((\partial\Omega) \cap A_R\big) \le C R^{n-1}, \tag{A.18}$$

for any $R > 0$, where $A_R = B_{8R} \setminus B_R$.

Proof. First of all, we show that for any $r \in (0, \kappa/2]$ and any $p \in \mathbb{R}^n$,

$$\mathcal{H}^{n-1}\big((\partial\Omega) \cap B_r(p)\big) \le C r^{n-1}. \tag{A.19}$$

To prove this, we may suppose that $(\partial\Omega) \cap B_r(p) \ne \varnothing$, otherwise we are done. Hence, let $q \in (\partial\Omega) \cap B_r(p)$. By assumption, there exists $i \in \{1, \cdots, N\}$ such that $q \in B_\kappa(x_i)$. We observe that if $y \in B_r(p)$ then

$$|y - x_i| \le |y - p| + |p - q| + |q - x_i| < r + r + \kappa \le 2\kappa,$$

hence $B_r(p) \subseteq B_\kappa(x_i)$.

Consequently, $(\partial\Omega) \cap B_r(p)$ lies in a Lipschitz graph, with Lipschitz seminorm controlled by K and thus

$$\mathcal{H}^{n-1}\big((\partial\Omega) \cap B_r(p)\big) \leq \int_{\{x' \in \mathbb{R}^{n-1} \text{ s.t. } |x'| \leq r\}} \sqrt{K^2+1}\, dx' \leq Cr^{n-1},$$

and this proves (A.19).

Now we complete the proof of (A.18). We distinguish two cases: either $R \leq \kappa/2$ or $R > \kappa/2$.

If $R \leq \kappa/2$, we cover $B_8 \setminus B_1$ by a family of balls of radius $1/4$. By scaling, this provides a finite number of balls of radius $R/4$ that cover A_R, say $B_{R/4}(q_1), \cdots, B_{R/4}(q_M)$ (notice that M is a fixed, universal number). Then, by (A.19),

$$\mathcal{H}^{n-1}\big((\partial\Omega) \cap A_R\big) \leq \sum_{i=1}^{M} \mathcal{H}^{n-1}\big((\partial\Omega) \cap B_{R/4}(q_i)\big) \leq CR^{n-1},$$

which proves (A.18) in this case.

Thus, we now deal with the case $R > \kappa/2$. For this, we consider a non overlapping partition of $\mathbb{R}^n$ into adjacent (closed) cubes of side κ/n (in jargon, a κ/n-net of $\mathbb{R}^n$). Notice that the number of these cubes needed to cover A_R in this case depends on κ (which is fixed for our purposes, since the constant C in (A.18) is allowed to depend on κ) but also on R, therefore we need a more careful argument to bound the number of such cubes that really play a role in our estimates. Indeed, we claim that

(A.20) the number of cubes which intersect $\partial\Omega$ is bounded by some $C_o > 0$ which depends only on D and κ.

To prove this, we may suppose that there is a cube Q_*, that intersects $\partial\Omega$, otherwise (A.20) is true and we are done. So let $P_* \in (\partial\Omega) \cap Q_*$. Let $\mathcal{F}_0 = \{Q_*\}$ and $U_0 = Q_*$. Then, we define $\mathcal{F}_1$ the set all the cubes adjacent to $U_0 = Q_*$, and we let

$$U_1 = \bigcup_{Q \in \mathcal{F}_0 \cup \mathcal{F}_1} Q.$$

Then, we let $\mathcal{F}_2$ the set of cubes adjacent to U_1 and we set

$$U_2 = \bigcup_{Q \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2} Q,$$

and so on, iteratively, we let $\mathcal{F}_{i+1}$ the set of cubes adjacent to U_i and

$$U_{i+1} = \bigcup_{Q \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{i+1}} Q.$$

Notice that U_i is a cube of side $(2i+1)\kappa/n$, that has Q_* "in its center", that is

$$\operatorname{dist}(\partial U_i, Q_*) = \inf_{a \in \partial U_i,\, b \in Q_*} |a - b| = i.$$

Also, if $Q_\sharp \in \mathcal{F}_i$ is such that $(\partial\Omega) \cap Q_\sharp \neq \varnothing$, namely there exists $P_\sharp \in (\partial\Omega) \cap Q_\sharp$, then we have that $|P_\sharp - P_*| \leq D$, thanks to the property of the diameter. Also,

$$\begin{aligned} & \operatorname{dist}(P_\sharp, Q_*) \geq \operatorname{dist}(\partial U_{i-1}, Q_*) = i - 1 \\ \text{and} \qquad & \sup_{a \in \partial Q_*} |P_* - a| \leq \kappa. \end{aligned}$$

Therefore

$$D \geq |P_\sharp - P_*| \geq i - 1 - \kappa,$$

hence $i \leq D + 1 + \kappa$.

This means that the cubes that intersect $\partial\Omega$ lie in U_i, with $i \leq D+1+\kappa$. Since U_i contains $(2i+1)^n/n^n$ cubes, the number of cubes of the net which intersect $\partial\Omega$ is at most $(2(D+1+\kappa)+1)^n/n^n$, which proves (A.20).

Furthermore, if Q is a cube of the family, we have that Q is contained in the ball of radius κ with the same center of Q: hence, by (A.19),

$$\mathcal{H}^{n-1}\big((\partial\Omega) \cap Q\big) \leq C\kappa^{n-1}.$$

From this and (A.20), we obtain that

$$\begin{aligned} \mathcal{H}^{n-1}\big((\partial\Omega) \cap A_R\big) &\leq \sum_{Q \text{ s.t. } (\partial\Omega)\cap Q \neq \varnothing} \mathcal{H}^{n-1}\big((\partial\Omega) \cap Q\big) \\ &\leq \sum_{Q \text{ s.t. } (\partial\Omega)\cap Q \neq \varnothing} C\kappa^{n-1} \leq C_o C \kappa^{n-1}. \end{aligned}$$

Then, since we are assuming in this case that $R > \kappa/2$,

$$\mathcal{H}^{n-1}\big((\partial\Omega) \cap A_R\big) \leq 2^{n-1} C_o C R^{n-1},$$

which proves (A.18) also in this case, up to renaming constants. ☐

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