Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Hölder-estimates for non-autonomous parabolic problems with rough data

Hannes Meinlschmidt[1], Joachim Rehberg[2]

submitted: January 12, 2015

[1] Technische Universität Darmstadt
Fachbereich Mathematik
Dolivostr. 15
64293 Darmstadt
Germany
E-Mail: meinlschmidt@mathematik.tu-darmstadt.de

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: joachim.rehberg@wias-berlin.de

No. 2064
Berlin 2015

2010 *Mathematics Subject Classification.* 35B65, 35K10, 35K15.

Key words and phrases. Linear, second order parabolic equations; non-smooth data; Hölder continuity of the solution.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

In this paper we establish Hölder estimates for solutions to non-autonomous parabolic equations on non-smooth domains which are complemented with mixed boundary conditions. The corresponding elliptic operators are of divergence type, the coefficient matrix of which depends only measurably on time. These results are in the tradition of the classical book of Ladyshenskaya et al. [39], which also serves as the starting point for our investigations.

1. **Introduction.** Parabolic equations are one of the most common features when modelling phenomena in science and engineering, see [2] and [12]. One of the main problems, however, is that the input to the equations is very often (highly) non-smooth: the corresponding domains are not smooth (often they are not even strong Lipschitz domains), the coefficient functions are definitely discontinuous, and the boundary conditions are mixed: on one part D of the boundary Dirichlet conditions are imposed, while on the complement N Neumann- or Robin conditions hold. In the meantime these phenomena are also well investigated – as long as the coefficients do not depend explicitly on time, see [36], [33], [35] and [6]. In this paper we intend to investigate non-autonomous equations which incorporate all the phenomena described above with the central aim being Hölder estimates. This is also classical ever since the monography [39], as long as mixed boundary conditions are not considered.

Unfortunately, those investigations contain – in their generality – some peculiarities which make it not easy to apply them to problems originating from the applications: First, the Hölder spaces under consideration, see [39, pg. 7], are *not* the classical ones – the oscillation of the function is only measured over the connected components of the intersection of the domain with suitable balls (what is indeed adequate in case of *general* Dirichlet boundary data). Secondly, the estimates affect distributional right hand sides which are represented as the (spatial) divergence of vector-valued L^p-functions. As is well-known, such representations are highly non-unique; in particular the zero-functional may be represented as the divergence of a non-zero vector valued function. Lastly, it is not quite clear how broad the admissable geometric setting *really* is: on one hand "piecewise C^1" is demanded, on the other the crucial "Condition A" ([39, pg. 9], compare also [38, Ch. II.B, Definition B.3]) – well-known from elliptic theory – comes into play.

Our intention is to deliver a text which

- clearly defines the underlying geometric concept for the domain Ω – thereby avoiding "Condition A",
- incorporates mixed boundary conditions within an appropriately defined framework,
- allows for right hand sides from $L^s(]T_0, T_1[, W_D^{-1,q}(\Omega))$ (for a precise definition, see Definition 2.6),
- avoids a global Lipschitz condition for Ω – at least insofar the Dirichlet boundary part D is concerned, compare also [26],
- gives a result in the formulation of *classical* Hölder spaces.

The paper is organized as follows: In the next chapter we first introduce some terminology and our geometrical assumptions on the domain Ω and the Dirichlet boundary part D. Then, as a starting point, we quote the classical result on the existence and uniqueness of solutions for non-autonomous parabolic equations in a

Hilbert spaces setting, namely the right-hand side f being taken from $L^2(J;V^*)$ yielding solutions from $L^2(J;V)\cap W^{1,2}(J;V^*)$, where $V\hookrightarrow H\hookrightarrow V^*$ is a Gelfand triplet. Afterwards the main result is announced, cf. Theorem 2.13. In Ch. 3 we give the proof of this theorem, starting with the quotation of several classical results from the book [39]. The idea is to take these in a setting – namely the half cube and the ball and *homogeneous* Dirichlet boundary conditions – where the inherent technical difficulties are still not appearing: here it is clear that the general suppositions on the domain posed in [39] are fulfilled and, concerning the results, falls back to the classical Hölder spaces. After establishing some preliminaries in Ch. 3.2, in Ch. 3.3 we establish permanence principles, such as localization of the equation, bi-Lipschitz transformations and reflection which allow us to treat the problems in the geometric setting established above. This results in problems of the same complexity as before on a half-cube or a cube, respectively. Now employing the classical Ladyshenskaya results, one is enabled to deduce the required Hölder results for the solution in space and time. Here the *homogeneous* Dirichlet condition allows us establish *global* Hölder continuity from the known Hölder continuity on the connected components of the intersection of Ω with suitable domains.
Up to this point, the considerations are restricted to initial value zero and, as mentioned before, homogeneous Dirichlet conditions. In Ch. 4 we deviate from this and admit nonzero initial values together with inhomogeneous Dirichlet data. Since one can show (cf. Theorem 4.3) that in our geometric setting the Dirichlet boundary part D is a $(n-1)$-set in the sense of Jonsson/Wallin [37], one may apply their deep restriction/extension results for Sobolev spaces. This allows again to prove Hölder regularity for the solution in space and time also in the inhomogeneous case. In Ch. 5 we give a typical application of the Hölder result by proving global existence and uniqueness for a corresponding nonlinear problem, whereas in Ch. 6, a link to optimal control theory is established. We end with some concluding remarks.

2. The result.

2.1. **Notations, general assumptions.** In this paper the symbol J always stands for the time interval $]T_0,T_1[$, which we assume as fixed. The roman letters $\mathrm{x},\mathrm{y},\mathrm{z}$ are reserved for points in $\mathbb{R}^n$ $(n>1)$, whereas the actual components of x will be denoted by italics $x_1,x_2,\dots,x_n$. For our model constellations, we define the n-dimensional unit cube $\widehat{K}:=\{\mathrm{x}\colon \|\mathrm{x}\|_\infty<1\}$, whereas by K we denote the strictly "lower" half cube $\widehat{K}\cap\{\mathrm{x}:\ x_n<0\}$. The upper plate of K is called $\Sigma=\widehat{K}\cap\{\mathrm{x}:\ x_n=0\}$ and Σ_0 is the "left half" of Σ, i.e., $\Sigma_0=\Sigma\cap\{\mathrm{x}:\ x_{n-1}\le 0\}$. Finally, by $\mathcal{H}_{n-1}$ we denote the $(n-1)$-dimensional Hausdorff measure, cf. [18, Ch. 2].

Definition 2.1. The symbol M_n denotes the set of *real* $n\times n$-matrices, and $M_n(\kappa_0,\kappa_1)$ stands for all matrices ϱ from M_n which satisfy the condition

$$\kappa_0\|\mathrm{z}\|^2\le\sum_{i,j=1}^n\varrho_{i,j}z_jz_i\le\kappa_1\|\mathrm{z}\|^2,\quad \mathrm{z}=(z_1,\dots,z_n)\in\mathbb{R}^n, \tag{2.1}$$

for some $\kappa_0,\kappa_1>0$. We denote the set of measurable mappings $\mu: J\times\Omega\to M_n$ (in the sequel usually identified with the mapping $J\ni t\mapsto\mu(t,\cdot)$) taking their values in the set $M_n(\kappa_0,\kappa_1)$ by $\mathcal{M}_n(\kappa_0,\kappa_1)$ in all what follows.

Moreover, throughout this article the following is supposed to hold.

Assumption 2.2. • *$\Omega \subseteq \mathbb{R}^n$ is a bounded domain and D (Dirichlet) is a closed subset of $\partial\Omega$ (which may be empty). In all what follows, $\partial\Omega \setminus D$ will be denoted by N (Neumann).*

• *The coefficient function μ belongs to $\mathcal{M}_n(\kappa_0, \kappa_1)$ for some fixed $\kappa_0, \kappa_1 > 0$.*

Remark 2.3. Concerning the notions "Lipschitz domain" and "domain with Lipschitz boundary" (synonymous: strong Lipschitz domain) we follow the terminology of Grisvard [27, Ch. 1.2], see also [40, Ch. 1.1.9].

Let us introduce the basic assumption on Ω and D which will define our geometrical framework and which is of fundamental importance in the sequel.

Assumption 2.4. *I) If $\mathrm{x} \in (\partial\Omega \setminus \overline{N})$, there is a domain $\mathcal{U}_{\mathrm{x}} =: \mathcal{U}$ with $\mathrm{x} \in \mathcal{U}$, such that $\mathcal{U} \cap N = \emptyset$ and $\mathcal{U} \cap \Omega$ has only finitely many connected components $V_1, \ldots, V_k$, where x is a limit point of each V_j. Moreover, for every $j \in \{1, \ldots, k\}$, there exists a number $\tau_j > 0$, an open neighbourhood U_j of x satisfying $V_j \subseteq U_j \subseteq \mathcal{U}$, and a bi-Lipschitz mapping ϕ_j, defined on an open neighbourhood of $\overline{U_j}$ into $\mathbb{R}^n$, such that $\phi_j(\mathrm{x}) = 0$, $\phi_j(U_j) = \tau_j \hat{K}$, $\phi_j(V_j) = \tau_j K$ and $\phi_j(\partial V_j \cap U_j) = \tau_j \Sigma$.*

II) For each point $\mathrm{x} \in \overline{N}$ there is an open neighbourhood $U_{\mathrm{x}} =: U$ of x, a number $\tau_{\mathrm{x}} =: \tau > 0$ and a bi-Lipschitz mapping $\phi_{\mathrm{x}} =: \phi$ from an open neighbourhood of $\overline{U}$ into $\mathbb{R}^n$, such that $\phi(\mathrm{x}) = 0 \in \mathbb{R}^n$, $\phi(U) = \tau\widehat{K}$, $\phi(U \cap \Omega) = \tau K$ and $\phi(\partial\Omega \cap U) = \tau\Sigma$.

a) If $\mathrm{x} \in N$, then U does not intersect D, i.e., $\phi(D \cap U) = \emptyset$.

b) If $\mathrm{x} \in \overline{N} \cap D$, then $\phi(D \cap U) = \tau\Sigma_0$.

III) Each of the occurring mappings ϕ is, in addition, volume-preserving.

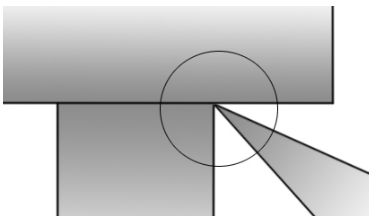

FIGURE 1. The figure shows (locally) an admissable geometric constellation around a Dirichlet point – here violating the Lipschitz condition of the domain

Remark 2.5. i) Primarily, Assumption 2.4 gives a typology of boundary points of Ω in the following sense: I) sets the conditions for points from the relative interior of the Dirichlet part, while IIa) is a condition for the Neumann boundary points and IIb) gives a condition for points from the border between Dirichlet and Neumann boundary part. In fact, this latter condition goes back to the paper of Gröger [28]. A simplifying topological characterization of Gröger's condition in case of space dimensions $n = 2$ and $n = 3$ is given in [32, Ch. 5].

ii) Note that Assumption 2.4 I) in particular demands that *every connected component* V_j of $\mathcal{U} \cap \Omega$ satisfies the assumptions for the Dirichlet boundary part of a Lipschitz domain on its own. Setting $V := U \cap \Omega$ in II), we find $\partial V \cap U = \partial(\Omega \cap U) \cap U = \partial\Omega \cap U$, which is the analogue to $\partial V_j \cap U_j$ in I) and shows compatibility of the conditions on the mappings on $\tau\Sigma$ in I) and II).

iii) The inclusions $\partial V_j \subset \overline{V_j} \subset \overline{U_j}$ imply the disjoint union $\partial V_j = (\partial V_j \cap U_j) \cup (\partial V_j \cap \partial U_j)$. Thus, $\partial V_j \cap U_j$ is a distinguished part of ∂V_j. Moreover, it is indeed not really necessary to demand the properties $\phi(U \cap \partial\Omega) = \tau\Sigma$ and $\phi_j(\partial V_j \cap U_j) = \tau_j\Sigma$ – they follow from the other ones by purely topological reasons. We have added them only to be at this point more suggestive, see also the previous item of this remark.

iv) In particular, all domains with Lipschitz boundary (*strong* Lipschitz domains) admit bi-Lipschitzian boundary charts which are volume preserving: if, after a shift and an orthogonal transformation, the domain lies locally beyond a graph of a Lipschitz function ψ, one defines

$$\phi(x_1, \ldots, x_n) = (x_1 - \psi(x_2, \ldots, x_n), x_2, \ldots, x_n).$$

This way, the mapping ϕ obviously is bi-Lipschitz and the determinant of its Jacobian is identically 1.

v) Note that the additional property *volume-preserving* also has been required in several similar contexts (see [25] and [29]). It turns out that the property bi-Lipschitz together with volume-preserving is not a too restrictive condition. In particular, there are bi-Lipschitzian, volume-preserving mappings – although not easy to construct – which map the ball onto the cylinder, the ball onto the cube and the ball onto the half ball, see [24], see also [19]. The general message is that this class of transformations has enough flexibility to map "non-smooth" objects onto smooth ones.

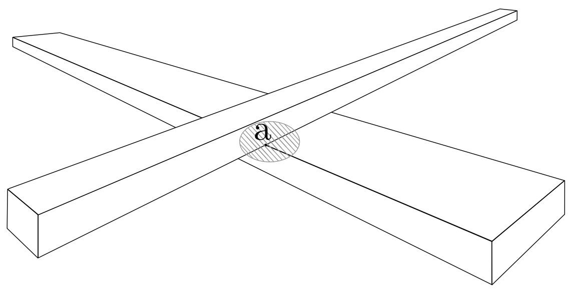

FIGURE 2. The – topologically regularized – double beam is the the prototype of a domain which is Lipschitzian, but *not* strong Lipschitzian. Moroever, a boundary chart around a may be constructed also as a *volume-preserving* one, cf. [33, Ch. 7].

In the following, all considered space are *real* ones.

Definition 2.6. Let Λ be a bounded open set, and let F be a closed part of $\partial\Lambda$. For $1 \le q < \infty$ we define $W^{1,q}_F(\Lambda)$ as the closure of

$$C^\infty_F(\Lambda) := \{\psi|_\Lambda : \psi \in C^\infty_0(\mathbb{R}^n),\ \mathrm{supp}(\psi) \cap F = \emptyset\} \tag{2.2}$$

in the *real* Sobolev space $W^{1,q}(\Lambda)$. If $F = \partial\Lambda$, then we write $W^{1,q}_F(\Lambda) = W^{1,q}_{\partial\Lambda}(\Lambda) = W^{1,q}_0(\Lambda)$.

If Λ is a Lipschitz domain and $F = \emptyset$, then $W^{1,q}_F(\Lambda)$ equals the usual Sobolev space $W^{1,q}(\Lambda)$. The latter follows from the fact that, for Lipschitz domains, the set

$C_\emptyset^\infty(\Lambda)$ is dense in $W^{1,q}(\Lambda)$, cf. [27, Thm. 1.4.2.1]. The space $W_F^{-1,q}(\Lambda)$ is defined as the space of continuous linear forms on $W_F^{1,q'}(\Lambda)$ for $1/q + 1/q' = 1$.

Definition 2.7. For $\mathcal{O} \subset \mathbb{R}^n$ and $\kappa \in]0,1]$, following [4, Ch. II.1.1], we define the Hölder space $C^\kappa(\mathcal{O})$ as follows:

$$C^\kappa(\mathcal{O}) := \Big\{\psi : \psi \text{ is a bounded function on } \mathcal{O}, \sup_{\mathrm{x},\mathrm{y}\in\mathcal{O},\mathrm{x}\neq\mathrm{y}} \frac{|\psi(\mathrm{x}) - \psi(\mathrm{y})|}{\|\mathrm{x}-\mathrm{y}\|^\kappa} < \infty\Big\}$$

with the norm

$$\|\psi\|_{C^\kappa(\mathcal{O})} = \|\psi\|_{C(\mathcal{O})} + |\psi|_{\kappa,\mathcal{O}} := \sup_{\mathrm{x}\in\mathcal{O}} |\psi(x)| + \sup_{\mathrm{x},\mathrm{y}\in\mathcal{O},\mathrm{x}\neq\mathrm{y}} \frac{|\psi(\mathrm{x}) - \psi(\mathrm{y})|}{\|\mathrm{x}-\mathrm{y}\|^\kappa}.$$

We call $|\cdot|_{\kappa,\mathcal{O}}$ the *Hölder seminorm.*

Remark 2.8.
i) It is clear that any function ψ from $C^\kappa(\mathcal{O})$ is necessarily uniformly continuous. Therefore, it admits a (uniquely determined) uniformly continuous extension $\hat\psi$ to the closure $\overline{\mathcal{O}}$, for which $|\psi|_{\kappa,\mathcal{O}} = |\hat\psi|_{\kappa,\overline{\mathcal{O}}}$ and thus $\|\psi\|_{C^\kappa(\mathcal{O})} = \|\hat\psi\|_{C^\kappa(\overline{\mathcal{O}})}$ holds.

ii) If $\mathcal{O} \subset \mathbb{R}^n$ is an open set, the condition of boundedness may be replaced by only essential boundedness.

iii) If $\mathcal{O}$ is bounded, then it suffices to have $\sup_{\mathrm{x},\mathrm{y}\in\mathcal{O},0<\|\mathrm{x}-\mathrm{y}\|<\varepsilon} \frac{|\psi(\mathrm{x})-\psi(\mathrm{y})|}{\|\mathrm{x}-\mathrm{y}\|^\kappa}$ for *one* $\varepsilon > 0$ under control in order to show Hölder continuity. Namely, one has for $\mathrm{x},\mathrm{y} \in \mathcal{O}$ with $\|\mathrm{x}-\mathrm{y}\| \geq \varepsilon$ the trivial estimate $|\psi(\mathrm{x})-\psi(\mathrm{y})| \leq \frac{2}{\varepsilon^\kappa} \sup|\psi|\|\mathrm{x}-\mathrm{y}\|^\kappa$.

iv) The reader should carefully notice that in [39, Ch. I.1] there are two notions of Hölder continuity in use, one coinciding with ours.

Furthermore, for the sake of clarity, we will write $\langle\cdot,\cdot\rangle_X$ for the dual pairing of elements of X and its dual X'. For a (vector-valued) function u, defined on J, we denote by u' its derivative in the sense of vector valued distributions, cf. [4, Ch. III.1] and define

$$W^{1,s}(J;X) := \{v : v, v' \in L^s(J;X)\}.$$

The symbol ∇ always stands for the *spatial* gradient – even if the corresponding function depends on space and time.

Definition 2.9. Let Λ be a bounded domain, and let $F \subseteq \partial\Lambda$ be closed. Let $\rho : \Lambda \to M_n$ be a bounded Lebesgue-measurable function. Then we define $-\nabla\cdot\rho\nabla + 1 : W_F^{1,2}(\Lambda) \to W_F^{-1,2}(\Lambda)$ by

$$\langle(-\nabla\cdot\rho\nabla + 1)\psi, \varphi\rangle_{W_F^{1,2}(\Lambda)} := \int_\Lambda \rho\nabla\psi\cdot\nabla\varphi + \psi\varphi \,\mathrm{d}\mathrm{x}, \quad \psi,\varphi \in W_F^{1,2}(\Lambda). \tag{2.3}$$

We maintain the notation of the operator when the range space is restricted to $W_F^{-1,q}(\Lambda)$ for $q > 2$. By Hölder's inquality, the domain of this restricted operator always contains the space $W_F^{1,q}(\Lambda) \supset C_F^\infty(\Lambda)$. For a bounded measurable function $\sigma\colon J\times\Lambda \to M_n$, we write $\mathcal{A}(\sigma)$ for the operator defined by $(\mathcal{A}(\sigma)u)(t) = -\nabla\cdot\sigma(t,\cdot)\nabla u(t) + u(t)$ for $u \in L^2(J; W_D^{1,2}(\Omega))$, taking its values in $L^2(J; W_D^{-1,2}(\Omega))$, with the analogous restriction conventions for the spatial operator as for the time-independent case.

2.2. **Formulation of the main result.** In order to establish the frame in which our main result can be formulated, we quote the following classical result, cf. [12, Ch. XVIII.3 and XVIII.4.2].

Proposition 2.10. *Suppose that $V \hookrightarrow H \hookrightarrow V'$ is a Gelfand triplet of* real *Hilbert spaces with dense embeddings. Let $\{\mathfrak{a}_t\}_{t\in J}$ be a family of bilinear forms on V the norms of which are uniformly bounded and such that each $\mathfrak{a}_t$ is coercive with a coercivity constant $\varkappa$, also uniformly in $t \in J$. Suppose that the mapping $J \ni t \mapsto \mathfrak{a}_t(\psi,\varphi)$ is measurable for all $\psi,\varphi \in V$. Then, for any $f \in L^2(J;V')$, there is a unique $u = u_f \in L^2(J;V) \cap W^{1,2}(J;V')$ such that $u(T_0) = 0$ and*

$$\langle u'(t), \psi\rangle_V + \mathfrak{a}_t(u(t),\psi) = \langle f(t),\psi\rangle_V, \quad \psi \in V \tag{2.4}$$

holds true for almost all $t \in J$. Moreover, u admits the following estimates:

$$\|u\|_{L^2(J;V)} \le \frac{1}{\varkappa}\|f\|_{L^2(J;V')}, \quad \|u\|_{C(\overline{J};H)} \le \sqrt{\frac{1}{\varkappa}}\|f\|_{L^2(J;V')}. \tag{2.5}$$

Thus, the mapping which assigns to the right hand side $f \in L^2(J;V')$ the solution u of (2.4) with initial value $u(T_0) = 0$ is well-defined and continuous from $L^2(J;V')$ into $L^2(J;V) \cap C(\overline{J};H)$, and its norm is not larger than $\frac{1}{\varkappa} + \sqrt{\frac{1}{\varkappa}}$.

Remark 2.11. Defining, for $t \in J$, the operator $A(t) : V \to V'$ by

$$\langle A(t)w,\psi\rangle_V = \mathfrak{a}_t(w,\psi), \quad w,\psi \in V, \tag{2.6}$$

equation (2.4) reads as

$$\langle u'(t),\psi\rangle_V + \langle A(t)u(t),\psi\rangle_V = \langle f(t),\psi\rangle_V, \ \psi \in V \tag{2.7}$$

for almost almost all $t \in J$.

In the following considerations using Proposition 2.10, the spaces $W^{1,2}_F(\Lambda)$ always play the role of V, and the form $\mathfrak{a}_t$ will be of type

$$W^{1,2}_F(\Lambda) \times W^{1,2}_F(\Lambda) \ni (\psi,\varphi) \mapsto \int_\Lambda \sigma(t,\cdot)\nabla\psi\cdot\nabla\varphi + \psi\varphi \,\mathrm{d}x \tag{2.8}$$

for some coefficient function $\sigma : J\times\Lambda \to M_n$. Clearly, the resulting operator $A(t)$ is then the corresponding divergence operator $-\nabla\cdot\sigma(t,\cdot)\nabla + 1$ on $W^{1,2}_F(\Lambda)$. Note that, vice-versa, $-\nabla\cdot\sigma(t,\cdot)\nabla+1$ also induces a family of forms $\mathfrak{a}_t$ on $W^{1,2}_F(\Lambda)\times W^{1,2}_F(\Lambda)$.

Remark 2.12. Let us point out that the following considerations may also be carried out for the operators $-\nabla\cdot\sigma(t,\cdot)\nabla$ alone, if $F \neq \emptyset$. The corresponding form (as in (2.8)) is then, via the Poincaré-inequality, still coercive on $W^{1,2}_F(\Lambda)$ (see [6, Rem. 3.4] for the Poincaré-inequality, see also Theorem 4.3), while the rest of the considerations remains untouched in its essence.

The subsequent theorem contains the main result of this paper.

Theorem 2.13. *Assume that Ω and D are given and fulfil Assumption 2.4, and let $\mu \in \mathcal{M}_n(\kappa_0,\kappa_1)$ for some $\kappa_0,\kappa_1 > 0$. Let $q > n$ and $s > 2(1-\frac{n}{q})^{-1}$ be fixed and $f \in L^s(J;W^{-1,q}_D(\Omega))$. Then the solution $u = u_f$ of the equation*

$$u'(t) - \nabla\cdot\mu(t,\cdot)\nabla u(t) + u(t) = f(t), \quad u(T_0) = 0 \tag{2.9}$$

in the sense of Proposition 2.10/Remark 2.11 exists and is unique. Moreover, let $\mathfrak{B}$ denote the unit ball in $L^s(J;W^{-1,q}_D(\Omega))$. Then the following holds true:

i) The supremum $\sup_{f\in\mathfrak{B}}\|u_f\|_{L^\infty(J\times\Omega)}$ is finite and depends exclusively on κ_0,κ_1.

ii) *There is an $\alpha > 0$, such that even $\sup_{f \in \mathfrak{B}} \|u_f\|_{C^\alpha(J\times\Omega)}$ is finite and depends exclusively on κ_0, κ_1. In other words: Let $(\partial_t + \mathcal{A}(\mu))^{-1}$ denote the linear operator which assigns to the right-hand side of the parabolic equation in (2.9) the solution $u = u_f$ with initial value $u_0 = 0$. Then the mapping*

$$(\partial_t + \mathcal{A}(\mu))^{-1} : L^s(J; W_D^{-1,q}(\Omega)) \to C^\alpha(\overline{J \times \Omega}) \tag{2.10}$$

is well-defined and continuous for some α. For fixed κ_0, κ_1, the mappings (2.10) are equicontinuous for all coefficient functions $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$.

Remark 2.14. It is straight-forward to check that for $q, s \geq 2$, $L^s(J; W_D^{-1,q}(\Omega))$ continuously injects into $L^2(J; W_D^{-1,2}(\Omega))$ with embedding constant $|\Omega|^{\frac{q-2}{2q}} |J|^{\frac{s-2}{2s}}$. In this sense, right-hand sides f from $L^s(J; W_D^{-1,q}(\Omega))$ are implicitly always to be understood as right-hand sides from $L^2(J; W_D^{-1,2}(\Omega))$ without further comment in the sequel.

3. **The proof.** Let us give the proof of Theorem 2.13. We first collect some classical results of Ladyzhenskaya et al. [39] adopted for our cause. The basis of our considerations will be Corollaries 3.5 and 3.7 which are based on space-time local estimates for so-called generalized solutions of corresponding equations in [39, Ch. III]. However, in order to use those, we invest quite some work and introduce a non-trivial localization-procedure for (2.9) which allows to transform the localized equation onto a very regular object, namely the lower half-cubes τK and (via reflection) the full cubes $\tau \hat{K}$ in such a way that the resulting equation still provides a generalized equation in the sense of Ladyzhenskaya.

3.1. **Classical results.** We begin by introducing the notion of a generalized equation. The crucial link to the concept of Lions is the space $V_2^{1,0}(J \times \Xi)$ introduced in the next definition, which corresponds to the spaces $L^2(J; V) \cap C(\overline{J}; H)$ in Proposition 2.10.

Definition 3.1. Let $\Xi \subset \mathbb{R}^n$ be a bounded Lipschitz domain.

i) Let $V_2^{1,0}(J \times \Xi)$ be the space $L^2(J; W^{1,2}(\Xi)) \cap C(\overline{J}; L^2(\Xi))$, equipped with the norm

$$v \mapsto \sup_{t \in \overline{J}} \|v(t,\cdot)\|_{L^2(\Xi)} + \Big(\int_J \int_\Xi v(t,\cdot)^2 + |\nabla v(t,\cdot)|^2 \mathrm{d}x \, \mathrm{d}t \Big)^{1/2}.$$

ii) Suppose that $\mathfrak{f} = (\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_n) \in L^2(J; L^2(\Xi; \mathbb{R}^{n+1}) \simeq L^2(J \times \Xi; \mathbb{R}^{n+1})$. We say that a function $u \in V_2^{1,0}(J \times \Xi)$ is a *generalized solution* of the equation

$$u' - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \Big(\mu_{ij} \frac{\partial u}{\partial x_j} \Big) + u = \sum_{k=1}^n \frac{\partial \mathfrak{f}_k}{\partial x_k} + \mathfrak{f}_0, \tag{3.1}$$

if for every $\vartheta \in W_0^{1,2}(J \times \Xi)$ it holds for almost all $T \in J$ the integral identity

$$\begin{aligned} 0 = \int_\Xi u(T,\mathrm{x})\vartheta(T,\mathrm{x}) \, \mathrm{d}x - \int_{T_0}^T \int_\Xi u \frac{\partial \vartheta}{\partial t} \, \mathrm{d}x\mathrm{d}t + \int_{T_0}^T \int_\Xi \sum_{i,j=1}^n \mu_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \vartheta}{\partial x_i} \, \mathrm{d}x\mathrm{d}t \\ + \int_{T_0}^T \int_\Xi u\vartheta \, \mathrm{d}x \, \mathrm{d}t - \int_{T_0}^T \int_\Xi \mathfrak{f}_0 \vartheta + \sum_{k=1}^n \mathfrak{f}_k \frac{\partial \vartheta}{\partial x_k} \, \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.2}$$

We denote the right-hand side in (3.2) by $\mathfrak{N}(u, \vartheta, t)$ for later reference. Finally, we say that $u \in V_2^{1,0}(J \times \Xi)$ is a *generalized solution* of the equation (3.1) *with initial value* $u_0 = 0$, if the integral identity (3.2) is satisfied even for all functions $\vartheta \in W^{1,2}_{\overline{J}\times\partial\Xi}(J \times \Xi)$.

Remark 3.2. Integrating the term $\int_{T_0}^{T}\int_\Xi u\frac{\partial\vartheta}{\partial t}\,d\mathrm{x}dt$ formally by parts with respect to time, the term $\int_\Xi u(T_0,\mathrm{x})\vartheta(T_0,\mathrm{x})\,d\mathrm{x}$ appears, which is not compensated by other terms in (3.2). Thus, if test functions ϑ are admitted which are nonzero on $\{T_0\}\times\Xi$, such as those from $W^{1,2}_{\overline{J}\times\partial\Xi}(J \times \Xi)$, this enforces $u(T_0,\cdot)$ to be the zero function – on a *formal* level.

The next results are in their essence space-time local estimates for generalized solutions if the right-hand side in (3.1) is regular enough. However, for initial value 0 we may re-obtain the estimates for the *whole* time interval J, see Corollaries 3.5 and 3.7.

Proposition 3.3. [39, Ch. III, Thm. 8.1] *Let* $\Xi \subset \mathbb{R}^n$ *be a bounded Lipschitz domain, and let* μ *be from* $\mathcal{M}_n(\kappa_0,\kappa_1)$*. Fix* $q > n$ *and* $s > 2(1-\frac{n}{q})^{-1}$*. Let the set* $\mathcal{F}$ *be given such that*

$$\mathcal{F} \subseteq \left\{\mathfrak{f} \in L^s(J;L^q(\Xi;\mathbb{R}^{n+1})) : \|\mathfrak{f}\|_{L^s(J;L^q(\Xi;\mathbb{R}^{n+1}))} \le C\right\}, \tag{3.3}$$

for some $C \ge 0$*. Moreover, assume that for every* $\mathfrak{f} \in \mathcal{F}$ *a generalized solution* $u = u_\mathfrak{f}$ *of* (3.1) *exists and* $\{u_\mathfrak{f} : \mathfrak{f} \in \mathcal{F}\}$ *is contained in a ball around* 0 *in* $V_2^{1,0}(J\times\Xi)$ *with radius* r_V*.*

i) Let $\Xi_0 \subset \Xi$ *be a subdomain which has a positive distance* $d < T_1 - T_0$ *to* $\partial\Xi$*. Then* $\sup_{\mathfrak{f}\in\mathcal{F}}\|u_\mathfrak{f}\|_{L^\infty(]T_0+d,T_1[\times\Xi_0)}$ *is finite, and depends only on* $n,\kappa_0,\kappa_1,r_V,d$ *and* C*, cf.* (2.1)*.*

ii) Let F *be a closed part of* $\partial\Xi$ *and let all* $u_\mathfrak{f}$ *belong to the space* $L^2(J;W^{1,2}_F(\Xi))$*. If a subdomain* Ξ_0 *of* Ξ *has a positive distance* $d < T_1 - T_0$ *to* $\partial\Xi\setminus F$*, then also* $\sup_{\mathfrak{f}\in\mathcal{F}}\|u_\mathfrak{f}\|_{L^\infty(]T_0+d,T_1[\times\Xi_0)}$ *is finite, and depends only on* κ_0,κ_1,d,r_V *and* C*.*

Remark 3.4. For Lipschitz domains Ξ the usual trace operator $\mathrm{tr} : W^{1,2}(\Xi) \to L^2(\partial\Xi;\mathcal{H}_{n-1})$ exists and is continuous, cf. [34, Ch. 3.1]. Hence, if F is any closed subset of $\partial\Xi$, then the mapping $W^{1,2}(\Xi) \ni \psi \mapsto \mathrm{tr}\,\psi|_F \in L^2(F;\mathcal{H}_{n-1})$ also is well-defined and continuous. Moreover, it is clear that any function $\psi \in W^{1,2}_F(\Xi)$ has trace 0 on F, since this is obviously true for all functions $\psi \in C^\infty_F(\Xi)$. This shows that, for any function $w \in L^2(J;W^{1,2}_F(\Xi))$, the function $J \ni t \mapsto \mathrm{tr}(w(t,\cdot))|_F$ belongs to $L^2(J;L^2(F,\mathcal{H}_{n-1})) \sim L^2(J\times F;dt\otimes\mathcal{H}_{n-1})$ and is *negligible* on $J\times F$ with respect to the measure $dt\otimes\mathcal{H}_{n-1}$. Namely, if $w \in L^2(J;W^{1,2}_F(\Xi))$, then, for almost every $t \in J$, the function $\mathrm{tr}(w(t,\cdot))|_F$ is a negligible one on F (with respect to the Hausdorff measure $\mathcal{H}_{d-1}$). In fact, this is the link to the suppositions in [39, Ch. III, Thm. 8.1].

Corollary 3.5. *Suppose the general conditions of Proposition 3.3 and consider the case of initial value* 0 *for the generalized solutions.*

i) Let $\Xi_0 \subset \Xi$ *be a subdomain which has a positive distance* d *to* $\partial\Xi$*. Then for the generalized solutions* $u_\mathfrak{f}$*, the supremum* $\sup_{\mathfrak{f}\in\mathcal{F}}\|u_\mathfrak{f}\|_{L^\infty(]T_0,T_1[\times\Xi_0)}$ *is finite and depends only on* $n,\kappa_0,\kappa_1,d,r_V$ *and* C*, cf.* (2.1) *and* (3.3)*.*

ii) Let F *be a closed part of* $\partial\Xi$ *and let all* $u_\mathfrak{f}$ *belong to the space* $L^2(J;W^{1,2}_F(\Xi))$*. If a subdomain* Ξ_0 *has a positive distance* d *to* $\partial\Xi\setminus F$ *then* $\sup_{\mathfrak{f}\in\mathcal{F}}\|u_\mathfrak{f}\|_{L^\infty(]T_0,T_1[\times\Xi_0)}$ *is finite and depends only on* κ_0,κ_1,d,r_V *and* C*.*

Proof. One associates to the problem (3.1) another one on the interval $J_0 :=]T_0 - d - 1, T_1[$ in the following manner: one defines a coefficient function $\check{\mu}$ on $J_0 \times \Xi$ by

$$\check{\mu}(t, \mathrm{x}) = \begin{cases} \frac{\kappa_0+\kappa_1}{2}\,\mathrm{id} & \text{if} \quad t \in J_0 \setminus J, \\ \mu(t, \mathrm{x}) & \text{else.} \end{cases}$$

Moreover, one defines a new right-hand side $\check{\mathfrak{f}}$ as 0 on $J_0 \setminus J$ and as $\mathfrak{f}$ on J and finds the solution $\check{u}$ on $J_0 \times \Xi$ with $u(T_0 - d - 1) = 0$. This solution $\check{u}$ is zero on $(J_0 \setminus J) \times \Xi$ and coincides with u on $J \times \Xi$. Applying Proposition 3.3 i) to the function $\check{u}$ one gets i). Point ii) is deduced analogously from ii) of the foregoing Proposition. □

Proposition 3.6. [39, Ch. III, Thm. 10.1] *Let $\Xi \subset \mathbb{R}^n$ be a bounded, convex domain (and, hence, a Lipschitz domain), and suppose $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$. Fix $q > n$ and $s > 2(1 - \frac{n}{q})^{-1}$. Assume that $\mathcal{F}$ is again a subset of the set in (3.3), such that for every $\mathfrak{f} \in \mathcal{F}$ a generalized solution $u = u_{\mathfrak{f}}$ of (3.1) exists and that this set of generalized solutions is contained in a ball around 0 in $L^\infty(J \times \Xi)$ with radius r_∞. Then there is an $\alpha > 0$ such that the following is true:*

- *i) For every subdomain $\Xi_0 \subset \Xi$ having a positive distance $d \in]0, T_1 - T_0[$ to the boundary $\partial\Xi$, $\sup_{\mathfrak{f} \in \mathcal{F}} \|u_{\mathfrak{f}}\|_{C^\alpha(]T_0+d,T_1[\times \Xi_0)}$ is finite and depends only on $n, \kappa_0, \kappa_1, d, r_\infty$ and C, cf. (2.1) and (3.3).*
- *ii) Let F be a closed part of $\partial\Xi$ and suppose that all $u_{\mathfrak{f}}$ belong to the space $L^2(J; W^{1,2}_F(\Xi))$. If a subdomain Ξ_0 of Ξ has a positive distance $d \in]0, T_1 - T_0[$ to $\partial\Xi \setminus F$, then the supremum $\sup_{\mathfrak{f} \in \mathcal{F}} \|u_{\mathfrak{f}}\|_{C^\alpha(]T_0+d,T_1[\times \Xi_0)}$ is finite and depends only on $n, \kappa_0, \kappa_1, d, r_\infty$ and C.*

Corollary 3.7. *Suppose the assumptions of Proposition 3.6 to hold and assume, additionally, that the initial value u_0 of the solution is zero. Then there is an $\alpha > 0$ such that the following is true:*

- *i) For every subdomain $\Xi_0 \subset \Xi$ having a positive distance d to the boundary $\partial\Xi$, $\sup_{\mathfrak{f} \in \mathcal{F}} \|u_{\mathfrak{f}}\|_{C^\alpha(]T_0,T_1[\times \Xi_0)}$ is finite and depends only on $n, \kappa_0, \kappa_1, d, r_\infty$ and C, cf. (2.1) and (3.3).*
- *ii) Let F be a closed part of $\partial\Xi$ and suppose that each $u_{\mathfrak{f}}$ belongs to the space $L^2(J; W^{1,2}_F(\Xi))$. Then, for any subdomain Ξ_0 with a positive distance d to $\partial\Xi \setminus F$, $\sup_{\mathfrak{f} \in \mathcal{F}} \|u_{\mathfrak{f}}\|_{C^\alpha(]T_0,T_1[\times \Xi_0)}$ is finite and depends only on $n, \kappa_0, \kappa_1, d, r_\infty$ and C.*

The proof works analogously to the one of Corollary 3.5.

Remark 3.8. In fact, the quoted result holds for much more general domains as convex ones. However, we have good reasons to restrict ourselves to this case:

- If Ξ is convex and $B \subset \mathbb{R}^n$ is a ball, then $\Xi \cap B$ is still convex and therefor always consists of only one component. Thus, one may deal with the classical notion of Hölder continuity – and not of the much more sophisticated one in [39, Ch. I]
- Secondly, if Ξ is convex, then every point $\mathrm{x} \in \partial\Xi$ admits a supporting hyperplane such that Ξ lies on one side of this hyperplane. Thus, for any ball $B \subset \mathbb{R}^n$ with center x, the intersection $\Xi \cap B$ has at most half the measure of B, what makes the crucial "Condition A" ([39, Ch. 1, p.9]) obviously fulfilled in our context, with the constant $\theta_0 = \frac{1}{2}$ – universal for all convex domains and all balls.

- We will need the result only in case of balls, cubes and half cubes, serving as our local model sets.

The next proposition establishes the link between generalized solutions and solutions in the sense of Proposition 2.10. For doing so, we restrict ourselves to the case of right hand sides which are step functions in time only (these being dense in the whole space under consideration). The reason is as follows: By a classical theorem, the elements f from $W_D^{-1,q}(\Omega)$ may be represented as the sum of the divergence of a $\mathbb{R}^n$-valued function $\mathfrak{f} \in L^q$ and $\mathfrak{f}$ itself. The problem is that this representation is highly non-unique and, the worse, not obviously linear. So we preferred to restrict ourselves to step functions and to use the corresponding representation theorem separately on any of the constancy intervals only.

Proposition 3.9. *Let $\Xi \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and F be a closed portion of the boundary $\partial\Xi$. Put $V = W_F^{1,2}(\Xi)$, $H := L^2(\Xi)$, such that $V' = W_F^{-1,2}(\Xi)$. Let a bounded, elliptic coefficient function σ on $J \times \Xi$ be given and put $\mathfrak{t}_t(\psi, \varphi) = \int_\Xi \sigma(t, \cdot)\nabla\psi \cdot \nabla\varphi + \psi\varphi \, \mathrm{d}x$ for $\psi, \varphi \in W_F^{1,2}(\Xi)$. Fix $q, s \geq 2$. Assume that $f \in L^s(J; W_F^{-1,q}(\Xi))$ is a step function, i.e., there exists a partition $(J_k)_k$ of J such that $f = \sum_k \chi_k f_k$ for $f_k \in W_F^{-1,q}(\Xi)$ – χ_k being the indicator function of the interval J_k.*

i) For every k, there is $\mathfrak{f}_k = (\mathfrak{f}_{k,0}, \ldots, \mathfrak{f}_{k,n}) \in L^q(\Xi; \mathbb{R}^{n+1})$ such that f_k is represented by

$$\langle f_k, \varphi \rangle_{W_F^{1,q'}(\Xi)} = \int_\Xi \mathfrak{f}_{k,0}\varphi - \sum_{j=1}^n \mathfrak{f}_{k,j} \frac{\partial \varphi}{\partial x_j} \, \mathrm{d}x, \quad \varphi \in W_F^{1,q'}(\Xi) \tag{3.4}$$

and with $\|\mathfrak{f}_k\|_{L^q(\Xi;\mathbb{R}^{n+1})} \leq 2\|f_k\|_{W_F^{-1,q}(\Xi)}$. Setting $\mathfrak{f} = \sum_k \chi_{J_k}\mathfrak{f}_k$, this consequently implies

$$\|\mathfrak{f}\|_{L^s(J;L^q(\Xi;\mathbb{R}^{n+1}))} \leq 2\|f\|_{L^s(J;W_F^{-1,q}(\Xi))}.$$

ii) The solution with initial value 0 of (2.4)/(2.7), there taking the forms $\mathfrak{t}_t$ and right-hand side f, is a generalized solution of (3.1) with $\mathfrak{f} = \sum_k \chi_{J_k}\mathfrak{f}_k$ and $\mu = \sigma$.

A proof of this is given in the Appendix.

3.2. **Preliminaries.** One of the main technical ingredients of our proof is a certain localization procedure of the equation (2.9). In contrast to [28] and many following papers it is *not* carried out by multiplying the solution with suitable cut-off functions and afterwards deriving a corresponding equation for the product. We only restrict the function to open subsets of the domain and deduce a corresponding equation for this restriction – in an adequate weak formulation. In fact, this idea was developed in [17] for elliptic problems.

The following lemmata allows us in the sequel to perform this in an appropriate manner. The first lemma covers the cases of neighbourhoods of interior points of Ω and from the Neumann boundary (i.e., satisfying case II) of Assumption 2.4).

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $D \subset \partial\Omega$ be a relatively closed subset. Let $U \subset \mathbb{R}^n$ be open. Set $\Lambda := U \cap \Omega$, $S := N \cap U$ and $E := \partial\Lambda \setminus S$.*

i) Then S is open in $\partial\Lambda$ and E is closed.

ii) Let $p \in [1, \infty)$. Then there exists a unique isometric map $\mathfrak{E}_U : W_E^{1,p}(\Lambda) \to W_D^{1,p}(\Omega)$ such that $\mathfrak{E}_U w$ is the extension of w to Ω by 0 for all $w \in C_E^\infty(\Lambda)$.

iii) Set $R := \overline{D \cap U}$. Then $R \subset \partial\Lambda$ and $R \subseteq D \cap E$. Moreover, $u|_\Lambda \in W^{1,p}_R(\Lambda)$, if $u \in W^{1,p}_D(\Omega)$. Thus, the restriction operator from $W^{1,p}_D(\Omega)$ is a continuous one into $W^{1,p}_R(\Lambda)$ with norm not larger than 1.

Proof. i) There exists an open $V \subset \mathbb{R}^n$ such that $(\partial\Omega \setminus D) = V \cap \partial\Omega$. Then $S = (\partial\Omega \setminus D) \cap U = U \cap V \cap \partial\Omega \subset U \cap V \cap \partial\Lambda$. But $\partial\Lambda \subset \partial\Omega \cup \partial U$. Therefore $U \cap V \cap \partial\Lambda \subset U \cap V \cap \partial\Omega$. So $(U \cap V) \cap \partial\Lambda = U \cap V \cap \partial\Omega = S$, and S is open in $\partial\Lambda$.
ii) Let $w \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} w \cap E = \emptyset$. Since

$$\overline{\Lambda} = \Lambda \cup \partial\Lambda = \Lambda \cup S \cup E \subset U \cup E$$

and $E \cap \operatorname{supp} w = \emptyset$ it follows that $\overline{\Lambda} \cap \operatorname{supp} w \subseteq U$. Therefore there exists an $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta|_{\overline{\Lambda} \cap \operatorname{supp} w} = 1$ and $\operatorname{supp} \eta \subset U$. Consider the function ηw. First, observe that $U \cap D \subset E$ – this results from the relations $U \cap D \subseteq U \cap \partial\Omega \subseteq \partial\Lambda$ and $U \cap D \cap S = U \cap \big(D \cap (\partial\Omega \setminus D)\big) = \emptyset$. Hence $\operatorname{supp}(\eta w) \cap D = \emptyset$ and $\eta w \in C_D^\infty(\Omega)$. Secondly, one has $(\eta w)|_\Lambda = w|_\Lambda$. Moreover, if $\mathrm{x} \in \Omega \setminus \Lambda$, then $\mathrm{x} \in U^c$ and $\eta(\mathrm{x}) = 0$. So

$$(\eta w)|_\Omega(\mathrm{x}) = \begin{cases} w|_\Lambda(x) & \text{if } \mathrm{x} \in \Lambda \\ 0 & \text{if } \mathrm{x} \notin \Lambda \end{cases}$$

for all $\mathrm{x} \in \Omega$. Hence

$$\|(\eta w)|_\Omega\|_{W^{1,p}_D(\Omega)} = \|(\eta w)|_\Omega\|_{W^{1,p}(\Omega)} = \|w|_\Lambda\|_{W^{1,p}(\Lambda)} = \|w|_\Lambda\|_{W^{1,p}_E(\Lambda)}.$$

Therefore there exists a unique isometric map $\mathfrak{E}_U : W^{1,p}_E(\Lambda) \to W^{1,p}_D(\Omega)$ such that $\mathfrak{E}_U w$ is the extension of w to Ω by 0 for all $w \in C_E^\infty(\Lambda)$.
iii) Observe that $D \cap U \subset \partial\Omega \cap U \subset \partial\Lambda$. Since $\partial\Lambda$ is closed, this gives $R \subset \partial\Lambda$. On the other hand, $R = \overline{D \cap U} \subset \overline{D} = D$, since D is closed. Let us show the assertion $R \subset E$: in ii) we have already proved $U \cap D \subseteq E$, what implies $R \subseteq E$, thanks to the closedness of E. Hence, if $u \in C_D^\infty(\Omega)$, then the restriction $u|_\Lambda$ belongs to $C_R^\infty(\Lambda)$ with the obvious estimate $\|u|_\Lambda\|_{W^{1,p}(\Lambda)} = \|u|_\Lambda\|_{W^{1,p}_R(\Lambda)} \le \|u\|_{W^{1,p}_D(\Omega)} = \|u\|_{W^{1,p}(\Omega)}$. □

In case I) in Assumption 2.4 the local model set is allowed to be disconnected. Nevertheless, one can also in this case find an adequate localization procedure. In the spirit of Remark 2.5, this relies on the localization procedure for each of the connected components.

Lemma 3.11. *Let $p \in [1, \infty[$. In the terminology of Assumption 2.4 I) the following holds true for each $j \in \{1, \dots, k\}$:*

i) There is an isometric operator $\mathfrak{E}_j$ which extends any function from $W^{1,p}_0(V_j)$ by 0 to a function from $W^{1,p}_0(\Omega) \subseteq W^{1,p}_D(\Omega)$.
ii) We have $\partial V_j \subseteq \partial(U_j \cap \Omega)$.
iii) Let $R_j = \overline{\partial V_j \cap U_j}$. Then $R_j \subset \partial V_j$ and one has $\psi|_{V_j} \in W^{1,p}_{R_j}(V_j)$, if $\psi \in W^{1,p}_D(\Omega)$.

Proof. i) The support of every function from $C_0^\infty(V_j)$ has a positive distance to $\partial\Omega$; thus the extension by zero leads to a function from $C_0^\infty(\Omega)$ in this case. The general claim follows by density.
ii) By the definition of V_j it is clear that ∂V_j is contained in $\overline{U_j \cap \Omega}$. Now suppose that a point $\mathrm{y} \in \partial V_j$ lies in $U_j \cap \Omega$ (i.e., not on $\partial(U_j \cap \Omega)$). Since $U_j \cap \Omega$ is open, we find an open ball B containing y which is still a subset of $U_j \cap \Omega$. By supposition,

y is a boundary point of V_j, hence $V_j \cap B \neq \emptyset$. Thus, the connectedness of both V_j and B implies that $V_j \cup B \supset V_j$ is also open and connected – and, hence, identical with V_j. But then $B \subset V_j$ which is a contradiction to y being a boundary point of V_j. So indeed $\partial V_j \subseteq \partial(U_j \cap \Omega)$.
iii) The inclusion $R_j \subset \partial V_j$ is obvious. Let $u \in W^{1,p}_D(\Omega)$. Applying Lemma 3.10 with $U = U_j$ shows that $u|_{U_j \cap \Omega} \in W^{1,p}_R(U_j \cap \Omega)$ for $R = \overline{D \cap U_j}$. We now restrict u further to V_j. By ii), we have $\partial V_j \subset \partial(U_j \cap \Omega)$. Together with $R \cap \partial V_j = \overline{U_j \cap \partial V_j}$ due to $U_j \cap \partial V_j \subseteq \mathcal{U} \cap \partial V_j \subset D$, we obtain $u|_{V_j} \in W^{1,p}_{R_j}(V_j)$. □

We aim lastly at equations on τK and $\tau \widehat{K}$ for localized equations in neighbourhoods of boundary points of Ω, to be achieved via the bi-Lipschitzian transformations occurring in Assumption 2.4. Hence it is, of course, of interest onto which sets the different boundary parts are mapped by these transformations:

Lemma 3.12. *Let* $\mathrm{x} \in \partial\Omega$.

- *If* x *satisfies Assumption 2.4 I), then for each* $j \in \{1, \dots, k\}$ *one has* $\phi_j(\partial V_j) = \partial(\tau_j K)$ *and, in the terminology of Lemma 3.11,* $\phi_j(R_j) = \phi_j(\overline{\partial V_j \cap U_j}) = \tau_j \overline{\Sigma}$.
- *If* x *satisfies Assumption 2.4 II), one has in the terminology of Lemma 3.10 (putting* $U := U_\mathrm{x}$ *and* $\phi := \phi_\mathrm{x}$*):*
  i) $\phi(E) = \partial(\tau K) \setminus \tau\Sigma$ *and* $\phi(R) = \emptyset$ *in case IIa),*
  ii) $\phi(E) = \partial(\tau K) \setminus (\tau\Sigma \setminus \tau\Sigma_0)$ *and* $\phi(R) = \tau\overline{\Sigma_0}$ *in case IIb).*

Proof. This is straight-forward from the mapping properties of the transformations ϕ_x and ϕ_j. □

It turns out that the model constellation in Assumption 2.4 IIb) is indeed suggestive, but not optimal for further analytical purpose. We show in the next lemma that it can be replaced by another one which is much more controllable later, cf. [32, Sect. 4.2].

Lemma 3.13. *For every* $\tau > 0$*, there exists a volume-preserving, bi-Lipschitzian mapping* $\varsigma_n : \mathbb{R}^n \to \mathbb{R}^n$ *that maps* τK *onto* τK*,* $\partial(\tau K) \setminus (\tau\Sigma \setminus \tau\Sigma_0)$ *onto* $\partial(\tau K) \setminus \tau\Sigma$ *and* $\tau\overline{\Sigma}_0$ *onto the set* $]-\tau, \tau[^{n-2} \times \{-\tau\} \times [-\tau, 0]$*. Finally,* $\varsigma_n(\frac{\tau}{2}K) = \tau\underline{K}$ *with* $\underline{K} :=]-\frac{1}{2}, \frac{1}{2}[^{n-2} \times]-1, 0[\times]-\frac{1}{2}, 0[$.

Proof. Let us start with the case $n = 2$, thereby focussing first on the case $\tau = 1$. We define on the lower halfspace $\{(x, y) \in \mathbb{R}^2 : y \le 0\}$

$$\xi_1(x, y) := \begin{cases} (x - y/2, y/2), & \text{if } x \le 0,\ y \ge x, \\ (x/2, -x/2 + y), & \text{if } x \le 0,\ y < x, \\ (x/2, x/2 + y), & \text{if } x > 0,\ y < -x, \\ (x + y/2, y/2), & \text{if } x > 0,\ y \ge -x. \end{cases}$$

Observing that ξ_1 acts as the identity on the x-axis, we may define ξ_1 on the upper half space $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ by $\xi_1(x, y) = (x, y/2)$. In this way we obtain a globally bi-Lipschitz transformation ξ_1 from $\mathbb{R}^2$ onto itself that transforms $K \cup \Sigma_0$ onto the triangle shown in Figure 3. Next we define the bi-Lipschitz mapping $\xi_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\xi_2(x, y) := \begin{cases} (x, x + 2y + 1), & \text{if } x \le 0, \\ (x, -x + 2y + 1), & \text{if } x > 0, \end{cases}$$

in order to get the geometric constellation in Figure 4. If ξ_3 is the (counter-

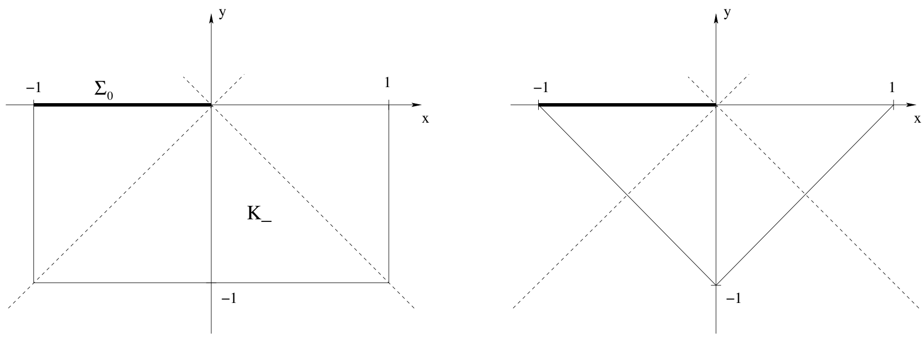

FIGURE 3. $K \cup \overline{\Sigma}_0$ and $\xi_1(K \cup \overline{\Sigma}_0)$

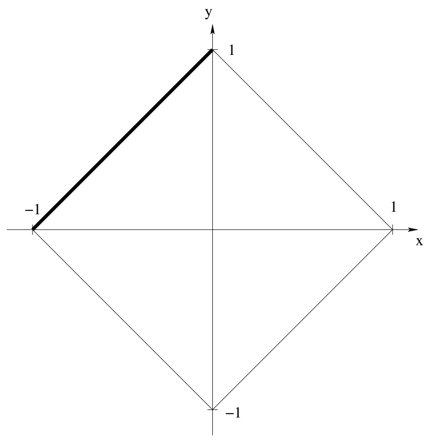

FIGURE 4. $\xi_2(\xi_1(K \cup \overline{\Sigma}_0))$

clockwise) rotation of $\pi/4$ around $0 \in \mathbb{R}^2$, we thus have achieved that $\xi := \xi_3\xi_2\xi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is bi-Lipschitzian and satisfies

$$\xi(K) = \frac{1}{\sqrt{2}}\widehat{K}, \quad \text{and} \quad \xi(\overline{\Sigma}_0) = \left\{\frac{-1}{\sqrt{2}}\right\} \times \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right].$$

Let $\xi_4 : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine mapping $(x, y) \mapsto (\sqrt{2}x, \frac{1}{\sqrt{2}}y - \frac{1}{2})$. Then $\varsigma_2 = \xi_4\xi$ is bi-Lipschitzian and maps K onto itself, $\partial K \setminus (\Sigma \setminus \Sigma_0)$ onto $\partial K \setminus \Sigma$, and $\overline{\Sigma_0}$ bi-Lipschitzian onto $\{-1\}\times[-1, 0]$. The assertion for $\underline{K}$ is verified by a straight forward calculation. As is easy to check, the determinant of the Jacobian is identically one almost everywhere. Hence, ς_2 is volume-preserving.
If $\tau \neq 1$, then one first applies the homothety $\mathrm{y} \mapsto \frac{1}{\tau}\mathrm{y}$, then the mapping ς_2 just constructed for the case $\tau = 1$ and afterwards the inverse homothety $\mathrm{y} \mapsto \tau\mathrm{y}$.
For $n \geq 3$, one simply puts $\varsigma_n(x_1, \ldots, x_n) := (x_1, \ldots, x_{n-2}, \varsigma_2(x_{n-1}, x_n))$. □

Corollary 3.14. *Suppose that Assumption 2.4 IIb) holds true. Then for every point* x *from* ∂D *(within* $\partial\Omega$*) there is a an open neighbourhood* U_{x}*, a positive number*

$\tau = \tau_{\mathrm{x}}$ and a bi-Lipschitzian, volume-preserving mapping from a neighbourhood of $\overline{U_{\mathrm{x}}}$ into $\mathbb{R}^n$, which maps $U_{\mathrm{x}} \cap \Omega$ onto τK, E onto $\partial(\tau K) \setminus \tau\Sigma$, and R onto the set $[-\tau, \tau]^{n-2} \times \{-\tau\} \times [-\tau, 0]$, where E, R are defined as in Lemma 3.10.

Proof. If one defines the asserted mapping as the composition $\varsigma_n \circ \phi_{\mathrm{x}}$, then the application of Lemma 3.12 and Lemma 3.13 gives the assertion. □

Having the bi-Lipschitz mappings ϕ and ς defined above at hand, we collect properties of bi-Lipschitzian transformations if applied to the typical data of parabolic equations as (2.9). It turns out that (volume-preserving) bi-Lipschitz mappings essentially preserve the structure of the underlying problem.

Proposition 3.15. *Let Λ be a bounded Lipschitz domain, and let F be a closed portion of its boundary. Assume that ζ is a bi-Lipschitzian mapping from a neighbourhood of $\overline{\Lambda}$ into $\mathbb{R}^n$. Define for any function $\varphi : \zeta(\Lambda) \to \mathbb{R}$ the function $\Phi\varphi : \Lambda \to \mathbb{R}$ by*

$$(\Phi\varphi)(\mathrm{x}) := \varphi(\zeta(\mathrm{x})) = (\varphi \circ \zeta)(\mathrm{x}), \quad \mathrm{x} \in \Lambda.$$

i) For every $\varphi \in W^{1,1}(\zeta(\Lambda))$, the (generalized) gradient of the function $\varphi \circ \zeta$ is calculated for almost all $\mathrm{x} \in \Lambda$ as follows:

$$\nabla\big(\varphi \circ \zeta\big)(\mathrm{x}) = \begin{pmatrix} \frac{\partial \zeta_1}{\partial x_1}(\mathrm{x}) & \dots & \frac{\partial \zeta_n}{\partial x_1}(\mathrm{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial \zeta_1}{\partial x_n}(\mathrm{x}) & \dots & \frac{\partial \zeta_n}{\partial x_n}(\mathrm{x}) \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi}{\partial x_1}(\zeta(\mathrm{x})) \\ \vdots \\ \frac{\partial \varphi}{\partial x_n}(\zeta(\mathrm{x})) \end{pmatrix} = (D\zeta)^T(\mathrm{x})\nabla\varphi(\zeta(\mathrm{x})).$$

ii) For every $p \in]1, \infty[$, the mapping Φ induces linear, topological isomorphisms

$$\Phi_{1,p} : W^{1,p}_{\zeta(F)}(\zeta(\Lambda)) \to W^{1,p}_F(\Lambda) \quad \text{and} \quad \Phi^*_{1,p'} : W^{-1,p}_F(\Lambda) \to W^{-1,p}_{\zeta(F)}(\zeta(\Lambda))$$

as well as $\Phi_p : L^p(\zeta(\Lambda)) \to L^p(\Lambda)$. These are consistent for different values of p.

iii) If $\alpha \in]0, 1[$, then Φ induces a topological isomorphism $\Phi_{0,\alpha}$ between $C^\alpha(\zeta(\Lambda))$ and $C^\alpha(\Lambda)$. The norms of $\Phi_{0,\alpha}$ and $\Phi^{-1}_{0,\alpha}$ only depend on the Lipschitz constants of ζ and ζ^{-1}.

iv) Let $\rho : \Lambda \to M_n$ be bounded and measurable. Then one has for every $p \in]1, \infty[$ and every pair $(\psi, \varphi) \in W^{1,p}(\zeta(\Lambda)) \times W^{1,p'}(\zeta(\Lambda))$ the identity

$$\int_\Lambda \rho\nabla(\psi \circ \zeta) \cdot \nabla(\varphi \circ \zeta)\, d\mathrm{x} = \int_{\zeta(\Lambda)} \rho_\zeta \nabla\psi \cdot \nabla\varphi\, d\mathrm{y}. \tag{3.5}$$

with

$$\rho_\zeta(\mathrm{y}) = \frac{1}{|\det(D\zeta)(\zeta^{-1}(\mathrm{y}))|}(D\zeta)(\zeta^{-1}(\mathrm{y}))\ \rho(\zeta^{-1}(\mathrm{y}))\ (D\zeta)^T(\zeta^{-1}(\mathrm{y})) \tag{3.6}$$

for almost all $\mathrm{y} \in \zeta(\Lambda)$. Here, $D\zeta$ denotes the Jacobian of ζ and $\det(D\zeta)$ the corresponding determinant.

v) Let $\mathfrak{l}_\zeta, \mathfrak{l}_{\zeta^{-1}}$ denote the Lipschitz constants of ζ and ζ^{-1}, respectively, and assume that ζ is volume preserving. If ρ takes its values in $M_n(\kappa_0, \kappa_1)$, then ρ_ζ takes its values in $M_n(\widehat{\kappa}_0, \widehat{\kappa}_1)$ where $\widehat{\kappa}_0 := \frac{\kappa_0}{\mathfrak{l}^2_{\zeta^{-1}}}$ and $\widehat{\kappa}_1 := \kappa_1 \mathfrak{l}^2_\zeta$.

Proof. For i) see [40, Ch. 1.1.7]. The proof of ii) is contained in [25, Thm. 2.7/2.10]. iii) is obvious. Assertion iv) can be deduced from i), for a complete proof see [31, Prop. 16]. v) First one observes that for a volume-preserving mapping ζ the function $|\det(D\zeta)(\cdot)|$ is identically 1, [18, Ch. 3]. Secondly, Rademacher's theorem shows that $\|D\zeta\|_{L^\infty(\Lambda;M_n)} \le \mathfrak{l}_\zeta$ and $\|D(\zeta^{-1})\|_{L^\infty(\zeta(\Lambda);M_n)} \le \mathfrak{l}_{\zeta^{-1}}$. With all this in mind, one easily calculates for almost all $\mathrm{y} \in \zeta(\Lambda)$ and all $\mathrm{z} \in \mathbb{R}^n$ as follows:

$$\begin{aligned}\|\rho_\zeta(\mathrm{y})\mathrm{z}\cdot\mathrm{z}\|_{\mathbb{R}^n} &= \|\rho(\zeta^{-1}\mathrm{y})(D\zeta)^T(\zeta^{-1}(\mathrm{y}))\mathrm{z}\cdot(D\zeta)^T(\zeta^{-1}(\mathrm{y}))\mathrm{z}\|_{\mathbb{R}^n}\\ &\le \kappa_1\|(D\zeta)^T(\zeta^{-1}(\mathrm{y}))\mathrm{z}\|^2_{\mathbb{R}^n} \le \kappa_1\mathfrak{l}_\zeta^2\|\mathrm{z}\|^2_{\mathbb{R}^n}.\end{aligned}$$

In order to deduce the lower bound, one first recalls the equality

$$(D\zeta)(\zeta^{-1}\mathrm{y}) = \big(D\zeta^{-1})(\mathrm{y})\big)^{-1}, \tag{3.7}$$

which holds for almost all $\mathrm{y} \in \zeta(\Lambda)$, see [18, Ch. 3.1.2 Cor. 1]. Having this at hand, one estimates for almost all $\mathrm{y} \in \zeta(\Lambda)$ and all $\mathrm{z} \in \mathbb{R}^n$

$$\begin{aligned}\|\rho_\zeta(\mathrm{y})\mathrm{z}\cdot\mathrm{z}\|_{\mathbb{R}^n} &= \|\rho(\zeta^{-1}\mathrm{y})(D\zeta)^T(\zeta^{-1}(\mathrm{y}))\mathrm{z}\cdot(D\zeta)^T(\zeta^{-1}(\mathrm{y}))\mathrm{z}\|_{\mathbb{R}^n}\\ &\ge \kappa_0\|(D\zeta)^T(\zeta^{-1}(\mathrm{y}))\mathrm{z}\|^2_{\mathbb{R}^n} = \kappa_0\|\big((D(\zeta^{-1}))^T(\mathrm{y})\big)^{-1}\mathrm{z}\|^2_{\mathbb{R}^n}\\ &\ge \frac{\kappa_0}{\operatorname{ess\,sup}_{\mathrm{y}\in\zeta(\Lambda)}\|D(\zeta^{-1})(\mathrm{y})\|^2}\|\mathrm{z}\|^2_{\mathbb{R}^n} \ge \frac{\kappa_0}{\mathfrak{l}^2_{\zeta^{-1}}}\|\mathrm{z}\|^2_{\mathbb{R}^n}.\end{aligned}$$ □

Remark 3.16. If μ is a coefficient function on $J \times \Lambda$ and $\zeta : \Lambda \to \Xi$ is bi-Lipschitzian, then we denote by μ_ζ the coefficient function $t \mapsto \mu_\zeta(t,\cdot)$ on $J \times \Xi$ given as in (3.6).

3.3. **Localization, transformation, reflection.** Now we have the principle ideas at hand and will first *localize* the parabolic equation suitably in order to consider it on smaller sets. The resulting equations are then *transformed* by bi-Lipschitzian mappings, corresponding of course to Assumption 2.4, to equations on the half cube K. In the case of points from the Neumann boundary part, one finally needs a reflection argument, which will be established in the last part of this subsection.

Having this in mind, let us now localize the equation

$$\langle u',\varphi\rangle_{W^{1,2}_D(\Omega)} - \langle\nabla\cdot\mu(t,\cdot)\nabla u,\varphi\rangle_{W^{1,2}_D(\Omega)} + \int_\Omega u\varphi\,\mathrm{d}\mathrm{x} = \langle f,\varphi\rangle_{W^{1,2}_D(\Omega)}, \quad \varphi\in W^{1,2}_D(\Omega), \tag{3.8}$$

where $f \in L^2(J;W^{-1,2}_D(\Omega))$. Note that a solution u to this equation belongs to the space $L^2(J;W^{1,2}_D(\Omega)) \cap W^{1,2}(J;W^{-1,2}_D(\Omega)) \hookrightarrow C(\overline{J};L^2(\Omega))$, cf. Proposition 2.10. Let us fix an arbitrary point $\mathrm{x} \in \overline{\Omega}$ and consider an open neighbourhood U of x. If $\mathrm{x} \in \Omega$, we assume $\overline{U} \subset \Omega$. We will now localize the equation around x according to the constructions from Lemmata 3.10 (for the first two cases) and 3.11 (the last case), respectively:

- If $\mathrm{x} \in \Omega$, set $\Lambda = U$, $E = \partial\Lambda$ and $R = \emptyset$.
- For $\mathrm{x} \in \overline{N}$, we choose $\Lambda = \Omega \cap U$ and E, R as in Lemma 3.10, i.e., $E = \partial\Lambda \setminus (N \cap U)$ and $R = \overline{D \cap U}$.
- In case of $\mathrm{x} \in D \setminus \overline{N}$, $\Omega \cap U$ may be disconnected with, say, k connected components V_j. We thus set $\Lambda_j = V_j$, $E_j = \partial V_j$ and $R_j = \overline{\partial V_j \cap U_j}$, where U_j is an open set with $V_j \subset U_j \subset U$, for each $j \in \{1,\dots,k\}$. The following localization procedure then has to be done for every $j \in \{1,\dots,k\}$. We will, however, omit the index j to simplify the notation.

In this terminology, one calculates for $w \in W^{1,2}_D(\Omega)$ and every $\varphi \in W^{1,2}_E(\Lambda)$

$$\begin{aligned}\langle -\nabla \cdot \rho|_\Lambda \nabla w|_\Lambda, \varphi\rangle_{W^{1,2}_E(\Lambda)} &= \int_\Lambda \rho|_\Lambda \nabla w|_\Lambda \cdot \nabla \varphi \, \mathrm{d}x \\ &= \int_\Omega \rho \nabla w \cdot \nabla(\mathfrak{E}_U \varphi)\, \mathrm{d}x = \langle -\nabla \cdot \rho \nabla w, (\mathfrak{E}_U \varphi)\rangle_{W^{1,2}_D(\Omega)}.\end{aligned} \tag{3.9}$$

Remark 3.17. The first term in (3.9) does *not* contain abuse of the above introduced notation in the following sense: for $w \in W^{1,2}_D(\Omega)$ the restriction $w|_\Lambda$ belongs to the space $W^{1,2}_R(\Lambda)$, cf. Lemma 3.10 iii) and 3.11 iii). The operator $-\nabla \cdot \rho|_\Lambda \nabla$ is well-defined from $W^{1,2}_R(\Lambda)$ to $W^{-1,2}_R(\Lambda)$, see Definition 2.9, giving $-\nabla \cdot \rho|_\Lambda \nabla w|_\Lambda \in W^{-1,2}_R(\Lambda)$. But R is contained in E, which yields $W^{1,2}_E(\Lambda) \hookrightarrow W^{1,2}_R(\Lambda)$ with isometric injection. Thus, $W^{1,2}_E(\Lambda) \ni \varphi \mapsto \langle -\nabla \cdot \rho|_\Lambda \nabla w|_\Lambda, \varphi\rangle_{W^{1,2}_E(\Lambda)}$ is to be understood as the restriction of the linear form $-\nabla \cdot \rho|_\Lambda \nabla w|_\Lambda \in W^{-1,2}_R(\Lambda)$ to the subspace $W^{1,2}_E(\Lambda) \subseteq W^{1,2}_R(\Lambda)$.

Assume now that a given function $u \in L^2(J; W^{1,2}_D(\Omega)) \cap W^{1,2}(J; W^{-1,2}_D(\Omega))$ satisfies (3.8) with $f \in L^2(J; W^{-1,2}_D(\Omega))$. From the identity

$$\int_\Lambda u(t)\varphi \, \mathrm{d}x = \int_\Omega u(t) \mathfrak{E}_U \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in W^{1,2}_E(\Lambda),$$

one easily deduces [12, Ch. XVIII.1.2 Prop. 7] that for all $\varphi \in W^{1,2}_E(\Lambda)$

$$\langle (u|_\Lambda)'(t), \varphi\rangle_{W^{1,2}_E(\Lambda)} = \frac{d}{dt}\int_\Lambda u|_\Lambda(t)\varphi\, \mathrm{d}x = \frac{d}{dt}\int_\Omega u(t)\mathfrak{E}_U\varphi \, \mathrm{d}x = \langle u'(t), \mathfrak{E}_U\varphi\rangle_{W^{1,2}_D(\Omega)}, \tag{3.10}$$

where the time derivative on the left hand side is taken in the sense of $W^{-1,2}_E(\Lambda)$-valued distributions and in the sense of $W^{-1,2}_D(\Omega)$-valued distributions on the right-hand side. Note carefully that everything is indeed in order since $\mathfrak{E}_U : W^{1,2}_E(\Lambda) \to W^{1,2}_D(\Omega)$ is well-defined and continuous, thanks to Lemma 3.10 ii) and Lemma 3.11 i). One step further, using (3.10) and (3.9) in case of $w = u(t)$ and $\rho = \mu(t, \cdot)$, one obtains for every $\varphi \in W^{1,2}_E(\Lambda)$ and almost every $t \in J$

$$\langle (u|_\Lambda)', \varphi\rangle_{W^{1,2}_E(\Lambda)} - \langle \nabla \cdot \mu|_\Lambda \nabla u|_\Lambda, \varphi\rangle_{W^{1,2}_E(\Lambda)} + \int_\Lambda u|_\Lambda \varphi \, \mathrm{d}x = \langle f, \mathfrak{E}_U \varphi\rangle_{W^{1,2}_D(\Omega)}. \tag{3.11}$$

For $q \in \,]1,\infty[$ and $g \in W^{-1,q}_D(\Omega)$, we denote the linear form $W^{1,q'}_E(\Lambda) \ni \varphi \mapsto \langle g, \mathfrak{E}_U\varphi\rangle$ by $g_U = \mathfrak{E}_U^* g$. One easily estimates

$$\|g_U\|_{W^{-1,q}_E(\Lambda)} \le \|\mathfrak{E}_U^*\|_{\mathcal{L}(W^{-1,q}_D(\Omega), W^{-1,q}_E(\Lambda))} \|g\|_{W^{-1,q}_D(\Omega)} \le \|g\|_{W^{-1,q}_D(\Omega)} \tag{3.12}$$

since $\mathfrak{E}_U$ is an isometry. This shows the following: the function $J \ni t \mapsto f_U(t)$, defining the right-hand side in (3.11), belongs to $L^2(J; W^{-1,2}_E(\Lambda))$, and its norm does not exceed $\|f\|_{L^2(J;W^{-1,2}_D(\Omega))}$. Analogously, if $q, s \in]2,\infty[$, and $f \in L^s(J; W^{-1,q}_D(\Omega))$, then $f_U \in L^s(J; W^{-1,q}_E(\Lambda))$ with a similar estimate. In this spirit, let us write (3.11) in the form

$$\langle (u|_\Lambda)', \varphi\rangle_{W^{1,2}_E(\Lambda)} + \langle -\nabla \cdot \mu|_\Lambda \nabla u|_\Lambda, \varphi\rangle_{W^{1,2}_E(\Lambda)} + \int_\Lambda u|_\Lambda \varphi \, \mathrm{d}x = \langle f_U, \varphi\rangle_{W^{1,2}_E(\Lambda)}. \tag{3.13}$$

Remark 3.18. In any case, the property $u \in L^2(J; W_D^{1,2}(\Omega)) \cap C(\overline{J}; L^2(\Omega))$ implies that we have $u|_\Lambda \in L^2(J; W_R^{1,2}(\Lambda)) \cap C(\overline{J}; L^2(\Lambda))$, and the corresponding $V_2^{1,0}$-norm of $u|_\Lambda$ is not larger as the $V_2^{1,0}$-norm of u, cf. Lemmata 3.10 and 3.11.

This completes the *localization* procedure so far: For every possible constellation in and around a point $\mathrm{x} \in \overline{\Omega}$, we have constructed a suitable local equation in $W_E^{-1,2}(\Lambda)$ which is satisfied by the global solution u. Next, we *transform* these local equations according to Assumption 2.4 using the properties of the transformations established in Proposition 3.15. Suppose from now on that for every point $\mathrm{x} \in \partial\Omega$, a neighbourhood U of x is given as declared in the fitting case in Assumption 2.4 and that Λ, E and R are chosen accordingly as in the localization procedure above (with the obvious adjustments).

We now exploit III) of Assumption 2.4, that is, for each case of boundary points x, there is a volume-preserving, bi-Lipschitzian mapping ζ from a neighbourhood of $\overline{\Lambda}$ onto a neighbourhood of the cube τK. Let us assume that E is mapped onto $E_\bullet \subset \partial(\tau K)$, and that R is mapped onto $R_\bullet \subset \partial(\tau K)$ – where ζ and $E_\bullet, R_\bullet$ will be specified later and, of course, in correspondence with Assumption 2.4, Lemma 3.12 and Corollary 3.14.

For almost all t, $u|_\Lambda(t) \in W_R^{1,2}(\Lambda)$ is of the form $v(t) \circ \zeta$ for a $v(t) \in W_{R_\bullet}^{1,2}(\tau K)$, just as $\varphi \in W_E^{1,2}(\Lambda)$ is of the form $\varphi = \psi \circ \zeta$ for some $\psi \in W_{E_\bullet}^{1,2}(\tau K)$, both thanks to Proposition 3.15 ii). Taking this into account, one obtains

$$\begin{aligned}\langle (u|'_\Lambda, \varphi\rangle_{W_E^{1,2}(\Lambda)} &= \frac{d}{dt}\langle u|_\Lambda, \varphi\rangle_{W_E^{1,2}(\Lambda)} = \frac{d}{dt}\langle v\circ\zeta, \psi\circ\zeta\rangle_{W_E^{1,2}(\Lambda)} = \frac{d}{dt}\int_\Lambda (v\circ\zeta)(\psi\circ\zeta)\,dy \\ &= \frac{d}{dt}\int_{\tau K} v\,\psi\,d\mathrm{x} = \langle v', \psi\rangle_{W_{E_\bullet}^{1,2}(\tau K)},\end{aligned} \tag{3.14}$$

since ζ is volume-preserving, i.e., $|\det(D\zeta)| = |\det(D\zeta^{-1})| \equiv 1$ almost everywhere. On the other hand, one gets for every $\varphi \in W_E^{1,2}(\Lambda)$ for almost every $t \in J$

$$\begin{aligned}\langle \nabla\cdot\mu(t,\cdot)|_\Lambda \nabla u|_\Lambda(t), \varphi\rangle_{W_E^{1,2}(\Lambda)} &= \langle \nabla\cdot\mu(t,\cdot)|_\Lambda \nabla\big(v(t)\circ\zeta\big), \psi\circ\zeta\rangle_{W_E^{1,2}(\Lambda)} \\ &= -\int_\Lambda \mu(t,\cdot)|_\Lambda \nabla\big(v(t)\circ\zeta\big)\cdot\nabla\big(\psi\circ\zeta\big)\,dy \\ &= -\int_{\tau K} \mu_\zeta(t,\cdot)\nabla v(t)\cdot\nabla\psi\,dy \\ &= \langle \nabla\cdot\mu_\zeta(t,\cdot)\nabla v(t), \psi\rangle_{W_{E_\bullet}^{1,2}(\tau K)},\end{aligned}$$

cf. Proposition 3.15 iv). Finally, for almost all $t \in J$, $\int_\Lambda u(t)|_\Lambda \varphi\,d\mathrm{x}$ is calculated to $\int_{\tau K} v(t)\psi\,d\mathrm{x}$ since ζ is volume-preserving. Hence, (3.13) leads to the following equation for the transformed function v:

$$\langle v', \psi\rangle_{W_{E_\bullet}^{1,2}(\tau K)} - \langle \nabla\cdot\mu_\zeta(t,\cdot)\nabla v, \psi\rangle_{W_{E_\bullet}^{1,2}(\tau K)} + \int_{\tau K} v(t)\psi\,d\mathrm{x} = \langle f_U, \psi\circ\zeta\rangle_{W_E^{1,2}(\Lambda)} \tag{3.15}$$

for $\psi \in W_{E_\bullet}^{1,2}(\tau K)$. In view of (3.12) one gets for every $\psi \in W_{E_\bullet}^{1,q'}(\tau K)$ and almost all $t \in J$

$$\begin{aligned}\Big|\langle f_U(t), \psi\circ\zeta\rangle_{W_E^{1,q'}(\Lambda)}\Big| &\le \|f_U(t)\|_{W_E^{-1,q}(\Lambda)}\|\psi\circ\zeta\|_{W_E^{1,q'}(\Lambda)} \\ &\le c\|f(t)\|_{W_D^{-1,q}(\Lambda)}\|\psi\|_{W_{E_\bullet}^{1,q'}(\tau K)},\end{aligned} \tag{3.16}$$

the constant c only depending on ζ, see Proposition 3.15i). Thus, for almost every $t \in J$, the linear form

$$W^{1,q'}_{E_\bullet}(\tau K) \ni \psi \mapsto \langle f_U(t), \psi \circ \zeta \rangle$$

belongs to $W^{-1,q}_{E_\bullet}(K)$. If one denotes this linear form by $g(t)$, then (3.16) shows the following: if f in (2.7), cf. also (2.9), even belongs to $L^s(J; W^{-1,q}_D(\Omega))$, then g is from $L^s(J; W^{-1,q}_{E_\bullet}(\tau K))$ and, additionally, fulfils the estimate

$$\|g\|_{L^s(J;W^{-1,q}_{E_\bullet}(\tau K))} \le c\|f\|_{L^s(J;W^{-1,q}_D(\Omega))}, \tag{3.17}$$

the constant c only depending on the mapping ζ. Expressing the right-hand side of (3.15) in this manner, we get the final equation for v on τK, namely

$$\langle v'(t), \psi\rangle_{W^{1,2}_{E_\bullet}(\tau K)} - \langle \nabla \cdot \mu_\zeta(t,\cdot)\nabla v(t), \psi\rangle_{W^{1,2}_{E_\bullet}(\tau K)} + \int_{\tau K} v(t)\psi \,\mathrm{d}x = \langle g(t), \psi\rangle_{W^{1,2}_{E_\bullet}(\tau K)} \tag{3.18}$$

for $\psi \in W^{1,2}_{E_\bullet}(\tau K)$.

Remark 3.19. Again, the property $u|_\Lambda \in L^2(J; W^{1,2}_R(\Lambda)) \cap C(\overline{J}; L^2(\Lambda))$ leads to v being from $L^2(J; W^{1,2}_{R_\bullet}(\tau K)) \cap C(\overline{J}; L^2(\tau K))$ inclusively a corresponding estimate – where the norm depends only on the bi-Lipschitz mapping ζ, cf. Lemma 3.15 ii). Moreover, (3.14) gives the inclusion $v \in W^{1,2}(J; W^{-1,2}_{E_\bullet}(\tau K))$ together with estimates for the corresponding norms.

Let us now specify the mapping ζ in dependence of the different cases in Assumption 2.4 and the conventions from the beginning of the localization procedure, defining the sets $E_\bullet = \zeta(E)$ and $R_\bullet = \zeta(R)$ correspondingly:

- In case I) one puts $\zeta_j := \phi_j$, thus obtaining
$$E_{j,\bullet} = \zeta_j(E_j) = \partial(\tau_j K) \quad \text{and} \quad R_{j,\bullet} = \zeta_j(R_j) = \tau_j\overline{\Sigma}, \tag{3.19}$$
for each $j \in \{1, \dots, k\}$, see Lemma 3.12.
- In case IIa), we set $\zeta = \phi_{\mathrm{x}}$, such that
$$E_\bullet = \zeta(E) = \partial(\tau K) \setminus \tau\Sigma \quad \text{and} \quad R_\bullet = \zeta(R) = \emptyset, \tag{3.20}$$
cf. Lemma 3.12.
- In case IIb) we choose $\zeta := \varsigma_n \circ \phi_{\mathrm{x}}$ and obtain, in view of Corollary 3.14,
$$E_\bullet = \zeta(E) = \partial(\tau K) \setminus \tau\Sigma \quad \text{and} \quad R_\bullet = \zeta(R) = [-\tau, \tau]^{n-2} \times \{-\tau\} \times [-\tau, 0]. \tag{3.21}$$
Observe that in this last case $\zeta(\mathrm{x}) = (0, \dots, 0, -\tau, 0)$.

Having the transformed equations on the half cubes with transformed boundary conditions at hand, we lastly introduce *reflection* for case II) from Assumption 2.4. Inspection of Corollaries 3.5 and 3.7 reveals why this is necessary: Both corollaries require a subdomain Ξ_0 which has a positive distance to the whole boundary $\partial\Xi$ or to the complement of the Dirichlet boundary part F. But in case II) of Assumption 2.4, after the localization and transformation procedure we end up with $\zeta(\mathrm{x})$ being a boundary point on the half square without prescribed Dirichlet boundary part (remember $\zeta(R) = \emptyset$ in case a)) and $\zeta(\mathrm{x})$ being at the boundary of the Dirichlet boundary part itself, respectively. Both cases do not admit a suitable neighbourhood of $\zeta(\mathrm{x})$ which would satisfy the assumptions of Corollaries 3.5 and 3.7. By reflecting the equation across the "upper" plate of the half cubes, we obtain $\zeta(\mathrm{x})$ being inner points of the whole cube and the (combined) Dirchlet boundary part, respectively, allowing to use the aforementioned corollaries.

Let us first define for $\mathrm{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ the symbol $\mathrm{x}_- := (x_1, \dots, x_{n-1}, -x_n)$, and for a $n \times n$ matrix ϱ, the matrix ϱ^- by

$$\varrho^-_{i,j} := \begin{cases} \varrho_{i,j}, & \text{if } i, j < n, \\ -\varrho_{i,j}, & \text{if } i = n \text{ and } j \neq n \text{ or } j = n \text{ and } i \neq n, \\ \varrho_{i,j}, & \text{if } i = j = n. \end{cases}$$

Corresponding to a coefficient function ρ on τK, we then define the coefficient function $\hat{\rho}$ on $\tau \widehat{K}$ by

$$\hat{\rho}(\mathrm{x}) := \begin{cases} \rho(\mathrm{x}), & \text{if } \mathrm{x} \in \tau K, \\ \big(\rho(\mathrm{x}_-)\big)^-, & \text{if } \mathrm{x}_- \in \tau K, \\ 1, & \text{if } \mathrm{x} \in \Sigma. \end{cases}$$

Finally, we define for $w \in L^1(\tau \widehat{K})$ the function w_- by $w_-(x) = w(x_-)$, and for $w \in L^1(\tau K)$ the (symmetrically) reflected function by

$$\mathfrak{E} : L^1(\tau K) \to L^1(\tau \widehat{K}), \qquad (\mathfrak{E} w)(\mathrm{x}) = \begin{cases} w(\mathrm{x}), & \text{if } \mathrm{x} \in \tau K, \\ w(\mathrm{x}_-), & \text{if } \mathrm{x}_- \in \tau K, \end{cases}$$

Lemma 3.20. *Let F be a closed subset of $\partial(\tau K) \setminus \tau\Sigma$, put $\widehat{F} := F \cup \{\mathrm{x} : \mathrm{x}_- \in F\}$ and assume $p \in [1, \infty[$. Then $w \in W^{1,p}_F(\tau K)$, if and only if $\mathfrak{E} w \in W^{1,p}_{\widehat{F}}(\tau \widehat{K})$.*

Proof. First, $\mathfrak{E} w \in W^{1,p}_{\widehat{F}}(\tau \widehat{K})$ trivially implies $w \in W^{1,p}_F(\tau K)$. In view of the converse assertion, it is known that $w \in W^{1,p}_F(\tau K) \subseteq W^{1,p}(\tau K)$ implies $\mathfrak{E} w \in W^{1,p}(\tau \widehat{K})$, see [23, Lemma 3.4]. Lastly, standard arguments show that $\mathfrak{E} w$ may be approximated in the $W^{1,p}$-norm by restrictions of $C_0^\infty(\mathbb{R}^n)$-functions the support of which avoids $\widehat{F}$. □

Let us next introduce an extension operator for distribution-type objects: For $p \in\,]1, \infty[$, define the extension operator $\mathfrak{S} : W^{-1,p}_F(\tau K) \to W^{-1,p}_{\widehat{F}}(\tau \widehat{K})$ by

$$\langle \mathfrak{S} f, \varphi \rangle_{W^{1,p'}_{\widehat{F}}(\tau \widehat{K})} = \langle f, \varphi|_{\tau K} + \varphi_-|_{\tau K} \rangle_{W^{1,p'}_F(\tau K)}, \quad \varphi \in W^{1,p'}_{\widehat{F}}(\tau \widehat{K}).$$

We immediately obtain the following properties:

Lemma 3.21. *Assume $p \in\,]1, \infty[$.*

i) If $\psi \in L^1(\tau K) \cap W^{-1,p}_F(\tau K)$, then $\mathfrak{S}\psi = \mathfrak{E}\psi$.
ii) For any closed subset $F \subseteq \partial\tau K \setminus \tau\Sigma$, the operator $\mathfrak{S} : W^{-1,p}_F(\tau K) \to W^{-1,p}_{\widehat{F}}(\tau \widehat{K})$ is continuous with norm not larger than 2.

Proof. One has for all $\varphi \in W^{1,p'}_{\widehat{F}}(\tau \widehat{K})$ the identity

$$\langle \mathfrak{S}\psi, \varphi \rangle_{W^{1,p'}_{\widehat{F}}(\tau \widehat{K})} = \int_{\tau K} \psi(\varphi|_{\tau K} + \varphi_-|_{\tau K}) \, \mathrm{d}\mathrm{x} = \int_{\tau \widehat{K}} \mathfrak{E}\psi \, \varphi \, \mathrm{d}\mathrm{x},$$

which proves the first point. Moreover, the operator under consideration is the adjoint of the continuous operator $W^{1,p'}_{\widehat{F}}(\tau \widehat{K}) \ni \varphi \mapsto (\varphi|_{\tau K} + \varphi_-|_{\tau K}) \in W^{1,p'}_F(\tau K)$, which implies both assertions from the second point. □

Lemma 3.22. *Let $E_\bullet, R_\bullet$ with $R_\bullet \subseteq E_\bullet$ be two closed subsets of $\partial(\tau K) \setminus \tau\Sigma$.*

i) *If* $w \in W^{1,2}_{R_\bullet}(\tau K)$ *satisfies*

$$\langle -\nabla \cdot \rho \nabla w + w, \psi\rangle_{W^{1,2}_{E_\bullet}(\tau K)} = \langle h, \psi\rangle_{W^{1,2}_{E_\bullet}(\tau K)}, \quad \psi \in W^{1,2}_{E_\bullet}(\tau K) \tag{3.22}$$

for some $h \in W^{-1,2}_{E_\bullet}(\tau K)$*, then*

$$\langle -\nabla \cdot \hat{\rho} \nabla (\mathfrak{E}w) + \mathfrak{E}w, \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})} = \langle \mathfrak{S}h, \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})}, \quad \varphi \in W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K}) \tag{3.23}$$

ii) *Suppose that* $v \in W^{1,2}(J; W^{-1,2}_{E_\bullet}(\tau K)) \cap L^2(J; W^{1,2}_{R_\bullet}(\tau K))$ *with initial value* $u_0 = 0$ *satisfies*

$$\langle v'(t), \psi\rangle_{W^{1,2}_{E_\bullet}(\tau K)} + \langle -\nabla \cdot \mu(t,\cdot)\nabla v(t) + v(t), \psi\rangle_{W^{1,2}_{E_\bullet}(\tau K)} = \langle g(t), \psi\rangle_{W^{1,2}_{E_\bullet}(\tau K)}, \tag{3.24}$$

for all $\psi \in W^{1,2}_{E_\bullet}(\tau K)$ *and some* $g \in L^2(J; W^{-1,2}_{E_\bullet}(\tau K))$ *and almost all* $t \in J$*. Then the function* $J \ni t \mapsto \mathfrak{E}v$ *is from* $W^{1,2}(J; W^{-1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})) \cap L^2(J; W^{1,2}_{\widehat{R}_\bullet}(\tau \widehat{K}))$ *and still has initial value 0. Finally, the function* $\mathfrak{E}v$ *satisfies*

$$\begin{aligned} \langle (\mathfrak{E}v)'(t), \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})} &+ \langle -\nabla \cdot \hat{\mu}(t,\cdot)\nabla(\mathfrak{E}v)(t) + (\mathfrak{E}v)(t), \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})} \\ &= \langle \mathfrak{S}g(t), \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})} \quad \text{for all } \varphi \in W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K}) \end{aligned} \tag{3.25}$$

for all almost all $t \in J$.

iii) *Assume* $s, q \geq 2$*. If* $g \in L^s(J; W^{-1,q}_{E_\bullet}(\tau K))$*, then* $\mathfrak{S}g \in L^s(J; W^{-1,q}_{\widehat{E}_\bullet}(\tau \widehat{K}))$*, and the norm of* $\mathfrak{S}g$ *is not larger than two times the norm of* g.

Proof. i) The assertion is obtained by the definitions of $\mathfrak{E}w$, $\mathfrak{S}h$, $\nabla \cdot \rho \nabla$, $\nabla \cdot \hat{\rho} \nabla$ and straightforward calculations, based on Proposition 3.15, when applied to the transformation $\mathrm{x} \mapsto \mathrm{x}_-$.

ii) The first two assertions follow from Lemmata 3.20 and 3.21; let us show that $\mathfrak{E}v$ indeed satisfies the correct equation: coming from (3.24),

$$\langle -\nabla \cdot \mu(t,\cdot)\nabla v(t) + v(t), \psi\rangle_{W^{1,2}_{E_\bullet}(K)} = \langle g(t), \psi\rangle_{W^{1,2}_{E_\bullet}(K)} - \langle v'(t), \psi\rangle_{W^{1,2}_{E_\bullet}(K)} \tag{3.26}$$

for $\psi \in W^{1,2}_{E_\bullet}(K)$ is for almost all $t \in J$ an equation of type (3.22). According to i), this leads to an equation

$$\begin{aligned} \langle -\nabla \cdot \widehat{\mu(t,\cdot)}\nabla(\mathfrak{E}v)(t) &+ (\mathfrak{E}v)(t), \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})} \\ &= \langle (\mathfrak{S}g)(t), \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})} - \langle (\mathfrak{S}v')(t), \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})}, \quad \varphi \in W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K}). \end{aligned} \tag{3.27}$$

Now one calculates for $\varphi \in W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})$

$$\begin{aligned} \langle (\mathfrak{S}v')(t), \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})} &= \langle v'(t), \varphi|_{\tau K} + \varphi_-|_{\tau K}\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})} \\ &= \frac{d}{dt}\int_{\tau K} v(t)(\varphi + \varphi_-)\, d\mathrm{x} \\ &= \frac{d}{dt}\int_{\tau \widehat{K}} (\mathfrak{E}v)(t)\, \varphi\, d\mathrm{x} = \langle (\mathfrak{E}v)'(t), \varphi\rangle_{W^{1,2}_{\widehat{E}_\bullet}(\tau \widehat{K})}, \end{aligned}$$

what gives the last assertion.

iii) The assertion follows immediately from Lemma 3.21 ii). $\square$

3.4. **The core of the proof.** Now we have all preparations at hand and will prove our main result, Theorem 2.13. The following lemma is the starting point for the usage of the foregoing results.

Lemma 3.23. *Let $\mathfrak{B}$ be the unit ball in $L^s(J;W_F^{-1,q}(\Omega))$. Then, for every $f\in\mathfrak{B}$, the solution $u=u_f$ of* (2.4)/(2.7) *is contained in a ball B around 0 in $V_2^{1,0}(\Omega)$ with radius*

$$r_V := \left(\frac{1}{\varkappa}+\sqrt{\frac{1}{\varkappa}}\right)|\Omega|^{\frac{q-2}{2q}}|J|^{\frac{s-2}{2s}},$$

here $\varkappa=\min(\kappa_0,1)$ being the (uniform) coercivity constant of the forms

$$W_D^{1,2}(\Omega)\times W_D^{1,2}(\Omega)\ni(\psi,\varphi)\mapsto\int_\Omega \mu(t,\cdot)\nabla\psi\cdot\nabla\varphi\,+\,\psi\varphi\,\mathrm{dx}.$$

Hence, for all coefficient functions μ admitting the same ellipticity constant κ_0, in particular all those from $\mathcal{M}(\kappa_0,\kappa_1)$, the radii r_V may be taken uniformly.

Proof. The unit ball $\mathfrak{B}$ is contained in the corresponding ball in $L^2(J;W_D^{-1,2}(\Omega))$ with radius $|\Omega|^{\frac{q-2}{2q}}|J|^{\frac{s-2}{2s}}$, cf. Remark 2.14. Then an application of Proposition 2.10, there using the triple $W_D^{-1,2}(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow W_D^{1,2}(\Omega)$, yields the desired result. □

We now proceed to construct a finite open covering of $\overline{\Omega}$ and to show uniform L^∞- and Hölder-bounds on the intersection of each of the covering sets with Ω. To this end, we localize the parabolic equation (2.9) with respect to a suitable neighbourhood of each point, transform the localized equations to such on the half cubes and reflect the problem to the whole cube, if necessary. This allows to use Corollaries 3.5 and 3.7, respectively, to deduce the wished-for estimates.

Choose for any point $\mathrm{x}\in\Omega$ a ball $B_\mathrm{x}^\bullet$ around x which satisfies $B_\mathrm{x}^\bullet\subset\Omega$ and has a positive distance to $\partial\Omega$. Define B_x as the ball with half the radius of $B_\mathrm{x}^\bullet$. Further, for every $\mathrm{y}\in\partial\Omega$, let U_y be an open neighbourhood of y which satisfies the conditions in Assumption 2.4. In case I of that assumption, we put $W_\mathrm{y}=\cap_j\phi_j^{-1}(\frac{\tau_j}{2}\widehat{K})$. If y fulfils case II of that assumption, then we put $W_\mathrm{y}=\phi_\mathrm{y}^{-1}\big(\frac{\tau_\mathrm{y}}{2}\widehat{K}\big)$, which implies $W_\mathrm{y}\cap\Omega=\phi_\mathrm{y}^{-1}\big(\frac{\tau_\mathrm{y}}{2}K\big)$. Obviously, the collection of the sets $\{B_\mathrm{x}\}_{\mathrm{x}\in\Omega}$ and $\{W_\mathrm{y}\}_{\mathrm{y}\in\partial\Omega}$ forms an open covering of $\overline{\Omega}$. Let $B_{\mathrm{x}_1},\dots,B_{\mathrm{x}_{m_0}},W_{\mathrm{y}_1},\dots,W_{\mathrm{y}_{m_1}}$ be a finite subcovering.

Before we continue, we need the following property of the sets W_y in case of Assumption 2.4 I):

Lemma 3.24. *In the situation of Assumption 2.4 I), with $W:=\bigcap_{j=1}^k\phi_j^{-1}(\frac{\tau_j}{2}\widehat{K})$ one has*

$$W\cap\Omega\subseteq\bigcup_{j=1}^k\phi_j^{-1}(\frac{\tau_j}{2}K),\tag{3.28}$$

the right hand side being a disjoint *union.*

Proof. Since $W\subset\bigcap_{j=1}^k U_j\subseteq\mathcal{U}$, we find

$$W\cap\Omega=W\cap\Omega\cap\mathcal{U}=W\cap\bigcup_{l=1}^k V_l=\bigcup_{j=1}^k(V_j\cap W)\subseteq\bigcup_{j=1}^k\Big(V_j\cap\phi_j^{-1}(\frac{\tau_j}{2}\widehat{K})\Big)$$
$$=\bigcup_{j=1}^k\Big(\phi_j^{-1}(\tau_jK)\cap\phi_j^{-1}(\frac{\tau_j}{2}\widehat{K})\Big)=\bigcup_{j=1}^k\phi_j^{-1}\Big(\tau_jK\cap\frac{\tau_j}{2}\widehat{K}\Big)=\bigcup_{j=1}^k\phi_j^{-1}(\frac{\tau_j}{2}K).\quad\square$$

Let $\mathfrak{B}$ be again the unit ball in $L^s(J;W_D^{-1,q}(\Omega))$, and let $\mathfrak{B}^{\mathrm{step}}$ denote the set of step functions in $\mathfrak{B}$.

Step 1: For every $f \in L^s(J;W_D^{-1,q}(\Omega)) \hookrightarrow L^2(J;W_D^{-1,2}(\Omega))$ a (unique) solution $u = u_f$ of (2.9) exists, cf. Proposition 2.10/Remark 2.11. The set of solutions $\{u_f : f \in \mathfrak{B}\}$ is bounded in $V_2^{1,0}(J \times \Omega)$, and the bound in this space can be taken uniformly with respect to all coefficient functions $\mu \in \mathcal{M}_n(\kappa_0,\kappa_1)$, cf. Lemma 3.23.

Step 2: We consider the restricted problem on each of the balls $B^\bullet_{\mathrm{x}_l}$ according to Ch. 3.3 (there setting $U = B^\bullet_{\mathrm{x}_l}$), cf. (3.13), where the right-hand side f_U in the restricted problem is still bounded by 1 for $f \in \mathfrak{B}$. For $f \in \mathfrak{B}^{\mathrm{step}}$, however, the solution u_f is a generalized solution of a corresponding generalized problem on $B^\bullet_{\mathrm{x}_l}$ with right-hand side $\mathfrak{f}_U$, cf. Proposition 3.9, $\mathfrak{f}_U$ still being a step function in time and contained in the ball with radius 2 in $L^s(J;L^q(B_{\mathrm{x}_l};\mathbb{R}^{n+1}))$. Thanks to Corollary 3.5, the functions $u_f|_{J\times B_{\mathrm{x}_l}}$ are essentially bounded, and the norms $\|u|_{J\times B_{\mathrm{x}_l}}\|_{L^\infty(J\times B_{\mathrm{x}_l})}$ are bounded uniformly in $f \in \mathfrak{B}^{\mathrm{step}}$ and in $\mu \in \mathcal{M}_n(\kappa_0,\kappa_1)$. This of course implies uniform boundedness for all $l \in \{1,\dots,m_0\}$.

Step 3: Let us now consider the boundary points, thereby temporarily fixing $\mathrm{y} = \mathrm{y}_l \in \partial\Omega$.
We start with case I) of Assumption 2.4): Intersecting Ω with U_{y}, the restriction of the function $u = u_f$ to each of the connected components V_j belongs to $W^{1,2}_{R_j}(V_j)$ when taking R_j as $\overline{\partial V_j \cap U_j}$, cf. Lemma 3.11. One obtains a restricted problem on V_j which is of the same quality as (2.9), cf. (3.13) with $\Lambda = V_j$ and $E = \partial V_j$. Further, we transform this resulting problem to a problem for the function $v_j := u|_{V_j} \circ \phi_j^{-1}$ on $\tau_j K$. According to (3.18)/(3.19), one ends up with an equation for the transformed function v_j on $\tau_j K$ with new right-hand side $g_j \in L^s(J;W_0^{-1,q}(\tau_j K))$, which is still a step function in time. By Proposition 3.9, v_j is then a generalized solution of the transformed equation (3.18) on $\tau_j K$ with right-hand side $\mathfrak{g}_j \in L^s(J;L^q(\tau_j K;\mathbb{R}^{n+1}))$ and coefficient function μ_{ϕ_j}. This is the setting for all $j \in \{1,\dots,k\}$. Let us show that we are in the situation to use Corollary 3.5 for each problem on V_j.

- The new right-hand sides $\mathfrak{g}_j$ may be estimated suitably with respect to the original ones, cf. (3.17) and Proposition 3.9, giving
$$\|\mathfrak{g}_j\|_{L^s(J;L^q(\tau_j K;\mathbb{R}^{n+1}))} \le 2\|g_j\|_{L^s(J;W_0^{-1,q}(\tau_j K))} \le 2c_j.$$
- The resulting transformed coefficient functions μ_{ϕ_j} on $J \times \tau_j K$ still admit uniform upper bounds $\hat\kappa_{1,j}$, and uniform ellipticity constants $\hat\kappa_{0,j}$, cf. Proposition 3.15 v).
- Moreover, it is clear that $\|u|_{J\times V_j}\|_{V_2^{1,0}(J\times V_j)}$ is not larger than $\|u\|_{V_2^{1,0}(J\times\Omega)}$, which was uniformly bounded over $\mathcal{M}_n(\kappa_0,\kappa_1)$ and with respect to $f \in \mathfrak{B}$ by the constant r_V thanks to Lemma 3.23. Proposition 3.15 ii) shows that $\|v_j\|_{V_2^{1,0}(J\times\tau_j K)}$ may be estimated by $\tilde c_j r_V$ for some constant $\tilde c_j$ depending on j via ϕ_j.
- By Remarks 3.18 and 3.19, we have $v_j \in L^2(J;W^{1,2}_{\tau_j\overline{\Sigma}}(\tau_j K))$.

Summing up, we have, for each j, coefficient functions from $\mathcal{M}_n(\hat\kappa_{0,j},\hat\kappa_{1,j})$ and right-hand sides $\mathfrak{g}_j$ contained in the $2c_j$-ball around 0 in $L^s(J;L^q(\tau_j K;\mathbb{R}^{n+1}))$ such that the generalized solutions v_j to all those right-hand sides are in turn contained in a ball with radius $\tilde c_j r_V$ in $V_2^{1,0}(J\times\tau_j K)$ and even belong to $L^2(J;W^{1,2}_{\tau_j\overline{\Sigma}}(\tau_j K))$.

Applying Corollary 3.5 ii) with the subdomain $\frac{\tau_j}{2}K$, we get L^∞-bounds on $J \times \frac{\tau_j}{2}K$ for every v_j which are uniform in $f \in \mathfrak{B}^{\text{step}}$ and $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$. Thanks to (3.28), this gives L^∞-bounds for u on $J \times (W_{\mathrm{y}} \cap \Omega)$, independent of $f \in \mathfrak{B}^{\text{step}}$ and $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$.

Next we will consider the case II) in Assumption 2.4. We abbreviate $\tau_{\mathrm{y}} =: \tau$. Localizing around y with respect to U_{y} according to Ch. 3.3 results in a problem for $u|_\Lambda$ in the form (3.13) with $\Lambda = U_{\mathrm{y}} \cap \Omega$ and $E = \partial\Lambda \setminus (N \cap U_{\mathrm{y}})$. By afterwards transforming the resulting problem via $\zeta = \phi_{\mathrm{y}}$ (case IIa)) and $\zeta = \varsigma_n \circ \phi_{\mathrm{y}}$ (case IIb)), one again ends up with a problem on τK as in (3.18), which we interpret as a generalized problem solved by the function $v = u|_\Lambda \circ \zeta$. We obtain analogous estimates and bounds, especially uniformly in $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$ and $f \in \mathfrak{B}^{\text{step}}$, for the coefficient function $\mu_{\phi_{\mathrm{y}}}$, right-hand side $\mathfrak{g}$ and solution $v \in V_2^{1,0}(J \times \tau K)$ as we did for each j in the previously handled case I). The following considerations require further distinguishing the assumptions.

In case IIa) of Assumption 2.4, we get $v \in L^2(J; W^{1,2}(\tau K))$ according to Remark 3.19 and (3.20). Here, the upper plate $\tau\Sigma$ is *disjoint* to the (transformed) Dirichlet boundary part (which in fact is even empty here, cf. (3.20)), permitting the direct application of Corollary 3.5 for a neighbourhood of $\phi_{\mathrm{y}}(\mathrm{y}) = 0$, since the latter is obviously also a boundary point of τK. However, we may reflect the problem across $\tau\Sigma$ according to Lemma 3.22, thus obtaining the corresponding equation (3.25) on $\tau\widehat{K}$ for the symmetrically reflected function $\mathfrak{E}v$. It is clear that the bounds for the data and the $V_2^{1,0}$-estimate for v carry over to $\tau\hat{K}$ in a straight forward manner, cf. Lemma 3.21 and the definition of the reflection operator $\mathfrak{E}$, and that $\phi_{\mathrm{y}}(\mathrm{y}) = 0 \in \tau\widehat{K}$ is an *interior* point in τK. Hence, we may apply Corollary 3.5 i) for the subdomain $\frac{\tau}{2}\widehat{K}$ and obtain an L^∞-bound for $\mathfrak{E}v$ on $J \times \frac{\tau}{2}\widehat{K}$, again uniformly in $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$ and $f \in \mathfrak{B}^{\text{step}}$. Obviously, this implies an L^∞-bound with the same property for u on $J \times (W_{\mathrm{y}} \cap \Omega) = J \times \phi^{-1}(\frac{\tau}{2}K)$.

In case IIb) of Assumption 2.4, where y sits at the boundary between Neumann- and Dirichlet boundary parts, Remark 3.19 and (3.21) give $v \in L^2(J; W^{1,2}_{R_\bullet}(\tau K))$ with $R_\bullet = [-\tau, \tau]^{n-2} \times \{-\tau\} \times [-\tau, 0] \subset \partial(\tau K)$ and $\zeta(\mathrm{y}) = (0, \dots, 0, -\tau, 0)$. Since this point is *not* an interior one of the Dirichlet boundary part $R_\bullet$, we also reflect this problem across $\tau\Sigma$ and, again, end up with a corresponding parabolic equation for the symmetrically extended function $\mathfrak{E}v$ on the set $\tau\widehat{K}$. The Dirichlet part of the extended solution $\mathfrak{E}v$ now equals $\widehat{R_\bullet} = [-\tau, \tau] \times \{-\tau\} \times [-\tau, \tau] \subset \partial(\tau\widehat{K})$, cf. Lemma 3.20. Recalling Lemma 3.13, we had

$$\tau\underline{K} := \varsigma_n\left(\frac{\tau}{2}K\right) = \left]-\frac{\tau}{2}, \frac{\tau}{2}\right[^{n-2} \times \left]-\tau, 0\right[\times \left]-\frac{\tau}{2}, 0\right[$$

and one observes that $\tau\underline{K}$ has the distance $\frac{\tau}{2}$ to the set

$$\partial(\tau\widehat{K}) \setminus \widehat{R_\bullet} = \partial(\tau\widehat{K}) \setminus \left([-\tau, \tau]^{n-2} \times \{-\tau\} \times [-\tau, \tau]\right).$$

Another application of Corollary 3.5 ii), this time for the subdomain $\tau\underline{K}$, gives an L^∞-bound for v on $J \times \tau\underline{K}$, and, correspondingly, on $J \times \zeta^{-1}(\tau\underline{K}) = J \times \phi_{\mathrm{y}}^{-1}(\frac{\tau}{2}K) = J \times (W_{\mathrm{y}} \cap \Omega)$, which again does not depend on $\mathfrak{f} \in \mathfrak{B}$, but only on κ_0, κ_1.

Hence we have L^∞-bounds on $J \times W_{\mathrm{y}_l} \cap \Omega$ for each $l \in \{1, \dots, m_1\}$ which then clearly implies L^∞-bounds uniform in l. Since the finite system $B_{\mathrm{x}_1}, \dots, B_{\mathrm{x}_{m_0}}, W_{\mathrm{y}_1} \cap \Omega, \dots, W_{\mathrm{y}_{m_1}} \cap \Omega$ is an open covering of Ω, this altogether gives L^∞-bounds on the whole set $J \times \Omega$, which are uniform for all $f \in \mathfrak{B}^{\text{step}}$ for the corresponding functions u_f and which do only depend on the constants κ_0, κ_1. This was the first point of

Theorem 2.13.

Step 4: Having the essential boundedness at hand, we will now establish the Hölder estimates by essentially re-iterating the considerations in the foregoing steps, this time investing the obtained uniform *global* L^∞-bounds instead of the $V_2^{1,0}$-estimates and then applying Corollary 3.7 instead of Corollary 3.5.

In detail: Both Step 2, which was the case of the balls B_{x_l}, and the considerations in case II) of Assumption 2.4 in Step 3 work exactly as above, using Corollary 3.7 this time. In case I) of Step 3, the situation is a bit more complicated and needs more care: Repeating the procedure outlined above to the point where Lemma 3.24 and (3.28) are used, one obtains the Hölder property for every transformed local solution v_j (including estimates uniform in $f \in \mathfrak{B}^{\mathrm{step}}$, depending only on κ_0, κ_1) on the set $J \times \frac{\tau_j}{2} K$ for each $j \in \{1, \dots, k\}$. Due to the disjoint union in (3.28), u can be represented as $u = \sum_{j=1}^{k} v_j \circ \phi_j$ on $J \times (\Omega \cap W_y)$. It is essential to observe, however, that this implies only Hölder continuity for u on each of the disjoint sets $J \times \phi_j^{-1}(\frac{\tau_j}{2} K) \subset J \times V_j$ on its own – it is not (yet) clear why the Hölder property should hold "across" different connected components. Let us note that this is exactly the result of Ladyshenskaya in [39]. In the sequel we will show that our setting allows to derive from this the required *global* Hölder estimates on the sets $J \times (\Omega \cap W_\mathrm{y})$.

Let us in the following identify the Hölderian function v_j, defined on $J \times \frac{\tau_j}{2} K$, with its unique Hölderian extension on $\overline{J} \times \frac{\tau_j}{2} \overline{K}$, cf. Remark 2.8. The crucial point is here that we imposed in our general ansatz a very special boundary value on the whole Dirichlet part D of the boundary – namely, 0. Indeed, the property $v_j \in L^2(J; W^{1,2}_{\tau_j \Sigma}(\tau_j K))$ implies that $v_j(t, \cdot)$ has trace 0 on $\frac{\tau_j}{2} \overline{\Sigma}$, i.e., vanishes there almost everywhere with respect to the boundary measure $\mathcal{H}_{n-1}$ for almost all $t \in J$, see Remark 3.4. However, $v_j(t, \cdot)$ is also a continuous function on $\frac{\tau_j}{2} \overline{K}$, and $\frac{\tau_j}{2} \overline{K}$ has a Lipschitz-boundary around 0, hence in fact $v_j(t, \cdot) \equiv 0$ on $\frac{\tau_j}{2} \overline{\Sigma}$ for almost all $t \in J$. But then, this time due to continuity in time, v_j must be identically 0 on the *whole* $\overline{J} \times \frac{\tau_j}{2} \overline{\Sigma}$. It is straight forward to verify that the continuation $\hat{v}_j$ of v_j to $J \times \frac{\tau_j}{2} \widehat{K}$ by zero is also Hölder continuous – with the same Hölder-norm as v_j on $J \times \frac{\tau_j}{2} \overline{K}$. This means we may extend u via $\hat{u} := \sum_j \hat{v}_j \circ \phi_j$ to the set $J \times W_\mathrm{y}$ (which indeed is an extension of $u = \sum_j v_j \circ \phi_j$ due to $\hat{v}_j = v_j$ and $\hat{v}_i = 0$ on $\phi_j^{-1}(\frac{\tau_j}{2} K)$ for $i \neq j$) and obtain a Hölder-continuous function, such that $u = \hat{u}|_{W_\mathrm{y} \cap \Omega}$ is also Hölderian on $W_\mathrm{y} \cap \Omega$ with the same estimates.

Let us inspect the corresponding Hölder bounds in some more detail: Let $t_1, t_2 \in J$ and $\mathrm{z}_1, \mathrm{z}_2$ be from two different connected components of $W_\mathrm{y} \cap \Omega$, that is, $\mathrm{z}_1 \in \phi_j^{-1}(\frac{\tau_j}{2} K) \cap W_\mathrm{y}$ and $\mathrm{z}_2 \in \phi_i^{-1}(\frac{\tau_i}{2} K) \cap W_\mathrm{y}$ for $j \neq i$ and let α_j, α_i be the degree of Hölder continuity of $\hat{v}_j$ and $\hat{v}_i$ on $J \times \frac{\tau_j}{2} \widehat{K}$ and $J \times \frac{\tau_i}{2} \widehat{K}$, respectively. We write

$$
\begin{aligned}
|u(t_1, \mathrm{z}_1) - u(t_2, \mathrm{z}_1)| &= |v_j(t_1, \phi_j(\mathrm{z}_1)) - v_i(t_2, \phi_i(\mathrm{z}_2))| \\
&= |\hat{v}_j(t_1, \phi_j(\mathrm{z}_1)) - \hat{v}_j(t_2, \phi_j(\mathrm{z}_2)) + \hat{v}_i(t_1, \phi_i(\mathrm{z}_1)) - \hat{v}_i(t_2, \phi_i(\mathrm{z}_2))| \\
&\leq (1 \vee \mathfrak{l}_{\phi_j}) |\hat{v}_j|_{\alpha_j} \|(t_1, \mathrm{z}_1) - (t_2, \mathrm{z}_2)\|^{\alpha_j} \\
&\qquad + (1 \vee \mathfrak{l}_{\phi_i}) |\hat{v}_i|_{\alpha_i} \|(t_1, \mathrm{z}_1) - (t_2, \mathrm{z}_2)\|^{\alpha_i}
\end{aligned}
$$

since $\hat{v}_j(t,\phi_j(\mathrm{z}_2)) = \hat{v}_i(t,\phi_i(\mathrm{z}_1)) = 0$ for all $t \in J$. This shows that the Hölder seminorm of u may be estimated as follows, using $\alpha^* = \min_{j\in\{1,\dots,k\}} \alpha_j = \alpha_{j^*}$:

$$|u|_{\alpha^*} \le \max_{j\in\{1,\dots,k\}} \Big((1 \vee \mathfrak{l}_{\phi_j})|\hat{v}_j|_{\alpha_j} \operatorname{diam}(J \times W_{\mathrm{y}} \cap \Omega)^{\alpha_j-\alpha^*} + (1 \vee \mathfrak{l}_{\phi_{j^*}})|\hat{v}_{j^*}|_{\alpha^*} \Big).$$

In particular, the Hölder seminorm estimate does not depend on all k of connected components of $W_{\mathrm{y}} \cap \Omega$ but only on two of those at once.

Now we have achieved the following: There exist constants $\alpha(\mathrm{x}_1), \dots, \alpha(\mathrm{x}_{m_0})$ and $\alpha(\mathrm{y}_1), \dots, \alpha(\mathrm{y}_{m_1})$, such that

$$\sup_{f\in\mathfrak{B}^{\mathrm{step}}} \|u_f\|_{C^{\alpha(\mathrm{x}_i)}(J\times B_{\mathrm{x}_i})} < \infty, \quad \text{and} \quad \sup_{f\in\mathfrak{B}^{\mathrm{step}}} \|u_f\|_{C^{\alpha(\mathrm{y}_l)}(J\times (W_{\mathrm{y}_l}\cap\Omega))} < \infty \tag{3.29}$$

for each $i \in \{1, \dots, m_0\}$ and $l \in \{1, \dots, m_1\}$, and these suprema are even uniform for all coefficient functions $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$. Diminishing the $\alpha(\mathrm{x}_i)$ and $\alpha(\mathrm{y}_l)$ in (3.29) to their common minimum, called α, (3.29) certainly remains true and we have Hölder-continuity of degree α on each of the sets $B_{\mathrm{x}_1}, \dots, B_{\mathrm{x}_{m_0}}, W_{\mathrm{y}_1} \cap \Omega, \dots, W_{\mathrm{y}_{m_1}} \cap \Omega$.

Step 5: In order to deduce *global* Hölder continuity from the previous considerations, we need the following

Lemma 3.25. *There exists an $\varepsilon > 0$ such that, for every $\mathrm{x} \in \Omega$, the balls in Ω with center x and radius not larger than ε lie completely in at least one of the sets B_{x_i} or W_{y_l}.*

Proof. Consider the function

$$\overline{\Omega} \ni \mathrm{y} \mapsto \varepsilon(\mathrm{y}) := \frac{1}{m_0 + m_1} \Big(\sum_{i=1}^{m_0} \operatorname{dist}(\mathrm{y}, \mathbb{R}^n \setminus B_{\mathrm{x}_i}) + \sum_{l=1}^{m_1} \operatorname{dist}(\mathrm{y}, \mathbb{R}^n \setminus W(\mathrm{y}_l)) \Big).$$

This function is continuous and strictly positive, since every $\mathrm{y} \in \overline{\Omega}$ is contained in at least one of the sets B_{x_i} or W_{y_l}. Therefore, it has to attain its minimum, say, $\varepsilon > 0$. Then it is straight forward to see that this ε fulfills the asserted condition, since at least one summand in the definition has to be bigger or equal to $\varepsilon(\mathrm{y})$ for each $\mathrm{y} \in \overline{\Omega}$. □

Now Lemma 3.25 in combination with Remark 2.8 iii) allows to fall back to the sets B_{x_i} and $W_{\mathrm{y}_l} \cap \Omega$ and thus implies *global* Hölder bounds on $J \times \Omega$, and this uniformly in $f \in \mathfrak{B}^{\mathrm{step}}$ and in $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$.

Step 6: The previous considerations show that, for each $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$, the linear mapping $(\partial_t + \mathcal{A}(\mu))^{-1}$ maps bounded sets in $L^s_{\mathrm{step}}(J; W^{-1,q}_D(\Omega))$ into bounded set in the space $C^\alpha(J \times \Omega)$, the bounds being uniform in κ_0, κ_1. Consequently, these mappings are equicontinuous with respect to $\mu \in \mathcal{M}_n(\kappa_0, \kappa_1)$ as mappings from $L^s_{\mathrm{step}}(J; W^{-1,q}_D(\Omega))$ into the space $C^\alpha(J \times \Omega)$. Since $L^s_{\mathrm{step}}(J; W^{-1,q}_D(\Omega))$ is dense in $L^s(J; W^{-1,q}_D(\Omega))$, they hence possess extensions to the whole $L^s(J; W^{-1,q}_D(\Omega))$ which are still equicontinuous. This was the claim in Theorem 2.13.

4. **Nonzero initial values and inhomogeneous Dirichlet boundary data.** Up to now, the fundamental difference between the approach in [39] and ours consists in the fact that here only the zero Dirichlet datum is allowed, which allowed to deduce *global* Hölder continuity for the solution (it is clear that also constant

nonzero data is admissible by obvious modifications). In this chapter we will show a way how to admit (nonconstant) nonzero Dirichlet data – without losing the classical Hölder property for the solution. We restrict ourselves to the case where the Dirichlet datum does *not* depend on time. Moreover, aiming at Hölder continuity for the solution in both time and space, it is clear that the initial value must admit the correct boundary behaviour. In particular, in this context one can never expect that a solution with initial value 0 admits a nonzero Dirichlet datum.

We start with the introduction of the fundamental property for this chapter. Recall that we denote the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}_{n-1}$.

Definition 4.1. We call a set $\mathcal{O} \subset \mathbb{R}^n$ a $(n-1)$-*set* if there exist constants $c_0, c_1 > 0$ such that for all $r \in \,]0,1[$ the inequality

$$c_0 r^{n-1} \leq \mathcal{H}_{n-1}(B(\mathrm{z}, r) \cap \mathcal{O}) \leq c_1 r^{n-1}, \quad \mathrm{z} \in \mathcal{O} \tag{4.1}$$

holds true, [37, Ch. II.1 and VIII.1].

Remark 4.2. i) It is clear that any finite union of $(n-1)$-sets is again a $(n-1)$-set.

ii) If $\mathcal{O} \subset \mathbb{R}^n$ is a $(n-1)$-set and ϕ is a bi-Lipschitzian mapping from a neighbourhood U of $\overline{\mathcal{O}}$ into $\mathbb{R}^n$, then $\phi(\mathcal{O})$ also is a a $(n-1)$-set, cf. [18, Ch. 2.4.1], see also [34, Ch. 3.1].

Having this at hand, we can prove our first preparatory lemma.

Theorem 4.3. *Let Ω and D satisfy Assumption 2.4. Then D is a $(n-1)$-set.*

Proof. Consider for each $\mathrm{z} \in D \setminus \overline{N}$ the domain $\mathcal{U}_{\mathrm{y}}$, the neighbourhoods $U_{\mathrm{z},1}, \ldots, U_{\mathrm{z},k}$ of the connected components $V_{\mathrm{z},1}, \ldots, V_{\mathrm{z},k}$ and $\phi_{\mathrm{z},1}, \ldots, \phi_{\mathrm{z},k}$, the bi-Lipschitz mappings from Assumption 2.4 I). For $\mathrm{z} \in \partial D$ we collect the bi-Lipschitz mapping ϕ_{z} and the neighbourhood U_{z} of z from case II) of Assumption 2.4.

For $\mathrm{z} \in D \setminus \overline{N}$ we define another neighbourhood W_{z} as follows: Let $\check{\tau}_{\mathrm{z}} \in \,]0, \tau_{\mathrm{z},1}[$ be a number such that

$$\phi_{\mathrm{z},1}^{-1}(\check{\tau}_{\mathrm{z}} \widehat{K}) \subseteq \bigcap_{j=2}^{k} U_{\mathrm{z},j} = \bigcap_{j=2}^{k} \phi_{\mathrm{z},j}^{-1}(\tau_{\mathrm{z},j} \widehat{K})$$

and define $W_{\mathrm{z}} := \phi_{\mathrm{z},1}^{-1}(\check{\tau}_{\mathrm{z}} \widehat{K})$ (this is well-defined since each $U_{\mathrm{z},j}$ is an open neighbourhood of z). Then the systems $\{U_{\mathrm{y}}\}_{\mathrm{y} \in \partial D}$ and $\{W_{\mathrm{z}}\}_{\mathrm{z} \in D \setminus \overline{N}}$ form an open covering of D from which we choose a finite subcovering $U_{\mathrm{y}_1}, \ldots, U_{\mathrm{y}_{m_2}}, W_{\mathrm{z}_1}, \ldots, W_{\mathrm{z}_{m_3}}$, which allows to write D in the form $D = \bigcup_{l=1}^{m_2} (U_{\mathrm{y}_l} \cap D) \cup \bigcup_{l=1}^{m_3} (W_{\mathrm{z}_l} \cap D)$. Thanks to the foregoing Remark 4.2, one has to show only that each of the sets $U_{\mathrm{y}_l} \cap D$ and $W_{\mathrm{z}_l} \cap D$ is a $(n-1)$-set. For the sets $D \cap U_{\mathrm{y}_l}$ this is immediate by Remark 4.2 ii) and the supposition on the mappings ϕ_{y_l}. For the sets W_{y_l} one has

$$D \cap W_{\mathrm{z}_l} = \bigcup_{j=1}^{k} \partial V_{\mathrm{z}_l,j} \cap W_{\mathrm{z}_l} \subseteq \bigcup_{j=1}^{k} \partial V_{\mathrm{z}_l,j} \cap V_{\mathrm{z}_l,j} \subseteq \bigcup_{j=1}^{k} \partial V_{\mathrm{z}_l,j} \cap \mathcal{U}_{\mathrm{y}_l}.$$

Let us now consider the terms $\partial V_{\mathrm{z}_l,j} \cap W_{\mathrm{z}_l}$, $j = 1, \ldots, k$ separately. From the definition of W_{z_l} it is clear that $\partial V_{\mathrm{z}_l,1} \cap W_{\mathrm{z}_l}$ is mapped by the bi-Lipschitzian transformation $\phi_{\mathrm{z}_l,1}$ onto the set $\check{\tau}\Sigma$. Thus, $\partial V_{\mathrm{z}_l,1} \cap W_{\mathrm{z}_1}$ is a $(n-1)$-set, thanks to Remark 4.2. This already assures the *lower* bound in (4.1) for the whole set $D \cap W_{\mathrm{z}_l}$. On the other hand, from the definition of W_{z_l} it follows that $\partial V_{\mathrm{z}_l,j} \cap W_{\mathrm{z}_l}$ is mapped by the bi-Lipschitzian mapping $\phi_{\mathrm{z}_l,j}$ onto a subset of $\tau_{\mathrm{z}_l,j}\Sigma$. Since $\tau_{\mathrm{z},j}\Sigma$ admits the upper bound in (4.1), its subset $\phi_{z_l,j}(\partial V_{z_l,j} \cap W_{z_l})$ surely also does so. Finally, the upper bound for $\partial V_{z_l,j} \cap W_{z_l}$ itself again follows from Remark 4.2. Hence, each of the sets $U_{y_l} \cap D$ and $W_{z_l} \cap D$ are a $(n-1)$-set, making D also a $(n-1)$-set. $\square$

First, let us, for a $(n-1)$-set $M \subset \mathbb{R}^n$, denote by $B^{p,p}_\beta(M)$ the usual Besov space on M with $p \in]1,\infty[$ and $\beta \in]0,1[$, cf. [37, Ch. V.1.1.]. Using the just established fact that D is an $(n-1)$-set (Lemma 4.3), we obtain the following characterization of traces for $W^{1,q}(\Omega)$-functions where $q \in]1,\infty[$ ([8, Thm. 5.1]):

Proposition 4.4. *There exists a linear, continuous "trace" operator* tr_D*, which maps* $W^{1,q}(\Omega)$ *onto* $B^{q,q}_{1-\frac{1}{q}}(D)$*. In particular, there is a linear continuous extension operator* $\mathfrak{E}_D$ *from* $B^{q,q}_{1-\frac{1}{q}}(D)$ *into* $W^{1,q}(\Omega)$*, such that* $\mathrm{tr}_D\,\mathfrak{E}_D$ *is the identity on* $B^{q,q}_{1-\frac{1}{q}}(D)$*.*

Remark 4.5. Note that for $q > n$, $W^{1,q}(\Omega)$-functions are Hölder continuous on Ω and thus in fact continuous up to the boundary of Ω, cf. Remark 2.8. So for $\psi \in W^{1,q}(\Omega)$, the pointwise restriction $\psi|_D$ is meaningful and indeed coincides with $\mathrm{tr}_D\,\psi$. We will use the notion $\psi|_D$ in the following.

Secondly, for $q \in [2,\infty[$, we define the operator $-\nabla\cdot\rho\nabla + 1 : W^{1,q}(\Omega) \to W^{-1,q}_D(\Omega)$ by

$$\langle\big(-\nabla\cdot\rho\nabla+1\big)\psi,\varphi\rangle_{W^{1,q'}_D(\Omega)} := \int_\Omega \rho\nabla\psi\cdot\nabla\varphi + \psi\varphi\,\mathrm{d}x, \quad \psi \in W^{1,q}(\Omega),\ \varphi \in W^{1,q'}_D(\Omega), \tag{4.2}$$

thereby extending Definition 2.9.

Let us now define the notion of a solution of a problem with inhomogeneuos Dirichlet-data:

Definition 4.6. Assume $q > n$ and let $u_0 \in W^{1,q}(\Omega)$ admit the D-trace ι, i.e., $u_0|_D = \iota$. Then we say that for $g \in L^2(J; W^{-1,2}_D(\Omega))$ the function $w = u + u_0 \in L^2(J; W^{1,2}(\Omega)) \cap W^{1,2}(J; W^{-1,2}_D(\Omega)))$ is a solution of the equation

$$w'(t) - \nabla\cdot\mu(t,\cdot)\nabla w(t) + w(t) = g(t), \qquad w(t)|_D = \iota, \quad u(T_0) = u_0 \tag{4.3}$$

if u satisfies (2.9) with $f(t) := g(t) + \nabla\cdot\mu(t,\cdot)\nabla u_0 - u_0$. Here, the divergence operators are meant as in (4.2).

Theorem 4.7. *Adopt Assumption 2.4 and suppose* $q > n$ *and* $s > 2(1-\frac{n}{q})^{-1}$*. Let* $\iota \in B^{q,q}_{1-\frac{1}{q}}$*. Assume that* $u_0 \in W^{1,q}(\Omega)$ *satisfies* $u_0|_D = \iota$ *and let* g *be from* $L^s(J; W^{-1,q}_D(\Omega))$*. Then (4.3) admits exactly one solution* w*, and this solution is even Hölder continuous in space and time. The Hölder norm of* w *is uniformly bounded within the class* $\mu \in \mathcal{M}_n(\kappa_0,\kappa_1)$ *for fixed* g*.*

Proof. Thanks to the assumption $u_0 \in W^{1,q}(\Omega)$, the function $t \mapsto \nabla\cdot\mu(t,\cdot)\nabla u_0 - u_0$ belongs to *any* space $L^s(J; W^{-1,q}_D(\Omega))$. Thus, by Theorem 2.13 there is exactly one solution u of the corresponding equation

$$w'(t) - \nabla\cdot\mu(t,\cdot)\nabla w(t) + w(t) = g(t) + \nabla\cdot\mu(t,\cdot)\nabla u_0 - u_0 \quad u(T_0) = 0$$

which is even Hölder continuous. Since u_0 is from $W^{1,q}(\Omega)$ with $q > n$, it is in particular Hölder continuous, hence $w = u + u_0$ also is. $\square$

Remark 4.8. Following the strategy to split off the initial value requires u_0 to be in the domain of $-\nabla \cdot \mu(t,\cdot)\nabla + 1$ for each $t \in J$, which in general is only to be achieved if $u_0 \in W^{1,q}(\Omega)$, cf. Definition 2.9. Hence, in view of Proposition 4.4, the space $B^{q,q}_{1-\frac{1}{q}}(D)$ for the boundary values on D is exactly the "optimal" one.

5. **Global solvability of a non-linear heat equation and optimal regularity for the solution.** We show an application of the results established in the foregoing chapters. More specifically, we employ Theorem 2.13 to establish unique *global* existence of a solution to a quasilinear equation in divergence-form. We fix the following assumptions on the (nonlinear) forcing terms in the following problem:

Assumption 5.1. *The function $\mathcal{F} : J \times C(\overline{\Omega}) \to W^{-1,q}_D(\Omega)$ is a Caratheodory function and, for $s \in]1,\infty[$, such that the superposition operator $w \mapsto [t \mapsto \mathcal{F}(t,w(t))]$ is continuous from every bounded subset of $C(\overline{J \times \Omega})$ to $L^s(J; W^{-1,q}_D(\Omega))$, with $\sup_{w \in C(\overline{\Omega})} \|\mathcal{F}(\cdot,w)\|_{L^s(J;W^{-1,q}_D(\Omega))}$ being bounded by a constant $C_{\mathcal{F}} < \infty$.*

Remark 5.2. Assumption 5.1 is satisfied for a Caratheodory function $\mathcal{F}$ if the boundedness assumption holds true and for every $R > 0$ there exists a function $L_R \in L^s(J; W^{-1,q}_D(\Omega))$ such that

$$\|\mathcal{F}(t,w_1) - \mathcal{F}(t,w_2)\|_{W^{-1,q}_D(\Omega)} \le L_R(t)\|w_1 - w_2\|_{C(\overline{\Omega})} \quad \text{for almost all } t \in J,$$

where $w_1, w_2 \in C(\overline{\Omega})$ with $\|w_1\|_{C(\overline{\Omega})}, \|w_2\|_{C(\overline{\Omega})} \le R$.

5.1. **A quasilinear heat-equation with optimal regularity for the solution.** Although we first have to introduce some auxiliary results for its proof (which, however, are of their own interest), this is the result:

Theorem 5.3. *Put $n = 3$ and adopt Assumption 2.4. Let ρ be a measurable coefficient function on Ω with values in $M_n(\kappa_0,\kappa_1)$. Assume that $\phi : \mathbb{R} \to [\underline{\phi}, \overline{\phi}]$, where $0 < \underline{\phi} \le \overline{\phi}$, is Lipschitz continuous on bounded sets. Suppose further that, for some $q > n$,*

$$-\nabla \cdot \rho\nabla + 1 : W^{1,q}_D(\Omega) \to W^{-1,q}_D(\Omega)$$

is a topological isomorphism. Let $s > 2(1-\frac{n}{q})^{-1}$ and $w_0 \in (W^{1,q}_D(\Omega), W^{-1,q}_D(\Omega))_{\frac{1}{s},s}$. Let moreover $\mathcal{F} : J \times C(\overline{\Omega}) \to W^{-1,q}_D(\Omega)$ satisfy the Assumption 5.1 for this s. Then there exists a global *solution $w \in W^{1,s}(J; W^{-1,q}_D(\Omega)) \cap L^s(J; W^{1,q}_D(\Omega))$ of the quasilinear equation*

$$w'(t) - \nabla \cdot \phi(w(t))\rho\nabla w(t) + w(t) = \mathcal{F}(t,w(t)), \quad w(T_0) = w_0. \tag{5.1}$$

If $\mathcal{F}$ even satisfies the assumptions in Remark 5.2, this solution is unique.

Let us first compare Theorem 5.3 with other well-known general existence- and uniqueness theorems for quasilinear equations such as [42, Thm 3.1], which allow for more general data but yield only *local* solutions. The trade-off we make for global solutions, at this point, is twofold: First, we restrict ourselves to divergence-type operators, and secondly the requirements for the (nonlinear) inhomogeneity are stricter – we have to require uniform boundedness over $C(\overline{\Omega})$ and a slightly stronger Lipschitz condition. However, we emphasize that even for right-hand sides not depending on the function itself, e.g. [42, Thm. 3.1] does not yield global solutions, while our theorem/proof nearly immediately does, cf. Corollary 5.8. Moreover, we have the requirement of space dimension $n = 3$, whose necessity is a bit hidden: it is needed to guarantee uniformity of the domains of each of the operators $-\nabla \cdot \phi(w)\rho\nabla$

for varying w – which in turn is a common assumption – by using invariance under pertubation by continuous functions of the assumed isomorphism property of $-\nabla \cdot \rho\nabla$, which is only available for space dimension up to 3.

For the proof of Theorem 5.3 we, amongst others, need the following

Lemma 5.4. *Let ρ be a measurable coefficient function on Ω. Adopt Assumption 2.4 and assume that $-\nabla\cdot\rho\nabla$ is a topological isomorphism between $W^{1,q}_D(\Omega)$ and $W^{-1,q}_D(\Omega)$ for some $q > n$.*

i) For $\theta \in \,]\frac{1}{2}+\frac{n}{2q}, 1[$, the real interpolation space $(W^{-1,q}_D(\Omega), W^{1,q}_D(\Omega))_{\theta,q}$ continuously injects into $C(\overline{\Omega})$.

ii) If $s > 2(1-\frac{n}{q})^{-1}$, then $W^{1,s}(J; W^{-1,q}_D(\Omega)) \cap L^s(J; W^{1,q}_D(\Omega))$ continuously injects into the space $C(\overline{J}; C(\overline{\Omega})) \simeq C(\overline{J\times\Omega})$.

Proof. i) First, Lemma 4.3 shows that, under Assumption 2.4, D is a $(n-1)$-set. Knowing this, [6, Thm. 11.5] tells us that the operator $-\nabla\cdot\rho\nabla + 1$, considered on $W^{-1,q}_D(\Omega)$, is a *positive* one (cf. [44, Ch. 1.14]) and that $\mathrm{dom}(-\nabla\cdot\rho\nabla+1)^{1/2} = L^q(\Omega)$, if $q \geq 2$ [6, Thm. 5.1]. Thus, the reiteration theorem for domains of positive operators (cf. [44, Ch. 1.15.4]) gives for θ as given in the suppositions:

$$
\begin{aligned}
&(W^{-1,q}_D(\Omega), W^{1,q}_D(\Omega))_{\theta,q} = (W^{-1,q}_D(\Omega), \mathrm{dom}(-\nabla\cdot\rho\nabla+1))_{\theta,q} \\
&\quad = (\mathrm{dom}(-\nabla\cdot\rho\nabla+1)^{1/2}, \mathrm{dom}(-\nabla\cdot\rho\nabla+1))_{2\theta-1,q} = (L^q(\Omega), W^{1,q}_D(\Omega))_{2\theta-1,q}.
\end{aligned}
$$

Secondly, recall that $W^{1,q}_D(\Omega)$ admits a continuous extension operator $\mathfrak{E} : W^{1,q}_D(\Omega) \to W^{1,q}(B)$, where $B \subset \mathbb{R}^n$ is an open ball containing $\overline{\Omega}$ cf. [6, Lem. 3.2]. Moreover, it is not hard to see that this extension operator $\mathfrak{E}$ simultaneously extends the functions form $L^q(\Omega)$ to function in $L^q(B)$, inclusively a corresponding estimate. Having this at hand, one can estimate for $\tau > \frac{n}{q}$ and every $\psi \in W^{1,q}_D(\Omega)$

$$
\begin{aligned}
\|\psi\|_{C(\overline{\Omega})} &\leq \|\mathfrak{E}\psi\|_{C(\overline{\Omega})} \leq c\,\|\mathfrak{E}\psi\|_{H^{\tau,q}(B)} \\
&\leq c\,\|\mathfrak{E}\psi\|^{\tau}_{W^{1,q}(B)}\|\mathfrak{E}\psi\|^{1-\tau}_{L^q(B)} \leq c\|\psi\|^{\tau}_{W^{1,q}_D(\Omega)}\|\psi\|^{1-\tau}_{L^q(\Omega)}, \qquad (5.2)
\end{aligned}
$$

cf. [44, Ch. 4.6.1/Ch. 4.3.1]. But it is well-known (cf. [44, Ch. 1.10.1] or [7, Ch. 5, Prop. 2.10]) that an inequality of type (5.2) is constitutive for the embedding $(L^q(\Omega), W^{1,q}_D(\Omega))_{\tau,1} \hookrightarrow C(\overline{\Omega})$. Moreover, it is clear that our supposition $\theta \in \,]\frac{1}{2}+\frac{n}{2q}, 1[$ implies $2\theta - 1 > \frac{n}{q}$. Let $\zeta \in \,]\frac{n}{q}, 2\theta-1[$. According to [44, Ch. 1.3.3], we find

$$
(W^{-1,q}_D(\Omega), W^{1,q}_D(\Omega))_{\theta,q} = (L^q(\Omega), W^{1,q}_D(\Omega))_{2\theta-1,q} \hookrightarrow (L^q(\Omega), W^{1,q}_D(\Omega))_{\zeta,1} \hookrightarrow C(\overline{\Omega}).
$$

ii) One knows the (classical) embedding

$$
W^{1,s}(J; W^{-1,q}_D(\Omega)) \cap L^s(J; W^{1,q}_D(\Omega)) \hookrightarrow C(\overline{J}; (W^{-1,q}_D(\Omega), W^{1,q}_D(\Omega))_{1-\frac{1}{s},s}), \qquad (5.3)
$$

cf. [4, III.4.10]. Thus, if $1-\frac{1}{s} > \frac{1}{2}+\frac{n}{2q}$, or, equivalently, $s > 2(1-\frac{n}{q})^{-1}$, then one has for any $\tau \in \,]\frac{1}{2}+\frac{n}{2q}, 1-\frac{1}{s}[$

$$
C(\overline{J}; (W^{-1,q}_D(\Omega), W^{1,q}_D(\Omega))_{1-\frac{1}{s},s}) \hookrightarrow C(\overline{J}; (W^{-1,q}_D(\Omega), W^{1,q}_D(\Omega))_{\tau,q}) \hookrightarrow C(\overline{J}; C(\overline{\Omega})),
$$

according to i). ☐

Recall from Definition 2.9 the operator $\mathcal{A}(\sigma)$ for a coefficient function $\sigma : J\times\Omega \to M_n$. We consider the special case $\sigma(t,x) = \varphi(t,x)\rho(x)$ for a coefficient function $\rho : \Omega \to M_n$ and measurable function $\varphi : J\times\Omega \to \mathbb{R}$, both bounded. Moreover, denote by γ_{T_0} the point evaluation in T_0, cf. (5.3). This gives rise to the continuous linear operator

$$(\partial_t + \mathcal{A}(\varphi\rho), \gamma_{T_0}) \; : \; W^{1,s}(J; W_D^{-1,q}(\Omega)) \cap L^s(J; W_D^{1,q}(\Omega)) \\ \to L^s(J; W_D^{-1,q}(\Omega)) \times (W_D^{1,q}(\Omega), W_D^{-1,q}(\Omega))_{\frac{1}{s},s}.$$

The following lemma establishes non-autonomous maximal parabolic $W_D^{-1,q}(\Omega)$-$W_D^{1,q}(\Omega)$ regularity of $\mathcal{A}(\varphi\rho)$ for continuous functions φ with positive lower bound.

Lemma 5.5. *Let $n = 3$ and $s \in]1, \infty[$. Suppose that ρ is a measurable coefficient function on Ω and that $\varphi \in C(\overline{J \times \Omega})$ with lower bound $0 < \underline{\varphi} < \varphi$. Adopt Assumption 2.4 and assume that $-\nabla \cdot \rho \nabla + 1$ is a topological isomorphism between $W_D^{1,q}(\Omega)$ and $W_D^{-1,q}(\Omega)$ for some $q > n$.*

i) For every $f \in L^s(J; W_D^{-1,q}(\Omega))$ and $u_0 \in (W_D^{1,q}(\Omega), W_D^{-1,q}(\Omega))_{\frac{1}{s},s}$, there exists a unique solution to the problem

$$u'(t) - \nabla \cdot \varphi(t)\rho\nabla u(t) + u(t) = f(t), \quad u(T_0) = u_0 \tag{5.4}$$

which is from $W^{1,s}(J; W_D^{-1,q}(\Omega)) \cap L^s(J; W_D^{1,q}(\Omega))$.

ii) The operator $(\partial_t + \mathcal{A}(\varphi\rho), \gamma_{T_0})$ is continuously invertible and the mapping $\varphi \mapsto (\partial_t + \mathcal{A}(\varphi\rho), \gamma_{T_0})^{-1}$ is continuous.

Proof. First, [6, Thm. 11.5] yields maximal parabolic regularity in $L^s(J; W_D^{-1,q}(\Omega))$ for each of the operators $-\nabla \cdot \varphi(t)\rho\nabla + 1$. For each $t \in J$, the operator $-\nabla \cdot \varphi(t)\rho\nabla + 1$ is still a topological isomorphism between $W_D^{1,q}(\Omega)$ and $W_D^{-1,q}(\Omega)$, see [15, Thm. 6.2] – note that this is the (only) point in the proof where space dimension $n = 3$ is the limiting factor. In particular, the domain of the operators is uniformly $W_D^{1,q}(\Omega)$. Since the mapping

$$t \mapsto -\nabla \cdot \varphi(t)\rho\nabla + 1 \in \mathcal{L}(W_D^{1,q}(\Omega); W_D^{-1,q}(\Omega)) \tag{5.5}$$

is continuous, [3, Thm. 7.1] shows existence of the unique solution u in the correct space. By [3, Thm. 3.1], this is equivalent to continuous invertibility of $(\partial_t + \mathcal{A}(\varphi\rho), \gamma_{T_0})^{-1}$. Due to

$$\sup_{t \in \overline{J}} \| -\nabla \cdot \varphi_k(t)\rho\nabla + \nabla \cdot \varphi(t)\rho\nabla\|_{\mathcal{L}(W_D^{1,q}(\Omega); W_D^{-1,q}(\Omega))} \leq \|\rho\|_{L^\infty} \|\varphi_k - \varphi\|_{C(\overline{J \times \Omega})},$$

the operators $\mathcal{A}(\varphi_k\rho)$ converge to $\mathcal{A}(\varphi\rho)$. This implies also convergence of $(\partial_t + \mathcal{A}(\varphi_k\rho), \gamma_{T_0})^{-1}$. $\square$

Remark 5.6. For initial value 0, the results of Lemma 5.5 may also be transferred to the operators $\partial_t + \mathcal{A}(\varphi\rho)$ on $W_0^{1,s}(J; W_D^{-1,q}(\Omega)) \cap L^s(J; W_D^{1,q}(\Omega))$ with values in $L^s(J; W_D^{-1,q}(\Omega))$ in a straightforward way (see also [3, Thm. 3.1]). For $s > 2(1 - \frac{3}{q})^{-1}$, the operators $(\partial_t + \mathcal{A}(\varphi\rho))^{-1}$ as just established and the one in Theorem 2.13 (with $\mu(t, \cdot) = \varphi(t)\rho(\cdot)$) then indeed agree on $L^s(J; W_D^{-1,q}(\Omega))$ and we directly obtain

$$W_0^{1,s}(J; W_D^{-1,q}(\Omega)) \cap L^s(J; W_D^{1,q}(\Omega)) \hookrightarrow C^\alpha(\overline{J \times \Omega}),$$

completing the usual collection of embeddings from Lemma 5.4.

It follows the proof of Theorem 5.3.

Proof. We choose an arbitrary function $u \in W^{1,s}(J;W_D^{-1,q}(\Omega)) \cap L^s(J;W_D^{1,q}(\Omega))$ with the initial value $u(T_0) = w_0$ (due to the very definition of the interpolation space which w_0 is from, this is always possible – we may, for instance, choose $t \mapsto e^{\nabla\cdot\rho\nabla(t-T_0)}w_0$). Note that, due to Lemma 5.4, u is a continuous function on $\overline{J\times\Omega}$. Set $w = u+v$. The equation under consideration then becomes an equation in v, since u is fixed, that is, we now have to solve

$$v' - \nabla\cdot\phi(u+v)\rho\nabla v + v = \mathcal{F}(u+v) - u' + \nabla\phi(u+v)\rho\nabla u - u, \quad v(T_0) = 0. \tag{5.6}$$

To this end, we consider for $\psi \in C(\overline{J\times\Omega})$ the equation

$$v' - \nabla\cdot\phi(u+\psi)\rho\nabla v + v = \mathcal{F}(u+\psi) - u' + \nabla\phi(u+\psi)\rho\nabla u - u, \quad v(T_0) = 0. \tag{5.7}$$

and define a function $\mathcal{T}(\psi) = v$, such that $v \in W^{1,2}(J;W_D^{-1,2}(\Omega)) \cap L^2(J;W_D^{1,2}(\Omega))$ solves (5.7) (this is well-defined due to Proposition 2.10). Clearly, a fixed point of $\mathcal{T}$ would yield the searched-for solution for (5.6). Let us construct an appropriate setting: First, the set of all right-hand sides in (5.7) is bounded in the space $L^s(J;W_D^{-1,q}(\Omega))$ – boundedness of $\mathcal{F}$ in $L^s(J;W_D^{-1,q}(\Omega))$ over $C(\overline{J\times\Omega})$ was an assumption – and for the divergence-term we estimate for every $t \in J$ as follows:

$$\begin{aligned}\|\nabla\phi(u(t)+\psi(t))\rho\nabla u(t)\|_{W_D^{-1,q}(\Omega)} &= \sup_{\|\zeta\|_{W_D^{1,q'}(\Omega)}=1} \left|\int_\Omega \phi(u(t)+\psi(t))\rho\nabla u(t)\cdot\nabla\zeta\,\mathrm{dx}\right| \\ &\le \overline{\phi}\,\|\rho\|_{L^\infty(\Omega)}\,\|u(t)\|_{W_D^{1,q}(\Omega)},\end{aligned}$$

hence

$$\|\nabla\phi(u+\psi)\rho\nabla u\|_{L^s(J;W_D^{-1,q}(\Omega)} \le \overline{\phi}\,\|\rho\|_{L^\infty(\Omega)}\,\|u\|_{L^s(J;W_D^{1,q}(\Omega))}, \tag{5.8}$$

which is independent of ψ. As u is fixed, this means the right-hand sides in (5.7) are contained in a ball around the origin in $L^s(J;W_D^{-1,q}(\Omega))$, say, of radius r. Now set

$$\begin{aligned}\mathcal{B} := \Big\{ v : v'(t) + \mathcal{A}(\zeta(t)\rho)v(t) = g(t),\ v(T_0) = 0,& \\ \text{with } \|g\|_{L^s(J;W_D^{-1,q}(\Omega))} \le r, \zeta \in C(\overline{J\times\Omega}) \text{ and } \underline{\phi} \le \zeta \le \overline{\phi}&\Big\}\end{aligned}$$

as a subset of $W^{1,2}(J;W_D^{-1,2}(\Omega)) \cap L^2(J;W_D^{1,2}(\Omega))$. Theorem 2.13 shows that $\mathcal{B}$ is in fact contained in a ball Q_α in some Hölder space $C^\alpha(\overline{J\times\Omega})$, which in turn is compactly included in some ball Q_c in $C(\overline{J\times\Omega})$. Clearly, $\mathcal{T}$ maps Q_c to $\mathcal{B} \subset Q_\alpha$ and the set $\{\mathcal{T}(\psi) : \psi \in Q_c\}$ is compact in Q_c. Hence, the Schauder fixed point theorem yields a fixed point $v = \mathcal{T}(v)$ in Q_c, provided we are able to show continuity of the mapping $\mathcal{T}$ from Q_c to Q_c. So:
The mapping $\psi \mapsto \phi(u+\psi)$ is continuous from $C(\overline{J\times\Omega})$ into itself by the Lipschitz assumption on ϕ, such that Lemma 5.5 implies that $\psi \mapsto (\partial_t + \mathcal{A}(\phi(u+\psi)\rho))^{-1}$ is continuous from $C(\overline{J\times\Omega})$ to the linear bounded operators from $L^s(J;W_D^{-1,q}(\Omega))$ to $W_0^{1,s}(J;W_D^{-1,q}(\Omega)) \cap L^s(J;W_D^{1,q}(\Omega))$, cf. Remark 5.6. Thanks to the assumptions on $\mathcal{F}$, $\psi \mapsto \mathcal{F}(\cdot, u(\cdot)+\psi(\cdot))$ is also a continuous map, hence the right-hand side $R(\psi)$ in (5.7) depends continuously on ψ (here one also uses the Lipschitz property of ϕ). For a sequence $\psi_k \to \psi$ in Q_c we find via Lemma 5.4

$$\begin{aligned}\|\mathcal{T}(\psi) - \mathcal{T}(\psi_k)\|_{C(\overline{J\times\Omega})} \le C\,\big\|&(\partial_t + \mathcal{A}(\phi(u(\cdot)+\psi(\cdot))\rho))^{-1}R(\psi) \\ &-(\partial_t + \mathcal{A}(\phi(u(\cdot)+\psi_k(\cdot))\rho))^{-1}R(\psi_k)\big\|_{W_0^{1,s}(J;W_D^{-1,q}(\Omega))\cap L^s(J;W_D^{1,q}(\Omega))}\end{aligned}$$

and a simple triangle argument shows that this goes to 0 as k goes to infinity since everything depends continuously on ψ. This is exactly the searched-for continuity of $\mathcal{T}$.

Finally, a fixed point v of $\mathcal{T}$ obviously solves (5.6) and is, thanks to Lemma 5.5, in fact from $W_0^{1,s}(J; W_D^{-1,q}(\Omega)) \cap L^s(J; W_D^{1,q}(\Omega))$, making $w := u + v$ a solution of (5.1) in the optimal space $W^{1,s}(J; W_D^{-1,q}(\Omega)) \cap L^s(J; W_D^{1,q}(\Omega))$. Concerning uniqueness, one observes that both the right-hand side $\mathcal{F}$ and the operator $w \mapsto -\nabla \cdot \phi(w)\rho\nabla w + w$ satisfy all assumptions in the theorem of Prüss [42, Thm. 3.1], if $\mathcal{F}$ satisfies the Lipschitz assumption in Remark 5.2. The quoted theorem then yields uniqueness of the solution. □

Corollary 5.7. *For fixed w_0, consider the set of admissible data $\{\rho, \phi, \mathcal{F}\}$ for the problem (5.1) as in the assumptions of Theorem 5.3, where $\kappa_0, \kappa_1, \underline{\phi}, \overline{\phi}$ and $C_{\mathcal{F}}$ are fixed. Then the set of associated solutions $w_{\rho,\phi,\mathcal{F}}$ is contained in a ball in some Hölder space $C^\alpha(\overline{J \times \Omega})$.*

Proof. Inspecting the proof of Theorem 5.3, one observes that the set $\mathcal{B}$ is always the same for all data $\{\phi, \mathcal{F}\}$ when $\underline{\phi}, \overline{\phi}$ and $C_{\mathcal{F}}$ are fixed, and that the bound of $\mathcal{B}$ in the Hölder space is also uniform in κ_0, κ_1 by Theorem 2.13. Hence, the size of the set Q_α is also uniform in $\kappa_0, \kappa_1, \underline{\phi}, \overline{\phi}$ and $C_{\mathcal{F}}$. □

If the forcing term $\mathcal{F}$ in fact does not depend on w, we still obtain the following useful result from Theorem 5.3 and Corollary 5.7.

Corollary 5.8. *Let the assumptions of Theorem 5.3 be satisfied, with $\mathcal{F}(t, \cdot) = f(t)$ for some $f \in L^s(J; W_D^{-1,q}(\Omega)$. Then, for every such $f \in L^s(J; W_D^{-1,q}(\Omega))$, there exists a unique global solution $w \in W^{1,s}(J; W_D^{-1,q}(\Omega)) \cap L^s(J; W_D^{1,q}(\Omega))$ of the equation*

$$w'(t) - \nabla \cdot \phi(w(t))\rho\nabla w(t) + w(t) = f(t), \quad w(T_0) = w_0.$$

In particular, the (nonlinear) solution operator, mapping f to w, transports bounded sets in $L^s(J; W_D^{-1,q}(\Omega))$ into bounded sets in $C^\alpha(\overline{J \times \Omega})$ for some $\alpha > 0$ and fixed w_0.

Remark 5.9. With Theorem 2.13 and essentially analogous techniques as displayed in this chapter, one might show existence of global Hölder-continuous solutions to semilinear equations with nonlinearities in the form as in Assumption 5.1, where the coefficient functions in the divergence-operator are only measurable in time. We omit the details.

6. **Applications to Optimal Control.** In this chapter we show that Hölder estimates, as established in various forms in the previous chapters, are not only interesting by their own right but may also put to good use in optimal control theory. The crucial point here is, of course, the compactness of bounded sets of Hölder functions in the space of continuous functions. We illustrate this in two ways, both of which translate weak convergence of the forcing terms to strong convergence of the associated solutions (or *states*) in the space of continuous functions. We do this for both a non-autonomous linear equation and a quasilinear equation as in Theorem 5.3. Applications in optimal control theory range from existence theory by standard arguments (see also Proposition 6.4 below) to second order sufficient conditions, see e.g. [10] or [13].

Let X and $\mathcal{X} \subset L^1(J;X)$ be reflexive Banach spaces and consider a continuous linear mapping $\mathcal{E} : \mathcal{X} \to L^s(J; W_D^{-1,q}(\Omega))$. The first result is an immediate consequence of Theorem 2.13 by noting that the operators $(\partial_t + \mathcal{A}(\mu))^{-1}$ are completely continuous from $L^s(J; W_D^{-1,q}(\Omega))$ to $C(\overline{J \times \Omega})$.

Proposition 6.1. *Adopt the assumptions of Theorem 2.13. Let $f \in L^s(J; W_D^{-1,q}(\Omega))$ and $u_k \rightharpoonup \bar{u}$ in $\mathcal{X}$. Then the solutions $w_k := w_{u_k}$ of*

$$w'(t) - \nabla\mu(t,\cdot)\nabla w(t) + w(t) = (\mathcal{E}u_k)(t) + f(t), \quad w(T_0) = 0$$

converge strongly *in $C(\overline{J \times \Omega})$ to $w_{\bar{u}}$.*

Note that affine-linear superposition operators for the control u are, in general, the best one can hope for in order to preserve weak convergence, see e.g. [9, Ex. 4.20].

Remark 6.2. Proposition 6.1 may also be extended to nonconstant Dirichlet data and/or initial data from $W^{1,q}(\Omega)$ as in Ch. 4 in a straightforward way. We did not carry this out for the sake of simplicity.

Next, we add a control to the right-hand sides of the quasilinear problem in Theorem 5.3 in the following way: Let $\mathfrak{F} : J \times C(\overline{\Omega}) \times X \to W_D^{-1,q}(\Omega)$ be such that

- i) for each $u \in \mathcal{X}$, $(t,w) \mapsto \mathfrak{F}(t,w,u(t))$ satisfies the assumptions in Assumption 5.1 with the bound $C_{\mathcal{F}}$ being uniform for u from bounded sets in $\mathcal{X}$,
- ii) the mapping $u \mapsto \mathfrak{F}(\cdot, w(\cdot), u(\cdot))$ is affine-linear and continuous for each fixed $w \in C(\overline{J \times \Omega})$.

Theorem 6.3. *Adopt the assumptions of Theorem 5.3 (the unique solutions case) and assume that the right-hand side is of the form $\mathfrak{F}$ as above. Let $u_k \rightharpoonup \bar{u}$ be a weakly convergent sequence in $\mathcal{X}$ and $w_0 \in (W_D^{1,q}(\Omega), W_D^{-1,q}(\Omega))_{\frac{1}{s},s}$. Then the solutions $w_k := w_{u_k}$ of*

$$w'(t) - \nabla \cdot \phi(w(t))\rho\nabla w(t) + w(t) = \mathfrak{F}(t, w(t), u_k(t)), \quad w(T_0) = w_0, \tag{6.1}$$

converge strongly *in $C(\overline{J \times \Omega})$ to $w_{\bar{u}}$.*

Proof. Without loss of generality, we assume $w_0 = 0$ in the proof. One arrives at this situation by repeating the "split-off"-procedure done at the beginning of the proof of Theorem 5.3 and the obvious modifications from thereon without changing the fundamental properties of the problem, as seen there.
The sequence $(u_k)_k$ is bounded in $\mathcal{X}$. Due to the choice of $s > 2(1 - \frac{n}{q})^{-1}$ and Lemma 5.4, we have $w_k \in C(\overline{J \times \Omega})$ for each k. The assumptions on $\mathfrak{F}$ and Corollary 5.7 then yield that the solutions w_k are from a bounded set in $C^\alpha(\overline{J \times \Omega})$ for some $\alpha > 0$. Hence, there is a subsequence $(w_{k_l})_l$ of $(w_k)_k$ such that $w_{k_l} \to \bar{w}$ in $C(\overline{J \times \Omega})$. We need to show that $\bar{w} = w_{\bar{u}}$. Re-inserting the newly found convergence of w_{k_l} in the equations shows that the right-hand sides $\mathfrak{F}(\cdot, w_{k_l}(\cdot), u_{k_l}(\cdot))$ now in fact converge weakly to $\mathfrak{F}(\cdot, \bar{w}(\cdot), \bar{u}(\cdot))$ in $L^s(J; W_D^{-1,q}(\Omega))$, while $(\partial_t + \mathcal{A}(\phi(w_{k_l})\rho))^{-1}$ goes go $(\partial_t + \mathcal{A}(\phi(\bar{w})\rho))^{-1}$ by virtue of Lemma 5.5. However, the operators $(\partial_t + \mathcal{A}(\phi(w_{k_l})\rho))^{-1}$ are even completely continuous from $L^s(J; W_D^{-1,q}(\Omega))$ to $C(\overline{J \times \Omega})$ – this is Theorem 2.13 via Remark 5.6 – and thus translate weak convergence to strong convergence in those spaces, even "diagonally", that is:

$$w_{k_l} = (\partial_t + \mathcal{A}(\phi(w_{k_l})\rho))^{-1}\mathfrak{F}(\cdot, w_{k_l}(\cdot), u_{k_l}(\cdot)) \longrightarrow (\partial_t + \mathcal{A}\phi(\bar{w})\rho))^{-1}\mathfrak{F}(\cdot, \bar{w}(\cdot), \bar{u}(\cdot)) = \bar{w}.$$

This limit implies that $\bar{w}$ solves

$$w'(t) - \nabla \cdot \phi(w(t))\rho\nabla w(t) + w(t) = \mathfrak{F}(t, w(t), \bar{u}(t)), \quad w(T_0) = 0,$$

which means exactly that $\bar{w} = w_{\bar{u}}$ by uniqueness of solutions. Since this procedure can be done for every subsequence with the same limit $w_{\bar{u}}$, the whole sequence (w_k) must converge. □

Let us briefly show how the previous result may be put to use: Let $\mathcal{X}_{\mathrm{ad}} \subseteq \mathcal{X}$ be closed and convex and let $J : C(\overline{J \times \Omega}) \times \mathcal{X}_{\mathrm{ad}} \to \mathbb{R}$ be continuous in the separated form $J(w, u) = J_1(w) + J_2(u)$. Assume that J_2 is coercive and convex on $\mathcal{X}_{\mathrm{ad}}$ and consider the problem

$$\min_{u \in \mathcal{X}_{\mathrm{ad}}} J(w, u) \quad \text{such that} \quad (6.1) \text{ holds.} \tag{QLOC}$$

We then obtain the following result by standard methods:

Proposition 6.4. *Suppose the assumptions of Theorem 6.3 and assume that a feasible point of* (QLOC) *exists. Then the problem* (QLOC) *has an optimal solution* $\bar{u} \in \mathcal{X}_{\mathrm{ad}}$.

A usual choice for the objective functional J would be

$$J(w, u) = \|w - w_d\|^2_{L^2(J;L^2(\Omega))} + \frac{\beta}{2}\|u\|^2_{\mathcal{U}},$$

where w_d is a given temperature distribution to be reached and $\beta > 0$ is a regularization parameter.

7. **Concluding Remarks.** It is not the intention of this paper to declare the concept of Ladyzhenskaja et. al in [39] to be outdated or not adequate any more. On the contrary, even nearly fifty years after it was first published, the results in [39] are still highly relevant – if not in their original form, then at least in a guiding and blue-print way, not accounting for the various hard facts it established. However, in view of the modern techniques for negative Sobolev spaces and Hölder spaces, an exposition of results in current, up-to-date mathematical "language" seems in order. In this sense, the preceding results could be seen as an adaption and translation of the classical results and deep insights in [39] to modern techniques.

We mention some open ends in the previous considerations:

The results presented may be transferred to complex spaces as long as the coefficient functions in the equations are real. In this case, one may consider the real- and imaginary parts in the considerations each on their own.

Moreover, the "next step" in the great scheme would surely be maximal parabolic L^p-regularity for non-autonomous equations with coeffients which are only measureable in time. While it is already known that maximal regularity for operators $A(\cdot)$ over an interval J implies maximal regularity for each of the autonomous operators $A(s)$ for $s \in J$, up to now mostly some sort of continuity of the time-dependence is assumed additionally in order to conclude maximal regularity, see e.g. [3] for the corresponding result (already used in Lemma 5.5) and an overview. There is also a sequence of related, very recent work [5], [14], [30] and [41] which follows Lions' Theorem 2.10 (see [12, Ch. XVIII.3]) in a slightly different direction (maximal regularity over the Hilbert space H). Also very recent is a positive result on maximal L^p regularity without any continuity assumptions on the time-dependence in [21].

Let us note that, in view of Ch. 6, maximal parabolic L^p-regularity for only measurably time-dependent non-autonomous evolution equations would allow for a concise treatment of optimal control problems subject to these equations.

Finally, it would certainly be interesting to know which degree of Hölder continuity one obtains in Theorem 2.13 in dependence on the coercivity-constant κ_0 and upper bound κ_1 of the associated coefficient matrix. Based on [16, Ch. 4] for the elliptic case and the lack of related results apart from [39], at least such known to the authors, this seems like a difficult question which might be worth investigating.

8. **Appendix.** In this Appendix we give the proof of Proposition 3.9. We start with the following

Lemma 8.1. *i) Let $\mathfrak{f} \in L^2(J; L^2(\Xi; \mathbb{R}^{n+1}))$ be given, and $\mathfrak{N}(u, \vartheta, t)$ be defined as in (3.2). Then, for every $u \in V_2^{1,0}(J \times \Xi)$, every $\vartheta \in W^{1,2}_{\overline{J} \times \partial\Xi}(J \times \Xi)$, and every $t \in J$, $\mathfrak{N}(u, \vartheta, t)$ is well-defined. Moreover, for every $u \in V_2^{1,0}(J \times \Xi)$ and every $t \in J$, the linear form $W^{1,2}_{\overline{J} \times \partial\Xi}(J \times \Xi) \ni \vartheta \mapsto \mathfrak{N}(u, \vartheta, t)$ is continuous.*

ii) The set $C^1(\overline{J}; W_0^{1,2}(\Xi))$ is dense in $W^{1,2}_{\overline{J} \times \partial\Xi}(J \times \Xi)$.

iii) For every Banach space X, the set $C^1(\overline{J}) \otimes X = \{\sum_{j=1}^k \eta_j \otimes v_j : \eta_j \in C^1(\overline{J}), v_j \in X\}$ is dense in $C^1(\overline{J}; X)$.

Proof. i) is straight forward, cf. also [39, Ch. III.1]. ii) It is known that $W^{1,2}(J \times \Xi)$ is isomorphic to $L^2(J; W^{1,2}(\Xi)) \cap W^{1,2}(J; L^2(\Xi))$ – algebraically and topologically, cf. [12, Ch. XVIII.1.3]. Restricting this isomorphism to the set $C^\infty_{\overline{J} \times \partial\Lambda}(J \times \Xi)$, which is dense in $W^{1,2}_{\overline{J} \times \partial\Xi}(J \times \Xi)$, one obtains that $W^{1,2}_{\overline{J} \times \partial\Xi}(J \times \partial\Xi)$ is isomorphic to $L^2(J; W_0^{1,2}(\Xi)) \cap W^{1,2}(J; L^2(\Xi))$. Moreover, it is also known that $C^1(\overline{J}; W_0^{1,2}(\Xi))$ is dense in $L^2(J; W_0^{1,2}(\Xi)) \cap W^{1,2}(J; L^2(\Xi))$, cf. [12, Ch. XVIII.1.3]. iii) is again straight forward. □

It follows the proof of Proposition 3.9.

Proof. i) Each continuous linear form on $W_F^{1,q'}(\Xi)$ may be extended by the Hahn-Banach theorem to a continuous linear form on the whole $W^{1,q'}(\Xi) = W_\emptyset^{1,q'}(\Xi)$ under the preservation of its norm. In case of $F = \emptyset$ the representation (3.4) is well-known, including the corresponding estimates, see [40, Ch. 1.1.14].
ii) By i) and the embedding $L^s(J; W_F^{-1,q}(\Xi)) \hookrightarrow L^2(J; W_F^{-1,2}(\Xi))$ the equation (2.4) can be written as

$$\begin{aligned} \langle u', v \rangle - \langle \nabla \cdot \mu \nabla u + u, v \rangle &= \frac{d}{dt} \int_\Xi uv \, \mathrm{dx} + \int_\Xi \mu \nabla u \cdot \nabla v + uv \, \mathrm{dx} \\ &= \langle f, v \rangle = \int_\Xi \mathfrak{g}_{k,0} v - \sum_{j=1}^n \mathfrak{g}_{k,j} \frac{\partial v}{\partial x_j} \, \mathrm{dx}, \end{aligned} \tag{8.1}$$

for all $v \in W_F^{1,2}(\Xi) \hookrightarrow W_F^{1,q'}(\Xi)$ and then for almost all $t \in J_k$. In particular, the test functions v may be chosen from $W_0^{1,2}(\Xi)$ – what we will do from now on. Take now any function $\eta \in C^1(\overline{J})$, multiply (8.1) with η and integrate from T_0 to $T \in J$;

one then obtains

$$
\begin{aligned}
\int_{T_0}^T \frac{d}{dt} \int_\Xi uv \,\mathrm{d}x\, \eta \,dt + \int_{T_0}^T \Big(\int_\Xi \mu \nabla u \cdot \nabla v + uv \,\mathrm{d}x \Big) \,\eta \,dt \\
= \int_\Xi u(T,\mathrm{x})(\eta \otimes v)(T,\mathrm{x}))\,\mathrm{d}x - \int_{T_0}^T \int_\Xi u \frac{\partial}{\partial t}(\eta \otimes v)\,\mathrm{d}x\,dt \\
+ \int_{T_0}^T \int_\Xi \mu \nabla u \cdot \nabla(\eta \otimes v) + u(\eta \otimes v)\,\mathrm{d}x\,dt \\
= \int_{T_0}^T \int_\Xi \Big(\sum_k \chi_{J_k} \mathfrak{g}_{k,0} \Big)(\eta \otimes v) - \int_{T_0}^T \sum_{j=1}^n \Big(\sum_k \chi_{J_k} \mathfrak{g}_{k,j} \Big) \frac{\partial(\eta \otimes v)}{\partial x_j}\,\mathrm{d}x\,dt.
\end{aligned}
$$

Clearly, this equality extends to the linear span of functions from $C^1(\overline{J}) \otimes W_0^{1,2}(\Xi)$ and, by continuity of $\vartheta \mapsto \mathfrak{N}(u,\vartheta,t)$ on $W^{1,2}_{\overline{J}\times\partial\Xi}(J \times \Xi)$ and density of $C^1(\overline{J}) \otimes W_0^{1,2}(\Xi)$ in the latter space, even to the whole $W^{1,2}_{\overline{J}\times\partial\Xi}(J \times \Xi)$. □

REFERENCES

[1] H. Amann, Parabolic evolution equations and nonlinear boundary conditions, *J. Differ. Equations* **72** (1988), 201–269.

[2] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: *Function spaces, differential operators and nonlinear analysis*, (eds. H.-J. Schmeisser et al.), Teubner, Stuttgart, 1993, 9–126.

[3] H. Amann, Maximal regularity for nonautonomous evolution equations, *Adv. Nonlinear Stud.* **4** (2004), 417–430.

[4] H. Amann, *Linear and quasilinear parabolic problems*, Birkhäuser, Basel-Boston-Berlin, 1995.

[5] W. Arendt, D. Dier, H. Laasri and E. M. Ouhabaz, Maximal regularity for evolution equations governed by non-autonomous forms, *Adv. Differential Equations* **19** (2014), 1043–1066.

[6] P. Auscher, N. Badr, R. Haller-Dintelmann and J. Rehberg, The square root problem for second order, divergence form operators with mixed boundary conditions on L^p, arXiv:1210.0780, accepted for *J. Evol. Eq.*.

[7] C. Bennett, R. Sharpley, *Interpolation of operators*, Academic Press, Boston etc., 1988.

[8] K. Brewster, D. Mitrea, I. Mitrea and M. Mitrea, Extending Sobolev functions with partially vanishing traces from locally (ε,δ)-domains and applications to mixed boundary problems, *J. Funct. Anal.* **266** (2014), 4314–4421.

[9] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[10] E. Casas, J.C. de los Reyes and F. Tröltzsch, Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints, *SIAM J. Control Optim.* **19** (2008), 616–643.

[11] P.G. Ciarlet, *The finite element method for elliptic problems*, North Holland, Amsterdam-New York-Oxford, 1978.

[12] R. Dautray, J.-L Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 5: Evolution problems I*, Springer, Berlin, 2000.

[13] J.C. de los Reyes, P. Merino, J. Rehberg and F. Tröltzsch, Optimality conditions for state-constrained PDE control problems with time-dependent controls, *Control Cybernet.* **37** (2008), 5–38.

[14] D. Dier, Non-autonomous maximal regularity for forms of bounded variation, arXiv:1406.2884.

[15] K. Disser, H.-C. Kaiser and J. Rehberg, Optimal Sobolev regularity for linear second-order divergence elliptic operators occuring in real-world problems, Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS) Preprint No. 1977 (2014).

[16] J. Elschner, J. Rehberg, and G. Schmidt, Optimal regularity for elliptic transmission problems including C^1 interfaces, *Interfaces Free Bound.* **9** (2007), 233-252.

[17] A.F.M. ter Elst and J. Rehberg, Hölder estimates for second-order operators with mixed boundary conditions, Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS) Preprint No. 1950 (2014), accepted for *J. Differ. Equations.*

[18] L.C. Evans, R.F. Gariepy, *Measure theory and fine properties of functions*, Studies in advanced mathematics, CRC Press, Boca Raton-New York-London-Tokyo, 1992.

[19] I. Fonseca, G. Parry, Equilibrium configurations of defective crystals, *Arch. Rat. Mech. Anal.* **120** (1992), 245–283.

[20] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974.

[21] Ch. Gallarati, M. Veraar, Maximal regularity for non-autonomous equations with measurable dependence on time, arXiv:1410.6394.

[22] M. Giaquinta, M. Struwe, An optimal regularity result for a class of quasilinear parabolic systems, *Manuscr. Math.* **36** (1981), 223–239.

[23] E. Giusti, *Metodi diretti nel calcolo delle variazioni*, Unione Matematica Italiana, Bologna, 1994.

[24] J.A. Griepentrog, W. Höppner, H.-C. Kaiser and J. Rehberg, A bi-Lipschitz continuous, volume preserving map from the unit ball onto a cube, *Note Mat.* **28** (2008), 177–193.

[25] J.A. Griepentrog, K. Gröger, H. C. Kaiser, J. Rehberg, Interpolation for function spaces related to mixed boundary value problems, *Math. Nachr.* **241** (2002), 110–120.

[26] J. Griepentrog, Maximal regularity for nonsmooth parabolic problems in Sobolev-Morrey spaces, *Adv. Differ. Equ.* **12** (2007), 1031-1078.

[27] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston, 1985.

[28] K. Gröger, A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations, *Math. Ann.* **283** (1989), 679–687.

[29] K. Gröger, $W^{1,p}$-estimates of solutions to evolution equations corresponding to nonsmooth second order elliptic differential operators, *Nonlinear Anal.* **18** (1992), 569–577.

[30] B. H. Haak, E. M. Ouhabaz, Maximal regularity for non-autonomous evolution equations, arXiv:1402.1136.

[31] R. Haller-Dintelmann, H.-C. Kaiser, J. Rehberg, Elliptic model problems including mixed boundary conditions and material heterogeneities, *J. Math. Pures Appl.* 89 (2008), 25–48.

[32] R. Haller-Dintelmann, C. Meyer, J. Rehberg and A. Schiela, Hölder continuity and optimal control for nonsmooth elliptic problems, *Appl. Math. Optim.* **60** (2009), 397–428.

[33] R. Haller-Dintelmann, J. Rehberg, Maximal parabolic regularity for divergence operators including mixed boundary conditions, *J. Differ. Equations* **247** (2009), 1354–1396.

[34] R. Haller-Dintelmann, J. Rehberg, Coercivity for elliptic operators and positivity of solutions on Lipschitz domains, *Arch. Math.* **95** (2010), 457–468.

[35] R. Haller-Dintelmann, J. Rehberg, Maximal parabolic regularity for divergence operators on distribution spaces, in *Parabolic problems: The Herbert Amann Festschrift* (eds. J. Escher, P. Guidotti, M. Hieber, P. Mucha, J. Prüss, Y. Shibata, G. Simonett, C. Walker, W. Zajaczkowski), Springer, Basel, 2011, 313–342.

[36] M. Hieber, J. Rehberg, Quasilinear parabolic systems with mixed boundary conditions on nonsmooth domains, *SIAM J. Math. Anal.* **40** (2008), 292–305.

[37] A. Jonsson, H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$, Harwood Academic Publishers, Chur-London-Paris-Utrecht-New York, 1984.

[38] D. Kinderlehrer, G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

[39] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva, *Linear and quasi-linear equations of parabolic type*, American Mathematical Society (AMS), Providence, RI, 1967.

[40] V.G. Maz'ya, *Sobolev spaces*, Springer, Berlin-Heidelberg-New York-Tokyo, 1985.

[41] E. M. Ouhabaz, Maximal regularity for non-autonomous evolution equations governed by forms having less regularity, arXiv:1411.0139.

[42] J. Prüss, Maximal regularity for evolution equations in L^p-spaces, *Conf. Semin. Mat. Univ. Bari* **285** (2002), 1–39.

[43] M. Struwe, On the Hölder continuity of bounded weak solutions of quasilinear parabolic systems, *Manuscr. Math.* **35** (1981), 125–145.

[44] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North Holland Publishing Company, Amsterdam-New York-Oxford, 1978.