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# Uniqueness for an inverse problem for a nonlinear parabolic system with an integral term by one-point Dirichlet data


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**Abstract**

We consider an inverse problem arising in laser-induced thermotherapy, a minimally invasive method for cancer treatment, in which cancer tissues is destroyed by coagulation. For the dosage planning quantitatively reliable numerical simulation are indispensable. To this end the identification of the thermal growth kinetics of the coagulated zone is of crucial importance. Mathematically, this problem is a nonlinear and nonlocal parabolic inverse heat source problem. We show in this paper that the temperature dependent thermal growth parameter can be identified uniquely from a one-point measurement.

# 1 Introduction

Laser-induced thermotherapy (LITT) is a minimally invasive cancer treatment. An applicator is placed in a tumour via a catheter and connected to a laser source (cf. Fig. 1). The energy of the laser light emitted from the surface of the applicator is absorbed by the biological tissue causing a rise in temperature. The laser power and treatment time is adjusted such that a temperature of around $60^oC$ is reached in a neighbourhood of the applicator. Driven by this rise in temperature the tissue is coagulated, a process which is governed by protein denaturation leading to the disruption of cell walls and eventually to the destruction of the tumour tissue. The deadened tissue remains in the body and is either decomposed or encapsulated.

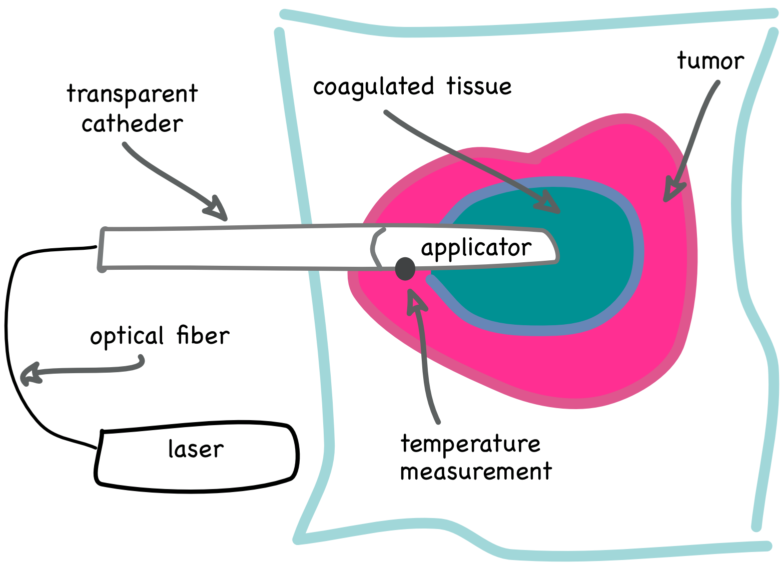


Figure 1: Sketch of laser-induced thermotheraphy treatment.

A detailed mathematical model for LITT is discussed in [5]. The most important part is a coagulation model coupled to the bioheat equation describing temperature changes $u(x, t)$ in the tumor tissue $\Omega$. In laser medicine, coagulation is defined as an optically visible irreversible cell destruction (necrosis) caused by the denaturation of proteins. In the spirit of [16], where an Arrhenius formalism model for protein denaturation is developed, the distribution of native tissue $z(x, t)$ representing the concentration of different proteins can be described as

$$\partial_t z(x,t) = -f(u(x,t))z(x,t), \quad x \in \Omega, t > 0 \tag{1.1a}$$

$$z(x, 0) = 1, \qquad x \in \Omega, \tag{1.1b}$$

where the non-negative function $f(u)$ describes the thermal part of the coagulation growth kinetics. Since the coagulation growth increases with growing temperature, we may assume $f$ to be monotonically increasing. Similar models are used in polymerization [1] and in solid-solid phase transitions [7]. The second physical quantity relevant for the treatment is the temperature $u(x, t)$ governed by the bio-heat equation. According to [16], for most of the biological tissues, the density $\rho$, the heat capacity $c_p$ and the thermal conductivity $k$ are almost constant in the relevant temperature interval between $37^oC$ and $70^oC$. Then the bio-heat equation reads as follows

$$\partial_t u(x, t) - \kappa \Delta u(x, t) = \tilde{\alpha}(z, x)(u_B - u), \quad x \in \Omega \times (0, T). \tag{1.2}$$

Here, $T$ is the end-time of the treatment and $\kappa = k/\rho c_p$ is the thermal diffusivity. Recalescence effect of the coagulation process and metabolic changes can be neglected in the energy balance (see [16]). The remaining heat source in the bio-heat process is the widely used Pennes model to describe the heat exchange due to blood perfusion in the tissue, where $u_B$ is the known temperature of the arterial blood. Since there are no (active) vessels in the coagulated zone, no perfusion takes place there, hence we can write

$$\tilde{\alpha}(z, x) = z\alpha(x) \tag{1.3}$$

where $\alpha(x)$ describes the perfusion in non-coagulated tissues.

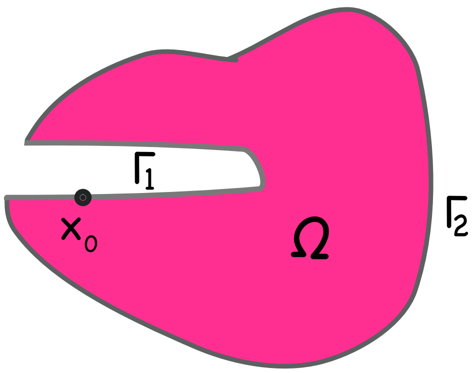


Figure 2: Domain and boundary parts.

The light is absorbed in a region around the catheter. The irradiation of laser light within the tissue can be described by the radiation transfer equation [5]. However, for our purposes it is sufficient to model it by a Neumann boundary condition, i.e., we have

$$-\kappa \partial_\nu u = h(x, t), \quad \text{in } \Gamma_1 \times (0, T) \tag{1.4a}$$
$$-\kappa \partial_\nu u = 0, \quad \text{in } \Gamma_2 \times (0, T) \tag{1.4b}$$

where $\Gamma_1$ is the boundary to the applicator and $\Gamma_2$ to the surrounding tissue, see Fig. 2. We also specify the initial temperature distribution to be the blood reference temperature, that is,

$$u(x, 0) = u_B, \quad x \in \Omega. \tag{1.5}$$

By the simplicity of the coagulation rate law (1.1) we easily obtain the solution

$$z(x, t) = \exp\left(-\int_0^t f(u(x, \tau))d\tau\right), \quad \text{in } \Omega \times (0, T). \tag{1.6}$$

Assuming $\alpha$ and $\kappa$ are spatially homogeneous, we norm them to one. Moreover, without loss of generality we assume $u_B = 0$ and accordingly the initial value $u(x, 0) = 0$. Then we arrive at the following nonlocal system:

$$\partial_t u = \Delta u - u \exp\left(-\int_0^t f(u(x,\xi))d\xi\right), \quad x \in \Omega,\, t > 0, \tag{1.7a}$$

$$\partial_\nu u(x,t) = \begin{cases} \varphi(x,t), & x \in \Gamma_1,\, t > 0, \\ 0, & x \in \Gamma_2,\, t > 0, \end{cases} \tag{1.7b}$$

$$u(x,0) = 0, \quad x \in \Omega. \tag{1.7c}$$

Here, we assume $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_1 \cup \Gamma_2}$, where $\Gamma_1, \Gamma_2$ are non-empty relatively open subsets of $\partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

The LITT treatment is guided using magnetic resonance imaging (MRI). Unfortunately, MRI is known to have either a good spatial or a good temporal resolution, making it difficult to predict the final size of the coagulated zone. Hence, there is a strong demand for computer simulations of LITT to support therapy planning and finding an optimal dosage. To obtain quantitatively satisfactory results, the estimation of tissue parameters is a crucial task. However, while the respective data for the bio-heat equation are by now available, the determination of the parameters in the coagulation model is still an important task, in our case this is the function $f(u)$. For technical reasons temperature measurements are only possible using a thermocouple placed inside the catheter, corresponding to a point on the boundary of the domain $\Omega$.

In mathematical terms the scope of this paper is thus to study the following **Inverse Problem**:
Let $x_0 \in \partial\Omega$ be arbitrarily fixed, $T > 0$ and $\varphi$ be suitably given. Then do one-point data $u(x_0, t)$, $0 < t < T$, uniquely determine $f$ in some interval?

Our problem is a type of inverse problem of determining nonlinear terms in a parabolic equation by a single measurement of boundary data. For such inverse problems for semilinear parabolic equations, we refer to [2, 3, 4, 11, 13, 15], for example. In those papers, one usually can prove the comparison principle, by which the uniqueness for the inverse problems is proved. Our system (1.7a) is involved with an integral term and the comparison principle is less trivial, but we can prove it (Lemma 2.3 in Section 2).

Regarding numerical methods for such inverse problems, we refer to [8, 17]. On the other hand, in the case where a governing equation does not allow the comparison principle, we do not know the uniqueness by one-point measurement data but the uniqueness is proved by more comprehensive data of solution in $\omega \times (0, T)$ with subdomain $\omega \subset \Omega$ (see [9]).

For the statement of our main result concerning the uniqueness, we introduce assumptions. Throughout this paper, we fix $\varphi$ which is sufficiently smooth and satisfies

$$\varphi(x, 0) = 0,\, x \in \partial\Omega, \quad \varphi(x,t) > 0,\, x \in \Gamma_1,\, t > 0.$$

Fixing two constants $L, \eta_0$ arbitrarily such that $0 < \eta_0 < L$, we define an admissible set $\mathcal{U}$ by

$$\mathcal{U} = \{f \geq 0 \text{ on } [0,\infty);\ f \text{ is analytic on } [0,\eta_0],\ \text{monotonically increasing, with } f(0) = 0, \}.$$

We assume the unique existence of the classical solution $u$ to (1.7a) – (1.7c), that is, $u \in C(\overline{\Omega} \times [0, T])$ satisfies

$$\frac{\partial u}{\partial x_j} \in C(\overline{\Omega} \times (0,T)), \quad \partial_t u, \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(\Omega \times (0,T)), \quad 1 \leq i, j \leq n. \tag{1.8}$$

We can prove the unique existence of the classical solution under suitable assumptions on $f$ and $\varphi$ by a standard method (e.g., [6]). However, in this paper, for concentrating on the inverse problem, we assume the unique existence of the classical solution to (1.7a) – (1.7c) on $[0, T]$ and denote it by $u_f = u_f(x,t)$.

Now we are ready to state

**Theorem 1.1.** *(Uniqueness for the inverse problem)*
*Let $f, g \in \mathcal{U}$, $x_0 \in \partial\Omega$ and $T > 0$ be arbitrarily fixed. If $u_f(x_0,t) = u_g(x_0,t)$ for $0 < t < T$, then $f = g$ on $[0, \eta_0]$.*

With a finer lower estimate of the fundamental solution (e.g., [2]), we can prove a conditional stability estimate in determining $f$. Similarly we can discuss the case where (1.7b) is replaced by a Dirichlet boundary condition and Neumann data as the extra observation data are adopted, but we omit the details.

The paper is composed of three sections. Section 2 is devoted to the comparison principle (Lemma 2.3) for (1.7a), which is interesting itself. In Section 3, we complete the proof of Theorem 1.1.

## 2 Comparison principle

Let the function $B = B(x,t)$ be smooth on $\overline{\Omega} \times [0,T]$. More precisely, $B$ should be in the Schauder space $C^{\gamma,\gamma/2}(\overline{\Omega} \times [0,T])$ with some $\gamma \in (0,1)$. For $0 < s < t < T$, let $U(x,t,y,s) = U_B(x,t,y,s)$ be the fundamental solution to $\partial_t U - \Delta U - B(x,t)U = 0$ in $\Omega \times (0,T)$ with the boundary condition $\partial_\nu U = 0$. Then

$$U(x,t,y,s) > 0 \quad \text{for } x, y \in \overline{\Omega} \text{ and } 0 < s < t \tag{2.1}$$

and the solution $v$ to

$$\begin{aligned} \partial_t v &= \Delta v + B(x,t)v + F \quad \text{in } \Omega \times (0,T), \\ v(x,0) &= 0, \quad x \in \Omega, \\ \partial_\nu v &= \varphi \quad \text{on } \partial\Omega \times (0,T), \end{aligned}$$

is represented by

$$\begin{aligned} v(x,t) = &\int_0^t \int_\Omega U(x,t,y,s)F(y,s)dyds \\ &+ \int_0^t \int_{\partial\Omega} U(x,t,y,s)\varphi(y,s)dS_y ds, \qquad x \in \overline{\Omega},\ 0 < t < T \end{aligned} \tag{2.2}$$

(see, e.g., [10]). First we show

**Lemma 2.1.** *Let a coagulation growth kinetic function $g \in \mathcal{U}$. Then $u_g(x,t) > 0$ for all $x \in \overline{\Omega}$ and $0 < t \le T$.*

*Proof.* For simplicity, we set $u = u_g$ and $B_0(x,t) = -\exp\left(-\int_0^t g(u_g(x,\xi))d\xi\right)$. Then we have

$$\partial_t u - \Delta u = B_0(x,t)u(x,t), \quad x \in \Omega,\ 0 < t < T.$$

Therefore with (2.2) for $U_{B_0}$, we have

$$u_g(x,t) = \int_0^t \int_{\Gamma_1} U_{B_0}(x,t,y,s)\varphi(y,s)dS_y ds, \qquad x \in \overline{\Omega},\ 0 < t < T.$$

Hence by (2.1) and $\varphi(\cdot,t) > 0$ on $\Gamma_1$ for $t > 0$, the proof is completed. □

**Lemma 2.2.** *(Uniform boundedness)*
*There exists a constant $M_0 > 0$ dependent on $\mathcal{U}$, $\Omega$, $T$, $\varphi$ and independent of choices of $g$, such that*

$$\|u_g\|_{L^\infty(\Omega \times (0,T))} \le M_0$$

*for all $g \in \mathcal{U}$.*

*Proof.* Let $U_0 = U_0(x,t,y,s)$ be the fundamental solution to $\partial_t U_0 - \Delta U_0 = 0$ in $\Omega \times (0,T)$ with the boundary condition $\partial_\nu U_0 = 0$. Then we have

$$u_g(x,t) = -\int_0^t \int_\Omega U_0(x,t,y,s) u_g(u,s) \exp\left(-\int_0^s g(u_g(y,\xi))d\xi\right) dyds$$
$$+ \int_0^t \int_{\Gamma_1} U_0(x,t,y,s)\varphi(y,s)dS_y ds, \quad x \in \overline{\Omega},\, 0 < t < T$$

(e.g., [10]). Noticing $g \in \mathcal{U}$ we have $g \ge 0$ and

$$\exp\left(-\int_0^s g(u_g(y,\xi))d\xi\right) \le 1, \quad y \in \overline{\Omega},\, 0 < s < T.$$

Therefore by (2.1) we obtain

$$\begin{aligned}\|u_g(\cdot,t)\|_{L^\infty(\Omega)} \le & \int_0^t \int_\Omega U_0(x,t,y,s)\|u_g(\cdot,s)\|_{L^\infty(\Omega)} dyds \\ & + \int_0^t \int_{\partial\Omega} U_0(x,t,y,s)dS_y ds \|\varphi\|_{L^\infty(\Gamma_1\times(0,T))} \\ =: & \, J_1(x,t) + J_2(x,t)\|\varphi\|_{L^\infty(\Gamma_1\times(0,T))}, \quad 0 < t < T. \end{aligned} \tag{2.3}$$

Since there holds

$$\int_\Omega U_0(x,t,y,s)dy \le Ce^{C(t-s)}, \quad x \in \Omega,\, 0 < s < t < T \tag{2.4}$$

(e.g., Itô [10], Chapter IV in Ladyzenskaja, Solonnikov and Ural'ceva [12]), we have

$$J_1(t) \le Ce^{2CT} \int_0^t \|u_g(\cdot,s)\|_{L^\infty(\Omega)} ds, \quad 0 < t < T. \tag{2.5}$$

Here and henceforth $C > 0$ denotes generic constants which are independent of choices of $g \in \mathcal{U}$, but dependent on $\Omega, \mathcal{U}, T, \Omega, \varphi$. Next we estimate $J_2(x,t) := \int_0^t \int_{\partial\Omega} U_0(x,t,y,s)dS_y ds$. We consider the solution $J_3$ to

$$\begin{cases} \partial_t J_3(x,t) = \Delta J_3(x,t), \quad x \in \Omega,\, 0 < t < T, \\ \partial_\nu J_3(x,t) = 1, \quad x \in \partial\Omega,\, 0 < t < T, \\ J_3(x,0) = 1, \quad x \in \Omega. \end{cases}$$

Then, by the a priori estimate (e.g., Theorem 5.3 (pp. 320-321) in [12]), we see that $\|J_3\|_{L^\infty(\Omega\times(0,T))} \le C$. Let $J_4 = J_4(x,t)$ satisfy

$$\begin{cases} \partial_t J_4(x,t) = \Delta J_4(x,t), \quad x \in \Omega,\, 0 < t < T, \\ \partial_\nu J_4(x,t) = 0, \quad x \in \partial\Omega,\, 0 < t < T, \\ J_4(x,0) = 1, \quad x \in \Omega. \end{cases}$$

Then (2.4) and $J_4(x,t) = \int_\Omega U_0(x,t,y,0)dy$ imply $\|J_4\|_{L^\infty(\Omega\times(0,T))} \le C$. Since $J_2 = J_3 + J_4$ in $\Omega \times (0,T)$, we have

$$\|J_2\|_{L^\infty(\Omega\times(0,T))} \le C. \tag{2.6}$$

Implementing (2.3), (2.5) and (2.6), we obtain

$$\|u_g(\cdot,t)\|_{L^\infty(\Omega)} \le C + C\int_0^t \|u_g(\cdot,s)\|_{L^\infty(\Omega)} ds, \quad 0 < t < T.$$

The Gronwall inequality completes the proof of Lemma 2.2. $\square$

Next, we prove

**Lemma 2.3.** *(Comparison principle)*
*Let $f, g \in \mathcal{U}$ and $f \le g$ on $[0, \eta_0]$. Then there exists a small constant $t_1 > 0$ such that*

$$u_f(x,t) \ge u_g(x,t), \qquad x \in \Omega,\ 0 < t < t_1.$$

*Proof.* Owing to the integral term in (1.7a), it is very difficult to prove the lemma directly. Therefore we construct $u_f$ by the contraction mapping theorem in a suitable set for the proof. Let $u = u_f - u_g$. Then

$$\begin{aligned}\partial_t u - \Delta u = & -u \exp\left(-\int_0^t f(u(x,\xi) + u_g(x,\xi))d\xi\right) \\ & - u_g\left(\exp\left(-\int_0^t f(u+u_g)d\xi\right) - \exp\left(-\int_0^t f(u_g)d\xi\right)\right) \\ & - u_g\left(\exp\left(-\int_0^t f(u_g)d\xi\right) - \exp\left(-\int_0^t g(u_g)d\xi\right)\right), \quad x \in \Omega,\ t > 0, \\ & \partial_\nu u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0\end{aligned}$$

and

$$u(x,0) = 0, \qquad x \in \Omega.$$

Therefore, although $u_f$ is already assumed to exist, we reconstruct $u_f$ as follows. First let $M > 0$ be an arbitrarily fixed constant. We set

$$\mathcal{W} = \{z \in C([0,t_1]; L^\infty(\Omega));\ z \ge 0 \quad \text{in } \Omega \times (0,t_1), \quad \|z\|_{C([0,t_1];L^\infty(\Omega))} \le M\}.$$

Later we will choose a specific constant $t_1 > 0$. Let the mapping $v = K(z)$, $z \in \mathcal{W}$, be defined by

$$\begin{aligned}\partial_t v - \Delta v \;=\; & -v \exp\left(-\int_0^t f(z(x,\xi) + u_g(x,\xi))d\xi\right) \\ & -u_g\left(\exp\left(-\int_0^t f(z+u_g)d\xi\right) - \exp\left(-\int_0^t f(u_g)d\xi\right)\right) \\ & -u_g\left(\exp\left(-\int_0^t f(u_g)d\xi\right) - \exp\left(-\int_0^t g(u_g)d\xi\right)\right), \ x \in \Omega,\ t > 0 \quad \text{(2.7a)}\end{aligned}$$

$$\partial_\nu v(x,t) = 0,\ x \in \partial\Omega,\ t > 0 \qquad \text{(2.7b)}$$

$$v(x,0) = 0, \qquad x \in \Omega. \qquad \text{(2.7c)}$$

We need to prove

$$K(z)(x,t) \ge 0, \quad (x,t) \in \Omega \times (0,t_1). \qquad \text{(2.8)}$$

Note that we can prove $u_g > 0$ in $\Omega \times (0,T)$ by Lemma 2.1. Let $z \in \mathcal{W}$. Set $B_1(x,t) = \exp\left(-\int_0^t f(z+u_g)d\xi\right)$. Then

$$\begin{aligned}& \partial_t v - \Delta v + B_1(x,t)v \\ = \; & -u_g\left(\exp\left(-\int_0^t f(z+u_g)d\xi\right) - \exp\left(-\int_0^t f(u_g)d\xi\right)\right) \\ - \; & u_g\left(\exp\left(-\int_0^t f(u_g)d\xi\right) - \exp\left(-\int_0^t g(u_g)d\xi\right)\right) \ge 0, \quad x \in \Omega,\ t > 0,\end{aligned}$$

$$\partial_\nu v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0$$

and

$$v(x,0) = 0, \qquad x \in \Omega.$$

Regarding the right-hand side of (2.7a) by $F \ge 0$ and applying (2.2) and in terms of (2.1), we have $v \ge 0$ in $\Omega \times (0,t_1)$. Hence $K(z) \ge 0$ in $\Omega \times (0,t_1)$.

Next we estimate $\|K(z)\|_{C([0,t_1];L^\infty(\Omega))}$ which satisfies $v = K(z)$ and

$$\begin{aligned}
&\partial_t v - \Delta v + B_1 v \\
= \;& -u_g\left(\exp\left(-\int_0^t f(z+u_g)d\xi\right) - \exp\left(-\int_0^t f(u_g)d\xi\right)\right) \\
- \;& u_g\left(\exp\left(-\int_0^t f(u_g)d\xi\right) - \exp\left(-\int_0^t g(u_g)d\xi\right)\right) := J_1 + J_2.
\end{aligned}$$

Henceforth $C > 0$ and $C_k > 0$, $k = 1,\ldots,7$, denote generic constants which are dependent on $f$ and $g$ but independent of choices of $z \in \mathcal{W}$. In terms of $f, g \in \mathcal{U}$, the mean value theorem and Lemma 2.2 yields

$$\begin{aligned}
\|J_1\|_{L^\infty(\Omega\times(0,t_1))} &\le C\left\|\int_0^t (f(z+u_g) - f(u_g))d\xi\right\|_{L^\infty(\Omega\times(0,t_1))} \\
&\le Ct_1\|f(z+u_g) - f(u_g)\|_{L^\infty(\Omega\times(0,t_1))} \le Ct_1\|z\|_{L^\infty(\Omega\times(0,t_1))}
\end{aligned}$$

and

$$\begin{aligned}
\|J_2\|_{L^\infty(\Omega\times(0,t_1))} &\le C\left\|\int_0^t (f(u_g) - g(u_g))d\xi\right\|_{L^\infty(\Omega\times(0,t_1))} \\
&\le Ct_1\|f(u_g) - g(u_g)\|_{L^\infty(\Omega\times(0,t_1))}.
\end{aligned}$$

Applying the fundamental solution $U = U_{B_1}$ to $\partial_t v - \Delta v + B_1 v = 0$ with the homogeneous Neumann boundary value condition, by (2.2) we obtain

$$\|v\|_{L^\infty(\Omega\times(0,t_1))} \le C_1\|J_1 + J_2\|_{L^\infty(\Omega\times(0,t_1))} \le C_2 t_1.$$

Select $t_1 > 0$ such that $C_2 t_1 \le M$. Then we verify that $K(z) \in \mathcal{W}$ if $z \in \mathcal{W}$.

Finally we estimate $\|K(z_1) - K(z_2)\|_{L^\infty(\Omega\times(0,t_1))}$ for $z_1, z_2 \in \mathcal{W}$. Let $v_1 = K(z_1)$ and $v_2 = K(z_2)$ and $v = v_1 - v_2$ to obtain

$$\begin{aligned}
\partial_t v - \Delta v &= \left\{-v_1 \exp\left(-\int_0^t f(z_1+u_g)d\xi\right) + v_2 \exp\left(-\int_0^t f(z_2+u_g)d\xi\right)\right\} \\
&- \left\{u_g \exp\left(-\int_0^t f(z_1+u_g)d\xi\right) - u_g \exp\left(-\int_0^t f(z_2+u_g)d\xi\right)\right\} \\
&:= I_1 + I_2.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 &= -v_1 \exp\left(-\int_0^t f(z_1+u_g)d\xi\right) + v_2 \exp\left(-\int_0^t f(z_1+u_g)d\xi\right) \\
&- v_2 \exp\left(-\int_0^t f(z_1+u_g)d\xi\right) + v_2 \exp\left(-\int_0^t f(z_2+u_g)d\xi\right)
\end{aligned}$$

$$
\begin{aligned}
&= -v \exp\left(-\int_0^t f(z_1+u_g)d\xi\right) \\
&\quad - v_2\left(\exp\left(-\int_0^t f(z_1+u_g)d\xi\right) - \exp\left(-\int_0^t f(z_2+u_g)d\xi\right)\right) \\
&:= -vB_2(x,t) + I_{12}
\end{aligned}
$$

and

$$
I_2 = -u_g\left(\exp\left(-\int_0^t f(z_1+u_g)d\xi\right) - \exp\left(-\int_0^t f(z_2+u_g)d\xi\right)\right).
$$

By the mean value theorem and Lemma 2.2, we have

$$
\begin{aligned}
\|I_{12}\|_{L^\infty(\Omega\times(0,t_1))} &\le C_3\left\|\int_0^t f(z_1+u_g)d\xi - \int_0^t f(z_2+u_g)d\xi\right\|_{L^\infty(\Omega\times(0,t_1))} \\
&\le C_4 t_1 \|z_1 - z_2\|_{L^\infty(\Omega\times(0,t_1))}
\end{aligned}
$$

and

$$
\begin{aligned}
\|I_2\|_{L^\infty(\Omega\times(0,t_1))} &\le C_5\left\|\exp\left(-\int_0^t f(z_1+u_g)d\xi\right) - \exp\left(-\int_0^t f(z_2+u_g)d\xi\right)\right\|_{L^\infty(\Omega\times(0,t_1))} \\
&\le C_6 t_1 \|z_1 - z_2\|_{L^\infty(\Omega\times(0,t_1))}.
\end{aligned}
$$

Using the fundamental solution to $\partial_t v - \Delta v + v\exp\left(-\int_0^t f(z_1+u_g)d\xi\right)$ with the homogeneous Neumann boundary condition, we similarly obtain

$$
\|v\|_{L^\infty(\Omega\times(0,t_1))} \le C_7 t_1 \|z_1 - z_2\|_{L^\infty(\Omega\times(0,t_1))}.
$$

Therefore we further choose $t_1 > 0$ such that $C_7 t_1 < 1$ and we see that $K : \mathcal{W} \longrightarrow \mathcal{W}$ is a contraction mapping. Hence $K$ possesses a unique fixed point $q$. By the uniqueness of the mild solution (e.g., [14]), it follows that $u_f = u_g + q$. By (2.1) we have $q \in \mathcal{U}$, in particular, $q \ge 0$ in $\Omega \times (0, t_1)$. Therefore $u_f \ge u_g$ in $\Omega \times (0, t_1)$. □

# 3 Proof of Theorem 1.1

We complete the proof of the main result.

Since $u_g(x_0, t) > 0$ for $0 < t < t_1$, and the intermediate value theorem, we have

$$
\{u_g(x_0,t);\, 0 \le t \le t_1\} = [0, \ell_0]
$$

with some constant $\ell_0 > 0$. We can choose $t_1$ small if necessary, and so $\ell_0 < \eta_0$ and thus $f$ and $g$ are analytic in $[0, \ell_0]$. Moreover, since $f - g$ is analytic in a neighborhood of $0$, the zeros of $f - g$ have no accumulation points, and we can choose small $\ell_1 > 0$ such that

$$
f(\xi) \ge g(\xi), \quad 0 \le \xi \le \ell_1
$$

or

$$
f(\xi) \le g(\xi), \quad 0 \le \xi \le \ell_1.
$$

Without loss of generality, we can assume that $f(\xi) \geq g(\xi), \quad 0 \leq \xi \leq \ell_1$ and denote $\ell = \min\{\ell_0, \ell_1\}$. Then

$$f(\xi) \geq g(\xi), \quad 0 \leq \xi \leq \ell. \tag{3.1}$$

Define

$$A = \max_{0 \leq \xi \leq \ell} |f(\xi) - g(\xi)| = \max_{0 \leq \xi \leq \ell} (f(\xi) - g(\xi)). \tag{3.2}$$

We choose $\eta \in [0, \ell]$ such that

$$A = (f - g)(\eta). \tag{3.3}$$

If $A = 0$, then the proof is already finished and so we can assume that $A > 0$. Moreover we can assume that $\eta > 0$. Otherwise we have $f(\cdot) = g(\cdot)$ in the non-empty interval $(0, \eta)$ with $\eta > 0$, which completes the proof by the analyticity of $f$ and $g$.

Since $\eta > 0$ and $\eta \in \{u_g(x_0, t);\ 0 \leq t \leq t_1\}$, there exists $t_0 \in (0, t_1]$ such that $u_g(x_0, t_0) = \eta$. We set $u = u_f - u_g$. Then

$$\begin{aligned}
& \partial_t u - \Delta u \\
= \quad & -u_f \exp\left(-\int_0^t f(u_f(x,\xi))d\xi\right) + u_g \exp\left(-\int_0^t f(u_f(x,\xi))d\xi\right) \\
- \quad & u_g \exp\left(-\int_0^t f(u_f(x,\xi))d\xi\right) + u_g \exp\left(-\int_0^t f(u_g(x,\xi))d\xi\right) \\
- \quad & u_g \exp\left(-\int_0^t f(u_g(x,\xi))d\xi\right) + u_g \exp\left(-\int_0^t g(u_g(x,\xi))d\xi\right) \\
= \quad & B_3(x,t)u(x,t) - u_g \left\{\exp\left(-\int_0^t f(u_f(x,\xi))d\xi\right) - \exp\left(-\int_0^t f(u_g(x,\xi))d\xi\right)\right\} \\
- \quad & u_g \left\{\exp\left(-\int_0^t f(u_g(x,\xi))d\xi\right) - \exp\left(-\int_0^t g(u_g(x,\xi))d\xi\right)\right\} \\
=: \quad & B_3 u + S_1 + S_2,
\end{aligned}$$

where we consequently set

$$\begin{aligned}
B_3(x,t) &= -\exp\left(-\int_0^t f(u_f(x,\xi))d\xi\right) \\
S_1 &= -u_g \left\{\exp\left(-\int_0^t f(u_f(x,\xi))d\xi\right) - \exp\left(-\int_0^t f(u_g(x,\xi))d\xi\right)\right\} \\
S_2 &= -u_g \left\{\exp\left(-\int_0^t f(u_g(x,\xi))d\xi\right) - \exp\left(-\int_0^t g(u_g(x,\xi))d\xi\right)\right\}.
\end{aligned}$$

Hence we rewrite the above equation by

$$\partial_t u - \Delta u - B_3(x,t)u = S_1 + S_2 \quad \text{in } \Omega \times (0, t_0), \tag{3.4a}$$

$$\partial_\nu u = 0 \quad \text{on } \partial\Omega \times (0, t_0), \tag{3.4b}$$

$$u(\cdot, 0) = 0 \quad \text{in } \Omega. \tag{3.4c}$$

Lemma 2.1 yields $u_g \geq 0$. By (3.1) and Lemma 2.3, we have $u_f \geq u_g$. Hence we have

$$S_1, S_2 \geq 0 \quad \text{in } \Omega \times (0, t_0). \tag{3.5}$$

Let $U = U_{B_3}$ be the fundamental solution for $\partial_t - \Delta - B_3$ with the homogeneous Neumann boundary condition. Then

$$u(x,t) = \int_0^t \int_\Omega U(x,t,y,s)S_1(y,s)dyds \\ + \int_0^t \int_\Omega U(x,t,y,s)S_2(y,s)dyds, \quad x \in \overline{\Omega},\ 0 < t < t_0.$$

Since $u(x_0,t) = 0$ for $0 < t < t_0$, we have

$$\int_0^{t_0} \int_\Omega U(x_0,t_0,y,s)S_1(y,s)dyds + \int_0^{t_0} \int_\Omega U(x_0,t_0,y,s)S_2(y,s)dyds = 0,\ x \in \partial\Omega,\ 0 < t < t_0.$$

Therefore (3.5) yields

$$\int_0^{t_0} \int_\Omega U(x_0,t_0,y,s)S_2(y,s)dyds = 0, \quad x \in \partial\Omega,\ 0 < t < t_0.$$

By (2.1), we have $S_2(x_0,t) = 0$ for almost all $t \in (0,t_0)$, that is,

$$u_g(x_0,t)\left\{\exp\left(-\int_0^t f(u_g)(x_0,\xi)d\xi\right) - \exp\left(-\int_0^t g(u_g)(x_0,\xi)d\xi\right)\right\} = 0$$

for almost all $t \in (0,t_0)$. Lemma 2.1 yields $u_g(x_0,t) > 0$ for $t > 0$, and

$$\exp\left(-\int_0^t f(u_g)(x_0,\xi)d\xi\right) = \exp\left(-\int_0^t g(u_g)(x_0,\xi)d\xi\right)$$

for almost all $0 < t < t_0$, in particular, $f(u_g(x_0,t)) = g(u_g(x_0,t))$ for $0 < t < t_0$. Therefore $f(u_g(x_0,t_0)) = g(u_g(x_0,t_0))$. Since $u_g(x_0,t_0) = \eta$, by (3.2) and (3.3) we see that $f(\xi) = g(\xi)$ for $0 \le \xi \le \ell$, which is a contradiction. Thus $A = 0$ holds, that is, the proof of the theorem is completed.

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