Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Singular limit of Allen–Cahn equation with constraints and its Lagrange multiplier

M. Hassan Farshbaf-Shaker[1], Takeshi Fukao[2], Noriaki Yamazaki[3]

submitted: December 16, 2014

[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Hassan.Farshbaf-Shaker@wias-berlin.de

[2] Kyoto University of Education
Department of Mathematics
1 Fujinomori
Fukakusa, Fushimi-ku
Kyoto, 612-8522
Japan
E-Mail: fukao@kyokyo-u.ac.jp

[3] Kanagawa University
Department of Mathematics
Faculty of Engineering
3-27-1 Rokkakubashi
Kanagawa-ku
Yokohama, 221-8686
Japan
E-Mail: noriaki@kanagawa-u.ac.jp

No. 2052
Berlin 2014

2010 *Mathematics Subject Classification.* 35K57, 35R35, 35B25.

Key words and phrases. Allen–Cahn equation, constraint, double obstacle, singular limit, Lagrange multiplier, subdifferential.

This research is supported by Grant-in-Aid No. 26400164 and No. 26400179, JSPS.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider the Allen-Cahn equation with constraint. Our constraint is the subdifferential of the indicator function on the closed interval, which is the multivalued function. In this paper we give the characterization of the Lagrange multiplier to our equation. Moreover, we consider the singular limit of our system and clarify the limit of the solution and the Lagrange multiplier to our problem.

1 Introduction

In this paper, for each $\varepsilon \in (0,1]$ we consider the following Allen-Cahn equation with constraint:

$$u^\varepsilon_t - \Delta u^\varepsilon + \frac{1}{\varepsilon^2}\partial I_{[-1,1]}(u^\varepsilon) \ni \frac{1}{\varepsilon^2}u^\varepsilon \ \text{ in } Q := (0,T)\times\Omega, \tag{1.1}$$

$$\frac{\partial u^\varepsilon}{\partial \nu} = 0 \ \text{ on } \Sigma := (0,T)\times\Gamma, \tag{1.2}$$

$$u^\varepsilon(0,x) = u^\varepsilon_0(x), \ \ x\in\Omega, \tag{1.3}$$

where $0 < T < +\infty$, Ω is a bounded domain in $\mathbb{R}^N$ $(1 \le N < +\infty)$ with smooth boundary $\Gamma := \partial\Omega$, ν is an outward normal vector on Γ and u^ε_0 is a given initial data. Also, $\partial I_{[-1,1]}(\cdot)$ is the subdifferential of the indicator function $I_{[-1,1]}(\cdot)$ on the closed interval $[-1,1]$ defined by

$$I_{[-1,1]}(z) := \begin{cases} 0, & \text{if } z\in[-1,1], \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4}$$

More precisely, $\partial I_{[-1,1]}(\cdot)$ is a set-valued mapping defined by

$$\partial I_{[-1,1]}(z) := \begin{cases} \emptyset & \text{if } z<-1 \text{ or } z>1, \\ [0,\infty) & \text{if } z=1, \\ \{0\} & \text{if } -1<z<1, \\ (-\infty,0] & \text{if } z=-1. \end{cases} \tag{1.5}$$

The Allen-Cahn equation was proposed to describe the macroscopic motion of phase boundaries. In the physical context, the function $u^\varepsilon = u^\varepsilon(t,x)$ in (P)$^\varepsilon$ is the nonconserved order parameter that characterizes the physical structure: $u^\varepsilon = 1$, $-1 < u^\varepsilon < 1$ and $u^\varepsilon = -1$ correspond respectively to pure liquid, mixture and pure solid.

There are vast literatures of Allen-Cahn equation with or without constraint. For such works, we refer to [1, 3, 7, 8, 9, 11, 15, 18], for instance. In particular, Chen and Elliott [8] considered the asymptotic behavior of the solution to (P)$^\varepsilon$ as $\varepsilon \to 0$. However, there was no information of an element of $\partial I_{[-1,1]}(u^\varepsilon)$ in [8].

In this paper, for each $\varepsilon \in (0,1]$ we consider an element $\lambda^\varepsilon \in \partial I_{[-1,1]}(u^\varepsilon)$, which is called the Lagrange multiplier to (P)$^\varepsilon$:={(1.1), (1.2), (1.3)}. Also, we investigate the limiting observation of λ^ε as

$\varepsilon \to 0$. Namely, we consider the singular limit of our system (P)$^{\varepsilon}$ and clarify the limiting of the solution u^{ε} and the Lagrange multiplier λ^{ε} to (P)$^{\varepsilon}$ as $\varepsilon \to 0$.

Recently, elliptic and parabolic variational inequalities were considered in connection with Lagrange multipliers (cf. [3, 4, 10, 13]). Note from the constraint that the notion of solution to (P)$^{\varepsilon}$ is given in variational sense (cf. Remark 2.1 below). Therefore, it is worthy considering the Lagrange multiplier to (P)$^{\varepsilon}$. Also, we are very interested in the limit of the Lagrange multiplier to (P)$^{\varepsilon}$ as $\varepsilon \to 0$.

This present paper aims to consider the Lagrange multiplier λ^{ε} and the singular limit of (P)$^{\varepsilon}$ as $\varepsilon \to 0$. The main novelties found in this paper are the following:

(i) We give the characterization of the Lagrange multiplier λ^{ε} to (P)$^{\varepsilon}$.

(ii) We show the convergence of the solution u^{ε} and the Lagrange multiplier λ^{ε} to (P)$^{\varepsilon}$ as $\varepsilon \to 0$.

(iii) We clarify the properties of the limit of u^{ε} and λ^{ε} as $\varepsilon \to 0$.

The plan of this paper is as follows. In Section 2, we state the main results in this paper. In Section 3 we recall the decomposition result of the subdifferential of convex functions. Also, we prove the main result (Theorem 2.1) concerning the existence-uniqueness of solutions to (P)$^{\varepsilon}$ and properties of the Lagrange multiplier λ^{ε}. In Section 4, we prove Theorem 2.2 corresponding to the item (ii) and (iii) listed in the above.

Notations and basic assumptions

Throughout this paper, for any reflexive Banach space B, we denote $|\cdot|_B$ the norm of B, and denote by B^* the dual space of B.

In particular, we put $H := L^2(\Omega)$ with usual real Hilbert space structure, and denote by $(\cdot,\cdot)_H$ the inner product in H. Also, we put $V := H^1(\Omega)$ with the usual norm

$$|z|_V := \left\{|z|_H^2 + |\nabla z|_H^2\right\}^{\frac{1}{2}}, \quad z \in V,$$

and denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V. By identifying H with its dual space, we have $V \subset H \subset V^*$ with compact and dense embeddings; then,

$$\langle u, v\rangle = (u, v)_H \text{ for } u \in H \text{ and } v \in V. \tag{1.6}$$

In the proof of Theorem 2.1, we use some techniques of proper (that is, not identically equal to infinity), l.s.c. (lower semi-continuous), convex functions and their subdifferentials, which are useful in the systematic study of variational inequalities. Therefore, let us outline some notations and definitions. For a proper, l.s.c. and convex function $\psi : H \to \mathbb{R} \cup \{+\infty\}$, the effective domain $D(\psi)$ is defined by

$$D(\psi) := \{z \in H;\ \psi(z) < \infty\}.$$

The subdifferential of ψ is a possibly multi-valued operator in H and is defined by $z^* \in \partial\psi(z)$ if and only if

$$z \in D(\psi) \quad \text{and} \quad (z^*, y - z)_H \le \psi(y) - \psi(z) \quad \text{for all } y \in H.$$

For various properties and related notions of the proper, l.s.c., convex function ψ and its subdifferential $\partial\psi$, we refer to a monograph by Brézis [5].

Next, let us give an assumption on initial data. Throughout this paper, we assume the following condition (A):

(A) $u_0^\varepsilon \in K := \{z \in V \ ; \ |z| \le 1$ a.e. in $\Omega\}$ for all $\varepsilon \in (0,1]$.

Finally, throughout this paper, $C_i = C_i(\cdot), i = 1, 2, 3, \cdots$, denotes positive (or nonnegative) constants depending only on its arguments.

2 Main results

We begin by giving the rigorous definition of solutions to our problem (P)$^\varepsilon$ ($\varepsilon \in (0,1]$).

Definition 2.1 *For each $\varepsilon \in (0,1]$, a function $u^\varepsilon : [0,T] \to H$ is called a solution to $(\mathrm{P})^\varepsilon$ on $[0,T]$, if the following conditions are satisfied:*

(i) $u^\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$.

(ii) There is a function $\lambda^\varepsilon \in L^2(0,T;H)$ with $\lambda^\varepsilon \in \partial I_{[-1,1]}(u^\varepsilon)$ a.e. in Q such that

$$(u_t^\varepsilon(t), z)_H + (\nabla u^\varepsilon(t), \nabla z)_H + \frac{1}{\varepsilon^2}\left(\lambda^\varepsilon(t), z\right)_H = \frac{1}{\varepsilon^2}\left(u^\varepsilon(t), z\right)_H$$

for all $z \in V$ and a.e. $t \in (0,T)$.

(iii) $u^\varepsilon(0) = u_0^\varepsilon$ *in* H.

We call λ^ε in (ii) a Lagrange multiplier to $(\mathrm{P})^\varepsilon$ on $[0,T]$.

Remark 2.1 *It follows from the constraint $\partial I_{[-1,1]}(\cdot)$ and (ii) of Definition 2.1 that the equation* (1.1) *is equivalent to the following variational inequality:*

$$\left(u_t^\varepsilon(t) - \frac{1}{\varepsilon^2}u^\varepsilon(t), u^\varepsilon(t) - z\right)_H + (\nabla u^\varepsilon(t), \nabla u^\varepsilon(t) - \nabla z)_H \le 0$$

for all $z \in K$ and a.e. $t \in (0,T)$.

Now, let us mention the first main result in this paper, which is concerned with the existence and basic property of the solution and the Lagrange multiplier to (P)$^\varepsilon$ on $[0,T]$.

Theorem 2.1 *Assume* (A). *Then, for each $\varepsilon \in (0,1]$, there exist a unique solution u^ε to $(\mathrm{P})^\varepsilon$ on $[0,T]$ and a Lagrange multiplier λ^ε in the sense of Definition 2.1 such that*

$$\lambda^\varepsilon(t,x) \begin{cases} \ge 0 & \text{on } \{(t,x) \in Q \ ; \ u^\varepsilon(t,x) = 1\}, \\ = 0 & \text{on } \{(t,x) \in Q \ ; \ -1 < u^\varepsilon(t,x) < 1\}, \\ \le 0 & \text{on } \{(t,x) \in Q \ ; \ u^\varepsilon(t,x) = -1\}. \end{cases} \tag{2.1}$$

In next Section 3, we give the proof of Theorem 2.1.

Next, we consider the limiting situation of (P)$^\varepsilon$ as $\varepsilon \to 0$. To do so, we use the following energy functional:

$$\mathscr{F}^\varepsilon(u) := \int_\Omega \left\{ \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} I_{[-1,1]}(u) + \frac{1-u^2}{2\varepsilon} \right\} dx, \ \ u \in V. \tag{2.2}$$

Now we state the second main result in this paper, which is concerned with the singular limit of (P)$^\varepsilon$ as $\varepsilon \to 0$:

Theorem 2.2 *Assume* (A)*. For each* $\varepsilon \in (0,1]$*, let* u^ε *be the unique solution to* (P)$^\varepsilon$ *on* $[0,T]$*. Also, let* λ^ε *be the Lagrange multiplier to* (P)$^\varepsilon$ *on* $[0,T]$ *in the sense of Definition 2.1. Assume that there are the function* $u_0 \in L^1(\Omega)$ *and a positive constant* M*, independent of* $\varepsilon \in [0,1]$*, satisfying* $u_0(x) = 1$ *or* -1 *for a.e.* $x \in \Omega$*,*

$$\sup_{\varepsilon \in [0,1]} \mathscr{F}^\varepsilon(u_0^\varepsilon) < M \tag{2.3}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega |u_0^\varepsilon(x) - u_0(x)| dx = 0. \tag{2.4}$$

Then, there are a subsequence $\{\varepsilon_k\}$ *of* $\{\varepsilon\}$ *with* $\varepsilon_k \searrow 0$ *as* $k \to \infty$*, the functions* $u \in L^2(0,T;H)$ *and* $\lambda^* \in L^2(0,T;V^*)$ *and a positive number* N_0*, independent of* $\varepsilon \in (0,1]$*, such that* $u(t,x)$ *takes only the values* 1 *or* -1 *for a.e.* $(t,x) \in (0,T) \times \Omega$*,*

$$\lim_{k \to \infty} u^{\varepsilon_k}(t,x) = u(t,x), \ \ \text{a.e. } (t,x) \in Q, \tag{2.5}$$

$$\int_\Omega |u(t_1,x) - u(t_2,x)|\, dx \le N_0 |t_1 - t_2|^{\frac{1}{2}}, \quad \forall t_1, t_2 \in [0,T], \tag{2.6}$$

$$\lim_{t \to 0} u(t,x) = u_0(x), \ \ \text{a.e. } x \in \Omega, \tag{2.7}$$

$$\int_\Omega |\nabla u(t)| \le N_0, \quad \text{a.e. } t \in (0,T) \tag{2.8}$$

and

$$\lambda^{\varepsilon_k} \longrightarrow \lambda^* \text{ weakly in } L^2(0,T;V^*) \ \text{ as } k \to \infty. \tag{2.9}$$

Moreover, $\lambda^* - u = 0$ *in* $L^2(0,T;V^*)$*, hence,*

$$\lambda^* = 1 \text{ on } \{(t,x) \in Q \,;\, u(t,x) = 1\} \tag{2.10}$$

and

$$\lambda^* = -1 \text{ on } \{(t,x) \in Q \,;\, u(t,x) = -1\}. \tag{2.11}$$

In Section 4 we prove Theorem 2.2 by using a priori estimates of u^ε and λ^ε.

3 Solvability of (P)$^\varepsilon$

In this section we consider (P)$^\varepsilon$ for each $\varepsilon \in (0,1]$. In fact, we study (P)$^\varepsilon$ by arguments similar to [14, 17], namely by the theory of abstract evolution equations governed by subdifferentials.

Now, we define a functional φ_0 on H by

$$\varphi_0(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & \text{if } z \in V, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.1}$$

Clearly, φ_0 is proper, l.s.c. and convex on H.

Also, we define the proper, l.s.c. and convex functional $\mathcal{I}_{[-1,1]}$ of H by

$$\mathcal{I}_{[-1,1]}(z) := \int_\Omega I_{[-1,1]}(z)dx \quad \text{for any } z \in H,$$

where $I_{[-1,1]}$ is the indicator function defined in (1.4).

Next, we consider the functional φ defined by the form:

$$\varphi(z) = \varphi_0(z) + \frac{1}{\varepsilon^2}\mathcal{I}_{[-1,1]}(z) \quad \text{for any } z \in H.$$

Clearly, φ is proper, l.s.c. and convex on H with the effective domain $D(\varphi) = K$, where K is the set defined in (A).

Here, we recall the following decomposition result of the subdifferential $\partial\varphi$.

Proposition 3.1 (cf. [6, Section 3], [17, Theorem 3.1]) *The subdifferential $\partial\varphi$ of φ is decomposed into the following form:*

$$\partial\varphi(z) = \partial\varphi_0(z) + \frac{1}{\varepsilon^2}\partial\mathcal{I}_{[-1,1]}(z) \text{ in } H \quad \text{for any } z \in H.$$

By arguments similar to [6, Section 3] and [17, Theorem 3.1], we can prove Proposition 3.1, so, omit its detailed proof.

Now, we prove Theorem 2.1 by using Proposition 3.1 and applying the abstract theory of nonlinear evolution equations associated with subdifferential $\partial\varphi$.

Proof of Theorem 2.1. By the similar arguments as in [14, Section 1], we can show the existence-uniqueness of a solution u^ε to (P)$^\varepsilon$ on $[0,T]$ for each $\varepsilon \in (0,1]$. In fact, we easily prove the uniqueness of solutions to (P)$^\varepsilon$ on $[0,T]$ by the quite standard arguments: monotonicity and Gronwall's inequality.

Now, we show the existence of solutions to (P)$^\varepsilon$ on $[0,T]$. We easily see that the problem (P)$^\varepsilon$ can be rewritten in an abstract framework of the form:

$$(\mathrm{CP})^\varepsilon \begin{cases} \dfrac{d}{dt}u^\varepsilon(t) + \partial\varphi(u^\varepsilon(t)) - \dfrac{1}{\varepsilon^2}u^\varepsilon(t) \ni 0 \ \text{ in } H, \ \text{for } t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon \ \text{ in } H. \end{cases} \tag{3.2}$$

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [6, 12, 16]), we can show the existence of a solution u^ε to (P)$^\varepsilon$ on $[0,T]$ for each $\varepsilon \in (0,1]$ in the variational sense (cf. Remark 2.1).

Also, note from Proposition 3.1 that (CP)$^\varepsilon$ is equivalent to the following:

$$\widetilde{(\mathrm{CP})}^\varepsilon \begin{cases} \dfrac{d}{dt}u^\varepsilon(t) + \partial\varphi_0(u^\varepsilon(t)) + \dfrac{1}{\varepsilon^2}\partial\mathcal{I}_{[-1,1]}(u^\varepsilon(t)) - \dfrac{1}{\varepsilon^2}u^\varepsilon(t) \ni 0 \ \text{ in } H, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon \ \text{ in } H. \end{cases} \tag{3.3}$$

Namely, there are functions $v^\varepsilon \in L^2(0,T;H)$ and $\lambda^\varepsilon \in L^2(0,T;H)$ such that $v^\varepsilon(t) \in \partial\varphi_0(u^\varepsilon(t))$ a.e. in $(0,T)$, $\lambda^\varepsilon \in \partial I_{[-1,1]}(u^\varepsilon)$ a.e. in Q and (3.3) holds in the following sense:

$$\frac{d}{dt}u^\varepsilon(t) + v^\varepsilon(t) + \frac{1}{\varepsilon^2}\lambda^\varepsilon(t) - \frac{1}{\varepsilon^2}u^\varepsilon(t) = 0 \text{ in } H, \text{ for } t > 0.$$

Thus, from the characterization of $\partial\varphi_0$, we easily see that u^ε is a solution to (P)$^\varepsilon$ on $[0,T]$ and λ^ε is the Lagrange multiplier to (P)$^\varepsilon$ on $[0,T]$ in the sense of Definition 2.1.

Taking account of the definition (1.5) of $\partial I_{[-1,1]}(\cdot)$, we conclude from $\lambda^\varepsilon \in \partial I_{[-1,1]}(u^\varepsilon)$ a.e. in Q that the signature result (2.1) of the Lagrange multiplier λ^ε holds. Thus, the proof of Theorem 2.1 has been completed. □

4 Singular limit of (P)$^\varepsilon$ as $\varepsilon \longrightarrow 0$

In this section we consider the singular limit of (P)$^\varepsilon$ as $\varepsilon \longrightarrow 0$. Then, we clarify the limit of the solution u^ε and the Lagrange multiplier λ^ε to (P)$^\varepsilon$ on $[0,T]$.

We begin by giving the uniform estimate of u^ε and λ^ε with respect to $\varepsilon \in (0,1]$.

Lemma 4.1 *Suppose all the same conditions in Theorem 2.2. For each $\varepsilon \in (0,1]$, let u^ε be the unique solution to $(\mathrm{P})^\varepsilon$ on $[0,T]$. Also, let λ^ε be the Lagrange multiplier to $(\mathrm{P})^\varepsilon$ on $[0,T]$ in the sense of Definition 2.1. Assume that there is a positive constant M, independent of $\varepsilon \in [0,1]$, satisfying*

$$\sup_{\varepsilon\in[0,1]} \mathscr{F}^\varepsilon(u_0^\varepsilon) < M.$$

Then, there is a positive number $N_1 > 0$, dependent on M and independent of $\varepsilon \in (0,1]$, such that

$$\varepsilon\int_0^T |u_t^\varepsilon(\tau)|_H^2 d\tau + \sup_{\tau\in[0,T]} \mathscr{F}^\varepsilon(u^\varepsilon(\tau)) + \int_0^T |\lambda^\varepsilon(\tau)|_{V^*}^2 d\tau \le N_1. \tag{4.1}$$

Proof. Multiplying (1.1) by $\varepsilon u_t^\varepsilon$, we get

$$\varepsilon|u_t^\varepsilon(\tau)|_H^2 + \frac{d}{d\tau}\mathscr{F}^\varepsilon(u^\varepsilon(\tau)) = 0 \quad \text{for a.e. } \tau > 0, \tag{4.2}$$

where $\mathscr{F}^\varepsilon(\cdot)$ is the functional defined in (2.2). By integrating (4.2) in τ over $[0,t]$ $(\subset [0,T])$, we get

$$\varepsilon\int_0^t |u_t^\varepsilon(\tau)|_H^2 d\tau + \mathscr{F}^\varepsilon(u^\varepsilon(t)) = \mathscr{F}^\varepsilon(u_0^\varepsilon) < M \quad \text{for all } t \in [0,T]. \tag{4.3}$$

Also, taking account of the constraint $\partial I_{[-1,1]}(\cdot)$ (cf. (1.5)), we easily see that

$$|u^\varepsilon| \le 1, \text{ a.e. in } Q. \tag{4.4}$$

By (1.6), (4.4) and (ii) of Definition 2.1, we see from Hölder inequality that:

$$\begin{aligned}
&\left|\int_0^T \langle\lambda^\varepsilon(t), z(t)\rangle dt\right| = \left|\int_0^T (\lambda^\varepsilon(t), z(t))_H dt\right| \\
&\le \int_0^T \left|(\varepsilon^2 u_t^\varepsilon(t), z(t))_H\right| dt + \int_0^T \left|\varepsilon^2 \left(\nabla u^\varepsilon(t), \nabla z(t)\right)_H\right| dt + \int_0^T \left|(u^\varepsilon(t), z(t))_H\right| dt \\
&\le \left(\varepsilon^2|u_t^\varepsilon|_{L^2(0,T;H)} + \varepsilon^2\sqrt{T}\sup_{t\in[0,T]}|\nabla u^\varepsilon(t)|_H + \sqrt{T|\Omega|}\right)|z|_{L^2(0,T;V)}
\end{aligned} \tag{4.5}$$

for any $z \in L^2(0,T;V)$, where $|\Omega|$ denotes the volume of Ω. Therefore, from $\varepsilon \in (0,1]$, (2.2), (4.3) and (4.5), we infer that:

$$|\lambda^\varepsilon|_{L^2(0,T;V^*)} \le \sqrt{M} + \sqrt{2MT} + \sqrt{T|\Omega|} \quad \text{for all } \varepsilon \in (0,1]. \tag{4.6}$$

From (4.3) and (4.6), we infer that the uniform estimate (4.1) holds for some positive constant N_1. Thus, the proof of Lemma 4.1 has been completed. □

Corollary 4.1 *Suppose all the same conditions in Lemma 4.1. For each $\varepsilon \in (0,1]$, let u^ε be the unique solution to $(\mathrm{P})^\varepsilon$ on $[0,T]$. Also, let $N_1 > 0$ be the positive number obtained in Lemma 4.1. Put*

$$h(s) := \frac{1}{\sqrt{2}} \int_{-1}^{s} \sqrt{1-\sigma^2}\, d\sigma \quad \text{for } s \in [-1,1]. \tag{4.7}$$

Then, the following estimates hold:

$$\sup_{t\in[0,T]} \int_\Omega |\nabla h(u^\varepsilon(t,x))| dx \le N_1 \tag{4.8}$$

and

$$\int_{t_1}^{t_2} \int_\Omega |(h(u^\varepsilon(t,x)))_t| dx dt \le N_1 (t_2 - t_1)^{\frac{1}{2}} \quad \text{for all } t_1, t_2 \text{ with } 0 \le t_1 < t_2 \le T. \tag{4.9}$$

Proof. First, note from (4.7) that $h'(s) = \sqrt{(1-s^2)/2}$ for $s \in [-1,1]$.

Now, we show the estimate (4.8). By (4.4) and the Schwarz inequality, we have:

$$\begin{aligned}
\int_\Omega |\nabla h(u^\varepsilon(t,x))| dx &= \int_\Omega |h'(u^\varepsilon(t,x))||\nabla u^\varepsilon(t,x)| dx \\
&\le \frac{1}{2\varepsilon} \int_\Omega \frac{1-(u^\varepsilon(t,x))^2}{2} dx + \frac{\varepsilon}{2} \int_\Omega |\nabla u^\varepsilon(t,x)|^2 dx \\
&\le \mathscr{F}^\varepsilon(u^\varepsilon(t)) \\
&\le N_1 \quad \text{for all } t \in [0,T].
\end{aligned}$$

Thus, (4.8) holds.

Next, we show (4.9). By (4.4) and Hölder inequality, we have:

$$\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega |(h(u^\varepsilon(t,x)))_t| dx dt &= \int_{t_1}^{t_2}\int_\Omega |h'(u^\varepsilon(t,x))||u_t^\varepsilon(t,x)| dx dt \\
&\le \left(\int_{t_1}^{t_2}\int_\Omega \frac{1-(u^\varepsilon(t,x))^2}{2} dx dt \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |u_t^\varepsilon(t)|_H^2 dt \right)^{\frac{1}{2}} \\
&\le (t_2 - t_1)^{\frac{1}{2}} \{ \varepsilon \sup_{t\in[t_1,t_2]} \mathscr{F}^\varepsilon(u^\varepsilon(t)) \}^{\frac{1}{2}} \frac{1}{\sqrt{\varepsilon}} N_1^{\frac{1}{2}} \\
&\le N_1 (t_2 - t_1)^{\frac{1}{2}}
\end{aligned}$$

for all t_1, t_2 with $0 \le t_1 < t_2 \le T$.

Thus, the proof of Corollary 4.1 has been completed. □

Now, we prove the main Theorem 2.2, which is concerned with the singular limit of (P)$^\varepsilon$ as $\varepsilon \to 0$.

Proof of Theorem 2.2. At first we show the existence of a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ and the function $u \in L^2(0,T;H)$ such that $u(t,x)$ takes only the values 1 or -1 for a.e. $(t,x) \in (0,T) \times \Omega$ and (2.5) holds.

By the definition of h (cf. (4.7)), we easily see that the function h is bounded on $[-1,1]$:

$$|h(s)| \leq C_h \quad \text{for all } s \in [-1,1] \tag{4.10}$$

for some positive constant $C_h > 0$. Therefore we infer from (4.4) and (4.10) that:

$$\int_0^T \int_\Omega |h(u^\varepsilon(t,x))| dxdt \leq T|\Omega|C_h. \tag{4.11}$$

Taking account of (4.8), (4.9) and (4.11), we see that $\{h(u^\varepsilon)\}$ is bounded in $BV((0,T)\times\Omega)$ uniformly in $\varepsilon \in (0,1]$, where $BV((0,T) \times \Omega)$ is the space of all bounded variation functions on $(0,T) \times \Omega$. Since $BV((0,T) \times \Omega)$ is compactly embedded into $L^1((0,T) \times \Omega)$ (cf. [2, Corollary 3.49]), there are subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ and the function $h^* \in BV((0,T) \times \Omega)$ such that $\varepsilon_k \to 0$ and

$$h(u^{\varepsilon_k}) \longrightarrow h^* \ \text{ in } L^1((0,T) \times \Omega) \ \text{ as } k \to \infty. \tag{4.12}$$

Therefore, taking a subsequence if necessary, we see that:

$$h(u^{\varepsilon_k}(t,x)) \longrightarrow h^*(t,x), \ \text{ a.e. } (t,x) \in (0,T) \times \Omega \ \text{ as } k \to \infty. \tag{4.13}$$

Since h is continuous and strictly increasing on $[-1,1]$ (cf. (4.7)), we can find a unique function $u(t,x)$ such that

$$h^*(t,x) = h(u(t,x)), \ \text{ a.e. } (t,x) \in (0,T) \times \Omega \tag{4.14}$$

and

$$u^{\varepsilon_k}(t,x) \longrightarrow u(t,x), \ \text{ a.e. } (t,x) \in (0,T) \times \Omega \ \text{ as } k \to \infty, \tag{4.15}$$

hence (2.5) holds. Clearly, it follows from (4.4) and (4.15) that:

$$|u| \leq 1, \ \text{ a.e. in } Q. \tag{4.16}$$

From (4.1), (4.4), (4.15), (4.16) and Lebesgue's dominated convergence theorem, we infer that:

$$\begin{aligned} 0 \leq \int_0^T \int_\Omega \left(1 - (u(t,x))^2\right) dxdt &= \lim_{k\to\infty} \int_0^T \int_\Omega \left(1 - (u^{\varepsilon_k}(t,x))^2\right) dxdt \\ &\leq 2N_1 T \lim_{k\to\infty} \varepsilon_k = 0, \end{aligned} \tag{4.17}$$

which implies that the limit function u of u^{ε_k} takes only the values 1 or -1 for a.e. $(t,x) \in (0,T)\times\Omega$.

Next, we show (2.6). Note from (4.9) that the following inequality holds:

$$\begin{aligned} \int_\Omega |h(u^{\varepsilon_k}(t_1,x)) - h(u^{\varepsilon_k}(t_2,x))| \, dx &\leq \int_\Omega \int_{t_1}^{t_2} |(h(u^{\varepsilon_k}(t,x)))_t| dtdx \\ &\leq N_1(t_2 - t_1)^{\frac{1}{2}} \end{aligned} \tag{4.18}$$

for all t_1, t_2 with $0 \leq t_1 < t_2 \leq T$.

Taking the limit in (4.18) as $k \to \infty$, we infer from (4.4), (4.10), (4.12)–(4.14) and Lebesgue's dominated convergence theorem that:

$$\int_\Omega |h(u(t_1,x)) - h(u(t_2,x))|\,dx \le N_1(t_2-t_1)^{\frac{1}{2}} \quad \text{for a.e. } t_1, t_2 \text{ with } 0 \le t_1 < t_2 \le T. \tag{4.19}$$

Since the limit function u of u^{ε_k} takes only the values 1 or -1 for a.e. on $(0,T)\times\Omega$, we easily see that:

$$|h(u(t_1,x)) - h(u(t_2,x))| = \frac{C_3}{2}|u(t_1,x)-u(t_2,x)| \quad \text{with } C_3 = h(1)-h(-1), \quad \text{a.e. } x \in \Omega \text{ and a.e. } t_1,t_2 \in (0,T). \tag{4.20}$$

Therefore, we observe from (4.19) and (4.20) that:

$$\int_\Omega |u(t_1,x)-u(t_2,x)|\,dx \le \frac{2N_1}{C_3}(t_2-t_1)^{\frac{1}{2}} \quad \text{for a.e. } t_1, t_2 \text{ with } 0 \le t_1 < t_2 \le T. \tag{4.21}$$

By (4.21), we can redefine the function u in order that $u(t) \in L^1(\Omega)$ is continuous with respect to $t \in [0,T]$. Therefore, (4.21) hold for all $t_1, t_2 \in [0,T]$ with $t_2 > t_1$, thus, (2.6) holds by putting $N_0 := (2N_1)/C_3$.

Next, we show (2.7). At first, we note from (2.4) that

$$u_0^{\varepsilon_k}(x) \longrightarrow u_0(x), \ \text{a.e. } x \in \Omega \ \text{ as } k \to \infty \tag{4.22}$$

by taking a subsequence if necessary. Therefore, by (2.3), (4.10), (4.22) and the continuity of h, we easily see from Lebesgue's dominated convergence theorem that:

$$\lim_{k\to\infty}\int_\Omega |h(u_0^{\varepsilon_k}(x)) - h(u_0(x))|\,dx = 0. \tag{4.23}$$

Now, putting $t_1 = 0$ in (4.18), we have:

$$\int_\Omega |h(u_0^{\varepsilon_k}(x)) - h(u^{\varepsilon_k}(t_2,x))|\,dx \le N_1 t_2^{\frac{1}{2}} \quad \text{for all } t_2 \in (0,T]. \tag{4.24}$$

Taking the limit in (4.24) as $k \to \infty$, we see from (4.12)–(4.15), (4.23) and (4.24) that:

$$\int_\Omega |h(u_0(x)) - h(u(t_2,x))|\,dx \le N_1 t_2^{\frac{1}{2}}, \quad \text{a.e. } t_2 \in (0,T]. \tag{4.25}$$

Since $u_0(x)$ takes only the values 1 or -1 for a.e. $x \in \Omega$ and $u(t) \in L^1(\Omega)$ is continuous with respect to $t \in [0,T]$, we easily see from (4.20) and (4.25) that:

$$\int_\Omega |u_0(x) - u(t_2,x)|\,dx \le \frac{2N_1}{C_3} t_2^{\frac{1}{2}} \quad \text{for all } t_2 \in (0,T]. \tag{4.26}$$

Thus, passing the limit $t_2 \to 0$ in (4.26), we observe that (2.7) holds.

Next, we show (2.8). By (4.8), (4.12) and the lower semicontinuity of the total variation under L^1-convergence (cf. [2, Proposition 3.6]), we observe that

$$\int_\Omega |\nabla h^*(t)| \le N_1, \quad \text{a.e. } t \in (0,T), \tag{4.27}$$

where $\int_\Omega |\nabla h^*(t)|$ is the total variation measure of $h^*(t)$. Since $u(t,x)$ takes only the values 1 or -1 for a.e. $(t,x) \in (0,T) \times \Omega$, we infer from (4.14) and (4.20) that

$$\int_\Omega |\nabla u(t)| \leq \frac{2N_1}{C_3}, \quad \text{a.e. } t \in (0,T). \tag{4.28}$$

Hence, (2.8) holds.

Finally, we show (2.9)–(2.11). By the uniform estimate (4.1), we easily see that there is a subsequence of $\{\varepsilon_k\}$ (which we denote ε_k for simplicity) and a function $\lambda^* \in L^2(0,T;V^*)$ satisfying (2.9).

From (4.4), (4.15) and Lebesgue's dominated convergence theorem, we infer that:

$$u^{\varepsilon_k} \longrightarrow u \ \text{ in } L^2(0,T;H) \ \text{ as } k \to \infty. \tag{4.29}$$

By (1.6), (4.1) and (ii) of Definition 2.1, we have that:

$$\begin{aligned} &\int_0^T \langle \lambda^{\varepsilon_k}(t) - u^{\varepsilon_k}(t), z(t)\rangle dt \left(= \int_0^T (\lambda^{\varepsilon_k}(t) - u^{\varepsilon_k}(t), z(t))_H dt \right) \\ \leq &\int_0^T \left| (\varepsilon_k^2 u_t^{\varepsilon_k}(t), z(t))_H \right| dt + \int_0^T \left| \varepsilon_k^2 (\nabla u^{\varepsilon_k}(t), \nabla z(t))_H \right| dt \\ \leq &\varepsilon_k^{\frac{3}{2}} \sqrt{N_1} |z|_{L^2(0,T;H)} + \varepsilon_k^{\frac{3}{2}} \sqrt{2TN_1} |z|_{L^2(0,T;V)} \\ &\qquad\qquad\qquad\qquad \text{for any } z \in L^2(0,T;V). \end{aligned}$$

From (2.9), (4.29) and the inequality as above, we see that

$$\int_0^T \langle \lambda^*(t) - u(t), z(t)\rangle dt = \lim_{k\to\infty} \int_0^T \langle \lambda^{\varepsilon_k}(t) - u^{\varepsilon_k}(t), z(t)\rangle dt \leq 0. \tag{4.30}$$

Since $z \in L^2(0,T;V)$ is arbitrary, we infer from (4.30) that

$$\lambda^* - u = 0 \left(\in L^2(0,T;H)\right) \ \text{ in } L^2(0,T;V^*). \tag{4.31}$$

Since the function $u \in L^2(0,T;H)$ takes only the values 1 or -1 for a.e. in $(0,T) \times \Omega$, we easily conclude from (4.31) that (2.10) and (2.11) hold. Thus, the proof of Theorem 2.2 has been completed. □

References

[1] S. Allen and J. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metall., **27**(1979), 1084–1095.

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[3] L. Blank, H. Garcke L. Sarbu and V. Styles, Primal-dual active set methods for Allen-Cahn variational inequalities with nonlocal constraints, Numer. Methods Partial Differential Equations, **29**(2013), 999–1030.

[4] V. Barbu and T. Precupanu, *Convexity and optimization in Banach spaces, Fourth edition,* Springer Monographs in Mathematics, Springer, Dordrecht, 2012.

[5] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[6] H. Brézis, M. G. Crandall and A. Pazy, Perturbations of nonlinear maximal monotone sets in Banach space, Comm. Pure Appl. Math., **23**(1970), 123–144.

[7] L. Bronsard and R.V. Kohn, Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics, J. Differential Equations, **90**(1991), 211–237.

[8] X. Chen and C. M. Elliott, Asymptotics for a parabolic double obstacle problem, Proc. Roy. Soc. London Ser. A, **444**(1994), 429–445.

[9] P. C. Fife, Dynamics of internal layers and diffusive interfaces, CBMS-NSF Regional Conf. Ser. in Appl. Math., **53**, SIAM, Philadelphia, 1988.

[10] T. Fukao and N. Kenmochi, Abstract theory of variational inequalities and Langrange multipliers, Adv. Math. Sci. Appl., **23**(2013), 545–574.

[11] A. Ito, Asymptotic stability of Allen-Cahn model for nonlinear Laplacian with constraints, Adv. Math. Sci. Appl., **9**(1999), 137–161.

[12] A. Ito, N. Yamazaki and N. Kenmochi, Attractors of nonlinear evolution systems generated by time-dependent subdifferentials in Hilbert spaces, *Dynamical systems and differential equations, Vol. I (Springfield, MO, 1996)*, Discrete Contin. Dynam. Systems 1998, Added Volume I, 327–349.

[13] K. Ito and K. Kunisch, *Lagrange multiplier approach to variational problems and applications*, Advances in Design and Control, **15**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[14] N. Kenmochi and K. Shirakawa, Stability for a parabolic variational inequality associated with total variation functional, Funkcial. Ekvac., **44**(2001), 119–137.

[15] T. Ohtsuka, K. Shirakawa and N. Yamazaki, Optimal control problem for Allen-Cahn type equation associated with total variation energy, Discrete Contin. Dyn. Syst. Ser. S, **5**(2012), 159–181.

[16] M. Ôtani, Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems, J. Differential Equations, **46**(1982), 268–299.

[17] K. Shirakawa and M. Kimura, Stability analysis for Allen-Cahn type equation associated with the total variation energy, Nonlinear Anal., **60**(2005), 257–282.

[18] Y. Tonegawa, Integrality of varifolds in the singular limit of reaction-diffusion equations, Hiroshima Math. J., **33**(2003), 323–341.