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On the effect of estimating the error density in nonparametric deconvolution

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ABSTRACT. It is quite common in the statistical literature on nonparametric deconvolution to assume that the error density is perfectly known. Since this seems to be unrealistic in many practical applications, we study the effect of estimating the unknown error density. We derive minimax rates of convergence and propose a modification of the usual kernel-based estimation scheme, which takes the uncertainty about the error density into account. A simulation study quantifies the possible gains by this new method in finite sample situations.

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1. Introduction

There is already a large amount of literature on nonparametric deconvolution. The most frequently used approach is the kernel method, which amounts to a damped or truncated division by the Fourier transform of the error density in the frequency domain; see Carroll and Hall (1988), Stefanski and Carroll (1990), Fan (1991a, 1991b, 1993) and Ruymgaart (1993). Some of these authors derive also minimax rates of convergence of such estimators in certain smoothness classes. Another approach based on a wavelet-vaguelette decomposition of the convolution operator, which is also appropriate in certain cases of inhomogeneous smoothness, is proposed in Donoho (1992). Sometimes authors also consider deconvolution on the circle; see e.g. van Rooij and Ruymgaart (1990). Sinusoids are then singular functions of the operator, regardless of the particular error density. This allows the application of orthonormal series methods that lead to a quite convenient analysis of the problem. However, this approach suffers from the drawback that only periodic densities can be adequately treated, which excludes many interesting functions and makes a widespread application of such methods impossible. Some examples for deconvolution problems are given in Carroll and Hall (1988); a real practical application is described in Mendelsohn and Rice (1982).

Closely related problems, which can be more or less reduced to nonparametric deconvolution, are estimation of mixing densities [Zhang (1990)] and nonparametric errors-in-variables regression [Fan and Truong (1993)]. There are also interesting and important higher-dimensional estimation problems, which are of deconvolution type, like image reconstruction from observations degraded by a Toeplitz transform [Hall (1990), Hall and Koch (1990)] and density estimation in computerized tomography [Johnstone and Silverman (1990), Kolaczyk (1994)].

In all of the abovementioned papers it is assumed that the convolution or convolution-like operator is exactly known. An approximate knowledge of it actually seems to be realistic in some cases of application, for example in density estimation in computer tomography. An amusing example of a perfectly known convolution operator is described on page 201 in Fuller (1987). However, very often one has only a rough idea about the convoluting density. A rather crude, but sometimes practicable method is to perform only partial deconvolution, that is to remove only a certain fraction of the error density from the data. This is studied in Stefanski and Carroll (1990), where the authors tried three error densities with a different behaviour at the origin and in the tails. Somewhat surprisingly, it turned out that the resulting estimators were not too different from each other as long as only a certain fraction of the (unknown) error density was removed. However, we can do much better, if we can draw some information about the convolution operator from an additional experiment. This situation has been studied by Horowitz in an econometric context and was described in a talk given at Humboldt University, Berlin, in 1995. As soon as one can estimate the error density consistently, one can also estimate the density of interest in an asymptotically consistent manner.

In the present paper we study this effect of estimating the error density f_ε more closely. We assume that information about f_ε is provided by an independent experiment. We propose a simple modification of the commonly used kernel estimators, which takes the uncertainty about the convoluting density into account. We quantify the additional error due to ignorance of f_ε and derive also a lower bound for the risk in this situation. It turns out that the modified version of a kernel estimator, which would be minimax in the case of known f_ε, is again minimax in the model with unknown f_ε. Similarly to Fan (1991a, 1993) we derive also minimax rates of convergence in the full problem.

2. A MODIFIED REGULARIZATION SCHEME FOR THE CASE OF UNKNOWN ERROR DENSITY

Suppose we have n i.i.d. random variables $X_1, \cdots, X_n$ distributed according to a density f_X. However, we do not observe the X_j's directly, but

$$Y_j = X_j + \varepsilon_j, \quad j = 1, \ldots, n, \tag{2.1}$$

where the ε_j's are i.i.d. with density f_ε, also independent of the X_j's.
We are interested in estimating f_X, but we do not know f_ε exactly. We assume that some knowledge about f_ε can be drawn from an additional experiment, where we observe

$$\varepsilon_j^* \sim f_\varepsilon, \quad j = 1, \ldots, N, \tag{2.2}$$

where the ε_j^*'s are again i.i.d., also independent of the Y_j's above.
In the case of known f_ε, the most frequently used approach is based on linear regularization in the frequency domain. Such an estimator has the form

$$\widehat{f}_X(x) = \frac{1}{2\pi} \int \exp(-i\omega x) K_n(\omega) \frac{\widehat{\varphi}_Y(\omega)}{\varphi_\varepsilon(\omega)}\, d\omega, \tag{2.3}$$

where $\varphi_\varepsilon(\omega) = E\exp(i\omega\varepsilon_1)$ is the Fourier transform of the density f_ε and

$$\widehat{\varphi}_Y(\omega) = n^{-1} \sum_{j=1}^{n} \exp(i\omega Y_j) \tag{2.4}$$

is the natural estimate of $\varphi_Y(\omega) = E\exp(i\omega Y_1)$. The smoothing kernel K_n is often taken as $K_n(\omega) = K(\omega/h_n)$ for some bandwidth $h_n \to \infty$, but other choices could also be reasonable in some particular cases for f_ε.
In our case of unknown f_ε, one might be tempted to replace φ_ε in (2.3) simply by

$$\widehat{\varphi}_\varepsilon(\omega) = N^{-1} \sum_{j=1}^{N} \exp(i\omega\varepsilon_j^*), \tag{2.5}$$

which gives the estimator

$$\widehat{\widehat{f}}_X(x) = \frac{1}{2\pi} \int \exp(-i\omega x) K_n(\omega) \frac{\widehat{\varphi}_Y(\omega)}{\widehat{\varphi}_\varepsilon(\omega)}\, d\omega. \tag{2.6}$$

This is actually done by Horowitz, and leads for an appropriate choice of the bandwidth h_n to a consistent estimator of f_X. However, the practical choice of h_n is now

even more complicated than in the deconvolution problem with known f_ε, since an appropriate value for h_n depends on the smoothness of both f_X and f_ε.
Here we consider an alternative method, which can be motivated as follows. It is clear that the characteristic function φ_ε can be estimated at each point ω with the rate $N^{-1/2}$. Hence, $\widehat{\varphi}_\varepsilon(\omega)$ is clearly a reasonable estimator of $\varphi_\varepsilon(\omega)$, if $|\varphi_\varepsilon(\omega)| \gg N^{-1/2}$. For frequencies ω with $|\varphi_\varepsilon(\omega)| \ll N^{-1/2}$, this estimator is no longer satisfying, since the noise of $\widehat{\varphi}_\varepsilon(\omega)$ is then of larger order than the size of the signal. Actually, we have to estimate $1/\varphi_\varepsilon(\omega)$ and therefore this problem is even more critical: very small values of $|\widehat{\varphi}_\varepsilon(\omega)|$ would lead to an instable estimator of f_X. Hence, one could wish to exclude all those frequencies ω from the estimator (2.3), for which $|\varphi_\varepsilon(\omega)| < N^{-1/2}$ holds. This is approximately reached by multiplying the integral in (2.3) with the indicator $I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})$, that is, we obtain the modified estimator

$$\widehat{f}^*_X(x) = \frac{1}{2\pi} \int \exp(-i\omega x) K_n(\omega) I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2}) \frac{\widehat{\varphi}_Y(\omega)}{\widehat{\varphi}_\varepsilon(\omega)}\, d\omega. \tag{2.7}$$

The following lemma shows that this is actually a reasonable strategy for estimating $1/\varphi_\varepsilon(\omega)$.

Lemma 2.1. *It holds*

$$E\left|\frac{I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})}{\widehat{\varphi}_\varepsilon(\omega)} - \frac{1}{\varphi_\varepsilon(\omega)}\right|^2 = O\left(\min\left\{\frac{N^{-1}}{|\varphi_\varepsilon(\omega)|^4}, \frac{1}{|\varphi_\varepsilon(\omega)|^2}\right\}\right).$$

In other words, the estimator $I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})/\widehat{\varphi}_\varepsilon(\omega)$ behaves just as we could hope: its risk attains the rate $N^{-1}|\varphi_\varepsilon(\omega)|^{-4}$ like the (nearly) ideal estimator $1/\widehat{\varphi}_\varepsilon(\omega)$ in the case of $|\varphi_\varepsilon(\omega)| \geq N^{-1/2}$, and attains the rate $|\varphi_\varepsilon(\omega)|^{-2}$ like the (nearly) ideal estimator which is identically zero, in the case of $|\varphi_\varepsilon(\omega)| < N^{-1/2}$.
An important implication of the above lemma is given by the next theorem.

Theorem 2.1. *There exists a universal constant* $C < \infty$, *such that*

$$E\|\widehat{f}^*_X - f_X\|^2_{L_2(\mathbb{R})} \leq C\left\{E\|\widehat{f}_X - f_X\|^2_{L_2(\mathbb{R})} + \int |K_n(\omega)|^2 |\varphi_X(\omega)|^2 \min\{N^{-1}|\varphi_\varepsilon(\omega)|^{-2}, 1\}\, d\omega\right\}. \tag{2.8}$$

We show in the next section that the second term on the right-hand side of (2.8) cannot be avoided; it is just a lower (rate-) bound for estimating f_X in the case of unknown f_ε under reasonable assumptions on the class within f_ε varies. It follows further, that our estimator $\widehat{f}^*_X$ is minimax in all smoothness classes $\mathcal{F}_X$ for f_X in which the estimator $\widehat{f}_X$ would attain the optimal rate of convergence for known f_ε.

From the practical point of view, the advantage of our new estimator $\widehat{f}^*_X$ over the estimation scheme (2.6) is, that we can use just the same bandwidth h_n as in the case of known f_ε. We think that the choice of h_n is now much less critical than without thresholding by $I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})$.
A data-driven bandwidth choice can be performed by some cross-validation technique in the Fourier domain, which was described in Stefanski and Carroll (1990) and Dey, Mair and Ruymgaart (1993). For the sake of a clear presentation we do not include this step in our considerations here.

3. Optimality of the method in smoothness classes

In this section we derive rates of convergence for the estimator $\widehat{\widehat{f}}_X^*$ in certain smoothness classes $\mathcal{F}_X$ and $\mathcal{F}_\varepsilon$ for f_X and f_ε, respectively. A lower bound for the rate in estimating f_X is obtained combining a lower bound for the problem with known f_ε with a lower bound, which captures the additional difficulty due to ignorance of f_ε. Since both bounds hold true simultaneously, we obtain a lower bound for the difficulty of the full problem. We show that this rate of convergence will be attained by our estimator $\widehat{\widehat{f}}_X^*$ for an appropriate choice of h_n.

3.1. The additional difficulty due to ignorance of the error density. First we fix certain smoothness classes of functions for f_X and f_ε. Fan (1991a, 1993) considers Hölder smoothness classes for f_X and obtained optimality of the kernel method in the case of known f_ε for the pointwise risk and for L_p-risk on a compact interval $[a,b]$. Fan (1991b) shows that the same rates are attainable for the global L_2-risk in some smoothness classes with integrated Hölder modulus of continuity. Here we consider also the L_2-risk on the whole real line; accordingly we consider the following class in a Bessel-potential space:

$$\mathcal{F}_X = \left\{ f \,\middle|\, f \text{ density}, \quad \int \left|\widetilde{f}(\omega)(1+|\omega|)^\beta\right|^2 d\omega \le C \right\},$$

where $\widetilde{f}(\omega) = \int \exp(ix\omega) f(x)\, dx$ is the Fourier transform of f. For integer β the set $\mathcal{F}_X$ is a usual L_2-Sobolev class; for β not an integer $\mathcal{F}_X$ is a set of functions with bounded norm in the Besov space $B^\beta_{2,2}$; see Triebel (1995, Section 2.3).

We do not want to fix a specific smoothness class for f_ε at this point. Let $\gamma(\omega) = |\widetilde{f_0}(\omega)|$ be the modulus of the Fourier transform of any density f_0 with the additional property that

$$C_1\gamma(\omega') \le \gamma(\omega) \le C_2\gamma(\omega') \tag{3.1}$$

holds for all ω, ω' with $|\omega-\omega'| \le 1$ and some constants $0 < C_1 \le C_2 < \infty$. Define, for any $C > 0$,

$$\widetilde{\mathcal{F}}_\varepsilon = \left\{ f \,\middle|\, f \text{ density}, \quad |\widetilde{f}(\omega)| \ge C\gamma(\omega) \right\}.$$

We consider this quite general class first, since the assumption of a "regular decay" of $\widetilde{f}(\omega)$ as considered in the next section excludes some interesting densities.

It is well-known (for details see the proofs in Fan (1991a, 1993)) that the hardest one-dimensional subproblem does not capture the full difficulty in estimating f_X in the case of known f_ε. In other words, there do not exist two sequences of densities $f_{X,1,n}, f_{X,2,n} \in \mathcal{F}_X$, which are statistically not consistently distinguishable and satisfy $\|f_{X,1,n} - f_{X,2,n}\|^2_{L_2(\mathbb{R})} \ge Cn^{-r}$, where n^{-r} is the optimal rate of convergence. A suitable lower bound can be derived considering composite hypotheses of growing dimension. In contrast, the *additional* difficulty due to ignorance of f_ε can be captured by an appropriate one-dimensional subproblem. The idea is now to find two pairs of densities $f_{\varepsilon,N,1}, f_{\varepsilon,N,2} \in \widetilde{\mathcal{F}}_\varepsilon$ and $f_{X,N,1}, f_{X,N,2} \in \mathcal{F}_X$ with

$$H\left(f_{\varepsilon,N,1}, f_{\varepsilon,N,2}\right) \le CN^{-1/2}, \tag{3.2}$$

where $H(f,g) = [\int(\sqrt{f} - \sqrt{g})^2]^{1/2}$ denotes the Hellinger distance,

$$f_{\varepsilon,N,1} * f_{X,N,1} \equiv f_{\varepsilon,N,2} * f_{X,N,2} \tag{3.3}$$

and the property that $\|f_{X,N,1} - f_{X,N,2}\|_{L_2(\mathbb{R})}$ is as large as possible (in order) among all densities satisfying (3.2) and (3.3). (3.2) and (3.3) will imply that one cannot consistently discriminate between the two experiments based on $f_{\varepsilon,N,1}, f_{X,N,1}$ and $f_{\varepsilon,N,2}, f_{X,N,2}$, respectively. The value of $\|f_{X,N,1} - f_{X,N,2}\|_{L_2(\mathbb{R})}$ provides then a lower bound for estimating f_X under the assumption $f_X \in \mathcal{F}_X$ and $f_\varepsilon \in \widetilde{\mathcal{F}}_\varepsilon$. This is formalized in the following theorem. The starting point for the search for these densities is the search for the least favourable frequency, that is

$$\omega_N := \arg\min_\omega \left\{(1+|\omega|)^{-2\beta} \min\{N^{-1}|\gamma(\omega)|^{-2}, 1\}\right\}. \tag{3.4}$$

Theorem 3.1. *Assume* (2.1) *and* (2.2) *and let* ω_N *be defined by* (3.4). *Then*

$$\inf_{\overline{f}_X} \sup_{f_X\in\mathcal{F}_X, f_\varepsilon\in\widetilde{\mathcal{F}}_\varepsilon} \left\{E\|\overline{f}_X - f_X\|^2_{L_2(\mathbb{R})}\right\} \geq C(1+|\omega_N|)^{-2\beta} \min\{N^{-1}|\gamma(\omega_N)|^{-2}, 1\}.$$

In conjunction with Theorem 2.1, we obtain immediately the optimality of $\widehat{\widehat{f}}{}^*_X$ in classes $\mathcal{F}_X$ where $\widehat{f}_X$ is minimax for the problem with known f_ε:

Corollary 3.1. *Assume* (2.1) *and* (2.2). *Let* $\widehat{f}_X$ *be minimax in* $\mathcal{F}_X$ *with known* f_ε, $|\widetilde{f}_\varepsilon(\omega)| \asymp |\gamma(\omega)|$, *where* γ *satisfies* (3.1). *Then*

$$\sup_{f_X\in\mathcal{F}_X, f_\varepsilon\in\widetilde{\mathcal{F}}_\varepsilon} \left\{E\|\widehat{\widehat{f}}{}^*_X - f_X\|^2_{L_2(\mathbb{R})}\right\} \leq C \inf_{\overline{f}_X} \sup_{f_X\in\mathcal{F}_X, f_\varepsilon\in\widetilde{\mathcal{F}}_\varepsilon} \left\{E\|\overline{f}_X - f_X\|^2_{L_2(\mathbb{R})}\right\}.$$

3.2. Rates of convergence in smoothness classes. So far we have concentrated on the quite general set $\widetilde{\mathcal{F}}_\varepsilon$ for f_ε. If we assume additionally some kind of "regular decay" for $|\widetilde{f}_\varepsilon(\omega)|$, we are able to derive rates of convergence depending on n and N. Let, for some $\alpha > 0$,

$$\mathcal{F}_\varepsilon = \left\{f \,\middle|\, f \text{ density}, \quad |\widetilde{f}(\omega)| \asymp (1+|\omega|)^{-\alpha}\right\}.$$

This assumption basically means that f_ε has about α derivatives. It turns out that this relation is satisfied for gamma distributions with shape parameter α, which contain for $\alpha = 1$ the exponential and for $\alpha = \kappa/2$ the chi-square distribution with κ degrees of freedom as special cases. Another example, which satisfies this condition with $\alpha = 2$, is the Laplace (double exponential) distribution.
First we state a lower bound, which characterizes the difficulty in estimating $f_X \in \mathcal{F}_X$ in the case of known $f_\varepsilon \in \mathcal{F}_\varepsilon$.

Theorem 3.2. *Assume observations* (2.1) *and that the error density* f_ε *is known. Then*

$$\inf_{\overline{f}_X} \sup_{f_X\in\mathcal{F}_X, f_\varepsilon\in\mathcal{F}_\varepsilon} \left\{E\|\overline{f}_X - f_X\|^2_{L_2(\mathbb{R})}\right\} \asymp n^{-2\beta/(2\beta+2\alpha+1)}.$$

According to Fan (1991b), a particular estimator, which attains the optimal rate of convergence, is given by

$$\widehat{f}_X(x) = \frac{1}{2\pi}\int \exp(-i\omega x)K(\omega/h_n)\frac{\widehat{\varphi}_Y(\omega)}{\varphi_\varepsilon(\omega)}\,d\omega, \tag{3.5}$$

where $h_n \asymp n^{1/(2\beta+2\alpha+1)}$ and the kernel function K satisfies

(A1) $K(\omega) = 1 + O(|\omega|^\beta)$ as $|\omega| \to 0$,
(A2) $\int |K(\omega)(1+|\omega|)^\alpha|^2\,d\omega < \infty$.

To derive a lower bound for the rate of convergence in the case of unknown f_ε, we only have to combine the lower bounds contained in Theorems 3.1 and 3.2. It can be easily seen that the lower bound given in Theorem 3.1 is of order $N^{-((\beta/\alpha)\wedge 1)}$, which gives the following assertion.

Corollary 3.2. *Assume that observations according to (2.1) and (2.2) are available. Then*

$$\inf_{\overline{f}_X}\ \sup_{f_X\in\mathcal{F}_X, f_\varepsilon\in\mathcal{F}_\varepsilon}\left\{E\|\overline{f}_X - f_X\|^2_{L_2(\mathbb{R})}\right\} \geq C\left(n^{-2\beta/(2\beta+2\alpha+1)} + N^{-((\beta/\alpha)\wedge 1)}\right).$$

In the case of unknown f_ε we consider the estimator

$$\widehat{\widehat{f}}{}^*_X(x) = \frac{1}{2\pi}\int \exp(-i\omega x)K(\omega/h_n)I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})\frac{\widehat{\varphi}_Y(\omega)}{\widehat{\varphi}_\varepsilon(\omega)}\,d\omega. \tag{3.6}$$

Theorem 3.2 and Corollary 3.1 imply the following assertion.

Theorem 3.3. *Assume that observations according to (2.1) and (2.2) are available. Further, assume (A1), (A2) and $h_n \asymp n^{1/(2\beta+2\alpha+1)}$. Then*

$$\sup_{f_X\in\mathcal{F}_X, f_\varepsilon\in\mathcal{F}_\varepsilon}\left\{E\|\widehat{\widehat{f}}{}^*_X - f_X\|^2_{L_2(\mathbb{R})}\right\} = O\left(n^{-2\beta/(2\beta+2\alpha+1)} + N^{-((\beta/\alpha)\wedge 1)}\right).$$

Remark 1. The uncertainty about φ_ε is modeled in a stochastic setting, which permits an estimator $\widehat{\varphi}_\varepsilon$ with $|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)| = O_P(N^{-1/2})$. Then an appropriate double asymptotics, that is $n, N \to \infty$, leads to certain rates of convergence. Note that analogous effects hold true in a nonstochastic setting as well, that is when one takes any nonrandom approximation $\widehat{\varphi}_\varepsilon$ to φ_ε. To fix this idea, assume that $|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)| \leq c_N$, $c_N \to 0$. Then we obtain analogous results to Theorems 2.1, 3.1 and 3.3.

Remark 2. We see from Theorem 3.3 that one obtains the usual rate of convergence in the special case of $n = N$. However, we stress again that it is important to apply the thresholding scheme (3.6) to guard against too small values of $\widehat{\varphi}_\varepsilon(\omega)$ which can occur with positive probability and which would lead to an unstable estimate of f_X.

4. SIMULATIONS

In this section we present some results of a simulation study, which support the claims made in the theoretical part of this paper.
We used as a convenient programming environment the *XploRe* system, which has been developed by W. Härdle and coworkers, and runs on personal computers. A description of this is contained in Härdle, Klinke and Turlach (1995).
For the sake of convenience we took convolutions of Laplace (double exponential) densities for f_X and f_ε. Let $f_0(x) = 0.5\exp(-|x|)$ be the standard Laplace density. Then its Fourier transform $\tilde{f}_0(\omega) = 1/(1+\omega^2)$ is a real function. The d-fold convolution of f_0 has a Fourier transform $1/(1+\omega^2)^d$, which makes the link to our smoothness classes obvious.
For the regularization step we used the uniform kernel $K(x) = I(|x| \le 1)$ and took this nonrandom bandwidth h_n, which is optimal for the estimator $\widehat{f}_\varepsilon$ with known error density f_ε. The explicit calculation of this bandwidth is quite simple in our particular example. The risk in estimating $\varphi_X(\omega)$ by $\widehat{\varphi}_X(\omega) = \widehat{\varphi}_Y(\omega)/\varphi_\varepsilon(\omega)$ is equal to

$$var(\widehat{\varphi}_X(\omega)) = var(\widehat{\varphi}_Y(\omega))/|\varphi_\varepsilon(\omega)|^2 = n^{-1}(1 - |\widehat{\varphi}_Y(\omega)|^2)|\varphi_\varepsilon(\omega)|^{-2}.$$

Since we use the uniform kernel, the only alternative is to estimate $\varphi_X(\omega)$ by zero, which gives a risk equal to $|\varphi_X(\omega)|^2$. Let f_X and f_ε be d_1-fold and d_2-fold convolutions of f_0, respectively. The optimal smoothing set, that is the set Ω_n of frequencies which should be included in the estimation process, is given as

$$\begin{aligned}\Omega_n &= \left\{\omega \,\middle|\, n^{-1}(1 - |\varphi_Y(\omega)|^2)|\varphi_\varepsilon(\omega)|^{-2} \le |\varphi_X(\omega)|^2\right\} \\ &= \left\{\omega \,\middle|\, 1 \le (n+1)|\varphi_Y(\omega)|^2\right\} \\ &= \left\{\omega \,\middle|\, 1 \le (n+1)(1+\omega^2)^{-2d_1-2d_2}\right\} \\ &= \left\{\omega \,\middle|\, |\omega| \le \sqrt{(n+1)^{1/(2d_1+2d_2)} - 1}\right\}.\end{aligned}$$

Accordingly, we set $h_n = \sqrt{(n+1)^{1/(2d_1+2d_2)} - 1}$.
We considered two cases, namely f_X smoother than f_ε and vice versa. The sample size of the main experiment was chosen to be $n = 200$. Since the effect of estimating f_ε can be expected to be large, if the sample size of the additional experiment is small compared to n, we took $N = 10$. In each case we generated 100 random samples using the Gaussian pseudo-random number generator from *XploRe*. To obtain the desired distributions of the random variables we used the fact that $X_1X_2 + X_3X_4$ is Laplace distributed, if $X_i \sim N(0,1)$ are independent; see Johnson, Kotz and Balakrishnan (1995, Section 24.6). In the following we present the results of the simulation experiments.

1) f_X smoother than f_ε

In this case we took $d_1 = 4$ and $d_2 = 2$. The (estimated) risks of the estimators $\widehat{f}_X$, $\widehat{\widehat{f}}_X$ and $\widehat{\widehat{f}}{}^*_X$ are shown in Table 1.
Typical realizations of the real part and the imaginary part of the Fourier transforms of these estimators as well as of these estimators themselves are shown in Figures 1a,

Table 1

estimator	L_2-risk
$\widehat{f}_X$	0.00257
$\widehat{\widehat{f}}_X$	0.01010
$\widehat{\widehat{f}}^*_X$	0.00828

1b and 1c, respectively. We displayed the "most typical" realization, which minimizes the sum of the differences between the L_2-losses and the L_2-risks of the three estimators. The thick line shows the truth, i.e. $Re(\varphi_X)$, $Im(\varphi_X)$ and f_X, respectively. The thin solid line refers to $\widehat{f}_X$, the dotted line to $\widehat{\widehat{f}}_X$ and the dashed line to $\widehat{\widehat{f}}^*_X$.

[Please insert Figures 1a, 1b and 1c about here.]

Figures 2a, 2b and 2c show the worst case for the uncorrected estimator $\widehat{\widehat{f}}_X$, that is this realization, where the L_2-loss for $\widehat{\widehat{f}}_X$ is maximal.

[Please insert Figures 2a, 2b and 2c about here.]

These pictures underline the necessity of taking the uncertainty about f_ε into account, which is done by our estimation scheme $\widehat{\widehat{f}}^*_X$.

2) f_ε smoother than f_X

We considered also a more difficult case of estimating f_X with $d_1 = 2$ and $d_2 = 4$. The estimated risks are shown in Table 2.

Table 2

estimator	L_2-risk
$\widehat{f}_X$	0.019037
$\widehat{\widehat{f}}_X$	0.086432
$\widehat{\widehat{f}}^*_X$	0.065923

As could be expected, the risks are considerably larger than in the case above, which reflects the fact that a smooth f_X and a nonsmooth f_ε are favorable. The most typical of the 100 realizations is shown in Figures 3a, 3b and 3c.

[Please insert Figures 3a, 3b and 3c about here.]

Finally, the worst case for $\widehat{\widehat{f}}_X$ is displayed in Figures 4a, 4b and 4c.

[Please insert Figures 4a, 4b and 4c about here.]

It shows even more drastically than in the first example the advantage of our new estimator $\widehat{\widehat{f}}^*_X$ over $\widehat{\widehat{f}}_X$.

5. Proofs

Proof of Lemma 2.1. We distinguish between two cases, $|\varphi_\varepsilon(\omega)| < 2N^{-1/2}$ and $|\varphi_\varepsilon(\omega)| \geq 2N^{-1/2}$.

(i) Let $|\varphi_\varepsilon(\omega)| < 2N^{-1/2}$. Then

$$
\begin{aligned}
& E\left|\frac{I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})}{\widehat{\varphi}_\varepsilon(\omega)} - \frac{1}{\varphi_\varepsilon(\omega)}\right|^2 \\
= \; & \frac{1}{|\varphi_\varepsilon(\omega)|^2} P\left(|\widehat{\varphi}_\varepsilon(\omega)| < N^{-1/2}\right) + EI\left(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2}\right) \frac{|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)|^2}{|\widehat{\varphi}_\varepsilon(\omega)|^2 |\varphi_\varepsilon(\omega)|^2} \\
= \; & O\left(1/|\varphi_\varepsilon(\omega)|^2\right).
\end{aligned}
$$

(ii) Let $|\varphi_\varepsilon(\omega)| \geq 2N^{-1/2}$. To derive the desired upper bound we apply Bernstein's inequality, which we quote for reader's convenience from Shorack and Wellner (1986, p. 855):
Let $Z_1, \ldots, Z_N$ be i.i.d. random variables with $EZ_1 = 0$ and $|Z_1| \leq K$ almost surely. Then, for $\bar{Z} = N^{-1} \sum Z_i$,

$$
\begin{aligned}
P(\bar{Z} > c) \; & \leq \; \exp\left(-\frac{Nc^2/2}{var(Z_1) + (Kc)/3}\right) \\
& \leq \; \exp\left(-\frac{c^2}{4\,var(\bar{Z})}\right) + \exp\left(-\frac{3Nc}{4K}\right)
\end{aligned}
$$

holds for arbitrary $c > 0$.
Now we have

$$
\begin{aligned}
P\left(|\widehat{\varphi}_\varepsilon(\omega)| < N^{-1/2}\right) \; & \leq \; P\left(|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)| > |\varphi_\varepsilon(\omega)| - N^{-1/2}\right) \\
& \leq \; P\left(|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)| > |\varphi_\varepsilon(\omega)|/2\right) \\
& \leq \; C\exp(-CN|\varphi_\varepsilon(\omega)|^2) \\
& = \; O\left(N^{-1}|\varphi_\varepsilon(\omega)|^{-2}\right).
\end{aligned}
$$

Using this and

$$
\frac{1}{|\widehat{\varphi}_\varepsilon(\omega)|^2} \leq \frac{1}{|\varphi_\varepsilon(\omega)|^2} + \frac{|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)|^2}{|\widehat{\varphi}_\varepsilon(\omega)|^2 |\varphi_\varepsilon(\omega)|^2},
$$

we obtain that

$$
\begin{aligned}
& E\left|\frac{I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})}{\widehat{\varphi}_\varepsilon(\omega)} - \frac{1}{\varphi_\varepsilon(\omega)}\right|^2 \\
= \;& \frac{1}{|\varphi_\varepsilon(\omega)|^2} P\left(|\widehat{\varphi}_\varepsilon(\omega)| < N^{-1/2}\right) + EI\left(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2}\right) \frac{|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)|^2}{|\widehat{\varphi}_\varepsilon(\omega)|^2 |\varphi_\varepsilon(\omega)|^2} \\
\leq \;& O\left(N^{-1}|\varphi_\varepsilon(\omega)|^{-4}\right) \\
& + \frac{E|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)|^2}{|\varphi_\varepsilon(\omega)|^4} + \frac{E|\widehat{\varphi}_\varepsilon(\omega) - \varphi_\varepsilon(\omega)|^4 N}{|\varphi_\varepsilon(\omega)|^4} \\
= \;& O\left(N^{-1}|\varphi_\varepsilon(\omega)|^{-4}\right).
\end{aligned}
$$

□

Proof of Theorem 2.1. Using the decomposition

$$
\begin{aligned}
& \widehat{\widehat{f}}{}^{*}_X(x) - f_X(x) \\
= \;& \widehat{f}_X(x) - f_X(x) \\
& + \frac{1}{2\pi}\int \exp(-ix\omega) K_n(\omega)[\widehat{\varphi}_Y(\omega) - \varphi_Y(\omega)] \left[\frac{I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})}{\widehat{\varphi}_\varepsilon(\omega)} - \frac{1}{\varphi_\varepsilon(\omega)}\right] d\omega \\
& + \frac{1}{2\pi}\int \exp(-ix\omega) K_n(\omega) \varphi_X(\omega)\varphi_\varepsilon(\omega) \left[\frac{I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})}{\widehat{\varphi}_\varepsilon(\omega)} - \frac{1}{\varphi_\varepsilon(\omega)}\right] d\omega \\
= \;& r_1(x) + r_2(x) + r_3(x)
\end{aligned}
$$

we obtain that

$$
E\|\widehat{\widehat{f}}{}^{*}_X - f_X\|^2 \leq 3\sum_{i=1}^{3} E\int |r_i(x)|^2 \, dx.
$$

The first summand on the right-hand side is just the risk of $\widehat{f}_X$. Using Parseval's identity we obtain, due to independence of $\widehat{\varphi}_Y$ and $\widehat{\varphi}_\varepsilon$, that

$$
\begin{aligned}
E\int |r_2(x)|^2 \, dx &= \frac{1}{2\pi} E\int |\widetilde{r}_2|^2 \, d\omega \\
&= \frac{1}{2\pi}\int |K_n(\omega)|^2 E\left|\widehat{\varphi}_Y(\omega) - \varphi_Y(\omega)\right|^2 E\left|\frac{I(|\widehat{\varphi}_\varepsilon(\omega)| \geq N^{-1/2})}{\widehat{\varphi}_\varepsilon(\omega)} - \frac{1}{\varphi_\varepsilon(\omega)}\right|^2 d\omega \\
&\leq C\int |K_n(\omega)|^2 \frac{E\left|\widehat{\varphi}_Y(\omega) - \varphi_Y(\omega)\right|^2}{|\varphi_\varepsilon(\omega)|^2} \, d\omega \\
&\leq C\, E\|\widehat{f}_X - f_X\|^2.
\end{aligned}
$$

Again by Parseval's identity we get that

$$E\int |r_3(x)|^2\,dx \;=\; \frac{1}{2\pi}E\int \left|K_n(\omega)\varphi_X(\omega)\varphi_\epsilon(\omega)\left[\frac{I(|\widehat{\varphi}_\epsilon(\omega)|\geq N^{-1/2})}{\widehat{\varphi}_\epsilon(\omega)} - \frac{1}{\varphi_\epsilon(\omega)}\right]\right|^2 d\omega$$
$$\leq\; C\int |K_n(\omega)|^2|\varphi_X(\omega)|^2\left[N^{-1}|\varphi_\epsilon(\omega)|^{-2}\wedge 1\right]\,d\omega,$$

which completes the proof. □

Proof of Theorem 3.1.
(i) Construction of $f_{\epsilon,N,i}$
Let f_0 be any density with $|\widetilde{f}_0(\omega)| \asymp \gamma(\omega)$.
If $\omega_N < 1$, we redefine $\omega_N := 1$. Now we set

$$f_{\epsilon,N,i}(x) \;=\; K_1\left[f_0(x) + \left(\frac{\sin(x)}{x}\right)^2\right] + (-1)^i\rho_N(x), \tag{5.1}$$

where

$$\rho_N(x) \;=\; K_2(N^{-1/2}\wedge|\gamma(\omega_N)|)\frac{\sin(\omega_N x)\sin(x)}{x}\left(\frac{\sin(x)}{x}\right)^2$$

and

$$K_1 \;=\; \left(\int\left[f_0(x) + \left(\frac{\sin(x)}{x}\right)^2\right]dx\right)^{-1}.$$

Since ρ_N is an odd function, we have

$$\int f_{\epsilon,N,i}(x)\,dx \;=\; 1, \quad i=1,2.$$

Moreover, if K_2 is chosen small enough, the $f_{\epsilon,N,i}$'s will be nonnegative and satisfy

$$|\widetilde{f}_{\epsilon,N,i}(\omega)| \;\geq\; C|\gamma(\omega)| \quad \forall\omega.$$

(ii) Construction of $f_{X,N,i}$
Let $c_N = |\gamma(\omega_N)|^{-1}(1+|\omega_N|)^{-\beta}$.
If $c_N \leq 1$, we define

$$f_{X,N,i}(x) \;=\; f_{\epsilon,N,(3-i)}(x)*\left\{\frac{2c_N}{\pi}\left(\frac{\sin(\omega_N x)\sin(x)}{x}\right)^2 + 2(1-c_N)\left(\frac{\sin(x)}{x}\right)^2\right\}, \quad i=1,2,$$

where "*" means convolution of the two functions on the right-hand side. The term in braces integrates to one; hence $f_{X,N,i}$ is a density. The convolution of $f_{\epsilon,N,i}$ with the function in braces has the effect that $f_{X,N,i}$ is bandlimited to frequencies around $\omega = \omega_N$ and $\omega = 0$.
If $c_N > 1$, we define the $f_{X,N,i}$'s in a slightly different way. Since $f_{X,N,i}$ must be densities, the L_1-norm of the term in braces must be one. On the other hand, it turns out that the L_2-norm of that term is responsible for the L_2-difference between $f_{X,N,1}$

and $f_{X,N,2}$. To maximize this difference we extract a smaller frequency band, that is we define

$$f_{X,N,i}(x) \;=\; f_{\varepsilon,N,(3-i)}(x) * \left\{ \frac{2c_N}{\pi} \left(\frac{\sin(\omega_N x)\sin(x/c_N)}{x} \right)^2 \right\}, \quad i = 1,2.$$

Hence, the $f_{X,N,i}$'s are in both cases densities, that is they are nonnegative and integrate to one. Further, it is easy to see that

$$\int \left| \tilde{f}_{X,N,i}(\omega)(1+|\omega|)^{\beta} \right|^2 d\omega \;=\; \int \left| \tilde{f}_{\varepsilon,N,(3-i)}(\omega)\widetilde{\{\ldots\}}(\omega)(1+|\omega|)^{\beta} \right|^2 d\omega \;\le\; C.$$

Moreover, we have that

$$\begin{aligned} \|f_{X,N,1} \;-\; f_{X,N,2}\|^2_{L_2(\mathbb{R})} \;&=\; \frac{1}{2\pi} \int \left| (\tilde{f}_{\varepsilon,N,1}(\omega) \;-\; \tilde{f}_{\varepsilon,N,2}(\omega))\widetilde{\{\ldots\}}(\omega) \right|^2 d\omega \\ &\ge\; C\left(N^{-1} \wedge |\gamma(\omega_N)|^2\right) |\gamma(\omega_N)|^{-2}(1 \;+\; |\omega_N|)^{-2\beta} \\ &=\; C\left(N^{-1}|\gamma(\omega_N)|^{-2} \wedge 1\right)(1 \;+\; |\omega_N|)^{-2\beta}. \end{aligned} \tag{5.2}$$

(iii) Asymptotic indistinguishability of the experiments

Now we state statistical indistinguishability of the experiment with $f_{\varepsilon,N,1}$ and $f_{X,N,1}$ from that with $f_{\varepsilon,N,2}$ and $f_{X,N,2}$.

First, we have

$$\begin{aligned} H^2(f_{\varepsilon,N,1}, f_{\varepsilon,N,2}) \;&=\; \int \frac{|f_{\varepsilon,N,1}(x) \;-\; f_{\varepsilon,N,2}(x)|^2}{(\sqrt{f_{\varepsilon,N,1}(x)} + \sqrt{f_{\varepsilon,N,2}(x)})^2}\, dx \\ &\le\; C\left(N^{-1} \wedge |\gamma(\omega_N)|^2\right) \int \left(\frac{\sin(\omega_N x)\sin(x)}{x} \right)^2 dx \\ &=\; O\left(N^{-1} \wedge |\gamma(\omega_N)|^2\right). \end{aligned}$$

We define for two densities f and g the Hellinger affinity $\rho(f,g) = \int \sqrt{f}\sqrt{g}$. Note that we have, according to (2.1) and (2.2), two sets of observations, which possibly provide information about f_X. Let f_i denote the joint density of $Y_1, \ldots, Y_n, \varepsilon_1^*, \ldots, \varepsilon_N^*$ under $f_{\varepsilon,N,i}$ and $f_{X,N,i}$. Since the observations Y_j and ε_j^* are independent, we obtain that

$$\begin{aligned} \rho(f_1, f_2) \;&=\; \rho\left((f_{\varepsilon,N,1} * f_{X,N,1})^{[n]}, (f_{\varepsilon,N,2} * f_{X,N,2})^{[n]}\right) \rho\left(f^{[N]}_{\varepsilon,N,1}, f^{[N]}_{\varepsilon,N,2}\right) \\ &=\; \rho^N\left(f_{\varepsilon,N,1}, f_{\varepsilon,N,2}\right) \\ &=\; \left(1 \;-\; \frac{1}{2}H^2(f_{\varepsilon,N,1}, f_{\varepsilon,N,2})\right)^N \\ &\ge\; (1 \;-\; C/N)^N \;\ge\; C \;>\; 0. \end{aligned} \tag{5.3}$$

(iv) A lower bound for the risk

On the other hand, we obtain for any estimator $\overline{f}_X$ that

$$\begin{aligned}
\rho(f_1, f_2) &\leq \int \frac{\|\overline{f}_X(y) - f_{X,N,1}\|}{\|f_{X,N,1} - f_{X,N,2}\|} \sqrt{f_1(y)}\sqrt{f_2(y)}\, dy \\
&\quad + \int \frac{\|\overline{f}_X(y) - f_{X,N,2}\|}{\|f_{X,N,1} - f_{X,N,2}\|} \sqrt{f_1(y)}\sqrt{f_2(y)}\, dy \\
&\leq \sqrt{\|f_{X,N,1} - f_{X,N,2}\|^{-2} \int \|\overline{f}_X(y) - f_{X,N,1}\|^2 f_1(y)\, dy} \\
&\quad + \sqrt{\|f_{X,N,1} - f_{X,N,2}\|^{-2} \int \|\overline{f}_X(y) - f_{X,N,2}\|^2 f_2(y)\, dy}. \qquad (5.4)
\end{aligned}$$

From (5.2) through (5.4) we obtain that

$$\begin{aligned}
&\inf_{\overline{f}_X} \max\left\{E_{f_{\varepsilon,N,1}, f_{X,N,1}} \|\overline{f}_X - f_{X,N,1}\|^2, E_{f_{\varepsilon,N,2}, f_{X,N,2}} \|\overline{f}_X - f_{X,N,2}\|^2\right\} \\
&\geq C\|f_{X,N,1} - f_{X,N,2}\|^2 \\
&\geq C\left(N^{-1}|\gamma(\omega_N)|^{-2} \wedge 1\right)(1 + |\omega_N|)^{-2\beta},
\end{aligned}$$

which proves the theorem. □

Proof of Corollary 3.1. The assertion follows from Theorem 2.1, which holds uniformly in f_X, Theorem 3.1 and the assumption that

$$\sup_{f_X \in \mathcal{F}_X, f_\varepsilon \in \widetilde{\mathcal{F}}_\varepsilon} \left\{E\|\widehat{f}_X - f_X\|^2\right\} \leq C \inf_{\overline{f}_X} \sup_{f_X \in \mathcal{F}_X, f_\varepsilon \in \widetilde{\mathcal{F}}_\varepsilon} \left\{E\|\overline{f}_X - f_X\|^2\right\}.$$

□

Proof of Theorem 3.2. Although the proof in Fan (1993) was given for Hölder classes and L_p-risk on a compact interval rather than for Bessel-potential classes and L_2-risk on the real line, we can derive a lower bound using the same calculations as in Fan's proof. An upper bound for the L_2-risk on the real line and smoothness classes with an integrated Hölder modulus of continuity was derived in Fan (1991b). It can be easily shown that analogous rates are valid in our Bessel-potential classes. □

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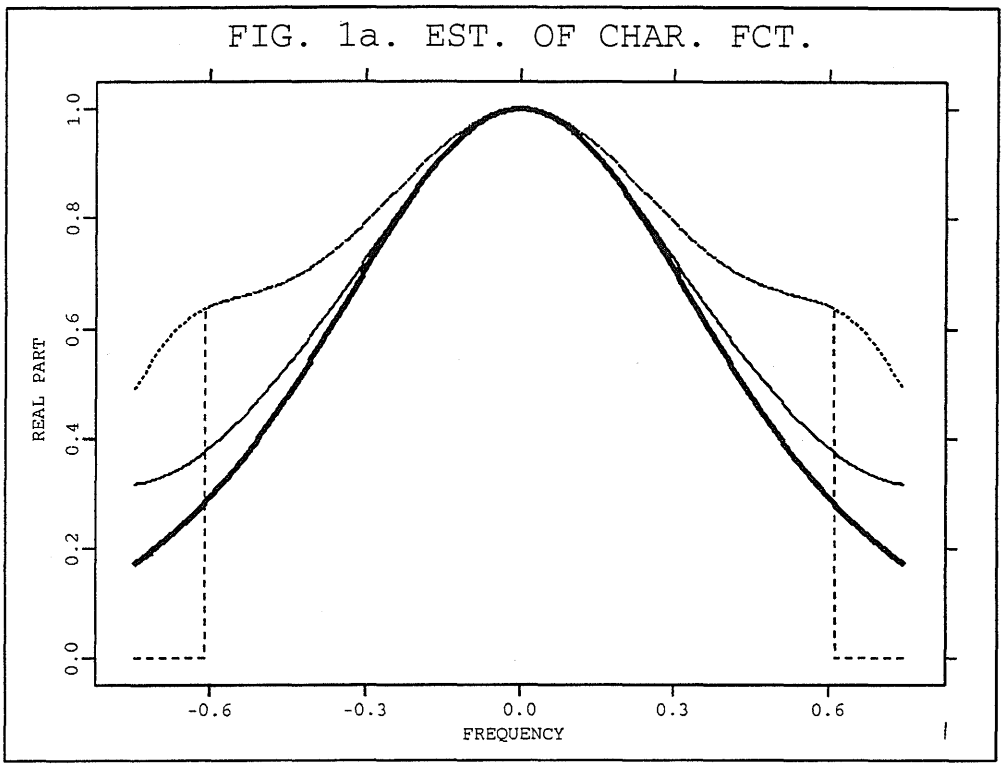


FIG. 1a. EST. OF CHAR. FCT.
REAL PART
0.0
0.2
0.4
0.6
0.8
1.0
-0.6
-0.3
0.0
0.3
0.6
FREQUENCY

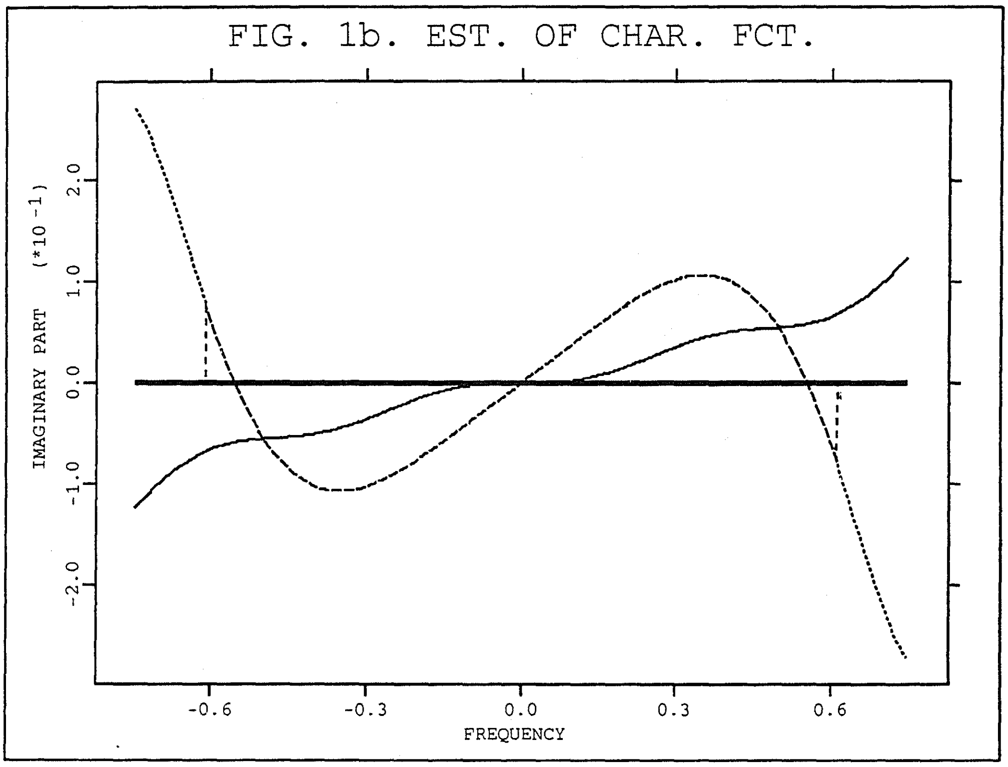


FIG. 1b. EST. OF CHAR. FCT.
IMAGINARY PART (*10 -1)
2.0
1.0
0.0
-1.0
-2.0
-0.6
-0.3
0.0
0.3
0.6
FREQUENCY

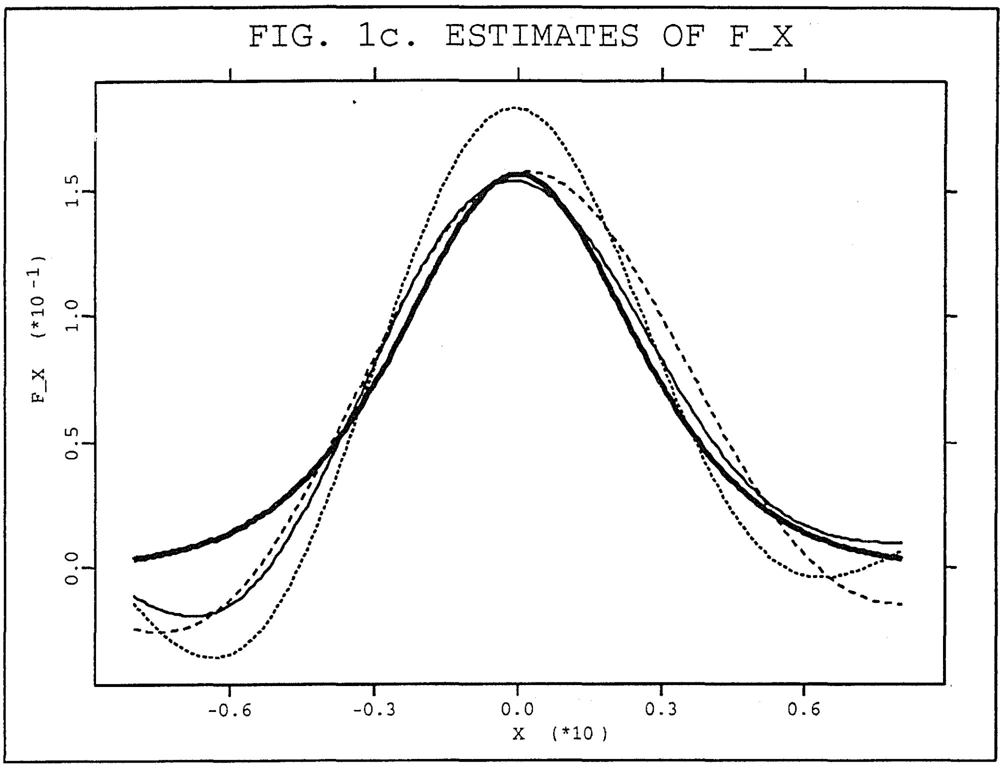
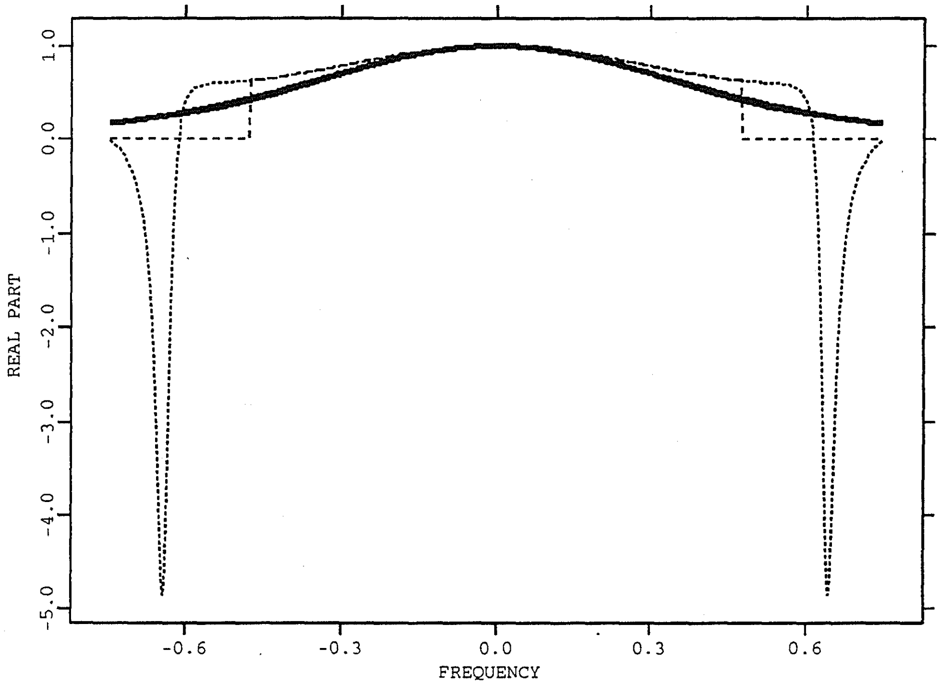


FIG. 1c. ESTIMATES OF F_X
F_X (*10 -1)
1.5
1.0
0.5
0.0
-0.6
-0.3
0.0
0.3
0.6
X (*10)

FIG. 2a. EST. OF CHAR. FCT.



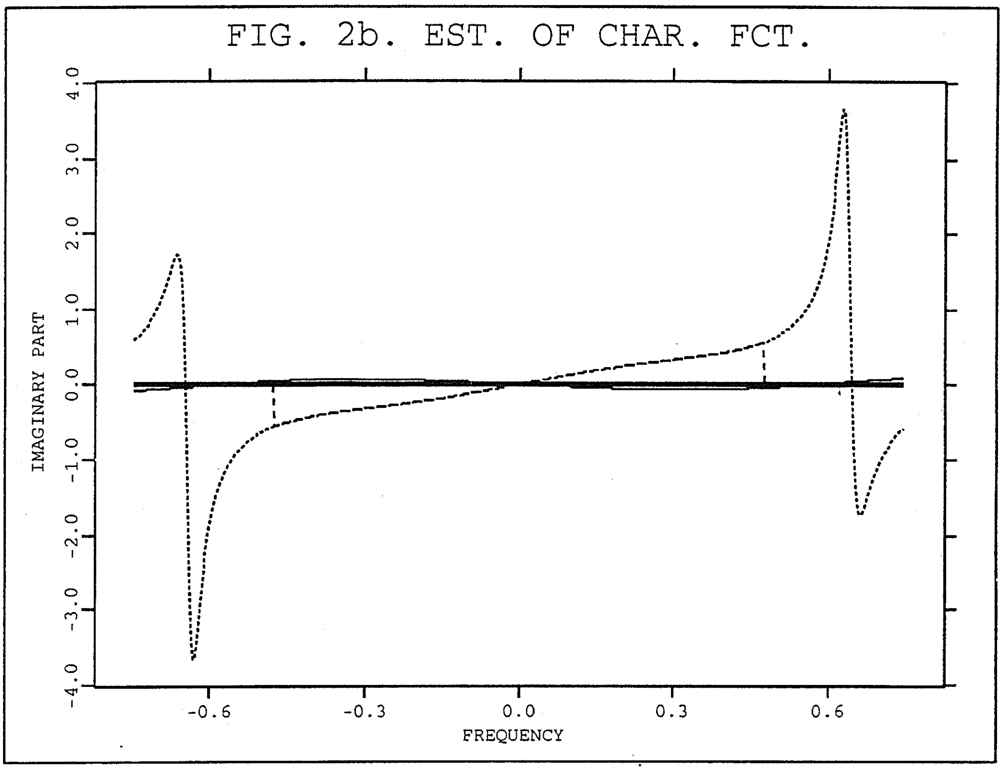


FIG. 2b. EST. OF CHAR. FCT.
IMAGINARY PART
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-0.6
-0.3
0.0
0.3
0.6
FREQUENCY

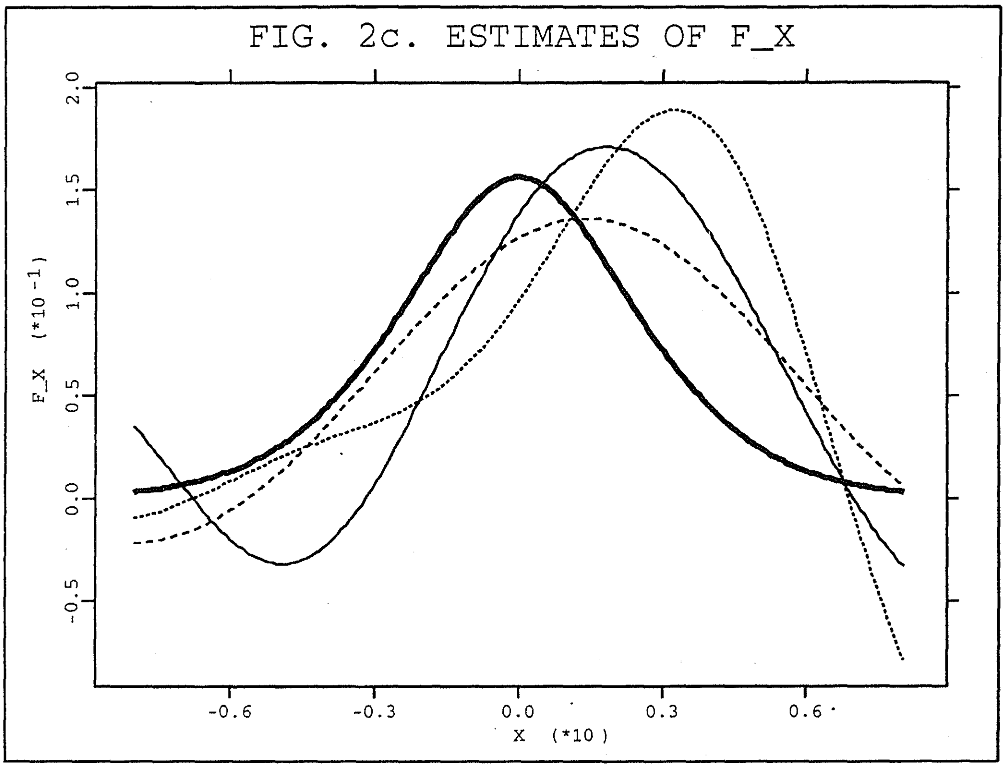


FIG. 2c. ESTIMATES OF F_X
F_X (*10 -1)
2.0
1.5
1.0
0.5
0.0
-0.5
-0.6
-0.3
0.0
0.3
0.6
X (*10)

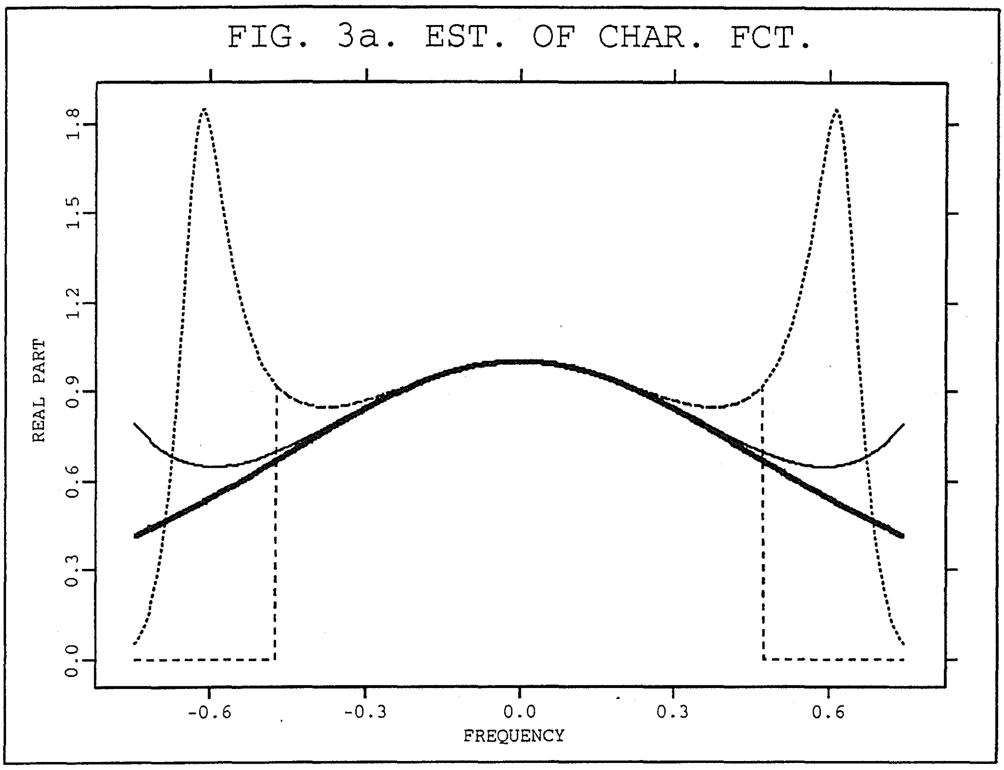


FIG. 3a. EST. OF CHAR. FCT.
REAL PART
0.0
0.3
0.6
0.9
1.2
1.5
1.8
-0.6
-0.3
0.0
0.3
0.6
FREQUENCY

FIG. 3b. EST. OF CHAR. FCT.

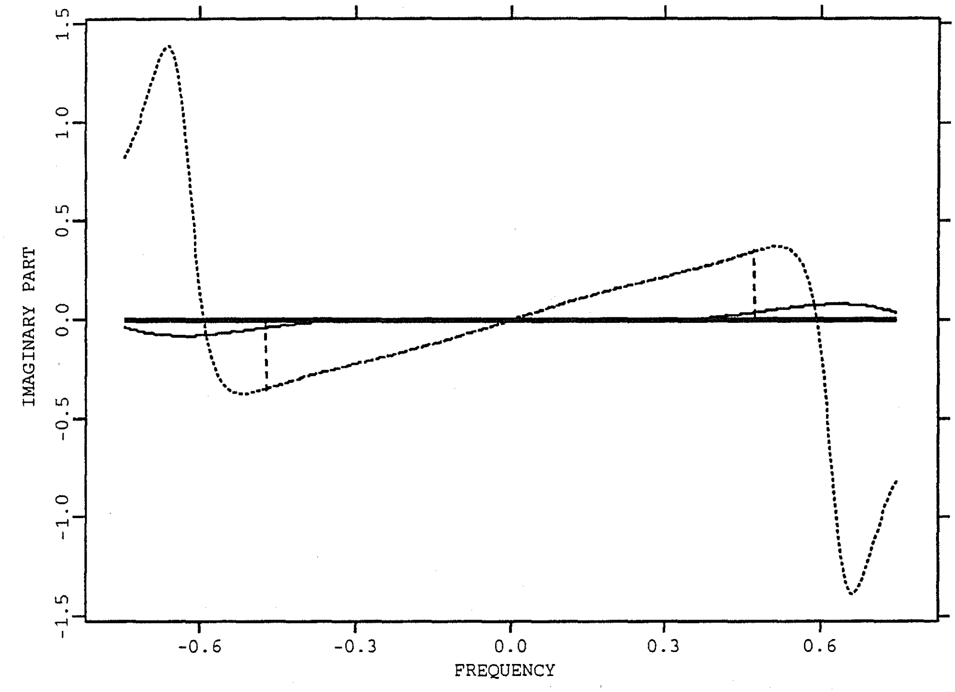
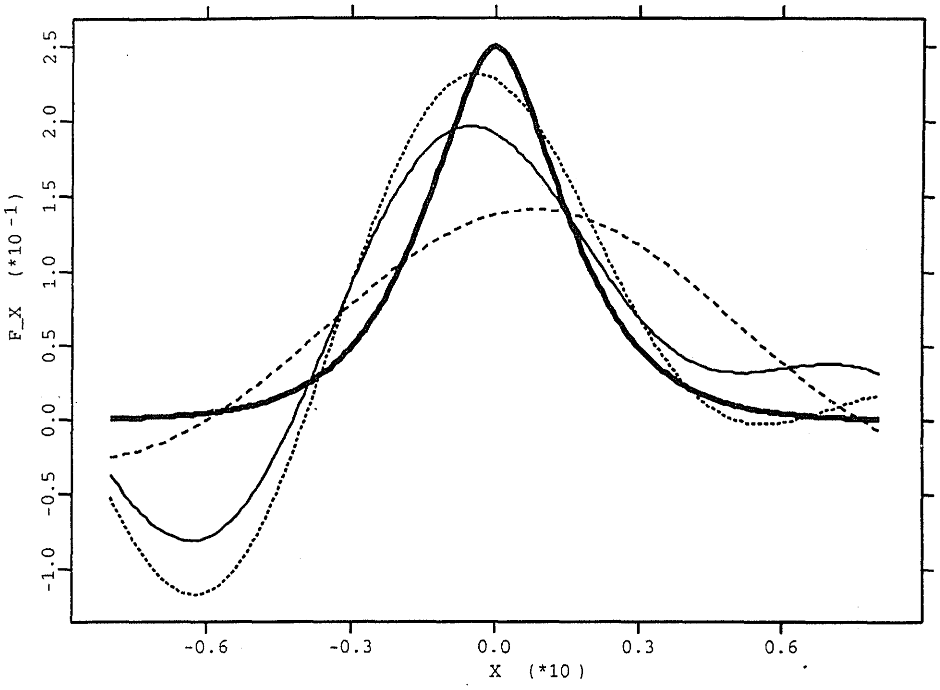


FIG. 3c. ESTIMATES OF F_X



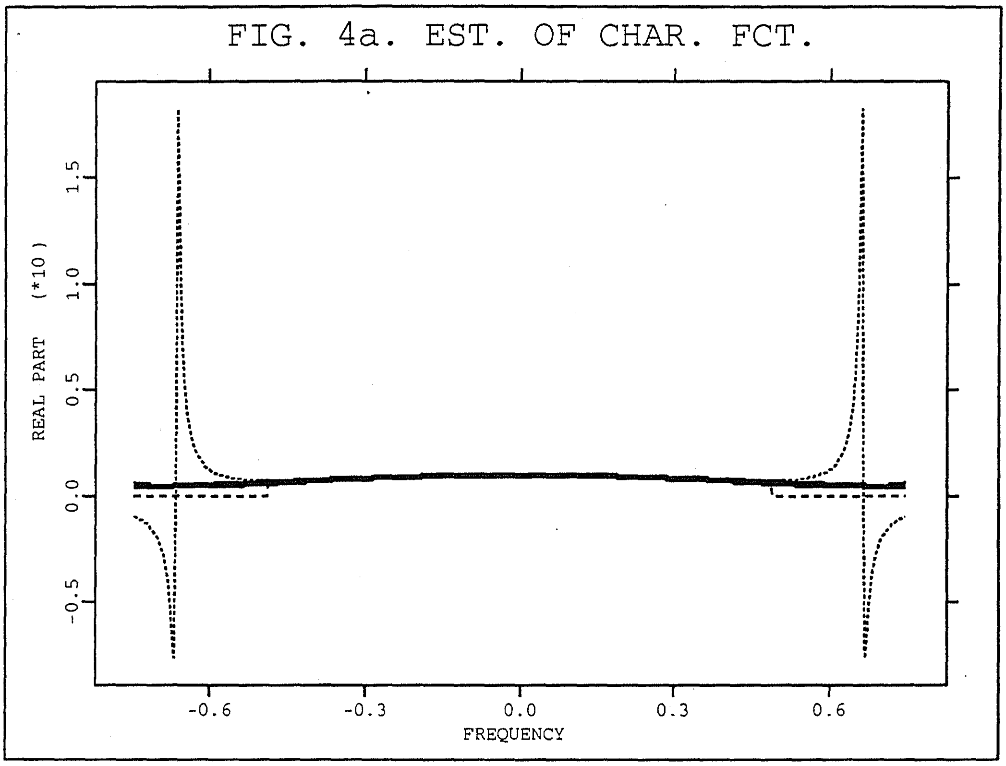


FIG. 4a. EST. OF CHAR. FCT.
REAL PART (*10)
-0.5
0.0
0.5
1.0
1.5
-0.6
-0.3
0.0
0.3
0.6
FREQUENCY

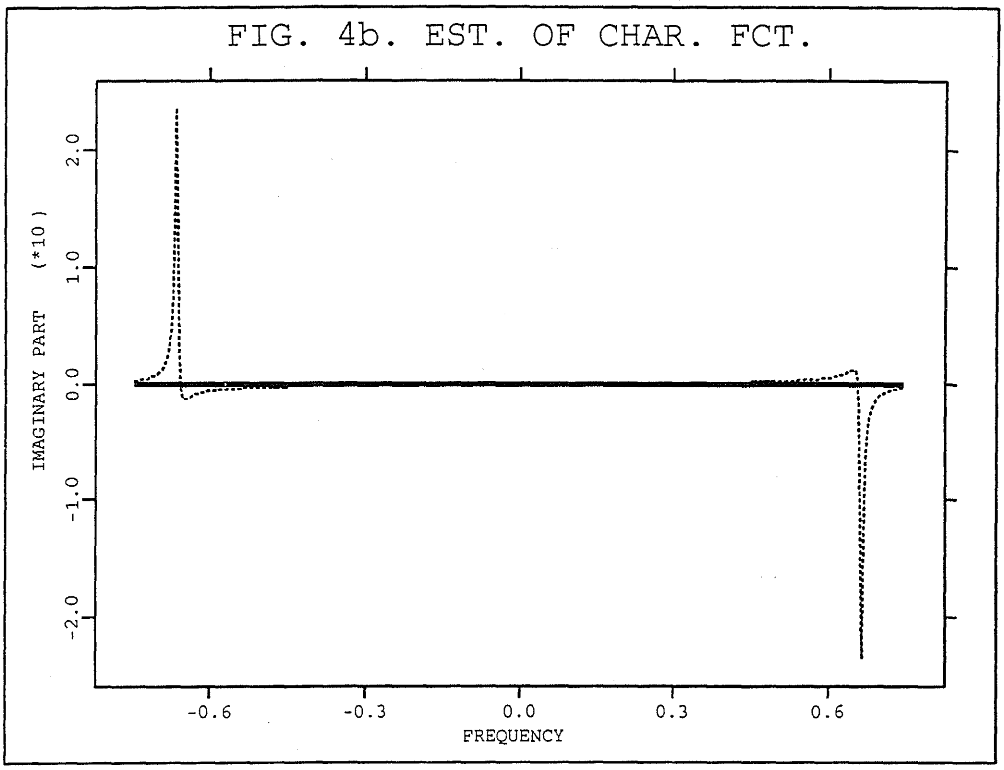
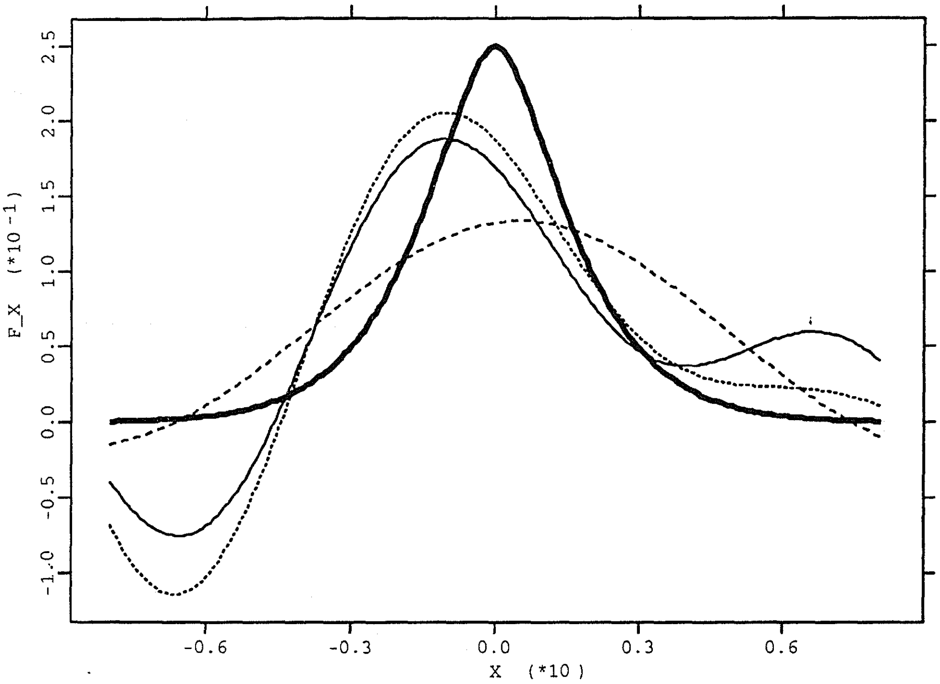


FIG. 4b. EST. OF CHAR. FCT.
IMAGINARY PART (*10)
2.0
1.0
0.0
-1.0
-2.0
-0.6
-0.3
0.0
0.3
0.6
FREQUENCY

FIG. 4c. ESTIMATES OF F_X



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